Frequently Asked Questions

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I'm frequently requested about precise references for various questions asked or problems posed in my papers. Here is a list of some of them.

Fixed Price Problem [Gab00, p. 50, Question I.8][1]:

Does there exist groups that don't have fixed price?

One can also mention [Gab10, Question 6.3].

Recall that a group is said to have *fixed price* if the relations $\mathcal{R}_\alpha$ for all of its free actions α have the same cost (on (X,μ) a probability space) [Gab00, p. 50, Déf. I.5 (4)].

Cost vs ℓ^2-Betti Numbers Problem For every pmp equivalence relation we have an inequality between its cost and its first (and zero-th) L^2-Betti number $C(\mathcal{R}) - 1 \geq \beta_1(\mathcal{R}) - \beta_0(\mathcal{R})$ [Gab02, Cor. 3.23, p. 128]. One doesn't know any example with strict inequality.

[Gab02, p. 129, l. +1][2]:

Does one always have the equality

$$C(\mathcal{R}) - 1 = \beta_1(\mathcal{R}) - \beta_0(\mathcal{R})?$$

See also [Gab10, Question 8.2].

Generalized Cost vs ℓ^2-Betti Numbers Problem [Gab02, p. 129, l. +3][3]:

More generally, does one always have the equality

$$\inf\{\alpha_n(\Sigma)-\alpha_{n-1}(\Sigma)+\cdots+(-1)^n\alpha_0(\Sigma)\} = \beta_n(\mathcal{R})-\beta_{n-1}(\mathcal{R})+\cdots+(-1)^n\beta_0(\mathcal{R})$$

where the infimum is taken over all the $(n-1)$-connected simplicial $\mathcal{R}$-complexes?

Treeability

[Gab00, p. 80, Question VI.2][4]:

Does there exist groups for which certain (free) actions are treeable and some others aren't?

[1] [Gab00, p. 50, Question I.8] "Existe-t-il des groupes qui ne soient pas à prix fixe ?"

[2] [Gab02, p. 129, l. +1] "Question. - On ne connaît aucun exemple d'inégalité stricte. A-t-on toujours l'égalité $C(\mathcal{R}) - 1 = \beta_1(\mathcal{R}) - \beta_0(\mathcal{R})$?"

[3] [Gab02, p. 129, l. +3] "Plus généralement, a-t-on toujours l'égalité $\inf\{\alpha_n(\Sigma)-\alpha_{n-1}(\Sigma)+\cdots+(-1)^n\alpha_0(\Sigma)\} = \beta_n(\mathcal{R})-\beta_{n-1}(\mathcal{R})+\cdots+(-1)^n\beta_0(\mathcal{R})$, où l'infimum est pris sur tous les $\mathcal{R}$-complexes simpliciaux $(n-1)$–connexes ?"

[4] [Gab00, p. 80, Question VI.2] "Existe-t-il des groupes dont certaines actions seraient arborables et d'autres non ?"

Locally Free Groups [Gab00, p. 87, Questions VI.17][5]:

What is the cost of *locally free groups* (i.e. those groups for which every finitely generated subgroup is a free group)?
Are they treeable?

Ergodic Dimension [Gab02, p. 98, l. -6][6]:

What is the ergodic dimension of the lattices in $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$?

Ergodic Dimension [Gab02, p. 98, l. -4][7]:

What is the ergodic dimension of the fundamental group of a compact acyclic manifold?

Ergodic Dimension [Gab02, p. 98, l. -2][8]:

Do all equivalence relations produced by the free pmp actions of a given group have the same geometric dimension?

Ergodic Dimension $= 1$ [Gab02, p. 145][9]:

The groups with ergodic dimension $= 1$ are they all in the measure equivalence classe (ME) of a free group (of $\mathbf{F}_\infty$, $\mathbf{F}_2$ ou de $\mathbb{Z}$)?
$\rightsquigarrow$ This problem has been solved positively by G. Hjorth in [Hjo06]. He proved that the treeable ergodic equivalence relations with integer cost are produced by a free action of a free group.

Crossed Product von Neumann Algebras The L^2-Betti numbers of an equivalence relation are invariants of the pair crossed product von Neumann algebra/ Cartan subalgebra.
[Gab02, p. 101, l. +3][10]:
Are they in fact invariants of the crossed product von Neumann algebra itself?

Non OE Actions of the Free Froup [Gab02, p. 102, l. +9][11]:

Can one build infinitely many non Orbit Equivalent free actions of the free group $\mathbf{F}_2$?
$\rightsquigarrow$ This problem has been solved positively in [GP05].

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[5] [Gab00, p. 87, Questions VI.17] "Quel est le coût des groupes qui sont localement libres (i. e. dont tous les sous-groupes de type fini sont libres) mais non libres ? Sont-ils arborables ?"

[6] [Gab02, p. 98] "Quelle est la dimension ergodique des réseaux de $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$?"

[7] [Gab02, p. 98] "Quelle est la dimension ergodique des groupes fondamentaux des variétés compactes acycliques ?"

[8] [Gab02, p. 98] "Toutes les actions libres d'un groupe produisent-elles des relations de même dimension géométrique ?"

[9] [Gab02, p. 145] Question : "Les groupes de dimension ergodique 1 sont-ils tous dans la classe de ME d'un groupe libre (de $\mathbf{F}_\infty$, $\mathbf{F}_2$ ou de $\mathbb{Z}$) ?"

[10] [Gab02, p. 101] Question : "Sont-ils en fait des invariants de l'algèbre de von Neumann $\mathcal{M}$?"

[11] [Gab02, p. 102] **Question** : "Peut-on construire une infinité d'actions libres non (OE) du groupe libre à deux générateurs ?"

Self Couplings

[Gab05, Question 2.8]:
Are there groups Γ such that the set of all indices of **ergodic** ME couplings of Γ with itself is discrete $\neq \{1\}$?

Measure Free-Factors of the Free Group

[Gab05, Question 3.10]:
What are all the measure free-factors of the free group $\mathbf{F}_2$?

Measure Free-Factors of the Free Group

It is known that if an amalgamated free product $\mathbf{F}_p *_{\mathbb{Z}} \mathbf{F}_q$ happens to be a free group, then $\mathbb{Z}$ is a free factor in one of $\mathbf{F}_p$ or $\mathbf{F}_q$ (see Bestvina-Feighn "Outer Limits", Ex. 4.2).
[Gab05, Question 3.11]:
Is it true that similarly if $\mathbf{F}_p *_{\mathbb{Z}} \mathbf{F}_q \overset{\mathrm{ME}}{\sim} \mathbf{F}_2$ then $\mathbb{Z}$ is a measure free-factor in $\mathbf{F}_p$ or $\mathbf{F}_q$?

Limit Groups

A limit group is a finitely generated group Γ that is ω-residually free, i.e. for every finite subset $K \subset \Gamma$ there exists a homomorphism $\Gamma \to F$ to a free group, that is injective on K.

[Gab05, Question 3.20] (Indeed this question was asked by Michah Sageev during a lecture I gave in Albany, NY 09/10/04):
It is a natural question to ask whether limit groups are ME to a free group.

A partial answer has been given by Bridson-Tweedale-Wilton [BTW07]: "Every elementarily free group is measure-equivalent to a free group".

Generalized von Neumann's Problem

[GL09, p. 539]:
Does every probability-measure-preserving free ergodic action of a non-amenable countable group contain an ergodic subrelation generated by a free action of a non-cyclic free group?

Generalized von Neumann's Problem

[GL09, p. 539]:
More generally: Does every standard countable probability-measure-preserving non-amenable ergodic equivalence relation contain a treeable non-amenable ergodic equivalence subrelation?
See also [Gab10, Question 10.8]

Subrelations of Bernoulli with Diffuse Ergodic Decomposition

[Gab10, Question 5.6]:
Let Γ be a countable group with a finite generating set S. Let $\pi : (X_0, \mu_0)^\Gamma \to [0,1]$ be any measure preserving map (i.e. $\pi_*(\otimes_\Gamma \mu_0) = \mathrm{Leb}$) and Φ_π be the "fiber-graphing" made of the restriction φ_s of $s \in S$ to the set $\{\omega \in (X_0, \mu_0)^\Gamma : \pi(s.\omega) = \pi(\omega)\}$. Is the equivalence relation generated by Φ_π finite?

$\rightsquigarrow$ This problem has been solved negatively by Péter Mester in the paper "A factor of i.i.d with uniform marginals and infinite clusters spanned by equal labels" (`http://arxiv.org/abs/1111.3067` and `http://gradworks.umi.com/34/56/3456485.html`).

Cost for Kazhdan Groups
[Gab10, Question 6.4]:
Does there exist a Kazhdan property (T) group with a p.m.p. free action of cost > 1?

Measurably Freely Indecoposability
[Gab10, Question 7.2]:
Produce a measurably freely indecomposable (MFI) group with first ℓ^2-Betti number satisfying $\beta_1 > 0$.

Freely Indecomposable Groups
[AG10, Question 4.16]:
Are there groups that admit some *freely indecomposable* and some *freely decomposable* free p.m.p. actions?

1 Additionnal Questions

Another Generalized von Neumann's Problem Can one find an example of a non-amenable countable group without any subgroup with positive first ℓ^2-Betti number ?

... To be continued ...

References

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