Vademecum on ℓ^2-Betti numbers

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Some properties of ℓ^2-Betti numbers.

1. (Finite index - Reciprocity Formula) If Λ is a finite index subgroup of Γ then
$$\beta_n^{(2)}(\Lambda) = [\Gamma : \Lambda]\ \beta_n^{(2)}(\Gamma),\ \forall\, n \geq 0.$$
2. (Künneth) For direct products, we have
$$\beta_n^{(2)}(\Gamma_1\ \times\ \Gamma_2) = \sum_{i+j=n} \beta_i^{(2)}(\Gamma_1)\, \beta_j^{(2)}(\Gamma_2),\ \forall\, n \geq 0.$$
3. (Poincaré Duality) For the fundamental group Γ of a closed aspherical manifold of dimension p, we have
$$\beta_n^{(2)}(\Gamma) = \beta_{n-p}^{(2)}(\Gamma),\ \forall\, p \geq n \geq 0.$$
4. (Euler-Poincaré-Atiyah formula) If Γ admits a compact classifying space K then
$$\sum_n (-1)^n \beta_n^{(2)}(\Gamma) =: \chi^{(2)}(\Gamma) = \chi(\Gamma) = \chi(K)$$
5. (Mayer-Vietoris + Cheeger-Gromov) For free products with amalgamation over an infinite amenable subgroup Γ_3, we have
$$\beta_n^{(2)}(\Gamma_1\ *_{\Gamma_3}\ \Gamma_2) = \beta_n^{(2)}(\Gamma_1) + \beta_n^{(2)}(\Gamma_2),\ \forall\, n \geq 0.$$
6. For free products, we have
$$\beta_1^{(2)}(\Gamma_1\ *\ \Gamma_2) = \beta_1^{(2)}(\Gamma_1) + \beta_1^{(2)}(\Gamma_2) + 1 - [\beta_0^{(2)}(\Gamma_1) + \beta_0^{(2)}(\Gamma_2)],$$
where the sum in the square bracket vanishes if the groups Γ_i, $i = 1, 2$ are infinite; and
$$\beta_n^{(2)}(\Gamma_1\ *\ \Gamma_2) = \beta_n^{(2)}(\Gamma_1) + \beta_n^{(2)}(\Gamma_2),\ \forall\, n \geq 2.$$

A list of ℓ^2 Betti Numbers. We give some information about the values of some ℓ^2 Betti numbers

Group Γ	$\beta_*^{(2)}(\Gamma)$
Γ finite	$(\frac{1}{\lvert\Gamma\rvert},\ 0,\ 0,\ \ldots)$
Γ generated by g elements	$\beta_1^{(2)}(\Gamma) \leq g - 1$
Γ infinite amenable	$(0,\ 0,\ 0,\ \ldots)$
$\mathbf{F}_n$	$(0,\ n-1,\ 0,\ \ldots)$
$\pi_1(S_g)$	$(0,\ 2g-2,\ 0,\ \ldots)$
Lattice in $\mathrm{SO}(p,q)$	$\beta_d^{(2)}(\Gamma) = \begin{cases} \chi^{(2)}(\Gamma) & \text{if } d = pq/2 \\ 0 & \text{otherwise} \end{cases}$
Lattice in $\mathrm{SL}(n,\mathbb{R})$, $n > 2$	$(0,\ 0,\ 0,\ \ldots)$
$\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_l}$	$\beta_d^{(2)}(\Gamma) = \begin{cases} \prod_{j=1}^{l}(p_j - 1) & \text{if } d = l \\ 0 & \text{otherwise} \end{cases}$
$(\mathbf{F}_m \times \mathbf{F}_n) * \mathbf{F}_k$	$(0,\ k,\ (m-1)(n-1),\ 0,\ \ldots)$
$(\bigoplus_{n\in\mathbb{N}} \mathbf{F}_2) \times \mathbb{Z}$	$(0,\ 0,\ 0,\ \ldots)$
one-relator group $\Gamma = \langle g_1, \cdots, g_k \vert r\rangle$ $r = w^m$ with max m	$\beta_d^{(2)}(\Gamma) = \begin{cases} k - 1 - \frac{1}{m} & \text{if } d = 1 \\ 0 & \text{otherwise} \end{cases}$
$\Gamma = \mathrm{MCG}(S_g)$ Mapping class group and B_j Bernoulli number	$\beta_d^{(2)}(\Gamma) = \begin{cases} \frac{\lvert B_{2g}\rvert}{4g(g-1)} & \text{if } d = 3g-3 \\ 0 & \text{otherwise} \end{cases}$