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Gauss–Bonnet Theorem for 2-Dimensional Foliations

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We show that the average Gaussian curvature of a 2-dimensional foliation without spherical leaves is nonpositive. The average is taken according to a harmonic measure.

1. INTRODUCTION

Let $\mathscr{F}$ be an oriented 2-dimensional foliation on a compact manifold M. Suppose we are given a Riemannian metric on the tangent bundle of $\mathscr{F}$ and let $k(x)$ be the Gaussian curvature at x of the leaf through the point x. If we assume that $\mathscr{F}$ admits a transverse invariant measure (see [5]), we can combine this transverse measure with the area form along the leaves to produce a global measure on M, denoted by μ. As an application of his foliated index theorem, Connes proved the following "Gauss–Bonnet type" theorem.

THEOREM [2]. *If the set of spherical leaves if μ-negligible, then the mean value of the curvature, i.e., $\int k\, d\mu$, is nonpositive.*

As Connes writes, this is fairly intuitive: "if there is enough positive curvature in the generic leaf, this leaf is forced to be closed and hence a sphere." The main disadvantage of this theorem is, however, that it requires the very strong assumption that the foliation $\mathscr{F}$ admits a transverse invariant measure. On the other hand, the heuristic argument above is not related to this assumption and one should be able to prove a similar theorem for any foliation. This is the purpose of this paper.

We shall use the notion of "harmonic measure" introduced by L. Garnett (see [3]). Given a metric on the tangent bundle of a foliation $\mathscr{F}$ on a compact manifold M, one can define a nonempty set of measures on M,

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called $\mathscr{F}$-harmonic measures, that describe the "statistical" behaviour of $\mathscr{F}$ (see Section 2 for a review of the definition and the main properties of these measures). Besides the fact that every foliation admits a nontrivial harmonic measure, this notion has the advantage of being a generalization of the notion of transverse invariant measure. Indeed, the combination of the volume along the leaves and a transverse invariant measure is always a harmonic measure.

We can now state our result.

MAIN THEOREM. *Let $\mathscr{F}$ be an oriented 2-dimensional foliation on a compact manifold M. Choose a Riemannian metric on the tangent bundle of $\mathscr{F}$ and denote by $k(x)$ the Gaussian curvature at x of the leaf $L(x)$ through the point x. Let μ be any $\mathscr{F}$-harmonic measure. If the set of spherical leaves is μ-negligible, then the integral $\int k\, d\mu$ of the Gaussian curvature is nonpositive.*

This result gives a positive answer to a question of Sullivan (see [3]). It turns out that our proof of the main theorem only uses Connes' theorem in the very special case where μ-almost all leaves are conformally flat. So, our proof can also be considered as a proof of "most cases" of Connes' theorem.

2. HARMONIC MEASURES

In this section, we recall the main results of [3]. Let $\mathscr{F}$ be any foliation on a compact manifold M equipped with a Riemannian metric on its tangent bundle. The smoothness assumption is very weak (see [3, p. 288]). For simplicity, we shall assume that both $\mathscr{F}$ and the Riemannian metric are of class C^3.

We can use the Laplace operators of the leaves to construct a global operator ${}^{\mathscr{F}}\Delta$ defined on functions $f: M \to \mathbb{R}$ that are C^2 along the leaves:

$$^{\mathscr{F}}\Delta f(x) = \Delta_{L(x)} f_{|L(x)}(x)$$

where $L(x)$ is the leaf through x and $\Delta_{L(x)}$ is the Laplace operator of the Riemannian manifold $L(x)$.

DEFINITION 2.1. A measure μ on M is called $\mathscr{F}$-harmonic if, for every continuous function $f: M \to \mathbb{R}$ which is C^2 along the leaves, the integral $\int {}^{\mathscr{F}}\Delta f\, d\mu$ is zero.

THEOREM 2.2 [3]. (1) *A compact foliated manifold always admits a nontrivial harmonic measure.*

(2) *A measure μ is harmonic if and only if, in any distinguished open set, μ can be disintegrated as a transversal sum of leaf measures, where every leaf measure is a positive harmonic function times the Riemannian volume of the leaf.*

Note that this positive harmonic function changes by a positive multiplicative constant when one changes the distinguished open set. Note as well that these harmonic functions are constants if and only if the measure μ is the combination of a transverse invariant measure and the volume along the leaves. This leads to the following results.

COROLLARY 2.3 [3]. *The measure on M obtained by combination of a transverse invariant measure and the volume along the leaves is always $\mathcal{F}$-harmonic. These special $\mathcal{F}$-harmonic measures are called "completely invariant measures."*

COROLLARY 2.4 [4]. *Suppose μ is an $\mathcal{F}$-harmonic measure such that for μ-almost every point x, the universal covering space $\tilde{L}(x)$ of $L(x)$ has no non-constant positive harmonic functions. Then μ is completely invariant.*

3. Proof of the Main Theorem

In this section, we prove the main theorem assuming a technical proposition that will be proven in Section 4.

From now on, we fix an oriented 2-dimensional foliation $\mathcal{F}$ on the compact manifold M. We choose a Riemannian metric g on the tangent bundle of $\mathcal{F}$ and we denote by $k(x)$ the Gaussian curvature of the leaf $L(x)$ through the point x. Finally, we choose an $\mathcal{F}$-harmonic measure μ on M.

The general idea is to change the metric conformally along $\mathcal{F}$ to produce a new metric of constant negative curvature for which the theorem is obvious. Recall that, if g is a Riemannian metric on an orientable surface L, then three cases are possible:

(i) L is a sphere S^2;

(ii) the universal covering space $\tilde{L}$ of L is conformally equivalent to the Euclidean plane E^2;

(iii) the universal covering space $\tilde{L}$ of L is conformally equivalent to the Poincaré disc D^2.

In case (iii), the conformal equivalence between $\tilde{L}$ and D^2 is unique up to isometries of D^2. In particular, the pull-back of the Poincaré metric of D^2 is a well-defined metric on $\tilde{L}$ which is obviously invariant by the deck transformations of the covering $\tilde{L} \to L$. In other words, there is a unique smooth function $\phi: L \to \mathbb{R}$ such that the metric $\exp(2\phi)\, g$ is complete and has curvature -1.

The following proposition will be proven in Section 4.

PROPOSITION 3.1. *Let $S \subset M$ be the closed $\mathcal{F}$-saturated set consisting of non-spherical leaves (this set is closed by the Reeb stability theorem). Let $\phi: S \to \mathbb{R} \cup \{-\infty\}$ be the map defined in the following manner:*

(1) *If $\tilde{L}(x)$ is conformally equivalent to E^2, we set $\phi(x) = -\infty$.*

(2) *If $\tilde{L}(x)$ is conformally equivalent to D^2, then $\phi_{|L(x)}$ is the unique function such that $\exp(2\phi_{|L(x)})\, g_{|L(x)}$ is complete and has curvature -1.*

Then ϕ is upper semi-continuous and smooth along the leaves of $\mathcal{F}$. Moreover, the gradient along the leaves ${}^{\mathcal{F}}\nabla\phi$ is bounded on S.

The proof of the main theorem will be decomposed into three steps.

Step 1. We can assume that either, for μ-almost every x, $\tilde{L}(x)$ is conformally equivalent to E^2 or, for μ-almost every x, $\tilde{L}(x)$ is conformally equivalent to D^2.

Indeed, let us consider the partition of M into three $\mathcal{F}$-saturated Borel sets:

$$M = M_1 \cup M_2 \cup M_3$$

where M_1 (resp. M_2) (resp. M_3) denotes the set of points x such that $\tilde{L}(x)$ is conformally equivalent to S^2 (resp. E^2) (resp. D^2). The assumption of the theorem is that $\mu(M_1) = 0$. So, we can write μ as a sum of two measures, μ_2 and μ_3, concentrated on M_2 and M_3. These new measures are obviously $\mathcal{F}$-harmonic.

Of course, if one proves that $\int k\, d\mu_2$ and $\int k\, d\mu_3$ are nonpositive, one gets the theorem by linearity.

Step 2. The main theorem is true if, for μ-almost every x, $\tilde{L}(x)$ is conformally equivalent to E^2.

This is just because, in this special case, the harmonic measure μ is completely invariant according to Corollary 2.4. So, the main theorem, in that case, follows from Connes' theorem.

Step 3. The main theorem is true if, for μ-almost every x, $\tilde{L}(x)$ is conformally equivalent to D^2.

If g is a metric on a surface L with curvature $k(x)$ and if $\phi: L \to \mathbb{R}$ is any smooth function, then the curvature $k'(x)$ of the metric $\exp(2\phi)\, g$ is given by the well-known formula:

$$k' = \exp(-2\phi)(k - \Delta\phi)$$

where Δ is the Laplace operator for the metric g.

If we apply this formula leaf by leaf for the function ϕ defined in Proposition 3.1, we get

$$-1 = \exp(-2\phi(x))(k(x) - {}^{\mathscr{F}}\!\Delta\phi(x)).$$

Therefore,

$$k(x) = {}^{\mathscr{F}}\!\Delta\phi(x) - \exp(2\phi(x)).$$

This formula has a meaning μ-almost everywhere because we assume that $\phi(x) \neq -\infty$ almost everywhere. Note that ϕ being upper semi-continuous, it is bounded from above on S so that $\exp(2\phi(x))$ is a positive bounded function on S. Consequently, ${}^{\mathscr{F}}\!\Delta\phi$ is also bounded because k is a continuous function. Therefore, $\exp(2\phi)$ and ${}^{\mathscr{F}}\!\Delta\phi$ are μ-integrable and we get

$$\int k\, d\mu = \int {}^{\mathscr{F}}\!\Delta\phi\, d\mu - \int \exp(2\phi)\, d\mu \leqslant \int {}^{\mathscr{F}}\!\Delta\phi\, d\mu.$$

If ϕ were a continuous function, smooth along the leaves, the integral $\int {}^{\mathscr{F}}\!\Delta\phi\, d\mu$ would be zero by definition of a harmonic measure. To complete the proof of the main theorem, it remains to show the following proposition:

PROPOSITION 3.2. *Let $\phi\colon M \to \mathbb{R}$ be a measurable function, C^2-along the leaves and such that $|{}^{\mathscr{F}}\!\Delta\phi|$ and ${}^{\mathscr{F}}\nabla\phi$ are bounded. Then the integral $\int {}^{\mathscr{F}}\!\Delta\phi\, d\mu$ is zero.*

Proof. Assume first of all that $\phi\colon M \to \mathbb{R}$ is a measurable function, smooth along the leaves and satisfying the following:

(i) $|{}^{\mathscr{F}}\!\Delta\phi|$ is bounded.

(ii) The support of ϕ is contained in a distinguished open set. Then ${}^{\mathscr{F}}\!\Delta\phi$ is μ-integrable by (i) and one can use the local description of harmonic measures (Theorem 2.2, (2)) to show that $\int {}^{\mathscr{F}}\!\Delta\phi\, d\mu = 0$.

Now, suppose that ϕ satisfies only condition (iii) below:

(iii) $|\phi|$, $|{}^{\mathscr{F}}\!\Delta\phi|$ and ${}^{\mathscr{F}}\nabla\phi$ are bounded. Choose a smooth partition of unity $(g_1, ..., g_n)$ subordinated to a covering of M by distinguished open sets. In order to show that $\int {}^{\mathscr{F}}\!\Delta\phi\, d\mu = 0$, it is enough to show that ${}^{\mathscr{F}}\!\Delta(g_i\phi)$

is bounded. Indeed, by the previous case, we shall have $\int {}^{\mathscr{F}}\Delta(g_i\phi)\,d\mu = 0$ and therefore $\int {}^{\mathscr{F}}\Delta\phi\,d\mu = 0$. Now, the boundedness of ${}^{\mathscr{F}}\Delta(g_i\phi)$ follows from the formula

$${}^{\mathscr{F}}\Delta(g_i\phi) = ({}^{\mathscr{F}}\Delta g_i)\phi + ({}^{\mathscr{F}}\Delta\phi)\, g_i + 2\langle {}^{\mathscr{F}}\nabla\phi, {}^{\mathscr{F}}\nabla g_i\rangle$$

and the boundedness of ${}^{\mathscr{F}}\Delta g_i$, ${}^{\mathscr{F}}\Delta\phi$, ϕ, g_i, ${}^{\mathscr{F}}\nabla\phi$ and ${}^{\mathscr{F}}\nabla g_i$.

Finally, we assume, as in the hypothesis of the proposition that ϕ satisfies:

(iv) $|{}^{\mathscr{F}}\Delta\phi|$ and ${}^{\mathscr{F}}\nabla\phi$ are bounded.

Let $f_n: \mathbb{R} \to \mathbb{R}$ be a sequence of C^∞-functions satisfying

(a) $f_n(x) = x$ if $|x| < |n|$.
(b) $f_n(x) = -n-1$ if $x \leqslant -n-1$
$f_n(x) = n+1$ if $x \geqslant n+1$.
(c) The first and second derivatives of f_n are uniformally bounded.

Now, we can estimate $\int {}^{\mathscr{F}}\Delta\phi\,d\mu$. Put $\phi_n = f_n \circ \phi$.

$$\begin{aligned}\int {}^{\mathscr{F}}\Delta\phi\,d\mu &= \int_{|\phi|<n} {}^{\mathscr{F}}\Delta\phi\,d\mu + \int_{|\phi|\geqslant n} {}^{\mathscr{F}}\Delta\phi\,d\mu \\ &= \int_M {}^{\mathscr{F}}\Delta\phi_n\,d\mu - \int_{|\phi|\geqslant n} {}^{\mathscr{F}}\Delta\phi_n\,d\mu + \int_{|\phi|\geqslant n} {}^{\mathscr{F}}\Delta\phi\,d\mu \\ &= A_n - B_n + C_n.\end{aligned}$$

We claim that ϕ_n satisfies (iii) and therefore $A_n = 0$. Indeed, $|\phi_n|$ is obviously bounded and the formulas

$${}^{\mathscr{F}}\Delta\phi_n = {}^{\mathscr{F}}\Delta(f_n \circ \phi) = (f_n'' \circ \phi)\,\|{}^{\mathscr{F}}\nabla\phi\|^2 + (f_n' \circ \phi)\,{}^{\mathscr{F}}\Delta\phi$$
$${}^{\mathscr{F}}\nabla\phi_n = {}^{\mathscr{F}}\nabla(f_n \circ \phi) = (f_n' \circ \phi)\,{}^{\mathscr{F}}\nabla\phi$$

show that $|{}^{\mathscr{F}}\Delta\phi_n|$ and ${}^{\mathscr{F}}\nabla\phi_n$ are bounded (actually uniformally in n).

Now, recall that ${}^{\mathscr{F}}\Delta\phi$ is bounded and that μ almost everywhere, ϕ is finite. Therefore C_n goes to zero as n goes to $+\infty$.

It remains to show that B_n tends to 0 as n goes to $+\infty$. This follows from the uniform boundedness of ${}^{\mathscr{F}}\Delta\phi_n$ (in n) shown by the previous formula and the fact that ϕ is finite μ-almost everywhere. ∎

4. Proof of Proposition 3.1

We first show that the function ϕ is upper semi-continuous. This will be proven in the following way; we shall construct, for every x in S, a family $\mathscr{C}_x$ of continuous functions f defined in neighbourhoods U_f of x and satisfying

(1) for all $f \in \mathscr{C}_x$ and $y \in U_f \cap S$, one has $f(y) \geqslant \phi(y)$;

(2) $\phi(x) = \inf_{f \in \mathscr{C}_x} f(x)$.

These properties obviously imply the upper semi-continuity of ϕ.

Let $\bar{D}^q$ denote the closed unit disc in $\mathbb{R}^q$ (q being the codimension of $\mathscr{F}$) and fix a point x in S. Consider the family of mappings $\theta: \bar{D}^2 \times \bar{D}^q \to M$ with the following properties

$$\begin{cases} \theta \text{ is a local diffeomorphism onto a neighbourhood of } x \\ \theta \text{ maps the plaques } \bar{D}^2 \times \{x\} \text{ into the leaves of } \mathscr{F}. \end{cases}$$

Because θ is a local diffeomorphism, one can pull back the metric g by θ and get a family of metrics $(\theta^* g)_z$ on $\bar{D}^2 = \bar{D}^2 \times \{z\}$ parametrized by the "transverse" parameter $z \in \bar{D}^q$. Each of these metrics, restricted to the open disc D^2, is conformally equivalent to the Poincaré metric on D^2. Therefore, there exists a function $h: D^2 \times D^q \to \mathbb{R}$ such that $\exp(2h)(\theta^* g)_z$ is isometric to the Poincaré metric. By our smoothness assumption, the family $(\theta^* g)_z$ is continuous and, consequently, h is also continuous (cf. [1]).

Let V be an open neighbourhood of $(0, 0) \in D^2 \times \bar{D}^q$ such that θ is a diffeomorphism of V onto $U = \theta(V)$. Define $f: U \to \mathbb{R}$ by $f(x) = h(\theta_{|U}^{-1}(y))$. We now define the family $\mathscr{C}_x$ as being the family of those functions f constructed in this way, corresponding to the different choices of θ and V.

We now verify (1) and (2).

Lemma 4.1. *Property* (1) *holds.*

Proof. Let us recall that Schwarz's lemma can be expressed in the following way; every conformal mapping from the Poincaré disc to itself has a dilatation bounded by one (by dilatation of a conformal map we mean the norm of its derivative). Now, the disc D^2 equipped with the metric $(\exp 2h)(\theta^* g)_z$ is isometric to the Poincaré disc and is conformally immersed in a leaf L of $\mathscr{F}$. Assume L is such that $\tilde{L}$ is conformally equivalent to D^2. Then, L equipped with the metric $\exp(2\phi)\, g_L$ is complete and has curvature -1 so that $\tilde{L}$ equipped with the lifted metric is isometric to the Poincaré disc.

If we lift $\theta_{|D^2 \times \{z\}}: D^2 \to L$ to $\tilde{L}$, we finally get a conformal immersion on a copy of the Poincaré disc into another copy of the Poincaré disc. The dilatation of this map is $\exp(\phi \circ \theta) \exp(-h)$ so that Schwarz's lemma implies that

$$h \geqslant \phi \circ \theta.$$

Therefore, by definition of f, one gets

$$f(y) \geqslant \phi(y).$$

This inequality has been established if $\tilde{L}$ is equivalent to D^2 that is if $\phi(y) \neq -\infty$. Otherwise, this inequality is obviously satisfied. ∎

LEMMA 4.2. *Property* (2) *holds.*

Proof. Assume first of all that $\phi(x) \neq -\infty$.

Consider a conformal diffeomorphism α between the Poincaré disc and the universal covering space $\tilde{L}(x)$, mapping 0 to a point $\tilde{x}$ bying above x. The restriction of α to the closed disc $\bar{D}_r$ of center 0 and (euclidean) radius r $(r < 1)$ is a conformal embedding of $\bar{D}_r$ into $\tilde{L}(x)$ that projects down to a conformal mapping α_r from $\bar{D}_r$ to $L(x)$ sending 0 to x. As it is well known, the simple connectivity of the compact disc $\bar{D}_r$ enables us to extend α_r to a map $\theta_r\colon \bar{D}_r \times D^q \to M$ satisfying (1) and (2) and such that $\theta_{r|\bar{D}^2_r \times \{0\}} = \alpha_r$. Now, we claim that the functions f_r associated to θ_r satisfy $\lim_{r \to 1} f_r(x) = \phi(x)$. This corresponds to the fact that the map $z \in D_r \to (1/r)\, z \in D^2$ is conformal and that its dilatation at 0 goes to 1 as r goes to 1.

When $\phi(x) = -\infty$, the proof is essentially the same. It suffices to replace D^2 by E^2 and to consider closed discs $\bar{D}_r$ whose radii go to $+\infty$. ∎

The smoothness of ϕ along the leaves is a consequence of the usual uniformization theorem. Therefore, in order to prove Proposition 3.1, it remains to show that the gradient along the leaves of ϕ is bounded in S. By compactness of S, it is enough to show that ${}^{\mathscr{F}}\nabla\phi$ is locally bounded.

Consider an embedding $i\colon \bar{D}^2 \times \bar{D}^q \to M$ whose image is a neighbourhood of a given point x and which maps the plaques $\bar{D}^2 \times \{z\}$ into the leaves of $\mathscr{F}$. For simplicity of notation, we shall identify a point $y \in \bar{D}^2 \times \bar{D}^q$ and its image $i(y)$. On each disc $D^2 \times \{z\}$ we have three natural metrics

(1) the metric g;

(2) the unique metric of the form $g_1 = \exp(2f)\, g$ which makes the disc $D^2 \times \{z\}$ isometric to the Poincaré disc. Here f is a smooth function on $D^2 \times D^q$;

(3) the metric $g_2 = \exp(2\phi)\, g$.

In order to show that ${}^{\mathscr{F}}\nabla_g \phi$ is bounded in a neighbourhood of x, it is enough to show that ${}^{\mathscr{F}}\nabla_g(\phi - f)$ is also bounded. Moreover, the gradients with respect to g and g_1 are related by

$$\nabla_{g_1} = \exp(-2f)\,\nabla_g.$$

Therefore, it is enough to show that the gradient $\nabla_{g_1}(\phi - f)$ is bounded in a neighbourhood of x. Now, consider the embedding of $(D^2 \times \{z\}, g_1)$ into $(L(x), \exp(2\phi)\, g)$. This embedding is conformal and $\exp(\phi - f)$ is its dilation. Noting that $(D^2 \times \{z\}, g_1)$ is isometric to the Poincaré disc and that $(L(x), \exp(2\phi)\, g)$ is a complete surface of curvature -1, the proposition will be proven by the following lemma.

LEMMA 4.3. *Let* $j\colon (D^2, g_1) \hookrightarrow (L, g_2)$ *be a conformal embedding of the Poincaré disc into a complete surface of curvature* -1. *Let* $\psi\colon D^2 \to \mathbb{R}$ *be the logarithm of the dilation of* j. *Then* $\|\nabla_{g_1}\psi\|$ *is bounded by* 2.

Proof. Lifting j to the universal covering space of L, we can assume that L is the Poincaré disc. Moreover, using isometries of the Poincaré disc, we can furthermore assume that the point where we want to evaluate $\nabla_{g_1}\psi$ is 0 and that $j(0) = 0$. Now, j is a holomorphic mapping from D^2 to D^2 such that $j(0) = 0$ and we want to estimate $\|\nabla_{g_1}\psi(0)\| = \|\nabla \operatorname{Log} |j'|(0)\| = |j''(0)/j'(0)|$. This number is bounded by 2 according to Koebe's theorem; if $j\colon D^2 \hookrightarrow \mathbb{C}$ is an injective holomorphic mapping such that $j(0) = 0$, then $|j''(0)/j'(0)| \leqslant 2$. ∎

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