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Lison JACOBONI

Propriétés métriques et probabilistes des groupes métabéliens

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Après avis des rapporteurs :
SÉBASTIEN GOUËZEL (CNRS & Université de Nantes)
LAURENT SALOFF-COSTE (Cornell University)

Jury de soutenance :

JÉRÉMIE BRIEUSSEL	(Université de Montpellier) Examinateur
YVES DE CORNULIER	(CNRS & Université Lyon 1) Codirecteur de thèse
ANNA ERSCHLER	(ENS) Examinatrice
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SÉBASTIEN GOUËZEL	(CNRS & Université de Nantes) Rapporteur
ROMAIN TESSERA	(CNRS & Université Paris-Sud) Directeur de thèse

Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8628), Bât. 425
Université Paris-Sud 11
91405 Orsay CEDEX

Propriétés métriques et probabilistes des groupes métabéliens de type fini

Lison Jacoboni

Résumé

Dans la première partie de cette thèse, on s'intéresse au comportement asymptotique de la probabilité de retour de la marche aléatoire dans les groupes métabéliens de type fini. Pour de tels groupes à croissance exponentielle, on obtient une caractérisation purement algébrique de ceux dont la probabilité de retour est la plus grande possible, c'est-à-dire équivalente à $\exp(-n^{\frac{1}{3}})$: la condition est de ne pas admettre de sous-groupe isomorphe à un groupe métabélien libre avec torsion de type fini, ou à un produit en couronne $\mathbb{Z} \wr \mathbb{Z}$. Une reformulation naturelle de cette caractérisation fait intervenir la dimension de Krull du groupe.

Dans ce but, on démontre pour les groupes métabéliens une version du fameux théorème de plongement de Kaloujnine et Krasner respectant la dimension de Krull. On est alors capable de construire des suites de couples de Følner dans un sur-groupe scindé, avec pour conséquence des bornes inférieures sur la probabilité de retour en fonction de la dimension de Krull.

Enfin, on met en évidence des sections spécifiques de ces groupes dont on montre dans certains cas qu'une puissance contient un groupe métabélien libre de type fini. Cela permet également de donner des bornes supérieures sur la probabilité de retour ne dépendant que de la dimension de Krull.

La seconde partie de cette thèse porte sur les profils isopérimétriques des groupes localement compacts compactement engendrés, qu'on utilise pour formuler une caractérisation de l'existence d'une suite de couples de Følner. On montre que le profil isopérimétrique augmente lorsque l'on passe au quotient et on obtient des constantes indépendantes de l'échelle choisie, améliorant des résultats précédents de Tessera. On en déduit que l'existence de couples de Følner passe au quotient par un sous-groupe normal fermé.

On établit également que l'existence de couples de Følner passe à un sous-groupe fermé, généralisant un résultat dû à Erschler pour les groupes de type fini. Cela permet d'obtenir une preuve plus auto-contenue du théorème principal de la première partie.

La troisième partie de cette thèse est issue d'un travail en commun avec Peter Kropholler. On y étudie la structure des groupes résolubles de rang sans torsion infini n'ayant pas de section isomorphe au produit en couronne de deux groupes cycliques infinis. On obtient comme corollaire qu'un groupe de type fini résoluble admettant une dimension de Krull et sans section isomorphe à $\mathbb{Z} \wr \mathbb{Z}$ est nécessairement de rang sans torsion fini.

Abstract

In the first part of this thesis, we investigate the asymptotic behaviour of the return probability of the random walk in finitely generated metabelian groups. For such groups with exponential volume growth, we obtain a characterization of metabelian groups whose return probability is the largest, namely equivalent to $\exp(-n^{\frac{1}{3}})$, in purely algebraic terms. The condition is not to admit a subgroup isomorphic either to a finitely generated free metabelian group with torsion or to a wreath product $\mathbb{Z} \wr \mathbb{Z}$. A natural reformulation of this characterization involves the Krull dimension of the group.

In order to do this, we elaborate on a famous embedding theorem of Kaloujnine and Krasner and establish a variation for metabelian groups that respects the Krull dimension. This allows to build sequences of Følner couples in a split overgroup, thus providing lower bounds for the return probability according to the Krull dimension.

Finally, we highlight and study specific sections of these groups, and show in certain cases that a power contains a finitely generated free metabelian group. As a consequence, we give upper bounds in terms of the Krull dimension on the return probability.

The second part of this thesis deals with isoperimetric profiles in locally compact compactly generated groups, that we use to characterize the existence of a sequence of Følner couples. We prove that the isoperimetric profil increases when going to a quotient in a uniform manner with respect to the scale, improving on a theorem of Tessera. From this, we deduce that the existence of a sequence of Følner couples goes to a quotient by a closed normal subgroup.

We also establish that a sequence of Følner couples persists in a closed subgroup, generalizing the corresponding result by Erschler for finitely generated groups. This allows to obtain a more self-contained proof of the main result of the first part of this thesis.

In the third chapter, we examine how the structure of soluble groups of infinite torsion-free rank with no section isomorphic to the wreath product of two infinite cyclic groups can be analysed. As a corollary, we obtain that if a finitely generated soluble group has a defined Krull dimension and has no sections isomorphic to the wreath product of two infinite cyclic groups, then it is a group of finite torsion-free rank.

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Introduction

"La cinquième planète était très curieuse. C'était la plus petite de toutes. Il y avait là juste assez de place pour loger un réverbère et un allumeur de réverbères. Le petit prince ne parvenait pas à s'expliquer à quoi pouvaient servir, quelque part dans le ciel, sur une planète sans maison, ni population, un réverbère et un allumeur de réverbères. "

- Antoine de Saint-Exupéry, *Le Petit Prince*

1. Groupes

Structure algébrique fondamentale de l'algèbre générale, la notion de groupe a émergé entre la fin du $XVIII^{\text{e}}$ et le début du XIX^{e} siècle. Elle trouve son origine dans l'étude des équations algébriques par Lagrange et Ruffini, étude développée ensuite par Galois qui donna un critère de résolubilité d'équations polynomiales particulières en matière de groupe de symétrie de leurs racines. Au-delà de ces études spécifiques, il faut sans doute attribuer à Cayley, en 1854, la définition abstraite d'un groupe telle que nous la connaissons aujourd'hui.

De la géométrie à la théorie des nombres, en passant, par la topologie et la théorie ergodique, les groupes sont apparus depuis dans de nombreux domaines mathématiques.

La théorie géométrique des groupes trouve son origine dès la fin des années 1960, au travers de nombreux travaux parmis lequels ceux de Milnor, Stallings, Mostow, Abel, Tits, Gromov. L'idée clef est d'étudier les groupes infinis comme des objets géométriques [1], avec notamment pour but d'établir des connexions entre les propriétés algébriques de ces groupes et les propriétés topologiques et géométriques des espaces sur lesquels ils opèrent.

1.1. Groupe vu comme espace métrique. Considérons un groupe Γ et donnons-nous une partie génératrice symétrique S de ce groupe. La *distance des mots* associée à S, notée d_S, est la métrique définie de la façon suivante : si x, y sont deux éléments du groupe, $d_S(x, y)$ est égal à la longueur du mot $x^{-1}y$ dans S. Le *graphe de Cayley* associé, noté $\mathrm{Cay}(\Gamma, S)$, est le graphe dont les sommets sont les éléments de G et dont les arêtes sont les couples $(x, xs) \in \Gamma^2$ avec $x \in \Gamma$ et $s \in S$. Si l'on donne à chaque arête la longueur 1, la métrique des mots d_S coïncide sur Γ avec la métrique géodésique sur le graphe de Cayley $\mathrm{Cay}(G, S)$.

La notion de quasi-isométrie fournit un cadre dans lequel cette métrique ne dépend pas de S et nous dit, informellement, que les propriétés de "grande échelle" du groupe ne dépendent pas de la partie génératrice à travers laquelle on le regarde.

1. C'est d'ailleurs le titre de [**Gro84**].

Définition 1.1.1 (Quasi-isométrie). Soient (X, d) et (Y, d') deux espaces métriques. Une application $\Phi : X \to Y$ est un *plongement quasi-isométrique de X dans Y* s'il existe des constantes $\lambda \geqslant 1$ et $C \geqslant 0$ telles que, pour tous $x, y \in X$,

$$\frac{1}{\lambda} d(x, y) - C \leqslant d'(\Phi(x), \Phi(y)) \leqslant \lambda d(x, y) + C.$$

L'application Φ est une *quasi-isométrie* s'il existe de plus une constante $D \geqslant 0$ telle que tout point de Y soit à distance au plus D de $\Phi(X)$.

S'il existe une quasi isométrie entre X et Y, ces deux espaces sont dits *quasi-isométriques.* C'est une relation d'équivalence entre espaces métriques.

Dans le cas d'un groupe de type fini Γ, muni d'une partie génératrice finie et symétrique S, l'injection de (Γ, d_S) dans son graphe de Cayley est une quasi-isométrie. De plus, si S' est une autre partie génératrice de Γ alors l'application identité de (Γ, d_S) dans $(\Gamma, d_{S'})$ est également une quasi-isométrie.

En conséquence, dans un groupe de type fini Γ, la métrique des mots est unique à quasi-isométrie près et toute propriété invariante par quasi-isométrie est en fait une propriété du *groupe* Γ.

On décrit dans le paragraphe suivant une famille d'exemples de groupes de type fini qui s'avérera centrale dans cette thèse.

1.2. Groupe de l'allumeur de réverbères, produit en couronne. Le groupe dit *de l'allumeur de réverbères* est le produit semi-direct G de $\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ par $\mathbb{Z}$, où $\mathbb{Z}$ agit par décalage. Ses éléments peuvent se représenter de la manière suivante : (η, y), où $\eta : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ est une fonction à support fini et $y \in \mathbb{Z}$. La loi de G est donnée par

$$(\eta, y)(\delta, z) = (\eta + \delta^y, y + z),$$

où $\delta^y(x) = \delta(x - y)$.

Ce groupe est engendré par $e = (0, 1), e^{-1}$ et $\mathbb{1} = (\delta_0, 0)$, où δ_0 est la fonction égale au générateur de $\mathbb{Z}/2\mathbb{Z}$ en 0 et nulle partout ailleurs.

Les fonctions à support fini de $\mathbb{Z}$ dans $\mathbb{Z}/2\mathbb{Z}$ correspondent aux polynômes à coefficients dans $\mathbb{Z}/2\mathbb{Z}$ en une indéterminée X inversible et le groupe G est également décrit par $G = \mathbb{F}_2[X, X^{-1}] \rtimes \mathbb{Z}$ où $\mathbb{Z}$ agit par multiplication par les puissances de X.

Le nom de ce groupe s'explique par l'image suivante : si l'on considère une allée bi-infinie de réverbères, un élément du groupe peut se voir comme la donnée de la configuration de cette allée (avec un nombre fini de lampes allumées) et de la position d'un allumeur de réverbères. La multiplication à droite par les éléments de S peut faire bouger l'allumeur d'un pas vers la gauche ou la droite, ou bien changer l'état de la lampe devant laquelle il est situé.

Plus généralement, le *produit en couronne* de deux groupes A et B, noté $A \wr B$ est le produit semi-direct

$$\bigoplus_B A_b \rtimes B$$

où les A_b sont des copies de A indexées par $b \in B$ et B agit par translation à gauche [2]. Un élément du groupe s'écrit (f, b) avec $f : B \to A$ une application à support fini et $b \in B$. La multiplication est donnée par : si $(f, b), (g, c) \in A \wr B$, alors

$$(f, b)(h, c) = (fh^b, bc)$$

où $h^b : B \to A$ est définie par $h^b(x) = h(b^{-1}x)$ pour tout $x \in A$.

Soit S_A une partie génératrice de A etS_B une partie génératrice de B. Une partie génératrice de $A \wr B$ est donnée par $\{(a\delta_e, 0); a \in S_A\} \cup \{(0, b); b \in S_B\}$.

En considérant un produit plutôt qu'une somme directe, on définit le produit en couronne *non restreint* de A par B : c'est le groupe

$$\prod_{b \in B} A_b \rtimes B,$$

qu'on notera $A \wr\wr B$.

Avec cette notation, le groupe dit de l'allumeur de réverbères s'écrit $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. Dans ce texte, on verra apparaître des produits en couronne de la forme $C \wr \mathbb{Z}^k$, avec C un groupe cyclique (fini ou infini), par exemple : l'allumeur de réverbères marchera donc sur un espace de dimension plus grande, et l'intensité des lampes allumées pourra varier.

Dans cette thèse, on s'intéresse aux propriétés métriques et probabilistes de certains groupes de type fini, les groupes métabéliens, possédant une structure particulière. À l'occasion de la définition de cette famille, on redonne quelques définitions classiques à propos de la structure des groupes.

1.3. Structure. Soit G un groupe. La *suite dérivée* de G est la suite décroissante de sous-groupes caractéristiques de G définie par récurrence de la façon suivante :

$$G^{(0)} = G \text{ et pour tout entier } i \geqslant 0, G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

Si cette suite dérivée est stationnaire à $\{e\}$, le groupe G est dit *résoluble*. Le plus petit entier naturel n tel que $G^{(n)} = \{e\}$ est la *classe de résolubilité* de G. Les groupes 1-résolubles sont les groupes abéliens, les groupes 2-résolubles sont aussi appelés *groupes métabéliens*.

La *suite centrale descendante* de G est la suite décroissante de sous-groupes caractéristiques de G définie par récurrence de la façon suivante :

$$G_1 = G \text{ et pour tout entier } i \geqslant 1, G_{i+1} = [G, G_i].$$

Si cette suite atteint $\{e\}$, le groupe G est dit *nilpotent*. Le plus petit entier naturel n tel que $G_{n+1} = \{e\}$ est la *classe de nilpotence* de G. Les groupes nilpotents sont résolubles.

Si $\mathfrak{P}$ est une propriété de groupe, un groupe G est dit *virtuellement* $\mathfrak{P}$ s'il admet un sous-groupe d'indice fini avec la propriété $\mathfrak{P}$.

Si $\mathfrak{P}$ et $\mathfrak{Q}$ sont deux propriétés de groupes, un groupe G est dit $\mathfrak{P}$-*par*-$\mathfrak{Q}$ s'il admet un sous-groupe normal N avec la propriété $\mathfrak{P}$ tel que le quotient G/N ait

2. Dans ce texte, on écrira toujours le groupe agissant à droite, il est à noter qu'il apparaît parfois à gauche chez certains auteurs.

la propriété $\mathfrak{Q}$. Par exemple, les groupes métabéliens sont abélien-par-abélien et un théorème de Mal'cev entraîne que les groupes linéaires résolubles sont virtuellement nilpotent-par-abélien [**Mal51**].

Un groupe G est dit *poly*-$\mathfrak{P}$ s'il existe une suite finie de sous-groupes

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

dont les facteurs G_{i+1}/G_i ont la propriété $\mathfrak{P}$. Par exemple, les groupes résolubles sont poly-abéliens.

On écrira *groupe polycyclique* pour désigner un groupe poly-cyclique. Cette classe de groupes contient les groupes nilpotents de type fini et est contenue dans la classe des groupes résolubles. On rappelle que les inclusions sont strictes : le groupe $\mathbb{Z}^2 \rtimes \mathbb{Z}$, où $\mathbb{Z}$ agit via la matrice

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

est polycyclique mais non nilpotent et le groupe de l'allumeur de réverbère est résoluble mais non polycyclique. Plus précisément, un groupe G est polycyclique si et seulement s'il est résoluble et vérifie la *condition maximale*, notée *max* (c'est-à-dire que toute suite croissante de sous-groupes de G est stationnaire ; on dit également que le groupe est *noethérien*).

Soit G, H deux groupes. On dit que H est une *section* de G s'il existe un sous-groupe K de G tel que K se surjecte sur H.

1.4. Groupes métabéliens. La première partie de cette thèse porte sur l'étude de la probabilité de retour des groupes métabéliens, définis ci-dessus comme étant les groupes 2-résolubles. De manière équivalente, un groupe G est métabélien si et seulement si son groupe dérivé $[G, G]$ est abélien, ou bien encore si et seulement s'il est une extension d'un groupe abélien par un autre.

1.4.1. *L'approche de Hall.* Une étude méthodique de ces groupes a été entreprise dès 1954 par Hall ([**Hal54**]).

Si G est un groupe métabélien, il existe deux groupes abéliens M et Q tels que G s'inscrive dans la suite exacte

$$M \hookrightarrow G \twoheadrightarrow Q.$$

On peut prendre par exemple pour M le groupe dérivé $[G, G]$ et pour Q l'abélianisé $G_{ab} = G/[G, G]$ de G.

L'observation de Hall est la suivante : l'action de G par conjugaison sur son sous-groupe normal M est en fait une action du quotient Q. Plus précisément, l'action d'un élément g de G ne dépend que de son image dans le quotient : si g' est un autre élément de G tel que $g' = gm$, avec $m \in M$, alors pour tout $x \in M$, on a $g'.x = g'xg'^{-1} = gxg^{-1} = g.x$ puisque le groupe M est abélien. On en déduit une action de l'anneau du groupe $\mathbb{Z}Q$ sur M : pour tous $x \in M$, $a_1, \ldots, a_k \in \mathbb{Z}$ et $g_1, \ldots g_k \in G$,

$$\left(\sum_{i=1}^{k} a_i g_i Q\right).x = \sum_{i=1}^{k} a_i (g_i x g_i^{-1}).$$

Ceci munit M d'une structure de $\mathbb{Z}Q$-module. Lorsque G est de type fini, Q est également de type fini et M est de type fini comme $\mathbb{Z}Q$-module. Une autre formulation est que M est *normalement de type fini dans G*, c'est-à-dire qu'il existe une partie finie de M telle que le sous-groupe normal qu'elle engendre soit M tout entier. Dans ce cas, l'anneau $\mathbb{Z}Q$ est *noethérien*, tout comme le module M, on reviendra sur cette notion dans le paragraphe suivant.

Ce point de vue crée une passerelle avec l'algèbre commutative classique, dont on peut espérer utiliser les théorèmes pour mieux comprendre la structure des groupes métabéliens de type fini. Il s'est révélé très fécond et a donné lieu à un grand nombre de résultats parmi lesquels :

(1) les groupes métabéliens de type fini satisfont la condition maximale pour les sous-groupes normaux. C'est une conséquence de la noethérianité de M.

(2) Tout groupe métabélien G de type fini est de présentation finie dans la catégorie des groupes métabéliens, c'est-à-dire que G peut être défini par un nombre fini de générateurs et de relations, à ajouter aux relations de la forme $[[x, y], [z, w]] = 1$. Par conséquent, il n'y a qu'un nombre dénombrable de classes d'isomorphismes de groupes métabéliens de type fini. De plus, bien qu'ils ne soient pas tous de présentation finie, ils se plongent tous dans un groupe métabélien de présentation finie.

(3) Les groupes métabéliens de type fini se comportent bien algorithmiquement : les problèmes du mot et de la conjugaison sont résolubles.

(4) Bieri et Strebel ont introduit un invariant géométrique permettant de distinguer les groupes métabéliens de présentation finie des autres ([**BS80, BS81**]).

Les deux premiers points sont dus à Hall. On renvoie à [**BMO12**, § 2] ou au livre [**LR04**] pour une liste plus complète, et un traitement détaillé.

1.4.2. *Noethérianité.* On revient dans cette partie sur la notion de module noethérien, qui apparaît dans un groupe métabélien de type fini, d'après l'observation de Hall.

Si A est un anneau commutatif et unitaire, un A-module M est dit *noethérien* s'il satisfait la condition de chaîne ascendante, à savoir que toute chaîne strictement croissante de sous-modules de M est finie. De façon équivalente, tous les sous-modules de M sont de type fini. Cette condition est l'une des propriétés introduites par Noether dans les années 20 et nommée ensuite en son honneur [3], l'utilisation adroite de celle-ci lui ayant permis d'obtenir des résultats puissants comme le théorème dit de Lasker-Noether, et d'aborder sous de nouvelles perpectives des questions de théorie de l'élimination (on peut citer entre autres [**Noe21, Noe23**]). Les conditions de chaines descendantes, elles aussi considérées par Noether, sont dites *artiniennes*. On dit qu'un groupe satisfait la propriété *min* si l'ensemble de ses sous-groupes est artinien.

Dans cette thèse, on fait grand usage de la structure des modules noethériens, détaillée dans la proposition suivante, et de son impact sur la structure d'un groupe métabélien.

3. Comme la plupart des conditions de chaîne ascendantes.

Proposition 1.4.1. *Soient R un anneau noethérien et M un R-module de type fini. Alors, il existe des sous-modules $M_0, M_1, \ldots, M_n$ de M tels que*

$$M = M_n > M_{n-1} > \cdots > M_1 > M_0 = 0 \tag{1.1}$$

et $M_{i+1}/M_i \simeq R/\mathcal{P}_i$, où $\mathcal{P}_i$ est un idéal premier de R.

1.4.3. *Groupes métabéliens (p)-libres.* Notre recherche de “briques de base” pour les groupes métabéliens nous amènera à considérer de tels groupes les plus libres possibles. Pour tout entier $d \geqslant 2$, le *groupe métabélien libre de rang d*, noté B_d est le quotient du groupe libre de rang d, F_d, par le second terme de sa série dérivée $F_d^{(2)} = [[F_d, F_d], [F_d, F_d]]$:

$$B_d = F_d/F_d^{(2)}.$$

De façon similaire, pour tout $n \in \mathbb{N}^*$, on définit le *groupe n-métabélien libre de rang d*, noté $B_d^{[n]}$ comme le groupe métabélien de rang d le plus libre possible et le groupe dérivé est d'exposant n, c'est-à-dire

$$B_d^{[n]} = B_d/[B_d, B_d]^n.$$

1.5. Groupes minimax. Si z est un nombre complexe non nul, $\mathbb{Z}\,[1/z]$ désigne le sous-anneau de $(\mathbb{C}, +)$ formé des nombres de la forme $\dfrac{n}{z^m}$, avec $n \in \mathbb{Z}, m \in \mathbb{N}$. Un groupe G est *quasi-cyclique* s'il existe un premier p tel que $G \simeq \mathbb{Z}[1/p]/\mathbb{Z}$. Ce dernier groupe est noté C_{p^∞}, c'est l'image par le morphisme $x \mapsto \exp(2i\pi x)$ de $\mathbb{Z}[1/p]$.

Un groupe max-par-min est appelé *minimax.* Par extension, on désigne également par *minimax* la classe des groupes poly-minimax, c'est-à-dire qu'un groupe G est minimax s'il existe une suite finie de sous-groupes

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

dont les facteurs G_{i+1}/G_i sont cycliques ou quasi-cycliques.

Le produit en couronne $F \wr \mathbb{Z}$, avec F un groupe fini, n'est pas minimax. Le théorème suivant de Kropholler énonce que c'est, en quelque sorte, la seule alternative possible pour les groupes résolubles de type fini non minimax.

Théorème 1.5.1 ([**Kro84**]). *Soit G un groupe résoluble de type fini. Alors soit G est minimax, soit G admet une section isomorphe à $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}$, pour un premier p.*

Pour les groupes résolubles, on définit le *rang sans torsion* (aussi appelé *longueur de Hirsch*) par la formule

$$h(G) = \sum_{i \geq 0} \dim_{\mathbb{Q}}(G^{(i)}/G^{(i+1)}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Les groupes résolubles de type fini et de rang sans torsion fini ont un sous-groupe normal localement fini tel que le quotient soit minimax [**LR04**, 5.2.1].

2. Marche aléatoire sur un groupe, probabilité de retour à l'origine

Soit G un groupe dénombrable. Étant donnée une mesure de probabilité μ sur ce groupe, on peut considérer la marche aléatoire sur G associée à μ : il s'agit du processus aléatoire $(S_i)_{i\geqslant 0}$, à valeurs dans G, défini de la façon suivante : partant d'un point S_0 arbitraire, on a, pour tout entier naturel n, $S_{n+1} = S_n X_{n+1}$, où $(X_i)_{i\geqslant 1}$

est une suite de variables aléatoires indépendantes et identiquement distribuées, de loi μ.

Ainsi, si la marche aléatoire est en x, la probabilité d'être en y à l'étape suivante est égale à $\mu(x^{-1}y)$. Cela définit implicitement une mesure de probabilité $\mathbb{P}_\mu$ sur $G^{\mathbb{N}}$ par

$$\mathbb{P}_\mu(S_n = y \mid X_0 = x) = \mu^{(n)}(x^{-1}y),$$

où $\mu^{(n)}$ désigne la convolée n-ième de μ. Rappelons que si f et h sont deux fonctions définies sur G, leur convolée est la fonction $f * h$ définie sur G par, pour tout $g \in G$, $f * h(g) = \sum_k f(k)h(k^{-1}g)$.

Pour que la marche aléatoire dise quelque chose du groupe et non pas d'un ensemble plus petit, on demandera également que le support de la mesure μ engendre G comme semi-groupe.

Si G est un groupe de type fini dont S est une partie génératrice finie et symétrique, un exemple fondamental est donné par la marche aléatoire associée à la distribution uniforme μ_S sur S. Cette marche aléatoire correspond à la marche aléatoire simple sur le graphe de Cayley $\mathrm{Cay}(G, S)$.

2.1. Probabilité de retour. Dans tout ce texte, on notera

$$p_{2n}^{\mu,G} = \mathbb{P}_\mu(S_{2n} = e \mid X_0 = e)$$

la *probabilité de retour* à l'origine e de G au temps $2n$. Le choix de considérer des temps pairs est fait pour éviter tout problème de parité [4] : par exemple, la marche aléatoire simple sur $\mathbb{Z}$, associée à la partie génératrice usuelle $\{-1, 1\}$, ne peut atteindre 0 aux temps impairs.

2.1.1. *Comportement asymptotique.* Si φ, ψ désignent deux fonctions monotones, on utilise la notation $\varphi \preccurlyeq \psi$ s'il existe des constantes c and C strictement positives telles que $c\varphi(Ct) \leq \psi(t)$ (éventuellement pour t dans $\mathbb{N}$ si l'on considère des fonctions à valeurs dans $\mathbb{N}$). Si la relation symétrique $\varphi \succcurlyeq \psi$ est également vérifiée, on écrit $\varphi \sim \psi$ et on dit que φ et ψ ont le même comportement asymptotique. C'est une relation d'équivalence.

2.2. Stabilités. Soient S, T deux parties finies, symétriques et génératrices de Γ. On désigne par p_{2t}^S, respectivement p_{2t}^T, la probabilité de retour de la marche aléatoire simple sur $\mathrm{Cay}(\Gamma, S)$, respectivement $\mathrm{Cay}(\Gamma, T)$. La question suivante est alors naturelle : a-t-on $p_{2t}^S \sim p_{2t}^T$? L'idée que le comportement asymptotique de la probabilité de retour ne dépend pas de la partie génératrice choisie, et même plus généralement qu'il est invariant par quasi-isométrie, est en effet au coeur du travail de Varopoulos au début des années 1980 (on renvoie par exemple à [**Var85a, Var85b, Var86, Var91, VSCC92a**] et aux références qu'ils contiennent), et apparaît également dans [**BLP77**] où l'invariance du comportement transient ou récurrent est établi. Si ces travaux apportent une réponse positive pour plusieurs classes de groupes – auxquelles nous reviendrons dans la suite – la réponse positive ne fut

4. Une autre possibilité de remédier à ce problème serait d'imposer que le support de la mesure contienne l'élément neutre.

donnée qu'en 2000 par Pittet et Saloff-Coste [5]. Ils établissent plus généralement la stabilité de la classe d'équivalence de la probabilité de retour par quasi-isométrie, comme énoncé ci-dessous.

Une mesure μ sur un groupe G est dite *symétrique* lorsque $\mu(g) = \mu(g^{-1})$ pour tout $g \in G$.

THÉORÈME 2.2.1 ([**PSC00**]). *Soient Γ_1, Γ_2 deux groupes de type fini. Pour $i = 1, 2$, soit μ_i une mesure de probabilité symétrique à support fini générateur sur Γ_i. Si Γ_1 et Γ_2 sont quasi isométriques alors*

$$p_{2n}^{\mu_1,\Gamma_1} \sim p_{2n}^{\mu_2,\Gamma_2}.$$

REMARQUE 2.2.1. Le théorème reste vrai si l'on suppose seulement la mesure μ symétrique et admettant un moment d'ordre 2, c'est-à-dire si $\sum_{g\in\Gamma} \mu(g)|g|^2 < \infty$ où $|\cdot|$ désigne une métrique des mots quelconque sur Γ.

Ainsi, on peut parler de *la* probabilité de retour d'un groupe Γ de type fini pour désigner la classe d'équivalence de la probabilité de retour de la marche aléatoire simple sur un (n'importe quel) graphe de Cayley de Γ. Cette quantité ne dépend que de la classe de quasi-isométrie de Γ et on la notera désormais (p_{2n}^{Γ}).

Une autre conséquence de [**PSC00**] est que la probabilité de retour ne change pas lorsque l'on passe à un sous-groupe d'indice fini ou quotiente par un sous-groupe normal fini. En revanche, elle augmente en général lorsque l'on passe à un sous-groupe ou à un quotient.

La théorie des marches aléatoires recouvre différentes questions. La plus naturelle est de se demander, pour un groupe de type fini donné, comment se comporte la marche aléatoire. Le problème inverse consiste à déterminer quelles sont les valeurs possibles pour les quantités qui décrivent la marche aléatoire – comme la probabilité de retour bien sûr, ou encore le profil isopérimétrique (dont on parlera plus bas), mais également la vitesse et l'entropie – et à construire des groupes qui les atteignent. Ce point de vue est notamment illustré par les travaux d'Amir, Virag, Brieussel, Bartholdi, Erschler et Zheng [**AV17, Bri15, Bar14, BZ16**] C'est le troisième point qui prédomine dans ce travail, et qui consiste à étudier les liens entre le comportement asymptotique de la probabilité de retour et du profil isopérimétrique, la structure algébrique du groupe G et les propriétés de grande échelle de l'espace métrique (G, d_S) telles que la croissance ou bien la moyennabilité.

3. Un bref historique

L'étude de la probabilité de retour commence dans les années 1920, avec le travail de Pólya et son étude de la récurrence de la marche aléatoire simple dans $\mathbb{Z}^d$. On rappelle qu'une marche aléatoire est *récurrente* si la probabilité de retour à l'origine en un temps fini est égale à un, *transiente* sinon. Pólya démontre que la marche aléatoire simple sur $\mathbb{Z}^d$ est récurrente si et seulement si la dimension de l'espace est inférieure ou égale à 2. Pour ce faire, il calcule explicitement la probabilité de retour à l'origine.

5. [**PSC00**] : "What is surprising is that the relatively simple proof that we will give has escaped notice until now."

Théorème 3.0.1 ([**Pó21**]). *La probabilité de retour de* $\mathbb{Z}^d$ *est équivalente à* $n^{-\frac{d}{2}}$. *En particulier, la marche aléatoire simple sur* $\mathbb{Z}^d$ *est récurrente si* $d \leqslant 2$, *et transiente sinon.*

Grâce aux propriétés de stabilité précédemment mentionnées, ce théorème permet de décrire l'asymptotique de la probabilité de retour des groupes abéliens de type fini. De nombreux résultats ont suivi, conséquence de l'étude des notions de croissance et de moyennabilité, que l'on rappelle dans les deux paragraphes suivants.

3.1. Croissance d'un groupe. Soit G un groupe de type fini et S une partie génératrice finie et symétrique de G. On suppose que S contient l'élément neutre e et on note S^n la boule de rayon n centrée en e.

Le groupe G est dit à *croissance polynomiale* s'il existe des constantes d, C telles que

$$|S^n| \leq Cn^d.$$

Si de plus il existe c tel que $|S^n| \geqslant cn^d$, G est dit à croissance polynomiale *de degré* d. Le groupe est dit à *croissance exponentielle* s'il existe C et $a > 1$ tels que

$$|S^n| \geqslant Ca^n.$$

Si $|S^n|$ est plus petit que toute exponentielle, on parle de groupe à *croissance sous-exponentielle*. Un groupe qui n'est à croissance ni polynomiale, ni exponentielle, est dit à *croissance intermédiaire*.

Par exemple, $\mathbb{Z}^d$ est à croissance polynomiale de degré d et le groupe libre de rang n est à croissance exponentielle (la boule de rayon k associée à la partie génératrice standard contient $2n(2n-1)^{k-1}$ éléments).

En 1968, dans des travaux indépendants, Milnor et Wolf montrent que les groupes virtuellement nilpotents sont à croissance polynomiale [**Mil68, Wol68**]. Au début des années 70, Bass et Guivarc'h [**Bas72, Gui71**] calculent, indépendamment aussi, le degré de croissance de ces groupes et donnent la formule suivante [6], exprimant ce degré de croissance à partir du *rang* des facteurs (plus grand nombre d'éléments sans torsion indépendants du groupe) de la suite centrale descendante :

$$d = \sum_{i=0}^{n-1} i \ \mathrm{rang}\left(G_{i+1}/G_i\right).$$

Milnor et Wolf obtiennent également le théorème suivant :

Théorème 3.1.1 ([**Mil68, Wol68**]). *Un groupe de type fini résoluble est à croissance polynomiale s'il est virtuellement nilpotent, et à croissance exponentielle sinon.*

Dans son article, Milnor pose la question suivante : la croissance d'un groupe de type fini est-elle toujours soit polynomiale soit exponentielle ? Cette question est restée ouverte jusqu'en 1984, date à laquelle Grigorchuk lui apporte une réponse négative.

Théorème 3.1.2 ([**Gri83**]). *Il existe un groupe de type fini à croissance intermédiaire.*

6. Désormais connue sous le nom de *formule de Bass-Guivarc'h*.

De nombreux exemples ont suivi, on sait même qu'il existe un continuum ([**Gri84**]).

L'étude de la croissance des groupes a donné lieu à de nombreux autres résultats dont le plus célèbre est certainement le théorème de Gromov établissant en quelque sorte, au début des années 80, la réciproque du résultat antérieur de Milnor et Wolf.

Théorème 3.1.3 ([**Gro81**]). *Un groupe de type fini est à croissance polynomiale si et seulement s'il est virtuellement nilpotent.*

Il n'existe donc pas de groupes à croissance polynomiale de degré non entier.

3.2. Moyennabilité. La notion de moyennabilité est apparue en 1929 à la suite des travaux de John von Neumann sur le paradoxe de Banach–Tarski. Ces deux mathématiciens prouvent en 1924 un théorème déconcertant que l'on peut résumer ainsi : il est possible de découper toute boule de l'espace à trois dimensions en un nombre fini de morceaux que l'on peut réarranger de sorte à former deux boules identiques à la première, à déplacements près, [**BT24**]. Rassurons-nous tout de suite : si ces morceaux sont en nombre fini, ils ne sont par contre pas mesurables.

Cet énoncé, qui contredit l'intuition liée à la notion de volume, est essentiellement un résultat de théorie de la mesure et il se traduit en terme d'inexistence d'une *moyenne invariante* par le groupe des déplacement de $\mathbb{R}^3$.

Définition 3.2.1 ([**Neu29**]). Soit G un groupe localement compact. On appelle *moyenne* toute forme linéaire positive sur $L^\infty(G)$ de norme 1.

Une moyenne m est dite *invariante à gauche* si pour toute fonction $f \in L^\infty(G)$ et pour tout $g \in G, m(g.f) = m(f)$, où $g.f$ désigne la fonction $x \mapsto f(g^{-1}x)$. Le groupe G est alors dit *moyennable* s'il admet une moyenne invariante à gauche.

On voit alors immédiatement que tous les groupes finis sont moyennables (il faut normaliser la mesure de comptage). En considérant la mesure de Haar, on démontre que les groupes compacts le sont également. L'existence d'une mesure invariante pour le groupe $\mathbb{Z}$ découle du théorème de Hahn-Banach. A contrario, le groupe libre engendré par deux éléments F_2 n'est pas moyennable. Le paradoxe de Banach-Tarski traduit le fait que le groupe d'isométries de $\mathbb{R}^3$ contient un tel groupe libre et n'est donc pas moyennable.

Savoir si contenir un sous-groupe libre à deux générateurs est, ou non, une caractérisation de la non-moyennabilité a longtemps été une question ouverte, formulée par von Neumann et Day. Du fait de l'alternative de Tits, qui affirme qu'un groupe linéaire de type fini contient un sous-groupe libre à deux générateurs ou bien est virtuellement résoluble, cette caractérisation est vraie pour les groupes linéaires. Olshanskii a cependant montré en 1980 qu'elle se révélait fausse dans le cas général en utilisant des groupes précédemment construits appelés monstres de Tarski [**Ol'79, Ol'80**]. Enfin, récemment, Monod a construit de nouveaux contres-exemples plus simples [**Mon13**].

En utilisant la définition, on peut montrer que la moyennabilité est une propriété stable par passage à un sous-groupe ou à un quotient, et par extension. De plus, si l'on se donne $G_1 \subset G_2 \subset \dots$ une suite de groupes moyennables alors la réunion $G := \cup_n G_n$ l'est aussi. Ces propriétés, ajoutées à la moyennabilité des groupes finis et de $\mathbb{Z}$, permettent de déduire que tout groupe abélien, puis virtuellement résoluble, est moyennable.

Il existe un grand nombre de définitions équivalentes de la moyennabilité faisant appel à des domaines divers et variés tels que l'analyse fonctionnelle et la théorie des représentations. Nous ne présenterons que l'une de ces formulations équivalentes, connue sous le nom de critère de Følner.

Définition 3.2.2. Un groupe dénombrable G satisfait la *condition de Følner* si pour toute partie finie $A \subset G$ et tout $\epsilon > 0$ il existe un sous-ensemble fini non vide $F \subset G$ tel que pour tout $a \in A$,

$$\frac{\mid aF \Delta F \mid}{\mid F \mid} \leq \epsilon,$$

où Δ est la notation ensembliste pour la différence symétrique.

Pour un groupe localement compact, on remplacera partie finie par partie compacte et cardinalité par mesure de Haar.

Pour un groupe discret G, on appelle *suite de Følner* toute suite de parties finies non vides F_n de G telle que pour tout $g \in G$,

$$\frac{\mid gF_n \Delta F_n \mid}{\mid F_n \mid} \to 0,$$

lorsque $n \to \infty$.

Cette notion fournit une définition équivalente : un groupe satisfait la condition de Følner si et seulement s'il admet une suite de Følner.

On peut interpréter ceci en considérant le graphe de Cayley de G par rapport à la partie A, de sommets les éléments de G et d'arêtes (g, ga) pour $g \in G, a \in A$. Le critère de Følner dit que pour tout sous-ensemble fini de sommets F, la frontière de F (le nombre d'arêtes sortantes) est très petite devant la taille de F.

Proposition 3.2.3 (Critère de Følner). *Un groupe est moyennable si et seulement s'il satisfait la condition de Følner.*

On définit parfois la moyennabilité via ce critère. Le résultat suivant tisse un lien avec la notion précédente : il s'agit d'un cas dans lequel les boules de rayon n contiennent une suite de Følner.

Proposition 3.2.4. *Un groupe à croissance sous-exponentielle est moyennable.*

3.3. Retour à la probabilité de retour. On énonce ici trois résultats reliant la probabilité de retour d'un groupe de type fini aux notions précédemment introduites de croissance et de moyennabilité.

3.3.1. *Groupes à croissance polynomiale.* Le comportement d'un groupe de type fini à croissance polynomiale, qui est, d'après le théorème de Gromov, virtuellement nilpotent, est en fait essentiellement similaire à celui des groupes abéliens de type fini de même croissance et la probabilité de retour ne fournit pas un invariant plus fin que le degré de croissance.

Théorème 3.3.1 ([**Var85b, Var86, VSCC92a**]). *Soit G un groupe de type fini et d un entier supérieur ou égal à 1. Alors G est à croissance polynomiale de degré d si et seulement si $p_{2n}^G \sim n^{-\frac{d}{2}}$.*

En particulier, la marche aléatoire simple sur un groupe G à croissance polynomiale est récurrente si et seulement si le degré de croissance du groupe G est inférieur ou égal à 2. On peut démontrer que dans ce cas, le groupe G est virtuellement $\mathbb{Z}$ ou $\mathbb{Z}^2$.

3.3.2. *Groupes non moyennables.* Si G est un groupe de type fini dont S est une partie génératrice finie et symétrique, il est facile de voir que $p_{2n}^G \succcurlyeq |S|^{-n}$. Dans sa thèse, Kesten caractérise les groupes pour lesquels la marche aléatoire admet ce comportement extrême et démontre le résultat suivant.

Théorème 3.3.2 ([**Kes59**]). *Un groupe de type fini G est non moyennable si et seulement si sa probabilité de retour décroit exponentiellement vite, c'est-à-dire $p_{2n}^G \sim \exp(-n)$.*

3.3.3. *Groupes à croissance exponentielle.* Pour les groupes à croissance exponentielle, Hebisch et Saloff-Coste ont établi une borne supérieure pour le comportement asymptotique de la probabilité de retour à l'origine.

Théorème 3.3.3 ([**HSC93**]). *Soit G un groupe de type fini à croissance exponentielle. Alors*

$$p_{2n}^G \preccurlyeq \exp\left(-n^{\frac{1}{3}}\right).$$

Cette borne supérieure est atteinte et, étant donné un groupe G, on dira que G a *grande probabilité de retour* lorsque $p_{2n}^G \succcurlyeq \exp(-n^{\frac{1}{3}})$ et on dira qu'il a *petite probabilité de retour* sinon.

Le premier exemple de groupe avec grande probabilité de retour a été donné par Varopoulos [**Var83a, Var85b**] : il s'agit du groupe $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ de l'allumeur de réverbères. Alexopoulos a établi la borne inférieure $p_{2n} \succcurlyeq \exp(-n^{\frac{1}{3}})$ pour les groupes polycycliques [**Ale92**]. Ce travail est ensuite généralisé par Pittet et Saloff-Coste qui démontrent dans [**PSC03**] que les groupes résolubles de type fini minimax et sans torsion [7] ont grande probabilitié de retour. Plus récemment, Kropholler et Lorensen ont montré le résultat suivant : tout groupe résoluble minimax est quotient d'un tel groupe sans torsion ([**KL15**]), permettant d'établir le résultat de [**PSC03**] dans le cas général.
Dans [**Tes13**], Tessera définit et étudie la classe des groupes géométriquement élémentairement moyennables, comprenant les sous-groupes discrets des groupes résolubles algébriques sur des corps locaux. Cette classe contient en particulier les groupes de type fini résolubles minimax et certains produits en couronne.

Néanmoins, ce comportement est loin d'être le seul possible en général et il existe des groupes moyennables à croissance exponentielle et petite probabilité de retour :

7. L'énoncé de [**PSC03**] ne précise pas sans torsion mais la preuve qu'ils donnent ne permet pas de traiter le cas général.

l'exemple le plus simple est donné par le produit en couronne $\mathbb{Z}\wr\mathbb{Z}$ dont la probabilité de retour est asymptotiquement équivalente à

$$\exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

Plus généralement, si F est un groupe fini et $d \geqslant 1$, le produit en couronne $F\wr\mathbb{Z}^d$ vérifie (**[Ers03, PSC02]**)

$$p_{2n}^{F\wr\mathbb{Z}^d} \sim \exp\left(-n^{\frac{d}{d+2}}\right).$$

Ainsi, pour tout $\epsilon > 0$, petit, il existe a tel que $1-\epsilon < a < 1$ et un groupe résoluble, et même métabélien, dont la probabilité de retour est équivalente à $\exp(-n^a)$.

Au regard de ces nombreux exemples, la question suivante est donc tout à fait naturelle.

QUESTION 1. *Comment caractériser les groupes moyennables à croissance exponentielle avec grande probabilité de retour ?*

4. Profil isopérimétrique

Un autre quantité intéressante, également invariante par quasi-isométrie, est celle de profil isopérimétrique. Dans cette section, on définit cette notion dans le cadre plus général des groupes localement compacts compactement engendrés et on mentionne certains liens avec la probabilité de retour. Notons que cette terminologie peut s'avérer ambigüe puisqu'elle est présente en de nombreux articles, avec des définitions parfois différentes.

Soit G un groupe localement compact compactement engendré. On munit G d'une mesure de Haar et on se donne S, une partie génératrice compacte et symétrique de G.

4.1. Quelques définitions. Si f est une fonction de $L^\infty(G)$, le *gradient* de f mesure les variations de f au voisinage d'un point :

$$|\nabla f|(x) = \sup_{s\in S}|f(x) - f(xs)|.$$

La définition la plus naturelle de *profil isopérimétrique* est sans doute la suivante :

$$J_G(r) = \sup_{\mu(A)\leqslant r} \frac{\mu(A)}{\mu(\partial_S A)},$$

où $\partial_S A = \{x \in A \mid \exists y \sim x, y \notin A\}$ désigne le bord de A.

On peut généraliser cette définition de la façon suivante. Pour une partie finie A de G, le profil isopérimétrique de G dans A est :

$$j_G(A) = \sup_f \frac{||f||_2}{|||\nabla f\,|||_2},$$

où le supremum porte sur les fonctions $f \in L^\infty(G)$ à support dans A. Le *profil isopérimétrique fonctionnel* de G est alors la fonction croissante définie par

$$j_G(r) = \sup_{\mu(A)\leqslant r} j_{G,}(A).$$

Lorsqu'il est nécessaire de préciser la norme utilisée, on parle de profil isopérimétrique L^2. Similairement, on peut définir un profil ispopérimétrique fonctionnel L^p pour tout $p \in]0, \infty[$. On peut remarquer qu'asymptotiquement, J_G et j_1 sont équivalents (voir Chapitre 2 pour plus de détails), ce qui justifie que le profil fonctionnel est bien une généralisation du profil classique.

4.2. Liens avec la probabilité de retour. Si G est un groupe à croissance polynomiale, G satisfait une inégalité isopérimétrique d-dimensionnelle et les résultats de Varopoulos s'énoncent dans ce contexte de la façon suivante : un groupe satisfaisant une telle inégalité isopérimétrique doit avoir une probabilité de retour au moins égale à $n^{-\frac{d}{2}}$.

Mentionnons également que Coulhon étudie dans [**Cou03**] les liens entre la géométrie à grande échelle et le comportement de la probabilité de retour. Il obtient en particulier que la connaissance du profil isopérimétrique entraîne à la connaissance de la probabilité de retour, sous une hypothèse technique. Avant de citer son théorème, on détaille cette condition.

Soit f une fonction définie sur $\mathbb{R}_+^*$, à valeurs dans lui-même, on pose $\mathcal{L}f = \frac{f}{f'}$. On dit que f *est régulière* lorsque

$$\mathcal{L}f(u) \geqslant \alpha \mathcal{L}f(t), \forall t > 0, \forall u \in [t, 2t],$$

pour un certain $\alpha > 0$. Cette condition est par exemple satisfaite par toutes les fonctions de la forme $f(t) = Ct^{-a}(\log(t+2))^{-b}e^{ct^d}$, pour tous $a, b, c, d, C \geqslant 0$.

THÉORÈME 4.2.1 ([**Cou03**], Théorème 3.7). *Soit γ la fonction définie sur $\mathbb{R}^+$ par*

$$t = \int_0^{\frac{1}{\gamma(t)}} (j_G(v))^2 \frac{\mathrm{d}v}{v}.$$

Si la fonction γ est régulière, alors il existe des constantes $C, c > 0$ telles que, pour tout n

$$\gamma(Ct) \leq p_{2n}^G \leq C\gamma(ct).$$

Ses théorèmes sont ensuite généralisés par Tessera aux cadres des espaces métriques mesurés, pour des gradients à échelle variable [**Tes08**].

Derrière ce théorème se cachent des inégalités fonctionnelles dépendant du profil isopérimétrique, lesquelles se relient aux bornes correspondantes sur la probabilité de retour. Nous ne rentrerons pas beaucoup dans ces détails ici et on renvoie au Chapitre 2 et aux références qui s'y trouvent, ainsi qu'aux survols [**PSC99, Tes16**] et à celui non publié de Pittet et Saloff-Coste [**PSC01**], pour de plus amples informations. La notion présentée dans la section suivante est apparue en lien avec ces inégalités.

5. Suites de couples de Følner

Dans l'article [**CGP01**], Coulhon, Grigor'yan et Pittet présentent une nouvelle méthode pour produire des bornes inférieures sur la probabilité de retour. Avec celle-ci, ils sont en mesure de retrouver des résultats précédemment mentionnés, mais aussi de donner de nouvelles estimées, notamment pour des groupes avec une structure de produit semi-direct.

A cette occasion, ils introduisent la notion de suite de couples de Følner. Cet ingrédient-clef de leur construction consiste en une suite de couples d'ensembles satisfaisant certaines hypothèses combinatoires et métriques. Leur construction rappelle par certains côtés celle des ensembles de Følner.

DÉFINITION 5.0.1 ([**CGP01**]). Soit G un groupe de type fini et S une partie génératrice finie symétrique de G. Le groupe G admet une *suite de paires de Følner*, adaptée à (la classe d'équivalence asymptotique de) la fonction $\mathcal{V}(n)$ s'il existe une suite (F'_n, F_n) de parties finies de G et une constante $C > 0$, tels que

(1) $F'_n S^n \subset F_n$.

(2) $|F_n| \preccurlyeq \mathcal{V}(n)$.

(3) $|F'_n| \geqslant C|F_n|$.

On dira que les paires sont *de taille exponentielle* dès lors que $\mathcal{V} \simeq \exp$.

L'existence d'une telle suite permet de montrer une inégalité fonctionnelle, dite *anti Faber-Kranh* que l'on sait relier à la probabilité de retour. Ils obtiennent le théorème suivant sous une hypothèse technique enlevée ensuite par Erschler.

THÉORÈME 5.0.1 ([**CGP01, Ers06**]). *Soit G un groupe admettant une suite de $\mathcal{V}$-paires de Følner, où $\mathcal{V}$ vérifie : $\mathcal{V} \in C^2(]1,+\infty[), \mathcal{V}' > 0$. Soit γ la fonction définie sur $\mathbb{R}^+$ par*

$$t = \int_{\mathcal{V}(1)}^{\gamma(t)} (\mathcal{V}^{-1}(v))^2 \frac{\mathrm{d}v}{v}.$$

Alors, si γ est régulière,

$$p_{2n}^G \succcurlyeq \frac{1}{\gamma(t)}.$$

On donne quelques bornes supérieures en fonction de $\mathcal{V}$.

$\mathcal{V}$	n^d	$\exp(n)$	$\exp(n^d)$
$p_{2n} \succcurlyeq$	$n^{-\frac{d}{2}}$	$\exp(-n^{\frac{1}{3}})$	$\exp(-n^{\frac{d}{d+2}})$

Dans le troisième chapitre de cette thèse, on s'intéressera aux profils isopérimétriques associés à des gradients à échelle compacte, pour des groupes localement compacts compactement engendrés et aux liens entre profil isopérimétrique et existence de paires de Følner.

6. Dimension de Krull

La notion la plus fondamentale en géométrie comme en topologie est probablement celle de dimension. Dans cette section, on considère son analogue algébrique, dont le rôle en algèbre commutative et en géométrie algébrique s'avère lui aussi essentiel. On renvoie le lecteur au Chapitre 8 du livre d'Eisenbud ([**Eis95**]) pour une introduction historique à la théorie de la dimension. Dans cette thèse, tous les anneaux considérés sont commutatifs et unitaires.

Mentionnons juste brièvement qu'au début du 20^e siècle, la définition naturelle et communément admise de dimension impliquait, par analogie avec la topologie, le degré de transcendance. Cette définition, suffisante lorsque l'on considère des algèbres de type fini sur un corps, ne prenait pas en compte de nombreux exemples, comme les anneaux d'entiers algébriques. D'autres définitions ad-hoc furent proposées, qui devinrent de moins en moins satisfaisantes à mesure que les constructions apparaissant en algèbre commutative se firent plus complexes. Dans les années 30, Krull proposa la définition de dimension qui suit et qui gardera son nom, justifiée notamment par des travaux de Noether sur les quotients d'anneaux de polynômes.

Definition 6.0.1. La dimension de Krull d'un anneau commutatif noethérien R, notée $\mathrm{Krull}(R)$, est le supremum des longueurs des chaînes d'idéaux premiers distincts de R :

$$\mathrm{Krull}(R) = \sup\{r \mid \exists \mathcal{P}_0, \dots, \mathcal{P}_r \text{ idéaux premiers distincts tels que } \mathcal{P}_0 \subset \cdots \subset \mathcal{P}_r\}.$$

On retrouve avec cette définition des résultats attendus, par exemple le fait que $\mathrm{Krull}(k[X_1, \dots, X_d]) = d$ si k est un corps, ou encore que $\mathrm{Krull}(A[X_1, \dots, X_d]) = d + \mathrm{Krull}(A)$ si A est un anneau noethérien.

On obtient alors une définition de dimension de Krull pour un A-module M, comme dimension du plus grand quotient de A agissant fidèlement sur M, à savoir $\mathrm{Krull}_A(M) = \mathrm{Krull}(A/\mathrm{Ann}(M))$, où $\mathrm{Ann}(M) = \{a \in A \mid aM = 0\}$ désigne l'annulateur de M.

Néanmoins, cette quantité peut-être infinie et ne se comporte pas toujours de la façon espérée hors du cadre des anneaux noethériens. En 1967, Gabriel et Rentschler ([**GR67**]) proposent une nouvelle définition de dimension pour un module à l'aide de la déviation, qu'ils introduisent, de l'ensemble de ces sous-modules.

Pour un ensemble partiellement ordonné, la *déviation* renvoie un ordinal qui mesure à quel point celui-ci est loin d'être artinien. D'abord définie pour des ordinaux finis, cette notion est peu après étendue à des ordinaux quelconques par Krause ([**Kra70**]). La définition est basée sur une récurrence transfinie, ainsi cette quantité n'existe pas toujours. Le caractère noethérien de l'ensemble partiellement ordonné fournit cependant une condition nécessaire d'existence.

Il s'avère que cette nouvelle définition de dimension coïncide avec la définition de Krull dans le cas d'un module noethérien de dimension finie et se révèle plus fine dans le cas infini. D'autre part, elle admet une généralisation naturelle au cadre des groupes : Tushev propose dans [**Tus03**] de définir la dimension de Krull d'un groupe comme la déviation de l'ensemble de ses sous-groupes normaux.

Pour plus de détails à ce propos et des énoncés précis, on renvoie à la seconde section du Chapitre 1, où l'on rappelle ces définitions avant d'étudier plus précisément la dimension de Krull des groupes métabéliens.

7. Résultats de cette thèse

Cette thèse comporte trois parties. La première partie (Chapitre 1) étudie les liens entre probabilité de retour et structure, pour les groupes métabéliens. Motivée par l'élaboration d'une preuve plus auto-contenue du théorème principal (voir Théorème 1 plus bas) de cette première partie, la seconde (Chapitre 2) s'intéresse aux liens entre profil isopérimétrique et existence de suites de couples de Følner. Le cadre est celui des groupes localement compacts compactement engendrés. Enfin, la troisième partie est un travail en commun avec Peter Kropholler autour de la structure des groupes résolubles de rang sans torsion infini n'admettant pas de section isomorphe au produit en couronne de deux groupes cycliques infinis.

7.1. Probabilité de retour. Dans le Chapitre 1, on donne une réponse à la question 1, dans le cas le plus simple, à savoir celui des groupes métabéliens de type fini. On obtient la caractérisation suivante :

Théorème 1. *Soit G un groupe métabélien de type fini. Alors,*

— *soit G contient $\mathbb{Z} \wr \mathbb{Z}$ ou $B_2^{[p]}$ pour un entier premier p,*

— *soit G a grande probabilité de retour.*

Pour obtenir la probabilité de retour du groupe p-métabélien libre de rang 2, on montre que les techniques élaborées par Saloff-Coste et Zheng ([**SCZ14**]) pour les groupes résolubles libres s'adaptent à ce cas. On obtient ainsi

Proposition 2. *Soit $d \geqslant 1$ et p un entier premier. La probabilité de retour du groupe p-métabélien libre $B_d^{[p]}$ est équivalente à*

$$\exp\left(-n^{\frac{d}{d+2}}\right).$$

On rappelle que la probabilité de retour du groupe $\mathbb{Z} \wr \mathbb{Z}$ est équivalente à $\exp(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}})$. Par conséquent, l'alternative du précédent théorème est exclusive. Un corollaire immédiat est donc l'existence d'un saut dans les probabilités de retour.

Corollaire 2. *Soit G un groupe métabélien de type fini. Alors,*

$$\textit{soit } p_{2n}^G \preccurlyeq \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right), \ \textit{soit } p_{2n}^G \succcurlyeq \exp\left(-n^{\frac{1}{3}}\right).$$

Un groupe métabélien G admet une suite exacte $M \hookrightarrow G \twoheadrightarrow Q$, avec M, Q abéliens et l'on a vu que M possède en outre une structure de $\mathbb{Z}Q$-module, noethérien lorsque G est de type fini. Les deux sous-groupes du Théorème 1 apparaissent lorsque l'on étudie la dimension de Krull de M. On montre que celle-ci s'avère (sauf cas trivial) être exactement la dimension de Krull du groupe G.

La proposition suivante est à rapprocher du théorème 1.5.1 dont elle précise en quelque sorte la seconde alternative.

Proposition 3. *Soit G un groupe métabélien de type fini et de dimension de Krull supérieure ou égale à 2. Alors, G contient $\mathbb{Z} \wr \mathbb{Z}$ ou $B_2^{[p]}$, pour un certain premier p.*

Lorsque l'extension est de plus scindée, on met en évidence des sous-groupes plus gros, de la forme $\mathbb{Z} \wr \mathbb{Z}^{d-1}$ ou $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^d$, où d est la dimension de Krull.

Le théorème 1 se reformule en termes de dimension de Krull de la façon suivante. La proposition précédente permet alors d'établir le sens direct.

Théorème 3 (Théorème 1, formulation équivalente). *Soit G un groupe métabélien de type fini. Alors G a grande probabilité de retour si et seulement si sa dimension de Krull est inférieure ou égale à 1.*

La stratégie pour établir l'implication réciproque possède deux étapes. Premièrement, on se ramène au cas scindé à l'aide d'un théorème de plongement respectant la dimension de Krull. Il s'agit d'une adaptation du fameux théorème de plongement de Kaloujinine et Krasner (voir par exemple [**KKM85**]).

Théorème 4. *Soit G un groupe métabélien, satisfaisant la suite exacte*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

avec M, Q abéliens.

Alors, G se plonge comme sous-groupe dans un groupe métabélien H scindé, de la forme $H = B \rtimes Q$ et de même dimension de Krull, de sorte que le plongement commute avec la projection sur Q.

De plus, si G est de type fini, on peut choisir H de type fini également.

On se ramène donc à des groupes de la forme $M \rtimes Q$. Le module M étant noethérien, il admet une suite de sous-modules $\{0\} = M_0 \leqslant M_1 \leqslant \dots M_n = M$ tels que $M_i/M_{i+1} \simeq \mathbb{Z}Q/\mathcal{P}_i$ pour un idéal premier $\mathcal{P}_i$ (ce résultat est rappelé plus haut, c'est également la Proposition 2.1.19 du Chapitre 1). L'étude se ramène ainsi à celle des groupes de la forme $\mathbb{Z}Q/\mathcal{P} \rtimes Q$ pour un idéal premier $\mathcal{P}$.

La seconde étape consiste alors en la construction de paires de Følner pour les groupes de dimension au plus 1 et de la forme $A \rtimes Q$, où A est un anneau de la forme précédente. Dans le cas sans torsion, on remarque que le groupe est de rang fini et on fait appel à la construction correspondante de Pittet et Saloff-Coste ([**PSC03**]). Dans le cas de torsion, on est en mesure de construire ces couples de Følner, quelle que soit la dimension de Krull du groupe. Pour revenir au groupe de départ $M \rtimes Q$, on établit un lemme technique de remontée utilisant le passage au quotient de l'existence de couples de Følner dans les groupes de type fini. Ce résultat de passage au quotient est généralisé dans le Chapitre 2 pour donner le Théorème 9 plus bas.

On obtient en particulier, après application des techniques de [**CGP01**], une borne inférieure fonction de la dimension pour les groupes métabéliens de type fini dont le groupe dérivé est de torsion.

Théorème 5. *Soit G un groupe métabélien de type fini dont le groupe dérivé $[G, G]$ est de torsion. Alors*

$$p_{2n}^G \succcurlyeq \exp\left(-n^{\frac{k}{k+2}}\right)$$

où k est la dimension de Krull de G.

Dans le cas scindé, l'étude des liens entre dimension de Krull et structure ayant fourni un sous-groupe admettant cette probabilité de retour, on a également :

Corollaire 6. *Soit G un groupe métabélien de type fini scindé dont le groupe dérivé $[G, G]$ est de torsion. On note k sa dimension de Krull. Alors,*

$$p_{2n}^G \asymp \exp\left(-n^{\frac{k}{k+2}}\right).$$

Enfin, on se pose la question des bornes supérieures. On précise alors la proposition 3 en établissant l'existence, de sections dites *sympathiques*, admettant un plongement dans un groupe d'allumeurs de réverbères de même dimension de Krull. On renvoie à la section 6 du Chapitre 1 pour plus de détails et notamment une définition de *section sympathique* (Remarque 6.1.2).

En reliant ces sections à un groupe métabélien libre dont le rang dépend de la dimension de Krull, on obtient les bornes suivantes :

Théorème 7.

(1) Soit G un groupe métabélien non abélien de type fini sans torsion et de dimension de Krull $k \geqslant 2$. Alors
$$p_{2n}^G \preccurlyeq \exp\left(-n^{\frac{k-1}{k+1}}(\log n)^{\frac{2}{k+1}}\right).$$

(2) Soit G un groupe métabélien non abélien de type fini et de dimension de Krull $k \geqslant 1$ dont le groupe dérivé est de torsion. Supposons que G admet une section sympathique dont la torsion ne divise pas k. Alors
$$p_{2n} \preccurlyeq \exp\left(-n^{\frac{k}{k+2}}\right).$$

7.2. Isopérimétrie et couples de Følner dans les groupes localement compacts compactement engendrés.

Le second chapitre de cette thèse s'intéresse aux liens entre profils isopérimétriques et existence de suites de couples de Følner. On y démontre la caractérisation suivante :

Proposition 4. *Soit G un groupe localement compact compactement engendré. Le groupe G admet une suite de couples de Følner adaptés à une fonction $\mathcal{L}$ si et seulement s'il existe $C > 0$ telle que*

$$\forall n, \forall m \geqslant \mathcal{L}(n),\ J_G^n(m) \geqslant C. \tag{7.1}$$

Ici, J_G^n désigne le profil isopérimétrique à l'échelle n, on renvoie à la deuxième section du Chapitre 2 pour une définition.

La probabilité de retour augmente asymptotiquement lorsque l'on passe à un quotient ou à un sous-groupe ([**PSC00**]). Il en est de même pour le profil isopérimétrique : Tessera le démontre dans [**Tes13**] dans le cadre des espaces métriques mesurés. Les suite de couples de Følner sont, on l'a vu, des outils importants pour obtenir des bornes inférieures pour la probabilité de retour. Il est naturel de se demander si leur existence perdure dans un quotient ou un sous-groupe. Pour le démontrer, une stratégie est de combiner la caractérisation aux résultats de Tessera. Malheureusement, les constantes de [**Tes13**] dépendent de l'échelle choisie pour le gradient et cela ne suffit pas. On démontre donc

Théorème 8. *Soit $Q = G/H$ le quotient d'un groupe G localement compact, compactement engendré et unimodulaire par un sous-groupe normal fermé H. On note $\pi : G \to Q$ la projection canonique. Alors, pour $1 \leq p < \infty$, il existe une constante b telle que, à toute échelle h et pour tout $r > 0$*
$$j_{G,p}^h(r) \leqslant b\ j_{Q,p}^h(br).$$

En combinant ce résultat avec la caractérisation, on obtient le corollaire souhaité.

Corollaire 9. *Soit Q le quotient d'un groupe de type fini G par un sous-groupe normal. Si G admet une suite de couples de Følner adaptée à une fonction $\mathcal{L}$, alors Q aussi.*

Le second théorème s'intéresse au cas d'un sous-groupe fermé et généralise le résultat correspondant d'Erschler pour les groupes de type fini ([**Ers06**]).

Théorème 10. *Soit H un sous-groupe fermé compactement engendré d'un groupe localement compact compactement engendré G. On suppose G et H unimodulaires. Si G admet une suite de couples de Følner adaptée à une fonction $\mathcal{L}$, alors Q aussi.*

Ce second théorème permet de construire des paires de Følner de taille exponentielle pour les groupes métabéliens de type fini sans torsion et de dimension 1, de la forme $A \rtimes Q$ mentionnée plus haut. On peut alors démontrer le théorème 1 sans faire appel à la construction générale de [**PSC03**].

Théorème 11. *Soit G un groupe métabélien de type fini sans torsion, de la forme $G = \mathbb{Z}Q/\mathcal{P} \rtimes Q$, où $\mathcal{P}$ est un idéal premier de $\mathbb{Z}Q$. On suppose que G est de dimension de Krull 1. Alors G admet une suite de couples de Følner de taille exponentielle. En particulier,*

$$p_{2n}^G \succcurlyeq \exp\left(-n^{\frac{1}{3}}\right).$$

7.3. Groupe résolubles de rang infini. La troisième partie de cette thèse est un travail avec Peter Kropoller ([**JK17**], en cours) et concerne les groupes résolubles de rang sans torsion infini. En 1984, Kropholler a donné un premier exemple de groupe résoluble de rang sans torsion infini n'admettant pas de section isomorphe à $\mathbb{Z} \wr \mathbb{Z}$ ([**Kro84**]). Dans ce travail, on étudie ces groupes et on établit le théorème de structure suivant :

Théorème 12. *Soit G un groupe résoluble de type fini, de rang sans torsion infini et n'admettant pas de section isomorphe à $\mathbb{Z} \wr \mathbb{Z}$. Alors G admet des sous-groupes $A \subset K$ and $(A_j)_{j\in\mathbb{N}}$ tels que :*

(1) Les sous-groupes A_i, A et K sont normaux.

(2) Le sous-groupe A est abélien sans torsion et de rang infini.

(3) Pour tout j, $A_j \subset A$, A_j est de rang fini, et A est le produit direct des A_j.

(4) K/A est localement fini.

(5) G/K est virtuellement minimax sans torsion.

(6) Pour tout j, $K/C_K(A_j)$ est fini.

(7) Pour tout sous-groupe H d'indice fini de K tel que H soit normal dans G, $C_A(H)$ est de rang fini.

En conséquence, on obtient un caractérisation pour les groupes résolubles admettant une dimension de Krull, de ceux de rang sans torsion infini.

Corollaire 13. *Soit G un groupe résoluble de type fini admettant une dimension de Krull. Alors, G est de rang sans torsion fini si et seulement si G n'admet pas de section isomorphe à $\mathbb{Z} \wr \mathbb{Z}$.*

Chapitre 1

Metabelian groups with large return probability

Ce chapitre est basé, à l'exception de la dernière partie (§6), sur la prépublication [**Jac16**].

1. Introduction

Let G be a countable group. For any probability measure μ on G, one can consider the *random walk on G driven by μ*: this is a sequence $(S_i)_{i\geq 0}$ of random variables valued in G such that S_0 is an arbitrary point of G and, for any $n \in \mathbb{N}$, $S_{n+1} = S_n X_{n+1}$, where $(X_i)_{i\geq 1}$ are independent and identically distributed random variables with probability distribution μ. If the current state is x, the probability of being in y at the next step is $\mu(x^{-1}y)$. This implicitly defines a probability measure $\mathbb{P}_\mu$ on $G^{\mathbb{N}}$ such that

$$\mathbb{P}_\mu(S_n = y \mid X_0 = x) = \mu^{(n)}(x^{-1}y),$$

where $\mu^{(n)}$ denotes the n-fold convolution of μ. Recall that if f and h are two functions on G, the convolution of f and h is $f * h(g) = \sum_k f(k)h(k^{-1}g)$.

An interesting class of examples arises from a probability measure whose support generates the whole group G as a semigroup. Within this class, a fundamental example is given by a finitely generated group G with $\mu = \mu_S$, the uniform distribution on a finite and symmetric generating set S. Such a probability measure satisfies $\mu_S(g) = \mu_S(g^{-1})$, for all $g \in G$. In general, one says that a probability measure μ is *symmetric* whenever this latter relation is true.

Let $p_{2n}^{\mu,G} = \mathbb{P}_\mu(X_{2n} = e \mid X_0 = e)$ denote the probability of return in $2n$ steps to the origin e of G (even times $2n$ are considered to avoid parity issues: namely, the simple random walk on $\mathbb{Z}$, with usual generating set, cannot reach 0 at odd times).

If φ, ψ denote two monotone functions, we use the notation $\varphi \precsim \psi$ if there exist positive constants c and C such that $c\varphi(Ct) \leq \psi(t)$ (possibly for t in $\mathbb{N}$ if $\mathbb{N}$-valued functions are considered). If the symmetric relation $\varphi \succsim \psi$ also holds, we write $\varphi \sim \psi$ and say that φ and ψ have the same asymptotic behaviour. This is an equivalence relation.

A theorem of Pittet and Saloff-Coste [**PSC00**] asserts that any two symmetric and finitely supported probability measures with generating support give rise to equivalent return probabilities. Let p_{2n}^G denote this invariant, dropping the G whenever the group is clear from the context. Also, if μ is a symmetric probability measure with generating support and finite second moment, that is $\sum_g |g|^2\mu(g) < \infty$, then $p_{2n}^{\mu,G}$ belongs to the class of p_{2n}^G.

Understanding how the random walk behaves allows to have insight into the large-scale geometry of the group. We give below a brief picture of what is known about p_{2n}^G, more details are to follow in part 3.1.

In his thesis, Kesten proved that non-amenable groups are characterized by the fact that they behave the worst, for their return probability decays exponentially fast ([**Kes59**]). On the other hand, a nilpotent group, which necessarily has polynomial growth of some definite degree d behaves like $\mathbb{Z}^d$: this follows from Varopoulos ([**Var85b**],[**VSCC92b**]).

What about amenable groups of exponential growth? Hebisch and Saloff-Coste ([**HSC93**]) proved that if G has exponential growth, then

$$p_{2n}^G \precsim \exp(-n^{\frac{1}{3}}). \tag{1.1}$$

Moreover, if G is a discrete subgroup of a connected Lie group, then G is amenable of exponential growth if and only if $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$.

However, the world of amenable groups of exponential growth can be very wild, and many other behaviours can occur. As an example, for any nontrivial finite group F, the return probability of the wreath product $F \wr \mathbb{Z}^d$ is

$$p_{2n}^{F \wr \mathbb{Z}^d} \sim \exp\left(-n^{\frac{d}{d+2}}\right).$$

Let us also mention the return probability of $\mathbb{Z} \wr \mathbb{Z}^d$:

$$p_{2n}^{\mathbb{Z} \wr \mathbb{Z}^d} \sim \exp\left(-n^{\frac{d}{d+2}}(\log n)^{\frac{2}{d+2}}\right). \tag{1.2}$$

We refer to the paper [**Ers06**] by Erschler for these examples (and many other).

It appears that, for every small positive ϵ, there exist $1-\epsilon < \alpha < 1$, and a solvable group whose return probability is equivalent to $\exp(-n^\alpha)$ (see also [**PSC02**]). These solvable groups can be chosen to be 2-step solvable (in other words *metabelian* or equivalently having abelian derived subgroup) and hence are relatively tame within the class of solvable groups of arbitrary derived length. Comparing with the situation for discrete subgroups of Lie groups, the following question arises naturally.

QUESTION. *Among finitely generated solvable groups, is it possible to characterize the groups whose return probability satisfies* $p_{2n} \succsim \exp(-n^{\frac{1}{3}})$ *?*

So far, (1.1) is known to be sharp for lamplighter group $F \wr \mathbb{Z}$, with F a nontrivial finite group ([**Var83a**]), for polycyclic groups ([**Ale92**]), for solvable Baumslag–Solitar ([**CGP01**]), and for solvable groups of finite Prüfer rank, ([**PSC00**],[**KL15**]). A group has *finite Prüfer rank* if there exists an integer r such that any finitely generated subgroup can be generated by at most r elements and the least such r is the rank. Lastly, Tessera proved in [**Tes13**] the reverse inequality for a class of groups containing all discrete subgroups of solvable algebraic groups over a local field. This class contains in particular lamplighter groups and torsion-free solvable groups of finite Prüfer rank.

Here, we answer the above question in the case of metabelian groups, showing a connexion between the Krull dimension of a metabelian group and the asymptotic behaviour of its return probability.

A metabelian group G is an extension of an abelian group by another abelian group, namely

$$[G,G] \hookrightarrow G \twoheadrightarrow G_{ab},$$

where we denote by $[G,G]$ the derived subgroup of G and by G_{ab} its abelianization $G/[G,G]$. The subgroup $[G,G]$ carries a natural structure of $\mathbb{Z}G_{ab}$-module, coming from the action by conjugation.

The notion of Krull dimension, introduced in **[GR67]**, plays an important role in the theory of rings and modules. In **[Tus03]**, Tushev extended this notion to groups. We recall the classical treatment of Krull dimension in non-commutative ring theory and its generalization to groups in section 2, after which we study it for metabelian groups.

For the moment, it is enough to mention that the Krull dimension of a metabelian group G, denoted Krull(G), is characterized by the following two properties (see Proposition 2.2.10):

— If the Krull dimension of the $\mathbb{Z}G_{ab}$-module $[G,G]$ is positive, then they are equal:

$$\mathrm{Krull}(G) = \mathrm{Krull}_{\mathbb{Z}G_{ab}}([G,G]).$$

— otherwise, when the module $[G,G]$ has Krull dimension 0 (ie is finite), Krull(G) is defined to be 1 if G is infinite, 0 if it is finite.

Theorem 1. *Let G be a finitely generated metabelian group of Krull dimension k. Then,*

$$k \leq 1 \quad \Leftrightarrow \quad p_{2n}^G \gtrsim \exp(-n^{\frac{1}{3}}).$$

Together with (1.1), this implies that the finitely generated metabelian groups with exponential growth and large return probability, that is $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$, are exactly those of dimension 0 or 1.

This is a consequence of studying the impact of Krull dimension on the structure of the group, which yields the more precise lower bound:

Theorem 2. *Let G be a metabelian group of Krull dimension k, $k \geq 1$. Assume that $[G,G]$ is torsion. Then,*

$$p_{2n}^G \gtrsim \exp(-n^{\frac{k}{k+2}}).$$

To obtain these estimates, we reduce to a more tractable case. A metabelian group G is *split* if it admits an exact sequence

$$M \hookrightarrow G \twoheadrightarrow Q,$$

that splits, where M and Q are abelian groups. The group G is therefore isomorphic to the semi-direct product $M \rtimes Q$.

The previous lower bound is actually optimal in that situation.

Proposition 3. *Let G be a finitely generated split metabelian group of Krull dimension k. If $[G,G]$ is torsion, then*

$$p_{2n}^G \sim \exp(-n^{\frac{k}{k+2}}).$$

To restrict to the split case, we use the following embedding theorem. A more detailed statement can be found in subsection 3.3.

THEOREM 4. *Every finitely generated metabelian group, which is the extension of an abelian group by a finitely generated abelian group Q, can be embedded inside a finitely generated split metabelian group $B \rtimes Q$, of the same Krull dimension, with B abelian.*

The other direction in the proof of Theorem 1 is a consequence of the existence of some special subgroups. We write $B_d^{(p)}$ for the free p-metabelian group of rank d, see part 2.3.3 for the definition.

PROPOSITION 5. *Let G be a metabelian group of Krull dimension k.*

If $k \geq 2$, then G has a subgroup isomorphic to either $\mathbb{Z} \wr \mathbb{Z}$, or to $B_2^{(p)}$ for some prime p.

Saloff-Coste and Zheng ([**SCZ14**]) computed the return probablity of the free metabelian group of rank d. We notice that, up to some adjustments, their method allows to compute that of the free p-metabelian group (see part 3.2).

PROPOSITION 6. *The return probability of the free k-metabelian group of rank d is equivalent to* $\exp(-n^{\frac{d}{d+2}})$.

Actually, we can rewrite Theorem 1 in a way involving these two subgroups.

THEOREM 7 (Equivalent version of Theorem 1). *Let G be a finitely generated metabelian group. Then,*

— *either G contains a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ or $B_2^{(p)}$ for some prime p,*

— *or* $p_{2n}^G \gtrsim \exp(-n^{\frac{1}{3}})$.

Together with (1.2), we note that there is a gap in the return probabilities of finitely generated metabelian groups.

COROLLARY 8. *Let G be a finitely generated metabelian group. Then, either*

$$p_{2n}^G \gtrsim \exp\left(-n^{\frac{1}{3}}\right),$$

or

$$p_{2n}^G \lesssim \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

Finally, we provide upper bounds for the return probability in terms of the Krull dimension. We refer to section 6 for the definition of a nice section.

THEOREM 9.

(1) Let G be a finitely generated non-abelian metabelian torsion-free group of Krull dimension $k \geq 2$. Then the return probability of G satisfies

$$p_{2n} \lesssim \exp\left(-n^{\frac{k-1}{k+1}}(\log n)^{\frac{2}{k+1}}\right).$$

(2) Let G be a finitely generated non-abelian metabelian group of Krull dimension $k \geq 1$ whose derived subgroup is torsion. Assume that G admits a nice section whose torsion does not divide k. Then the return probability of G satisfies

$$p_{2n} \lesssim \exp\left(-n^{\frac{k}{k+2}}\right).$$

Organization. In section 2, we recall the definition and some useful properties of the Krull dimension of a module or a group. Then, we establish basic facts in the metabelian case. Section 3 contains preliminary results: the adaptation to the torsion case of computations by Saloff-Coste and Zheng ([**SCZ14**]) of the return probability of the free metabelian group, and the proof of Theorem 4, which is a variation of an embedding theorem of Kaloujinine and Krasner. Proposition 5 is proved in section 4. In section 5, we study sequences of Følner couples in order to apply the machinery of [**CGP01**] for lower bounds on the return probability: we state in particular that such sequences go to a quotient and explain how to construct them for specific extensions. Finally, in section 6, we highlight and study specific sections of finitely generated groups, and show in certain cases that they contain a power of a finitely generated free metabelian group. As a consequence, we give upper bounds in terms of the Krull dimension on the return probability.

2. Krull dimension

Before defining the Krull dimension of a metabelian group, we recall the definition for a module. In this paper, all rings considered will be **commutative with one**.

2.1. Krull dimension of a module. We mention two equivalent definitions: the first one will be generalized later to define the Krull dimension of a group, while the second will be sometimes easier to handle.

2.1.1. *Krull dimension interpreted as the deviation of a poset.* We follow [**MR87**].

Let A be a poset. If $a \leq b$, let $[a,b] = \{x \in A \mid a \leq x \leq b\}$. This is a subposet of A, called *interval*, or *factor*, from a to b. A *descending chain* is a chain $(a_i)_i$ of elements of A such that $a_1 \geq a_2 \geq \dots$, and the intervals $[a_{i+1}, a_i]$ are the *factors of the chain.* If every such descending chain is eventually constant, we say that A satisfy the *descending chain condition.* Similarly, one defines the *ascending chain condition.* A poset A is *trivial* if $a \leq b$ implies $a = b$, for all $a, b \in A$.

Definition 2.1.1. The *deviation* of A, denoted $\operatorname{dev} A$, if it exists, is

— $-\infty$, if A is empty or trivial,

— 0, if A is non-trivial and satisfies the descending chain condition,

— and in general by induction: $\operatorname{dev} A$ is defined and equal to an ordinal n, provided $\operatorname{dev} A$ is not equal to m for every $m < n$, and in any descending chain of A, all but finitely many factors have deviation defined and less than n.

Note that a poset may not have a deviation. A sufficient condition is

Proposition 2.1.2 ([**MR87**]). *Any poset with the ascending chain condition has a deviation.*

We may now define the Krull dimension of a module as the deviation of a natural associated poset.

Definition 2.1.3. Let M be a R-module. Denote by $\mathcal{L}_R(M)$ the poset of R-submodules of M. The *Krull dimension of M as a R-module* is defined as

$$\operatorname{Krull}_R(M) = \operatorname{dev} \mathcal{L}_R(M),$$

whenever it exists. Otherwise, M *does not have a Krull dimension.* We may forget the reference to the ring whenever it is clear from the context.

REMARKS 2.1.4.

(1) If A is a ring, the Krull dimension of A denotes the Krull dimension of the A-module A, written $\mathrm{Krull}(A)$.

(2) A module satisfying the descending chain condition is called *artinian.* Hence, modules of dimension 0 are just artinian modules.

LEMMA 2.1.5 ([**MR87**]). *If N is a submodule of M, then*

$$\mathrm{Krull}(M) = \max\{\mathrm{Krull}(N), \mathrm{Krull}(M/N)\}.$$

2.1.2. *Krull dimension interpreted as the dimension of a faithful ring.*

For commutative Noetherian rings, there is an equivalent way of defining of the Krull dimension in terms of length of chains of prime ideals. One can then derive an equivalent definition for the Krull dimension of a module over a commutative Noetherian ring. These equivalent definitions will sometimes turn out to be more tractable. References for this are [**Eis95**] and [**MR87**].

PROPOSITION 2.1.6 ([**MR87**] , Theorem 4.8). *Let A be a commutative Noetherian ring. The Krull dimension of A, when finite, is equal to the supremum among the r such that there exists a chain*

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_r$$

of prime ideals in A.

REMARK 2.1.7. For simplicity, the equivalent definition just stated is for *finite* Krull dimension. It is possible to refine it using induction so as to obtain an ordinal and get a general equivalent definition for Krull dimension (see [**MR87**] 6.4).

EXAMPLES 2.1.8.

(1) Any field has dimension 0, and any principal ideal domain that is not a field has dimension 1.

(2) Polynomial rings over a Noetherian ring A satisfy:

$$\mathrm{Krull}(A[X_1, \ldots, X_n]) = \mathrm{Krull}(A) + n.$$

Considering invertible variables does not change the dimension:

$$\mathrm{Krull}(A[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]) = \mathrm{Krull}(A) + n.$$

Given fields $K \subset L$, the *degree of transcendence* of L over K will be denoted by $\mathrm{trdeg}_K L$. For any module M and any element c in M, we denote by M_c the localization of M with respect to $\{c^n; n \in \mathbb{N}\}$.

The second part of example 2.1.8 actually generalizes: there is a connexion between Krull dimension and degree of transcendence. Before stating it, we recall Noether's normalization theorem ([**Noe26**], see [**Bos12**] for this generalized version).

THEOREM 2.1.9 (Noether's normalization theorem).

Let A be an integral domain and let $A \subset R$ be an extension of A such that R is finitely generated as an A-algebra.

Then, there is a non-zero element c in A and elements $z_1, \ldots, z_k$ in R_c algebraically independent over A_c such that R_c is a finitely generated $A_c[z_1, \ldots, z_k]$-module.

REMARK 2.1.10. The subring $A_c[z_1, \ldots, z_k]$ is isomorphic to a polynomial ring over A_c. Note that k may be zero.

Moreover, if $A = K$ is a field, then $K_c = K$ for every non-zero element c and the theorem says that every finitely generated K-algebra is a finitely generated module over a polynomial ring over K. The number k of indeterminates is the Krull dimension of R.

Krull dimension behaves well with respect to extension, as stated in the next proposition.

PROPOSITION 2.1.11 ([**Eis95**]). *Let A be an integral domain, K its fraction field and $\mathbb{P}$ the prime subfield of K.*

(1) If A is finitely generated as a $\mathbb{P}$-algebra, then,

$$\mathrm{Krull}(A) = \mathrm{trdeg}_{\mathbb{P}} K.$$

(2) If A is finitely generated as a $\mathbb{Z}$-algebra, then

$$\mathrm{Krull}(A) = \mathrm{trdeg}_{\mathbb{Q}} K + 1.$$

This point of view gives another way to compute the Krull dimension of a module, as the dimension of a faithful ring.

PROPOSITION 2.1.12 ([**MR87**]). *Let A be a commutative Noetherian ring and M a finitely generated A-module. Then,*

$$\mathrm{Krull}_A(M) = \mathrm{Krull}\left(A/Ann(M)\right), \tag{2.1}$$

where $Ann(M)$ denotes the annihilator of M, that is $Ann(M) = \{a \in A \mid aM = 0\}$.

2.1.3. *Refinements.* It is useful to distinguish the contributions to Krull dimension that are made by torsion modules and by torsion-free modules. Let M be a module and $T(M)$ be the torsion subgroup of the group M. The group $T(M)$ also carries a $\mathbb{Z}Q$-module structure, hence has finite exponent, and is a direct factor.

LEMMA 2.1.13 ([**Hal59**]). *Let M be a Noetherian module and denote by $T(M)$ the torsion subgroup of M. Then, $T(M)$ is a submodule of M and there is a torsion-free subgroup N of M such that, as groups , $M = T(M) \oplus N$.*

In particular, the group N is isomorphic to $M/T(M)$.

DEFINITION 2.1.14. The *torsion-free Krull dimension* of M is $\mathrm{Krull}^0(M) = \mathrm{Krull}(M/T(M))$ and the *torsion Krull dimension* of M is $\mathrm{Krull}^t(M) = \mathrm{Krull}(T(M))$.

REMARK 2.1.15. Proposition 2.1.5 implies that $\mathrm{Krull}(M) = \max\{\mathrm{Krull}^0(M), \mathrm{Krull}^t(M)\}$.

2.1.4. *Krull dimension and associated prime ideals.*

We end this subsection collecting some lemmas about Krull dimension which will be needed hereafter.

DEFINITION 2.1.16. Let M be a R-module. A prime ideal $\mathcal{P}$ of R is an *associated prime ideal of* M if it is the annihilator of some element in M. We denote by $\mathrm{Ass}_R(M)$ the set of associated primes of M.

REMARK 2.1.17. Equivalently, a prime $\mathcal{P}$ is an associated prime ideal of M if $R/\mathcal{P}$ is a submodule of M. Note that all the associated prime ideals of M contain the annihilator of M.

LEMMA 2.1.18 ([**Eis95**]). *Let R be a Noetherian ring. Let M, N and Q be finitely generated R-modules fitting together in a short exact sequence*

$$N \hookrightarrow M \overset{p}{\twoheadrightarrow} Q.$$

Assume $\mathrm{Krull}(N) < d$ *and that there exists an ideal $\mathcal{P}$ such that, for every $x \in Q - \{0\}$, $Ann(x) = \mathcal{P}$, with* $\mathrm{Krull}(R/\mathcal{P}) = d$.

Then $\mathcal{P}$ is an associated prime ideal of M. In particular, M has a submodule isomorphic to $R/\mathcal{P}$.

PROOF. Denote by I be the annihilator of N: $\mathrm{Krull}(R/I) < d$ implies $I \nsubseteq \mathcal{P}$. Let $x \in I$, $x \notin \mathcal{P}$. The module xM has nontrivial image in Q and one can check that $N \cap xM = \{0\}$. Hence $xM \simeq p(xM) \subset Q$ and elements of xM have annihilator $\mathcal{P}$. □

The next structure result will be very useful in the sequel. This shows that Noetherian modules are built up from modules isomorphic to $R/\mathcal{P}$ where $\mathcal{P}$ is prime.

PROPOSITION 2.1.19 ([**Eis95**]). *Let R be a Noetherian ring, M be a finitely generated R-module. Then, there exist $M_0, M_1, \ldots, M_n$ submodules of M such that*

$$M = M_n > M_{n-1} > \cdots > M_1 > M_0 = 0 \tag{2.2}$$

and $M_{i+1}/M_i \simeq R/\mathcal{P}_i$, where $\mathcal{P}_i$ is a prime ideal of R.

As a consequence, and it is probably well-known, the Krull dimension of a Noetherian module only depends on its associated prime ideals.

PROPOSITION 2.1.20. *Let R be a Noetherian ring, M be a finitely generated R-module. Then, the Krull dimension of M is attained by $R/\mathcal{P}$ for some associated prime $\mathcal{P}$ of M.*

PROOF. Proposition 2.1.19 gives a decomposition of M as a tower of rings of the form $R/\mathcal{P}_i$, and Proposition 2.1.5 implies that

$$\mathrm{Krull}(M) = \max_i \{\mathrm{Krull}(R/\mathcal{P}_i)\}.$$

Look at the minimal i such that the Krull dimension of M_{i+1}/M_i is the Krull dimension of M. We can apply Lemma 2.1.18 to the exact sequence

$$M_i \hookrightarrow M_{i+1} \twoheadrightarrow M_{i+1}/M_i.$$

□

2.2. Krull dimension of a group.

2.2.1. *Definition.* Tushev defined in [**Tus03**] the Krull dimension of a group analogously to that of a module. For a group G, let $\mathcal{N}(G)$ be the poset of all normal subgroups of G.

Definition 2.2.1 ([**Tus03**]). Let G be a group. We say that G *admits a Krull dimension* whenever the poset $\mathcal{N}(G)$ admits a deviation. In this case, we set

$$\mathrm{Krull}(G) = \mathrm{dev}\,\mathcal{N}(G).$$

Otherwise, G *does not admit a Krull dimension.*

Examples 2.2.2.

(1) A finite group has Krull dimension 0.

(2) $\mathbb{Z}$ has Krull dimension 1. Indeed, a decreasing sequence of subgroups of $\mathbb{Z}$ is a sequence $(c_n\mathbb{Z})_n$, where c_n belongs to $\mathbb{N}$ and $c_n \mid c_{n+1}$. The factors are finite, except when c_n is nonzero and c_{n+1} is zero. This can only happen once.

Remark 2.2.3. If the group is abelian, its Krull dimension coincide with its Krull dimension as a $\mathbb{Z}$-module.

Morevover, if K, H are subgroups of an abelian group, the deviation of the factor $[H, K]$ is exactly the dimension of the quotient group K/H.

Lemma 2.2.4. *A finitely generated abelian group has Krull dimension zero if it is finite, or one if it is infinite.*

Proof. Such a group G is isomorphic to $\mathbb{Z}^d \times F$, for some integer d and some finite group F.

Let $G_0 < ... < G_m$ be a series of subgroups of G in which there are n infinite factors. For each infinite factor G_j/G_{j-1} choose an element x_j in G_j which has infinite order modulo G_{j-1}. Then these x_j taken together generate a free abelian group of rank n and hence $n \leq d$.

One can derive an alternative proof from Remark 2.2.3 above. By (2.1), the Krull dimension of G is the Krull dimension of $\mathbb{Z}/\mathrm{Ann}(G)$, when $\mathrm{Ann}(G)$ denotes the annihilator of the $\mathbb{Z}$-module G. Therefore $\mathrm{Krull}(G)$ is 1 if the group is infinite, or 0 if it is finite. □

A reformulation of Proposition 2.1.2 yields:

Proposition 2.2.5. *If G satisfies the maximal condition on normal subgroups, then G has a Krull dimension.*

This is not a necessary condition: the group $\mathbb{Z}[\frac{1}{p}]$ does admit a Krull dimension, equal to 1.

2.2.2. *Krull dimension of a G-group.*

To study the Krull dimension of a metabelian group and link it with the Krull dimension of certain submodules of the group, we need the following broader notion. Let G, H be two groups. H is said to be a *G-group* if there is an action of G on H containing the inner automorphisms of H. If H satisfies an exact sequence $M \hookrightarrow H \twoheadrightarrow Q$, with abelian groups M and Q, then the induced actions on M and Q endow them with a structure of G-groups.

Definition 2.2.6. Let H be a G-group for some group G. Denote by $\mathcal{N}_G(H)$ the subposet of subgroups of H that are stable under the action of G. We say that H *admits a Krull dimension as a G-group* whenever the poset $\mathcal{N}_G(H)$ admits a deviation. In this case, we set

$$\mathrm{Krull}_G(H) = \mathrm{dev}\,\mathcal{N}_G(H),$$

Otherwise, H *does not admit a Krull dimension as a G-group.*

Remarks 2.2.7.

(1) The poset $\mathcal{N}_G(H)$ is a subposet of $\mathcal{N}(H)$, therefore $\mathrm{Krull}_G(H) \leq \mathrm{Krull}(H)$.

(2) The Krull dimension of G as a G-group for the conjugation action is indeed the Krull dimension of G.

(3) If K, L are elements of $\mathcal{N}_G(H)$, with $K \subset L$, then

$$\mathrm{dev}[K, L] = \mathrm{Krull}_G(L/K),$$

where $[K, L]$ is the the segment between K and L in $\mathcal{N}_G(H)$ on the left-hand side, and L/K denotes the G-group for the induced action of G on the right-hand side.

As a consequence, if N is a subgroup of a G-group H, stable under the action of G, then $\mathrm{Krull}_G(N) \leq \mathrm{Krull}_G(H)$.

Moreover, if Q is a quotient of a G-group H by a G-stable subgroup, then the action of G on H induces a structure of G-group on Q and again $\mathrm{Krull}_G(H) \geq \mathrm{Krull}_G(Q)$.

Lemma 2.2.8. *Let*

$$M \hookrightarrow H \twoheadrightarrow Q$$

be a sequence of G-groups. Then,

$$\mathrm{Krull}_G(H) = \max\{\mathrm{Krull}_G(M), \mathrm{Krull}_G(Q)\}.$$

Proof. The fact that $\mathrm{Krull}_G(H) \geq \max\{\mathrm{Krull}_G(M), \mathrm{Krull}_G(Q)\}$ follows from the remarks.

To show the reverse inequality, we use induction on the ordinal

$$q = \max\{\mathrm{Krull}_G(M), \mathrm{Krull}_G(Q)\}.$$

It holds if q is $-\infty$ or 0. Assume that the reverse inequality is true for any such extension where the corresponding maximum is strictlty less than q.

Let $(M_n)_n$ be a decreasing sequence of $\mathcal{N}_G(H)$. We need to show that all but finitely many of the factors $[M_{n+1}, M_n]$ have deviation less than q. Equivalently, we need to show that all but finitely many of the G-groups M_n/M_{n+1} have dimension less than q. Denote by p the projection of H into Q and set $I_n = M_n \cap M$ and $P_n = p(M_n)$: these are decreasing sequences of $\mathcal{N}_G(M)$ and $\mathcal{N}_G(Q)$ respectively. By definition of q, all but finitely many of the factors of these sequences have deviation less than q.

To apply the induction hypothesis, consider $S_n = M_{n+1}I_n$ in $\mathcal{N}_G(H)$. We have

$$S_n/M_{n+1} \hookrightarrow M_n/M_{n+1} \overset{p}{\twoheadrightarrow} M_n/S_n$$

and $M_n/S_n \simeq P_n/P_{n+1}$ and $S_n/M_{n+1} \simeq I_n/I_{n+1}$, hence they have dimension less than q:

$$\mathrm{Krull}_G(M_n/M_{n+1}) \leq \max\{\mathrm{Krull}_G(I_n/I_{n+1}), \mathrm{Krull}_G(P_n/P_{n+1})\} < q.$$

□

COROLLARY 2.2.9. *Let H, Q be two groups so that Q is a quotient of H. Then,*

$$\mathrm{Krull}(H) \geq \mathrm{Krull}(Q).$$

PROOF. If $G = H$, then $\mathrm{Krull}(H) \geq \mathrm{Krull}_H(Q)$. Morevover the poset $\mathcal{N}_H(Q)$ is isomorphic to $\mathcal{N}(Q)$. Hence the corollary. □

2.2.3. *Krull dimension of a metabelian group.*

The derived series $(G^{(j)})_j$ of a group G is defined inductively by $G^{(0)} = G$ and $G^{(j+1)} = [G^{(j)}, G^{(j)}]$. The group G is *solvable* if $G^{(j)} = \{e\}$ for some j, and the smallest such j is the *derived length of* G. Recall that a metabelian group G is a solvable group with derived length 2. We will denote by G_{ab} the abelianization $G/[G, G]$ of G. The group G fits into the exact sequence

$$[G, G] \hookrightarrow G \twoheadrightarrow G_{ab}.$$

We now consider a general extension of one abelian group by another. Let G be a metabelian group such that

$$M \hookrightarrow G \twoheadrightarrow Q,$$

where M and Q are abelian groups. The group G acts on M by conjugation, and the action of an element g in G only depends on its projection in the quotient group Q. Hence, it induces an action from Q on M, endowing M with a structure of $\mathbb{Z}Q$-module. If Q is finitely generated then $\mathbb{Z}Q$ is Noetherian and if G is finitely generated then both $\mathbb{Z}Q$ and the $\mathbb{Z}Q$-module M are Noetherian.

If G is a finitely generated metabelian group, note that Q is virtually $\mathbb{Z}^d$, for some d. Hence, the group G has a finite index metabelian subgroup G_1 with torsion-free quotient $\mathbb{Z}^d$:

$$M \hookrightarrow G_1 \twoheadrightarrow \mathbb{Z}^d, \tag{2.3}$$

and the ring $\mathbb{Z}\mathbb{Z}^d$ identifies with $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$.

PROPOSITION 2.2.10. *Let G be a metabelian group, admitting an exact sequence of the form*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

where M and Q are abelian groups. The Krull dimension of M as a G-group for the conjugation action coincide with the Krull dimension of M as a $\mathbb{Z}Q$-module.

Then, if $\mathrm{Krull}(M) > 0$, *we have* $\mathrm{Krull}(G) = \mathrm{Krull}(M)$. *Otherwise,* $\mathrm{Krull}(G)$ *is* 0 *if the group is finite,* 1 *if not.*

PROOF. This follows from Lemmas 2.2.4 and 2.2.8 , applied to the action of G on itself by conjugation. Note that $\mathrm{Krull}_G(Q)$ is the Krull dimension of the group Q. □

As a consequence, the Krull dimension of a metabelian group G is the Krull dimension of $[G,G]$ as a $\mathbb{Z}G_{ab}$-module, except when the former is zero and the group infinite: we retain especially

$$\mathrm{Krull}(G) = \mathrm{Krull}_{\mathbb{Z}G_{ab}}([G,G]), \text{ if positive.}$$

In particular, if the dimension is at least 2, we may use *any* exact sequence expressing G as an extension of an abelian group by another one to compute the Krull dimension of G as the Krull dimension of the module involved.

Remark 2.2.11. An easy consequence of this proposition is that finitely generated metabelian groups have finite Krull dimension.

In general, a metabelian group G admits a Krull dimension if and only if it has a finite series of normal subgroups each of whose factor meets the maximal or the minimal condition for G-invariant subgroups [**Tus03**]. Recall that a G-group H is said to satisfy the *maximal (minimal) condition for G-invariant subgroups* if every descending (ascending) chain of G-invariant subgroups eventually terminates.

Up to passing to a finite index subgroup, we may consider exact sequences with torsion-free abelian quotient as (2.3).

Proposition 2.2.12. *Let G be a finitely generated metabelian group. Consider an exact sequence*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

with M and Q abelian.

There exists a subgroup G' of finite index in G such that

(1) $\mathrm{Krull}(G) = \mathrm{Krull}(G')$.

(2) *G' is an extension of M by a finitely generated free abelian group.*

The proof of the proposition requires the following lemmas.

Lemma 2.2.13. *Let $A \supset B$ be two rings such that A is finitely generated as a B-module. Then,*

$$\mathrm{Krull}(A) = \mathrm{Krull}(B) = \mathrm{Krull}_B(A).$$

Proof. Proposition 9.2 of [**Eis95**] states that $\mathrm{Krull}(A) = \mathrm{Krull}(B)$. We compare this quantity with $\mathrm{Krull}_B(A)$. First, $\mathrm{Krull}_B(A) = \mathrm{dev}\,\mathcal{L}_B(A) \geq \mathrm{dev}\,\mathcal{L}_A(A) = \mathrm{Krull}(A)$. On the other hand, $\mathrm{Krull}_B(A) \leq \mathrm{Krull}(B) = \mathrm{Krull}(A)$. □

Lemma 2.2.14. *Let Q be a finitely generated group and Q' be a subgroup of finite index in Q. Let M be a noetherian $\mathbb{Z}Q$-module. Then*

$$\mathrm{Krull}_{\mathbb{Z}Q}(M) = \mathrm{Krull}_{\mathbb{Z}Q'}(M).$$

Proof. Take a decomposition of M as a $\mathbb{Z}Q$-module, as given in (2.2): $M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset \{0\}$, where $M_{i+1}/M_i = \mathbb{Z}Q/\mathcal{P}_i$, for some prime $\mathcal{P}_i$. Then,

$$\mathrm{Krull}_{\mathbb{Z}Q}(M) = \max_i\{\mathrm{Krull}(\mathbb{Z}Q/\mathcal{P}_i)\}.$$

Lemma 2.2.13 above implies that

$$\mathrm{Krull}(\mathbb{Z}Q/\mathcal{P}_i) = \mathrm{Krull}_{\mathbb{Z}Q'/(\mathcal{P}_i\cap Q')}(\mathbb{Z}Q/\mathcal{P}_i) = \mathrm{Krull}_{\mathbb{Z}Q'}(\mathbb{Z}Q/\mathcal{P}_i)$$

and $\max_i\{\mathrm{Krull}_{\mathbb{Z}Q'}(\mathbb{Z}Q/\mathcal{P}_i)\} = \mathrm{Krull}_{\mathbb{Z}Q'}(M)$. □

Proof of Proposition 2.2.12 . Let $p : G \to Q$ be the projection. The group Q is a finitely generated abelian group: we may write $Q = Q' \times T$ where T is a finite abelian group and Q' is finitely generated free abelian. Take $G' = p^{-1}(Q')$, it has finite index in G and

$$M \hookrightarrow G' \twoheadrightarrow Q'.$$

Hence, we are left to show that $\mathrm{Krull}(G) = \mathrm{Krull}(G')$. By lemma 2.2.14 above, $\mathrm{Krull}_{\mathbb{Z}Q}(M) = \mathrm{Krull}_{\mathbb{Z}Q'}(M)$. When this dimension in nonzero, Proposition 2.2.10 ensures that we are done. When it is zero, $\mathrm{Krull}(G)$ is 0 if the group is finite, 1 if not. As G' has finite index, we have the same dichotomy. □

Lemma 2.2.15. *Let G be a metabelian group. Assume that G satisfies an exact sequence*

$$M \hookrightarrow G \twoheadrightarrow Q$$

with M and Q abelian groups.

- *If* $\mathrm{Krull}^0(M) \geq 2$, *then*

  $$\mathrm{Krull}^0(M) = \mathrm{Krull}(G/T(G))$$

  where $T(G)$ denotes the torsion subgroup of G. As a consequence, for any other exact sequence

  $$\hat{M} \hookrightarrow G \twoheadrightarrow \hat{Q}$$

  satisfied by G, $\mathrm{Krull}^0(\hat{M}) = \mathrm{Krull}^0(M)$ and the latter does not depend on the choice of the exact sequence satisfied by G.
- *Denote by M_0 the derived subgroup of G and by Q_0 its abelianization. If* $\mathrm{Krull}^t(M) \geq 1$, *then*

  $$\mathrm{Krull}^t(M) = \mathrm{Krull}^t(M_0)$$

  and the latter does not depend on the choice of the exact sequence satisfied by G.

Proof. Assume first that $\mathrm{Krull}^0(M) \geq 2$. As in the proof of Proposition 2.2.12 above, write $Q = Q' \times T$, where T is a finite abelian group and Q' is finitely generated free abelian. The exact sequence

$$M \hookrightarrow G \twoheadrightarrow Q$$

provides an exact sequence satisfied by the quotient group $G/T(G)$, where $T(G)$ denotes the torsion subgroup of G and $\pi : G \to Q$ is the canonical projection:

$$M/(T(G) \cap M) \hookrightarrow G/T(G) \twoheadrightarrow Q/\pi(T(G))$$

Note that $T(G) \cap M = T(M)$ and the left-hand side is actually $M/T(M)$. As $\pi(T(G)) \subset T, Q/\pi(T(G))$ is isomorphic to $Q' \times T'$ for some finite abelian group T'. Using Lemma 2.2.14, we have

$$\mathrm{Krull}^0(M) = \mathrm{Krull}_{\mathbb{Z}Q}(M/T(M)) = \mathrm{Krull}_{\mathbb{Z}Q'}(M/T(M)) = \mathrm{Krull}_{\mathbb{Z}(Q' \times T')}(M/T(M)).$$

Hence the latter is greater or equal to 2, therefore equal to $\mathrm{Krull}(G/T(G))$ and $\mathrm{Krull}^0(M) = \mathrm{Krull}(G/T(G))$ as announced.

The proof of the consequence is similar.

Assume now that $\mathrm{Krull}^t(M) \geq 1$. As M_0 is the derived subgroup of G, it is a $\mathbb{Z}Q$-module and $\mathrm{Krull}_{\mathbb{Z}Q}(M_0) = \mathrm{Krull}_{\mathbb{Z}Q_0}(M_0)$. One have $M_0 \subset M$ and consequently

$M_0^t = M^t \cap M_0$. The quotient M^t/M_0^t is isomorphic to a subgroup of the torsion subgroup of Q_0^t, hence has Krull dimension $-\infty$ or 0 as a $\mathbb{Z}Q$-module. Therefore

$$\mathrm{Krull}^t(M) = \mathrm{Krull}_{\mathbb{Z}Q}(M^t) = \mathrm{Krull}_{\mathbb{Z}Q}(M_0^t) = \mathrm{Krull}_{\mathbb{Z}Q_0}(M_0^t) = \mathrm{Krull}^t(M_0).$$

□

As in the case of modules, we may define the following refinements. The previous lemma ensures that this is well-defined.

DEFINITION 2.2.16. Let G be a metabelian group. Assume that G satisfies an exact sequence

$$M \hookrightarrow G \twoheadrightarrow Q$$

with M and Q abelian groups.

- If $\mathrm{Krull}^0(M) \geq 2$, the *torsion-free Krull dimension* of G is defined as $\mathrm{Krull}^0(G) = \mathrm{Krull}^0(M) = \mathrm{Krull}(G/T(G))$.
  If $\mathrm{Krull}^0(M) \leq 1$, set $\mathrm{Krull}^0(G) = 1$.
- If $\mathrm{Krull}^t(M) \geq 1$, the *torsion Krull dimension* of G is defined as $\mathrm{Krull}^t(G) = \mathrm{Krull}^t(M)$.
  If $\mathrm{Krull}^t(M) \leq 0$, set $\mathrm{Krull}^t(G) = 0$.

As a consequence, if G satisfies an exact sequence $M \hookrightarrow G \twoheadrightarrow Q$ such that $\mathrm{Krull}_{\mathbb{Z}Q}(M) \geq 2$, we have

$$\mathrm{Krull}(G) = \max\{\mathrm{Krull}^0(G), \mathrm{Krull}^t(G)\}.$$

EXAMPLE 2.2.17. To explain the dichotomies in the previous two definitions, consider the following example, suggested by Gouëzel. Let $G = \mathbb{Z} \oplus ((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^k)$. The group G has Krull dimension k. It satisfies the exact sequence

$$\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z} \hookrightarrow G \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}^k.$$

Set $M_1 = \bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z}$. Then, $\mathrm{Krull}^0(M_1) = \mathrm{Krull}(0) = -\infty$ and $\mathrm{Krull}^t(M_1) = \mathrm{Krull}(M_1) = k$.

The group G also satisfies the exact sequence

$$\mathbb{Z} \oplus \left(\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z} \right) \hookrightarrow G \twoheadrightarrow \mathbb{Z}^k.$$

Set $M_2 = \mathbb{Z} \oplus (\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z})$. Then, $\mathrm{Krull}^0(M_2) = \mathrm{Krull}(\mathbb{Z}) = 1$ and $\mathrm{Krull}^t(M_2) = \mathrm{Krull}(\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z}) = k$.

The group $H = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z} \wr \mathbb{Z}^k)$ provides a similar example for the torsion Krull dimension.

2.3. Examples. We study now three classes of examples. First, we consider small dimensional metabelian groups and study the rank of the torsion-free ones. Then, in the second and third paragraphs, we give the Krull dimension of some metabelian wreath products and of the free (p-)metabelian groups (see the definitions below). These two last classes will appear to be fundamental in the sequel.

2.3.1. *Small dimensional metabelian groups.*

Finitely generated metabelian groups of Krull dimension 0 are finite, as stated in Proposition 2.2.10. When the dimension is 1 and the group is torsion-free, we can also say something about the structure. Recall that a group has *finite Prüfer rank* if there exists an integer r such that any finitely generated subgroup can be generated by at most r elements. The least such r is the *Prüfer rank* of the group.

PROPOSITION 2.3.1. *Let G be a finitely generated torsion-free metabelian group of dimension one. Then, it has finite Prüfer rank.*

This Proposition is false if we do not assume that the group is torsion-free: the lamplighter is a counter-example, as it has dimension 1 but infinite Prüfer rank. Moreover, we shall see later in Section 4 that a finitely generated metabelian group of dimension 1 whose derived subgroup is torsion has a subgroup isomorphic to a lamplighter.

PROOF. Let G be a finitely generated torsion-free metabelian group of dimension 1. By Proposition 2.2.12, up to passing to a finite index subgroup, we may assume that G fits inside an exact sequence

$$M \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d,$$

for some integer d, and abelian group M. The group M is a Noetherian $\mathbb{Z}\mathbb{Z}^d$-module and, by Proposition 2.1.19, it admits an increasing sequence of submodules M_i whose factors have the form $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$, for $\mathcal{P}$ a prime ideal.

We first consider the case where M is a ring. It has characteristic zero, and Proposition 2.1.11 implies that the transcendental degree of its fraction field over $\mathbb{Q}$ is zero, so M is an algebraic number field. Algebraic number fields have finite Prüfer rank. Hence, M has finite Prüfer rank.

In the general case, we look at a decomposition of M: the subquotients M_{i+1}/M_i have dimension either zero or one. If the dimension is one, it is enough to show that the characteristic is not positive. By contradiction, suppose M_{k+1}/M_k has positive characteristic p and dimension 1. $M_{k+1}/M_k \simeq \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$, with $\mathcal{P}$ prime, hence p is prime . Then, the transcendental degree of the fraction field of M_{k+1}/M_k is 1, and Proposition 4.1.3 will imply that it contains $\mathbb{F}_p[X^{\pm 1}]$. Pulling back the transcendental element X in M_{k+1}: we still get a transcendental element, which contradicts the fact that M is torsion-free of dimension 1.

The property of being of finite Prüfer rank is stable under extension, and finitely generated abelian groups have finite Prüfer rank. Hence, as M has finite Prüfer rank, G has finite Prüfer rank as well. □

2.3.2. *Wreath products.*

Let K and H be countable groups. The wreath product $K \wr H$ with base group H is the semi-direct product $(\oplus_{h\in H} K_h) \rtimes H$ where K_h are copies of K indexed by H, and H acts by translation of the indices. More precisely, one may identify elements in $\oplus_{h\in H} K_h$ with finitely supported functions from H to K. Thus, for any two elements $(f, h), (f, h')$ in the group, their multiplication is given by

$$(f, h)(f', h') = (ff'(h^{-1}\bullet), hh').$$

Similarly, the unrestricted wreath product of K by H, denoted by $K \wr\wr H$ is the semi-direct product $(\sum_H K_h) \rtimes H$.

The most classical example of a wreath product is the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. This group is generated by two elements $(0,1)$ and $(\delta_0, 0)$, where δ_0 is the function from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ taking value 1 in 0, and vanishing elsewhere. This group can be described in the following way: imagine a bi-infinite pathway of lamps, one for each integer. Each one of those lamps can be on or off, and a lamplighter is walking along the line. An element of the group corresponds to a configuration for the lamps, with only finitely many of them lit, and a position for the lamplighter. The first generator $(0,1)$ changes the position of the lamplighter while the second generator $(0, \delta_0)$ has him change the state of the lamp at his current position. A configuration is a function from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ and corresponds to an element in $\mathbb{F}_2[X, X^{-1}]$. This subgroup has dimension 1 as a $\mathbb{Z}[X, X^{-1}]$-module.

In other words, the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ has an elementary abelian subgroup of dimension 1 such that the quotient is $\mathbb{Z}$. Hence, its Krull dimension is 1.

Other examples are:

(1) $\text{Krull}(\mathbb{Z} \wr \mathbb{Z}^d) = d + 1$.

(2) $\text{Krull}(F \wr \mathbb{Z}^d) = d$, where F is a finite group.

2.3.3. *Free (p-)metabelian groups.*

Denote by F_d the free group of rank d. Let F_d' be its derived subgroup and F_d'' the second term of its derived series.

The *free metabelian group of rank d* is

$$B_d^{(p)} := F_d / F_d''$$

and the *free p-metabelian group of rank d* is

$$B_d^{(p)} := F_d / F_d'' (F_d')^p.$$

$B_d^{(p)}$ is the freest metabelian group whose derived subgroup has exponent p and is the quotient of the free metabelian group of rank d, B_d by the p-powers of its commutators.

These two groups have abelianization $\mathbb{Z}^d$ and looking at the associated exact sequence, one may compute their dimensions:

$$\text{Krull}(B_d) = d + 1, \text{ and } \text{Krull}(B_d^{(p)}) = d.$$

3. Preliminaries

3.1. Generalities on the return probability. We review some known properties and known behaviours for p_{2n} to provide a larger picture to the reader.

3.1.1. *Stability properties.* ([**PSC00**])

(1) Let H be a finitely generated subgroup of G. Then $p_{2n}^H \gtrsim p_{2n}^G$.

(2) Let Q be a quotient of G. Then $p_{2n}^Q \gtrsim p_{2n}^G$.

If the index of the subgroup, respectively the kernel of the quotient, is finite, then the inequality is an equivalence.

As a consequence, when studying the return probability of a finitely generated metabelian group of a given Krull dimension, we may assume that its admits an exact sequence such as (2.3), up to passing to a finite index subgroup (see Proposition 2.2.12).

3.1.2. *Known behaviours: amenability and growth.* The following results were mentioned in the introduction.

(1) G has polynomial growth of degree d iff $p_{2n}^G \sim n^{\frac{-d}{2}}$ (see [**Var87**] and [**HSC93**]).
(2) G is non-amenable iff $p_{2n}^G \sim \exp(-n)$ ([**Kes59**]).
(3) If G has exponential growth then $p_{2n}^G \precsim \exp(-n^{\frac{1}{3}})$ ([**HSC93**]). Moreover, if G is a discrete subgroup of a connected Lie group, there is an equivalence: G is amenable with exponential growth iff $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$.
   These three previous behaviours are the only possible ones in this case.

In the introduction, we mentioned that outside the world of discrete subgroups of connected Lie groups, far more behaviours happen. Here are some other examples.

3.1.3. *Known behaviours: among solvable groups.*

(1) Lamplighter groups:
   — $\mathbb{Z} \wr \mathbb{Z}^d : p_{2n}^{\mathbb{Z}\wr\mathbb{Z}^d} \sim \exp(-n^{\frac{d}{d+2}}(\log n)^{\frac{2}{d+2}})$,
   — $F \wr \mathbb{Z}^d : p_{2n}^{F\wr\mathbb{Z}^d} \sim \exp(-n^{\frac{d}{d+2}})$ (see [**Ers06**]).
(2) Free solvable groups on d generators:
   — Free metabelian group B_d: $p_{2n}^{B_d} \sim \exp(-n^{\frac{d}{d+2}}(\log n)^{\frac{2}{d+2}})$,
   — Free r-solvable group$(r > 2)$ $B_{d,r}$: $p_{2n}^{B_{d,r}} \sim \exp(-n(\frac{\log_{[r-1]} n}{\log_{[r-2]} n})^{\frac{2}{d}})$ ([**SCZ14**]).

3.2. Return probability of $B_d^{(p)}$.

In [**SCZ14**], Saloff-Coste and Zheng computed the asymptotic of the return probability of the free solvable group of rank d (and, in particular, that of the free metabelian group of rank d), using a method based on the Magnus embedding. In this part, we explain why it is possible to use their techniques to get the return probability of the free p-metabelian group, obtained by adding p-torsion to the derived subgroup. The only thing to do is to check that the Magnus embedding behaves well in this case: this was done by Bachmuth in [**Bac67**].

Recall that the Magnus embedding allows to embed groups of the form $F_d/[N, N]$, with N a normal subgroup of F_d, into the wreath product $\mathbb{Z}^d \wr (F_d/N)$ ([**Mag39**]).

In the case of B_d, the free metabelian group of rank d, this applies to $N = F_d'$. The resulting wreath product is $\mathbb{Z}^d \wr \mathbb{Z}^d$, that is,the semi-direct product $\oplus_{\mathbb{Z}^d}\mathbb{Z}^d \rtimes \mathbb{Z}^d$, where $\mathbb{Z}^d$ acts by shift on $\oplus_{\mathbb{Z}^d}\mathbb{Z}^d$.

The embedding takes the following form. Let $s_1, \dots, s_d$ denote the classical generators of F_d, and $a_1, \dots, a_d$ their images in the abelianization $\mathbb{Z}^d$.

Consider matrices of the form

$$\begin{pmatrix} b & m \\ 0 & 1 \end{pmatrix}$$

with b in $\mathbb{Z}^d$ and $m \in M$, the free $\mathbb{Z}(\mathbb{Z}^d)$-module of rank d with basis $(e_i)_{i=1,\dots,d}$. This is a matricial representation of $\mathbb{Z}^d \wr \mathbb{Z}^d$. Let i be the extension to a homomorphism of

$$s_i \mapsto \begin{pmatrix} a_i & e_i \\ 0 & 1 \end{pmatrix}.$$

Magnus proved that the kernel of this homomorphism is $[[F_d, F_d], [F_d, F_d]]$ ([**Mag39**]). Therefore, i induces an embedding from B_d to $\mathbb{Z}^d \wr \mathbb{Z}^d$.

Bachmuth studied this representation in the case of free k-metabelian groups, for any positive integer k. He proved that moding out by k in the derived subgroup corresponds to moding out by k in M, that is

PROPOSITION 3.2.1 ([**Bac67**], Proposition 1). *The Magnus embedding induces an embedding of the free k-metabelian group of rank d, $B_d^{(k)}$ into $(\mathbb{Z}/k\mathbb{Z})^d \wr \mathbb{Z}^d$.*

A lower bound for $p_{2n}^{B_d^{(k)}}$ follows directly from the comparison with $(\mathbb{Z}/k\mathbb{Z})^d \wr \mathbb{Z}^d$:

$$p_{2n}^{B_d^{(k)}} \gtrsim \exp(-n^{\frac{d}{d+2}}).$$

In [**SCZ14**], Saloff-Coste and Zheng used the Magnus embedding to produce upper bounds for the return probability of groups of the form $F_d/[N, N]$, for some normal subgroup N of F_d.

They introduced the notion of exclusive pair in such a group $F_d/[N, N]$, made of a subgroup Γ together with an element ρ of the derived subgroup. The pair is designed so that the images of Γ and ρ in the $\mathbb{Z}$-module M should have minimal interaction (we refer to [**SCZ14**], §4 for a precise definition). From this, they derived a comparison ([**SCZ14**], Theorem 4.13): the return probability of $F_d/[N, N]$ is smaller than the return probability of the subgroup $\mathbb{Z} \wr \bar{\Gamma}$ in $\mathbb{Z}^d \wr (F_d/N)$, where $\bar{\Gamma}$ is the image of Γ in F_d/N.

In the case of the free metabelian group B_d, Γ can be chosen so that $\bar{\Gamma}$ has finite index in $\mathbb{Z}^d$ (more generally, this is possible whenever F_d/N is nilpotent).

Their techniques still apply to the free k-metabelian groups, if one consider $\mathbb{Z}/k\mathbb{Z}$-modules instead of $\mathbb{Z}$-modules and yield

$$p_{2n}^{B_d^{(k)}} \lesssim p_{2n}^{\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}^d} \lesssim \exp(-n^{\frac{d}{d+2}}).$$

As a conclusion, we get

PROPOSITION 3.2.2. *The return probability of the free k-metabelian group of rank d is equivalent to* $\exp(-n^{\frac{d}{d+2}})$.

3.3. An embedding theorem for metabelian groups that preserves the Krull dimension. In this section, we consider a finitely generated metabelian group G, extension of an abelian group M by a finitely generated abelian group Q. Recall that M also carries a structure of $\mathbb{Z}Q$-module.

A famous theorem of Kaloujinine and Krasner allows to embed G in a split metabelian group.

THEOREM 3.3.1 (see [**KKM85**]). *Any extension of a group A by a group B embeds in the non-restricted wreath product $A \wr\wr B$ of A with B.*

We first recall their construction and then explain how to modify it so as to preserve the Krull dimension.

3.3.1. *The embedding of Kaloujinine and Krasner.*

Fix a section $s : Q \to G$ of the group. The embedding $i : G \hookrightarrow M \wr\wr Q = \sum_Q M \rtimes Q$ is given by

$$g \mapsto (f_g, \bar{g}).$$

where $\bar{g}$ is the image of g in the quotient Q and

$$f_g : \begin{cases} Q \to M \\ q \mapsto s(gq)^{-1} g s(q). \end{cases}$$

Note that the embedding does depend on s. This embedding commutes with the projection on Q, and its restriction to M is $\mathbb{Z}Q$-equivariant. Let S be a finite generating set for G and $\pi : M \wr\wr Q \to \sum_Q M$ denote the canonical projection. Set $T = \pi(i(S))$, it is a finite subset of the base of the wreath product $M \wr\wr Q$. The subgroup $\hat{G}$, generated by T and Q in $M \wr\wr Q$, contains $i(G)$ and is the semi-direct product $B \rtimes Q$, where B is the submodule of $\sum_Q M$ generated by T.

Therefore, G embeds in the finitely generated split metabelian group $\hat{G}$.

3.3.2. *Respecting the Krull dimension.*

We now explain how to modify this embedding so that the Krull dimension of the target group equals $\mathrm{Krull}(G)$.

Let

$$\mathcal{S} = \{C \subset B \text{ submodule } \mid C \cap i(G) = \{0\}\}.$$

Zorn's lemma provides us a maximal element C_0 of $\mathcal{S}$. Set $B_0 = B/C_0$. By construction, G embeds in $B_0 \rtimes Q$. We still denote by i the composition of i with the projection onto $B_0 \rtimes Q$. Note that every submodule of B_0 now intersects $i(G)$.

Claim. The associated prime ideals of B_0 and M are the same.

Indeed, $\mathrm{Ass}(M) \subset \mathrm{Ass}(B_0)$, because $M \subset B$. On the other hand, if $\mathcal{P}$ is an associated prime of B_0, then $\mathbb{Z}Q/\mathcal{P}$ is a submodule of B_0, hence does intersect $i(G)$: there exists g in G such that $i(g) \in \mathbb{Z}Q/\mathcal{P}$. More precisely, g projects trivially onto Q and thus belongs to M. It generates a submodule isomorphic to $\mathbb{Z}Q/\mathcal{P}$ in M.

Lemma 2.1.20 then implies that M and B_0 have the same Krull dimension. Now (see Proposition 2.2.10), if this dimension is positive, this is also the dimension of G and $B_0 \rtimes Q$, and we are done. Otherwise, M and B_0 have dimension 0. If G is infinite, so is $B_0 \rtimes Q$, and they have dimension 1. The last case arises when $\mathrm{Krull}(G) = \mathrm{Krull}(M) = 0$, that is G is finite: $B_0 \rtimes Q$ is finite as well, hence has dimension 0.

We just proved

THEOREM 3.3.2. *Let G be a metabelian group, given as an extension*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

with M and Q abelian.

Then, there exists an embedding of G inside a split metabelian group $B \rtimes Q$, commuting with the projection on Q, such that M and B have the same associated prime ideals. In particular, $\mathrm{Krull}(G) = \mathrm{Krull}(B \rtimes Q)$.

Moreover, if G is finitely generated, $B \rtimes Q$ can be chosen to be finitely generated as well.

4. Special subgroups of metabelian groups

The purpose of this section is to prove

PROPOSITION 4.0.1. *Let G be a metabelian group. Assume G has Krull dimension at least 2. Then, G has a subgroup isomorphic to either $\mathbb{Z} \wr \mathbb{Z}$, or to $B_2^{(p)}$ for some prime p.*

The first option happens whenever $\mathrm{Krull}^0(G) \geq 2$, and the consequence for the return probability of G is

$$p_{2n}^G \precsim \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

The second option happens whenever $\mathrm{Krull}^t(G) \geq 2$, and yields

$$p_{2n}^G \precsim \exp\left(-n^{\frac{1}{2}}\right).$$

4.1. Looking for transcendental elements. We first show that a group of dimension k contains a maximal polynomial ring of the same dimension.

PROPOSITION 4.1.1. *Let G be a finitely generated metabelian group, satisfying*

$$M \hookrightarrow G \twoheadrightarrow Q, \tag{4.1}$$

with M abelian and Q free abelian. Assume that G has Krull dimension $k \geq 2$.

Then, M contains a ring isomorphic to $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{k-1}^{\pm 1}]$ or $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_k^{\pm 1}]$, for some prime p. The first option happens whenever $\mathrm{Krull}^0(G) = k$, the second whenever $\mathrm{Krull}^t(G) = k$.

REMARK 4.1.2. According to its proof, this proposition rewrites as follows:

- If $\mathrm{Krull}^0(G) = k$, there exists $Q_0 \subset Q$, with $Q_0 \simeq \mathbb{Z}^{k-1}$, and $m \in A$ such that $\mathbb{Z}Q_0.m \simeq \mathbb{Z}Q_0$ as $\mathbb{Z}Q_0$-modules.
- If $\mathrm{Krull}^t(G) = k$, there exists $Q_0 \subset Q$, with $Q_0 \simeq \mathbb{Z}^k$, and $m \in A$ such that $\mathbb{Z}Q_0.m \simeq (\mathbb{Z}/p\mathbb{Z})Q_0$ as $(\mathbb{Z}/p\mathbb{Z})Q_0$-modules, for some prime divisor p of n.

To prove Proposition 4.1.1, it will be enough to deal with the case of M being a ring, thanks to Propositions 2.1.19 and 2.1.20 on the structure of M.

4.1.1. *The case of a ring.*
In this paragraph, let Q be a finitely generated free abelian group and A be a ring isomorphic to $\mathbb{Z}Q/\mathcal{P}$, for some prime ideal $\mathcal{P}$. The group Q acts on A by multiplication by the corresponding monomial. The characteristic of A is either a prime p, or zero. Denote by K be the fraction field of A, and by $\mathbb{P}$ its prime field.

PROPOSITION 4.1.3. *Let D be the degree of transcendence of K over $\mathbb{P}$.*

Then A contains a family of D transcendental monomials which is algebraically free. The subring generated by this family is, according to the characteristic, either of the form

$$\mathbb{Z}[X_1^{\pm 1}, \dots, X_D^{\pm 1}] \quad \text{or} \quad \mathbb{Z}/p\mathbb{Z}[X_1^{\pm 1}, \dots, X_D^{\pm 1}],$$

for some prime p.

As a consequence, if A has Krull dimension d, we have the following dichotomy:

— *when A has characteristic p, A has a subring isomorphic to $\mathbb{Z}/p\mathbb{Z}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$.*
— *when A has characteristic zero, A has a subring isomorphic to $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{d-1}^{\pm 1}]$.*

Lemma 4.1.4. *Let C be a subfield of K. If the degree of transcendence of K over C is positive then there is a transcendental monomial in A.*

Proof. By hypothesis, there exists in K an element which is transcendent over C. We can write it $\frac{t_1}{t_2}$, with $t_1, t_2 \in A$. The set of all elements of K that are algebraic over C is a subfield of K, therefore one of the t_i's has to be transcendental. This transcendental element belongs to A and is a polynomial modulo $\mathcal{P}$, hence a linear combination of monomials. Again, one of its monomials has to be transcendental. □

We may now proceed to the proof of the proposition.

Proof of Proposition 4.1.3. The proof is by induction on D.
Applying the previous lemma with $C = \mathbb{P}$ yields a monomial $m_1 \in A$, transcendental over $\mathbb{P}$. Let $k \leq D$. Suppose we have constructed an algebraically free family $(m_1, \dots, m_{k-1})$ of transcendental monomials of A over $\mathbb{P}$. Then, the transcendence degree of K over $\mathbb{P}(m_1, \dots, m_{k-1})$ is $D - k + 1 > 0$. We apply the lemma again with $C = \mathbb{P}(m_1, \dots, m_{k-1})$ to get m_k. □

4.1.2. *Proof of proposition 4.1.1.*

Proof. The group G fits into an exact sequence such as (4.1) with quotient $Q \simeq \mathbb{Z}^d$, for some d. As a consequence of Lemma 2.1.13, one can write the module M as $M = T(M) \oplus M_0$, where $T(M)$ is the torsion subgroup of M and M_0 is torsion-free. We have

$$\mathrm{Krull}^0(G) = \mathrm{Krull}(M/T(M)) \text{ and } \mathrm{Krull}^t(G) = \mathrm{Krull}(T(M)).$$

Proposition 2.1.20 implies that $T(M)$, respectively $M/T(M)$, contains a ring isomorphic to $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ of Krull dimension $\mathrm{Krull}(T(M))$, respectively $\mathrm{Krull}(M/T(M))$.
Conclusion then follows from Propositions 2.1.11 and 4.1.3, up to pulling back the polynomial ring in the second case. □

4.2. Wreath products inside metabelian groups. We use the results of the previous subsection to exhibit wreath products as subgroups of some (split) metabelian groups.
4.2.1. *In split metabelian groups.*

Proposition 4.2.1. *Let G be a split finitely generated metabelian group, that is $G = M \rtimes Q$, with M and Q abelian. Let k be the Krull dimension of G and assume $k \geq 1$. Then,*

(1) if $\mathrm{Krull}^0(G) = k$, then G has a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}^{k-1}$. As a consequence,

$$p_{2n}^G \precsim p_{2n}^{\mathbb{Z} \wr \mathbb{Z}^{k-1}} \sim \exp\left(-n^{\frac{k-1}{k+1}} (\log n)^{\frac{2}{k+1}}\right).$$

(2) otherwise, there exists a prime p such that G has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$. As a consequence,

$$p_{2n}^G \precsim p_{2n}^{(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k} \sim \exp\left(-n^{\frac{k}{k+2}}\right).$$

PROOF. Up to passing to a finite index subgroup, we may assume that Q is a finitely generated free abelian group (Proposition 2.2.12). This does not change the dimension nor the return probability. In both cases, Proposition 4.1.1 implies that M contains a ring isomorphic to either $\mathbb{Z}[X_1^{\pm 1}, \dots X_{k-1}^{\pm 1}]$, or to $\mathbb{F}_p[X_1^{\pm 1}, \dots X_k^{\pm 1}]$ for some prime p. We denote this ring by B.

Recall that these algebraically free transcendental elements are actually monomials of $Q \simeq \mathbb{Z}^d$. The subgroup of G generated by B and these monomials is either $\mathbb{Z} \wr \mathbb{Z}^{k-1}$ or $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$. □

4.2.2. *In general metabelian groups.*

In the general case, we may not be able to lift commutatively the transcendental elements obtained as we do in Proposition 4.2.1.

For instance, in the free metabelian group B_2, the images of the two generators of $\mathbb{Z}^2$ by any section s from $\mathbb{Z}^2$ to B_2 will never commute.

We still have the following corollary.

COROLLARY 4.2.2. *Let G be a metabelian group. If $\mathrm{Krull}^0(G) \geq 2$, then G has a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ and*

$$p_{2n}^G \precsim p_{2n}^{\mathbb{Z}\wr\mathbb{Z}} \sim \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

This proves the first part of Proposition 4.0.1. The next part deals with the case $\mathrm{Krull}^t(G) \geq 2$.

4.3. Extensions of torsion modules by $\mathbb{Z}^2$. Let p be a prime. The p-metabelian free group of rank 2, $B_2^{(p)}$ fits into an exact sequence

$$[B_2^{(p)}, B_2^{(p)}] \hookrightarrow B_2^{(p)} \twoheadrightarrow \mathbb{Z}^2. \tag{4.2}$$

Denote by α and β its two generators. As a group, $[B_2^{(p)}, B_2^{(p)}]$ is generated by conjugates of $[\alpha, \beta]$. Hence, it is generated by $[\alpha, \beta]$ as a $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$-module and $[B_2^{(p)}, B_2^{(p)}]$ is a cyclic module. As it is also a torsion-free module, it is isomorphic to $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$.

LEMMA 4.3.1. *Let H be an extension of $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ by $\mathbb{Z}^2$, with the usual action of $\mathbb{Z}^2$ on $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$. Then H is metabelian and has a subgroup isomorphic to $B_2^{(p)}$.*

PROOF. H is metabelian because the action is not trivial.

Take a, b in H so that they do not commute and so that their projections in $\mathbb{Z}^2$ generate $\mathbb{Z}^2$. We claim that $\langle a, b\rangle$ is isomorphic to $B_2^{(p)}$. Indeed, $\langle a, b\rangle$ is a metabelian group generated by two elements. Its derived subgroup is the ideal $([a, b])$ in $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$, which is isomorphic to $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$. It satisfies

$$\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}] \hookrightarrow \langle a, b\rangle \twoheadrightarrow \mathbb{Z}^2. \tag{4.3}$$

Moreover, it is a quotient of the free p-metabelian group of rank 2 : there exists

$$p : B_2^{(p)} \twoheadrightarrow \langle a, b\rangle.$$

This p is a morphism of extensions (4.2) to (4.3), hence an isomorphism. □

REMARK 4.3.2. If the extension splits, the group H has a subgroup isomorphic to the bigger group $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^2$.

We may now complete the proof of Proposition 4.0.1.

PROOF OF PROPOSITION 4.0.1. Let G be a finitely generated metabelian group. Again, we can reduce by Proposition 2.2.12 to the case of G fitting in an exact sequence such as (4.1) with torsion-free quotient $\mathbb{Z}^d$, for some d . Assume that $\text{Krull}(G) \geq 2$. There are two cases:

(1) $\text{Krull}^0(G) \geq 2$. Then, Corollary 4.2.2 implies that G has a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.
(2) $\text{Krull}^t(G) \geq 2$. Let
$$M \hookrightarrow G \twoheadrightarrow Q$$
with M abelian, and Q torsion-free abelian of finite rank. Then, M has a submodule isomorphic to $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$, for some prime p (see Proposition 4.1.1). The elements X and Y come from monomials of $Q \simeq \mathbb{Z}^d$. The subgroup generated by the ring $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ and lifts of the two corresponding monomials is an extension of $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ by $\mathbb{Z}^2$, with the usual action of $\mathbb{Z}^2$. We conclude with an application of Lemma 4.3.1.

Consequences in term of return probability follow from computations for $\mathbb{Z} \wr \mathbb{Z}$ (see [**Ers06**]) and $B_2^{(p)}$ (see Proposition 3.2.2). □

5. Return probability lower bounds via the construction of sequences of Følner couples

5.1. Følner couples and return probability.

DEFINITION 5.1.1 (see [**CGP01**] and [**Ers06**]). Let G be a finitely generated metabelian group. We denote by S a finite and symmetric generating set of G, and consider the associated word distance. Let $\mathcal{V}$ be a positive continuous increasing function on $[1, +\infty)$ whose inverse is defined on $[\mathcal{V}(1), +\infty)$. We say that G admits *a sequence of Følner couples adapted to* $\mathcal{V}$ if there exists a sequence $(\Omega_m, \Omega'_m)_{m\in\mathbb{N}}$ of pairs of non-empty finite sets $\Omega'_m \subset \Omega_m$ in G, with $\#\Omega_m \nearrow \infty$ such that

(1) $\#\Omega'_m$ is a positive proportion of $\#\Omega_m : \#\Omega'_m \geq c_0 \#\Omega_m$.
(2) Ω'_m lies linearly inside $\#\Omega_m : \Omega'_m S^m \subset \Omega_m$.
(3) $\mathcal{V}$ controls the size: $\#\Omega_m \leq \mathcal{V}(m)$.

When there exists a positive constant C such that $\mathcal{V}(t) = Ce^{Ct}$, we should say that G admits a sequence of Følner couples *of exponential size.*

REMARK 5.1.2. Note that if $(\Omega_m, \Omega'_m)_m$ is a sequence of Følner couples adapted to $\mathcal{V}$, so is the sequence $(\Omega'_m S^m, \Omega'_m)_m$. It is in some sense the smallest one and we will refer to it as the *smaller version* of $(\Omega_m, \Omega'_m)_m$.

Given this, one can deduce a lower bound for the return probability, depending on $\mathcal{V}$ (see Coulhon, Grigor'yan and Pittet [**CGP01**], as well as Erschler [**Ers06**] for more general statements). We only recall the corollary that we need.

COROLLARY 5.1.3 ([**CGP01**]). *If a group G admits a sequence of Følner couples adapted to a function of the form $\mathcal{V}(t) = C\exp(Ct^d)$, then*
$$p_{2n}^G \gtrsim \exp(-n^{\frac{d}{d+2}}).$$

5.2. Følner couples for split metabelian groups. In this part, we prove Theorem 2. For any finitely generated metabelian group, Proposition 2.2.12 provides us with a finite index subgroup G, of the same Krull dimension, admitting an exact sequence

$$M \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d, \tag{5.1}$$

with M abelian. The return probability of G is equivalent to that of the initial group. Morevover, as the return probability increases when going to a subgroup, Theorem 3.3.2 imply that, when looking for lower bounds on the return probability for metabelian groups, we can reduce to the split case.

Therefore, it will be enough to prove Theorem 2 in the split case, namely for a finitely generated metabelian group G of the form $G = M \rtimes \mathbb{Z}^d$, with M abelian. Our goal is then to produce sequences of Følner couples for such a group G so as to apply Corollary 5.1.3. These sequences will take the following form.

Definition 5.2.1. Let $G = A \rtimes B$ be a finitely generated semi-direct product. Equip G with a generating set $S = S_A \sqcup S_B$ where $S_A \subset A$ generates A as a $\mathbb{Z}B$-module and $S_B \subset B$ generates B. We should say that G admits a sequence of *split* Følner couples if it has a sequence of Følner couples of the form $(\Omega_m \times F_m, \Omega'_m \times F'_m)_m$, with $\Omega_m, \Omega'_m \subset A$ and $F_m, F'_m \subset B$.

Remark 5.2.2. In the previous definition, the sequence $(F_m, F'_m)_m$ is a sequence of Følner couples for B. Moreover, $\#\Omega'_m$ is a positive proportion of $\#\Omega_m$.

5.2.1. *When M is a ring of positive prime characteristic.*

We deal first with the case of M being a ring $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ of positive characteristic, with $\mathcal{P}$ a prime ideal. Hence, in this part, M is $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$, for some prime p and some prime ideal $\mathcal{P}$. Denote by π the canonical projection

$$\pi : \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}] \to \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}.$$

and by B_m the polynomials of degree bounded by m, that is

$$B_m = \{P \in \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}] \mid \operatorname{Supp}(P) \subset [\![-m, m]\!]^d\}. \tag{5.2}$$

Note that B_m, as well as $\pi(B_m)$ are abelian groups.

The following proposition is straightforward.

Proposition 5.2.3. *The sequence*

$$(\Omega_m, \Omega'_m)_m = (\pi(B_{2m}) \rtimes [\![-2m, 2m]\!]^d, \pi(B_{2m}) \rtimes [\![-m, m]\!]^d)_m$$

is a sequence of split Følner couples adapted to $\mathcal{V}(m) = \#B_{2m} \leq C\exp(Cm^d)$.

Remarks 5.2.4. Obviously, this $\mathcal{V}$ is not optimal, although it gives, by Corollary 5.1.3, a lower bound on the return probability depending only on the rank of G_{ab}:

$$p_{2n}^G \gtrsim \exp(-n^{\frac{d}{d+2}}).$$

The following lemma aims to improve it, making use of Noether's normalization theorem 2.1.9 to control the size.

Lemma 5.2.5. *There exists a constant C such that*

$$\#\pi(B_m) \leq C\exp(Cm^k), \tag{5.3}$$

where k is the Krull dimension of $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$.

PROOF. By Noether's normalization theorem 2.1.9, there exists elements $z_1, \dots, z_k$ in $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$ giving it the structure of a finitely generated module over its subring $\mathbb{F}_p[z_1, \dots, z_k]$. Hence, each element $X_i^{\pm 1}$ is integral over $\mathbb{F}_p[z_1, \dots, z_k]$ and is a root of a monic polynomial with coefficients in $\mathbb{F}_p[z_1, \dots, z_k]$. We may assume that these polynomials have the same degree D. Let N be such that the support of all coefficients appearing in these polynomials lies in $[\![-N, N]\!]^k$.

Then, for X being one of the $X_i^{\pm 1}$, we may write each power X^{D+r} as a polynomial in $X, \dots X^{D-1}$, whose coefficients have support inside $[\![-rN, rN]\!]^k$.

Therefore, we can write any element of $\pi(B_m)$ as a linear combination of monomials $X_1^{n_1}, \dots X_d^{n_d}$ with $(n_1, \dots, n_d) \in [-D, D]^d$ and coefficients which are elements of $\mathbb{F}_p[z_1, \dots, z_k]$, with support in $[\![-Cm, Cm]\!]^k$, for some constant C. □

5.2.2. *Følner sequences for extensions.* In this part, we recall and prove Corollary 6 stated in the introduction.

COROLLARY 5.2.6. *Let G be a finitely generated split metabelian group, whose derived subgroup is torsion. Assume G has Krull dimension $k \geq 1$. Then*

$$p_{2n} \asymp \exp(-n^{\frac{k}{k+2}}).$$

PROOF. As mentionned at the beginning of this subsection, we may assume that G fits inside an exact sequence such as (5.1), with torsion-free quotient.

The upper bound comes from Proposition 4.2.1. We are left with the lower bound: it is enough (see Corollary 5.1.3) to show that G admits a sequence of Følner couples adapted to $\mathcal{V}(m) = C\exp(Cm^k)$, for some C.

The proof of this fact uses recurrence along the decomposition (2.2) of the subgroup M as an increasing sequence of submodules. Initialization is given by Proposition 5.2.3 and Lemma 5.2.5, and iteration follows from the next lemma. □

LEMMA 5.2.7. *Let $G = M \rtimes \mathbb{Z}^d$ be a split metabelian group, with M a torsion $\mathbb{Z}\mathbb{Z}^d$-module of Krull dimension at most k.*

Assume that

(1) *there exists a submodule M_1 in M such that $M_1 \rtimes \mathbb{Z}^d$ admits a split sequence of Følner couples $(\Omega_m \rtimes [\![-2m, 2m]\!]^d, \Omega'_m \rtimes [\![-m, m]\!]^d)_m$ adapted to $\mathcal{V}(m) = C\exp(Cm^k)$, where Ω_m and Ω'_m are abelian subgroups of M_1,*

(2) *M/M_1 is a ring of the form $\mathbb{Z}(\mathbb{Z}^d)/\mathcal{P}$, with $\mathcal{P}$ a prime ideal.*

Then, the group G admits a split sequence of Følner couples adapted to a function $\mathcal{V}'(m) = C'\exp(C'm^k)$, of the form $(\Gamma_m \rtimes [\![-2m, 2m]\!]^d, \Gamma'_m \rtimes [\![-m, m]\!]^d)_m$. Moreover, the projections Γ_m, Γ'_m on M are abelian subgroups.

PROOF. We have $M/M_1 = \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ for some prime ideal $\mathcal{P}$, and this module has dimension at most k. Its characteristic is a prime p. Let 1_Q be a lift of the unit of M/M_1 in M. The $\mathbb{Z}\mathbb{Z}^d$-module M_2 generated by 1_Q surjects onto M/M_1, and is of the form $\mathbb{Z}\mathbb{Z}^d/I$, for some ideal $I \subset \mathcal{P}$. Because M is a torsion module, M_2 is actually a ring of characteristic n, for some multiple n of p.

We cannot apply directly Noether's normalization theorem to M_2. To get around this, note that the projection $M_2 \twoheadrightarrow M/M_1$ have finite fibres of cardinality $\frac{n}{p}$. Hence, we can do as in part 5.2.1: consider the finite subgroup B_m in $\mathbb{Z}\mathbb{Z}^d$ (defined in (5.2)). Noether's normalization theorem provides us with an upper bound on the cardinality of its projection $\pi(B_m)$ into M/M_1, where $\pi : \mathbb{Z}\mathbb{Z}^d \twoheadrightarrow M/M_1$. Let Λ_m be the pullback of $\pi(B_{2m})$ in M_2. It is also the projection of B_{2m} in $M_2 = \mathbb{Z}\mathbb{Z}^d/I$. Because the fibres are finite, the cardinality of Λ_m satisfies (5.3), for some constant C_1:

$$\#\Lambda_m \leq C_1 \exp(C_1 m^k) = \mathcal{V}_1(m).$$

Note that Λ_m is an abelian subgroup of M.

We fix a generating set S for G as follows:

$$S = S_1 \cup \{(\pm 1_Q, 0)\},$$

such that $S_1 = S_1' \cup \{(0, \pm e_1), \dots (0, \pm e_d)\}$ with S_1' a finite and symmetric generating set for the finitely generated $\mathbb{Z}(\mathbb{Z}^d)$-module M_1 and $(e_i)_{i=1}^d$ being the canonical basis of $\mathbb{Z}^d$. We choose S_1 so as to be compatible with the sequence of Følner couples of $M_1 \rtimes \mathbb{Z}^d$.

Consider the following sequence of couples indexed by $m \in \mathbb{N}$:

$$F_m = \{P.1_Q + \omega \mid P \in \Lambda_m, \omega \in \Omega_m\} \rtimes [\![-2m, 2m]\!]^d$$

$$F_m' = \{P.1_Q + \omega \mid P \in \Lambda_m, \omega \in \Omega_m'\} \rtimes [\![-m, m]\!]^d.$$

We claim that this is a sequence of Følner couples for the group G, adapted to $\mathcal{V}'(m) = C' \exp(C' m^k)$, for some constant C'. Note that the projections on M are again abelian subgroups. Indeed, $\#F_m \leq \mathcal{V}_1(m) \#\Omega_m (4m+1)^d \leq C' \exp(C' m^k)$. The choice of a sequence of Følner couples for Ω_m and Ω_m' implies $\#\Omega_m' \geq c \#\Omega_m$, and that $F_m' S^m \subset F_m$.

The last condition, namely $\frac{\#F_m'}{\#F_m} \geq c$, follows from the following straightforward fact, applied to $H = M$, $\Omega = \Omega_m$, $\Omega' = \Omega_m$ and $X = \Lambda_m$. ☐

Fact 5.2.8. *Let H be a group and $\Omega' \subset \Omega$ be two subgroups of H. Assume that X is a subset of H so that ΩX and $\Omega' X$ are subgroups. Then*

$$[\Omega X : \Omega' X] \leq [\Omega : \Omega'].$$

We may now complete the proof of Theorem 2.

Proof of Theorem 2. As the return probability increases when taking subgroups, the theorem follows from Corollary 5.2.6 and Theorem 3.3.2. ☐

5.3. Proof of Theorem 1. The proof of Theorem 1 requires to manage with the torsion submodule and torsion-free quotient of M so as to deal with both cases. Therefore, Lemma 5.2.7, that provides Følner couples for extension in the torsion case, will not suffice.

We first prove a proposition stating that admitting Følner couples descends to the quotient in finitely generated groups. The fact that it goes to a subgroup is due to Erschler [**Ers06**].

Proposition 5.3.1. *Let G be a finitely generated group and Q be a quotient of G.*

Assume that G admits a sequence of Følner couples adapted to a function $\mathcal{V}$. Then, so does Q.

Proof. Equip G with a finite and symmetric generating set S and H with the projection T of S to the quotient.

Denote by $(\Omega_n, \Omega'_n)_n$ the given sequence of Følner couples. If h is a function from a discrete group K to $\mathbb{R}$, $\|h\|_1 = \sum_{k\in K}|h(k)|$ designates its L^1-norm.

Set $f_n = 1_{\Omega'_n}$ and $|\nabla_n f_n|(g) = \sup_{x\in S^n}|f_n(g) - f_n(gx)|$. The latter gradient measures the variations of f_n along S^n. We have, for some constant c:

$$\||\nabla_n f_n|\|_1 \leq c\|f_n\|_1.$$

Let $\widehat{f_n}$ be the function from Q to $\mathbb{R}$ defined by

$$\widehat{f_n}(q) = \sum_{g\in\pi^{-1}(q)} f(g).$$

The norm is preserved: $\|\widehat{f_n}\|_1 = \|f_n\|_1 = \#\Omega'_n$. Similarly, consider the gradient $|\widehat{\nabla}_n\widehat{f_n}|(q) = \sup_{x\in T^n}|\widehat{f_n}(q) - \widehat{f_n}(qx)|$. Then, we have: $\||\widehat{\nabla}_n\widehat{f_n}|\|_1 \leq \||\nabla_n f_n|\|_1 \leq c\|\widehat{f_n}\|_1$.

For a subset A of G we denote by $\partial_n A$ the n-boundary of A, that is: $\partial_n A = AS^n \cap A^C S^n$. The following version of the co-area formula (see for instance **[Tes08]** (3.1) for a proof):

$$\frac{1}{2}\int_{\mathbb{R}^+} \#\partial_n\{f > t\}\mathrm{dt} \leq \|\nabla_n f\|_1 \leq \int_{\mathbb{R}^+} \#\partial_n\{f > t\}\mathrm{dt},$$

implies that there exist $t \geq 0$ and $C > 0$, such that

$$\#(\partial_n\{\widehat{f_n} > t\}) \leq C\#\{\widehat{f_n} > t\}.$$

Set $\widehat{\Omega'_n} = \{\widehat{f_n} > t\}$ and $\widehat{\Omega_n} = \widehat{\Omega'_n} \cup \partial_n\widehat{\Omega}'_n$. We claim that $(\widehat{\Omega_n}, \widehat{\Omega'_n})$ is a sequence of Følner couples for Q adapted to $\mathcal{V}$. □

Remark 5.3.2. It is possible to elaborate on the technique appearing in this proof and to generalize this lemma to the context of locally compact compactly generated groups. The good setting for this seems to be the theory of isoperimetric profiles of groups through an approach similar to **[Tes08, Tes13]**. This will be the content of the next chapter, and hopefully, of a forthcoming paper **[Jac17]**.

Corollary 5.3.3. *Let $G = M \rtimes Q$ be a finitely generated semi-direct product and $H = M' \rtimes Q$, with M' a quotient of M as a $\mathbb{Z}Q$-module.*

Assume G admits a sequence of split Følner couples of exponential size whose projection onto Q is $(F_n, F'_n)_n$, then H admits a sequence of split Følner couples of exponential size that projects onto $(F'_n S_Q^n, F'_n)_n$.

Proof. Write $(\Omega_n, \Omega'_n) = (A_n \times F_n, A'_n \times F'_n)_n$ for the sequence of split Følner couples of G and denote by π the projection of the $\mathbb{Z}Q$-module M onto M'. Coming back to the proof of the proposition above, a computation gives: for any $(m', q) \in H$,

$$\widehat{f_n}(m', q) = \begin{cases} \#(A'_n \cap \pi^{-1}(m')) \text{ if } q \in F'_n \\ 0 \text{ otherwise.} \end{cases}$$

. Hence, $\widehat{\Omega'_n} = \widehat{A'_n} \times F'_n$ for some $\widehat{A'_n} \subset \pi(A'_n)$ and by construction the projection onto Q of $\widehat{\Omega_n}$ is $F'_n S_Q^n$. □

The next lemma will allow to combine Følner couples along the decomposition of a finitely generated module.

LEMMA 5.3.4. *Let Q be a finitely generated abelian group and M be a finitely generated $\mathbb{Z}Q$-module. Let S_Q be a finite generating set for Q and S_M be a finite generating set for the $\mathbb{Z}Q$-module M. Assume that*

(1) M has submodules M_1 and M_2 so that M_2 is a cyclic $\mathbb{Z}Q$-module and its projection onto M/M_1 is the whole of M/M_1.

(2) The group $M_1 \rtimes Q$ admits a sequence of split Følner couples of exponential size that projects onto a sequence $(F_n, F_n')_n$ in Q.

(3) The group $M_2 \rtimes Q$ admits a sequence of split Følner couples of exponential size that projects onto a sequence $(F_n, F_n')_n$ in Q.

Then, the group $M \rtimes Q$ admits a sequence of Følner couples of exponential size that projects onto the sequence $(F_n' S_Q^n, F_n')_n$ in Q.

PROOF. The module $M_1 \times M_2$ surjects onto M, hence the group $H = (M_1 \times M_2) \rtimes Q$, where Q acts diagonally, surjects onto $M \rtimes Q$. By Corollary 5.3.3, it is enough to construct a sequence of split Følner couples for H, whose projection onto Q is $(F_n, F_n')_n$.

Write $(\Omega_n \times F_n, \Omega_n' \times F_n')_n$, resp. $(\Lambda_n \times G_n, \Lambda_n' \times F_n')_n$ the sequence of Følner couples of $M_1 \rtimes Q$, resp. $M_2 \rtimes Q$, and set for all n

$$\Delta_n = (\Omega_n \times \Lambda_n) \times F_n, \qquad \Delta_n' = (\Omega_n' \times \Lambda_n') \times F_n'.$$

The sequence $(\Delta_n, \Delta_n')_n$ is a sequence of Følner couples of exponential size, that projects onto the sequence $(F_n, F_n')_n$ in Q. □

The proof of Theorem 1 makes use the following ingredient of the proof of the main theorem in Pittet and Saloff-Coste [**PSC03**].

PROPOSITION 5.3.5 ([**PSC03**], Proposition 7.8). *Let Γ be a finitely generated torsion-free soluble group of finite Prüfer rank satisfying*

$$N \hookrightarrow \Gamma \twoheadrightarrow A,$$

with N nilpotent and torsion-free and A free abelian of finite rank d.

Then, Γ admits a sequence of Følner couples of exponential size. If the extension splits, so does the Følner sequence. Moreover, the projection onto A of this sequence is the classical Følner sequence $([\![-2n, 2n]\!]^d, [\![-n, n]\!]^d)_n$ of Følner couples of A.

PROOF OF THEOREM 1. The reverse implication follows from Proposition 4.0.1.

To prove the direct implication, let G be a finitely generated metabelian group of Krull dimension at most 1. We wish to prove a lower bound on its return probability, therefore, by Proposition 2.2.12 and Theorem 3.3.2, we may assume that G is a semi-direct product $M \rtimes Q$ where M is abelian and Q is free abelian. Equip G with a generating set $S = S_M \sqcup S_Q$ where $S_M \subset M$ generates M as a $\mathbb{Z}Q$-module and $S_Q \subset Q$ generates the group Q. Denote by d the rank of Q and set $F = (F_n, F_n')_n = ([\![-2n, 2n]\!]^d, [\![-n, n]\!]^d)_n$ the classical sequence of Følner couples for Q.

Proposition 2.1.19 provides us with submodules of M

$$\{0\} = M_0 \leq M_1 \leq \cdots \leq M_n = M$$

such that M_{i+1}/M_i is isomorphic to $\mathbb{Z}Q/\mathcal{P}_i$ with $\mathcal{P}_i$ a prime ideal of $\mathbb{Z}Q$. Each of these cyclic modules $\mathbb{Z}Q/\mathcal{P}_i$ has dimension at most 1 and by Proposition 5.2.3 and Lemma 5.2.5 for the torsion case, and Propositions 2.3.1 and 5.3.5 for the torsion-free case, the groups $\mathbb{Z}Q/\mathcal{P}_i \rtimes Q$ admits Følner couples of exponential size whose projection onto Q is F.

We combine them, or possibly their smaller version as defined in Remark 5.1.2, by iterative applications of Lemma 5.3.4 to obtain a sequence of Følner couples of exponential size in G. The conclusion then follows from 5.1.3. □

5.4. Application to the L^2-isoperimetric profile. (see for instance [**Cou00**]) The L^2-*isoperimetric profile* of a finitely generated group G, equipped with a finite and symmetric generating set S, is the non-decreasing function

$$j_G(v) = \sup_{\#A \leq v} \sup_{f \in L^2(A)} \frac{\|f\|_2}{\sup_{s \in S} \|f - f(s^{-1}.)\|_2},$$

where $L^2(A)$ denotes the set of functions $f : G \to \mathbb{R}$ supported in A.

We recall the following theorem due to Coulhon and Grigor'yan [**Gri94, Cou96**]. Tessera then generalized it to the setting of metric measure spaces in [**Tes08**].

Theorem 5.4.1. *Let G be a finitely generated group. Then the isoperimetric profile of G satisfies $j_G(v) \sim \ln v$ if and only if $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$.*

Hence, our main theorem may be stated as follows:

Corollary 5.4.2. *Let G be a finitely generated metabelian group of exponential growth. Let d be the Krull dimension of G. Then,*

$$j_G(v) \sim \ln v \Leftrightarrow d \leq 1.$$

6. Section of metabelian groups and upper bounds for the return probability

In this section, we describe a method to obtain upper bounds on the return probability of a metabelian group, depending only on its Krull dimension.

When passing to a subgroup, or going to a quotient, the return probability increases. Hence, to establish an upper bound for this quantity, we may perform either of these operations. We do so in the first part. Below, in the second part, we will produce upper bounds for the return probability of the section obtained. This rests on studying some quotients of the free (p)-metabelian group whose derived subgroup has rank 1.

6.1. Sections. We describe a way to obtain a section of a metabelian group respecting the Krull dimension. The section obtained is an extension of a torsion-free module of rank 1 over a polynomial ring by a finitely generated free abelian group (see Lemma 6.1.1 below). We note in the last paragraph that it embeds inside a wreath product of the same Krull dimension.

6.1.1. *When the derived subgroup is torsion.* Let G be a finitely generated metabelian non-abelian group of Krull dimension $k \geq 2$, whose derived subgroup is torsion. Up to passing to a finite index subgroup (see Proposition 2.2.12), we may assume that G fits in

$$A \hookrightarrow G \twoheadrightarrow Q$$

with Q free abelian of rank $d \geq k$ and A a finitely generated $\mathbb{Z}Q$-module, whose underlying additive group is torsion. Because G is finitely generated and $\mathbb{Z}Q$ is a noetherian ring, A is a finitely generated noetherian module and has uniform torsion, that is there exists $n \in \mathbb{N}$ such that $nA = \{0\}$.

A reformulation of Proposition 4.1.1 is that there exists $Q_0 \subset Q$, with $Q_0 \simeq \mathbb{Z}^k$, and $m \in A$ such that $\mathbb{Z}Q_0.m \simeq (\mathbb{Z}/p\mathbb{Z})Q_0$ as $(\mathbb{Z}/p\mathbb{Z})Q_0$-modules, for some prime divisor p of n. Denote by L_0 this $\mathbb{Z}Q_0$-module generated by m.

Consider the subgroup G_1 of G generated by m and pull-back $g_1, \ldots, g_k$ of generators of Q_0 in G. This subgroup satisfies

$$M' \hookrightarrow G_1 \twoheadrightarrow Q_0$$

where M' is a $\mathbb{Z}Q_0$-module containing L_0. Note that $M = M'/pM'$ is now a $(\mathbb{Z}/p\mathbb{Z})Q_0$-module containing L_0. We denote by G_2 the corresponding quotient of G_1.

Let C be a $(\mathbb{Z}/p\mathbb{Z})Q_0$-submodule of M maximal among the submodules of M having trivial intersection with L_0 and look at the quotient group $H := G_2/C$. It satisfies

$$L \hookrightarrow H \twoheadrightarrow Q_0$$

with L an *essential extension* for L_0, that is: for every submodules B of L, if $B \cap L_0 = \{0\}$ then $B = \{0\}$. In particular, L is a torsion-free $(\mathbb{Z}/p\mathbb{Z})Q_0$-module of rank 1.

6.1.2. *When the group is torsion-free.* Let G be a finitely generated metabelian non-abelian group of Krull dimension $k \geq 2$. Assume G is torsion-free. We can argue similarly as above, without the considerations about the torsion. One difference relates to the reformulation of Proposition 4.1.1. If G satisfies $A \hookrightarrow G \twoheadrightarrow Q$ with Q free abelian of rank $d \geq k-1$ and A a finitely generated $\mathbb{Z}Q$-module, it reformulates as: there exists $Q_0 \subset Q$, with $Q_0 \simeq \mathbb{Z}^{k-1}$, and $m \in A$ such that $\mathbb{Z}Q_0.m \simeq \mathbb{Z}Q_0$ as $\mathbb{Z}Q_0$-modules.

Following the argument yiedls a finitely generated non-abelian metabelian section H of G, with Krull dimension k satisfying

$$L \hookrightarrow H \twoheadrightarrow Q_0$$

where $Q_0 \simeq \mathbb{Z}^{k-1}$ and L is a torsion-free $\mathbb{Z}Q_0$-module of rank 1.

6.1.3. *Conclusion.* We summarize in the next lemma the content of the last two paragraphs.

LEMMA 6.1.1. *Let G be a finitely generated non-abelian metabelian group of Krull dimension $k \geq 2$. Then there exists a finitely generated non-abelian metabelian section H of G with Krull dimension k satisfying*

$$L \hookrightarrow H \twoheadrightarrow Q$$

where

1. *(1) If the derived subgroup of G is torsion: $Q \simeq \mathbb{Z}^k$ and L is a torsion-free $(\mathbb{Z}/p\mathbb{Z})Q$-module of rank* 1 *.*
2. *(2) If G is torsion-free and $k \geq 2$: $Q \simeq \mathbb{Z}^{k-1}$ and L is a torsion-free $\mathbb{Z}Q$-module of rank* 1*.*

Remark 6.1.2. In general, a finitely generated non-abelian metabelian group has sections of each of the forms listed above, where k is respectively the torsion or torsion-free Krull dimension of the group.

Such a section is, in some sense, minimal among the sections of maximal Krull dimension of G. In the following, we will refer to it as a *nice section.*

6.1.4. *Inside wreath products.* We prove in this paragraph that the above sections embed inside wreath products of the same Krull dimension.

Lemma 6.1.3. *Let G be a finitely generated non-abelian metabelian group of Krull dimension $k \geq 2$, whose derived subgroup is torsion. Then, G has a section which embeds in $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$, for some prime p.*

Similarly, let H be a finitely generated non-abelian metabelian torsion-free group of Krull dimension $k \geq 3$. Then, H has a section which embeds in $\mathbb{Z} \wr \mathbb{Z}^{k-1}$.

Proof. We prove the first statement. The second one can be proved similarly.

By Lemma 6.1.1, G has a finitely generated non-abelian metabelian section H of Krull dimension k satisfying $L \hookrightarrow H \twoheadrightarrow Q$ where $Q \simeq \mathbb{Z}^k$ and L is a torsion-free $(\mathbb{Z}/p\mathbb{Z})Q$-module of rank 1 . Moreover, L has an essential cyclic submodule L_0, with $L_0 \simeq (\mathbb{Z}/p\mathbb{Z})Q$, as $(\mathbb{Z}/p\mathbb{Z})Q$-modules. Denote by K the fraction field of $(\mathbb{Z}/p\mathbb{Z})Q$. By injectivity of K, there exists a morphism $L \to K$ such that the following diagram commute

$$\begin{array}{ccc} L_0 & \hookrightarrow & L \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

Because L_0 is an essential submodule, this map is injective and L embeds in K. Hence, we can push forward the extension and get

$$\begin{array}{ccccc} L & \hookrightarrow & H & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ K & \hookrightarrow & \widehat{H} & \twoheadrightarrow & Q \end{array}$$

The latter extension is split: indeed, the cohomology group $H^2(Q, K)$ is reduced to $\{0\}$.

We now exhibit the wreath product. Fix a sequence $(f_n)_n$ of elements of $(\mathbb{Z}/p\mathbb{Z})Q$ so that the following holds

(1) for every i in $\mathbb{N}$, $f_i \mid f_{i+1}$.

(2) for every f in $(\mathbb{Z}/p\mathbb{Z})Q$, there exists an integer i so that $f \mid f_i$.

We have

$$\frac{1}{f_i}(\mathbb{Z}/p\mathbb{Z})Q \subset \frac{1}{f_{i+1}}(\mathbb{Z}/p\mathbb{Z})Q$$

and K is the nondecreasing union of the $\frac{1}{f_i}(\mathbb{Z}/p\mathbb{Z})Q$. As L is a finitely generated $(\mathbb{Z}/p\mathbb{Z})Q$-module, there exists a f such that $L \subset \frac{1}{f}(\mathbb{Z}/p\mathbb{Z})Q$. As a consequence, H actually sits in the following subgroup of $\widehat{H}$:

$$\frac{1}{f}(\mathbb{Z}/p\mathbb{Z})Q \hookrightarrow H \twoheadrightarrow Q,$$

isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$. □

6.2. Upper bound. We now study the sections given by Lemma 6.1.1 in order to obtain upper bounds for the return probability of the initial group.

Let G be a finitely generated metabelian non-abelian group of Krull dimension $k \geq 2$ satisfying

$$L \hookrightarrow G \twoheadrightarrow Q$$

where either

(1) $Q \simeq \mathbb{Z}^k$ and L is a torsion-free $(\mathbb{Z}/p\mathbb{Z})Q$-module of rank 1,

(2) or $k \geq 2$, $Q \simeq \mathbb{Z}^{k-1}$ and L is a torsion-free $\mathbb{Z}Q$-module of rank 1.

If this extension splits, then it contains either $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^k$ or $\mathbb{Z} \wr \mathbb{Z}^{k-1}$ and we can derive the corresponding upper bound for the return probability of G. If it does not split, then the subgroup generated by any pull-back of generators of Q still satisfies the exact sequence above and is either k-generated or $(k-1)$-generated. We still denote it G. Hence, G is a quotient of either $B_k^{(p)}$ or B_{k-1}. Denote by H this latter group. We have the commuting diagram

$$\begin{array}{ccccc} [H,H] & \hookrightarrow & H & \twoheadrightarrow & Q \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow \\ L & \hookrightarrow & G & \twoheadrightarrow & Q \end{array}$$

Let N be the kernel of this morphism, we have $N \subset [H,H]$ and we are left to study the extensions of Q with a rank 1 quotient of the derived subgroup $[H,H]$.

We start with the torsion-free case, that is $H = B_k$.

PROPOSITION 6.2.1. *The free metabelian B_k group embeds as a subgroup of the power $(B_k/N)^{k!}$.*

PROOF. Consider the action of the symmetric group $\mathbf{S}_k$ on B_k permuting the generators. The derived subgroup $M = [B_k, B_k]$ has rank $d-1$ ([**Cor11**]) and is invariant under this action. Consider the diagonal homomorphism

$$B_k \to \prod_{\sigma \in \mathbf{S}_k} B_k/\sigma(N).$$

Lemma 6.2.2 below precisely says that this homomorphism is injective. Since $B_k/\sigma(N)$ is isomorphic to B_k/N, this proves the proposition. □

LEMMA 6.2.2. *The intersection $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N)$ is reduced to $\{0\}$.*

The proof of the lemma requires the following fact.

FACT. *For every $k \geq 0$, the group $\mathbf{S}_k$ has no faithful complex representation of dimension $\leq k-2$. In particular, every faithful complex representation of dimension $k-1$ is irreducible.*

PROOF. This is trivial for $k \leq 1$. For $k \geq 2$ distinct of 4, tables (see [**Gro**] and [**Bur11**] Note C) shows that the only irreducible representations of dimension $< k-1$ are the trivial and signature 1-dimensional representations, and are thus not faithful. For $k = 4$, there is an additional one, 2-dimensional, factoring through a quotient of order 6, completing the proof. The second statement immediately follows. □

PROOF OF LEMMA 6.2.2. Let K be the fraction field of $\mathbb{Z}Q$. By flatness of K, we have an exact sequence

$$0 \to N \otimes K \to M \otimes K \to M/N \otimes K \to 0.$$

Since M/N is torsion-free of rank 1, we deduce that $N \otimes K$ has positive codimension in $M \otimes K$. The intersection $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N \otimes K)$ is a $\mathbf{S}_k$-invariant subspace of positive codimension in $M \otimes K$. By the fact, it is therefore reduced to $\{0\}$. Since M is embedded into $M \otimes K$, it follows that $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N) = \{0\}$. □

Combined with Lemma 6.1.1(2), we get the following estimate.

COROLLARY 6.2.3. *Let G be a finitely generated non-abelian metabelian torsion-free group of Krull dimension $k \geq 2$. Then the return probability of G satisfies*

$$p_{2n} \preccurlyeq \exp\left(-n^{\frac{k-1}{k+1}} (\log n)^{\frac{2}{k+1}}\right).$$

REMARK 6.2.4. The representation above is the standard representation for the symmetric group. Indeed, by Magnus' embedding, B_d embeds in the matrix group

$$\left\{ \begin{pmatrix} b & m \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z}^d, m \in M \right\}$$

where M denotes the free $\mathbb{Z}(\mathbb{Z}^k)$-module of rank d with basis $(e_i)_{i=1,\dots,k}$, via

$$s_i \mapsto \begin{pmatrix} a_i & e_i \\ 0 & 1 \end{pmatrix}.$$

Here, $s_1, \dots, s_k$ denotes the generators of B_k and a_i is the image of s_i in the abelianization $\mathbb{Z}^k$. The symmetric group acts on M by permuting the elements of the basis.

The derived subgroup of B_k embeds in the submodule $N_0 = \{\sum b_i e_i \mid \sum b_i(1 - X_i) = 0\}$ and they both have rank $d-1$ ([**Cor11**]). Therefore, their tensor product by K are equal and we denote by V this subspace of $M \otimes K$. Up to multiplying the e_i's by the scalars $(1-X_i)^{-1}$, V is the subspace of vectors $\sum b_i e_i$ satisfying $\sum b_i = 0$.

REMARK 6.2.5. Proposition 6.2.1 remains true in positive characteristic p, as long as p does not divide k. Indeed, in this setting the standard representation is irreducible: any $v \in V \smallsetminus \{0\}$ has coordinates x_i and x_j such that $x_i \neq x_j$. Set $a = x_i - x_j \neq 0$. Then

$$a^{-1}(x - (i,j).x) = (0, \ldots, \underset{\substack{\uparrow \\ i}}{-1}, 0, \ldots, \underset{\substack{\uparrow \\ j}}{1}, \ldots, 0)$$

generates a basis of V.

COROLLARY 6.2.6. *Let G be a finitely generated non-abelian metabelian group of Krull dimension $k \geq 2$ whose derived subgroup is torsion. Assume that G admits a nice section whose torsion does not divide k.*

Then the return probability of G satisfies

$$p_{2n} \preccurlyeq \exp\left(-n^{\frac{k}{k+2}}\right).$$

REMARK 6.2.7. Recall that, if G is a finitely generated metabelian group such that $[G,G]$ is torsion, then the torsion is uniform. Let $n \geq 1$ be the smallest integer so that $n[G,G] = \{1\}$. A sufficient condition for G to admit a nice section whose torsion does not divide its Krull dimension k is therefore to assume that n and k are coprime.

6.2.1. *When the group only has nice sections whose characteristic divides the Krull dimension.* Let G be a finitely generated metabelian group of Krull dimension k whose derived subgroup is torsion. Assume that Corollary 6.2.6 does not apply, that is G only admits nice sections whose characteristic divides k.

Following the previous lines of reasoning, we consider again the standard representation V of $\mathbb{S}_k$ (see Remark 6.2.4 for the notation) and now suppose that p divides k. Therefore, there is one more invariant subspace: the line D, generated by the vector $(1, \ldots, 1) \in V_0$. We explain in Remark 6.2.4 how this representation arises from the Magnus' embedding of $B_k^{(p)}$. In this setting, D is the span of the vector

$$x = \left(\prod_{j \neq 1}(1 - X_j), \ldots, \prod_{j \neq i}(1 - X_j), \ldots, \prod_{j \neq k}(1 - X_j)\right).$$

LEMMA 6.2.8. *The derived subgroup $[B_k^{(p)}, B_k^{(p)}]$ of the free p-metabelian group of rank k is a free $(\mathbb{Z}/p\mathbb{Z}(\mathbb{Z}^k))$-module of rank $k-1$.*

PROOF. Set $M = [B_k^{(p)}, B_k^{(p)}]$. By **[Cor11]**, M is torsion-free of rank $k-1$. The elements $[s_1, s_i]$, with $i \geq 2$, generate the module M. Hence, M is free. $\square$

The vector x is a linear combination of the $[s_1, s_i]$, for $i \geq 2$, namely

$$x = \sum_{i=2}^{k} \prod_{j \neq 1, i} (1 - X_j)[s_1, s_i].$$

Recall that we would like is to study quotients of $B_k^{(p)}$ with kernel $N \subset M$ such that M/N has rank one and to obtain upper bounds on their return probability. To reason as in Proposition 6.2.1 above, note that the intersection $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N)$ is either D or trivial.

When it is trivial, one can conclude on the return probability as above. When this intersection is D, one can conclude that the quotient $B_k^{(p)}/D$ embeds in $G^{k!}$ and one needs to understand the return probability (or at least get an upper bound on it) of this new group.

6.3. Free metabelian groups as subgroups of wreath products. Let G be a finitely generated non-abelian metabelian group. In the last two subsections, we exhibited a section of G that embeds in a wreath product and so that a power of it contains a free (p-)metabelian group as a subgroup. All groups mentionned here have the same Krull dimension. To complete the picture, we note, as a corollary of the Magnus' embedding, that the free (p-)metabelian group embeds in a wreath product of the same Krull dimension.

Corollary 6.3.1. *Let $d \geqslant 2$. The free metabelian group B_d embeds in the wreath product $\mathbb{Z} \wr \mathbb{Z}^d$. Similarly, for any prime p, the free p-metabelian group $B_d^{(p)}$ embeds in the wreath product $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^d$.*

Proof. By Magnus' embedding, the free metabelian group embeds in the wreath product $\mathbb{Z}^d \wr \mathbb{Z}^d$ and the free p-metabelian group embeds in $(\mathbb{Z}/p\mathbb{Z})^d \wr \mathbb{Z}^d$. Hence, it is enough to prove that $\mathbb{Z}^d \wr \mathbb{Z}^d$ embeds in $\mathbb{Z} \wr \mathbb{Z}^d$, and respectively that $(\mathbb{Z}/p\mathbb{Z})^d \wr \mathbb{Z}^d$ embeds in $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^d$. We indicate the proof in the torsion-free case, the proof for the torsion case is similar.

Denote by B the base of the wreath product $G = \mathbb{Z} \wr \mathbb{Z}^d$. The group satisfies the split extension

$$B \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d.$$

Consider the finite index subgroup H of G mapping to $Q = \mathbb{Z}^{d-1} \times d\mathbb{Z}$ in the quotient. Under the action of Q, the base B becomes a free $\mathbb{Z}Q$-module of rank d. Therefore H is isomorphic to the wreath product $\mathbb{Z}^d \wr \mathbb{Z}^d$. □

Chapitre 2

Isoperimetry and Følner couples

In this chapter, we explain how to relate the existence of a sequence of Følner couples in a group to its L^1-isoperimetric profile. As an application, we show that the existence of Følner couples descends to a quotient group or a closed subgroup. Then, we use it to obtain a lower bound on the return probability of torsion-free split metabelian groups of Krull dimension 1, without referring to the construction in [**PSC03**]. This provides a more self-contained proof of the main theorem of Chapter 1. In some sense, this chapter is thus an expanded version of part 5.3 of the previous chapter.

Some of the material presented here originates in the results developed in [**Tes08, Tes13**] in the context of metric measure spaces. Nevertheless, as we need to know precisely the dependence with respect to the scale to be able to apply our characterization, we recall some of the proofs to be able to keep track of the constants or provide more subtle statements.

1. Introduction

Let G be a locally compact, compactly generated group. Let λ be a left Haar measure on G. For any probability measure μ on G, we consider the random walk on G driven by μ and starting at the identity element e. Calling this process $(S_i)_{i\geq 0}$, we have $S_0 = e$ and $S_{i+1} = S_i X_{i+1}$ where $(X_i)_{i\geq 1}$ are independent and identically distributed G-valued random variables with probability distribution μ. If the current state is x, the probability of being in y at the next step is $\mu(x^{-1}y)$.

In the following, we assume that μ is symmetric, that is $\mu(U) = \mu(U^{-1})$, for every Borel subset U, and that the group G is unimodular. Moreover, we require μ to be absolutely continuous with respect to the Haar measure, with continuous density, and that the support of μ is compact and generates G.

Recall that if μ and ν are two probability measures on G, the convolution product $\mu \star \nu$ is the push-forward measure under the map

$$\begin{cases} G \times G \to G \\ (g, g') \mapsto gg'. \end{cases}$$

Observe that the distribution of S_i is the i-fold convolution product $\mu^{\star i} = \underbrace{\mu \star \cdots \star \mu}_{i}$.

The measure μ acts by convolution on $L^1(G, \lambda)$, we denote by P_μ this operator:

$$P_\mu : \begin{cases} L^1(G) \to L^2(G) \\ f \mapsto f \star \mu. \end{cases}$$

Set $\phi(n) = \|P_\mu^n\|_{1\to 2}^2$.

Asymptotic behaviour. If φ, ψ denote two monotone functions, we use the notation $\varphi \preccurlyeq \psi$ if there exist positive constants c and C such that $c\varphi(Ct) \leq \psi(t)$ (possibly for t in $\mathbb{N}$ if functions defined on $\mathbb{N}$ are considered). If the symmetric relation $\varphi \succcurlyeq \psi$ also holds, we write $\varphi \sim \psi$ and say that φ and ψ have the same asymptotic behaviour. This is an equivalence relation. As we need to know precisely the constants and their dependance, we should make in this chapter a sparse use of this notation.

In this setting, the asymptotic behavior of $\phi(n)$ does not depend on the choice of such a measure μ. Alternatively, $\phi(n)$ corresponds to the value at the identity element of the density of $\mu^{\star 2n}$ with respect to the Haar measure (see for example the survey by Tessera [**Tes16**]). When the group is finitely generated, $\phi(n)$ is the probability of being back in the identity element after $2n$ steps [1]. For these reasons, we refer to $\phi(n)$ as the *return probability* of G.

We quickly recall known results about the behavior of the return probability, that were mentioned with more details in Chapter 1. We also refer to the survey [**PSC01**] where most of these results are gathered.

Kesten's theorem states that G is non-amenable if and only if $\phi(n) \succcurlyeq \exp(-n)$ ([**Kes59**]). For virtually nilpotent groups, one can prove that $\phi(n) \asymp n^{-\frac{D}{2}}$ if and only if the group has polynomial growth, and this follows essentially from the seminal work of Varopoulos [**Var83a, Var83b, Var85b, Var87**]. Provided G is an amenable Lie group, the following dichotomy, takes place for the volume growth function:

— either G has polynomial growth of exponent D, and $\phi(n) \asymp n^{-\frac{D}{2}}$,

— either G has exponential growth and $\phi(n) \asymp \exp(-n^{\frac{1}{3}})$.

This dichotomy for the volume growth function was described by Guivarc'h and Jenkins ([**Gui73, Jen73**]). Before explaining the two corresponding behaviours for the return probability, let us mention a few words about the situation with discrete groups.

If G is a finitely generated group of exponential growth, one can only claim the upper bound

$$\phi(n) \preccurlyeq \exp(-n^{\frac{1}{3}})$$

by a result of Hebisch and Saloff-Coste ([**HSC93**]). The corresponding lower bound, which we refer to as a *large return probability behaviour*, fails in general: even if we restrict ourselves to the solvable case, there is a wider range of possible return probabilities than in the case of Lie groups. Actually, even among the class of metabelian groups, it is possible to find a group with return probability smaller than $\exp(-n^\alpha)$, for any $\frac{1}{3} \leq \alpha < 1$. Nevertheless, Alexopoulos established in [**Ale92**] that polycyclic groups do have this large return probability behavior.

In the previous chapter, we focused on the interesting question of characterizing those groups with exponential growth admitting this matching lower bound and we refer to this chapter for more information about it. Here we just want to tell a few words about the proof of the aforementioned bounds.

1. Even times $2n$ are considered to avoid parity issues: namely, the simple random walk on $\mathbb{Z}$, with usual generating set, cannot reach 0 at odd times.

The upper bounds are now well-understood and have a clear geometric interpretations. The proof goes as follows:first, one establishes some isoperimetric inequality for groups relating to the volume growth function ([**CSC93**]). The second step consists in deducing a Faber–Krahn inequality. Such an inequality provides a lower bound for the smallest eigenvalue of the Dirichlet problem depending on the measure of the domain. This latter inequality is known to be closely related to upper bounds on the return probability ([**Gri94**]). We refer to [**CGP01**, Sections 2 and 4] for a nice exposition on this matter.

Alternatively, in the polynomial growth case, we now know elementary proofs. For instance, it is possible to go from polynomial lower estimates on the volume growth to the correct upper estimate on the return probability, devising a proof (see [**Heb92, HSC93**]) that does not appeal to Gromov's polynomial growth theorem ([**Gro81**]). A simple proof for the lower bound was given by Barlow, Coulhon and Grigor'yan ([**BCG01**]), while the classical proof used Gaussian upper bounds for the heat kernel ([**Dav89, Str92, HSC93**]).

In the exponential growth case, the proof of the upper bound makes use of structure results for Lie groups and polycyclic groups as well as certain properties of Brownian motion in $\mathbb{R}^d$ and random walks in $\mathbb{Z}^d$ ([**HSC93**]).

In [**CGP01**], Coulhon, Grigor'yan and Pittet introduced a new method for obtaining return probability lower bounds on non-compact Lie groups and infinite discrete groups. Using this method, they were able to recover the results mentioned above for unimodular amenable Lie groups and for certain classes of discrete groups, including polycyclic groups and Baumslag-Solitar groups, as well as to obtain new results for some groups with a semi-direct product or a wreath product structure.

Their method provides a more direct approach and relies on two ingredients. The fisrt one is a technique for obtaining lower bounds based on the so-called anti-Faber-Krahn inequalities, from [**CG97**]. The second one is to construct sequences of so-called Følner couples (see Definition 3.1.2), that are couples of sets satisfying certain volume and distance requirements and whose existence implies an anti-Faber-Krahn inequality. In some sense, the construction of these couples reminds that of Følner sets on groups.

1.1. Results. In this chapter, we consider locally compact compactly generated groups. Our first result relates the existence of a sequence of Følner couples in a group to its L^1-isoperimetric profile.

PROPOSITION 1. *The group G admits a $\mathcal{L}$-sequence of Følner couples if and only if there exists $C > 0$ such that*

$$\forall n, \forall m \geq \mathcal{L}(n),\ J_G^n(m) \geq C, \tag{1.1}$$

where J_G^n denotes the isoperimetric profile at scale n (see part 2.4.2).

Our motivation for this proposition is to establish that admitting a sequence of Følner couples goes to a quotient by a closed subgroup or to a closed subgroup. To do so, we need to understand the behavior of the isoperimetric profile when going to a quotient by a closed subgroup or to a closed subgroup. Tessera proved in [**Tes13**], in the context of metric measure spaces, that it increases in both cases.

Unfortunately the constants he obtained depends on the scale of the gradient and one cannot apply directly the characterization above. Hence, we prove a stronger version of these results.

THEOREM 2. *Let $Q = G/H$ be the quotient of a locally compact, compactly generated unimodular group G by a closed normal subgroup H. Let $\pi : G \to Q$ denote the canonical projection. Then, for any $1 \leq p < +\infty$, there exists a constant b such that for any scale h and for any positive r*

$$j^h_{G,p}(r) \leq b\ j^h_{Q,p}(br).$$

As a corollary, we obtain that the existence of a sequence of Følner couples goes to a quotient by a closed subgroup. This was already stated for finitely generated groups in the previous chapter.

COROLLARY 3. *Let Q be a quotient of a finitely generated group G by a normal subgroup H. If G admits a sequence of Følner couples adapted to a function $\mathcal{L}$, then so does Q.*

We also establish that Følner couples go to a closed subgroup, generalizing the corresponding result for finitely generated groups by Erschler [**Ers06**].

THEOREM 4. *Let H be a closed compactly generated subgroup of a locally compact compactly generated group G. Assume G and H are unimodular.*
If G admits a $\mathcal{L}$-sequence of Følner couples, then so does H.

As an application, we produce Følner couples for finitely generated split torsion-free metabelian groups of Krull dimension 1:this provides a proof of the main theorem of the previous chapter that does not use the construction for torsion-free soluble groups of finite Prüfer rank from [**PSC03**].

THEOREM 5. *Let $G = A \rtimes \mathbb{Z}^d$ be a finitely generated group, so that $A = \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ is a ring of Krull dimension 1 and characteristic zero, where $\mathcal{P}$ is a prime ideal. Then, G admits a split sequence of Følner couples of exponential size.*

Whenever it is possible, we consider gradient and isoperimetric profiles at compact scale. This slightly generalizes the study initiated by Tessera in [**Tes08**], where scales were balls.

2. Preliminaries

2.1. Setting and some notations.

Let G be a locally compact, compactly generated unimodular group, equipped with a left-invariant Haar measure μ. Denote by S a compact and symmetric generating set for G, that is $G = \cup_{n\in\mathbb{N}} S^n$.

Here, for any two subsets A, B of G, AB denotes the set $\{ab \mid a \in A, b \in B\}$ and A^n is the set $AA \ldots A$, where A occurs n times.

Equip G with the left-invariant word metric d_S associated to S, such that $d_S(g,h) = \inf\{n \mid g^{-1}h \in S^n\}$. With these notations, gS^n is the ball of radius n centered around an element g, also denoted $B_G(g,n)$.

The triple (G, μ, d_S) defines a metric measure space, in the view of [**Tes08**].

Doubling constant at scale K. If K is a compact symmetric subset of G containing 1, let C_K denote the doubling constant at scale K, that is

$$C_K = \frac{\mu(K^2)}{\mu(K)}.$$

We begin with the following notion, generalizing to any compact symmetric subset K of G the gradient at scale h defined by Tessera ([**Tes08**], paragraph 2.1).

2.2. Gradient at scale K.

Let K be a compact symmetric subset of G. Given a function $f \in L^\infty(G)$, the norm of the gradient at scale K will measure the variations of f along K :

$$|\nabla f\,|_K\,(x) = \sup_{k\in K}|f(x) - f(xk)|. \tag{2.1}$$

The definition in [**Tes08**] corresponds to K being $B_G(1,h)$ the ball of radius h, centered around identity.

2.3. K-boundary.

Let K be a compact symmetric subset of G. The *K-boundary* of a measurable subset Ω is

$$\partial_K\Omega = \Omega K \cap \Omega^C K. \tag{2.2}$$

The following lemma explains the choice made for the gradient:

LEMMA 2.3.1. *Assume that K is a compact and symmetric subset of G containing the identity element. For every measurable subset Ω of G, for any $1 \leq p < +\infty$,*

$$\mu(\partial_K\Omega) = \int_G |\nabla 1_\Omega|_K(x)\mathrm{d}\mu(x) = \||\nabla 1_\Omega|_K\|_1.$$

PROOF. It is enough to prove that $1_{\partial_K\Omega} \equiv |\nabla 1_\Omega|_K$. Let $g \in G$. If $g \notin \partial_K\Omega$, then $g \notin \Omega K$ or $g \notin \Omega^C K$. In the first case, none of the K-translates of g belongs to Ω. In the second case, all of them belong to Ω. Therefore, in both cases, $|\nabla 1_\Omega|_K(g) = 0$. Assume now that $g \in \partial_K\Omega$. Similarly, considering if g belongs to Ω or its complement, one can show that there is a K-translate of g belonging to Ω^C or its complement Ω respectively. Therefore $|\nabla 1_\Omega|_K(x) = 1$. □

2.4. Isoperimetric profiles at scale K.

This terminology of "isoperimetric profile" is ambiguous as there exist various different definitions in the literature. Here, we consider the following setting.

2.4.1. *Functional isoperimetric profile.* Let Ω be a measurable subset of G. For any $1 \le p \le \infty$, define the *functional L^p-isoperimetric profile at scale K inside* Ω by

$$j_{G,p}^K(\Omega) = \sup_f \frac{||f||_p}{|||\nabla f\ |_K||_p}, \tag{2.3}$$

where the supremum is taken over all f in $L^\infty(G)$ with support inside Ω. The *functional L^p-isoperimetric profile at scale K* is then the non-decreasing function

$$j_{G,p}^K(r) = \sup_{\mu(\Omega)\le r} j_{G,p}^K(\Omega). \tag{2.4}$$

In the following, j_G^K will denote the L^1-isoperimetric profile at scale K. We may omit to mention the group whenever it is clear from the context.

Remark 2.4.1. Triangular inequality implies, for every $x \in G$ and every $k \in K$,

$$|\nabla f|_K(x) \le |\nabla f|_{K^2}(x) \le |\nabla f|_K(x) + |\nabla f|_K(xk).$$

Integrating over x yields $|||\nabla f|_K||_p \le |||\nabla f|_{K^2}||_p \le 2|||\nabla f|_K||_p$. As a consequence,

$$\frac{j_p^K}{2} \le j_p^{K^2} \le j_p^K.$$

Remark 2.4.2. If ρ denotes the action of G by right-translation on functions, given by $\rho(g)f(x) = f(xg)$, then one can rewrite the functional isoperimetric profile as

$$j_{G,p}^K(r) = \sup_{|\mathrm{Supp}(f)|\le r} \frac{||f||_p}{||\sup_{k\in K}|f - \rho(k)f|||_p}$$

2.4.2. *Isoperimetric profile.*
Considering measurable subsets, one can also define the *isoperimetric profile at scale K*

$$J_G^K(r) = \sup_{\mu(\Omega)\le r} \frac{\mu(\Omega)}{\mu(\partial_K\Omega)}, \tag{2.5}$$

dropping the G when it is clear from the context.

2.5. Isoperimetric inequalities at scale K.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. The group G satisfies an isoperimetric inequality (I_φ^K) at scale K if there exist positive constants C, C' such that for every measurable subset Ω,

$$\frac{\mu(\partial_K\Omega)}{\mu(\Omega)} \ge \frac{1}{C\varphi(C'\mu(\Omega))}. \tag{I_φ^K}$$

If we want to remember the constants, we should say that G satisfies (I_φ^K, C, C').

By definition, G always satisfies $(I_{J_G^K}^K, 1, 1)$. Conversely, if G satisfies (I_φ^K), then $\varphi \succcurlyeq J_G^K$.

2.6. Sobolev inequalities at scale K.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function, K a compact and symmetric subset of G, and let $p \in [1, +\infty]$.

As introduced in [**Cou03**], we say that G satisfies a Sobolev inequality $(S^K_{\varphi,p})$ at scale K if there exist positive constants C, C', depending only on K and φ such that

$$(S^K_{\varphi,p}) \qquad ||f||_p \leq C\varphi(C'\mu(\Omega))|||\nabla f|_K||_p$$

where Ω ranges over all compacts subsets of G and $f \in L^\infty(\Omega)$. We should say that G satisfies $(S^K_{\varphi,p})$, or $(S^K_{\varphi,p}, C, C')$ if we need to remember the constants. The Sobolev inequality $(S^K_{\varphi,1})$ will be denoted (S^K_φ).

By definition, G always satisfies $(S^K_{j^K_{G,p},p}, 1, 1)$. Conversely, if $(S^K_{\varphi,p})$ holds then $\varphi \succcurlyeq j^K_{G,p}$.

Whenever $1 \leq p \leq q < +\infty$, the Sobolev inequality $(S^K_{\varphi,p})$ implies $(S^K_{\varphi,q})$. In the Riemannian setting, the proof goes back to [**Cou03**, Proposition 2.1]. In general, the proof uses Hölder as in the Riemannian case combined with Lemma 4 in [**Ms97**].

2.7. Equivalence of Sobolev and isoperimetric inequalities.

We start recalling equivalence of Sobolev inequalities and isoperimetric inequalities at scale K.

Proposition 2.7.1. *The inequality (S^K_φ) is equivalent to the isoperimetric inequality (I^K_φ). More precisely*

$$(S^K_\varphi, C, C') \Rightarrow (I^K_\varphi, C, C') \Rightarrow (S^K_\varphi, 2C, C').$$

The usual proof easily extends to our context, using the following version of the coarea formula, stated at scale $B_G(1, h)$ in [**Tes08**].

Lemma 2.7.2 (Coarea formula at scale K). *Let f be a non-negative function defined on G. Then,*

$$\frac{1}{2}\int_{\mathbb{R}^+} \mu(\partial_K(\{f \geq t\})\mathrm{d}t \leq \int_G |\nabla f \mid_K (x)\mathrm{d}\mu(x) \leq \int_{\mathbb{R}^+} \mu(\partial_K(\{f \geq t\})\mathrm{d}t.$$

Proof. The proof given in [**Tes08**], section 3.1, generalizes to a compact symmetric scale K. □

Proof of Proposition 2.7.1. Assume (S^K_φ) holds. Applying it with $f = 1_\Omega$, with Ω a compact subset of G yields (I^K_φ), and the constants are not impacted.

Now assume that (I^K_φ, C, C') holds. Hence, for every measurable subset Λ, one has

$$\frac{\mu(\partial_K \Lambda)}{\mu(\Lambda)} \geq \frac{1}{C\varphi(C'\mu(\Lambda))}.$$

Let Ω be a compact subset of G, $f \in L^\infty(\Omega)$ and $t > 0$. The set $\{f \geq t\}$ is a subset of Ω. As φ increases

$$\mu(\partial_K(\{f \geq t\}) \geq \frac{\mu(\{f \geq t\})}{C\varphi(C'\mu(\{f \geq t\}))} \geq \frac{\mu(\{f \geq t\})}{C\varphi(C'\mu(\Omega))}.$$

The coarea formula then implies

$$|||\nabla f|||_1 \geq \frac{1}{2C\varphi(C'\mu(\Omega))} \int_{R+} \mu(\{|f| \geq t\})\mathrm{d}t \geq \frac{||f||_1}{2C\varphi(C'\mu(\Omega))},$$

hence $(S^K_\varphi, 2C, C')$ holds. □

As a corollary, the isoperimetric profiles have the same asymptotic behaviour.

Corollary 2.7.3. *We have*

$$J^K_G(r) \leq j^K_G(r) \leq 2J^K_G(r).$$

Proof. The left-hand side inequality arises when taking $f = 1_\Omega$, with $\mu(\Omega) \leq r$.

To prove the right-hand side, we use the fact that $(I^K_{J_K}, 1, 1)$ always holds. Hence, by proposition 2.7.1 $(S^K_{J_K}, 2, 1)$ holds and by definition of the functional isoperimetric profile, $j_K \leq 2J_K$. □

Remark 2.7.4. We could have defined the isoperimetric profile using left-translation, namely

$$\sup_{|\mathrm{Supp}(f)| \leq r} \frac{||f||_p}{||\sup\limits_{k \in K} |f - \lambda(k)f|||_p},$$

where λ is the action of G by left-translation, that is $\lambda(g)f(x) = f(g^{-1}x)$. Discussion and comparison between these two choices appears in **[Tes13]**, section 4.

Notation. In the following, we will denote with by $|\nabla f\,|_n$ the gradient at scale S^n and use the term n-boundary as well as the corresponding notation for the S^n-boundary. Finally, $j^n_{G,p}$ and J^n_G will be the isoperimetric profiles at scale S^n.

2.8. Thickness.

Our study of isoperimetry, and Følner pairs, for subgroups or quotients, will sometimes require to understand how the measure of projection or restriction of sets relates to the original measure. To obtain such a control, it will be more convenient to work with balls. For a subset $Z \subset G$, we denote by $[Z]_v$ the v-neighbourhood of Z, that is $[Z]_v = \{x \in G \mid d(x, Z) \leq v\}$.

Definition 2.8.1. A subset A of a metric space is called *h-thick* if it is a union of closed balls of radius h.

The following lemma from **[Tes08]**, follows from standard covering argument.

Lemma 2.8.2. [**Tes08**, Proposition 8.4] *Let G be a locally compact group and μ be a Haar measure on G. Then, there exists $C_{u,v}$ such that for any u-thick subset A and for any v*

$$\mu([A]_v) \leq C_{u,v}\mu(A)$$

Lemma 2.8.3. *Let $Q = G/H$ be the quotient of a locally compact, compactly generated unimodular group G by a closed normal subgroup H. Let $\pi : G \to Q$ be the canonical projection and denote by μ, resp. ν, the Haar measure on G, resp. G/H.*

Then, for any $u > 0$, there exists a constant a_u such that, for any u-thick measurable subset A of G,

$$\nu(\pi(A)) \leq a_u\mu(A).$$

PROOF. Since A is u-thick, there exists a subset $\mathcal{A}$ in A such that $A = \cup_{x\in\mathcal{A}} B(x,u)$. Let Z be a maximal u-separated subset of $\mathcal{A}$. Then, u-thickness of A and maximality implies that $[Z]_{u/3} \subset A \subset [Z]_{2u}$, and the balls $B(x, \frac{u}{3})$ are disjoint. Set

$$C_u = \frac{\mu(B(\bullet, 2u))}{\mu(B(\bullet, \frac{u}{3}))} \text{ and } C'_u = \frac{\nu(B_Q(\bullet, 2u))}{\mu(B_G(\bullet, 2u))},$$

whose values are independent of the centre of the balls. Going to the quotient, we have

$$\begin{aligned}
\nu(\pi(A)) &\leq \nu(\pi([Z]_{2u})) \leq \sum_{z\in Z} \nu(B_Q(\pi(z), 2u)) \\
&\leq C'_u \sum_{z\in Z} \mu(B_G(z, 2u)) \\
&\leq C_u C'_u \sum_{z\in Z} \mu\left(B_G\left(z, \frac{u}{3}\right)\right) = C_u C'_u \mu([Z]_{u/3}) \\
&\leq C_u C'_u \mu(A).
\end{aligned}$$

□

LEMMA 2.8.4. *Let H be a closed generated subgroup in a locally compact group G. Endowed G with a left-invariance distance d_G and H with the associated induced distance d_H. Equip G, resp. H, with Haar measures μ, resp. ν.*

Then, for any $u > 0$, there exists a constant c_u such that, for any subset A of G which u-thick,

$$\nu(A \cap H) \leq c_u \mu(A).$$

PROOF. Pick a maximal 2-separated (for d_H) subset Z of $A \cap H$. By maximality, we have

$$[A \cap H]_1 = \{h \in H \mid d_H(h, A \cap H) \leq 1\} \subset [Z]_3 = \{h \in H \mid d_H(h, Z) \leq 3\}$$

and separability implies that the balls $B_G(z, 1)$, for $z \in Z$, are disjoint subset of G. Set

$$C = \frac{\nu(B_H(\bullet, 3))}{\nu(B_H(\bullet, 1))} \text{ and } C' = \frac{\nu(B_H(\bullet, 1))}{\mu(B_G(\bullet, 1))},$$

by left-invariance, they are indeed independent of the center of the balls considered. Then,

$$\begin{aligned}
\nu(A \cap H) &\leq \nu([A \cap H]_1) \leq \nu([Z]_3) \leq \sum_{z\in Z} \nu(B_H(z, 3)) \\
&\leq C \sum_{z\in Z} \nu(B_H(z, 1)) \leq CC' \sum_{z\in Z} \mu(B_G(z, 1)) \leq CC' \mu([A]_1) \\
&\leq c_u \mu(A).
\end{aligned}$$

where we use the fact that A is u-thick and Lemma 2.8.2 in the last inequality.

□

PROPOSITION 2.8.5. *Let $1 \leq p < \infty$. Then, for every $f \in L^\infty(G)$ and every $h \geq \frac{8}{3}$, there is a function $\tilde{f} \in L^\infty(G)$ whose support is included in a 1-thick subset*

$\tilde{A}$ *such that* $\mu(\tilde{A}) \leq \alpha_p \mu(\mathrm{Supp}(f))$*, and*

$$\frac{||f||_p}{|||\nabla f|_h||_p} \leq 5^p \frac{||\tilde{f}||_p}{|||\nabla \tilde{f}|_{\frac{h}{2}}||_p}.$$

This proposition origins in [**Tes08**, Proposition 8.3] and allows to restrict to functions with thick support. Indeed, a similar result is proved in [**Tes08**] for any scale h: it is better in terms of the thickness of $\mathrm{Supp}(\tilde{f})$ but the constant b depends on h. Here, we prove a more subtle statement for big enough h without this dependance. The proof makes use of the following easy lemma, of which we give a proof for completeness.

LEMMA 2.8.6. *Let* a, a', b, b' *and* β *be real numbers such that* $b, b' > 0$ *and*

$$\frac{a+a'}{b+b'} \leq \beta \text{ and } \frac{a'}{b'} > \beta.$$

Then

$$\frac{a}{b} \leq \beta.$$

PROOF. We prove the contraposition. Assume $\frac{a}{b} > \beta$ and $\frac{a'}{b'} > \beta$. Then

$$\frac{a+a'}{b+b'} > \frac{\frac{a}{b}b'^{-1} + \frac{a'}{b'}b^{-1}}{b^{-1}+b'^{-1}} > \beta.$$

□

PROOF. Fix a function f in $L^\infty(G)$ and set $A = \mathrm{Supp}(f)$. Let $n > 1$, whose value we shall choose below and define a subset Ω by

$$\Omega = \left\{ x \in A \mid \frac{\mu(B(x,1) \cap A)}{\mu(B(x,1))} \geq \frac{1}{n} \right\} \text{ and } A_1 = [\Omega]_1.$$

Set $A_2 = A \setminus (A_1 \cap A)$, $f_1 = f.\mathbb{1}_{A_1}$ and $f_2 = f - f_1 = f.\mathbb{1}_{A_2}$. It is clear that

$$||f||_p^p = ||f_1||_p^p + ||f_2||_p^p. \tag{2.6}$$

The support of f_1 is included in the 1-thick subset A_1.

CLAIM 1. There exists a C such that

$$\mu(A_1) \leq C\mu(A).$$

PROOF OF THE CLAIM. To prove this inequality, let Z be a maximal 3-separated subset of Ω. Then $[Z]_1 \subset A_1 \subset [Z]_4$ and the balls $B(z,1)$ for $z \in Z$ are disjoint. We have

$$\mu(A_1) \leq \sum_{z\in Z} \mu(B(z,4)) \leq C \sum_{z\in Z} \mu(B(z,1)) \leq Cn \sum_{z\in Z} \mu(B(z,1)\cap A) \leq Cn\mu(A).$$

□

To apply the lemma above, we need the corresponding inequalities: these are the objects of the following claims.

CLAIM 2. We have

$$|||\nabla f_2|_{\frac{h}{2}}||_p \geq \frac{\alpha}{2} n ||f_2||_p. \tag{2.7}$$

Proof of the claim. To prove the claim, let Z be a maximal 1-separated subset of A_2. By maximality,

$$A_2 \subset \bigcup_{z \in Z} B(z,1),$$

and by separation the balls $B(z, \frac{1}{3})$ for $z \in Z$ are disjoint. We have

$$||f_2||_p^p \leq \sum_{z \in Z} \int_{B(z,1)} |f_2|^p \mathrm{d}\mu \leq \sum_{z \in Z} \mu(A \cap B(z,1)) \sup_{B(z,1)} |f_2|.$$

On the other hand,

$$\begin{aligned} |||\nabla f_2|_{\frac{h}{2}}||_p^p &\geq \sum_{z \in Z} \int_{B(z,\frac{1}{3})} |\nabla f_2|_{\frac{h}{2}}^p \mathrm{d}\mu \geq \sum_{z \in Z} \mu\left(B(z, \frac{1}{3})\right) \inf_{B(z,\frac{1}{3})} |\nabla f_2|_{\frac{h}{2}} \\ &\geq \alpha \sum_{z \in Z} \mu\left(B(z,1)\right) \inf_{B(z,\frac{1}{3})} |\nabla f_2|_{\frac{h}{2}}. \end{aligned}$$

For $z \in Z$, let us compare the above infimum and supremum.We consider a slightly different gradient:for any function g and scale k define

$$|\tilde{\nabla} g|_k(x) = \sup_{y,y' \in B(x,k)} |g(y) - g(y')|.$$

By triangular inequality, $|\tilde{\nabla} g|_k(x) \leq 2|\nabla g|_k(x)$. Moreover, for any $x \in B(z, \frac{1}{3})$,

$$|\tilde{\nabla} f_2|_{\frac{h}{2}}(x) \geq \sup_{B(x,\frac{h}{2})} |f_2| \geq \sup_{B(z,1)} |f_2|$$

as $B(x, \frac{h}{2})$ contains $B(z,1)$ if $h \geq \frac{8}{3}$ and the latter meets A^C.

As a consequence,

$$\frac{|||\nabla f_2|_{\frac{h}{2}}||_p^p}{||f_2||_p^p} \geq \frac{\alpha}{2} n$$

and this completes the proof of the claim. □

Claim 3. We have

$$|||\nabla f_1|_{\frac{h}{2}}||_p^p + |||\nabla f_2|_{\frac{h}{2}}||_p^p \leq 5^p |||\nabla f|_h||_p^p. \tag{2.8}$$

Proof of the claim. To prove the claim, we prove that, for every x,

$$|\nabla f_1|_{\frac{h}{2}}(x) + |\nabla f_2|_{\frac{h}{2}}(x) \leq 5|\nabla f|_h(x).$$

Note that, as $f_1 = f - f_2$, triangular inequality implies that $|\nabla f_1|_{\frac{h}{2}}(x) \leq |\nabla f_2|_{\frac{h}{2}}(x) + |\nabla f|_{\frac{h}{2}}(x)$. Hence, it is enough to prove that $|\nabla f_2|_{\frac{h}{2}}(x) \leq 2|\nabla f|_h(x)$.

If $x \notin A$, this is clear. Assume now that $x \in A$. We distinguish two cases. First, assume that $d(x, A^C) < h$. The ball $B(x,h)$ meets A^C, therefore

$$|\nabla f|_h(x) = \max\{|f(x)|, \sup_{y \in B(x,h)} |f(x) - f(y)|\}.$$

In this case

$$|\nabla f_2|_{\frac{h}{2}}(x) = \begin{cases} \sup_{B(x,\frac{h}{2})}|f_2| \text{ if } x \notin A_2 \\ \sup_{y\in B(x,\frac{h}{2})}|f(x) - f_2(y)| \text{ otherwise} \end{cases}$$
$$\leq 2\max\{|f(x)|, \sup_{y\in B(x,\frac{h}{2})} |f(x) - f(y)|\} \leq 2|\nabla f|_h(x).$$

Secondly, assume that $d(x, A^C) \geq h$. Then, for any y in $B(x, \frac{h}{2})$, as $h \geq \frac{8}{3}$, the ball $B(y, 1)$ is included in A. Consequently, $|\nabla f_2|_{\frac{h}{2}}(x) = 0$ and this ends the proof of the claim. □

Combining inequalities 2.6 and 2.8, we obtain

$$\frac{|||\nabla f_1|_{\frac{h}{2}}||_p^p + |||\nabla f_2|_{\frac{h}{2}}||_p^p}{||f_1||_p^p + ||f_2||_p^p} \leq 5^p \frac{|||\nabla f|_h||_p^p}{||f||_p^p}. \tag{2.9}$$

We now choose n such that

$$n \geq \frac{2.5^p|||\nabla f|_h||_p^p}{\alpha||f||_p^p}$$

and apply Lemma 2.8.6 to (2.9) and (2.7) to conclude. □

Isoperimetric profile and quotient.

In this part, we show that the functional isoperimetric profile at scale h increases when going to a quotient. This was already proved by Tessera in [**Tes13**, Proposition 5.5], although the constant he obtained depends on the scale. Here, our constant is uniform with respect to the scale.

THEOREM 2.8.7. *Let $Q = G/H$ be the quotient of a locally compact, compactly generated unimodular group G by a closed normal subgroup H. Let $\pi : G \to Q$ denote the canonical projection. Then, for any $1 \leq p < +\infty$, there exists a constant $b = b(p, G, H)$ such that for any scale $h \geq \frac{8}{3}$ and for any positive r,*

$$j_{G,p}^h(r) \leq b\, j_{Q,p}^{\frac{h}{2}}(br).$$

The proof we give follows part of the lines of [**Tes13**] but do not use exactly the same definition of gradient. Indeed, the proof given in [**Tes13**] would not give a constant independent of h, as discussed in the appendix.

PROOF. Consider Q as the group of right H-cosets of G. Take the image $T = \pi(S)$ as a compact symmetric generating set for Q, so that the map $\pi : (G, S) \to (Q, T)$ is 1-Lipschitz. Equip G and Q with left Haar measures μ and ν and take a Haar measure σ on H such that, for every continuous compactly supported function f on G,

$$\int_G f(g)\mathrm{d}\mu(g) = \int_Q \left(\int_H f(hg)\mathrm{d}\sigma(h)\right) \mathrm{d}\nu(Hg).$$

Let $1 \leq p < \infty$ and $\Sigma : C_0(G) \to C_0(Q)$ defined by

$$\Sigma(f)(Hg) = \left(\int_H |f(hg)|^p\mathrm{d}\sigma(h)\right)^{\frac{1}{p}}.$$

The application Σ preserve the L^p-norm and the support of $\Sigma(f)$ is the projection on Q of the support of f. Fix $r > 0$ and let $f \in L^\infty(G)$ such that $\mu(\mathrm{Supp}(f)) \leq r$. Let $g \in G$. By definition of the gradient and choice of T,

$$|\nabla\Sigma(f)|_h(Hg) = \sup_{k\in S^h} |\Sigma(f)(Hg) - \Sigma(f)(Hgk)|.$$

Hence,

$$|\nabla\Sigma(f)|_h(Hg) = \sup_{k\in S^h} \left| \left(\int_H |f(hg)|^p \mathrm{d}\sigma(h)\right)^{\frac{1}{p}} - \left(\int_H |f(hgk)|^p \mathrm{d}\sigma(h)\right)^{\frac{1}{p}} \right|$$

$$\leq \left(\int_H (\sup_{k\in S^h} |f(hg) - f(hgk)|)^p \mathrm{d}\sigma(h)\right)^{\frac{1}{p}} = \left(\int_H \left(|\nabla f|_h(hg)\right)^p d\sigma(h)\right)^{\frac{1}{p}}.$$

Integrating on Q yields, by definition of the measure σ,

$$|||\nabla\Sigma(f)|_h||_p \leq |||\nabla f|_h||_p.$$

As a consequence, we have the comparison

$$\frac{||f||_p}{|||\nabla f|_h||_p} \leq \frac{||\Sigma(f)||_p}{|||\nabla\Sigma(f)|_h||_p}.$$

To derive the conclusion for the functional isoperimetric profiles of G at scale h and G/H at scale h, it is necessary to relate $\nu(\mathrm{Supp}(\Sigma(f)))$ with $\mu(\mathrm{Supp}(f))$. To do so, we use Proposition 2.8.5 to restrict to functions with thick support, at the cost of a factor b. The support is now included in a 1-thick subset whose measure is bounded above by αr. Then, Lemma 2.8.3 allows to compare the measure of the support with the measure of its projection to the quotient. This completes the proof of the theorem. □

3. Følner couples in locally compact compactly generated groups

Sequences of Følner couples are tools introduced in **[CGP01]** to produce lower bounds on the return probability. In this section, we recall the definition and explain how to characterize their existence in terms of isoperimetric profile. We then apply this characterization to establish that the existence of such a sequence goes to a quotient. Finally, we address the case of subgroups.

3.1. Definition.

Definition 3.1.1 (**[CGP01]**). We say that G admits a *sequence of Følner couples* adapted to the class of a function $\mathcal{L}(n)$ if there exists a sequence (F'_n, F_n) of compact subsets of G and $C > 0$, such that

(1) $F'_n S^n \subset F_n$.

(2) $\mu(F_n) \preccurlyeq \mathcal{L}(n)$.

(3) $\mu(F'_n) \geq C\mu(F_n)$.

To specify the function, we use the term $\mathcal{L}$-sequence of Følner couples. We should say that the couples have exponential size whenever $\mathcal{L} \simeq \exp$.

Example 3.1.2 ([**CGP01**]). Let G be the Baumslag-Solitar $BS(q) :=< x, y \mid xyx^{-1} = y^q >$. It is isomorphic to the semi-direct product $\mathbb{Z}[\frac{1}{q}] \rtimes \mathbb{Z}$ and is metabelian, but non polycyclic as it contains the non-finitely generated subgroup $\mathbb{Z}[\frac{1}{q}]$. A generating set for G is given by $(\pm 1, 0), (0, \pm 1)$. Let

$$\Omega_n = \{u \in \mathbb{Z} \mid |u| < (2q)^n\}$$
$$\Omega'_n = \{u \in \mathbb{Z} \mid |u| < 3(2^n - n)q^n\}$$

Then, if $F_n = \Omega_n \times [\![0, 3n]\!]$ and $F'_n = \Omega'_n \times [\![n, 2n]\!]$, the sequence $(F_n, F'_n)_n$ is a sequence of Følner couples associated to $\mathcal{L}(n) = \exp(n)$.

Remark 3.1.3. Existence of Følner couples adapted to the class of a given function $\mathcal{L}$ does not depend on the choice of the generating set for G.

Interest for the existence of sequences of Følner couples lies in the following application: existence of Følner couples yields a lower bound for the return probability (see Coulhon, Grigor'yan and Pittet [**CGP01**], as well as Erschler [**Ers06**]). We only give one corollary.

Corollary 3.1.4. *If the group G admits a sequence of Følner couples adapted to a function of the form $\mathcal{V}(t) = C\exp(Ct^d)$, then*

$$p_{2n}^G \succcurlyeq \exp(-n^{\frac{d}{d+2}}).$$

3.2. A characterization in terms of isoperimetry.

Proposition 3.2.1. *The group G admits a $\mathcal{L}$-sequence of Følner couples if and only if there exists $C > 0$ such that*

$$\forall n, \forall m \geq \mathcal{L}(n),\ J_G^n(m) \geq C. \tag{3.1}$$

Proof. To prove the direct implication, it is enough to find a positive constant C and, for every n, a measurable subset G_n of measure at most $\mathcal{L}(n)$ satisfying

$$\frac{\mu(G_n)}{\mu(\partial_n G_n)} \geq C. \tag{3.2}$$

Take C as given in the definition of Følner couples and set $G_n = F'_n$. Part (1) of the definition implies that $\partial_n F'_n \subset F_n$.

To prove the reverse implication, fix some small positive ϵ such that $C - \epsilon > 0$. For every n, let G_n be a compact subset of mesure at most $\mathcal{L}(n)$ such that:

$$\frac{\mu(G_n)}{\mu(\partial_n G_n)} \geq C - \epsilon.$$

Set $F_n = G_n \cup \partial_n G_n$ and $F'_n = G_n$. Clearly, $F'_n S_n \subset F_n$.

One can check that $\mu(F_n) \leq \mu(G_n) + \mu(\partial_n G_n) \leq (1 + (C - \epsilon)^{-1})\mu(G_n) \preccurlyeq \mathcal{L}(n)$. Similarly

$$\frac{\mu(F'_n)}{\mu(F_n)} \geq \frac{1}{1 + (C - \epsilon)^{-1}}.$$

□

3.3. Følner couples and quotient.

We now use the characterization to produce sequences of Følner couples in a quotient group.

Corollary 3.3.1. *Let $Q = G/H$ be the quotient of a locally compact, compactly generated unimodular group G by a closed normal subgroup H. If G admits a sequence of Følner couples adapted to a function $\mathcal{L}$, then so does Q.*

In the previous chapter, we proved it for finitely generated groups, in a less natural way (see Proposition 5.10, Chapter 1).

Proof. It readily follows from the combination of Proposition 3.2.1 and Theorem 2.8.7. $\square$

3.4. Følner couples and closed subgroups.

3.4.1. *Inheritance.* Having Følner couples in finitely generated groups is inherited by finitely generated subgroups, as proved by Erschler in [**Ers03**], Lemma 4 (her argument for Følner functions also deals with sequences of Følner couples). We prove a similar statement in the case of locally compact compactly generated groups, following the same lines of reasoning: the only thing to do is to deal with the measures.

Theorem 3.4.1. *Let H be a closed compactly generated subgroup of a locally compact compactly generated group G. Assume G and H are unimodular.*

If G admits a $\mathcal{L}$-sequence of Følner couples, then so does H.

Proof. We equip G and H with left Haar measures μ and ν. Let λ be a G-invariant σ-finite measure on the quotient G/H. Up to normalization, we can assume that for every integrable compactly supported function f on G,

$$\int_G f(g)\mathrm{d}\mu(g) = \int_{G/H}\left(\int_H f(gh)\mathrm{d}\nu(h)\right)\mathrm{d}\lambda(gH). \tag{3.3}$$

Let S, resp. S_H, be a compact generating set for G, resp. H, with $S_H \subset G$. Let $(F_n, F'_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}$-Følner couples of G. We may assume that F_n is 1-thick, for example by replacing it with F'_nS^n. Fix a measurable set Q of representatives of G/H in G and set, for every n in $\mathbb{N}$ and every q in Q, $F_{n,q} = q^{-1}F_n \cap H$, and $F'_{n,q} = q^{-1}F'_n \cap H$, so that

$$F_n = \bigcup_{q\in Q} qF_{n,q} \text{ and } F'_n = \bigcup_{q\in Q} qF'_{n,q}.$$

By construction, for every q in Q and every n in $\mathbb{N}$, $F'_{n,q} \subset F_{n,q}$ and $F'_{n,q}S_H^n \subset q^{-1}F'_nS^n \cap H \subset F_{n,q}$. By 3.3 and pigeon-hole principle, there exists a C such that for every n, there is a q_n in Q such that $\nu(F'_{n,q_n}) \geq C\nu(F_{n,q_n})$.

The size condition $\nu(F_{n,q}) \preccurlyeq \mathcal{L}(n)$ follows from Lemma 2.8.4 as F_n is a thick subset of G.

Finally, $(F_{n,q_n}, F'_{n,q_n})_{n\in\mathbb{N}}$ provides the desired sequence of Følner couples. $\square$

REMARK 3.4.2. In the first point of Definition 3.1.1 of Følner couples, we require $F'_n S^n \subset F_n$. This rewrites it as $d(F'_n, (F_n)^C) \geq n$, where d denotes the word distance associated to the generating set S.

It is actually possible to define Følner couples for any other distance. With this in mind, note that the previous proof actually provides Følner couples in H for the induced distance, which is smaller than the word distance.

3.4.2. *Split couples.* When working with semi-direct products, it will be natural to consider couples that respect the splitting.

DEFINITION 3.4.3. Let G be a locally compact compactly generated group with two subgroups H, K so that H is abelian and $G = H \rtimes K$ is a semi-direct product. Equip G with the metric associated to a generating set $S = S_H \cup S_K$ where S_H is a finite generating set for the $\mathbb{Z}K-$module H and S_K is a finite generating set for the group K. Let $(F_n, F'_n)_n$ be a sequence of Følner couples of G. We say that this sequence *splits* if there exists A_n, A'_n, finite subsets of H and B_n, B'_n finite subsets of K such that, as sets, $F_n = A_n \times B_n$ and $F'_n = A'_n \times B'_n$ for every n.

REMARK 3.4.4. The sequence of couples $(B'_n, B_n)_n$ is a sequence of Følner couples for K.

In the proof of the previous proposition, the couples obtained are cosets of the original couples. Hence, we may state the following corollary.

COROLLARY 3.4.5. *Let G be a locally compact compactly generated group with two subgroups H, K so that H is abelian and $G = H \rtimes K$ is a semi-direct product. Let $L = H' \rtimes K$ be a closed compactly generated subgroup of G such that H' is a subgroup of H.*

Assume that G admits a sequence of split $\mathcal{L}-$Følner couples whose projection on K is the sequence $(K'_n, K_n)_n$. Then L admits a sequence of split $\mathcal{L}$-Følner couples whose projection on K is $(K'_n, K_n)_n$.

4. Construction of Følner couples for split torsion-free metabelian groups

4.1. The reason.

Recall that a finitely generated group G is said to have *large return probability* if $p_{2n}^G \succcurlyeq \exp(-n^{\frac{1}{3}})$. Section 5.3 of chapter 1 contains the proof of the main result of this chapter, namely

THEOREM (Theorem 1 in Chapter 1). *Let G be a finitely generated metabelian group. Then,*

$$G \textit{ has large return probability } \Leftrightarrow G \textit{ has Krull dimension } 0 \textit{ or } 1.$$

We refer to the previous chapter for more details about the Krull dimension of a group. Remind that the direct implication follows from looking at the impact of the Krull dimension on the structure of the group.

If G is a finitely generated metabelian group, we denote by M and Q two abelian groups so that Q is finitely generated and G is an extension of M by Q. The proof of the reverse implication makes use of the machinery of sequences of Følner couples designed by Coulhon, Grigor'yan and Pittet in [**CGP01**] and, in view of Corollary 3.1.4, it is enough to provide such a sequence of couples of exponential size. Let us now outline this construction.

First, we restrict to the split case with quotient Q being finitely generated free abelian, so that $G = M \rtimes Q$. The splitting will persist in the couples we propose to construct.

Secondly, we decompose M as a tower of submodules $\{0\} = M_0 \leq M_1 \leq \cdots \leq M_n = M$ and study the groups $M_{i+1}/M_i \rtimes Q$. This is possible by Proposition 2.1.19, Chapter 1. Whenever G has Krull dimension at most 1, so have they. Recall that M_{i+1}/M_i is isomorphic to $\mathbb{Z}Q/\mathcal{P}_i$ for some prime ideal $\mathcal{P}_i$, thus has characteristic either zero or prime.

In the torsion case, we construct a sequence of split Følner couples of exponential size, see part 5.2.1, Chapter 1. In the torsion-free case, we show that the group has finite Prüfer rank (that is, there is a bound on the number of generators needed to generate any finitely generated subgroup) and use a general construction of (split) Følner couples for finitely generated solvable groups of finite rank due to Pittet and Saloff-Coste [**PSC03**]. Along the way, we manage to prescribe the projection on Q.

Then, we combine all these couples through the lifting Lemma 5.3.4, Chapter 1. This uses in particular the fact that the existence of (split) Følner couples in finitely generated groups descends to a quotient group (Proposition 5.3.1, Chapter 1).

In the following subsection, we intend to construct sequences of Følner couples in the torsion-free case, thus providing a more self-contained proof of the theorem above, as it will no longer require the construction from [**PSC03**].

4.2. The construction.

We consider a finitely generated metabelian group $G = M \rtimes Q$ of Krull dimension at most 1, where M is isomorphic to a ring $\mathbb{Z}Q/\mathcal{P}$ of characteristic zero, for some prime ideal $\mathcal{P}$, and Q is finitely generated free abelian. Thus, G is actually torsion-free of Krull dimension 1. We intend to construct sequences of Følner couples of exponential size in G with prescribed projection on Q.

THEOREM 4.2.1. *Let $G = A \rtimes \mathbb{Z}^d$ be a finitely generated group, so that $A = \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ is a ring of Krull dimension 1 and characteristic zero, where $\mathcal{P}$ is a prime ideal. Then, G admits a split sequence of Følner couples of exponential size that projects onto the Følner couples $([\![-n, n]\!]^d, [\![-2n, 2n]\!]^d)_n$ on $\mathbb{Z}^d$.*

PROOF. As A has Krull dimension 1, Noether's normalization theorem (Theorem 2.1.9, Chapter 1) implies that there exists $c \in \mathbb{Z}$ such that A_c contains $\mathbb{Z}[\frac{1}{c}]$ and is finite as a $\mathbb{Z}[\frac{1}{c}]$−module. The fraction field of A_c is a finite extension of $\mathbb{Q}$. By the primitive element theorem, it is a simple extension of $\mathbb{Q}$: let α be a complex number so that $\mathrm{Frac}(A_c) = \mathbb{Q}(\alpha)$. The diagonal embedding

$$A_c \hookrightarrow \prod_{p|c} \mathbb{Q}_p(\alpha) \times \mathbb{C} = B,$$

where the product runs over the primes p dividing c, is a discrete embedding. The action of $\mathbb{Z}^d$ on A extends to a diagonal action on B.

Claim 1. *The group $B \rtimes \mathbb{Z}^d$ admits a sequence of split Følner couples of exponential size that project onto the Følner couples $(\llbracket -n,n\rrbracket^d, \llbracket -2n,2n\rrbracket^d)_n$ on $\mathbb{Z}^d$.*

As G is a subgroup of $B\rtimes\mathbb{Z}^d$, Corollary 3.4.5 yields a sequence of split Følner couples of exponential size for G that projects onto the Følner couples $(\llbracket -n,n\rrbracket^d, \llbracket -2n,2n\rrbracket^d)_n$ on $\mathbb{Z}^d$, hence the theorem. □

Proof of the Claim. Recall that

$$B \rtimes \mathbb{Z}^d = \left(\prod_{p|c} \mathbb{Q}_p(\alpha) \times \mathbb{C}\right) \rtimes \mathbb{Z}^d,$$

where $\mathbb{Z}^d$ acts diagonally. We consider the generating set $S = (\prod_{p|c} S_p \times B_{\mathbb{C}}(1)) \rtimes \{e_1,\ldots,e_d\}$, where $B_{\mathbb{C}}(a) = \{z\in\mathbb{C} \mid |z| < a\}$ for any real number a and $e_1,\ldots,e_d$ denote the canonical generators of $\mathbb{Z}^d$.

Denote by $(B'_n, B_n)_n$ the prescribed sequence of Følner couples for $\mathbb{Z}^d$. To construct the desired Følner couples, we exhibit Følner couples for the groups $\mathbb{C}\rtimes\mathbb{Z}^d$ and $\mathbb{Q}_p(\alpha)\times\mathbb{Z}^d$ and take a product.

We start with the group $\mathbb{C}\rtimes\mathbb{Z}^d$. Each generator e_i of $\mathbb{Z}^d$ acts multiplicatively on $\mathbb{C}$ and we still denote by e_i the corresponding complex number, so that $e_i.x = e_i x$ for every $x\in\mathbb{C}$. Set $M = \max\{|e_i|, |e_i|^{-1}; i = 1\ldots d\}^d$ and

$$F'_{0,n} = B_{\mathbb{C}}(nM^{2n}), \quad F_{0,n} = B_{\mathbb{C}}(2nM^{2n}).$$

Then, $(F'_{0,n}\times B'_n, F_{0,n}\times B_n)_n$ is a sequence of Følner couples for $\mathbb{C}\times\mathbb{Z}^d$.

In the groups $\mathbb{Q}_p(\alpha)\times\mathbb{Z}^d$, for $p \mid c$, we argue similarly: again, we still denote by e_i the element in $\mathbb{Q}_p(\alpha)$ such that for every $x\in\mathbb{Q}_p(\alpha)$, $e_i.x = e_i x$. Let k be a radius such that for every i, $e_i^d, e_i^{-d} \in p^{-k}\mathbb{Z}_p(\alpha)$. Set $F'_{p,n} = F_{p,n} = p^{-kn}\mathbb{Z}_p$. Then, $(F'_{p,n}\times B'_n, F_{p,n}\times B_n)_n$ is a sequence of Følner couples for $\mathbb{Q}_p(\alpha)\times\mathbb{Z}^d$.

Finally, define

$$F_n = \left(\prod_{p|c} F_{p,n}\times F_{0,n}\right)\times B_n \text{ and } F'_n = \left(\prod_{p|c} F'_{p,n}\times F'_{0,n}\right)\times B'_n,$$

this is the desired sequence of Følner couples for $B\rtimes\mathbb{Z}^d$. □

4.3. Følner couples for extensions.

We end this section mentioning the following application of Theorem 3.3.1, that generalizes Lemma 5.3.4 in Chapter 1 to locally compact compactly generated groups and general sizes.

Lemma 4.3.1. *Let $G = M\rtimes Q$ be a locally compact compactly generated split metabelian group, with M and Q abelian groups. Let S_Q be a compact generating set for Q and S_M be a compact generating set for the $\mathbb{Z}Q$-module M. Fix a $\mathcal{K}$-sequence of Følner couples $F = (F'_n, F_n)_n$ for Q. Assume that*

(1) M has submodules M_1 and M_2 so that M_2 is a cyclic $\mathbb{Z}Q$-module and its projection onto M/M_1 is the whole of M/M_1.

(2) The group $M_1\rtimes Q$ admits a $\mathcal{L}$-sequence of split Følner couples that projects onto the sequence F in Q.

(3) The group $M_2 \rtimes Q$ admits a $\mathcal{M}$-sequence of split Følner couples that projects onto the sequence F in Q.

(4) $\frac{\mathcal{LM}}{\mathcal{K}} \preccurlyeq \mathcal{N}$.

Then, the group $M \rtimes Q$ admits a $\mathcal{N}$-sequence of Følner couples that projects onto the sequence $(F'_n S_Q^n, F'_n)_n$ in Q.

PROOF. The module $M_1 \times M_2$ surjects onto M, hence the group $H = (M_1 \times M_2) \rtimes Q$, where Q acts diagonally, surjects onto $M \rtimes Q$. By Corollary 3.3.1, it is enough to construct Følner pairs for H, whose projection onto Q is $(F'_n, F_n)_n$.

Write $(\Omega'_n, \Omega_n)_n$, resp. $(\Lambda'_n \times F'_n, \Lambda_n \times G_n)_n$ the sequence of Følner couples of $M_1 \rtimes Q$, resp. $M_2 \rtimes Q$, and set for all n

$$\Delta_n = \Omega_n \times \Lambda_n, \qquad \Delta'_n = \Omega'_n \times \Lambda'_n.$$

The sequence $(\Delta'_n, \Delta_n)_n$ is a sequence of Følner couples of exponential size, that projects onto the sequence $(F_n, F'_n)_n$ in Q. □

Appendix: Discussions around another equivalent definition for the isoperimetric profile

Coming back to the isoperimetric profiles defined in (2.4), the following proposition, adapted from [**Tes13**] Proposition 4.1, allows to take the supremum outside, and yields an asymptotically equivalent isoperimetric profile. However, the constant that appears here depends on the compact K, hence this profile is not appropriate if one wishes to consider growing compacts.

PROPOSITION 4.3.2. *We have*

$$j_{G,p}^K(r) \leq \sup_{\mu(\mathrm{Supp}(f))\leq r} \frac{||f||_p}{\sup\limits_{k\in K}||f-\rho(k)f||_p} \leq \; 4C_K \; j_{G,p}^K(r),$$

where C_K denotes the doubling constant of G at scale K.

PROOF. The left-hand side inequality follows from the fact that

$$\sup_{k\in K}||f-\rho(k)f||_p \leq ||\sup_{k\in K}|f-\rho(K)f|||_p.$$

To prove the right-hand side inequality, note that

$$\begin{aligned}\left\|\left(\frac{1}{\mu(K^2)}\int_{K^2}|f-\rho(k)f|^p\mathrm{d}\mu\right)^{\frac{1}{p}}\right\|_p^p &= \frac{1}{\mu(K^2)}\int_{K^2}||f-\rho(k)f||_p^p\mathrm{d}\mu(k)\\ &\leq \sup_{k\in K^2}\;||f-\rho(k)f||_p^p.\end{aligned}$$

Hence

$$\sup_{|\mathrm{Supp}(f)|\leq r} \frac{||f||_p}{\sup\limits_{k\in K^2}\;||f-\rho(k)f||_p} \leq \sup_{|\mathrm{Supp}(f)|\leq r} \frac{||f||_p}{\left\|\frac{1}{\mu(K^2)}\int_{K^2}|f-\rho(k)f|\mathrm{d}\mu\right\|_p}.$$

Recall that $(S_{j_p^K,p}^K,1,1)$ always holds. Proposition 7.2 in [**Tes08**] implies that there exists a constant L_K such that $(S_{j_p^K,p},L_K,1)$ holds for the gradient appearing in the above right-hand side denominator. Moreover, one can take L_K to be twice the doubling constant at scale K:$L_K = 2\frac{\mu(K^2)}{\mu(K)}$. Indeed, Proposition 7.2 in [**Tes08**] is stated for scale being balls but generalizes directly to any compact symmetric scale K. As a consequence,

$$\sup_{|\mathrm{Supp}(f)|\leq r} \frac{||f||_p}{\sup\limits_{k\in K^2}\;||f-\rho(k)f||_p} \leq L_K\; j_p^K(r).$$

To conclude, just note that, by triangular inequality,

$$\sup_{|\mathrm{Supp}(f)|\leq r} \frac{||f||_p}{\sup\limits_{k\in K^2}\;||f-\rho(k)f||_p} \geq \frac{1}{2}\sup_{|\mathrm{Supp}(f)|\leq r} \frac{||f||_p}{\sup\limits_{k\in K}\;||f-\rho(k)f||_p}.$$

Hence, we get the desired inequality. □

REMARK 4.3.3. The previous proposition is still valid when considering left-translation instead of right-translations, that is replacing ρ by λ.

Remark 4.3.4. The proposition above shows that this "outside" profile may yield constants that blow out as the compact K grows. To prove that this indeed happens, one should exhibit examples for which it is possible to compute the optimal constants, that do explode for growing compacts.

Chapitre 3

Soluble groups with no $\mathbb{Z} \wr \mathbb{Z}$ sections

Ce chapitre est basé sur un travail en cours avec Peter Kropholler ([**JK17**]).

1. Introduction

In this chapter, we examine how the structure of soluble groups of infinite torsion-free rank with no section isomorphic to the wreath product of two infinite cyclic groups can be analysed. As a corollary, we obtain that if a finitely generated soluble group has a defined Krull dimension and has no sections isomorphic to the wreath product of two infinite cyclic groups then it is a group of finite torsion-free rank.

2. Notation and background

2.1. Classes of soluble groups. The terms of the derived series of a group G are denoted $G^{(n)}$, inductively defined with $G^{(0)} = G$ and $G^{(n+1)}$ being the commutator subgroup $[G^{(n)}, G^{(n)}]$. The soluble groups are those for which some term of the derived series is trivial and the derived length is the length of the derived series. Recall that a group G is *soluble and minimax* provided it has a series $\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ in which the factors are cyclic or quasicyclic. By a *quasicyclic* group, we mean a group C_{p^∞}, where p is a prime number, isomorphic to the group of p-power roots of unity in the field $\mathbb{C}$ of complex numbers. For a useful alternative point of view, the exponential map $z \mapsto e^{2\pi i z}$ identifies the additive group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ with C_{p^∞}. The terminology *Prüfer p-group* is often used to mean the quasicyclic group C_{p^∞}. For brevity, we write $\mathfrak{M}$ for the class of soluble minimax groups.

For soluble groups the *Hirsch length* or *torsion-free rank* is defined by the formula $h(G) = \sum_{i\geq 0} \dim_{\mathbb{Q}} G^{(i)}/G^{(i+1)} \otimes_{\mathbb{Z}} \mathbb{Q}$. Finitely generated soluble groups of finite torsion-free rank possess a locally finite normal subgroup such that the quotient belongs to $\mathfrak{M}$. We write $\tau(G)$ for the largest normal locally finite subgroup of G.

Let $\mathfrak{X}$ denote the class of soluble groups of finite torsion-free rank, and let $\mathfrak{X}_q$ be the subclass of those having torsion-free rank q. Let $\mathfrak{A}_0$ denote the class of torsion-free abelian groups. For classes of groups $\mathfrak{Y}$ and $\mathfrak{Z}$, we write $\mathfrak{Y}\mathfrak{Z}$ for the class of groups that have a normal $\mathfrak{Y}$-subgroup with corresponding quotient in $\mathfrak{Z}$. If G is a finitely generated soluble group not in $\mathfrak{X}$ then it is a triviality to choose a quotient of G that belongs to $\mathfrak{A}_0\mathfrak{X}$ but not $\mathfrak{X}$. For example, let d be least such that $G^{(d-1)}/G^{(d)}$ has infinite torsion free rank and let $H/G^{(d)}$ be the torsion subgroup of $G^{(d-1)}/G^{(d)}$. Then G/H belongs to $\mathfrak{A}_0\mathfrak{X}\backslash\mathfrak{X}$. While it is not in general possible to choose a quotient that is *just-non-*$\mathfrak{X}$ we can nevertheless find quotients enjoying certain key properties. We write $\mathrm{Fitt}(G)$ for the join of the normal nilpotent subgroups of G. This is the Fitting subgroup. Fitting's lemma states that the join of two nilpotent normal subgroups is nilpotent and consequently the Fitting subgroup is locally nilpotent. It is the directed union of the nilpotent normal subgroups and it contains every subnormal nilpotent subgroup.

THEOREM A. *Every finitely generated soluble group of infinite torsion-free rank has a quotient G with the following properties:*

(i) The Fitting subgroup F of G is torsion-free abelian of infinite rank.

(ii) The factor group G/F has finite torsion-free rank.

(iii) Every non-trivial normal subgroup of G meets F.

(iv) F is self-centralizing.

(v) If K is a normal subgroup of G such that KF/F is not locally finite then G/K has finite torsion-free rank.

Part (i) should be compared with the standard fact that the Fitting subgroup of a just-infinite or just-non-polycyclic group is abelian.

To prove the Proposition we need the following facts about nilpotent groups. Here and subsequently, $\zeta(K)$ denote the centre of the group K. We write $\gamma_i(K)$ for the ith term of the lower central series of K, that is $\gamma_1(K) = K$ and inductively $\gamma_{i+1}(K) = [\gamma_i(K), K]$. For nilpotent groups the lower series terminates in 1 by definition and the *class* of a nilpotent group is its length.

LEMMA 2.1.1. *Let K be a nilpotent group.*

(i) The set of elements of finite order in K is a subgroup.

(ii) If K is torsion-free then so is $K/\zeta(K)$.

(iii) If $K/\zeta(K)$ is torsion then $[K, K]$ is torsion.

(iv) If K is torsion-free and possesses an abelian normal subgroup A such that K/A is a torsion group then K is abelian.

PROOF. Parts (i), (ii) and (iii) are standard results and we refer the reader to [**Rob96**] for these and further background. For readers' convenience we include an argument to prove (iv). Since K is torsion-free, so is $K/\zeta(K)$ by (ii). By induction on class we may assume the result true of $K/\zeta(K)$ so we reduce at once to the case when $K/\zeta(K)$ is abelian. In this case, for any $a \in A$, the map $x \mapsto [x, a]$ is a homomorphism from K to $\zeta(K)$. For any x there is an $m \geq 1$ with $x^m \in A$ and the homomorphism evaluates to 1 this power of x. Since K is torsion-free it follows that the homomorphism is trivial and hence A lies in the centre of K, and $K/\zeta(K)$ (being a quotient of K/A) is torsion. Now (iii) implies that $[K, K]$ is torsion and since K is torsion-free the result follows. □

The following further result about arbitrary groups is extremely important in analysing the structure of nilpotent groups.

LEMMA 2.1.2. *Let K be a group. Then for each i there is a natural surjective homomorphism*

$$\underbrace{K/[K, K] \otimes \cdots \otimes K/[K, K]}_{i} \to \gamma_i(K)/\gamma_{i+1}(K).$$

In particular, if K is nilpotent and $\mathfrak{Z}$ is a class of groups such that quotients of $\mathfrak{Z}$-groups are in $\mathfrak{Z}$ and tensor products of abelian $\mathfrak{Z}$-groups are in $\mathfrak{Z}$ then K belongs to $\mathfrak{Z}$ if and only if $K/[K, K]$ belongs to $\mathfrak{Z}$.

This result, including the application to classes of groups $\mathfrak{Z}$ with the stated closure properties can be found in Robinson's book [**Rob96**].

Corollary 2.1.3. *If K is a nilpotent group of infinite torsion-free rank then $K/[K,K]$ has infinite torsion-free rank.*

Proof of Theorem A. We may replace G by a quotient that lies in $\mathfrak{A}_0\mathfrak{X}_q$, and that has infinite torsion-free rank, and so that q is as small as possible amongst quotients of G with these properties. Quotienting by $\tau(G)$ we may also assume that $\tau(G) = 1$. Let A be an abelian normal subgroup such that $G/A \in \mathfrak{X}_q$.

If N is a nilpotent normal subgroup of G then $K := NA$ is nilpotent of infinite torsion-free rank and hence $K/[K,K]$ has infinite torsion-free rank by Corollary 2.1.3. It follows that G/K has torsion-free rank q and so K/A is torsion. By Lemma 2.1.1(iv) K is abelian and therefore N is abelian. Hence every nilpotent normal subgroup of G is abelian and (i) follows. The remaining parts are then automatically true. □

2.2. The set of rational numbers. The set of rational numbers has two roles in this paper. First it is the prime field of characteristic zero and we denote this by $\mathbb{Q}$. Secondly it is a countable dense linear order and when in this guise we denote it by $\mathbf{Q}$. In general a poset is a set with a reflexive antisymmetric and transitive relation $\leq$. We shall write $x < y$ to mean ($x \leq y$ and $x \neq y$). We also freely use the notation $x > y$ and $x \geq y$ to mean $y < x$ and $y \leq x$ respectively. The interval notation $[x, y]$ is used for the set $\{z;\ x \leq z \leq y\}$. The poset $\mathbf{Q}$ enjoys a special role on account of

Cantor's Theorem (Theorem 9.3 of [**BMMN97**]). *Every countable dense linearly ordered set without endpoints is order-isomorphic to* $\mathbf{Q}$.

We refer the reader to Chapter 9 of [**BMMN97**] for a careful introduction to Cantor's theorem and its ramifications.

2.3. On deviation and Krull dimension. We write $\mathrm{dev}(\mathcal{S})$ for the deviation of a poset $\mathcal{S}$. The definition can be found in [**MR87**] and can be stated like this:

(i) $\mathrm{dev}(\mathcal{S}) = -\infty$ if $\mathcal{S}$ is *trivial* (meaning that $a \leq b \implies a = b$ for all $a, b \in S$).

(ii) $\mathrm{dev}(\mathcal{S}) = 0$ if $\mathcal{S}$ is non-trivial and artinian.

(iii) And in general by transfinite induction: $\mathrm{dev}(\mathcal{S})$ is defined and equal to the ordinal α if $\mathcal{S}$ does not have defined deviation β for any predecessor β of α and, in every strictly descending chain $x_0 > x_1 > x_2 > \dots$, all but finitely many of the intervals $[x_{i+1}, x_i]$ have deviation defined and preceding α.

For a group G we write $\mathrm{dev}(G)$ for the deviation of the poset of subgroups of G and we write $\mathrm{Krull}(G)$ for the deviation of the poset of normal subgroups of G. This last is known as the *Krull dimension* of G [**Tus03**]. In ring theory, the Krull dimension of a module over a ring is defined to be the deviation of the poset of its submodules.

If a group G acts on a group M so that the action contains the inner automorphisms of M, we write $\mathrm{Krull}_G(M)$ for the Krull dimension of M as a G-group, defined as the deviation of the subposet of subgroups of M that are stable under the action of G. Similarly, if a group H acts on M, we write $\mathrm{dev}_H(M)$ for the deviation of M as an H-group, which is the deviation of the subposet of subgroups of M that are stable under the action of G.

The next proposition studies how deviation and Krull dimension of G behave with respects to extensions. It is stated in Chapter 1, Lemma 2.2.8 for the Krull dimension. The argument for the deviation is the same.

LEMMA 2.3.1. *Let*

$$M \hookrightarrow G \overset{p}{\twoheadrightarrow} Q$$

be a sequence of H-groups. Then,

$$\begin{aligned} \mathrm{Krull}_H(G) &= \max\{\mathrm{Krull}_H(M), \mathrm{Krull}_H(Q)\}, \\ \mathrm{dev}_H(G) &= \max\{\mathrm{dev}_H(M), \mathrm{dev}_H(Q)\}. \end{aligned}$$

LEMMA 2.3.2. *Let $\mathcal{S}$ be a poset. Then* $\mathrm{dev}(\mathcal{S})$ *exists if and only if $\mathcal{S}$ has no subposet isomorphic to* $\mathbf{Q}$.

PROOF. McConnell and Robson supply a statement and proof in [**MR87**, Chapter 6, §1.13], but using the poset of dyadic numbers $\mathbb{Z}[1/2]$ instead of $\mathbf{Q}$. Cantor's theorem allows us to reconcile theirs with ours since it implies that $\mathbf{D}$ minus endpoints is order-isomorphic to $\mathbf{Q}$. □

The Krull dimension of a nilpotent group can be expressed in terms of the dimension of the factors of its lower central series, and is equal to its deviation.

PROPOSITION 2.3.3. *Let N be a nilpotent group. Then,*

$$\mathrm{Krull}(N) = \max_{1\leqslant i\leqslant n}\{\mathrm{Krull}(N_i/N_{i+1})\} = \max_{1\leqslant i\leqslant n}\{\mathrm{dev}(N_i/N_{i+1})\} = \mathrm{dev}(N),$$

where n denotes the nilpotency length of N and the groups N_i form the lower central series of N, defined inductively by $N_1 = N$ and, for $i \geqslant 1, N_{i+1} = [N, N_i]$.

PROOF. The proof is by induction on the nilpotency length. The last term N_n of the lower central series is abelian. Consider the short exact sequence

$$N_n \rightarrowtail N \twoheadrightarrow N/N_n.$$

It follows from Lemma 2.3.1 that $\mathrm{Krull}(N) = \max\{\mathrm{Krull}_N(N_n), \mathrm{Krull}_N(N/N_n)\}$. By induction,

$$\mathrm{Krull}(N) = \max_{i=1\ldots n}\{\mathrm{Krull}_N(N_i/N_{i+1})\}.$$

The action of N on the factors N_i/N_{i+1} is trivial. Indeed, let $n \in N$ and n_iN_{i+1} an element of the factor, where $n_i \in N_i$. Then, $n.n_iN_{i+1} = nn_in^{-1}N_{i+1} = n_i[n_i^{-1}, n]N_{i+1} = n_iN_{i+1}$. Therefore, we get the formula stated above. □

LEMMA 2.3.4. *Let G be a nilpotent group. Then the following are equivalent:*

(i) $\mathrm{dev}(G)$ *exists.*

(ii) $\mathrm{dev}(G) \leq 1$.

(iii) $\mathrm{Krull}(G)$ *exists.*

(iv) $\mathrm{Krull}(G) \leq 1$.

(v) G is minimax.

Proof. Proposition 2.3.3 above and stability under extension of the minimax property imply that it is enough to prove the lemma for abelian groups. Hence, we may assume that G is abelian. Note that (i) and (iii) are the same, as well as (ii) and (iv). We shall prove that $(v) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (v)$.

If G is a minimax abelian group, then it is max-by-min. Finitely generated abelian groups have deviation 0 or 1 (Chapter 1, Lemma 2.2.4), hence G has deviation less or equal to 1. It follows at once that $(ii) \Rightarrow (i)$, and $(i) \Rightarrow (v)$ is proved in ([**BCGS14**], Lemma 4.6). □

An alternative proof of this lemma can be devised by employing the variation on Corollary 2.3 that says G is minimax if $G/[G,G]$ is minimax.

The following similar lemma already appeared in the paper [**Tus03**] by Tushev. We provide a different proof.

Lemma 2.3.5 ([**Tus03**]). *Let G be a soluble group. Then the following are equivalent:*

(i) $\mathrm{dev}(G)$ *exists.*

(ii) $\mathrm{dev}(G) \leq 1$.

(iii) G is minimax.

Proof. If G is abelian, this is the content of the previous lemma. If G is soluble, the result follows by induction on its derived length. □

Example 2.3.6. Let $\mathbb{Z}$ act on the polynomial ring $\mathbb{Z}[X]$ by $f(X){*}m := f(X{+}m)$. Then $\mathbb{Z}[X] \rtimes_* \mathbb{Z}$ is metabelian, locally nilpotent and has

(i) Krull dimension 2,

(ii) infinite torsion-free rank, and

(iii) no sections isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

Proof. Denote by G the group $\mathbb{Z}[X] \rtimes_* \mathbb{Z}$.

(i) The group G is metabelian, we have the following short exact sequence $\mathbb{Z}[X] \rightarrowtail G \twoheadrightarrow \mathbb{Z}$. Therefore, by Chapter 1, Proposition 2.2.10 its Krull dimension is the Krull dimension of the $\mathbb{Z}\mathbb{Z}$-module $\mathbb{Z}[X]$. As the annihilator of $\mathbb{Z}[X]$ in the group ring $\mathbb{Z}\mathbb{Z}$ is trivial, this dimension equals 2.

(ii) G has infinite torsion-free rank, as it contains $\mathbb{Z}[X]$ which is abelian of infinite torsion-free rank.

(iii) Existence of a section of G isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ would contradict local nilpotency.

□

3. The Main Structure Theorem

Let $\mathfrak{V}$ denote the class of finitely generated groups in $\mathfrak{A}_0\mathfrak{X} \setminus \mathfrak{X}$ and let $\mathfrak{U}$ denote the class of those $\mathfrak{V}$-groups that have no $\mathbb{Z} \wr \mathbb{Z}$ sections. Our goal in this section is to provide a description of the groups in $\mathfrak{U}$. Of course all such groups have a quotient satisfying all the conclusions of the Theorem A, but our structure theorem applies to arbitrary $\mathfrak{U}$-groups.

Theorem B. *Let G be a $\mathfrak{U}$-group. Then G has subgroups $A \subset K$ and $(A_j)_{j\in\mathbb{N}}$ such that the following hold:*

(i) All the subgroups A_i, A and K are normal.

(ii) A is torsion-free abelian of infinite rank.

(iii) For each j, $A_j \subset A$, A_j has finite rank, and A is the direct product of the A_j.

(iv) K/A is locally finite.

(v) G/K is a virtually torsion-free $\mathfrak{M}$-group.

(vi) For each j, $K/C_K(A_j)$ is finite.

(vii) For each subgroup H of finite index in K that is normal in G, $C_A(H)$ has finite rank.

Corollary 1. *Let G be a finitely generated soluble group with Krull dimension. Then G has finite torsion-free rank if and only if G has no sections isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.*

Example 2.3.6 shows that finite generation is an essential hypothesis here.

Proof. The direct implication follows from the fact that $\mathbb{Z} \wr \mathbb{Z}$ has infinite torsion-free rank.

We prove the reverse implication. Let G be a finitely generated soluble group with Krull dimension that has no section isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. We proceed by induction on the length of the derived series of G. Let B be the last non-trivial term of this series. By induction, G/B has finite torsion-free rank. We aim to prove that B has finite torsion-free rank as well.

Quotienting out the torsion subgroup of B has no impact on the torsion-free rank, thus we may assume that B is a torsion-free abelian normal subgroup of G.

By contradiction, if B were not of finite torsion-free rank, the group G would belong to the class $\mathfrak{U}$. Therefore, the description given in Theorem B would provide a torsion-free abelian normal subgroup A of G, of infinite rank, such that A is the direct product of infinitely many $\mathbb{Z}Q$-modules, where Q stands for the quotient group G/A. Hence A would have infinite uniform dimension, and consequently would not admit a Krull dimension, by (6.2.6 in [**MR87**]). This contradicts the existence of the Krull dimension of G. □

The theorem has the following consequence for random walks on soluble linear groups. We refer to Chapter 1 for background and definitions.

Corollary 2. *Let G be a finitely generated soluble linear group. Then either G has large return probability or G has a section isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. By results of Mal'cev [**Mal51**] and Schur, the group G has a finite index subgroup H which is virtually torsion-free nilpotent-by-abelian. By Remarque 2.0.4, (*ii*), the group H, admits a Krull dimension. The dichotomy of Corollary 1 above applies : either H has finite torsion-free rank or H has a section isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. Therefore, either G has finite rank or G has a section isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. The lower bound for the return probability of finitely generated soluble groups of finite rank is due to Pittet and Saloff-Coste [**PSC03**]. □

Proof of the Structure Theorem.

LEMMA 3.0.1. *Let Q be a soluble group with a locally finite normal subgroup K such that Q/K is minimax. Let M be a $\mathbb{Q}Q$-module on which K acts trivially and which is locally finite-dimensional. Then for any cohomology class $\xi \in H^n(Q,M)$ there exists a finite-dimensional submodule L of M such that ξ lies in the image of the map $H^n(Q,L) \to H^n(Q,M)$ induced by the inclusion of L in M.*

This is a minor extension of Proposition 4 in [**Kro84**]. The example below shows that the assumption that K acts trivially on M cannot be dropped.

PROOF. Let $(M_j)_{j\in J}$ be a sequence of increasing finite-dimensional submodules of M with $\cup_{j\in J} M_j = M$. Triviality of the K-action implies that M and the M_j's are $\mathbb{Q}Q/K$-modules, and that

$$H^q(K,M_j) = \begin{cases} M_j \text{ if } q=0, \\ 0 \text{ otherwise,} \end{cases} \quad \text{and } H^q(K,M) = \begin{cases} M \text{ if } q=0, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore the spectral sequence

$$\lim_{\to} H^p(Q/K, H^q(K,M_j)) \Rightarrow \lim_{\to} H^{p+q}(Q,M_j)$$

collapses when $q>0$, and takes value $\lim_{\to} H^p(Q/K,M_j)$ when $q=0$. We also have the following spectral sequence

$$H^p(Q/K,H^q(K,M)) \Rightarrow H^{p+q}(Q,M),$$

which collapses for $q>0$ and takes value $H^p(Q/K,M)$ when $q=0$. In addition, there is a natural map from the first of these to the second. As a consequence, it is sufficient to prove that this natural map

$$\lim_{\to} H^p(Q/K,M_j) \to H^p(Q/K,M)$$

is an isomorphism for all p. As Q/K is minimax and M_j is finite-dimensional, this follows from ([**Kro84**], proposition 4). □

EXAMPLE 3.0.2. Let $Q = K = \mathbb{F}_2[t]$. For every natural number j, let M_j be a finite-dimensional $\mathbb{Q}K$-module on which

$$\bigoplus_{l\leqslant j} \mathbb{F}_2 t^l$$

acts trivially and such that $M_j^K = 0$. As an example, one can take $\mathbb{Q}$ with the following K-action: for every i in $\mathbb{Z}$ and every x in $\mathbb{Q}$,

$$t^i.x = \begin{cases} -x \text{ provided } i=j+1, \\ x \text{ otherwise.} \end{cases}$$

We have the following exact sequence

$$\bigoplus_{j\in\mathbb{N}} M_j \rightarrowtail \prod_{j\in\mathbb{N}} M_j \twoheadrightarrow X$$

where X denotes the quotient of the product of the $M_j's$ by their direct sum. By construction, K acts trivially on X. The long-exact sequence of cohomology

$$\left(\bigoplus_{j\in\mathbb{N}} M_j\right)^K \rightarrowtail \left(\prod_{j\in\mathbb{N}} M_j\right)^K \to X^K \to H^1\left(K, \bigoplus_{j\in\mathbb{N}} M_j\right) \to H^1\left(K, \prod_{j\in\mathbb{N}} M_j\right)$$

has many simplifications. First, note that the two left terms are trivial, and the third one is actually X. Moreover, by [**Bie81**],

$$H^1(K, \prod M_j) = \prod H^1(K, M_j) = 0$$

because each term of the product is zero. Therefore, $H^1(K, \oplus M_j)$ is isomorphic to X, whereas the direct sum $\bigoplus_j H^1(K, M_j)$ is trivial.

Let G be a group, and M a $\mathbb{Z}G$-module. We shall say that M is a *constrained module* if and only if for each $g \in G$, and $m \in M, m.Z\langle g\rangle$ has finite abelian section rank. Similarly, if k is a field, a kG-module will be called constrained if and only if for each $g \in G$ it is locally finite-dimensional as a $Q\langle g\rangle$-module. For a given group ring, the class of constrained modules is both section and extension closed. These definitions were introduced in [**Kro84**] by Kropholler, who proved that a finitely generated soluble group with no section isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \wr Z$ is minimax.

Proposition 3.0.3 ([**Kro85**], Lemma 3.3). *Let Q be a finitely generated soluble group of finite torsion-free rank and let M be a constrained $\mathbb{Q}Q$-module. Then M is locally finite-dimensional.*

Lemma 3.0.4. *Let Q be a group, T be a normal subgroup of Q and V be a $\mathbb{Q}Q$-module. Assume that V is completely reducible as a $\mathbb{Q}T$-module. Denote by Λ the set of isomorphisms classes of simple $\mathbb{Q}T$-submodules of V and set, for every $\lambda \in \Lambda$ and for every orbit $\sigma \in \Lambda/Q$,*

$$V_\lambda = \sum_{\substack{S \leqslant V \\ S \simeq_{\mathbb{Q}T} \lambda}} S \text{ and } W_\sigma = \sum_{\lambda\in\sigma} V_\lambda.$$

Then W_σ is the $\mathbb{Q}Q$-submodule of V generated by V_λ and

$$V = \bigoplus_{\sigma\in\Lambda/Q} W_\sigma.$$

Proof. By construction, W_σ is a $\mathbb{Q}Q$-submodule of V and $\sum W_\sigma = V$. Assume that σ and γ are such that W_σ and W_γ intersect non-trivially. Then this intersection contains a simple $\mathbb{Q}T$-submodule $S \simeq \lambda$ for some $\lambda \in \Lambda$ and $W_\sigma = W_\gamma$. □

We may now proceed to the proof of the Structure theorem.

Proof of the Structure Theorem. As G belongs to the class $\mathfrak{U}$, it has a normal torsion-free abelian subgroup A with infinite torsion-free rank such that the quotient $Q = G/A$ is a finitely generated soluble group of finite torsion-free rank. The group Q has a locally finite normal subgroup $T = K/A$ such that the quotient $Q/T = G/K$ belongs to $\mathfrak{M}$. Let $\xi \in H^2(Q, A)$ be the cohomology class corresponding to this extension.

Denote by V the tensor product $A \otimes \mathbb{Q}$. Since G has no section isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, it follows that V must be a constrained $\mathbb{Q}Q$-module. Then, by proposition 3.0.3, V is locally finite-dimensional.

Claim 1. The module V is a direct sum of simple $\mathbb{Q}T$-modules.

To prove this claim, consider

$$\mathcal{X} = \{X \subset V \mid X.\mathbb{Q}T \simeq \bigoplus_{x \in X} x.\mathbb{Q}T \text{ and for each } x \text{ in } X, x.\mathbb{Q}T \text{ is simple } \}.$$

Zorn's lemma provides a maximal element X in $\mathcal{X}$. If $X.\mathbb{Q}T \neq V$, choose $v \in V \setminus X.\mathbb{Q}T$. By Maschke's theorem, $v.\mathbb{Q}T$ decomposes as a direct sum of simple $\mathbb{Q}T$-modules, and at least one of them is not contained in $X.\mathbb{Q}T$. Therefore, by changing the choice of v if necessary, we may assume that $v.\mathbb{Q}T$ is simple. Set $X' = X \cup \{v\}$. The set X' belongs to $\mathcal{X}$ and that is a contradiction. Therefore $X.\mathbb{Q}T = V$.

Lemma 3.0.4 applies and allows to write V as the direct sum of the $\mathbb{Q}Q$-modules $V = \oplus_\sigma W_\sigma$ where σ runs along the orbits of the action of Q on the set Λ of isomorphisms classes of simple $\mathbb{Q}T$-submodules of V and

$$W_\sigma = \sum_{\lambda \in \sigma} V_\lambda, \text{ where } V_\lambda = \sum_{\substack{S \leqslant V \\ S \simeq_{\mathbb{Q}T} \lambda}} S.$$

Claim 2. The $\mathbb{Q}Q$-modules W_σ are finite-dimensional over $\mathbb{Q}$.

Take a simple $\mathbb{Q}T$-submodule S of V and consider the $\mathbb{Q}Q$-module $S.\mathbb{Q}Q$: by local finiteness, it is finite-dimensional over $\mathbb{Q}$. Set $T_0 = C_T(S.\mathbb{Q}Q)$. The subgroup T_0 is normal in Q and has finite index in T. Denote by σ the orbit of the isomorphism class $\lambda \in \Lambda$ of the submodule S. It follows that the module W_σ is acted on trivially by T_0, hence $W_\sigma \subset V^{T_0}$. Denote by $[V, T_0]$ the span of the elements $v(t-1)$ for $v \in V$ and $t \in T_0$. It intersects V^{T_0} trivially as $V/[V, T_0]$ is the biggest quotient of V on which T_0 acts trivially. Therefore, the second claim follows from

Claim 3. The quotient $V/[V, T_0]$ is finite-dimensional.

To prove this third claim, first note that Q/T_0 is minimax. Let $\hat{\xi}$ be the image of ξ in $H^2(Q, V/[V, T_0])$. Lemma 3.0.1 provides a finite-dimensional submodule $L/[V, T_0]$ of $V/[V, T_0]$ such that $\hat{\xi}$ lies in the image of the map $H^2(Q, L/[V, T_0]) \to H^2(Q, V/[V, T_0])$ induced by the inclusion of $L/[V, T_0]$ in $V/[V, T_0]$. Consequently, $\hat{\xi}$ goes to zero in $H^2(Q, V/L)$. As G is finitely generated and the extension

$$A/A \cap L \rightarrowtail G/L \twoheadrightarrow Q$$

splits, we have $V = L$ and therefore $V/[V, T_0]$ is finite-dimensional. This ends the proof of the claim.

Consequently, W_σ is finite-dimensional and V is a direct sum of finite-dimensional $\mathbb{Q}Q$-modules V_i. Set $A_i = V_i \cap A$, this is a normal subgroup of G with finite rank. The group A contains the infinite direct sum of the A_i's and the corresponding quotient $A/(\oplus A_i)$ is torsion. Hence, we may replace A with $\oplus A_i$. This proves $(i) - (vi)$.

The proof of (vii) is similar to the third claim: if H is a finite index subgroup of K, it acts trivially on $C_A(H)$ and similarly, $V/[V, H]$ is finite-dimensional.

□

APPENDIX A

More on Krull dimension

In this appendix, we recall the obstruction for a group not to admit a Krull dimension. We also give formulas for the Krull dimension of nilpotent-by-abelian or abelian-by-nilpotent groups involving only modules, applying Lemma 2.28 of Chapter 1.

We refer to Chapter 1, section 2 for details and notations about posets and Krull dimension.

1. Groups without Krull dimension

Recall that we defined the Krull dimension of a group G as being the deviation of the poset of its normal subgroups, whenever it exists (see section 2 in Chapter 1 for details). Otherwise, we say that G does not admit a Krull dimension.

We discuss in this subsection groups that does not admit a Krull dimension. Recall the following fact, probably well-known, stated in Chapter 3, Lemma 2.3.2.

LEMMA 1.0.1. *The Krull dimension of a group G exists if and only if the poset of normal subgroups of G has no subposet isomorphic to* $\mathbf{Q}$.

A basic well-known example is given by free groups.

PROPOSITION 1.0.2. *Let $n \geqslant 2$. The free group F_n of rank n does not admit a Krull dimension.*

PROOF. For $n \geqslant 2$, let a, b be two generators of F_n. Consider, for every $k \geqslant 1$, the words $R_k = [a^k b a^{-k}, b]$ [1]. One can see that, for every n, R_n does not belong to the normal subgroup generated by $\{R_m \mid m \neq n\}$. Hence, one can consider the map

$$\Phi : \begin{cases} \mathcal{P}(\mathbb{N}^*) \to \mathcal{N}(F_2) \\ I \mapsto \langle\langle R_n ; n \in I \rangle\rangle \end{cases}$$

Independence of the R_n implies injectivity of Φ, which is also preserves inclusion. Therefore, the poset $\mathcal{N}(F_2)$ contains $(\mathcal{P}(\mathbb{N}^*), \subset)$ as a subposet. The latter is isomorphic to the poset $(\{0,1\}^{\mathbb{N}^*}, \leqslant)$, which has a subposet isomorphic to the interval $[0,1]$, hence contains $\mathbb{Q} \cap [0,1]$. Lemma 1.0.1 above allows to conclude. □

COROLLARY 1.0.3. *A group surjecting on a nonabelian free group does not admit a Krull dimension.*

1. These words appear in a presentation of the wreath product $\mathbb{Z} \wr \mathbb{Z}$ which is not finitely presented ([**Bau61**]).

2. Krull dimension of metanilpotent groups

Recall the following, about the Krull dimension of nilpotent groups.

LEMMA 2.0.1 (Lemma 2.3.4, Chapter 3). *Let G be a nilpotent group. Then the following are equivalent:*

(i) $\mathrm{Krull}(G)$ *exists.*

(ii) $\mathrm{Krull}(G) \leq 1$.

(iii) G *is minimax.*

REMARK 2.0.2. Therefore, the Krull dimension of a nilpotent group N

— is $-\infty$ if $N = \{0\}$,

— is 0 if N is non-trivial artinian,

— is 1 if M is minimax non-artinian,

— is not defined otherwise.

The Krull dimension of metanilpotent groups can be expressed using particular module sections.

PROPOSITION 2.0.3. *Let G be a metanilpotent group, that is*

$$N \hookrightarrow G \twoheadrightarrow P,$$

where N and P are nilpotent. Then,

$$\mathrm{Krull}(G) = \max\left\{ \max_{i=1\ldots n}\left\{\mathrm{Krull}_{\mathbb{Z}P}(N_i/N_{i+1})\right\}, \max_{j=1\ldots p}\left\{\mathrm{Krull}(P_i/P_{i+1})\right\}\right\},$$

where n, resp. p, denote the nilpotency length of N, resp. P, and the groups N_i, resp. P_j, form the lower central series of N, resp. P, defined inductively by $N_1 = N$ and, for $i \geqslant 1, N_{i+1} = [N, N_i]$, and similarly for P and the P_j's.

PROOF. Lemma 2.2.8 from Chapter 1 applied to the action of G by conjugation yields

$$\mathrm{Krull}(G) = \max\{\mathrm{Krull}_G(N), \mathrm{Krull}_G(P)\}.$$

First, note that the G-action on P is actually a P-action and $\mathrm{Krull}_G(P) = \mathrm{Krull}(P)$. The desired formula for this last term is given in Proposition 2.3.3 from Chapter 3. Hence, we are left with studying $\mathrm{Krull}_G(N)$.

Using the decomposition of N, we get

$$\mathrm{Krull}_G(N) = \max_{i=1\ldots n}\{\mathrm{Krull}_G(N_i/N_{i+1})\}.$$

We claim that the G-action on N_i/N_{i+1} induces an action of the quotient P, for $1 \leq i \leq n$. Indeed, let xN_{i+1} be an element of N_i/N_{i+1} and g, g' two elements of G such that $g' = gn$, for some n in N. We have $g'.xN_{i+1} = g'x(g')^{-1}N_{i+1} = g.(nxn^{-1}N_{i+1}) = g.xN_{i+1}$, where the last equality uses $nxn^{-1} = x[x^{-1}, n] \in xN_{i+1}$. Hence, the action of an element of G only depends on its image on the quotient P. Moreover, the groups N_i/N_{i+1} are abelian groups, hence their dimension as P-groups is equal to their dimension as $\mathbb{Z}P$-modules.

This proves the formula. □

REMARKS 2.0.4.

(i) When N is abelian (so that G is abelian-by-nilpotent), we have

$$\mathrm{Krull}(G) = \max\left\{\mathrm{Krull}_{\mathbb{Z}P}(N), \max_{i=1\ldots n}\left\{\mathrm{Krull}(P_i/P_{i+1})\right\}\right\}.$$

If G is moreover finitely generated, N is a finitely generated module over a Noetherian ring, hence is Noetherian. Indeed, this is a result, due to Hall [**Hal54**] that the integral group ring of a polycyclic group is Noetherian. Hence, G admits a Krull dimension.

(ii) When P is abelian (so that G is nilpotent-by-abelian), we have

$$\mathrm{Krull}(G) = \max\left\{\mathrm{Krull}(P), \max_{i=1\ldots n}\left\{\mathrm{Krull}_{\mathbb{Z}P}(N_i/N_{i+1})\right\}\right\}.$$

If G is moreover finitely generated, $\mathrm{Krull}(P)$ is either 0 or 1 (see Lemma 2.2.4 in Chapter 1) and the $\mathbb{Z}P$-modules N_i/N_{i+1} are finitely generated, hence Noetherian. As a consequence, G admits a Krull dimension.

Bibliographie

[Ale92] George Alexopoulos. A lower estimates for central probabilities on polycyclic groups. *Canad. J. Math.*, 44 :897–910, 1992.

[AV17] Gideon Amir and Bálint Virág. Speed exponents of random walks on groups. *International Mathematics Research Notices*, 2017(9) :2567–2598, 2017.

[Bac67] S. Bachmuth. Automorphisms of a class of metabelian groups. *Transactions of the American Mathematical Society*, 127(2) :284–293, 1967.

[Bar14] Erschler Anna Bartholdi, Laurent. Groups of given intermediate word growth. *Annales de l'institut Fourier*, 64(5) :2003–2036, 2014.

[Bas72] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc. (3)*, 25 :603–614, 1972.

[Bau61] Gilbert Baumslag. Wreath products and finitely presented groups. *Mathematische Zeitschrift*, 75(1) :22–28, 1961.

[BCG01] Martin Barlow, Thierry Coulhon, and Alexander Grigor'yan. Manifolds and graphs with slow heat kernel decay. *Invent. Math.*, 144(3) :609–649, 2001.

[BCGS14] Robert Bieri, Yves Cornulier, Luc Guyot, and Ralph Strebel. Infinite presentability of groups and condensation. *J. Inst. Math. Jussieu*, 13(4) :811–848, 2014.

[Bie81] R. Bieri. *Homological dimension of discrete groups.* Queen Mary College mathematics notes. Mathematics Dept., Queen Mary College, University of London, 1981.

[BLP77] Paolo Baldi, Noël Lohoué, and Jacques Peyrière. Sur la classification des groupes récurrents. *C. R. Acad. Sci. Paris Sér. A-B*, 285(16) :A1103–A1104, 1977.

[BMMN97] Meenaxi Bhattacharjee, Dugald Macpherson, Rögnvaldur G. Möller, and Peter M. Neumann. *Notes on infinite permutation groups*, volume 12 of *Texts and Readings in Mathematics.* Hindustan Book Agency, New Delhi ; co-published by Springer-Verlag, Berlin, 1997. Lecture Notes in Mathematics, 1698.

[BMO12] Gilbert Baumslag, Roman Mikhailov, and Kent E. Orr. A new look at finitely generated metabelian groups. In *Computational and combinatorial group theory and cryptography*, volume 582 of *Contemp. Math.*, pages 21–37. Amer. Math. Soc., Providence, RI, 2012.

[Bos12] S. Bosch. *Algebraic Geometry and Commutative Algebra.* Universitext. Springer London, 2012.

[Bri15] Jérémie Brieussel. About the speed of random walks on solvable groups. *arXiv*, arXiv :1505.03294, 2015.

[BS80] Robert Bieri and Ralph Strebel. Valuations and finitely presented metabelian groups. *Proc. London Math. Soc. (3)*, 41(3) :439–464, 1980.

[BS81] Robert Bieri and Ralph Strebel. A geometric invariant for modules over an abelian group. *J. Reine Angew. Math.*, 322 :170–189, 1981.

[BT24] Stefan Banach and Alfred Tarski. Sur la décomposition des ensembles de points en parties respectivement congruentes. *Fundamenta Mathematicae*, 6(1) :244–277, 1924.

[Bur11] W. Burnside. *Theory of Groups of Finite Order.* second edition. Cambridge University Press, 1911.

[BZ16] Jérémie Brieussel and Tianyi Zheng. Speed of random walks, isoperimetry and compression of finitely generated groups. *arxiv*, arXiv :1510.08040v2, 2016.

[CG97] Thierry Coulhon and Alexander Grigor'yan. On-diagonal lower bounds for heat kernels and Markov chains. *Duke Math. J.*, 89(1) :133–199, 1997.

[CGP01] Thierry Coulhon, Alexander Grigor'yan, and Christophe Pittet. A geometric approach to on-diagonal heat kernel lower bounds on groups. *Ann. Inst. Fourier*, 51(6) :1763–1827, 2001.

[Cor11] Yves Cornulier. On the Cantor-Bendixson rank of metabelian groups. *Ann. Inst. Fourier*, 61(2) :593–618, 2011.

[Cou96] Thierry Coulhon. Ultracontractivity and Nash type inequalities. *J. Funct. Anal.*, 141(2) :510–539, 1996.

[Cou00] Thierry Coulhon. Random walks and geometry on infinite graphs. In *Lecture notes on analysis in metric spaces (Trento, 1999)*, Appunti Corsi Tenuti Docenti Sc., pages 5–36. Scuola Norm. Sup., Pisa, 2000.

[Cou03] Thierry Coulhon. Heat kernel and isoperimetry on non-compact Riemannian manifolds. In *Heat kernels and analysis on manifolds, graphs, and metric spaces. Lecture notes from a quarter program on heat kernels, random walks, and analysis on manifolds and graphs, April 16–July 13, 2002, Paris, France*, pages 65–99. Providence, RI : American Mathematical Society (AMS), 2003.

[CSC93] Thierry Coulhon and Laurent Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana*, 9(2) :293–314, 1993.

[Dav89] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[Eis95] David Eisenbud. *Commutative Algebra : With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[Ers03] Anna Erschler. On isoperimetric profiles of finitely generated groups. *Geometriae Dedicata*, 100(1) :157–171, 2003.

[Ers06] Anna Erschler. Isoperimetry for wreath products of markov chains and multiplicity of selfintersections of random walks. *Probability Theory and Related Fields*, 136(4) :560–586, 2006.

[GR67] P. Gabriel and R. Rentschler. Sur la dimension des anneaux et ensembles ordonnes. *C. R. Acad. Sci., Paris, Sér. A*, 265 :712–715, 1967.

[Gri83] R. I. Grigorchuk. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR*, 271(1) :30–33, 1983.

[Gri84] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5) :939–985, 1984.

[Gri94] Alexander Grigor'yan. Heat kernel upper bounds on a complete non-compact manifold. *Rev. Mat. Iberoamericana*, 10(2) :395–452, 1994.

[Gro] Groupprops. Degrees of irreducible representations for symmetric groups of small degree. `https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_symmetric_groups#Particular_cases`. Accessed : 2017-06-01.

[Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53) :53–73, 1981.

[Gro84] Mikhael Gromov. Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 385–392. PWN, Warsaw, 1984.

[Gui71] Yves Guivarc'h. Groupes de Lie à croissance polynomiale. *C. R. Acad. Sci. Paris Sér. A-B*, 272 :A1695–A1696, 1971.

[Gui73] Yves Guivarc'h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France*, 101 :333–379, 1973.

[Hal54] P. Hall. Finiteness conditions for soluble groups. *Proceedings of the London Mathematical Society*, s3-4(1) :419–436, 1954.

[Hal59] P. Hall. On the finiteness of certain soluble groups. *Proceedings of the London Mathematical Society*, s3-9(4) :595–622, 1959.

[Heb92] Waldemar Hebisch. On heat kernels on Lie groups. *Math. Z.*, 210(4) :593–605, 1992.

[HSC93] Waldek Hebisch and Laurent Saloff-Coste. Gaussian estimates for markov chains and random walks on groups. *Ann. Probab.*, 21(2) :673–709, 1993.

[Jac16] Lison Jacoboni. Metabelian groups with large return probability. *arXiv :1611.08026, submitted*, 2016.

[Jac17] Lison Jacoboni. Isoperimetry and følner pairs in locally compact compactly generated groups. *in preparation*, 2017.

[Jen73] J. W. Jenkins. Growth of connected locally compact groups. *J. Functional Analysis*, 12 :113–127, 1973.

[JK17] Lison Jacoboni and Peter Kropholler. Soluble groups with no $F \wr F$. *in preparation*, 2017.

[Kes59] Harry Kesten. Symmetric random walks on groups. *Transactions of the American Mathematical Society*, 92(2) :336–354, 1959.

[KKM85] Mikhaïl Kargapolov, Vladimir Kotliar, and Iouri Merzliakov. *Éléments de la théorie des groupes*. Traduit du russe. Mir, 1985.

[KL15] Peter Kropholler and Karl Lorensen. Torsion-free covers for solvable minimax groups. *http ://arxiv.org/abs/1510.07583*, 2015.

[Kra70] Günter Krause. On the Krull-dimension of lift noetherian left Matlis-rings. *Math. Z.*, 118 :207–214, 1970.

[Kro84] P.H. Kropholler. On finitely generated soluble groups with no large wreath product sections. *Proceedings of the London Mathematical Society*, 49(1) :155–169, 1984.

[Kro85] P. H. Kropholler. A note on the cohomology of metabelian groups. *Math. Proc. Cambridge Philos. Soc.*, 98(3) :437–445, 1985.

[LR04] John C. Lennox and Derek J. S. Robinson. *The theory of infinite soluble groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2004.

[Mag39] Wilhelm Magnus. On a theorem for marshall hall. *Annals of Mathematics*, 40(4) :764–768, 1939.

[Mal51] A. I. Mal'cev. On some classes of infinite soluble groups. *Mat. Sbornik N.S.*, 28(70) :567–588, 1951.

[Mil68] John Milnor. Growth of finitely generated solvable groups. *J. Differential Geometry*, 2 :447–449, 1968.

[Mon13] Nicolas Monod. Groups of piecewise projective homeomorphisms. *Proc. Natl. Acad. Sci. USA*, 110(12) :4524–4527, 2013.

[MR87] John C. McConnell and James C. Robson. *Noncommutative Noetherian Rings*. Wiley series in pure and applied mathematics. Wiley, 1987.

[Ms97] Ji ˇ ríMatou ˇ sek. On embedding expanders into l_p spaces. *Israel J. Math.*, 102 :189–197, 1997.

[Neu29] J. Neumann. Zur allgemeinen theorie des masses. *Fundamenta Mathematicae*, 13(1) :73–116, 1929.

[Noe21] Emmy Noether. Idealtheorie in Ringbereichen. *Math. Ann.*, 83(1-2) :24–66, 1921.

[Noe23] Emmy Noether. Eliminationstheorie und allgemeine Idealtheorie. *Math. Ann.*, 90(3-4) :229–261, 1923.

[Noe26] Emmy Noether. Der endlichkeitssatz der invarianten endlicher linearer gruppen der charakteristik p. *Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen, Mathematisch-Physikalische Klasse*, 1926 :28–35, 1926.

[Ol'79] A. Ju. Ol'sanskii. Infinite groups with cyclic subgroups. *Dokl. Akad. Nauk SSSR*, 245(4) :785–787, 1979.

[Ol'80] A. Ju. Ol'sanskii. On the question of the existence of an invariant mean on a group. *Uspekhi Mat. Nauk*, 35(4(214)) :199–200, 1980.

[PSC99] Christophe Pittet and Laurent Saloff-Coste. Amenable groups, isoperimetric profiles and random walks. In *Geometric group theory down under (Canberra, 1996)*, pages 293–316. de Gruyter, Berlin, 1999.

[PSC00] Christophe Pittet and Laurent Saloff-Coste. On the stability of the behavior of random walks on groups. *The Journal of Geometric Analysis*, 10(4) :713–737, 2000.

[PSC01] Christophe Pittet and Laurent Saloff-Coste. A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on cayley graphs, with examples. *unpublished, available at http ://www.math.cornell.edu/ lsc/articles.html (Accessed : 08.28.2017)*, 2001.

[PSC02] Christophe Pittet and Laurent Saloff-Coste. On random walks on wreath products. *Ann. Probab.*, 30(2) :948–977, 2002.

[PSC03] Christophe Pittet and Laurent Saloff-Coste. Random walks on finite rank solvable groups. *Journal of the European Mathematical Society*, 5(4) :313–342, 2003.

[Pó21] G. Pólya. Über eine aufgabe der wahrscheinlichkeitsrechnung betreffend die irrfahrt im straßennetz. *Mathematische Annalen*, 84 :149–160, 1921.

[Rob96] D. Robinson. *A Course in the Theory of Groups*. Graduate Texts in Mathematics. Springer New York, 1996.

[SCZ14] Laurent Saloff-Coste and Tianyi Zheng. Random walks on free solvable groups. *Mathematische Zeitschrift*, 279(3) :811–848, 2014.

[Str92] Daniel W. Stroock. Estimates on the heat kernel for second order divergence form operators. In *Probability theory (Singapore, 1989)*, pages 29–44. de Gruyter, Berlin, 1992.

[Tes08] Romain Tessera. Large scale sobolev inequalities on metric measure spaces and applications. *Revista Matemática Iberoamericana*, 24(3) :825–864, 2008.

[Tes13] Romain Tessera. Isoperimetric profile and random walks on locally compact solvable groups. *Rev. Math. Iberoam.*, 29(2) :715–737, 2013.

[Tes16] Romain Tessera. The large-scale geometry of locally compact solvable groups. *Internat. J. Algebra Comput.*, 26(2) :249–281, 2016.

[Tus03] A. V. Tushev. On deviation in groups. *Illinois J. Math.*, 47(1-2) :539–550, 2003.

[Var83a] Nicholas Varopoulos. A potential theoretic property of solvable groups. *Bull. Sci. Math.*, (108) :263–273, 1983.

[Var83b] Nicholas Th. Varopoulos. Random walks on soluble groups. *Bull. Sci. Math. (2)*, 107(4) :337–344, 1983.

[Var85a] N. Th. Varopoulos. Isoperimetric inequalities and Markov chains. *J. Funct. Anal.*, 63(2) :215–239, 1985.

[Var85b] Nicholas Varopoulos. Théorie du potentiel su les groupes nilpotents. *C. R. Acad. Sci. Paris*, pages 143–144, 1985.

[Var86] Nicholas Th. Varopoulos. Théorie du potentiel sur des groupes et des variétés. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(6) :203–205, 1986.

[Var87] N. Th. Varopoulos. Convolution powers on locally compact groups. *Bull. Sci. Math. (2)*, 111(4) :333–342, 1987.

[Var91] Nicholas Th. Varopoulos. Analysis and geometry on groups. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 951–957. Math. Soc. Japan, Tokyo, 1991.

[VSCC92a] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[VSCC92b] Nicholas Varopoulos, Laurent Saloff-Coste, and Thierry Coulhon. *Analysis and Geometry on Groups*. Cambridge Tracts in Mathematics. Cambridge University Press, 1992.

[Wol68] Joseph A. Wolf. Growth of finitely generated solvable groups and curvature of Riemanniann manifolds. *J. Differential Geometry*, 2 :421–446, 1968.

université PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

Titre : Propriétés métriques et probabilistes des groupes métabéliens de type fini

Mots Clefs : groupes métabéliens, probabilité de retour, profil isopérimétrique, paires de Følner

Résumé : Dans la première partie, on étudie la probabilité de retour des groupes métabéliens de type fini. On donne une caractérisation des tels groupes avec grande probabilité de retour en des termes purement algébriques, à l'aide de la dimension de Krull. Cela nécessite, pour les groupes métabéliens, une variation d'un théorème de Kaloujnine et Krasner qui respecte cette dimension. Au passage, on obtient des bornes inférieures et supérieures sur la probabilité de retour des groupes métabéliens en fonction de la dimension de Krull.
La seconde partie concerne les profils isopérimétriques des groupes localement compacts compactement engendrés, qu'on utilise pour caractériser l'existence d'une suite de paires de Følner. On démontre que le profil isopérimétrique augmente lorsqu'on passe au quotient, avec des constantes indépendantes de l'échelle, améliorant un théorème de Tessera. Combinant les deux, on obtient que l'existence de suites de paires de Følner passe au quotient. On montre qu'elle passe au sous-groupe fermé, généralisant un résultat correspondant d'Erschler pour les groupes de type fini. Cela permet d'obtenir une preuve plus auto-contenue du théorème principal de la première partie.
La troisième partie est un travail en commun avec Kropholler dans lequel on étudie la structure des groupes résolubles de rang sans torsion infini n'ayant pas de section isomorphe à $\mathbb{Z} \wr \mathbb{Z}$. On en déduit qu'en présence d'une dimension de Krull, ce type de section est la seule obstruction à la finitude du rang sans torsion.

Title : Metric and probabilistic properties of finitely generated metabelian groups

Keys words : metabelian group, return probability, isoperimetric profie, Følner couples

Abstract : In the fist part, we study the return probability of finitely generated metabelian groups. We give a characterization of such groups with large return probability in purely algebraic terms, namely the Krull dimension of the group. To do so, we establish, for metabelian groups, a variation of a famous embedding theorem of Kaloujnine and Krasner that respects this dimension. Along the way, we obtain lower and upper bounds on the return probability of metabelian groups according to their dimension.
The second part of this thesis deals with isoperimetric profiles of locally compact compactly generated groups, that we use to characterize the existence of sequences of Følner couples. We generalize at a compact scale previous results of Tessera, in particular that they increase when going to a quotient group, so as to state in more generality a result from the first part, namely that the existence of Følner couples goes to a quotient group. We also prove that it goes to a closed subgroup. This allows to obtains a more self-contained proof of the main result of the first part of this thesis.
The third part is a joint work with Kropholler in which we study the structure of soluble groups of infinite torsion-free rank with no $\mathbb{Z} \wr \mathbb{Z}$. As a corollary, we obtain that a finitely generated soluble group with Krull dimension has finite torsion-free rank if and only if it has no $\mathbb{Z} \wr \mathbb{Z}$.