Lecture 5: $\mathbb{SL}_2(\mathbb{R})$, part 3

Gabriel Dospinescu

CNRS, ENS Lyon

Goal

(I) In this lecture we want to discuss a beautiful application of the theory developed so far to the spectral theory of a compact Riemann surface X of genus ≥ 2. By the uniformization theorem, any such surface is a quotient $X \simeq \Gamma\backslash\mathscr{H}$ with Γ a co-compact lattice in $\mathbb{PSL}_2(\mathbb{R}) = \mathbb{SL}_2(\mathbb{R})/\{\pm 1\}$ having no nontrivial torsion points.

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(II) We can associate to X two collections of real numbers: one coming directly from the geometry of X, namely the set of lengths of closed geodesics on X, and the second one coming from spectral theory, namely the eigenvalues of the Laplace-Beltrami operator on X. Our goal in this lecture is to study the relation between these sets.

Closed geodesics and their lengths

(I) Before doing that let's define more carefully the two sets. Each $\gamma \in \Gamma \setminus \{1\}$ is hyperbolic, i.e. satisfies $|\mathrm{tr}(\gamma)| > 2$, thus we can define

$$l(\gamma) = 2\mathrm{arccosh}(\frac{|\mathrm{tr}(\gamma)|}{2}).$$

Note that $l(\gamma)$ depends only on the conjugacy class of γ in $\mathbb{PSL}_2(\mathbb{R})$.

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Note that $l(\gamma)$ depends only on the conjugacy class of γ in $\mathbb{PSL}_2(\mathbb{R})$.

(II) More geometrically, the action of γ on $\mathscr{H}$ is conjugated to $z \to e^{l(\gamma)}z$. There is a unique geodesic in $\mathscr{H}$ stabilized by γ, called the axis $a(\gamma)$ of γ. It is naturally oriented, by going from the unique repulsive fixed point of γ to the unique attractive fixed point (both points being on $a(\gamma)$). Then $l(\gamma)$ is the length of the oriented closed geodesic $\pi(a(\gamma))$ on X, where $\pi : \mathscr{H} \to X$ is the canonical projection.

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(III) Let $\mathscr{L}_X$ be the multi-set of lengths of all primitive closed geodesics on X, taken with multiplicities.

Laplacian spectrum

(I) The G-invariant hyperbolic measure $d\mu(z) = dxdy/y^2$ on $\mathscr{H}$ descends to X and we can form $L^2(X) = L^2(X, d\mu(x))$, with

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}d\mu(x).$$

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(II) The Laplace-Beltrami operator Δ on $C^\infty(\mathscr{H})$

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

commutes with the action of G and descends therefore to an operator Δ on $C^\infty(X)$.

Laplacian spectrum

(I) We can thus see Δ as an unbounded operator on $L^2(X)$ and try to study its spectrum. One checks using Stokes' formula that $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$ for $f, g \in C^\infty(X)$ and that

$$\langle \Delta f, f\rangle \geq 0$$

for all $f \in C^\infty(X)$, with equality if and only if f is constant.

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(II) In particular all eigenvalues of Δ on $C^\infty(X)$ are ≥ 0 and the eigenvalue 0 occurs with multiplicity 1.

(III) We will see that $L^2(X)$ has an orthonormal basis consisting of eigenvalues of Δ and each eigenspace is finite dimensional. Let $\Delta(X)$ be the set of eigenvalues of Δ on $C^\infty(X)$, each eigenvalue occurring with a multiplicity equal to the dimension of the eigenspace.

The Selberg trace formula

(I) We can now state the amazing theorem we're looking for:

Theorem (Selberg's trace formula for compact hyperbolic curves) Let $g \in C_c^\infty(\mathbb{R})$ be an even function and let $h = \hat{g}$ be its Fourier transform, thus $h(x) = \int_{\mathbb{R}} e^{-ixt} g(t)dt$. Then

$$\sum_{\lambda \in \Delta(X)} h(\sqrt{\lambda - \frac{1}{4}}) = \frac{\mathrm{area}(X)}{2\pi} \int_0^\infty x h(x) \tanh(\pi x) dx$$

$$+ \frac{1}{4\pi} \sum_{l \in \mathscr{L}_X} \sum_{n \geq 1} \frac{l}{\sinh \frac{nl}{2}} \hat{h}(nl),$$

all sums and integrals being absolutely convergent.

Note that the statement makes sense, i.e. it is independent of the choice of the square root of $\lambda - \frac{1}{4}$, since h is even.

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(II) First, Huber's theorem: two compact hyperbolic surfaces X, X' are isospectral (i.e. $\Delta(X) = \Delta(X')$) if and only if $\mathscr{L}_X = \mathscr{L}_{X'}$. Next, McKean's theorem: for a given X there are only finitely many X' up to isometry which are isospectral to X.

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(III) Weyl's estimate: if $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ is the sequence of all eigenvalues of Δ, then

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{\mathrm{area}(X)}.$$

The Selberg trace formula

(I) Once one has the Weyl estimate we can refine the trace formula (by an approximation argument) by allowing any even holomorphic function h on the domain $|\mathrm{Im}(z)| < \frac{1}{2} + \varepsilon$ such that $h(z) = O((1 + |z|^2)^{-1-\varepsilon})$ (for some $\varepsilon > 0$).

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(II) One then obtains (with work!) the prime geodesic theorem, analog of the prime number theorem: the number of $l \in \mathscr{L}_X$ with $e^l \leq x$ is asymptotically $x/\log x$ as $x \to \infty$.

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(III) Finally, the Selberg zeta function

$$Z_X(s) = \prod_{l \in \mathscr{L}_X} \prod_{n \geq 0} (1 - e^{-l(s+n)}),$$

a priori convergent for $\mathrm{Re}(s) > 1$, extends to a holomorphic function on $\mathbb{C}$ satisfying a functional equation $Z_X(s) = G(s) Z_X(1-s)$ for an explicit, but rather complicated function G.

"L'île aux enfants": Casimir

(I) To prove the trace formula, we will reformulate the problem in terms of representation theory and use a very general Selberg trace formula for compact quotients, coupled with a fine study of the Casimir operator and of the spherical Hecke algebra of G.

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(III) A first key observation is that we can identify (since K is compact)

$$L^2(X) \simeq L^2(\Gamma\backslash G)^K.$$

Thus our problem is closed related to the study of $L^2(\Gamma\backslash G)$ and that of K-invariants in unitary representations of G.

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(I) Passing to K-invariants in the GGPS decomposition

$$L^2(\Gamma\backslash G) \simeq \widehat{\bigoplus_{\pi\in\hat{G}}} \pi^{\oplus m(\pi)}$$

and letting

$$\hat{G}^{\mathrm{sph}} = \{\pi \in \hat{G} \,|\, \pi^K \neq 0\}$$

yields

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(II) The classification theorem describes $\hat{G}^{\mathrm{sph}}$ completely: it consists of the unitary principal series attached to characters $a \to |a|^s$ with $s \in i\mathbb{R}_+$, and of the complementary series of parameter $s \in (0,1)$. Call these representations simply π_s with $s \in i\mathbb{R}_+ \cup (0,1)$.

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(I) A second key observation (which is not really an observation, but rather a brutal computation that I will skip) is that the Casimir operator $\mathscr{C} \in Z(U(\mathfrak{g}))$ acting on $C^\infty(G)$ descends (by invariance) to an operator on $C^\infty(\mathscr{H}) \simeq C^\infty(G)^K$ and this is precisely 2Δ:

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(II) It turns out that $\mathscr{C}$ acts on the smooth vectors π^∞ of each $\pi \in \hat{G}$ by a scalar. For instance, $\mathscr{C}$ acts by $\frac{1-s^2}{2}$ on π_s^∞, as one can easily check by hand. In particular the eigenvalue of $\mathscr{C}$ determines $s \in i\mathbb{R}_+ \cup (0,1)$ uniquely.

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(III) Another key fact, which we will prove soon is that $\dim \pi^K = 1$ for $\pi \in \hat{G}^{\mathrm{sph}}$, and each $v \in \pi^K$ is smooth and an eigenvector of $\mathscr{C}$.

"L'île aux enfants": Casimir

(I) Combining the previous observations gives

Theorem $L^2(X)$ has an ON-basis consisting of smooth functions that are eigenvectors of $\mathscr{C}$ and thus of Δ.

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(I) We want to express $m(\pi_s)$ in terms of the eigenvalue $\frac{1-s^2}{4}$ only. For this consider the space

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(III) This follows immediately from the decomposition

$$L^2(X) \simeq \bigoplus_{s \in i\mathbb{R}_{\geq 0} \cup (0,1)} (\mathbb{C} f_s)^{\oplus m(\pi_s)},$$

induced by the GGPS decomposition, passage to K-invariants and the previous results.

Gelfand pairs

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(II) Let G be a locally compact unimodular group and let K be a compact subgroup. We let dk be the unique probability Haar measure on G and dg a Haar measure on G. Let $C_c(G//K)$ be the space of continuous compactly supported functions on G which are bi-K-invariant, i.e. $f(k_1 g k_2) = f(g)$ for $g \in G$, $k_1, k_2 \in K$.

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(III) We can construct elements of $C_c(G//K)$ by starting with an arbitrary $f \in C_c(G)$ and considering

$$f_K(x) = \int_{K^2} f(k_1 x k_2) dk_1 dk_2.$$

Gelfand pairs

(I) Define $L^1(G//K)$ and $C_c^\infty(G//K)$ (if G is a Lie group) in the obvious way. One easily checks that $C_c(G//K)$ and $L^1(G//K)$ are algebras for the convolution product and $C_c(G//K)$ is dense in $L^1(G//K)$.

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(II) If $V \in \mathrm{Rep}(G)$, then for all $v \in V$ and $f \in C_c(G//K)$ we have $f.v \in V^K$, since

$$k.(f.v) = \int_G f(g)kg.vdg = \int_G f(kg)kg.vdg =$$

$$\int_G f(g)g.vdg = f.v.$$

In particular V^K becomes a module over $C_c(G//K)$.

Gelfand pairs

(I) We say that (G, K) is a **Gelfand pair** if $C_c(G//K)$ is commutative. This is equivalent to saying that $L^1(G//K)$ is commutative. A key source of Gelfand pairs comes from the following beautiful and easy result.

Theorem (Gelfand's trick) Suppose that there is an automorphism $\iota : G \to G$ with $\iota \circ \iota = \mathrm{id}$ and $\iota(x) \in Kx^{-1}K$ for $x \in G$. Then (G, K) is a Gelfand pair.

Gelfand pairs

(I) For instance if $G = \mathbb{SL}_2(\mathbb{R})$ and $K = \mathbb{SO}_2(\mathbb{R})$ we can take $\iota(x)$ the inverse of the transpose of x. The condition comes down to $x^T \in KxK$ for all $x \in G$. This follows from the Cartan decomposition $G = KAK$ (exercise), which reduces everything to the case $x \in A$, but then $x = x^T$ and we are done. This kind of argument generalizes to real reductive groups and their maximal compact subgroups.

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(II) The proof of the theorem is simple and beautiful. If $f \in C_c(G)$ let $\bar{f}(x) = f(\iota(x))$ and $\tilde{f}(x) = f(x^{-1})$. The hypothesis implies that $\bar{f} = \tilde{f}$ for $f \in C_c(G//K)$. On the other hand, the uniqueness (up to scalar) of the Haar measure gives the existence of a constant $c > 0$ such that $\int_G \bar{f}(x)dx = c \int_G f(x)dx$ for all $f \in C_c(G)$. Since $\iota^2 = 1$, we have $c^2 = 1$, thus $c = 1$. This easily implies that $\bar{f} * \bar{g} = \overline{f * g}$. On the other hand, the unimodularity of G yields $g \tilde{*} f = \tilde{f} * \tilde{g}$.

Gelfand pairs

(I) Since $\bar{f} = \tilde{f}$ for $f \in C_c(G//K)$, we conclude that for all $f, g \in C_c(G//K)$

$$\overline{f * g} = \bar{f} * \bar{g} = \tilde{f} * \tilde{g} = g \,\tilde{*}\, f = \overline{g * f},$$

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(II) For us the most important application of Gelfand pairs is the following beautiful result (whose converse also holds, but is quite a bit more delicate, using the Gelfand-Raikov theorem which we haven't discussed). Let

$$G^{\mathrm{sph}} = \{\pi \in \hat{G} \,|\, \pi^K \neq 0\}.$$

Theorem If (G, K) is a Gelfand pair and $V \in G^{\mathrm{sph}}$, then $\dim V^K = 1$ and there is a morphism of algebras $\chi_\pi : C_c(G//K) \to \mathbb{C}$ such that $f.v = \chi_\pi(f)v$ for $v \in V^K$ and $f \in C_c(G//K)$.

Gelfand pairs

(I) Of course, it suffices to prove that $\dim V^K = 1$. We claim that V^K is irreducible under $C_c(G//K)$ when $V \in \hat{G}$, i.e. for any $v \in V^K \setminus \{0\}$ the closure of $C_c(G//K).v$ is V^K. Pick any $w \in V^K$ and $\varepsilon > 0$. By irreducibility of V, $C_c(G).v$ is dense in V, thus we can find $f \in C_c(G)$ with $||f.v - w|| < \varepsilon$.

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(II) Since v is K-invariant, a simple calculation yields

$$f_K.v = \int_G \int_{K^2} f(k_1 x k_2) x.v dx =$$

$$\int_K \int_G f(k_1 x) x.v dx dk_1 = \int_K k.(f.v) dk.$$

Gelfand pairs

(I) Since $v \to \int_K k.vdk$ is the orthogonal projection of V onto V^K (lecture 2), we deduce that

$$||f_K.v - w|| \leq ||f.v - w|| \leq \varepsilon$$

and since $f_K \in C_c(G//K)$, the claim is proved.

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(II) Now since by assumption $L^1(G//K)$ is a commutative Banach algebra with a natural involution $f \to (g \to \overline{f(g^{-1})})$, an argument as in the proof of Schur's lemma (lecture 2) shows that the only irreducible unitary reps. of $L^1(G/K)$ are 1-dimensional, thus dim $V^K \leq 1$ and we are done.

Gelfand pairs

(I) Suppose now that G is a real Lie group and (G, K) is a Gelfand pair. If $\pi \in \hat{G}$, consider the restriction $\chi_\pi : \mathrm{Sph} \to \mathbb{C}$ of $\chi_\pi : C_c(G//K) \to \mathbb{C}$ to the spherical Hecke algebra $\mathrm{Sph} = C_c^\infty(G//K)$.

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(II) It is important to interpret $\chi_\pi(f)$ as a trace. Namely, the operator $T_f : \pi \to \pi, v \to f.v$ has image inside π^K, thus it is trivially of trace class and $\mathrm{tr}(T_f) = \chi_\pi(f)$.

The spherical unitary dual and Hecke algebra

(I) Let's come back to earth and get our hands dirty with $G = \mathbb{SL}_2(\mathbb{R})$. Keep the usual notations A, N, K, etc. We want to make χ_π as explicit as possible for $\pi \in \hat{G}$.

The spherical unitary dual and Hecke algebra

(I) Let's come back to earth and get our hands dirty with $G = \mathbb{SL}_2(\mathbb{R})$. Keep the usual notations A, N, K, etc. We want to make χ_π as explicit as possible for $\pi \in \hat{G}$.

(II) Recall that π_s is realised as a space of functions on G, and elements of π_s^K correspond to certain functions on $G/K \simeq \mathscr{H}$. The explicit description of $\pi_s|_K$ (lecture 1) then shows that

$$\pi_s^K = \mathbb{C} f_s,$$

where the spherical vector f_s is the function on $\mathscr{H}$

$$f_s(z) = \mathrm{Im}(z)^{\frac{1+s}{2}}.$$

The spherical unitary dual and Hecke algebra

(I) Taking $f \in \mathrm{Sph}$ and evaluating at i the identity $f.f_s = \chi_{\pi_s}(f) f_s$, we obtain

$$\chi_{\pi_s}(f) = \int_G f(g) f_s(g.i) dg.$$

The spherical unitary dual and Hecke algebra

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$$\chi_{\pi_s}(f) = \int_G f(g)f_s(g.i)dg.$$

(II) The Haar measure decomposes with respect to the Iwasawa decomposition $G = ANK$

$$\int_G F(g)dg = \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} F(\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k)dkdudx.$$

The spherical unitary dual and Hecke algebra

(I) If F is right K-invariant, this simplifies to

$$\int_G F(g)dg = \int_{\mathbb{R}}\int_{\mathbb{R}} F(\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})dudx.$$

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(II) We conclude that

$$\chi_{\pi_s}(f) = \int_{\mathbb{R}^2} f(\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})e^{u\frac{1+s}{2}}dudx.$$

The spherical unitary dual and Hecke algebra

(I) Introducing the **Harish-Chandra transform** of f

$$HC(f)(u) = \int_{\mathbb{R}} f(\begin{pmatrix} e^{u/2} & x \\ 0 & e^{-u/2} \end{pmatrix})dx$$

$$= e^{u/2} \int_{\mathbb{R}} f(\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})dx$$

and the Fourier transform $\hat{g}(u) = \int_{\mathbb{R}} g(x)e^{iux}dx$, we can rewrite

$$\chi_{\pi_s}(f) = \widehat{HC(f)}(\frac{s}{2i}).$$

The spherical unitary dual and Hecke algebra

(I) W can describe very nicely Sph thanks to:

Theorem (Harish-Chandra) The map $f \to HC(f)$ is an isomorphism (of vector spaces)

$$\mathrm{Sph} \simeq C_c^\infty(\mathbb{R})^{\mathrm{even}} := \{f \in C_c^\infty(\mathbb{R}) | \, f(x) = f(-x).\}$$

Moreover we have the "Fourier inversion" formula

$$f(1) = \frac{1}{2\pi} \int_0^\infty r\widehat{HC(f)}(r) \tanh(\pi r) dr.$$

The spherical unitary dual and Hecke algebra

(I) Any $f \in \mathrm{Sph}$ is determined by its restriction to A, since $G = KAK$. Moreover if $f(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) = u(a)$, then $u(a) = u(a^{-1})$ since f is bi-K-invariant and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}.$$

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(II) Now a funny real analysis exercise shows that there is $F \in C_c^\infty([1,\infty))$ such that

$$f(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) = F(\frac{a^2 + a^{-2}}{2}).$$

The spherical unitary dual and Hecke algebra

(I) It follows from here that

$$f(g) = F(\frac{\mathrm{tr}(gg^T)}{2})$$

for all $g \in G$: both terms are in Sph and they have the same restriction to A.

The spherical unitary dual and Hecke algebra

(I) It follows from here that

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for all $g \in G$: both terms are in Sph and they have the same restriction to A.

(II) We deduce that $F \to f_F = (g \to F(\frac{\mathrm{tr}(gg^T)}{2}))$ gives an isomorphism of vector spaces

$$C_c^\infty([1,\infty)) \simeq \mathrm{Sph}$$

and

$$HC(f_F)(u) = \int_{\mathbb{R}} F(\cosh(u) + \frac{x^2}{2})dx.$$

It is therefore clear that $HC(f) \in C_c^\infty(\mathbb{R})^{\mathrm{even}}$.

The spherical unitary dual and Hecke algebra

(I) In order to prove the first part, it suffices to prove that the **Abel transform**

$$A(F)(a) = \int_{\mathbb{R}} F(a + \frac{x^2}{2})dx$$

gives an isomorphism

$$C_c^\infty([1, \infty)) \simeq C_c^\infty([1, \infty)).$$

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gives an isomorphism

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(II) For this it suffices (exercise: why?) to check that

$$F(a) = -\frac{1}{2\pi} \int_{\mathbb{R}} A(F)'(a + \frac{x^2}{2})dx.$$

Indeed, we have (polar coordinates!)

$$\int_{\mathbb{R}} A(F)'(a + \frac{x^2}{2})dx = \int_{\mathbb{R}} \int_{\mathbb{R}} F'(a + \frac{x^2 + y^2}{2})dxdy =$$

$$= 2\pi \int_0^\infty F'(a + \frac{r^2}{2})rdr = 2\pi \int_a^\infty F'(x)dx = -2\pi F(a).$$

The spherical unitary dual and Hecke algebra

(I) We conclude the proof using that even C_c^∞ functions on $\mathbb{R}$ are related to C_c^∞ functions on $[1, \infty)$ by $g(x) = F(\cosh(x))$ (exercise). This finishes the first part.

The spherical unitary dual and Hecke algebra

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(II) For the Fourier inversion formula let $g = HC(f)$ and $f(g) = F(\frac{\mathrm{tr}(gg^T)}{2})$, so that

$$g(u) = \int_{\mathbb{R}} F(\cosh u + \frac{x^2}{2})dx = A(F)(\cosh u).$$

It follows that (make $x = e^{t/2} - e^{-t/2}$)

$$f(1) = F(1) = -\frac{1}{2\pi}\int_{\mathbb{R}} A(F)'(1+\frac{x^2}{2})dx =$$

$$= -\frac{1}{2\pi}\int_{\mathbb{R}} A(F)'(\cosh t)\cosh(t/2)dt =$$

$$-\frac{1}{2\pi}\int_{\mathbb{R}} g'(t)\frac{\cosh(t/2)}{\sinh t}dt = -\frac{1}{2\pi}\int_{\mathbb{R}}\frac{g'(t)}{e^{t/2}-e^{-t/2}}dt.$$

The spherical unitary dual and Hecke algebra

(I) Since g is even, Fourier inversion gives

$$g(x) = \frac{1}{\pi}\int_0^\infty \hat{g}(u)e^{-iut}du,\ g'(x) = -\frac{i}{\pi}\int_0^\infty u\hat{g}(u)e^{-iut}du$$

and

$$f(1) = \frac{i}{2\pi^2}\int_0^\infty u\hat{g}(u)\int_{\mathbb{R}} \frac{e^{-iut}}{e^{t/2} - e^{-t/2}}dt.$$

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(II) Thus we are done if we prove that

$$\int_{\mathbb{R}} \frac{e^{-iut}}{e^{t/2}-e^{-t/2}}dt = -i\pi\tanh(\pi u),\; u > 0.$$

The spherical unitary dual and Hecke algebra

(I) This can be proved using the residue formula, but we can also argue via Poisson summation:

$$\int_{\mathbb{R}} \frac{e^{-iut}}{e^{t/2} - e^{-t/2}} dt = -2i \int_0^\infty \frac{\sin(ut)}{e^{t/2}(1 - e^{-t})} dt =$$

$$-2i \sum_{n \geq 0} \int_0^\infty \mathrm{Im}(e^{iut}) e^{-(n+1/2)t} dt$$

$$= -2i \sum_{n \geq 0} \frac{u}{u^2 + (n+1/2)^2} = -i \sum_{n \in \mathbb{Z}} \frac{u}{u^2 + (n+1/2)^2}.$$

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(II) Now observe that $\hat{f}_a(x) = \frac{2a}{a^2+x^2}$ where $f_a(x) = e^{-a|x|}$ and apply Poisson summation to obtain

$$\sum_{n \in \mathbb{Z}} \frac{u}{u^2 + (n+1/2)^2} = \pi \sum_{n \in \mathbb{Z}} e^{-2\pi u|n|} e^{i\pi n} = \pi \tanh(\pi u).$$

Trace formula for compact quotients

(I) Let G be a unimodular real Lie group and let Γ be a discrete **co-compact** subgroup of G. Fix a Haar measure dg on G. We have already seen that we can decompose

$$L^2(\Gamma\backslash G) \simeq \widehat{\bigoplus_{\pi\in\hat{G}}} \pi^{\oplus m(\pi)}$$

with $m(\pi) \in \mathbb{Z}_{\geq 0}$.

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with $m(\pi) \in \mathbb{Z}_{\geq 0}$.

(II) Moreover, we saw that each $f \in C_c^\infty(G)$ defines an operator $T_f = f * \varphi$ on $L^2(\Gamma \backslash G)$, which is Hilbert-Schmidt and even (thanks to the Dixmier-Malliavin theorem) of trace class. Our goal will be to compute this trace in two different ways: in representation-theoretic terms using the previous decomposition, and "geometrically", using orbital integrals on G.

Trace formula for compact quotients

(I) The representation-theoretic computation is "trivial": each $\pi \in \hat{G}$ for which $m(\pi) > 0$ is a sub-representation of $L^2(\Gamma\backslash G)$ and T_f preserves π, thus the restriction of T_f to π is of trace class. Moreover, picking an ON-basis in each π we immediately obtain

$$\mathrm{tr}(T_f) = \sum_{\pi \in \hat{G}} m(\pi) \mathrm{tr}\pi(f),$$

where we write $\pi(f) = T_f|\pi$ for the restriction of T_f to π.

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where we write $\pi(f) = T_f|\pi$ for the restriction of T_f to π.

(II) We study now the "geometric" part. Recall that

$$T_f(\varphi)(x) = \int_{\Gamma\backslash G} K_f(x,y)\varphi(y)dy$$

where

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \in C^\infty(\Gamma\backslash G \times \Gamma\backslash G).$$

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(I) First, let us prove the

Theorem We have

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(II) By Dixmier-Malliavin, WLOG $f = f_1 * f_2$ with $f_1, f_2 \in C_c^\infty(G)$. Then $T_f = T_{f_1}T_{f_2}$ and if e_i is an ON-basis of $L^2(\Gamma\backslash G)$ then letting $f_1^*(g) = \overline{f_1(g^{-1})}$ (so $T_{f_1}^* = T_{f_1^*}$)

$$\mathrm{tr}(T_f) = \sum_i \langle T_{f_1}T_{f_2}e_i, e_i\rangle = \sum_i \langle T_{f_2}e_i, T_{f_1}^* e_i\rangle$$

$$= \sum_{i,j} \langle T_{f_2}e_i, e_j\rangle \overline{\langle T_{f_1^*}e_i, e_j\rangle}.$$

Trace formula for compact quotients

(I) On the other hand a direct calculation shows that

$$\langle T_{f_2} e_i, e_j \rangle = \int_{\Gamma \backslash G \times \Gamma \backslash G} K_{f_2}(x,y) e_i(y) \overline{e_j(x)} dxdy = \langle K_{f_2}, e_j \otimes \overline{e_i} \rangle,$$

the latter product being in $L^2(\Gamma \backslash G \times \Gamma \backslash G)$. Similarly for $\langle T_{f_1^*} e_i, e_j \rangle$.

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the latter product being in $L^2(\Gamma\backslash G \times \Gamma\backslash G)$. Similarly for $\langle T_{f_1^*} e_i, e_j \rangle$.

(II) Since the $e_j \otimes \overline{e_i}$ form an ON-basis of $L^2(\Gamma\backslash G \times \Gamma\backslash G)$, we conclude that

$$\mathrm{tr}(T_f) = \langle K_{f_2}, K_{f_1^*} \rangle = \int_{\Gamma\backslash G\times\Gamma\backslash G} K_{f_2}(x,y)\overline{K_{f_1^*}(x,y)}dxdy.$$

Since $K_{f_1^*}(x,y) = \overline{K_{f_1}(y,x)}$, we finally obtain

$$\mathrm{tr}(T_f) = \int_{\Gamma\backslash G\times\Gamma\backslash G} K_{f_2}(x,y) K_{f_1}(y,x)dxdy.$$

Trace formula for compact quotients

(I) Now writing the equality $T_f = T_{f_1} T_{f_2}$ in terms of K_{f_1}, K_{f_2}, K_f immediately yields (equality of continuous functions...)

$$K_f(x,y) = \int_{\Gamma \backslash G} K_{f_1}(x,z) K_{f_2}(z,y) dz,$$

thus we conclude that

$$\mathrm{tr}(T_f) = \int_{\Gamma \backslash G} K_f(x,x) dx.$$

Trace formula for compact quotients

(I) We want to split

$$\int_G K_f(x,x)dx = \int_G (\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x))dx$$

according to conjugacy classes in Γ.

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according to conjugacy classes in Γ.

(II) To justify the various operations we will do, it is convenient to isolate certain topological properties that are fairly simple to prove and left to the reader. Let Γ_γ, resp. G_γ be the centralizer of γ in Γ, resp. G. Thus $\Gamma_\gamma = G_\gamma \cap \Gamma$ and so we have a natural bijection

$$\Gamma\backslash\Gamma G_\gamma \simeq \Gamma_\gamma\backslash G_\gamma.$$

One easily checks that ΓG_γ is closed in G, its image $\Gamma\backslash\Gamma G_\gamma$ in $\Gamma\backslash G$ is closed, thus compact, and the previous bijection is a homeomorphism. In particular Γ_γ is a co-compact lattice in G_γ and this implies that G_γ is unimodular.

Trace formula for compact quotients

(I) Next, let $\{\Gamma\}$ be a set of representatives for the Γ-conjugacy classes of elements of G. If $\gamma \in \Gamma$ let $cc_G(\gamma) = \{x\gamma x^{-1} | \, x \in G\}$ be the conjugacy class of γ in G.

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(II) One easily checks that

$$\coprod_{\gamma \in \{\Gamma\}} (\Gamma_\gamma \backslash G \times \{\gamma\}) \to G, \, (\Gamma_\gamma x, \gamma) \to x\gamma x^{-1}$$

is a proper map (i.e. the inverse image of a compact set is compact), thus a closed map, and from here one deduces that $cc_G(\gamma)$ is closed in G and for any compact set $K \subset G$ there are only finitely many $\gamma \in \{\Gamma\}$ such that $cc_G(\gamma) \cap K \neq \emptyset$.

Trace formula for compact quotients

(I) This being said, we can safely write (recall that $f \in C_c^\infty(G)$)

$$\int_{\Gamma\backslash G}(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x))dx = \int_{\Gamma\backslash G}\sum_{\gamma\in\{\Gamma\}}\sum_{g\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}g^{-1}\gamma gx)dx =$$

$$= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx.$$

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$$= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx.$$

(II) On the other hand,

$$\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx = \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f((gh)^{-1}\gamma g h)dgdh$$

$$= \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx.$$

Trace formula for compact quotients

(I) In the above formula one starts by choosing a Haar measure on G_γ, then takes the quotient measure on $G_\gamma \backslash G$ and on $\Gamma_\gamma \backslash G_\gamma$ (we put the counting measure on Γ and its subgroups). Combining the two expressions for $\mathrm{tr}(T_f)$ yields:

Theorem (Selberg's trace formula for compact quotients)
If Γ is a co-compact lattice in a real Lie group G, then for all $f \in C_c^\infty(G)$

$$\sum_{\pi \in \hat{G}} m(\pi, \Gamma)\mathrm{tr}(\pi(f)) = \sum_{\gamma \in \{\Gamma\}} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f),$$

where

$$m(\pi, \Gamma) = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma \backslash G))$$

and

$$O_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx.$$

Trace formula for compact quotients

(I) Let's suppose that G is abelian. By Schur's lemma, $\hat{G}$ consists of all unitary (continuous of course) characters $\chi : G \to \mathbb{S}^1$ of G. One checks very easily that $m(\chi, \Gamma) = 1$ if $\chi(\Gamma) = \{1\}$ and 0 otherwise. Let's compute $\mathrm{tr}(\chi(f))$.

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(II) If v is a nonzero vector in the space of χ, we have

$$\chi(f).v = \int_G f(g)g.vdg = \int_G f(g)\chi(g)vdg = \hat{f}(\chi^{-1})v,$$

thus $\mathrm{tr}(\pi(\chi)) = \hat{f}(\chi^{-1})$, with

$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}dg.$$

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$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}dg.$$

(III) On the other hand $O_\gamma(f) = f(\gamma)$ and so the trace formula yields a general abelian Poisson summation formula

$$\sum_{\chi \in \hat{G}, \chi(\Gamma) = \{1\}} \hat{f}(\chi) = \mathrm{vol}(\Gamma \backslash G) \sum_{\gamma \in \Gamma} f(\gamma).$$