Lecture 3: $\mathbb{SL}_2(\mathbb{R})$, part 2

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Goal

(I) The goal of this very technical lecture is to prove that $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ has a discrete decomposition for any lattice Γ in $G = \mathbb{SL}_2(\mathbb{R})$, and that cuspidal automorphic forms are rapidly decreasing near cusps.

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(II) This requires a very careful study of growth conditions on $\Gamma\backslash G$, and the key ingredient is finding reasonable fundamental domains, or approximations of such things, for the action of Γ on $\mathscr{H}$.

Fundamental domains

(I) If a group G acts on a topological space X, a fundamental domain for G acting on X is an open subset $\Omega \subset X$ such that $X = \cup_{g \in G} g.\overline{\Omega}$ and the various translates $g.\Omega$ are pairwise disjoint. The standard example is the following classical result (cf. any book on modular forms for the proof)

Theorem (Gauss) The set $\mathscr{F} = \{z \in \mathbb{C} \mid |z| > 1, |\mathrm{Re}(z)| < 1/2\}$ is a fundamental domain for the action of $\mathbb{SL}_2(\mathbb{Z})$ on $\mathscr{H}$.

One easily checks that $\mathscr{F}$ has finite (hyperbolic) area, and this implies that $\mathbb{SL}_2(\mathbb{Z})$ is indeed a lattice in G (something we never really checked before!).

Fundamental domains

(I) As an application, let's consider a finite index subgroup Γ in $\mathbb{SL}_2(\mathbb{Z})$ and $f \in M_k(\Gamma)$. Then an immediate calculation shows that

$$\varphi_f : \mathscr{H} \to \mathbb{R},\ z \to |f(z)|y^{k/2}$$

is Γ-invariant, more precisely $\varphi_{f|_k g}(z) = f(g.z)$ for $g \in G$. We claim that φ_f is bounded when $f \in S_k(\Gamma)$.

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(II) Indeed, write $\mathbb{SL}_2(\mathbb{Z}) = \coprod_{i=1}^{k} \Gamma\gamma_i$ and $D = \overline{\mathscr{F}}$, so that $\mathscr{H} = \cup_i \cup_{\gamma\in\Gamma} \gamma\gamma_i D$. Thus it suffices to check that $\varphi_{f|_k\gamma_i}$ is bounded on D for all i.

Fundamental domains

(I) But $f_i := f|_k\gamma_i \in S_k(\gamma_i^{-1}\Gamma\gamma_i)$ and the q-expansion at ∞ shows that $f_i(x+iy) = O(e^{-cy})$ for some $c > 0$, as $y \to \infty$, uniformly in $z = x + iy \in D$. Thus $\varphi_{f_i}(z)$ tends to 0 as $z \to \infty$ in D, so we are done.

Theorem (Hecke's bound) Let $f(z) = \sum_{n \geq 0} a_n e^{2i\pi nz/h}$ be the q-expansion at ∞ of $f \in S_k(\Gamma)$. Then $a_n = O(n^{k/2})$, more precisely

$$\sum_{n \leq x} |a_n|^2 = O(x^k),\ x \to \infty.$$

Fundamental domains

(I) The proof is very simple: write $|\varphi_f(z)| \leq C$ for all z, so $|f(x+iy)| \leq Cy^{-k/2}$. Plancherel's formula yields (for a suitable constant c)

$$\sum_{n \geq 1} |a_n|^2 e^{-4\pi n y/h} = c \int_0^h |f(x+iy)|^2 dx \leq c' y^{-k}.$$

Take $y = 1/N$ to get $\sum_{n \leq N} |a_n|^2 \leq c'' N^k$.

Theorem We have $S_0(\Gamma) = 0$ and $M_0(\Gamma) = \mathbb{C}$.

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Theorem We have $S_0(\Gamma) = 0$ and $M_0(\Gamma) = \mathbb{C}$.

(II) If we use that $X(\Gamma)$ is a compact Riemann surface, this is clear. Without this input, note that for $f \in S_0(\Gamma)$ the function $\varphi_f = |f|$ is bounded and tends to 0 at ∞, thus has a maximum on $\mathscr{F}$. By the maximum principle f is constant and since f vanishes at ∞, $f = 0$. Actually the same argument works even if we only assume that $f \in M_0(\Gamma)$.

Siegel sets

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(II) Pick $z \in \partial\mathscr{H}$ and let $P = \pm A_P N_P = G_z$ be the associated parabolic of G. The action of A_P on $\mathrm{Lie}(N_P)$ defines a character $\alpha = \alpha_P : A_P \to \mathbb{R}_{>0}$, thus $aYa^{-1} = \alpha(a)Y$ for $a \in A_P$ and $Y \in \mathrm{Lie}(N_P)$. If $P = B$ is the standard Borel subgroup, then $\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = t^2$. If $t > 0$, let

$$A_{P,t} = \{a \in A_P \,|\, \alpha_P(a) > t\}.$$

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$$A_{P,t} = \{a \in A_P | \, \alpha_P(a) > t\}.$$

(III) A **Siegel set at** P is a set of the form

$$\Sigma = \omega A_{P,t} K \subset G$$

for some $t > 0$ and some compact set $\omega \subset N_P$. The image of Σ in $\mathscr{H} \simeq G/K$ is called a **Siegel set at** z.

Siegel sets

(I) Let us make a few useful remarks. First, since $N_P \times A_P \times K \to G$ is a homeomorphism, any compact subset of G is contained in some Siegel set at P.

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(II) Next, if $z = \infty$ and $\omega = \begin{pmatrix} 1 & [-c, c] \\ 0 & 1 \end{pmatrix}$ with $c > 0$, the associated Siegel set at z is

$$\Sigma = \{x + iy \in \mathscr{H} \,|\, |x| \leq c, y > t\}.$$

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(III) If $k \in K$ and Σ is a Siegel set for P, with fixed point $z \in \partial \mathscr{H}$ then $k.\Sigma$ is a Siegel set for kPk^{-1}, with fixed point $k.z$, so we can always reduce to the previous situation.

Siegel sets

(I) Let $z \in C(\Gamma)$ and $\pi : \mathscr{H} \cup C(\Gamma) \to X(\Gamma)$ the natural projection. Using the previous remarks, one easily checks that sets of the form $\pi(\{z\} \cup \Sigma)$ form a basis of neighborhoods of $\pi(z)$ in $X(\Gamma)$, when Σ varies among Siegel sets at z.

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(II) Since Γ is a lattice in G, by Siegel's theorem $\Gamma \backslash CP(\Gamma)$ is finite. Choose a set of representatives $P_1, ..., P_l$ for this set.

Theorem There are Siegel sets Σ_i at P_i such that

$$G = \Gamma \bullet (\cup_{i=1}^{l} \Sigma_i).$$

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Theorem There are Siegel sets Σ_i at P_i such that

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(III) The proof follows easily from the compactness of $X(\Gamma)$ and the previous geometric remarks.

Siegel sets

(I) We will constantly use the following simple but useful result. Fix a Siegel set Σ at some parabolic P, and write $x = n(x)a(x)k(x)$ with respect to the Iwasawa decomposition $N_P \times A_P \times K \simeq G$.

Lemma As x varies in Σ, $a(x)^{-1}x$ stays in a compact set and $||x||^2$ behaves like $\alpha_P(a(x))$, i.e. there are constants $c_1, c_2 > 0$ such that for all $x \in \Sigma$

$$c_1 \leq \frac{||x||^2}{\alpha_P(a(x))} \leq c_2.$$

(II) By conjugating, WLOG $P = B$, so that $\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = t^2$.

Siegel sets

(I) Write $a(x) = \begin{pmatrix} t_x & 0 \\ 0 & t_x^{-1} \end{pmatrix}$. Then $a(x)^{-1} n(x) a(x) = \begin{pmatrix} 1 & u_x t_x^{-2} \\ 0 & 1 \end{pmatrix}$ if $n_x = \begin{pmatrix} 1 & u_x \\ 0 & 1 \end{pmatrix}$.

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(II) Since u_x stays in a compact and t_x is bounded from below on Σ, this gives the first part. For the second, by the first part $||x||$ behaves like $||a(x)||$, so it suffices to check that $||a(x)||$ behaves like t_x, which again follows from the fact that t_x is bounded from below on Σ by definition.

Siegel sets

(I) If $P \in CP(\Gamma)$ and Σ is a Siegel set at P, we say that $f : \Sigma \to \mathbb{C}$ is **moderate growth**(resp **rapidly decreasing**) if there exists $d \geq 1$ (resp. for all integers d) such that $\sup_{x \in \Sigma} \alpha(a(x))^{-d}|f(x)| < \infty$. By the previous lemma, one could replace $\alpha(a(x))$ with $||x||$ and get equivalent definitions.

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(II) The following result reduces many global problems to problems at individual cusps of $X(\Gamma)$. The proof is slightly tricky.

Theorem Let $\Sigma_1, ..., \Sigma_l$ be Siegel sets such that $\Gamma(\cup \Sigma_i) = G$. A function f on $\Gamma \backslash G$ has moderate growth on G if and only if f has moderate growth on each Σ_i.

(III) The only delicate part is showing that if f has MG on Σ_i for all i, then f has MG on G.

Siegel sets

(I) So assume that $|f(x)| \leq c||x||^N$ for $x \in \cup_i \Sigma_i$, for suitable c, N. Pick $g \in G$ and write $g = \gamma u$ for some $u \in \Sigma_i$ and $\gamma \in \Gamma$. Then
$$|f(g)| = |f(u)| \leq c||u||^N.$$

(II) So it suffices to check that $||u|| \leq c'||\gamma u||$ for all $u \in \Sigma_i$ and $\gamma \in \Gamma$, for a suitable c'. By the useful lemma it suffices to have an estimate $||a(x)|| \leq c'||\gamma a(x)||$ for $x \in \Sigma_i$. Conjugating everything WLOG $P = B$. Write $a(x) = \begin{pmatrix} t_x & 0 \\ 0 & t_x^{-1} \end{pmatrix}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We need
$$t_x^2 + 1/t_x^2 \leq c''(t_x^2(a^2 + c^2) + (b^2 + d^2)/t_x^2).$$

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$$t_x^2 + 1/t_x^2 \leq c''(t_x^2(a^2 + c^2) + (b^2 + d^2)/t_x^2).$$

(III) Since t_x has a positive lower bound, we win if we can prove that c cannot be too small, unless it is 0 (we have already seen in the last lecture that if $c = 0$, then $a^2 = 1$). This is clear when $\Gamma \subset \mathbb{SL}_2(\mathbb{Z})$, but tricky in general.

Siegel sets

(I) Say $\Gamma \cap \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$, we will show that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfies $|ch| < 1$, then $c = 0$.

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(II) Indeed, suppose that $|ch| < 1$ and define $\gamma_0 = \gamma$ and $\gamma_{n+1} = \gamma_n \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \gamma_n^{-1}$, then an amusing real analysis exercise shows that $\gamma_n \to \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. Since Γ is discrete, $\gamma_n = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for n large enough, and then easily $c = 0$.

The first fundamental estimate

(I) We're going to use several times the following very effective estimate:

Theorem There is $N \geq 1$ such that for all $\alpha \in C_c^\infty(G)$ there is $c_\alpha > 0$ with

$$|f * \alpha(x)| \leq c_\alpha ||x||^N_\bullet ||f||_{L^1}, \forall f \in L^1(\Gamma \backslash G), x \in G.$$

In particular $f * \alpha$ has moderate growth for any $\alpha \in C_c^\infty(G)$ and $f \in L^1(\Gamma \backslash G)$, with uniform exponent!

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In particular $f * \alpha$ has moderate growth for any $\alpha \in C_c^\infty(G)$ and $f \in L^1(\Gamma \backslash G)$, with uniform exponent!

(II) By the usual trick we have, with $K(x, y) = \sum_{\gamma \in \Gamma} |\alpha(y^{-1}\gamma x)|$

$$|(f * \alpha)(x)| \leq \int_G |f(y)||\alpha(y^{-1}x)|dy = \int_{\Gamma \backslash G} |f(y)|K(x, y)dy.$$

The first fundamental estimate

(I) It suffices therefore to have a bound $K(x,y) \leq c||x||^N$ with c depending only on α, not on f and x. But if $U = \mathrm{Supp}(\alpha)$ (a compact set), then

$$K(x,y) \leq ||\alpha||_\infty \sum_{\gamma\in\Gamma} 1_{y^{-1}\gamma x\in U}$$

and we saw in the previous lecture that this is bounded uniformly by $c||x||^N$.

The second fundamental estimate

(I) The key technical result of this lecture is the following rather awful-looking statement. Fix $P \in CP(\Gamma)$, and let $N = N_P$ and $\Gamma_N = \Gamma \cap N$. Recall that for $u \in C(\Gamma_N \backslash G)$ the constant term at P is

$$u(g) = \int_{\Gamma_N \backslash N} u(ng) dn.$$

Theorem (second fundamental estimate) Let Σ be a Siegel set at P. For any $d \geq 1$ there are $D_1, ..., D_k \in U(\mathfrak{g})$ such that for all $f \in C^\infty(\Gamma_N \backslash G)$, $x \in \Sigma$

$$|f(x) - f_P(x)| \leq ||x||^{-d} \sum_{i=1}^{k} |D_i f|_P(x).$$

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(II) We leave the proof for the end of the lecture, and focus on the applications first. Keep P and Σ as in the theorem.

Cusp forms are rapidly decreasing

(I) Using the previous results, we are ready to prove the fundamental:

Theorem Let Σ be a Siegel set at some $P \in CP(\Gamma)$. Any $f \in A_{\mathrm{cusp}}(\Gamma)$ is rapidly decreasing on Σ.

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Theorem Let Σ be a Siegel set at some $P \in CP(\Gamma)$. Any $f \in A_{\mathrm{cusp}}(\Gamma)$ is rapidly decreasing on Σ.

(II) We saw in the last lecture that f has uniform moderate growth, i.e. there is N such that for all $D \in U(\mathfrak{g})$ we have $|Df(g)| \leq c_D ||g||^N$ for all g. This allows us to bound $|D_i f(g)| \leq c||g||^N$ with D_i as in the second fundamental estimate (for a given $d \geq 1$). Since $\Gamma_N \backslash N$ is compact, this gives an estimate $|D_i f|_P(x) \leq c||x||^N$ for $x \in \Sigma$ and thus

$$|f(x)| \leq c||x||^{N-d}$$

on Σ. Since N is fixed and d is arbitrary, we are done.

A key estimate

(I) For this lecture, the most important application of all previous results is the following technical but useful:

Theorem For any $\alpha \in C_c^\infty(G)$ there is c_α such that for all $f \in L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ and all $g \in G$

$$||f * \alpha||_\infty \leq c_\alpha ||f||_{L^2(\Gamma\backslash G)}.$$

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$$||f * \alpha||_\infty \leq c_\alpha ||f||_{L^2(\Gamma\backslash G)}.$$

(II) Since $\Gamma\backslash G$ is covered by finitely many Siegel sets at cuspidal parabolic subgroups, it is enough to prove the lemma with g varying in a given Siegel set Σ at $P \in CP(\Gamma)$.

A key estimate

(I) Fix now $\alpha \in C_c^\infty(G)$. A simple computation shows that

$$(f * \alpha)_P = f_P * \alpha = 0.$$

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(II) Now pick $N \geq 1$ so that (first fundamental estimate) for any $\beta \in C_c^\infty(G)$ we have

$$\sup_{x \in G, f \in L^1(\Gamma \backslash G)} \frac{|(f * \beta)(x)|}{||x||^N{}_\bullet ||f||_{L^1}} < \infty. \quad (1)$$

A key estimate

(I) Combining the previous observations with the second fundamental estimate (applied to $f * \alpha$ and $d = N$) yields $D_1, ..., D_k \in U(\mathfrak{g})$ so that for all $x \in \Sigma$

$$|f * \alpha(x)| \leq ||x||^{-N} \sum_{i=1}^{k} |f * (D_i \alpha)|_P(x) \qquad (2).$$

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(II) Taking $\beta = D_i\alpha$ in (1) yields c so that for all $f \in L^1(\Gamma \backslash G)$ and $1 \leq i \leq k$ we have $|f * (D_i\alpha)(x)| \leq c_i ||x||^N ||f||_{L^1}$ for all $x \in G$. Since $L^2 \subset L^1$ is a continuous injection (Cauchy-Schwarz coupled with $\int_{\Gamma \backslash G} dg < \infty$), it follows that there is c such that for all $f \in L^2$ and all i and $x \in G$

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$$|f * (D_i\alpha)(x)| \leq c||x||^N||f||_{L^2}.$$

(III) Again the compactness of $\Gamma_N\backslash N$ yields an estimate $|D_i f|_P(x) \leq C||x||^N$ for $x \in \Sigma$ and we are done thanks to (2).

GGPS in the non co-compact case

(I) Recall that $C_c^\infty(G)$ acts on any object $V \in \mathrm{Rep}(G)$ by $f.v = \int_G f(g)g.vdg$ and when V is a space of functions on G, the induced operator $T_f : \varphi \to f.\varphi$ is simply $f * \varphi$.

Theorem (Gelfand, Graev, Piatetski-Shapiro) For any $\alpha \in C_c^\infty(G)$ the operator T_α is Hilbert-Schmidt, thus compact on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Hence $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ has a discrete decomposition

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G) \simeq \widehat{\bigoplus}_{\pi\in\hat{G}} \pi \otimes \mathrm{Hom}_G(\pi, L^2_{\mathrm{cusp}}(\Gamma\backslash G))$$

with $\mathrm{Hom}_G(\pi, L^2_{\mathrm{cusp}}(\Gamma\backslash G))$ finite dimensional vector spaces.

Combining this with the Dixmier-Malliavin theorem, it follows that T_α is actually of trace class.

GGPS in the non co-compact case

(I) The previous theorem combined with Riesz' theorem show that for any $g \in \Gamma\backslash G$ there is $K_g \in L^2_{\mathrm{cusp}}$ with $T_\alpha(f)(g) = \langle f, K_g \rangle$ for all $f \in L^2_{\mathrm{cusp}}$. Moreover $||K_g||_{L^2} \leq c_\alpha$, thus $g \to K_g$ is bounded. The tricky thing is that we don't know that setting $K(g, x) = K_g(x)$ gives a measurable function.

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(II) We prove first that $\Gamma\backslash G \to L^2_{\mathrm{cusp}}, g \to K_g$ is continuous. Fix g and $\varepsilon > 0$. We need to show that

$$|T_\alpha(f)(g) - T_\alpha(f)(g')| \leq \varepsilon ||f||_{L^2}$$

for all $f \in L^2_{\mathrm{cusp}}$ if g' is close enough to g.

GGPS in the non co-compact case

(I) It suffices for this to have a bound for each $X \in \mathfrak{g}$

$$||X.T_\alpha(f)||_\infty \leq c_X ||f||_{L^2}$$

with c_X independent of f. But $X.T_\alpha(f) = X.(f * \alpha) = f * (X.\alpha)$, so it suffices to apply the previous theorem to $X\alpha \in C_c^\infty(G)$.

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(II) Since $g \to K_g$ is continuous and bounded, we can define a continuous linear form on $L^2(\Gamma\backslash G \times \Gamma\backslash G)$ by

$$U(\varphi) := \int_{\Gamma\backslash G} \langle \varphi(g, \bullet), K_g \rangle dg,$$

where $\varphi(g, \bullet) : x \to \varphi(g, x)$ (by Fubini $g \to \varphi(g, \bullet)$ is in $L^2(\Gamma\backslash G, L^2(\Gamma\backslash G))$, so U is well-defined).

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(I) Applying Riesz we obtain some $K' \in L^2(\Gamma\backslash G \times \Gamma\backslash G)$ such that $U(\varphi) = \langle \varphi, K' \rangle$ for all φ. Taking $\varphi(x,y) = u(x)f(y)$ with $u \in C_c^\infty(\Gamma\backslash G)$ and expanding everything yields

$$\int_{\Gamma\backslash G} u(g) T_\alpha(f)(g) = \int_{\Gamma\backslash G} \langle u(g)f, K_g \rangle dg =$$

$$\int_{\Gamma\backslash G} \langle \varphi(g,\bullet), K_g \rangle dg = \int_{\Gamma\backslash G \times \Gamma\backslash G} u(g)f(y)\overline{K'(g,y)}dy =$$

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(II) Varying u finally exhibits exhibits T_α as a HS operator

$$T_\alpha(f)(x) = \int_{\Gamma\backslash G} f(y) \overline{K'(x,y)} dy.$$

Proof of the second fundamental estimate

(I) Fix $P \in CP(\Gamma)$ and write for simplicity $A := A_P$ and $N := N_P$. Recall the character $\alpha = \alpha_P : A \to \mathbb{R}_{>0}$ such that $aYa^{-1} = \alpha(a)Y$ for $Y \in \mathrm{Lie}(N)$, and that $N \times A \times K \to G$ is a diffeomorphism, so we can write $x = n(x)a(x)k(x)$ with $n(x) \in N, a(x) \in A, k(x) \in K$.

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(III) Fix $x \in \Sigma$ and consider the smooth 1-periodic map $u(t) = f(e^{tY}x)$.

Proof of the second fundamental estimate

(I) Note that $f(x) = u(0)$ and

$$f_P(x) = \int_{\Gamma_N \backslash N} f(nx)dn = \int_{\mathbb{Z}\backslash\mathbb{R}} f(e^{tY}x)dt = \int_0^1 u(t)dt.$$

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(III) Replacing u by $u - u(0)$, WLOG $u(0) = 0$, so $u(1) = 0$. Now use repeated integrations by parts to get

$$|\int_0^1 u(t)dt| = |\pm \int_0^1 \frac{t^d}{d!} u^{(d)}(t)dt| \leq \frac{1}{d!} \int_0^1 |u^{(d)}(t)|dt.$$

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(I) Next, we compute

$$u'(t) = \frac{d}{du}|_{u=0} f(e^{tY} x x^{-1} e^{uY} x) =$$

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(II) By the very useful lemma on Siegel sets we can write $x = a(x)y(x)$, with $y(x)$ in a compact set, and by definition of α we obtain

$$D_x = \alpha(a(x))^{-d}(y(x)^{-1}Yy(x))^d.$$

Proof of the second fundamental estimate

(I) Take a basis D_i $(1 \leq i \leq k)$ of the subspace of $U(\mathfrak{g})$ spanned by all $X_1 ... X_d$ with $X_i \in \mathfrak{g}$. Then

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(II) Hence there is c such that for all f and $x \in \Sigma$

$$|(D_x.f)(e^{tY}x)| \leq c\alpha(a(x))^{-d} \sum_{i=1}^{k} |D_i.f(e^{tY}x)|.$$

Putting everything together we get

$$|(f - f_P)(x)| \leq c\alpha(a(x))^{-d} \sum_{i=1}^{k} \int_{\Gamma_N \backslash N} |D_i.f(nx)| dn.$$

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(III) We conclude recalling that $\alpha(a(x))$ is approximately $||x||^2$ on Σ, and $||x|| \geq 1$ for all x.

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(III) Prove that $A_{\mathrm{cusp}}(\Gamma) \subset L^2(\Gamma \backslash G)$.