TRANSLATORS INTRODUCTION

W. E. Grosso

What you are about to read is a rough translation of the "Green Book" on Hyperbolic Group Theory. At some point in my graduate school career, I realised that

(1) I knew almost no French.
(2) I knew almost no TeX.
(3) I had never systematically worked through a foundational article on Hyperbolic Group Theory

After the epiphany ended, I began working on this translation.

I called this a translation a "rough translation" because I learned French by translating this book. As I became more fluent, I began to translate more freely, translating colloquial French into colloquial English. Unfortunately, this means that the text is sometimes stilted (strict translation) and sometimes veers into Long Island Native Prose (a peculiar subdialect of American English).

At this point, it is traditional for me, as translator, to include a sentence taking full responsibility for any and all errors in this document. I refuse to do so. However, as a sop to those who wish for somebody to blame, I offer the following statement:

Any and all errors are strictly your fault

The gist of this is: if you spot an error, you are responsible for it. At which point you can either wallow in guilt, or you can do something constructive like e-mail me with complete details of the error. My e-mail address is william@ math.berkeley.edu. If that e-mail address doesn't work, contact John Stallings (stall@ math.berkeley.edu). He will probably know how to contact me.

Cordially,

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

OVERVIEW (CHAPTER 1)

Etienne Ghys and Pierre de la Harpe

Abstract. The first part of this chapter is a solicitor's plea for the geometric approach to groups of finite type, e.g. groups which are finitely generated. Hyperbolic groups are defined in §4, where we present the results which are central to our exposition. The last section evokes a few important points from Gromov's article which are not presented in these notes.

1. The Objective

We attribute to M. Gromov the assertion that a theorem which is valid for all groups is either trivial or without importance. It is true that the more spectacular results to date in combinatorial group theory have more often than not been obtained by restricting to particular classes of groups. We cite as examples the well developed theories of abelian, nilpotent, polycyclic, solvable and amenable groups. While the interest in these groups is undoubtedly sincere, it is nonetheless clear that these categories are very restricted. Who would dare to pretend that a majority of interesting groups are solvable, for example ?

In addition to these "minorities", which have a simple structure and are well understood, we can list a number of examples of groups whose study has proven to be both necessary and fruitful: free groups, surface groups, braid groups... Although these particular groups are very important, it is necessary, above all else, to reconsider and change our idea of the behavior of a typical infinite, finitely generated group.

Given combinatorial group theory's exemplary resistance to generality, the theory of hyperbolic groups is an extremely satisfying compromise. In this theory, one defines a vast category of finitely presented groups for which one can, nevertheless, prove surprisingly precise results about the structure of the group. Further down, we will give a definition and some important properties of these groups. Very roughly, they behave as if they are the fundamental groups of negatively curved, compact manifolds. Even if they are not all of this type, at least they all have a number of crucial properties.

Indeed, hyperbolic groups were not the first tentative step in this direction. The most elaborate example is, without doubt, given by Small Cancellation Theory. We have dedicated an appendix of these notes to this theory, which has become an important part of hyperbolic group theory. Meanwhile, the theory of small cancellation groups remains important for at least two reasons. The first is that it

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

constitutes one of the principal motivations for hyperbolic group theory and remains a significant source of examples. The second is that, strictly speaking, not all small cancellation groups are hyperbolic; for example, C'(1/5) groups (see definition 2 in the appendix by R. Strebel) are not hyperbolic. However, one quickly realizes that small cancellation groups are quite peculiar. For example, the fundamental group of a negatively curved, compact manifold of dimension 3 or greater is never a small cancellation group because small cancellation groups have rational cohomological dimension at most 2. It is inconvenient that Small Cancellation Theory is not contained within hyperbolic group theory.

While the theory of hyperbolic groups covers a vast area, it does not contain all the groups which we find interesting! One grave defect in the theory is that uniform lattices in real semi-simple Lie groups with real rank at least 2 are never hyperbolic; these groups are undeniably interesting. Perhaps one should consider the study of hyperbolic groups as a stopping place, before a still larger class of groups which M. Gromov has named "semi-hyperbolic" and for which no-one has given a good definition. Typical examples of semi-hyperbolic groups are already clear: in addition to the lattices mentioned above, fundamental groups of non-positively curved compact manifolds should be semi-hyperbolic.

The chapters which follow are devoted to an exposition of part of Gromov's theory. We think that hyperbolic group theory is fundamental in combinatorial group theory because, after all, it gives a picture of the global structure of many finitely presented groups. Undoubtedly, one of the most interesting aspects of the theory is the generality of its methods, on which one has some hope of basing the future theory of semi-hyperbolic groups.

2. The Method

The method is (of course) geometric. The main idea is that a finitely generated group has, in addition to its algebraic structure, a geometric structure and, more precisely, a metric. The study of a group will be done through the underlying metric space.

Let Γ be a finitely generated group and S a finite set of generators. For simplicity, we assume throughout that S doesn't contain the identity element of Γ and that S is symmetric, by which we mean that for every element γ of the group which is in S, γ^{-1} is also in S.

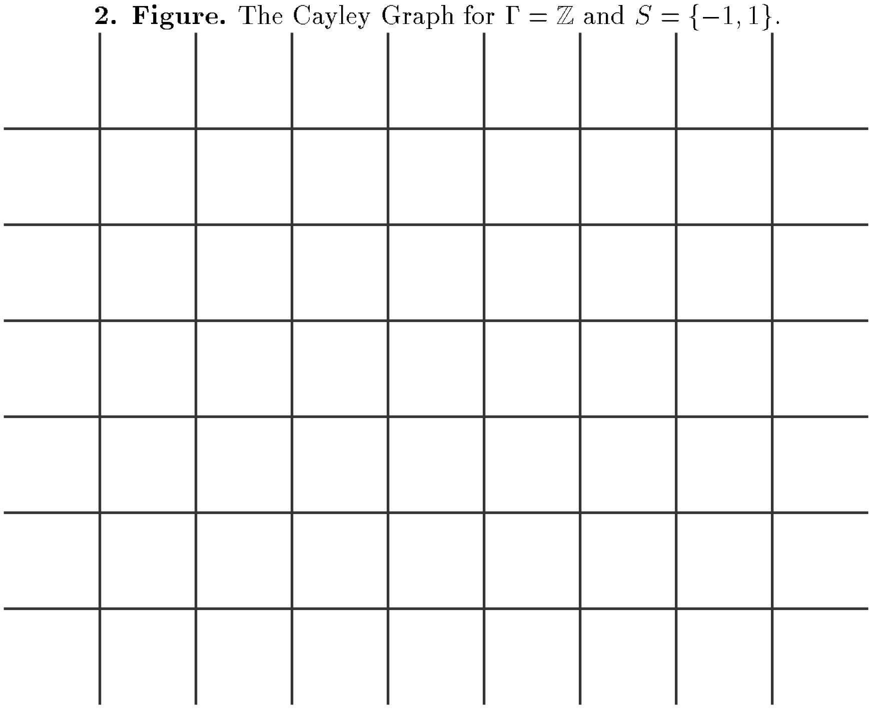
If γ is an element of the group Γ, we denote the minimal number of generators from S necessary to write the element γ by $l_S(\gamma)$, and call this the length of γ relative to S. If γ_1 and γ_2 are elements of Γ, we use $d_S(\gamma_1, \gamma_2)$ to denote the value $l_S({\gamma_1}^{-1}\gamma_2)$ and call this the distance between γ_1 and γ_2 relative to S. The reader can easily verify the following fact:

1. Proposition. *The function d_s is a metric on Γ and it is invariant under left actions of Γ.*

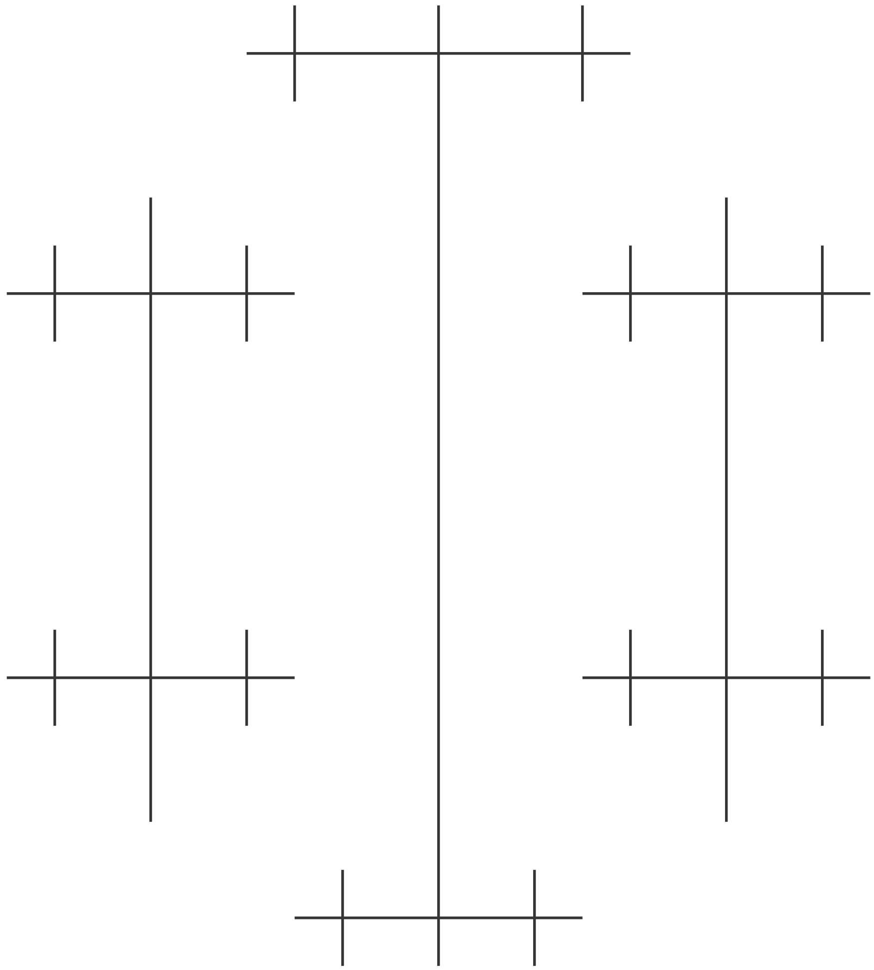
There are two possible criticisms of this distance. First, it depends on the choice of generators; second, it only assumes integer values–the metric space that it defines is consequently discrete.

The second point is not a serious problem– at worst, it might hamper our intuition; moreover there is an easy and graceful solution: the classical construction of a Cayley Graph. It is a non-oriented graph (this means it has a geometric realisation as a 1-dimensional simplicial complex) without loops (consisting of 1 edge) or multiple edges, denoted by $G(\Gamma, S)$. The vertices are the elements of Γ and an edge connects γ_1 and γ_2 if $d_S(\gamma_1, \gamma_2) = 1$, this means that ${\gamma_1}^{-1}\gamma_2) \in S$. It is clear that left actions of Γ upon itself let us define an action of Γ on $G(\Gamma, S)$ which is a simplicial automorphism. One sometimes furnishes the edges with riemannian metrics that make all of them isometric to a fixed interval of length 1 (and such that the riemannian metrics are invariant under the action of Γ). One then defines a metric on $G(\Gamma, S)$ by considering the shortest length along the paths connecting two given points. In this way, $G(\Gamma, S)$ becomes an arc-connected metric space and the natural immersion of (Γ, d_S) is an isometry. Although its introduction is "superfluous", the Cayley Graph nevertheless enables visualisation of Γ. Figures 2-4 above show simple examples; others follow.

2. Figure. The Cayley Graph for $\Gamma = \mathbb{Z}$ and $S = \{-1, 1\}$.



3. Figure. The Cayley Graph for $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ and$S = \{\pm(1,0), \pm(0,1)\}$.



4. Figure. The Cayley Graph for the free group on two generators $a, b$and generating set $S = \{a^{\pm 1}, b^{\pm 1}\}$.

5. Example : The Heisenberg Group. It is the group Γ generated by 3 generators a,b,c subject to the relations

$$ca = ac \qquad cb = bc \qquad bab^{-1}a^{-1} = c$$

This is the simplest example of a non-abelian, nilpotent group. All the elements of have the form $a^m b^n c^p$ with $(m, n, p) \in \mathbb{Z}^3$. The map :

$$\Phi : a^m b^n c^p \to \begin{pmatrix} 1 & n & p \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$$

is a homomorphism of Γ into $SL_3(\mathbb{Z})$. It is a fact that all elements of Γ can be uniquely written in the form $a^m b^n c^p$ and that Φ is an isomorphism onto its image. The center of Γ is the infinite cyclic group generated by c and the quotient of Γ by its center is a free abelian group of rank two generated by the images of a and b. One can represent Γ by the points from the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$. The formulas:

$$(a^m b^n c^p)a = a^{m+1} b^n c^{p+n}$$
$$(a^m b^n c^p)b = a^m b^{n+1} c^p$$
$$(a^m b^n c^p)c = a^m b^n c^{p+1}$$

indicate how we connect the points of $\mathbb{Z}^3$ to obtain the Cayley Graph G(Γ,S). Each cross-section n =constant is a copy of the Cayley Graph of $\mathbb{Z} \oplus \mathbb{Z}$ as represented in figure 3, but the point (m, n, p) is also connected to the neighboring slices at $(m + 1, n, p + n)$ and $(m, n - 1, p - m)$, as indicated in the figure on the following page.

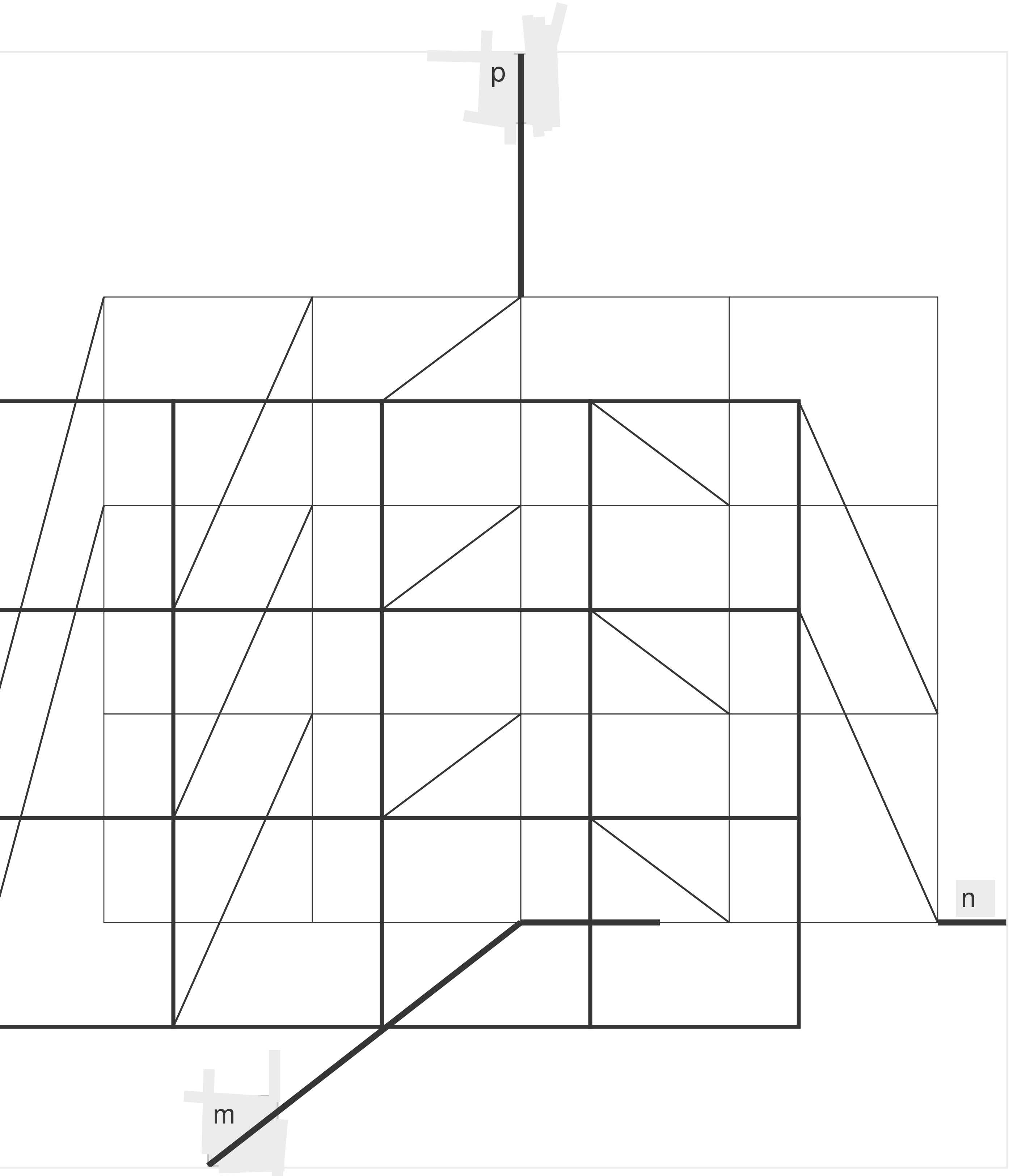
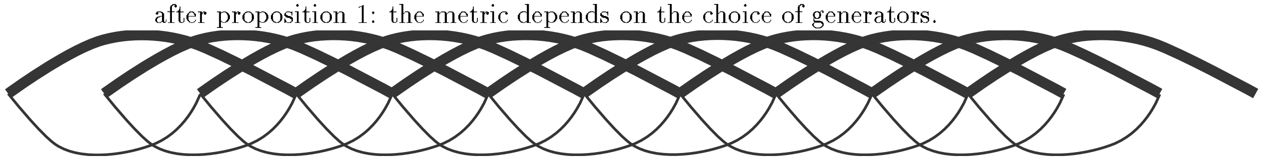

p
n
m

Figure for example 5

Before contemplating other examples, we will answer the second criticism made after proposition 1: the metric depends on the choice of generators.



6. Figure. Choose the system of generators $\{\pm 2, \pm 3\}$ in the group $\mathbb{Z}$. The associated Cayley Graph looks this this.

Of course, this graph is different from the one in figure 2, but there is a strong resemblance when "viewed from afar". To make this notion of resemblance precise, we introduce the following definition:

7. Definition. Let (X, d) and (X', d') be two metric spaces. We say that they are quasi-isometric if there exist maps $f : X \to X'$ and $g : X' \to X$ and also constants $\lambda > 0$, $C > 0$ such that:

$$d'(f(x), f(y)) \leq \lambda d(x, y) + C \forall x, y \in X$$
$$d(g(x'), g(y')) \leq \lambda d'(x', y') + C \forall x', y' \in X'$$
$$d'(g(f(x)), x) \leq C \forall x, y \in X$$
$$d'(f(g(x')), x') \leq C \forall x', y' \in X'$$

The first two inequalities mean that f and g are "lipschitz with ratio λ over large distances". As for the second two, they mean that f and g are "almost inverse" to each other.

8. Exercise. Verify that quasi-isometry is an equivalence relation among metric spaces.

9. Examples. Equip $\mathbb{R}$ with the usual metric and give $\mathbb{Z}$ the metric induced from the immersion into $\mathbb{R}$. This immersion, and the map which sends numbers to their whole part, permit us to show that $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric (although not homeomorphic). More generally, if Γ is a group generated by a finite set S, the group Γ equipped with the usual metric is quasi-isometric to the Cayley Graph $G(\Gamma, S)$ equipped with the metric we have described.

If S and S' are two finite sets of generators for the group Γ, the two associated metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are quasi-isometric. As a matter of fact, if λ_1 is the maximum of $l_{S'}$ on S and λ_2 is the maximum of l_S on S', then it is clear that $d_{S'} \leq \lambda_1 d_S$ and $d_S \leq \lambda_2 d_{S'}$

To summarize, every finitely generated group Γ can be associated with a metric space which is well defined up to quasi- isometry: this space is the group is equipped with d_S– it is unimportant which finite system of generators is used (as long as $e \notin S$ and $S = S^{-1}$). Consequently, all quasi-isometry invariants of (Γ, d_S) determine invariants of the group . This is a characteristic of the approach we follow: If two groups are quasi- isometric then the theory will not be able to distinguish them.

3. Quasi-Isometries and Isomorphisms

We begin by testing out examples of quasi-isometries among groups and we see that it is remarkably well-adapted for our purpose, neither too loose nor too strict. The general idea is that, although quasi-isometry identifies non-isomorphic groups with each other, it nonetheless preserves some of the algebraic structure. The first examples spell out the difference between isomorphism and quasi-isometry. Recall that all the groups considered here are finitely generated.

10. Example. All finite groups are quasi-isometric to each other. This result follows from the fact that two bounded metric spaces are quasi-isometric to each other.

11.Proposition. *If Γ_1 is a finite index subgroup of the group Γ, then Γ_1 and Γ are quasi-isometric.*

One can directly show this proposition by choosing suitable systems of generators inside Γ_1 and Γ. However, we prefer to refer to the proof in chapter 3, where it is a consequence of a more general result concerning group actions (proposition 3.19).

12. Corollary. *Commensurable groups, e.g. those which contain finite index subgroups which are isomorphic, are quasi-isometric.*

13. Examples. A non-abelian free group on k generators ($k \geq 2$) is isomorphic to a finite index subgroup of the free group on two generators. Therefore, two non-abelian free groups with distinct ranks are quasi-isometric, although not isomorphic.

In the same way, we see that the group $PSL_2(\mathbb{Z})$ is isomorphic to the free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ and contains a free subgroup of index 6. Consequently, the group $PSL_2(\mathbb{Z})$ is quasi-isometric to the free group on two generators. For that matter, when "we look carefully" at the drawing below of the Cayley Graph (for the natural generators), $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is quasi-isometric to the drawing in figure 4.

However, quasi-isometry does not always reduce to commensurability. Here is one way to show this.

14. Proposition. *Suppose M is a compact Riemannian manifold. The fundamental group of M is quasi-isometric to the universal cover of M .*

For the proof, see Corollary 21 of Chapter 3.

15. Corollary. *There exist groups which are quasi- isometric and not commensurable.*

Proof. Let M_1 and M_2 be two compact three dimensional Riemannian manifolds with curvature -1 and such that the ratio of their volumes is irrational (see [THU] for examples). The fundamental groups of M_1 and M_2 are quasi-isometric because, by proposition 14, they are quasi-isometric to three dimensional hyperbolic space. If these groups were commensurable, M_1 and M_2 would have finite covers with isomorphic fundamental groups. These covers are isometric by Mostow rigidity. This contradicts the irrationality of the ratio of the volumes of M_1 and M_2. □

The numbers which follow show that two quasi-isometric groups, even if they are not isomorphic, often have very similar algebraic properties.

16. Exercise. Suppose Γ is a finitely generated group; we choose a finite system S of generators and write $|x-y|$ for $d_S(x,y)$ for all $x, y \in \Gamma$. We give $\mathbb{Z}$ the usual metric. If Γ and $\mathbb{Z}$ are quasi-isometric, show that Γ possesses a finite index cyclic subgroup.

Hints:

(1) Show that there exists a surjective map $f : \Gamma \to \mathbb{Z}$ and two constants $\lambda > 0$, $C > 0$ such that

(A) $$\frac{1}{\lambda}|x-y-C \leq |f(x)-f(y)| \leq \lambda|x-y|+C$$

for all $x, y \in \Gamma$. For all $n \in \mathbb{Z}$, choose $x_n \in f^{-1}(n)$. Show that there exists $D \geq 0$ such that

(B) $$\frac{1}{\lambda^2}|x_p-x_q|-D \leq |f(\gamma x_p)-f(\gamma x_q) \leq \lambda^2|x_p-x_q|+D$$

for all $p, q \in \mathbb{Z}$. In particular, there exists $E \geq 0$ such that

(C) $$|f(\gamma x_{n+1})-f(\gamma x_n)| \leq E$$

for all $\gamma \in \Gamma$ and $n \in \mathbb{Z}$.

(2) Suppose $\gamma \in \Gamma$. Show that (B) and (C) imply that there exists $\epsilon_\gamma \in \{1,-1\}$ such that

(D) $$\lim_{k\to\infty} f(\gamma x_k) = \epsilon_\gamma \infty \quad \text{and} \quad \lim_{k\to\infty} f(\gamma x_{-k}) = -\epsilon_\gamma \infty$$

Replacing Γ by a subgroup of index two, show that we can assume $\epsilon_\gamma = 1$ for every $\gamma \in \Gamma$.

(3) Show that there exists a constant $G \geq 0$ such that

(E) $$f(\gamma x_k) \geq f(\gamma x_0) - G \quad \text{and} \quad f(\gamma x_-k) \leq f(\gamma x_0) + G$$

for all $\gamma \in \Gamma$ and $k \in \mathbb{Z}$.

(Suppose there is a constant $H \geq 0$, an element $\gamma \in \Gamma$ and an integer $k \geq 0$ such that $f(\gamma x_k) < f(\gamma x_0) - H$. Use 2 to maximise H as a function of k. By (C) and (D), there is $j \geq k$ such that $|f(\gamma x_j) - f(\gamma x_0)| \leq E$, and (B) provides an upper bound on j. This gives an upper bound on H.)

(4) Henceforth, we assume that $f(e) = 0$ and that $x_0 = e$ (if necessary, replace f by $x \mapsto f(x) - f(e)$). Show there exists a constant $K > 0$ with the following property: for all $\gamma \in \Gamma$ such that $f(\gamma) > K$ and $f(\gamma^{-1}) < -K$, we have $f(\gamma^{n+1}) > f(\gamma^n)$ and $f(\gamma^{-n-1}) < f(\gamma^{-n})$ for all $n > 0$.

(First suppose $x \in \Gamma$ and $k \geq 0$ are such that $f(x) = k$. For all $n > 0$, one can maximise $|f(x^{n+1}) - f(x^n x_k)|$ and minimise $f(x^n x_k) - f(x^n)$ by using (A) and (E). Conclude

(F) $$f(x^{n+1}) \geq f(x^n) - \big(G - \big(\lambda^2 + 1\big)\, C\big)$$

Verify, on the other hand,

(G) $$|f(x^{n+1}) - f(x^n)| \geq \frac{1}{\lambda}|x - e| - C \geq \frac{1}{\lambda^2}(k - C) - C$$

For k large enough, deduce from (F) and (G) that $f(x^{n+1}) > f(x^n)$. Suppose also that $Y\ in \Gamma$ and $k \geq 0$ are such that $f(y) = -k$. Verify in the same way that, if k is large enough, $f(y^{-n-1}) < f(y^{-n})$ for all $n \geq 0$.)

(5) Choose $\gamma \in \Gamma$ so that $f(\gamma) \geq K$ and $f(\gamma^{-1}) \leq -K$, in such a way that the subgroup Γ_0 of Γ generated by γ is infinite cyclic. Deduce from (A) that there exists an $R \geq 0$ with the following property: for all $x \in \Gamma$, there exists $n \in \mathbb{Z}$ such that $|\gamma^{-n} x - e| \leq R$. Deduce that $[\Gamma : \Gamma_0] < \infty$.

17. Theorem. *For all integers $n \geq 1$, a group which is quasi-isometric to $\mathbb{Z}^n$ contains a finite index subgroup isomorphic to $\mathbb{Z}^n$.*

We would be interested in seeing an elementary proof of theorem 17. Lacking such a thing, we limit ourselves to hinting at a very onerous argument after theorem 19; but first we must broach the notion of the growth rate of a group.

Suppose Γ is a group generated by a finite set S. We define for all whole numbers N

$$\beta(N) = \text{Card}\{\gamma \in \Gamma \ : \ l_S(\gamma) \leq N\}$$

This function $\beta : \mathbb{N} \to \mathbb{N}$ is the growth function of Γ relative to S. If $d \geq 0$ is an integer, we say that Γ has polynomial growth of degree d if there exist two constants C_1 and C_2 such that $C_1 N^d \leq \beta(N) \leq C_2 N^d$ for all $N \in \mathbb{N}$; it is easy to verify that this property is a quasi-isometry invariant. It follows, for example, that $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not quasi-isometric if $m \neq n$.

18. Example. The Heisenberg group has polynomial growth of degree 4. We sketch the calculations, using the notation of example 5.

First, one shows by induction on N that, if $l_S(a^m b^n c^p) \leq N$, then $|m| \leq N$, $|n| \leq N$, and $|p| \leq N^2$. It follows that $\beta(N) \leq (2N+1)^4$. Also, utilising $bab^{-1}a^{-1} = c$, and therefore $b^q a^q b^{-q} a^{-q} = c^{q^2}$, we see that $l_S(c^p)$ is on the order of $\sqrt{p}$; continuing, for all $(m,n,p) \in \mathbb{Z}^3$ such that $|m| \leq N$, $|n| \leq N$, and $|p| \leq N^2$, the length $l_S(a^m b^n c^p)$ is less than $l_S(a^m) + l_S(b^n) + l_S(c^p)$, and also less that a product of N times a suitable constant. It follows that $\beta(N) \leq kN^4$ for a suitable constant k.

Therefore, the Heisenberg group is not quasi-isometric to $\mathbb{Z}^n$ if $n \leq 3$ or $n \geq 5$. One can further show that it is not quasi-isometric to $\mathbb{Z}^4$. More precisely, to any finitely generated nilpotent group , one associates a simply connected graded Lie Group $\mathrm{Gr}(\Gamma)$ and a "Carnot-Caratheodory" metric d_{cc} on $\mathrm{Gr}(\Gamma)$ and shows (see [PA1], [PA2]) that the pair $\mathrm{Gr}(\Gamma), d_{cc}$ only depends on the quasi-isometry class of Γ. It is easy to verify that $\mathrm{Gr}(\Gamma)$ is $\mathbb{R}^n$ if Γ is $\mathbb{Z}^n$, and the group of real matrices of the form

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

if Γ is the Heisenberg Group of example 5. In particular, the Heisenberg group is not quasi-isometric to a finitely generated abelian group.

An important theorem of Gromov's [GR3] characterizes groups of polynomial growth:

19. Theorem. *A group has polynomial growth rate if and only if it contains a nilpotent subgroup of finite index.*

In particular, a group which is quasi-isometric to $\mathbb{Z}^n$ contains a nilpotent subgroup of finite index. Hence, to prove theorem 17, we need to find a "geometric invariant" of abelian groups amongst nilpotent groups. This could add elegance to isoperimetric inequalities, but we will not develop this aspect of the theory here.

Theorems 17 and 19 show that commutativity and nilpotence, properties which are algebraic, are also geometric properties.

Open Problem. Does a group which is quasi-isometric to a solvable group contain a solvable finite index subgroup ? A similar, perhaps easier, problem is the formulation for polycyclic groups. And also for torsion free groups.

In [Gr2] and [Gr4], M. Gromov gives other example of algebraic properties which conceal a geometric aspect.

20. Theorem. *Let Γ_1 and Γ_2 be two quasi-isometric torsion free groups. If Γ_1 is a non-trivial free product of two groups, then so is Γ_1.*

The proof appeals to a famous result of Stallings which says that a torsion free group is a non-trivial free product if and only if it has an infinite number of ends. It is clear that the number of ends is a quasi-isometry invariant (see proposition 7.17).

Here is an analagous result, which we prove in chapter 7.

21. Theorem. *A group which is quasi-isometric to a free group contains a free subgroup of finite index.*

In anticipation of §4, we also mention a result that appears in [GR4] and [CA2]. In the first reference, Gromov cites Mostow, Margulis, and Tukia. In the second, Cannon cites joint work with Cooper. Look also at the end of chapter 11 of [PA2].

22. Theorem. *Let $n \geq 2$ be an integer and Γ a torsion free group. If Γ is quasi-isometric to a co-compact discrete subgroup of the lie group $O(n,1)$, then Γ is isomorphic to a co- compact discrete subgroup.*

We could continue with our enumeration of "geometric" properties of finitely generated groups. We do not know an important property of finitely generated groups which is not a quasi-isometry invariant. Here are 2 more examples in the form of exercises.

23. Exercise. Show that a group which is quasi-isometric to a finitely presented group is finitely presented.

Hint. Let Γ be generated by a finite set S and let $\mathrm{G}(\Gamma, S)$ be the corresponding Cayley Graph. For all real numbers $R > 0$, we use $L(R)$ to denote the set of nooses of the type $\phi\psi\phi^{-1}$. where ϕ is a path joing the basepoint to a point $x \in \mathrm{G}(\Gamma, S)$ and ψ is a loop of length ¡ R based at x. First show that Γ is finitely presented if and only if $L(R)$ generates the fundamental group of $\mathrm{G}(\Gamma, S)$ for large enough R.

24. Exercise. Show that a group which is quasi-isometric to an amenable group is amenable.

Hint. Let Γ and S be as above. The boundary of a finite subset X of Γ is the set $\partial_S X$ of points $x \in X$ such that there exists $y \in \Gamma - X$ with $d_S(x,y) = 1$. The isoperimetric constant of Γ is defined by

$$h_S(\Gamma) = \inf \left\{ \frac{\mathrm{Card}(\partial_S X)}{\mathrm{Card}(X)} \quad : \quad X \subset \Gamma \text{ and } X \text{ finite} \right\}$$

and we say that Γ is amenable if $h_S(\Gamma) = 0$. Verify that the vanishing of the isoperimetric constant is independent of the choice of S; consequently that amenability is a quasi-isometry invariant.

In addition to the exercises, use the above definition to show that a group with polynomial growth is amenable and a free group is not. Finally, we note that there are a number of other equivalent definitions of amenability; see for example [EYM] for a general introduction, as well as pages 446-448 in [CO] for the equivalence (due to Fölner) in the case of discrete groups of the above definition to the definition given using "averaging invariants".

4. Presentation of Results

After we have convinced you that it can be useful to consider finitely generated groups as metric spaces, we will define hyperbolic groups. We start with several definitions concerning a metric space (X, d). For the distance between two points in X, we usually write $|x - y|$ instead of $d(x, y)$.

25. Definitions. Let x_0, x_1 be two points in X and let $a = |x_0 - x_1|$ be their distance. A *geodesic segment* originating at x_0 and ending at x_1 is an isometry $g : [0, a] \to X$ such that $g(0) = x_0$ and $g(a) = x_1$. We sometimes say that g is a *parametrised geodesic segment* and that the image of g is a *geometric geodesic segment* (or, abusing terminology, a geodesic segment).

We say that X is a geodesic space if for each pair of points x_0, $x_1 \in X$, there is a geodesic segment $[0, |x_0 - x1|] \to X$ with endpoints x_0, x_1 (we don't demand that there be only one such segment).

A geodesic triangle with vertices $x, y, z \in X$ is the union of 3 geodesic segments joining pairs of vertices. We allow degenerate cases, like for example a triangle where the points y and z are the same but the segments from x to y and from x to z are distinct.

26. Examples and Notation. Let Γ be a group generated by a finite set S. The Cayley Graph $\mathrm{G}(\Gamma, S)$ defined in §2 is a geodesic space. If Γ is not freely generated by S, then $\mathrm{G}(\Gamma, S)$ possesses a circuit and therefore contains pairs of points between which there exist several geodesic segments.

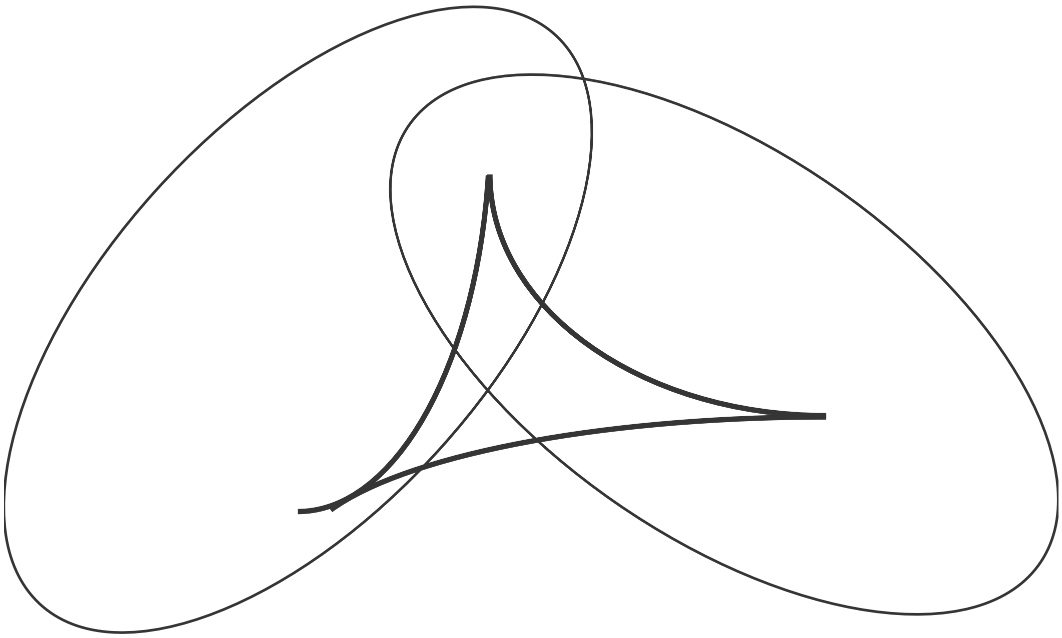
A complete riemannian manifold is a geodesic space (the Hopf-Rinow theorem). The example of antipodal points on spheres shows that one can have several geodesic segments with the same endpoints. Of course, there is the remarkable case where each pair of points determines a unique geodesic segment (simply connected manifolds with non-positive curvature).

Although two points x_0, x_1 in a geodesic space do not, in general, determine a unique segment, it is nevertheless convenient to use $[x_0, x_1]$ to denote a geodesic segment with endpoints x_0 and x_1.

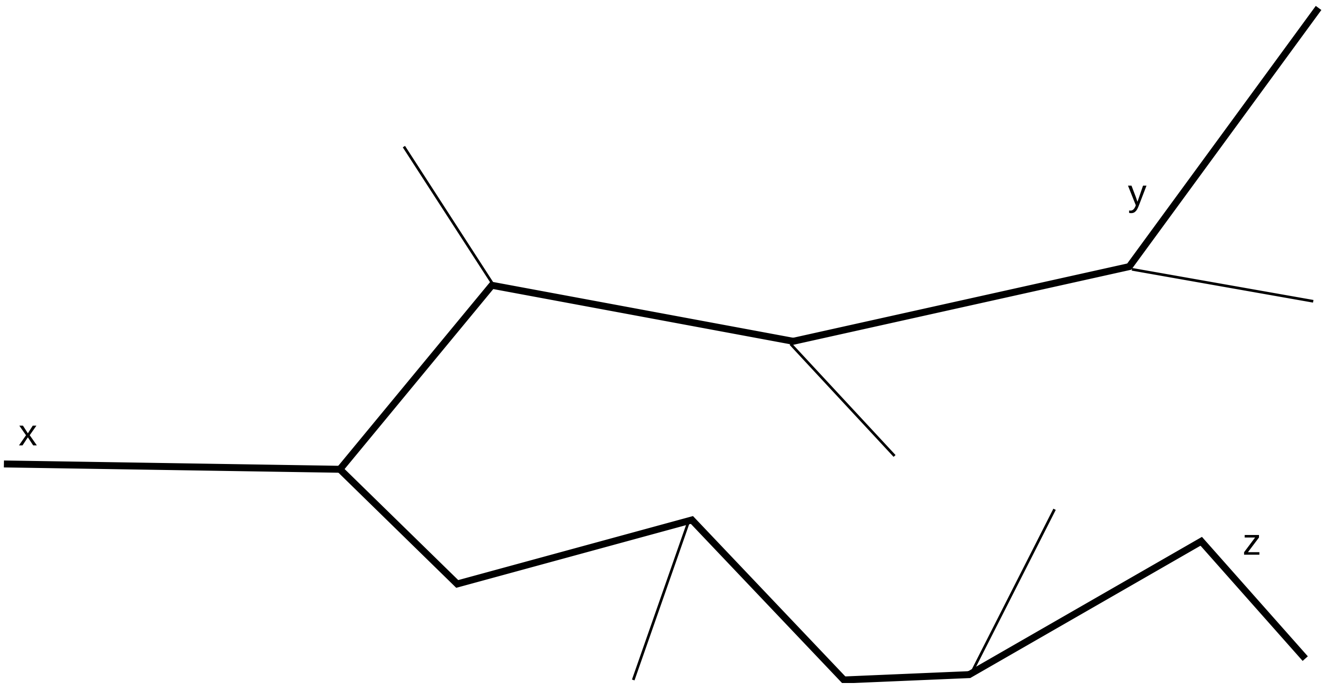
27. Definition. Given a number $\delta \geq 0$, a geodesic metric space X satisfies the Rips condition with constant δ if, for all geodesic triangles $\triangle$ in X, the distance between any point on a side of $\triangle$ and the union of the other two sides is less than δ. Formally:

For all $\triangle = [x, y] \bigcup [y, z] \bigcup [z, x]$
and for all $u \in [y, z]$, we have $d(u, [x, y] \bigcup [z, x]) \leq \delta$

A geodesic space X is called hyperbolic is there exists a number $\delta \geq 0$ such that X satisfies the Rips condition with constant δ.



28. Example: Trees. Suppose T is a simplicial tree given a metric such that each edge is isometric to the real segment $[0,1]$ and so that the distance between 2 points is the greatest lower bound of the lengths of paths joing the points. Any triangle in T is degenerate in the sense that each of its edges is contained in the union of the others. Therefore a tree is hyperbolic since it satisfies definition 27 with $\delta = 0$.



In order for it to enter our general framework, it is important that the notion of hyperbolicity is a quasi-isometry invariant. Although this is the case, one finds that the proof of this assertion is not easy; we give it in chapter 5.

29. Theorem. *If two geodesic metric spaces are quasi- isometric and one is hyperbolic, then so is the other one.*

This theorem justifies the following definition.

30. Definition. A finitely generated group Γ is *hyperbolic* if the Cayley Graph defined by Γ and a finite system of generators of Γ is hyperbolic.

Theorem 29 also shows that the Cayley Graph $G(\Gamma, S)$ is hyperbolic for all finite systems of generators S if it is for a particular system S_0. It also shows that 2 quasi-isometric groups are either both hyperbolic or neither is. The first examples of hyperbolic groups are now immediate from looking at numbers 4 (Cayley Graph of a free group) and 28.

31. Proposition. *Any free group is hyperbolic.*

Number 27 is one possible definition of hyperbolicity in a metric space, but is not the definition best adapted to all of our needs. For this reason we give several equivalent definitions in chapter 2 – we hope these aid the reader in familiarising herself with this notion. Afterwards, since free groups are not sufficient long term examples, we dedicate the main part of chapter 3 to the principal source of examples, which, moreover, suggested the name hyperbolic.

32. Theorem. *The fundamental group of a compact riemannian manifold with negative curvature is hyperbolic.*

A second family of examples, already mentioned in §1, is the subject of the appendix.

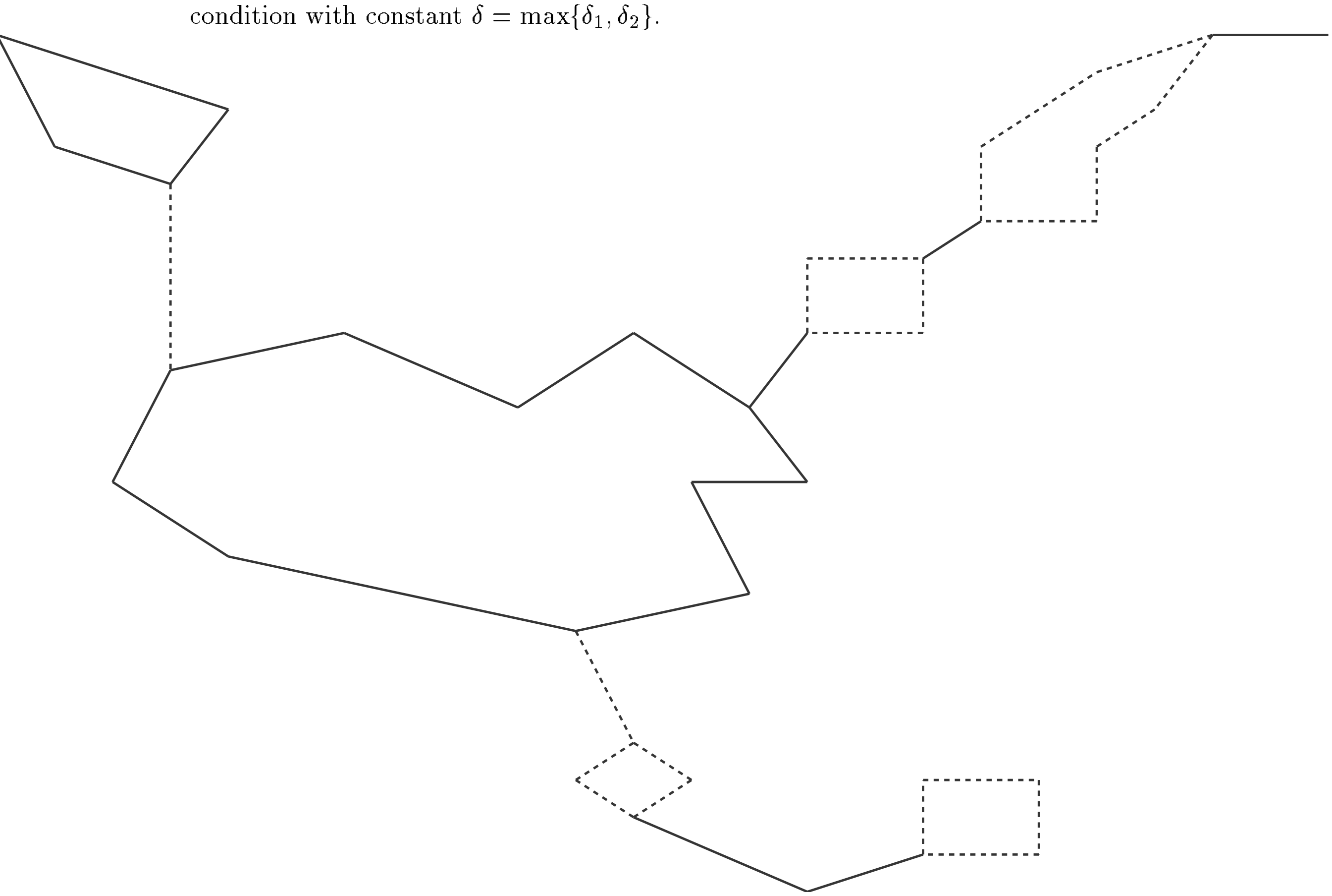
33. Theorem. *Suppose* Γ *admits a finite presentation satisfying the small cancellation condition* $C'(\frac{1}{6})$ *or the conditions* $C'(\frac{1}{4})$ *and* $T(4)$. *The group* Γ *is hyperbolic.*

We note, in passing, that the two families are essentially disjoint. In fact, the only groups covered by both Theorem 32 and Theorem 33 are surface groups from orientable surfaces of genus 2 and non-orientable surfaces of genus 3.

Meanwhile, hyperbolic groups are far more than these two families. We point out an important example of this fact.

34. Exercise. The free product of hyperbolic groups is hyperbolic.

Hint. We are given numbers $\delta_i \geq 0$ and groups Γ_i generated by finite subsets S_i such that the metric spaces $\mathrm{G}(\Gamma_i, S_i)$ satisfy the Rips condition with constant δ_i $(i = 1, 2)$. Then $S = S_1 \bigcup S_2$ generates $\Gamma = \Gamma_1 * \Gamma_2$. Let $\triangle = [x, y] \bigcup [y, z] \bigcup [z, x]$ be a geodesic triangle in $\mathrm{G}(\Gamma, S)$. Using the normal form of elements in a free product, verify that $\triangle$ decomposes into geodesic triangles, each of which is isometric to a geodesic triangle in $\mathrm{G}(\Gamma_1, S_1)$ or $\mathrm{G}(\Gamma_2, S_2)$. Deduce that $\mathrm{G}(\Gamma, S)$ satisfies the Rips condition with constant $\delta = \max\{\delta_1, \delta_2\}$.



To show the reader the extent of the class covered by his theory, M. Gromov used a statistic. We consider all presentations of groups with p generators and q relators which are reduced words in the generators. Given positive numbers $n_1, \ldots, n_q$, we define $N(p, n_1, \ldots, n_q)$ to be the number of presentations which have relators of length $n1, \ldots, nq$ respectively, and $N_h(p, n_1, \ldots, n_q)$ to be the number of these which are hyperbolic. The statement below by M. Gromov was not accompanied by a proof, not even a sketch of a proof, and we hesitate about its status: Theorem ? Conjecture ? We risk the term below

35. Theorem without proof. *As $n_1, \ldots, n_q$ tend towards infinity, the ratio*

$$\frac{N_h(p, n_1, \ldots, n_q)}{N(p, n_1, \ldots, n_q)}$$

tends towards 1. This is the meaning of "a group taken at random is probably hyperbolic".

We have separated the properties we show about hyperbolic groups into 3 categories. First, in chapter 4, we show the properties which result directly from the definition, of which these are the main ones:

36. Theorem. *Let Γ be a hyperbolic group. There exists a finite dimensional polyhedron P and a simplicial action of Γ on P such that*

(1) *The stabiliser of every simplex is finite.*
(2) *The quotient of P by Γ is compact.*
(3) *P is contractible.*

It is an impressive consequence that Γ is finitely presented and has finite dimensional rational cohomology.

The second category of properties is the subject of chapters 6 through 8. First, we recall the following construction: Let M be a negatively curved compact n-manifold and $\tilde{M}$ the universal cover of M. Then there exists a natural compactification of $\tilde{M}$ in which the $\partial\tilde{M}$, the complement of $\tilde{M}$, is formed from "limit points" of geodesics. The boundary at infinity $\tilde{M}$ is is homeomorphic to a sphere of dimension $n-1$ on which the fundamental group of M acts by homeomorphisms, and one studies $\pi_1(M)$ via this "action at infinity" (see, for example, §8 of [BGS]). It is a remarkable fact that an important part of this step generalises to hyperbolic groups.

37. Theorem. *Let Γ be a hyperbolic group. There exists a compact metric space $\partial\Gamma$, called the boundary of Γ, on which Γ acts by quasi-conformal homeomorphisms (this term will be defined in chapter 6). Furthermore, this space $\partial\Gamma$ is well-defined up to quasi-conformal homeomorphism by the quasi-isometry class of Γ. When Γ is the fundamental group of a negatively curved n-manifold, $\partial\Gamma$ is homeomorphic to the $(n-1)$ sphere mentioned above.*

The theorem will be stated more precisely in chapter 7. We emphasize that $\partial\Gamma$ in general does not resemble a sphere: it could be a Cantor set or a much more complicated compact space which can be thought of as a universal curve. Whatever it is, in general $\partial\Gamma$ is used in the same way as when Γ is the fundamental group of a negatively curved manifold. We also mention two applications: the first is a proof of theorem 21; the second occurs in chapter 8 and is stated as follows:

38. Theorem. *Let Γ be a hyperbolic group. The centraliser in Γ of any element of infinite order contains a cyclic subgroup of finite index, and any amenable subgroup of contains a cyclic subgroup of finite index. If, in addition, Γ is not amenable, then Γ contains a non-abelian free group.*

It is a fundamental property of hyperbolic spaces that they "resemble trees" in a sense which will be made clear in chapter 2. Indeed, experience shows that it is always advantageous to test the truth of a "hyperbolic statement" or the efficiency of a "hyperbolic proof" in the particular case of a tree. We hope to show that this holds for the boundary in chapter 6, devoted to the boundary of a tree.

The third category mentioned after number 35 is broached in chapter 9. It concerns another generalisation of properties already known for fundamental groups of negatively curved manifolds (results due to J. Cannon). Putting it loosely (we will be precise in chapter 9), we show that the elements of a hyperbolic group help to describe a finite Markov chain. In this introduction, we only cite a consequence of this "Markov property":

39. Theorem. *Let S be a finite system of generators for a hyperbolic group Γ. For all integers $n \geq 0$, we use $\sigma(n)$ to denote the number of elements of Γ satisfying the equation $l_S(\gamma) = n$. Then the series*

$$\zeta(t) = \sum_{n=0}^{\infty} \sigma(n) t^n$$

is a rational function in the variable t.

Once the basic techniques are firmly in place, the theory of hyperbolic groups is remarkably flexible and effective. To display these qualities, we have chosen to develop the construction done by M.Gromov of an infinite, finitely generated group, all of whose elements are torsion. This construction was not the first (see, for example [Go1] and [Gri]), but it has the advantage of being extremely natural.

Before we describe this construction and some examples, we introduce (in chapter 10), a general notion of negatively curved polyhedra. These are polyhedra whose faces are given negatively curved riemannian metrics, and which satisfy a combinatorial condition requiring that the "amount of curvature" at every point is negative. The fundamental group of a finite, negatively curved polyhedron is a hyperbolic group, from which we get new examples of such groups. Even as the notion of a manifold generalises to that of an orbifold, we introduce (in chapter 11), a notion of orbi-space which generalises that of polyhedron. Further, we show that an *negatively curved, compact orbi-space* naturally possess a fundamental group which is a hyperbolic group.

Focusing these tools, we obtain the following result:

40. Theorem. *Let M be a compact manifold, or more generally, a negatively curved, finite polyhedron. The fundamental group of M possesses an infinite quotient all of whose elements are torsion.*

It is true that this theorem doesn't give an example of an infinite, finitely generated group where the orders of all the elements are bounded (to know that such groups exist, see [Adi], [Ol2], and [Gup]). Meanwhile, we can hope that perfecting the method of M.Gromov will obtain such groups.

We cite two consequences of theorem 40. First, it permits an answer to a question posed in [Day], and already implicit in the work of Von Neumann [VNe]

41.Theorem. *There exist finitely generated groups which are not amenable and which don't contain any free, non-abelian subgroups.*

Still, this result was already known [Ol1], but by a more difficult to understand method. A second consequence of theorem 40 has to do with property T of Kazhdan, which is (for infinite groups) a much stronger property than non-amenability. All of the examples known before the theory of hyperbolic groups to have property T were finitely presented. From now on, we have the following result.

42. Theorem. *There are an uncountable number of finitely generated groups which possess property T. In particular, there exist some which are not finitely presented.*

5. What These Notes Don't Cover

M. Gromov's article is too rich; we had to make choices. To end this overview, we will rapidly describe three major ideas which we, for lack of time and courage, have not developed.

The first concerns isoperimetric inequalities. Let Γ be a finitely presented group and let X be a finite polyhedron with fundamental group isomorphic to Γ. Let γ be a simplicial loop contained in the one-skeleton of the universal cover $\tilde{X}$ of X; we use $l(\gamma)$ to denote its length (by which we mean the number of 1-simplices in it). Since $\tilde{X}$ is simply connected, there is a simplical disk with boundary γ. We use $A(\gamma)$ to denote the minimal area (this means the number of two-simplices) of such a disk. We say that Γ satisfies a linear isoperimetric inequality if there exists a positive constant C such that, for all γ, we have $A(\gamma) \leq Cl(\gamma)$. It is not hard to verify that this property is independent of the choice of X.

43. Theorem. *[Gr5] If Γ is a finitely presented group, the following conditions are equivalent:*

(1) Γ *is a hyperbolic group.*
(2) Γ *satisfies a linear isoperimetric inequality.*

The implication (1)$\Rightarrow$(2) follows from arguments similar to those of chapter 4. M. Gromov shows the converse (a very delicate argument) by using Riemann's uniformisation theorem and analytic techniques borrowed from the theory of minimal surfaces. An appendix of [GeS] gives a completely elementary proof of this result which doesn't use "analysis". We have also received the notes of B. Bowditch on this subject.

A second point that we would have liked to cover is the construction of *geodesic flows in a hyperbolic group.* Suppose V is a compact manifold with a negatively curved riemannian metric g. We see ([Gr1]) that the geodesic flow of (V, g), acting on the unit tangent bundle $T_1(V)$ of V, only depends on the fundamental group of V. More precisely, let (V', g') be another compact, negatively curved manifold and let ϕ be an isomorphism between the fundamental groups of V and V'. Then, there is a homeomorphism $h(\phi)$ between $T_1(V)$ and $T_1(V')$ sending the orbits of the geodesic flow of V to those of the geodesic flow of V'. Of course, $h(\phi)$ does not conjugate the flows because the lengths of periodic geodesics in V and V' may be different.

M. Gromov proposes a construction of the geodesic flow starting from the fundamental groups. Let Γ be any hyperbolic group (Γ doesn't have to be the fundamental group of a negatively curved manifold). M. Gromov constructs ([Gr5], section 8.3), a space $T_1(\Gamma)$, well defined up to homeomorphism, and a "geodesic flow" on $T_1(\Gamma)$. This flow is only defined up to topological equivalence (this means up to homeomorphisms preserving the orbits). In this way, a hyperbolic group is attached to a dynamical system. This dynamical system satisfies all the properties of geodesic flows on negatively curved manifolds: hyperbolic topology, stable and unstable leaves, etc. In this way, the theory of hyperbolic groups linked to that of Anosov-Smale...

Finally, we regret that these notes have not described a magnificent construction of M. Gromov which lets us obtain numerous examples. Given any polyhedron, he gives a procedure to change the topology and associate a new, negatively curved polyhedron to it. If the original polyhedron was a manifold, the new polyhedron is also a manifold, naturally cobordant to the original. Since it follows from the ideas in chapter 10 that a negatively curved polyhedron is an Eilenberg-MacLane Space, we obtain new examples of compact manifolds which are Eilenberg-MacLane spaces. By way of example, we cite two results of M. Gromov.

44. Theorem. *[Gr5] All compact manifolds are cobordant to a manifold which is an Eilenberg-MacLane Space.*

45. Theorem. *[Gr5] If two manifolds are Eilenberg-MacLane spaces and are cobordant, then they are cobordant by a space which is also an Eilenberg-MacLane space.*

The interested reader can find a description of these constructions, as well as some interesting developments in [DaJ].

Translated by WILLIAM GROSSO

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CA 94720

HYPERBOLIC METRIC SPACES (CHAPTER 2)

Etienne Ghys and Pierre de la Harpe

Abstract. The first section contains a definition of hyperbolic spaces and indicates an important way that such a space resembles a tree (tangent subcone at infinity, proposition 11). The second section treats approximating trees to a hyperbolic space (theorem 12). The third is dedicated to several other equivalent definitions of hyperbolicity valid for geodesic metric spaces (subtleties of triangles); in particular, we will show in corollary 22, that the Poincare disk is hyperbolic (!!).

1. A Definition of Hyperbolicity

Let X be a metric space. We denote the distance between two points $y, z \in X$ by $d(y,z)$ or $|y-z|$.

1. Definitions. Given a base point $x \in X$, the Gromov Product of two points $y, z \in X$ is

$$(y|z)_x = \frac{1}{2}\{|y-x| + |z-x| - |z-y|\}$$

We also write $(y|z)$ when there is no ambiguity about the base point. The triangle inequality shows that

$$0 \leq (y|z)_x \leq \min\{|y-x|, |z-x|\}$$

For example, if x, y, and z are colinear, then $(y|z)_x$ is 0 if x is between y and z (this means $|y-z| = |y-x| + |x-z|$) and $(y|z)_x = \min\{|y-x|, |z-x|\}$ otherwise.

To explain the geometric significance of the Gromov Product, we introduce the notion of a metric tree. Let T be a tree, by which we mean a connected graph without cycles, and let $|T|$ be its geometric realisation. Choose a real number $l(a) > 0$ for each edge a of $|T|$. Then, there exists a unique metric on $|T|$ (up to isometry) which is maximal with respect to the following property:

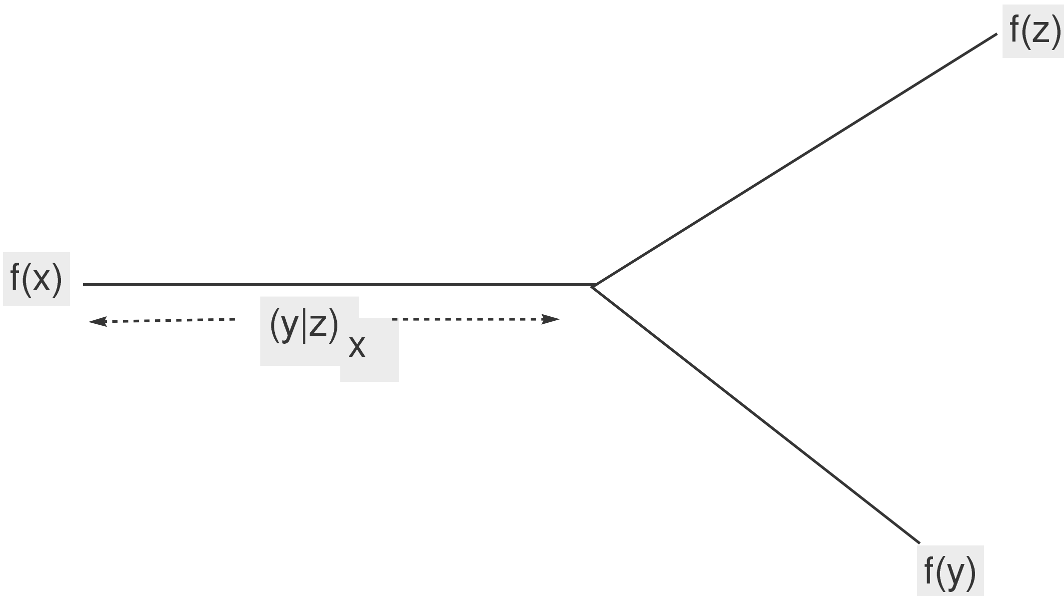
The edge a is isometric to the interval $[0, l(a)]$ of $\mathbb{R}$.

The space $|T|$, with this metric, is called a "metric tree".

We call a metric tree made up of three edges beginning at a common vertex a *tripod*. A tripod is characterised up to isometry by the length of its edges; we allow the degenerate case where some of the lengths can be zero.

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

2. Proposition. *Let x, y, z be three points in a metric space. There exists a tripod T and an isometry $f : \{x, y, z\} \to T$ with image the three endpoints of T. Furthermore, $(y|z)_x$ is the length of the edge of T which ends in the image of x.*



The proof is clear. This proposition shows that the metric relationship between *three* points in any metric space is the same as in a tree. For *four* points, on the other hand, one can use the following definition, which Gromov has shown is very interesting.

3. Definition. Let δ be a non-negative real number. We say a metric space X is δ-hyperbolic if

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta$$

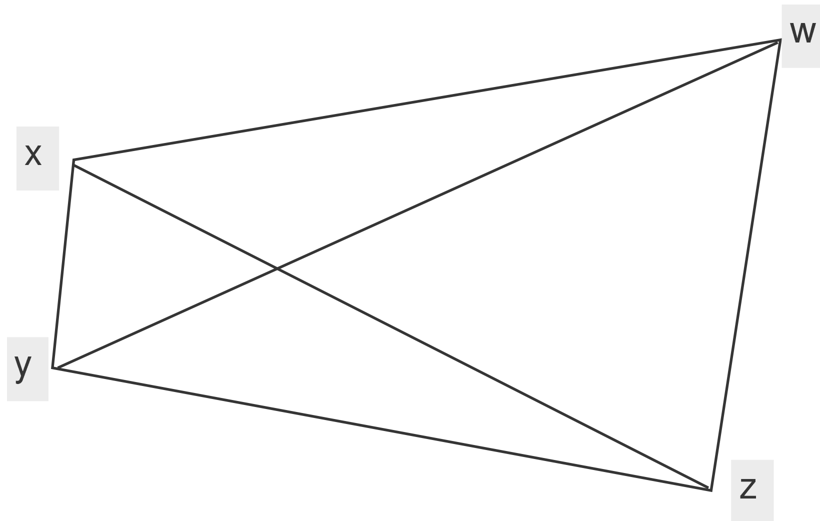
for all $w, x, y, z \in X$. We say that X is hyperbolic if the value of δ is irrelevant (the equivalence between this definition and that of chapter 1 is shown in proposition 21).

4. Reformulation. Let $\delta \geq 0$ and $w, x, y, z \in X$. We replace the three Gromov products in the previous inequality with their values. We obtain the equivalent inequality:

$$|x - z| + |y - w| \leq \max\{|x - y| + |z - w|, |x - w| + |z - y|\} + 2\delta$$

We can think of w, x, y, and z as four vertices which define 6 distances among themselves. There are 3 ways to group these points into two pairs and three sums of two distances.

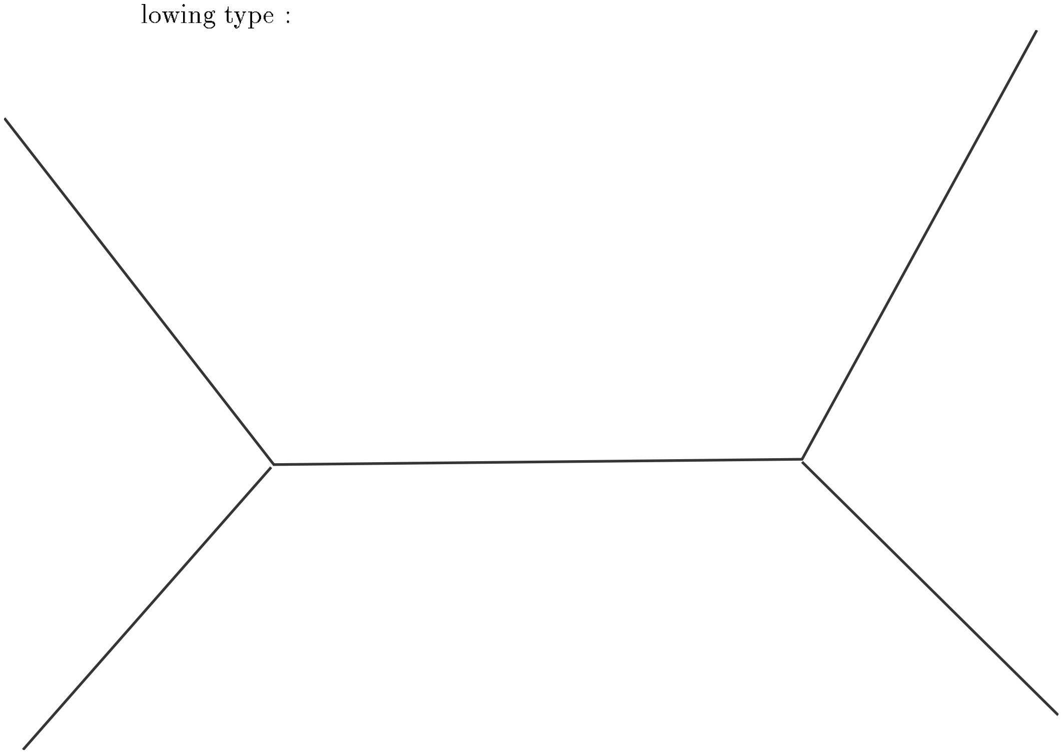
$$r = |x - w| + |z - y|$$
$$l = |x - y| + |z - w|$$
$$j = |x - z| + |y - w|$$



If the notation is such that $r \leq l \leq j$ (the reader can think of "regular", "large" and "jumbo"), a δ-hyperbolic space is characterised by $j \leq l + 2\delta$; this means that "the longest distance does not exceed the median by more than 2δ".

5. Examples.

(1) The most important example is given by metric trees, which are 0-hyperbolic. It is obvious that the quadrilateral structures of these spaces are of the following type :



(It is understood that some edges might be degenerate). We note that all subspaces of a δ-hyperbolic space are also δ-hyperbolic. Thus, all free groups are 0-hyperbolic for the word metric defined on a free system of generators.

(2) It is clear that any metric space with diameter $\delta < \infty$ is δ-hyperbolic. In what follows, we will not be interested in any examples of this type.
(3) Suppose X and Y are two geodesic metric spaces (definition 1.25). We suppose that there is a quasi-isometry $X \to Y$ and that Y is hyperbolic. Then X is hyperbolic (chapter 5).
(4) Suppose X is a compact negatively curved Riemannian manifold, $\tilde{X}$ is its universal cover, Γ its fundamental group, and S a finite set of generators for Γ. Then $\tilde{X}$ and Γ are hyperbolic for the canonical Riemannian metric and for the word metric respectively (see chapter 3). One can generalise these examples to the case where X is a polyhedron or even an orbi-space (see chapter 11).
(5) Suppose Γ is a small cancellation group satifying the conditions C'(1/6) and given the word metric. Then Γ is hyperbolic (see the appendix).
(6) On the other hand, Euclidean space $\mathbb{R}^2$ is not hyperbolic. More generally, if X is a metric space which admits a self-similarity with ratio $\lambda > 1$ and contains 4 points w, x, y, and z with:

$$(x|z)_w < \min\{(x|y)_w, (y|z)_w\}$$

then X is not hyperbolic.

Example 5 admits a partial converse. We first state it in a simple case.

6. Proposition. *Suppose F is a finite 0-hyperbolic metric space. Then there is an immersion of F into a metric tree.*

Proof. Let w be a base point in F. For all $x \in F$, we write $|x|$ for $|x-w|$. We denote the disjoint union of real segments $[0, |x|]$ for $x \in F - w$ by $\tilde{T}$. Given $t \in [0, |x|]$ and $t' \in [0, |x'|]$, we set

$$t \sim t' \qquad \text{if } t = t' \leq (x|x')_w$$

This defines an equivalence relation; in fact, if $t \sim t' \sim t''$, then

$$t = t'' \leq \min\{(x|x')_w, (x'|x'')_w\} \leq (x|x')_w$$

in view of the hyperbolicity of F. Let T be the quotient of $\tilde{T}$ by this relation. The map $\tilde{T} \times \tilde{T}$ defined by

$$(t, t') \mapsto t + t' - 2\min\{t, t', (x|x')_w\}$$

(where $t \in [0, |x|]$ and $t' \in [0, |x'|]$) induces a metric on T which makes it into a metric tree. The desired immersion sends $x \in F$ to the class in T of $|x| \in [0, |x|] \subset \tilde{T}$. $\square$

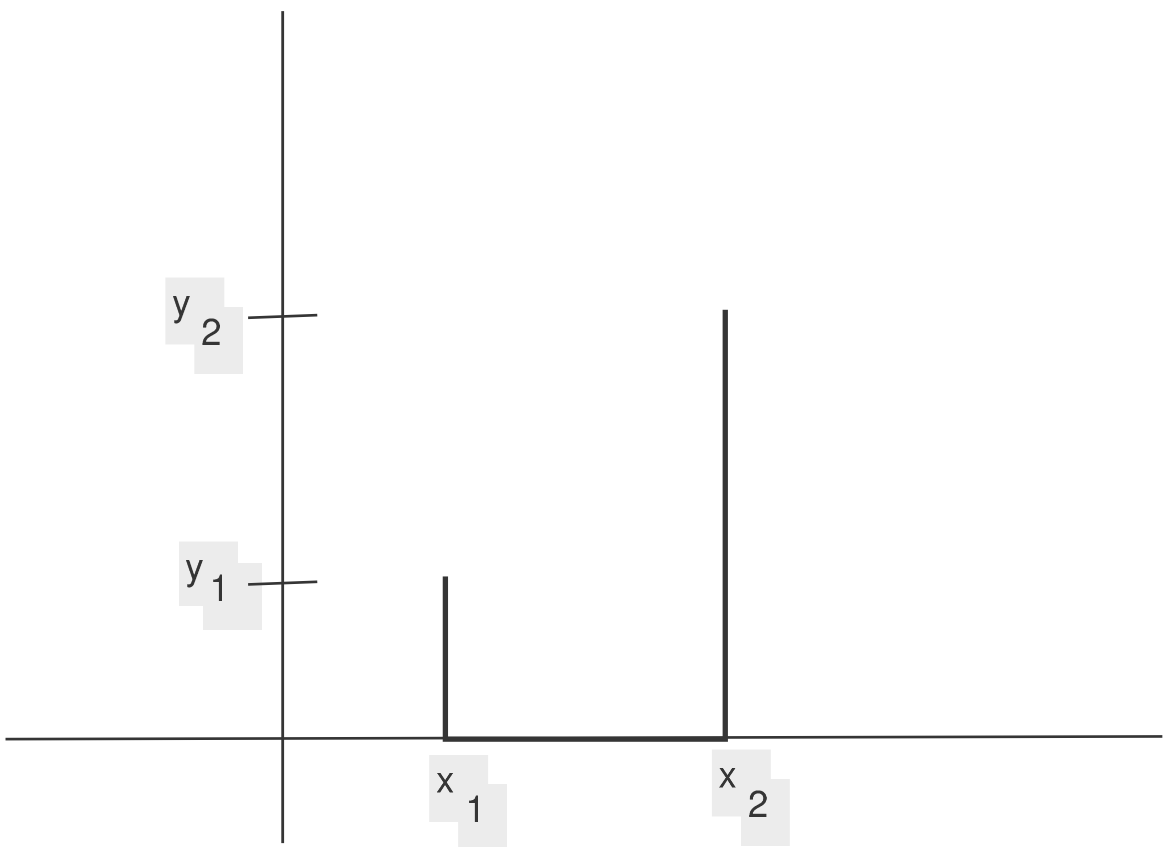
It is clear that this proof only used the finitude of F to guarantee that the space T, the quotient of $\tilde{T}$, is a metric tree.

7. Definition. A metric space T is a real tree if it meets the two conditions below:

(1) Any two distinct points are the endpoints of a geodesic segment.
(2) If two segments have exactly one endpoint in common,then their union is a geodesic segment.

Here is a simple example of a real tree which is not a metric tree. Let the metric d on $\mathbb{R}^2$ be defined in the following way:

$$d((x_1,y_1),(x_2,y_2)) = |y_1 - y_2| \quad \text{if } x_1 = x_2$$
$$= |y_1| + |x_1 - x_2| + |y_2| \quad \text{if} x_1 \neq x_2$$



One can consult [Sha] for a study of real trees.

We return to proposition 6; it is not difficult to assure oneself that the class of real trees is suited to a complete converse of example 5 (i). We give it below as an exercise.

8. Exercise. Show that a metric space is 0-hyperbolic if and only if it is isometric to a subset of a real tree. Show that a geodesic metric space is 0-hyperbolic if and only if it is a real tree.

In the rest of these notes, we do not use the theory of real trees. In fact, we have to content ourselves with defining a real tree as a geodesic 0-hyperbolic space (see theorem 3.17 of [AlB]).

9. Definition. Let X be a metric space. We say that a metric space T is a tangent sub-cone of X at infinity if it is a geodesic space and if the following condition is satisfied:

For all finite subsets $\{t_1, ..., t_k\}$ of T, there exists an infinite sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers tending towards zero and k infinite sequences $(x_{i,n})_{n\geq 1}$ of points in X with $i = 1, \ldots, k$ such that

$$\lim_{n\to\infty} \frac{\epsilon_n |x_{i,n} - x_{j,n}|}{|t_i - t_j|} = 1 \qquad i, j = 1, \ldots, k$$

10. Examples.

(1) A segment is a tangent sub-cone at infinity of $\mathbb{R}$.
(2) Let D^2 be the Poincare disk of constant curvature -1. Let k be a whole number and T a star-shaped tree made of k segments (of arbitrary lengths) having a point t_0 in common. Then T is a tangent sub-cone at infinity for D^2. (if, for example, the set is made from t_0 and $l \leq k$ points which are the same distance from t_0, consider, in D^2, a point x_0 and l sequences of points $(x_{i,n})_{n\geq 1}$, where $x_{1,n}, \ldots, x_{l,n}$ are the vertices of a regular polygon in D^2 centered at x_0 and with n sides.).
(3) If X has finite diameter, any tangent subcone at infinity is a point.

11. Proposition. *Let X be a metric space. If X is hyperbolic, any tangent sub-cone at infinity is a real tree.*

Proof. Let T be a tangent sub-cone at infinity. Since T is, by definition, geodesic, exercise 8 implies that it suffices to show that T is 0-hyperbolic.

Let $t_0, t_1, t_2, t_3 \in T$. Suppose $(\epsilon_n)_{n\geq 1}$ and $(x_{j,n})_{n\geq 1}$ are as in definition 9. with $j = 0, 1, 2, 3$. Since X is hyperbolic, there is a $\delta \geq 0$ such that

$$(x_{1,n}|x_{3,n})_{x_{0,n}} \geq \min\{(x_{1,n}|x_{2,n})_{x_{0,n}}, (x_{2,n}|x_{3,n})_{x_{0,n}}\} - \delta$$

for all $n \geq 1$. Since $\lim_{n\to\infty} \frac{\epsilon_n |x_{i,n} - x_{j,n}|}{|t_i - t_j|} = 1$, we therefore have

$$(t_1|t_3)_{t_0} \geq \min\{(t_1|t_2)_{t_0}, (t_2|t_3)_{t_0}\}$$

□

This proposition shows that "at ∞, X looks like a tree". M. Gromov has also stated the reciprocal of proposition 11, and F. Paulin has sent us a proof.

2. Approximating Trees

Proposition 6 showed that a finite 0-hyperbolic space is (isometric to) a subset of a tree. If F is a finite δ-hyperbolic space with $\delta > 0$, the same assertion is almost true, up to a constant that depends on δ and the *logarithm* of the cardinality

$|F|$ of F. The theorem below state this assertion precisely and also gives us a generalisation. The proof of this theorem constitutes this section. We define a *subray* with origin x in a metric space Y to be a subspace $X \subset Y$ containing x such that there is an isometric immersion $(X, x) \to (\mathbb{R}^+, 0)$.

In this section, we are given a constant $\delta \geq 0$. If x is a point in a metric space with a base point w, we define $|x| = |x - w|$.

12. Theorem. *Let F be a δ-hyperbolic metric space with base point w and let k be a positive integer.*

(1) *If $|F| \leq 2^k + 2$, there is a pointed finite metric tree T and a map $\Phi : F \to T$ such that*

A) Φ preserves distance to the base point:

$|\Phi(x)| = |x|$ *for every* $x \in F$

B)$|y - x| - 2k\delta \leq |\Phi(y) - \Phi(x)| < |y - x|$ *for all* $x, y \in F$.

(2) *Suppose there exist subrays F_i, originating at w_i in F, with $i = 1, \dots, n$ and $n \leq 2^k$, such that $F = \bigcup_{i=1}{}^n F_i$. We define $c = \max_{i=1}{}^n |w_i|$. Then there is a pointed real tree T and a map $\Phi : F \to T$ with property (A) above as well as:*

$$B')|y - x| - 2(k+1)\delta - c \leq |\Phi(y) - \Phi(x)| < |y - x| \text{for all} x, y \in F$$

13. Remarks.

(1) If $|F| = 3$, the theorem follows from proposition 3 and doesn't use the hypothesis of hyperbolicity. If $|F| = 4$, the theorem is essentially a repetition of the definition of hyperbolicity.
(2) If two points x and y are collinear with the base point and if x is between w and y (i.e. if $|y| = |x| + |y - x|$), then $\Phi(x)$ and $\Phi(y)$ are collinear with the base point of T and $\Phi(x)$ is between $\Phi(w)$ and $\Phi(y)$. This follows easily from properties (1) and (2).
(3) As we have already mentioned, we don't utilise the theory of real trees. The only property of the space T that we use in point (2) is that it is geodesic and 0-hyperbolic. Every finite subtree of T is contained in a metric tree.
(4) In statement (2) of the theorem, F is not necessarily finite. Nevertheless, suppose $F = w, x_1, \dots, x_n$ is a finite δ-hyperbolic space, with $|F| = n + 1 \leq 2^k + 1$. Theorem 12.2 implies a statement essentially equivalent to theorem 12.1. Below, we give a proof of theorem 12.2, and let the reader fiddle with the proof of theorem 12.1.

The proof of the theorem is preceded by two lemmas, in which we use the same notation.

14.i. Lemma (short version). *We suppose that $|F| \leq 2^k + 2$. Let L be a whole number and $x_1, \dots, x_L$ be a sequence of points of F (repetitions are allowed). Then*

$$(x_1|x_L) \geq \min_{2 \leq i \leq L}(x_{i-2}|x_i) - k\delta$$

(note that the term k depends on the cardinality of F and not on L).

Proof. If there exists an index j such that $x_j = w$ (the base point of F), then $(x_j|x_{j\pm 1}) = 0$ and there is nothing to show. From now on, we suppose $x_j \neq w$ for $j \in \{1, \dots, L\}$ and that $L \geq 4$.

We first suppose $L \leq 2^k + 1$. The proof is by induction on k. We set $K = [\frac{L}{2}]$, so $2 \leq K \leq 2^{k-1} + 1$ and $2 \leq L - K + 1 \leq 2^{k-1} + 1$. By the induction hypothesis:

$$(x_1|x_K) \geq \min_{2 \leq j \leq K}(x_{j-1}|x_j) - (k-1)\delta$$

$$(x_K|x_L) \geq \min_{K+1 \leq j \leq L}(x_{j-1}|x_j) - (k-1)\delta$$

Continuing

$$(x_1|x_L) \geq \min\{(x_1|x_K), (x_K|x_L)\} - \delta \geq \min_{2 \leq j \leq L}(x_{j-1}|x_j) - k\delta$$

Now suppose $L > 2k + 1$. Then there exist $p, q \in \{1, \dots, L\}$ with $x_p = x_q$ and $x_1, \dots, x_p, x_{q+1}, \dots, x_L$ of length at most $2^k + 1$ (if $q = L$, we replace x_p, x_{q+1} by x_{p-1}, x_L). The preceding argument permits the conclusion. □

14.ii. Lemma (extended dance remix). *The hypothesis on F are the same as in statement (2) of theorem 12. Let L be a whole number and $x_1, \dots, x_L$ a sequence of points in F. Then*

$$(x_1|x_L) \geq \min_{2 \leq i \leq L}(x_{i-1}|x_i) - (k+1)\delta - 2c$$

Proof. Suppose $x, y \in F$ and $i \in \{1, \dots, n\}$. On the one hand, $(x|y) \leq \min\{|x|, |y|\}$ by the triangle inequality, on the other hand

$$(x|y)_{w_i} - c \leq (x|y) \leq (x|y)_{w_i} + c$$

In particular, let $p, q \in \{1, \dots, L\}$ and $i \in \{1, \dots, n\}$ with $p < q$ and $x_p, x_q \in F_i$. Then $(x_p|x_q)_{w_i} \leq \min\{|x_p - w_i|, |x_q - w_i|\}$ and

$$(*) \qquad \begin{aligned} (x_p|x_q) &\geq \min\{|x_p|, |x_q|\} - 2c \\ &\geq \min\{(x_p|X_{p+1}), (x_{q-1}|X_q)\} - 2c \\ &\geq \min_{p+1 \leq j \leq q}\{(x_{j-1}|X_j)\} - 2c \end{aligned}$$

Furthermore, we can choose p and q so that x_p and x_q are the only points in the sequence $x_1, \dots, x_p, x_q, \dots, x_L$ which are in F_i.

We apply the preceding argument several times, as follows. First, if there is $j > 1$ with x_1 and x_j in the same F_i, we consider the largest such j and replace the sequence $x_1, \dots, x_L$ by $x_1, x_j, \dots, x_L$; if there is no such j, we don't do anything. Next, we do the same thing with the second term of the new sequence. And so on.

In the end, we obtain a sub-sequence $y_1 = x_1, y_2, \dots, y_M = x_L$ with the following two properties. For each i, there is at most two y_j in F_i. And, if there are two, they are consecutive: $y_j, y_{j+1} \in F_i$. In particular, $M \leq 2n$.

The argument in the short version of the lemma shows that

$$(x_1|x_L) \geq \min_{2 \leq j \leq M}(y_{j-1}|y_j) - (k-1)\delta$$

It follows from (*) that

$$(x_1|x_L) \geq \min_{2 \leq j \leq M}(x_{j-1}|x_j) - (k-1)\delta - 2c$$

whence the lemma. □

15. Lemma. *We define two maps of $F \times F$ into $\mathbb{R}_+$ by*
$(x|y)' = sup\{\min_{2 \le j \le L}(x_{j-1}|x_j)\}$
with the supremum taken over all chains $x = x_1, \dots, x_L = y$
$|x-y|' = |x| + |y| - 2(x|y)'$
and we set $\delta' = (k+1)\delta - 2c$. Then

1. $(x,y) \mapsto |x-y|'$*is a psuedo-metric on F.*
2. $|x-y| - 2\delta' \le |x-y|' \le |x-y|$ *for all $x, y \in F$.*
3. $|x|' = |x|$ *for all $x \in F$ (with $|x|' = |x-w|'$, where w is the base point of F).*
4. *The psuedo metric is 0-hyperbolic:*
 $(x|z)' \ge \min\{(x|y)', (y|z)'\}$ *for all $x, y, z \in F$.*

(Note that we do not exclude the possibility of two distinct points $x, y \in F$ with $|x-y|' = 0$.)

Proof.

1. We need to verify the triangle inequality. Let x, y, z be three points in F. The definition of the psuedometric implies that $|x-z|' \le |x-y|' + |y-z|'$ if and only if
$$(x|y)' + (y|z)' \le |y| + (x|z)'$$
We assume, WLOG, that $(x|y)' \le (y|z)'$.
Let $\epsilon > 0$. We choose sequences $x = u_1, u_2, \dots, u_L = y$ and $y = v_0, v_1, \dots, v_M = z$ with
$$(x|y)' \le \min_{2 \le j \le L}(u_{j-1}|u_j) + \epsilon$$
$$(y|z)' \le \min_{1 \le j \le M}(v_{j-1}|v_j) + \epsilon$$
We use $z_1, \dots, z_{L+M}$ to denote the sequence from x to z obtained by concatenating the two chosen sequences. Then
$$(x|z)' \ge \min_{2 \le j \le L+M}(z_{j-1}|z_j) \ge \min\{(x|y)', (y|z)'\} - \epsilon$$
Since we can do this for any $\epsilon > 0$, we have
$$(x|z)' \ge \min\{(x|y)', (y|z)'\} = (x|y)' \qquad (**)$$
On the other hand,
$$|y| \ge (y|v_1) = (v_0|v_1) \ge (y|z) - \epsilon$$
for all $\epsilon \ge 0$, hence $|y| \ge (y|z)$. This last inequality, along with (**), implies (*).
2. Let $x, y \in F$. We have $(x|y)' \ge (x|y)$, hence $|x-y|' \le |x-y|$. In view of lemma 14, we also have $(x|y) \ge (x|y)' - (k+1)\delta - 2c$, hence
$$|x-y|' \ge |x| + |y| + 2(x|y) - 2(k+1)\delta - 4c = |x-y| - 2(k+1)\delta - 4c$$
3. For each $x \in F$, we have $(x|w) = 0$, hence also $(x|w)' = 0$ and therefore $|x| = |x|'$.
4. Follows from the proof of (1).

□

Proof of 12.2. Let F' be the quotient of F by the equivalence relation

$$x \sim y \text{ if } |x-y|' = 0$$

Then $|\cdot|'$ defines a distance on F'. Let T be the real tree associated to F' as in proposition 6. The composition Φ of the natural maps $F \to F'$ and $F' \to T'$ has the desired properties. □

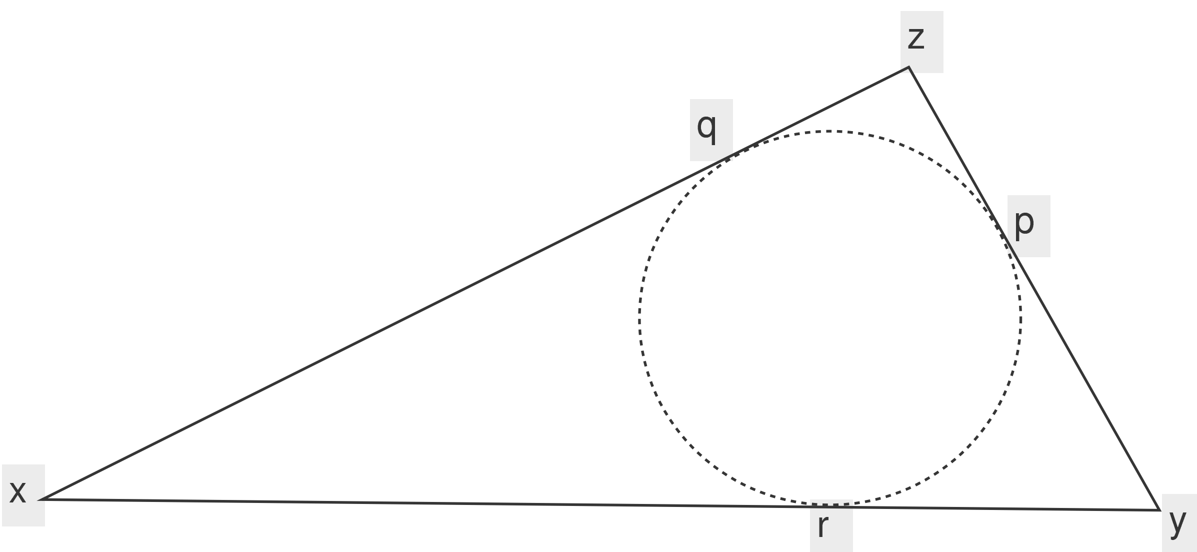
3. Definitions in terms of Triangles

Let x_0, X_1 be two points in a metric space X at a distance d from each other. If there exists a geodesic segment $g : [0, d] \to X$ from x_0 to x_1 (definition 1.25), recall that we often denote the image of g by $[x_0, x_1]$. This is abuse of notation, inasmuch as g is not, in general, uniquely defined by x_0 and x_1, but it is nonetheless very useful.

16. Definition Let $\triangle = [x,y] \bigcup [y,z] \bigcup [z,x]$ be a geodesic triangle in a metric space X. Let $T_\triangle$ be the tripod and $f_\triangle$ the isometry from proposition 2. Then $f_\triangle$ admits a unique extension (also denoted $f_\triangle$) to $\triangle$ which, restricted to each edge of $\triangle$, is an isometry. Given a real number $\delta \geq 0$, we say that the triangle $\triangle$ is δ-thin if $|u-v| \leq \delta$ for all $u, v \in \triangle$ with $f_\triangle(u) = f_\triangle(v)$ (an equivalent condition is that $|u-v| \leq |f_\triangle(u) - f_\triangle(v)| + \delta$ for all $u, v \in \triangle$).

17. Lemma. *Let $\triangle = [x,y] \bigcup [y,z] \bigcup [z,x]$ be a geodesic triangle in a metric space X; we use $d(,)$ to denote the distance in X.*

(1) *We have $(y|z)_x \leq d(x, [y,z])$*
(2) *If, furthermore, $\triangle$ is δ-thin, then $d(x,[y,z]) \leq (y|z)_x + \delta$*



Proof. We use $p \in [y,z]$, $q \in [z,x]$, and $r \in [x,y]$ to denote the three points in $\triangle$ mapped by $f_\triangle$ to the center of the tripod. We have $|q-x| = |r-x| = (y|z)_x$. We now show (1).

Take $w \in [y,z]$ with $|w - x| = d(x,[y,z])$. There is a point $w' \in [x,y] \bigcup [z,x]$ which has the same image as w under $f_\triangle$. WLOG, we assume $w' \in [z,x]$. Then

$$(y|z)_x \leq |w' - x| = |z - x| - |z - w| \leq |x - w| = d(x,[y,z])$$

Under the hypothesis of (2), we also have

$$d(x,[y,z]) \leq d(x,q) + d(p,q) \leq (y|z)_x + \delta$$

□

18. Definitions. Let $\triangle$ and $f_\triangle$ be as in definition 16. The *inscribed triple* of $\triangle$ is the preimage of the center of the tripod and the *insize* of $\triangle$ is the diameter of the inscribed triple.

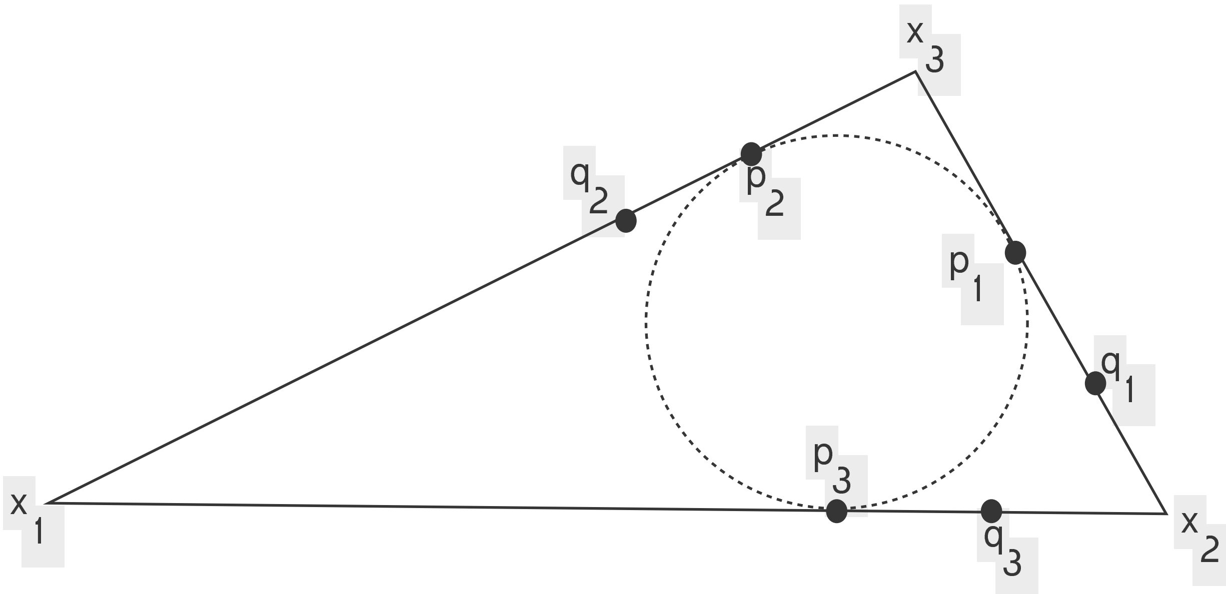
The *minsize* of $\triangle$ is the minimum of the diameters of set $\{u,v,w\}$ with $u \in [y,z]$, $v \in [z,x]$, and $w \in [x,y]$.

19. Remarks.

(1) Let X be a simply connected surface given a complete riemannian structure with constant non-positive curvature, and let $\triangle$ be a geodesic triangle in X. The points of the inscribed triple are the tangent points of $\triangle$ with and inscribed circle; the insize of $\triangle$ is less than the diameter of this circle.
(2) Let X, Y be two metric spaces and let $g : X \to Y$ be a map which takes geodesics to geodesics. Suppose that g is a contraction ($|g(x) - g(x')| \leq |x - x'|$ for all $x, x' \in X$). If $\triangle$ is a geodesic triangle in X, the minsize of $g(\triangle)$ is less than the minsize of $\triangle$. It follows from the lemma below that the insize of $g(\triangle)$ is less than 4 times the insize of $\triangle$.

20. Lemma. *Let $\triangle = [x_1,x_2] \bigcup [x_2,x_3] \bigcup [x_3,x_1]$ be a geodesic with minsize δ and insize δ'. Then $\delta \leq \delta' \leq 4\delta$.*

Proof. The inequality $\delta \leq \delta'$ follows from the definitions; it remains to show that $\delta' \leq 4\delta$.



Suppose $p_1 \in [x_2, x_3]$, $p_2 \in [x_3, x_1]$, and $p_3 \in [x1, x_2]$ are the points of the inscribed triple of $\triangle$. Let $q_1 \in [x_2, x_3]$, $q_2 \in [x_3, x_1]$, and $q_3 \in [x1, x_2]$ be the points such that $\delta = \mathrm{diam}\{q_1, q_2, q_3\}$.

Let (i, j, k) be a permutation of $(1, 2, 3)$. We set

$$a_i = |x_i - x_j| \qquad b_{i,k} = |p_i - x_k| \qquad c_{i,k} = |q_i - x_k|$$

Then

$$b_{i,k} = b_{j,k} = \frac{1}{2}(a_i + a_j - a_k)$$
$$b_{i,j} + b_{i,k} = c_{i,j} c_{i,k} = a_i$$

by definition. We have $|q_i - q_j| \leq \delta$, whence, by the triangle inequality applied to $\{q_1, q_2, q_3\}$, we have $|c_{i,k} - c_{j,k}| \leq \delta$. We therefore get, in succession,

$$(*) \qquad \begin{aligned} &2b_{i,k} = a_i + a_j - a_k = c_{i,j} + c_{i,k} + b_{j,i} + b_{j,k} - c_{k,i} - c_{k,j} \\ &c_{i,j} + c_{i,k} + b_{k,i} - b_{i,k} - c_{k,i} - c_{k,j} = 0 \\ &|c_{i,k} - b_{i,k} - c_{k,i} + b_{k,i}| = |c_{i,j} - c_{k,j}| \leq \delta \end{aligned}$$

From now on, we assume that (i, j, k) is a circular permutation of $(1, 2, 3)$ and define

$$d_i = c_{i,j} - b_{i,j} = -(c_{i,j} - b_{i,k})$$

On one hand, $|d_i| = |p_i - q_i|$. On the other hand, the inequality (*) (and the related inequalities obtained via circular permutations of (i, j, k)) can be written

$$| - d_i - d_k| \leq \delta$$

$$|-d_j - d_i| \leq \delta$$

$$|-d_k - d_j| \leq \delta$$

In this way,

$$|d_i| = \frac{1}{2}|d_i + d_j + d_i + d_k - d_j - d_k| \leq \frac{3}{2}\delta$$

Thus, one finally has

$$|p_j - p_k| \leq |p_j - q_j| + |q_j - q_k| + |q_k - p_k| \leq 4\delta$$

This says that $\delta' \leq 4\delta$. □

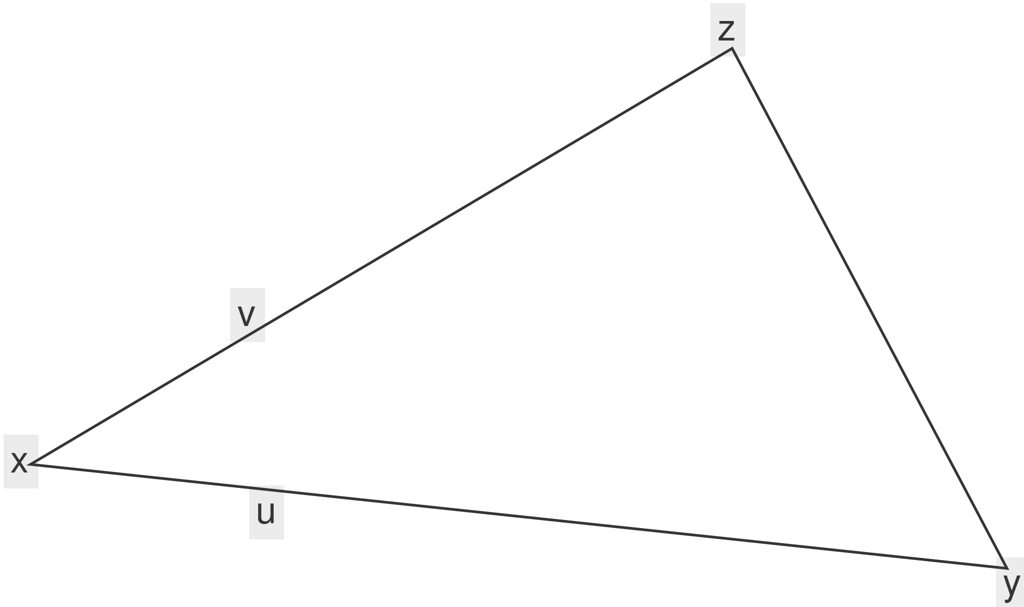
21. Proposition. *Let X be a geodesic metric space. We consider the properties below on X, where δ is a positive real number.*
(P_1, δ) The space X is δ-hyperbolic (definition 3).
(P_2, δ) All geodesic triangles in X are δ-thin.
(P_3, δ) The space X satisfies the Rips condition with constant δ (definition 1.25) : For every geodesic triangle $\triangle = [x,y] \bigcup [y,z] \bigcup [z,x] \subset X$, and for all $u \in [x,y]$, we have $d(u, [y,z] \bigcup [z,x]) \leq \delta$.
(P_4, δ) Every geodesic triangle in X has insize less than δ.
(P_5, δ) Every geodesic triangle in X has minsize less than δ. Then, except for a slight change in constants, the properties (P_1, δ) to (P_5, δ) are equivalent. More precisely, for all $i, j \in \{1, \ldots, 5\}$, there is a constant $c_{i,j}$ with $1 \leq c_{i,j} \leq 4$ such that the following assertion is true:
Let $\delta, \delta' \geq 0$ be given with $\delta' = c_{i,j}\delta$; If X satisfies
(P_i, δ), then X also satifies (P_j, δ).

Proof. The implications $(\mathrm{P}_2, \delta) \Rightarrow (\mathrm{P}_3, \delta)$ and $(\mathrm{P}_2, \delta) \Rightarrow (\mathrm{P}_4, \delta) \Rightarrow (\mathrm{P}_5, \delta)$ are obvious; the implication $(\mathrm{P}_5, \delta) \Rightarrow (\mathrm{P}_4, 4\delta)$ follows from lemma 20. We show the following :

$(\mathrm{P}_1, \delta) \Rightarrow (\mathrm{P}_2, 4\delta)$ $(\mathrm{P}_2, \delta) \Rightarrow (\mathrm{P}_1, 2\delta)$

$(\mathrm{P}_3, \delta) \Rightarrow (\mathrm{P}_2, 4\delta)$ $(\mathrm{P}_4, \delta) \Rightarrow (\mathrm{P}_2, 2\delta)$ □

Proof of. $(\mathrm{P}_1, \delta) \Rightarrow (\mathrm{P}_2, 4\delta)$. Let $\triangle = [x,y] \bigcup [y,z] \bigcup [z,x]$ be a geodesic triangle in X and $f : \triangle \to T$ the map from definition 16. Suppose $u, v \in \triangle$ with $u \neq v$ and $f(u) = f(v)$. We must show $|u - v| \leq 4\delta$.



WLOG, $u \in [x, y]$ and $v \in [z, x]$. if $t = |x - u|$, we have
$|f(x) - f(u)| = |f(x) - f(v)| = t \leq (y|z)_x$
$(u|y)_x = (v|y)_x = t$ so

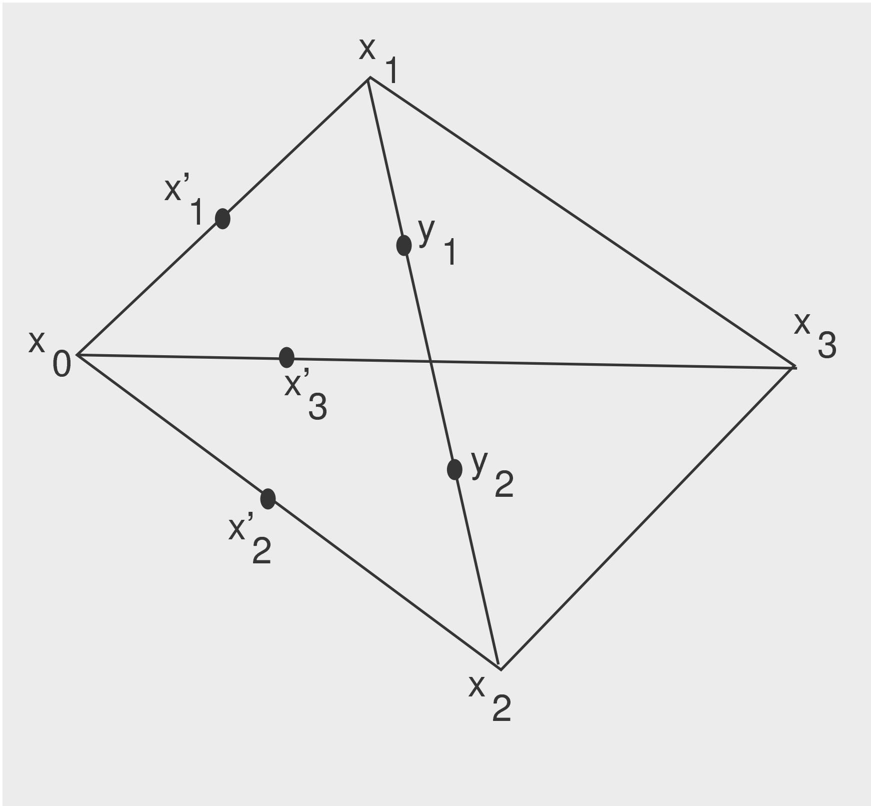
$$(u|v)_x \geq \min\{(u|y)_x, (y|z)_x, (z|v)_x\} - 2\delta = t - 2\delta$$

But $(u|v)_x = t - \frac{1}{2}|u - v|$, and therefore

$$|u - v| \leq 2t - 2(t - 2\delta) = 4\delta$$

□

Proof of $(P_2, \delta) \Rightarrow (P_1, 2\delta)$. We consider 4 points $x_0, x_1, x_2, x_3 \in X$ and six geodesic segments connecting them two by two.



The Gromov Products are taken relative to x_0. We define $t = \min\{(x_1|x_3), (x_2|x_3)\}$ and we must show that $(x_1|x_2) \geq t - 2\delta$.

Furthermore, we may assume that $t \geq (x_1|x_2)$ as without this there is nothing to show.

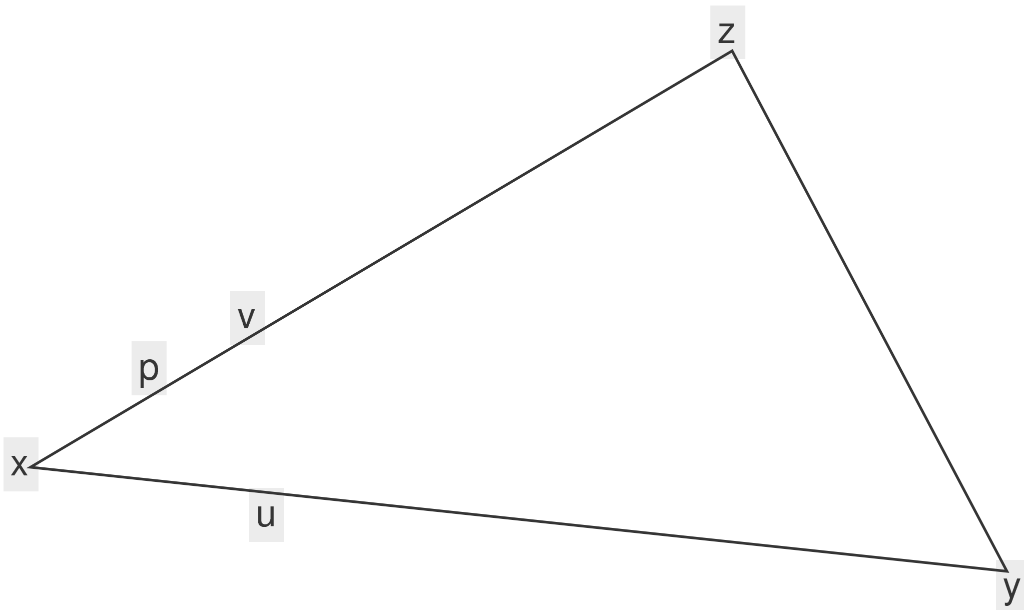
For $j \in \{1, 2, 3\}$, we denote the point on $[x_0, x_j]$ with distance t from x_0 by x'_j. For $j \in \{1, 2\}$, we denote the map of definition 16 on the triangle $[x_0, x_j] \bigcup [x_j, x_3] \bigcup [x_1, x_0]$ by f_{0j3}. We have $|x'_j - x_0| = |x'_3 - x_0| \leq (x_j|x_3)$ and hence $f_{0j3}(x'_j) = f_{0j3}(x'_3)$ and $|x'_j - x_3| \leq \delta$. It follows that $|x'_1 - x'_2| \leq 2\delta$.

Since $t \geq (x_1|x_2)$, there is a point $y_j \in [x_1, x_2]$ with $f_{012}(x'_j) = f_{012}(y_j)$; this implies $|x'_j - y_j| \leq \delta$. We have

$$\begin{aligned} 2\delta \geq |x'_1 - x'_2| \geq |y_1 - y_2| - 2\delta &= |x_1 - x_2| - |x_1 - y_1| - |x_2 - y_2| - 2\delta \\ &= |x_1 - x_2| - (|x_1 - x_0| - |x'_1 - x_0|) - (|x_2 - x_0| - |x'_2 - x_0|) - 2\delta \\ &= 2t - 2(x_1|x_2) - 2\delta \end{aligned}$$

and therefore $(x_1|x_2) \geq t - 2\delta$. □

Proof of $(P_3, \delta) \Rightarrow (P_2, 4\delta)$. Suppose the implication is false.



Then there is a geodesic triangle $\triangle = [x,y]\bigcup[y,z]\bigcup[z,x]$ in X and points $u \in [x,y]$, $v \in [z,x]$ with
$|u-x| = |v-x| = t < (y|z)_x$
$|u-v| > 4\delta$
But

$$\begin{aligned} d(v,[x,y]) &= \min\{d(v,[x,u]), d(v,[u,y])\} \\ &\geq \min\{(x|u)_v, (u|y)_v\} \end{aligned}$$

by lemma 17, and

$$\begin{aligned} 2(x|u)_v &= |u-v| \\ 2(u|y)_v &= |u-v| + |y-v| - (|x-y| - |x-u|) \\ &= |u-v| + (|y-v| + |v-x| - |x-y|) \geq |u-v| \end{aligned}$$

hence

$$d(v,[x,y]) \geq \frac{1}{2}|u-v| > 2\delta$$

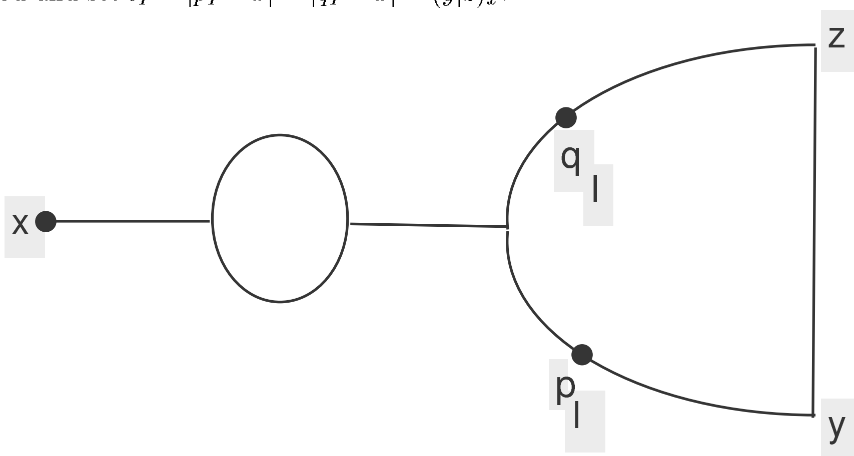
In particular, $|v-x| > 2\delta$ and there exists $p \in [x,v]$ with $|p-v| = \delta$. We have

$$\begin{aligned} d(p,[x,y]) &\geq d(v,[x,y]) - |v-p| > \delta \\ d(p,[y,z]) &\geq d(x,[y,z]) - |x-p| \geq (y|z)_x - |x-p| \\ &> t - |x-p| = |v-x| - |x-p| = |p-v| = \delta \end{aligned}$$

Therefore $d(p,[x,y]\bigcup[y,z]) > \delta$, which contradicts the hypothesis. □

Proof of $(P_4,\delta) \Rightarrow (P_2, 2\delta)$.

WLOG, $\delta > 0$. Let $\triangle = [x,y] \bigcup [y,z] \bigcup [z,x]$ be a geodesic triangle in X. We use $p_I \in [x,y]$ and $q_I \in [z,x]$ to denote the points mapped by $f_\triangle$ to the center of the tripod and set $t_I = |p_I - x| = |q_I - x| = (y|z)_x$.



For every number $s \in [0, t_I]$, we denote the point in $[x,y]$ with distance s from x by p_s and denote the point in $[z,x]$ with distance s from x by q_s. We must show that $|q_s - p_s| \le 2\delta$.

Step One. We use induction to define a sequence of points $(p_i)_{i \ge 1}$ in $[x, p_I]$ and a sequence of points $(q_i)_{i\ge 1}$ in $[x, q_i]$. First, set $p_1 = p_I$ and $q_1 = q_I$. Next, if p_i and q_i are already defined, we consider the canonical map f_i frmo the triangle $[x,p_i] \bigcup [p_i, q_i] \bigcup [q_i, x]$ to the corresponding tripod T_i. Then $p_{i+1} \in [x, p_i]$ and $q_{i+1} \in [x, q_i]$ are the points mapped by f_i to the center of T_i. For each $i \ge 1$, we have $|p_i - q_i| \le \delta$ by hypothesis, hence

$$(*) \qquad |p_i - p_{i+1}| + |q_i - q_{i+1}| = |p_i - q_i| \le \delta$$

Since $|x - p_{i+1}| = |x - q_{i+1}| \le |x - p_i| = |x - q_i|$, the p_i's have a limit $p_\infty \in [x, p_I]$ and the q_i's have a limit $q_\infty \in [x, q_I]$. We set $t' = |x - p_\infty|$ and observe that $p_\infty = q_\infty$. Let $s \in [t'_I, t_I]$. There is $i \ge 1$ with $|p_i - p_s| = |q_i - q_s| \le \frac{1}{2}\delta$ by (*). If $t'_I \le \delta$, there is nothing more to show.

Step Two. If $t'_I > \delta$, we define $t_{II} = t'_I - \frac{\delta}{2}$ and use $p_{II} \in [x, p_I]$, $q_{II} \in [x_{q_I}]$ to denote the points with distance t_{II} from x. The preceding argument produces sequences $(p_i)_{i\ge 1}{}^{(}2)$ and $(q_i)_{i \ge 1}{}^{(}2)$ which convergely, respectively, to $p_\infty{}^{(}2) = q_\infty{}^{(}2)$ with distance t'_{II} from x and, in this way, we show that $|p_s - q_s| \le 2\delta$ for all $s \in [t'_{II}, t_{II}]$.

In this way, in a finite number of steps (less than $\frac{\delta}{2} t_I + 1$), we show that $|p_s - q_s| \le 2\delta$ for all $s \in [0, t_I]$. □

22. Corollary and Example. *The Poincare Disk D^2 is a hyperbolic space.*

Proof. Let δ be a disk in D^2 with area π. The Gauss-Bonnet formula shows that the area of any geodesic triangle $\triangle$ in D^2 is less than π. It follows that the diameter of an inscribed circle is less than δ, and therefore, that $\triangle$ is δ-thin. □

23. Exercise. Show that every triangle in D^2 is δ-thin with $\delta = \log(3)$.

We end this chapter by examining the relationship between hyperbolicity and convexity.

24. Definition. Let X be a geodesic metric space. We consider a number $\delta \geq 0$ and a geodesic segment $g : [0, \delta] \to X$ with origin x_0 and endpoint x_1. The *natural parametrisation* of this segment is the map $t \mapsto x_t$ from $[0, 1]$ into the image of g defined by $|x_t - x_0| = t|x_1 - x|0|$ for all $t \in [0, 1]$.

Given a real number $\delta \geq 0$, we say that X is δ-convex if

$$|x_t - y_t| \leq (1-t)|x_0 - y_0| + t|x_1 - y_1| + \delta$$

for every pair $[x_0, x_1]$ and $[y_0, y_1]$ of naturally parametrised geodesic segments in X.

A 0-convex space is called *convex*. A complete, non-positively curved riemannian manifold is convex (see theorem 1.3 of [BGS]).

25. Proposition. *A geodesic space with δ-thin triangles is 2δ-convex.*

Proof. Let $[x_0, x_1]$ and $[y_0, y_1]$ be two naturally paramtrised geodesic segments in X.

Special Case. Suppose $x_0 = y_0$. Let T be the tripod and $f : [x_0, x_1] \bigcup [y_0, y_1] \to T$ as in definition 16. Since T is convex, we have

$$|x_t - y_t| \leq |f(x_t) - f(y_t)| + \delta \leq t|f(x_1) - f(y_1)| + \delta = t|x_1 - y_1| + \delta$$

General Case. Let $t \mapsto z_t$ be the naturally parametrised geodesic segment such that $z_0 = x_0$ and $z_1 = y_1$. Then

$$|x_t - y_t| \leq |x_t - z_t| + |z_t - y_t| \leq t|x_1 - y_1| + \delta + (1-\delta)|x_0 - y_0| + \delta$$

in view of the special case. $\square$

The converse of this proposition is not true! For example, the Euclidean plane is convex and not hyperbolic.

Translated by WILLIAM GROSSO

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CA 94720

NEGATIVELY CURVED SPACES AND HYPERBOLIC GROUPS (CHAPTER 3)

M. Troyanov

Abstract. We have seen in 2.22 that the Poincare disk is a hyperbolic space in the sense of Gromov. Using classical comparison techniques (reviewed in §1) we generalise this result to manifolds with negative curvature; more precisely, we show that all complete, simply connected riemannian manifolds with curvature less than a number $k < 0$ are hyperbolic. In §3, we show that the fundamental group of a compact riemannian manifold M is quasi-isometric to the universal cover $\tilde{M}$. We deduce (thanks to the invariance of hyperbolicity under quasi-isometry established in chapter 5) that the fundamental group of a compact negatively curved riemannian manifold is a hyperbolic group (theorem 24). The last section of this chapter presents several ways to define "convexity"

1. Comparison of Geometries

In this section, we discuss the method of "comparison of geometries" due essentially to A.D. Aleksandrov (cf [Ale], [ABN]).

We denote the usual plane with constant curvature $k \leq 0$ by H_k. When $k = 0$, this is the Euclidean plane and a homothety of the Poincare disk when $k < 0$. We simply use H for the plane H_{-1}.

Let X be a geodesic space and let $\triangle = [w, x] \cup [x, y] \cup [y, w]$ be a geodesic triangle in X (definition 1.25). We say that a triangle $\triangle^* = [w^*, x^*] \cup [x^*, y^*] \cup [y^*, w^*]$ in H_k is a comparison triangle for $\triangle$ if

$$|x^* - w^*| = |x - w| \qquad |y^* - x^*| = |y - x| \qquad |w^* - y^*| = |w - y|$$

It is easy to see that a comparison triangle always exists and it is unique up to an isometry of H_k. It is convenient to introduce the *comparison map* between a triangle $\triangle$ of X and a comparison triangle $\triangle^*$ of H_k; it is the map $f : \triangle \to \triangle^*$ such that $f(w) = w^*$, $f(y) = y^*$, $f(x) = x^*$, and f, restricted to each edge of $\triangle$, is an isometry. If $z \in \triangle$, we often denote $f(z) \in \triangle^*$ by z^* and name this the "point corresponding" to z.

Following Aleksandrov, we also introduce the notion of *comparison angles.* Suppose $g : [0, a] \to X$ and $h : [0, b] \to X$ are two geodesic segments parametrised by arc length and beginning at some point $w \in X$. For all s, t such that $0 \leq s \leq a$ and $0 \leq t \leq b$, we choose a segment $[g(s), h(t)]$ and we denote the triangle $g([0, s]) \cup [g(s), h(t)] \cup f([0, t])$ by $\triangle(s, t)$. We use $\alpha^k{}_{g,h}(s, t)$ (or, more simply,

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

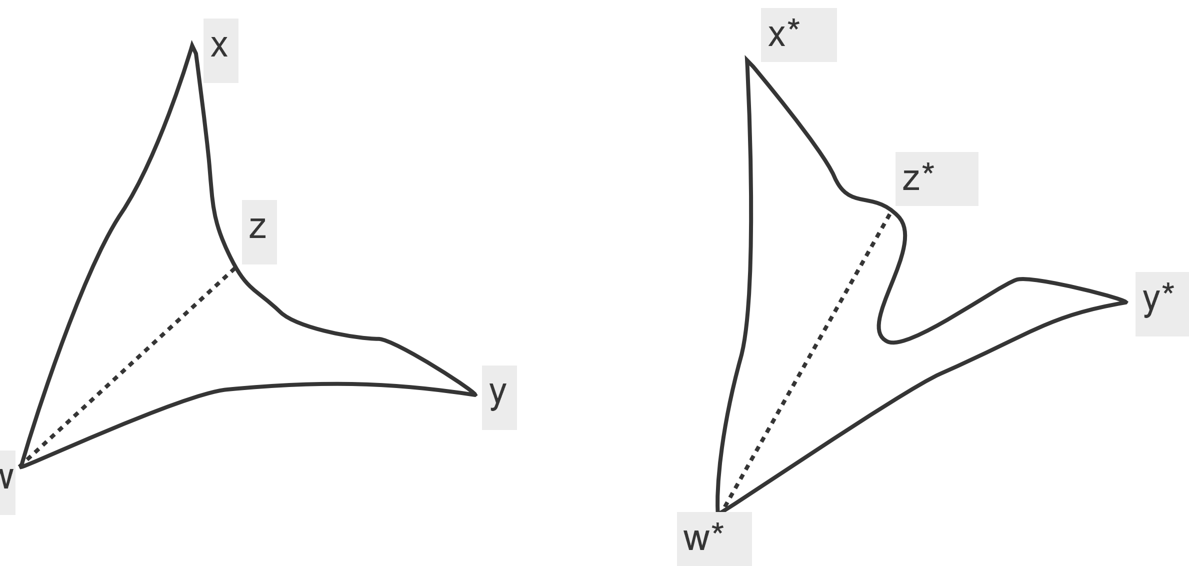
$\alpha(s,t)$ if g, h, and k are fixed) to denote the angle at w^* of a comparison triangle $\triangle^* \subset H_k$ of $\triangle(s,t)$. We call $\alpha^k{}_{g,h}(s,t)$ the comparison angle of the segments g and H (note that this "angle" is a function of 2 variables, and this function is constant in the special case when $X = H_k$). We observe that this definition is independent of the choice of segment $[g(s), h(t)]$.

We define three criteria for comparing the geometry of a geodesic space X and that of H_k

1. Condition C. We say that X satisfies the condition (C_k) if, for every triangle $\triangle = [w,x] \cup [x,y] \cup [y,w]$ of X and every point $z \in [x,y]$, we have

$$|z - w| \leq |z^* - w^*|$$

where $\triangle^* \subset H_k$ is a comparison triangle for $\triangle$ and z^* is the point of $\triangle^*$ corresponding to z.

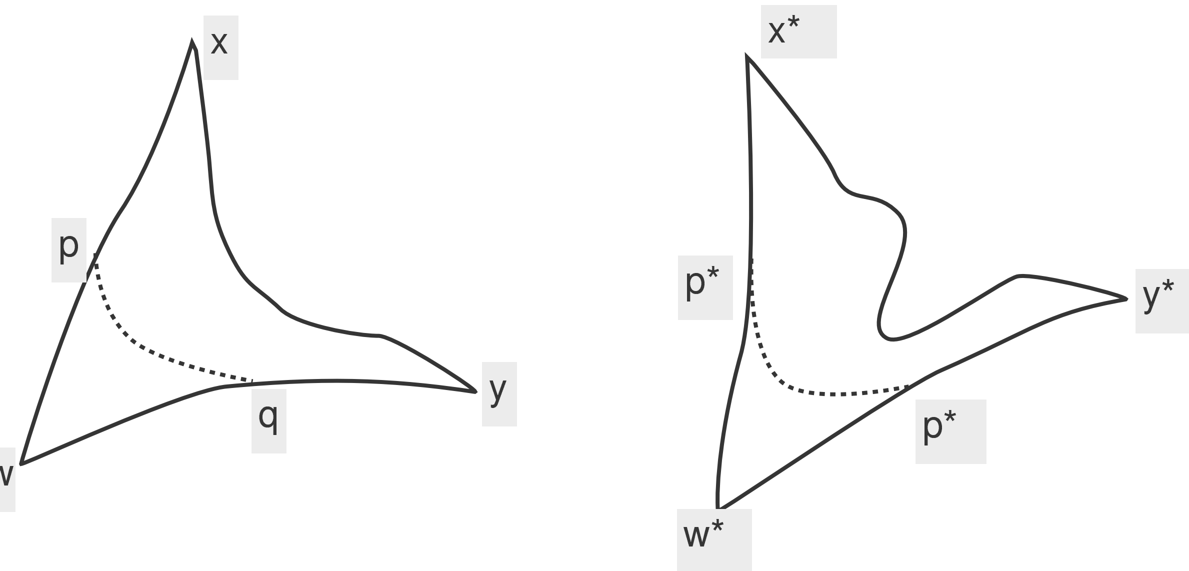


2. Condition A. The space X meets the criterion (A_k) if $\alpha^k{}_{g,h}(s,t)$ is a monotone non-decreasing function of its arguments s,t for all geodesic segments $g : [0,a] \to X, h : [0,b] \to X$ beginning from the same point $w \in X$.

3. Condition T. We say that X satisfies condition (T_k) if, for every triangle $\triangle = [w,x] \cup [x,y] \cup [y,w]$ of X and all points $p \in [w,x], q \in [w,y]$ we have

$$|p - q| \leq |p^* - q^*|$$

where $\triangle^* \subset H_k$ is a comparison triangle and p^*, q^* are the points of $\triangle^*$ corresponding to p, q



4. Theorem. *Let X be a geodesic space and $k \leq 0$. The conditions $(C_k), (A_k)$ and (T_k) are equivalent.*

5. Definition. We say that X is CAT_k if the criteria $(C_k), (A_k)$ and (T_k) are satisfied. (CAT can be read as "criteria of Camparison of Aleksandrov-Topoganov").

6. Example. If X is a metric tree, then X satisfies CAT_k for all k.

We sometimes says that X has *curvature bounded by k* if every point in X possesses a convex neighborhood satifying CAT_k.

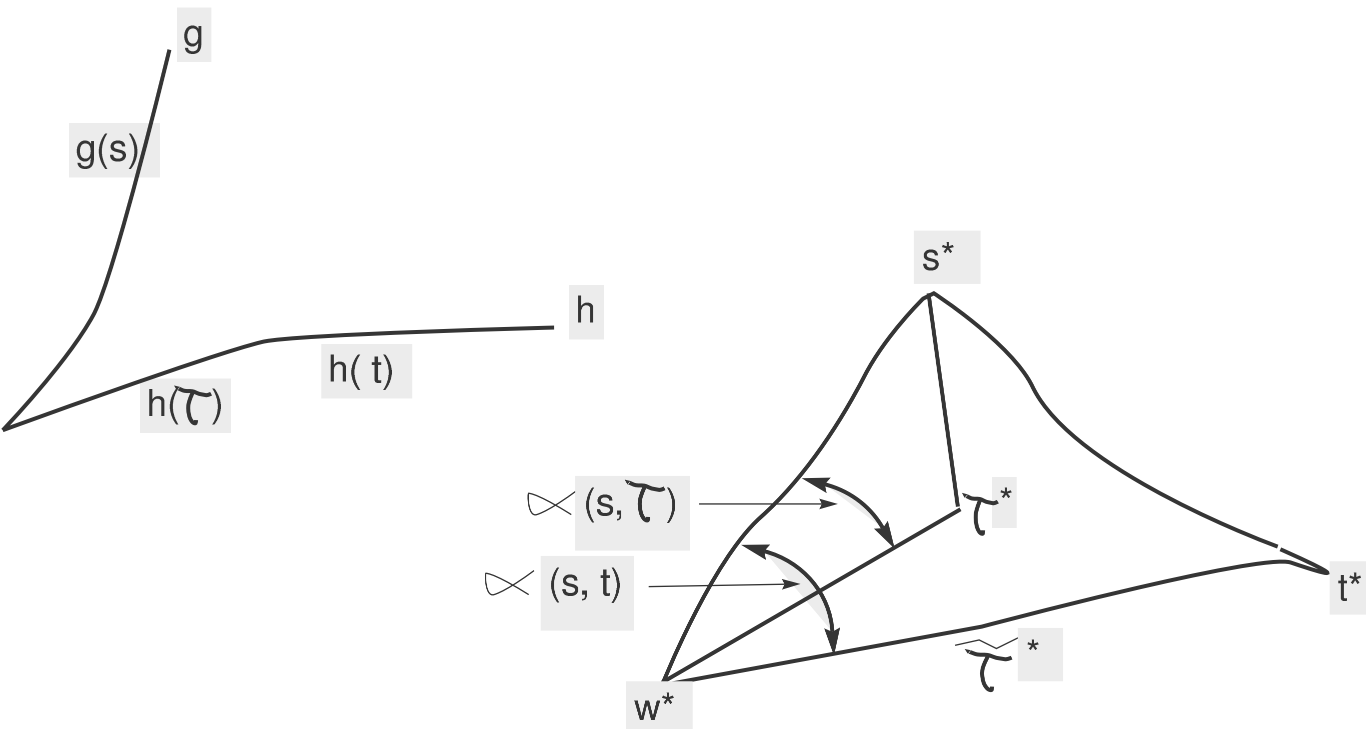
7. Proposition. *Suppose X is a geodesic space which is CAT_k for a fixed $k < 0$. ThenX is hyperbolic.*

Proof. We know that there exists a $\delta \geq 0$ such that all the triangles in H_k are $\delta - thin$ (corollary 2.22). It follows immediately from condition (T_k), that the triangles of X are also $\delta - thin$; proposition 2.22 then shows the hyperbolicity of X. $\square$

Proof of Theorem 4. To simplify notation, in this proof we assume that $k = -1$; there are no essential changes in the other cases.

$(C_k) \Rightarrow (A_k)$. Consider a point $w \in X$ and two geodesic segments $g : [0, a] \to X, h : [0, b] \to X$ beginning at w. Suppose s, τ, t are such that $0 \leq s \leq a$ and $0 \leq \tau \leq t \leq b$. We denote the triangle with vertices $w, g(s), h(t)$ by $\triangle(s, t)$. We must show that $\alpha(s, \tau) \leq \alpha(s, t)$ when $\alpha = \alpha^{-1}{}_{g,h}(s, t)$ is the comparison angle of the segments g and h. We choose 5 points $w^*, s^*, t^*, \tau^*, \tilde{\tau}^*$ in the Poincare disk such that

(1) w^*, s^*, t^* are the vertices of a comparison triangle for $\triangle(s, t)$.
(2) w^*, s^*, τ^* are the vertices of a comparison triangle for $\triangle(s, \tau)$.
(3) $\tilde{\tau}$ is a point on the segment $[w^*, t^*]$ such that $|w^* - t^*| = |w^* - \tilde{\tau}^*| = \tau$.



By definition, $\alpha(s,\tau)$ is the angle at w^* of the segments $[w^*,s^*]$ and $[w^*,\tau^*]$. These segments have length s and τ.; we therefore have the cosine formula

$$\cos(\alpha(s,\tau)) = \frac{\cosh(s) cosh(\tau) - cosh(|s^* - \tau^*|)}{\sinh(s)\sinh(\tau)}$$

On the other hand, $\alpha(s,t)$ is angle at w^* of the segments $[w^*,s^*]$ and $[w^*,t^*]$ which is also the angle at w^* of the segments $[w^*,s^*]$ and $[w^*,\tilde{\tau}^*]$; these segments also have lengths s and τ. Therefore

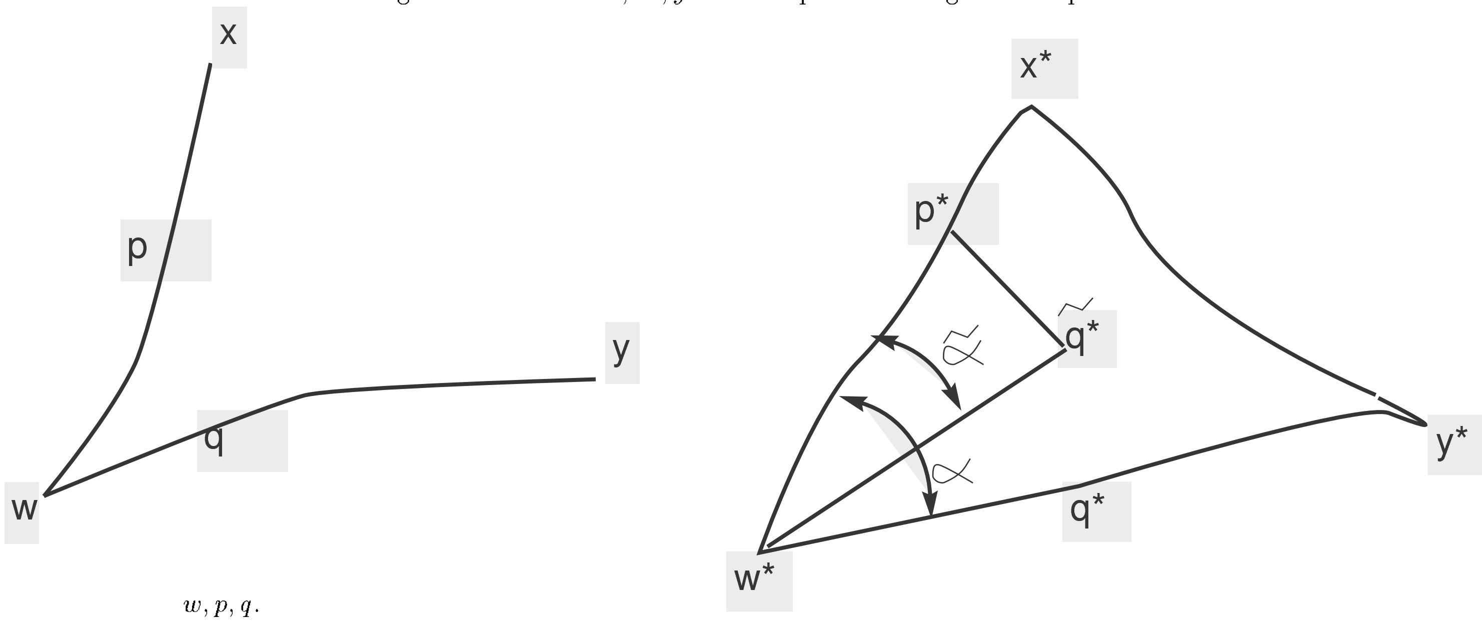
$$\cos(\alpha(s,t)) = \frac{\cosh(s)\cosh(\tau) - \cosh(|s^* - \tau^*|)}{\sinh(s)\sinh(\tau)}$$

Now the condition C_{-1} implies

$$|s^* - \tilde{\tau}^*| \geq |s^* - \tau^*| (= |g(s) - g(t)|)$$

It follows that $\cos(\alpha(s,\tau)) \geq \cos(\alpha(s,t))$ and therefore $\alpha(s,\tau) \leq \alpha(s,t)$.

$(C_k) \Rightarrow (A_k)$. The proof is the same type of argument (trigonometry shows us that an inequality in angles forces an inequality in distances). Let $\triangle = [w,x] \cup [x,y] \cup [y,w]$ be a triangle in X and $p \in [w,x]$, $q \in [w,y]$ two points in $\triangle$. Consider a comparison triangle $\triangle^*$ with vertices w^*, x^*, y^* in the Poincare disk. We denote the points corresponding to p and q by $p^*, q^* \in \triangle^*$. Suppose also that $\tilde{q}^*$ is a point in H such that the triangle with vertices w^*, x^*, y^* is a comparison triangle for the points



w, p, q.
Denote the angle at w^* of the triangle $\triangle^*$ by α and the angle of the segment $[w^*, p^*]$ and $[w^*, \tilde{q}^*]$ by $\tilde{\alpha}$. The condition A_{-1} forces $\tilde{\alpha} \leq \alpha$. Observe, on the other hand, that α is also the angle at w^* between the segments $[w^*, p^*]$ and $[w^*, q^*]$, therefore

$$\cos(\alpha) = \frac{\cosh(|w^* - p^*|)\cosh(|w^* - q^*|) - \cosh(|p^* - q^*|)}{\sinh(|w^* - p^*|)\sinh(|w^* - q^*|)}$$

and

$$\cos(\alpha) = \frac{\cosh(|w^* - p^*|)\cosh(|w^* - \tilde{q}^*|) - \cosh(|p^* - \tilde{q}^*|)}{\sinh(|w^* - p^*|)\sinh(|w^* - \tilde{q}^*|)}$$

Since $|w^* - q^*| = |w^* - \tilde{q}^*$ and $\cos(\tilde{\alpha}) \geq \cos(\alpha)$, we see that

$$|p - q| = |p^* - \tilde{q}^*| \leq |p^* - q^*|$$

$(T_k) \Rightarrow (C_k)$. Obvious. □

8. Remark. If $k > 0$ then we define the conditions (C_k) and (T_k) as when $k \leq 0$, but only test those triangles with perimeter $\leq \frac{2\pi}{\sqrt{k}}$ (the comparison triangles are constructed on the sphere of radius $\frac{1}{\sqrt{k}}$ in$\mathbb{R}^3$). Similarly, to define (A_k), we only consider those segments $g : [0, a] \to X$ and $h : [0, b] \to X$ such that $a + B + |g(a) - h(b)| \leq \frac{2\pi}{\sqrt{k}}$. Theorem 4 is still true when $k > 0$.

2. The Case of Riemannian Manifolds

The goal of this section is to give a complete proof of the theorem, due to Aleksandrov, below (the theorem follows, for example, from theorem 3.2 and corollary 5.1 of [ABN]).

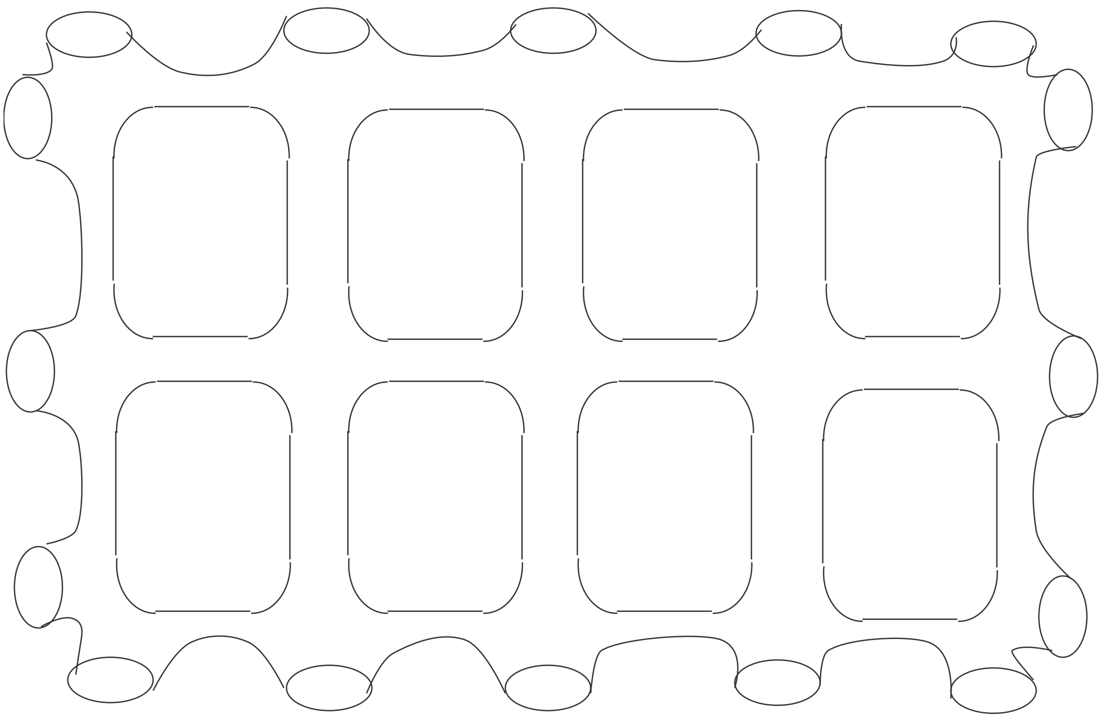
9. Theorem. *Let (M, g) be a simply connected, complete riemannian manifold which has sectional curvature $K \leq k \leq 0$. Then M is CAT_k.*

By proposition 7, it immediately follows that

10. Corollary. *A simply connected, complete, riemannian manifold with curvature $K \leq k < 0$ is hyperbolic (in the sense of Gromov).*

11. Remarks.

(1) Theorem 9 implies that all CAT_k geodesic spaces are also $CAT_{k'}$ for all $k' > k$ (indeed, the theorem tells us that H_k is $CAT_{k'}$, therefore $(T_k) \Rightarrow (T_{k'})$).
(2) In the theorem, and the corollary, one can replace M by any convex subset of M.
(3) The theorem (but not the corollary), if also true if $k > 0$. CAT_k is then defined as in remark 8.
(4) If we omit the hypothesis that M is simply connected, the theorem is clearly false (a complete, negatively curved riemannian manifold which is not simply connected always has a simple closed geodesic and therefore never meets CAT_0).
(5) Corollary 10 is also false when M is not simply connected. Here is an example of a complete negatively curved riemannian manifold which is not hyperbolic: Let Q be the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = 0$ and either x or y is an integer. Then there is an infinite grating invariant under the group $\mathbb{Z}^2$ (acting by translations).Let $U \subset \mathbb{R}^3$ be a $\mathbb{Z}^2$ invariant neighborhood of Q such that for all $u \in U$, $d(u, Q) \leq \frac{1}{4}$. Let $M = \partial U$; M is a smooth surface of infinite genus which is quasi-isometric to $\mathbb{Z}^2$ (or to $\mathbb{R}^2$) and is therefore not hyperbolic (theorem 5.12). Meanwhile, $M/\mathbb{Z}^2$ is a closed surface of genus 2; consequently M has a riemannian metric of constant negative curvature This example generalises (via corollary 21, below) to covers of a compact, negatively curved manifold with covering group $\mathbb{Z}^m$ $(m \geq 2)$.



The proof of theorem 9 relies on Aleksandrov's comparison theorem. Although it is a classical theorem, it is not in introductory textbooks. We find it, for example, in [Ale], [Gve, thm 3.9] or [Tsu]. Grove calls it the "Inverse Topogonov Theorem".

12. Theorem (Aleksandrov Comparison Theorem). *Suppose (M, g) is a simply connected, complete riemannian manifold with sectional curvature $K \leq k < 0$. let $\triangle$ be a geodesic triangle in M and let $\triangle^*$ be a comparison triangle in H_k; we denote the angles of $\triangle$ by α, β, γ and the corresponding angles of $\triangle^*$ by $\alpha^*, \beta^*, \gamma^*$. Then*

$$\alpha \leq \alpha^* \qquad \beta \leq \beta^* \qquad \gamma \leq \gamma^*$$

To prove this theorem, we need the following lemma.

13. Lemma. *Let (M, g) be a complete riemannian manifold and let $V = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq r \leq \infty, 0 \leq \theta \leq a\}$. Choose a point $w \in M$. as well as a C^2 curve $\lambda : [0, a] \to T_w M$ such that $|\lambda(\theta)| = 1$ and $|\frac{\partial \lambda}{\partial \theta}| = 1$ for all θ. Let $\phi : V \to M$ be the map defined by $\phi(r, \theta) = \exp_w(r\lambda(\theta))$. Then*

$$\phi^*(g) = dr^2 + p^2(r, \theta) d\theta^2$$

where $p : V \to R$ is a function such that $p(0, \theta) = 0$ and $\frac{\partial p}{\partial r} = 1$ for all θ.

Proof of lemma 13. We denote $\phi(V) \subset M$ by F; let R and T be the vector fields on F defined by $R = d\phi(\frac{\partial}{\partial r})$ and $T = d\phi(\frac{\partial}{\partial \theta})$. Then, we have

$$\phi^*(g) = q(R, R) dr^2 + 2g(R, T) dr d\theta + g(T, T) d\theta^2$$

By the definition of the exponential map, we have $g(R,R) = 1$. Gauss' lemma (cf. [Gve], lemma 2.11) says that $g(R,T) = 0$. The function $p(r,\theta) \equiv g_{\phi(r,\theta)}(T,T)$ vanishes at $r = 0$ since $d_{\phi(0,\theta)}$ is 0.

It remains to show that $\frac{\partial p}{\partial r}(0,\theta) = 1$. Let $U \subset M$ be a neighborhood of w on which $\exp_w$ psosses an inverse $\log_w : U \to T_wM$. And let $\xi_1, \dots, \xi_n$ be a system of orthonormal coordinates for T_wM. The $x_i = \xi_i \circ \log_w$ defines a system of coordinates for U (the "riemann normal coordinates"). Recall (cf: [Sp2], proposition 1 of chapter 4, page 159), the the metric on M can be written, in these coordinates, as

$$g_x = \Sigma dx_i{}^2 + \Sigma h_{i,j}(x) dx_i dx_j$$

where $h_{i,j} = o(x)$. Define $\lambda_i : [0,a] \to \mathbb{R}$ $(i = 1,\dots,n)$ by$\exp_w(\lambda(\theta)) = (\lambda_1(\theta), \lambda_2(\theta), \dots, \lambda_n(\theta))$ and $x_i(r,\theta) \equiv r\lambda_i(\theta)$. We now have

$$\phi(r,\theta) = (x_1(r,\theta), x_2(r,\theta), \dots, x_n(r,\theta))$$

Consequently

$$dx_i^2 = \lambda_i^2(\theta)dr^2 + r\frac{d}{d\theta}(\lambda_i^2(\theta)drd\theta + r^2\frac{d\lambda_i}{d\theta}d\theta^2)$$

Now,

$$\begin{aligned}\Sigma dx_i^2 &= \left(\Sigma\lambda_i^2(\theta)\right) + r\frac{d}{d\theta}\left(\Sigma\lambda_i^2(\theta)\right) drd\theta + r^2\Sigma\frac{d\lambda_i}{d\theta}d\theta^2 \\ &= dr^2 + r^2 d\theta^2\end{aligned}$$

We also have

$$dx_i dx_j = \lambda_i\lambda_j dr^2 + r\frac{d}{d\theta}(l\lambda_i\lambda_j)drd\theta + r^2\left(\frac{d\lambda_i}{d\theta}\frac{d\lambda_j}{d\theta}\right)d\theta^2$$

Hence, if

$$\Sigma h_{i,j}(x)dx_i dx_j = Adr^2 + Bdrd\theta + Cd\theta^2$$

Then, we see that $C(r,\theta) = r^2 u(r,\theta)$ where $u(r,\theta) = O(r)$ (since $h_{i,j} = o(x)$). So, $p(r,\theta) = \sqrt{r^2(1+u(r,\theta))}$ which satisfies $\frac{\partial p}{\partial r}(0,\theta) = 1$. □

14. Proof of theorem 12.

The idea of the proof is to define a ruled surface (e.g. one foliated by geodesic segments) F in M containing $\triangle$ as well as a contraction mapping F to the Poincare Disk. We can then compare the angles of $\triangle$ to those of $\triangle^*$ suing this map.

We use w, x, y to denote the vertices of $\triangle$ and α to denote the angle At w. Recall (Cartan-Hadamard) that the map $exp_w : T_wM \to M$ is a diffeomorphism and denote its inverse by $\log_w : M \to T_wM$. We also denote the norm of a vector $\xi \in T_wM$ by $|\xi|$. We introduce the following objects:

L is the geodesic segment $[x,y] \subset M$.

Σ is the unit sphere of T_wM.

$\xi = \log_w(x) \in T_wM \xi_1 = \frac{\xi}{|\xi|} \in \Sigma$.

$\eta = \log_w(y) \in T_wM \quad \eta_1 = \frac{\eta}{|\eta|} \in \Sigma.$

$\Lambda = \log_w(L) \subset T_wM \quad \Lambda_1 = \left\{\frac{\zeta}{|\zeta|} : \zeta \in \Lambda\right\} \subset \Sigma.$

The curve Λ_1 is a simple and differentiable curve in Σ.Let α_1 denote its length and let $\lambda : [0, \alpha_1] \to \Lambda_1$ be its parametrisation by arc-length (such that $\lambda(0) = \xi_1$ and $\lambda(\alpha_1) = \eta_1$). We have:

(a) $$\alpha_1 \geq \alpha$$

In fact, α is the angle between ξ and η. Hence, it is the distance on the unit sphere Σ of T_wM between ξ_1 and η_1. Therefore, α is less than α_1 (which is the length of the curve Λ_1 joining ξ_1 to η_1).

Let $V \equiv \{(r, \theta) \in \mathbb{R}^2 : 0 \leq r \leq \infty, 0 \leq \theta \leq \alpha_1\}$ and consider the map $\phi : V \to M$ defined by $\phi(r, \theta) = exp_w(r\lambda(\theta))$. Lemma 13 then implies that the metric on $F \equiv \phi(V)$ by the immersion of $F \subset M$ can be written, in (r, θ) coordinates, as

(b) $$ds^2 = dr^2 + p(r, \theta)^2 d\theta^2$$

where $p(0, \theta) = 0$ and $\frac{\partial p}{\partial r}(0, \theta) = 1$ for all $\theta \in [0, \alpha_1]$.

The curvature of F satisfies

(c) $$K \leq k$$

In fact, F is a ruled surface in M (i.e. foliated by geodesics of M), and we can apply Synge's inequality (cf [Pre] or [Sp3], corollary 7 of chapter 1).

Recall (cf [Sp2] page 119) that the curvature of F is given by the equation

$$\frac{\partial^2 p(r, \theta)}{\partial r^2} = -K(r, \theta)p(r, \theta)$$

(where $K(r, \theta)$ is the curvature of F). Since $p(0, \theta) = 0$, $\frac{\partial p}{\partial r}(0, \theta) = 1$, and $K \leq k$, Sturm's comparison theorem (cf. [Har]), gives us the following estimate for $p(r, \theta)$

(d) $$p(r, \theta) \geq \frac{sh(\sqrt{K}r)}{\sqrt{K}}$$

We now construct a map $\psi : F \to H_k$; to do this, we use $w^*, x^*, y^* \in H_k$ to denote the vertices of $\triangle^*$ and introduce a polar system of coordinates $\phi^* : [0, \infty) \times \mathbb{R}/2\pi\mathbb{Z} \to H_k$ such that $w* = \phi^*(0, 0)$ and $x^* = \phi^*(|x - w|, 0)$. Next, we define ψ by $\psi(\phi(r, \theta)) = \phi^*(r, \theta)$. In particular, we observe that $\psi(w) = w^*$ and $\psi(x) = x^*$. We use $\tilde{y^*}$ to denote $\psi(y)$ (in general, $\tilde{y^*} \neq y^*$). Since the metric on H_k is, in polar coordinates, $dr^2 + \frac{1}{k}sh^2(\sqrt{k}r)d\theta^2$, inequality (d) implies that the map $\psi : F \to H_k$ is a contraction. Therefore, we have,

(e) $$|x^* - \tilde{y^*}| \leq |x - y| = |x^* - y^*|$$

On the other hand, it is clear that the segments $[w^*, x^*]$ and $[w^*, \tilde{y^*}]$ meet at w^* at an angle of α_1. Therefore, we have

$$cos(\alpha_1) = \frac{ch(\sqrt{k}|x^* - w^*|)ch(\sqrt{k}|\tilde{y^*} - w^*|) - ch(\sqrt{k}|x^* - \tilde{y^*}|)}{sh(\sqrt{k}|x^* - w^*)sh(\sqrt{k}|\tilde{y^*} - w^*|)}$$

and

$$cos(\alpha^*) = \frac{ch(\sqrt{k}|x^* - w^*|)ch(\sqrt{k}|y^* - w^*|) - ch(\sqrt{k}|x^* - y^*|)}{sh(\sqrt{k}|x^* - w^*)sh(\sqrt{k}|y^* - w^*|)}$$

By (a) and (e), we therefore have $\alpha^* \geq \alpha_1 \geq \alpha$. $\square$

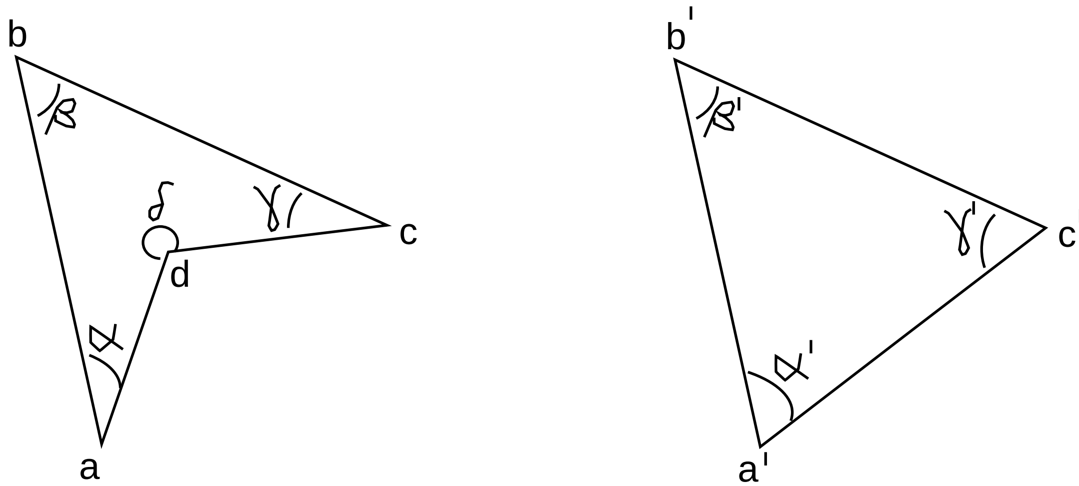
The proof of theorem 9 also uses a lemma, due to Aleksandrov, on quadrilaterals.

15. Lemmma. *[Ale] Let Q be a quadrialteral in h_k with vertices a, b, c and d. Consider a triangle $\triangle' \subset H_k$ with vertices a', b', c'such that*

$$|a' - b'| = |a - b| \quad |b' - c'| = |b - c| \quad |a' - c'| = |a - d| + |d - c|$$

Denote the angles in Q at the corners a, b, c, d by $\alpha, \beta, \gamma, \delta$ and use α', β', γ' for those in $\triangle'$. Assume that $\delta \geq \pi$. Then

$$\alpha \leq \alpha' \quad \beta \leq \beta' \quad \gamma \leq \gamma'$$



Proof of lemma 15. It is obvious that $|a - c| \leq |a' - c'|$ and the cosine formula then implies that $\beta' \geq \beta$. We show that $\alpha \leq \alpha'$. Pretend that Q is an articulated system with a and b anchored. We can then define a movement of this system $c(t), d(t)$ such that $c(0) = c(1) = c$; $d(0) = d$ and $d(1)$ is the reflection of d with respect to the line ac (a and b are fixed, as well as the lengths of the 4 segments). We may assume that during this movement, the angle at a, $\alpha(t)$, increases continuously. There is an intermediate time t_0 where the points $a, d(t_0), c(t_0)$ are collinear. Then the triangle with vertices $a, b, c(t_0)$ is isometric to $\triangle'$; it follows that

$$\alpha' = \alpha(t_0) \geq \alpha(0) = \alpha$$

The proof that $\gamma \geq \gamma'$ is similar. □

16. Proof of Theorem 9. We assume that $k = -1$ and leave the other cases to the reader. Let $g : [0, a] \to M$, $h : [0, b] \to M$ be two geodesic segments beginning at the same point $w \in M$. We must show that $\alpha(s, t) = \alpha^{-1}{}_{g,h}(s, t)$ is a monotone non-decreasing function of s and t (the (A_{-1}) criterion).

We assume that we are given three numbers s, t, τ such that $0 \leq s \leq a$ and $0 \leq \tau \leq t \leq b$. We consider 4 points w^*, s^*, t^*, τ^* in the hyperbolic plane H such that

(1) w^*, s^*, t^* are the vertices of a comparison triangle for w, $g(s)$, $h(\tau)$.
(2) τ^*, t^*, s^* are the vertices of a comparison triangle for $h(\tau)$, $h(t)$, $g(s)$.
(3) w^* and t^* are situated on opposite sides of the line $s^*\tau^*$.

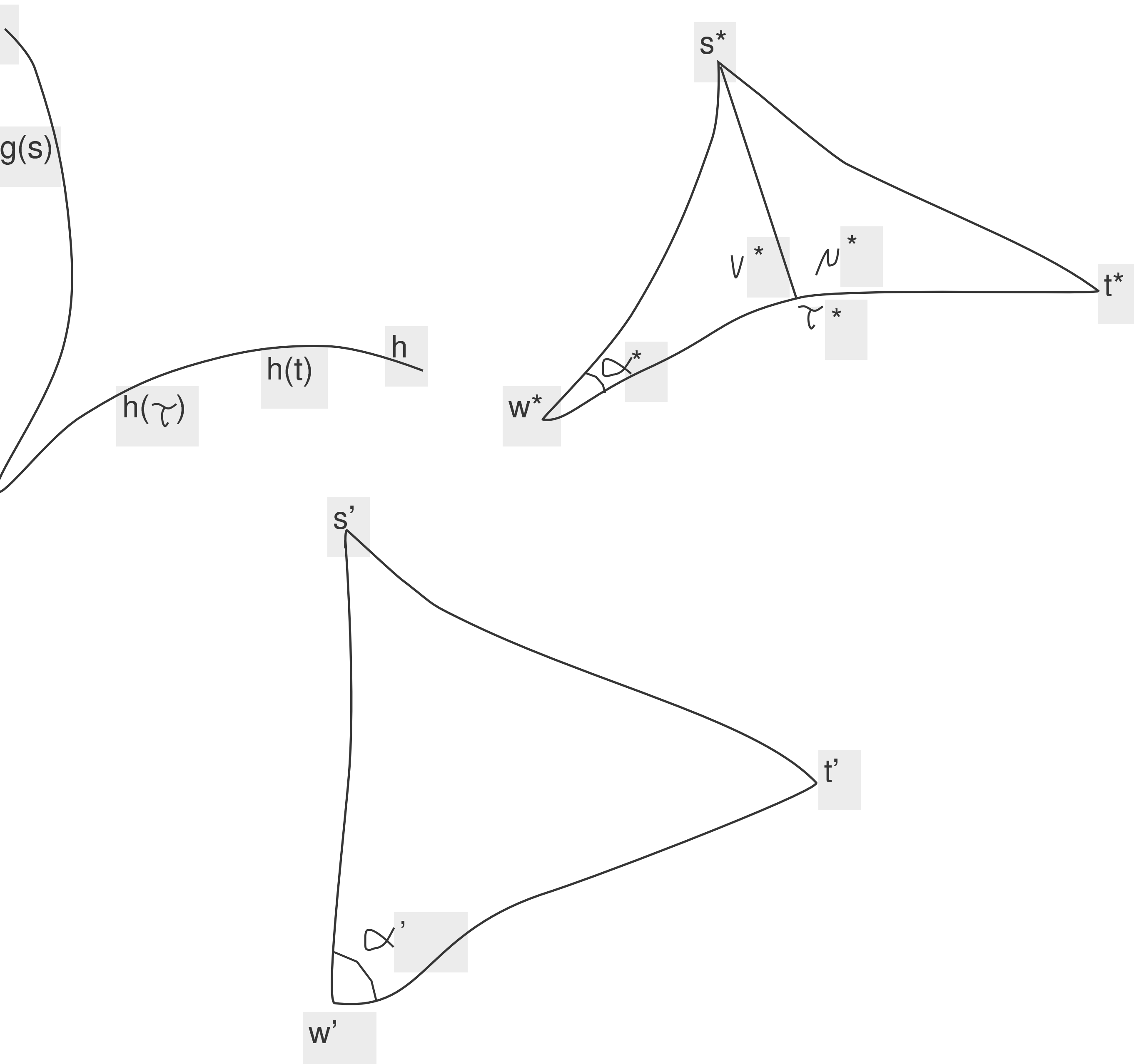
We also consider the triangle $\triangle' \subset H$ with vertices w', s', t' such that $\triangle'$ is a comparison triangle for w, $g(s)$, $h(t)$.

Some notation:
α^* is the angle at w^* of the triangle with vertices w^*, s^*, τ^*.
α' is the angle at w' of the triangle with vertices w', s', τ'.
ν is the angle at $h(\tau)$ of the triangle with vertices w, $g(s)$, $h(t)$.
ν^* is the angle at τ^* of the triangle with vertices w^*, s^*, t^*.
μ is the angle at $h(\tau)$ of the triangle with vertices $h(\tau)$, $h(t)$, $g(s)$.
μ^* is the angle at τ^* of the triangle with vertices τ^*, t^*, s^*.

Then, we have $\alpha^* = \alpha(s, \tau)$ and $\alpha' = \alpha(s, t)$. By theorem 12, we have $\nu \leq \nu^*$ and $\mu \leq \mu^*$. Hence, w^*, s^*, t^*, and τ^* are the vertices of a quadrilateral Q in H where the angle at τ^* is $\nu^* + \mu^* \geq \nu + \mu = \pi$. On the other hadn, $|w' - s'| = |w^* - s^*|$, $|s' - t'| = |s^* - t^*|$ and $|w' - t'| = |w * - \tau^*| + |\tau^* - t^*|$. It follows from lemma 15 (applied to Q and $\triangle'$), that

$$\alpha^* \leq \alpha'$$

Hence, we have shown that if $\tau \leq t$, then $\alpha(s, \tau) \leq \alpha(s, t)$. $\square$



3. Actions of Discrete Groups

17. Definitions. A metric space is *proper* if all bounded balls are compact. A discrete group Γ acting on a topological space X acts *properly* if for every compact set $C \subset X$, the set of $\gamma \in \Gamma$ such that $\gamma C \cap C \neq \emptyset$ is finite.

18. Lemma. *Let X be a geodesic space. Then X is proper if and only if X is locally comact and complete.*

This lemma is proved in [GLP] (theorem 1.10). We note that the lemma is false if X is not geodesic. For example, let $X = \mathbb{R}$ with the metric $d(x,y) = \min\{1, |x-y|\}$; then X is locally compact and complete, but X is not proper (any ball of radius > 1 is not compact).

The following result is, essentially, lemma 2 of [Mil]; see also theorem 1 of [Ca1] and propositions 3.22 and 3.23 of [GLP].

19. Proposition. *Let X be a proper geodesic space and let Γ be a discrete group of isometries of X acting properly on X. If $\Gamma \backslash X$ is compact, then Γ is finitely generated and Γ, given the word metric (relative to any finite system of generators), is quasi-isometric to X.*

Proof of Proposition 19. Since Γ acts by isometries, $\Gamma\backslash X$ is a metric space. Its diameter R is finite (because $\Gamma\backslash X$ is compact and proper). Let x_0 be a point in X and set

$$B = \{x \in X : |x - x_0| \leq R\} \quad \text{and} \quad S = \{\gamma \in \Gamma : \gamma \neq e \text{ and } \gamma B \cap B \neq \emptyset\}$$

Observe that B is compact (because X is proper), and that $\{\gamma B\}_{\gamma\in\Gamma}$ is a covering of X. Furthermore $S = S^{-1}$ and S is finite (because Γ acts properly). Now set $\lambda = \sup\{|x_0 - \gamma x_0| : \gamma \in S\}$ and $r = \inf\{d(B, \gamma B) : \gamma \in \Gamma - (S \bigcup \{e\})\}$.

We show that $r > 0$. The set $E =\sim \{\gamma \in \Gamma - (S\bigcup\{e\}) : |x_0 - x\gamma x_0| \leq 4R\}$ is finite (because Γ acts properly), and therefore $r' = \inf\{d(B, \gamma B) : \gamma \in E\}$ is non-zero (the infinum is attained becase E is finite). If E is empty, we set $r' = \infty$. Then, it is easy to see that $r \geq \min\{r', R\}$. We observe that r has the following property.

(*) If there exists $y, z \in X, \sigma \in \Gamma$such that $y \in B, z \in \sigma B$ and $|y - zz| \leq r$ then$\sigma \in S \bigcup e$

The proposition follows from the following assertions:

(1) S generates Γ and, for all $\gamma \in \Gamma$, we have :$d_S(1,\gamma) \leq \frac{1}{r}|x_0 - \gamma x_0| + 1$.
(2) $|x_0 - \gamma x_0| \leq \lambda d_S(1,\gamma)$ for all $\gamma \in \Gamma$
(3) The orbit $\Gamma x_0 \subset X$ is quasi-isometric to X.

In fact, (1) and (1) imply that Γ is quasi-isometric to X.

To prove (1), we consider an element $\gamma \in \Gamma$ and let k be the smallest integer such that $|x_0 - \gamma x_0| \leq kr + R$. We choose a segment $[x_0, \gamma x_0] \subset X$, as well as points $x_1, x_2, \ldots, x_{k+1} = \gamma x_0$ on $[x_0, \gamma x_0]$ such that $|x_0 - x_1| \leq R$ and $|x_i - x_{i+1}| < r$ for $1 \leq i \leq k$. Since $X = \Gamma B$, for each $i = 1, 2, \ldots, k+1$, we can find an element $\gamma_i \in \Gamma$ such that $x_i \in \gamma_i B$. We can choose $\gamma_1 = \text{id}$ and $\gamma_{k+1} = \gamma$. Set $\sigma_i = {\gamma_i}^{-1}\gamma_{i+1}$; we then have $\gamma = \sigma_1\sigma_2\ldots\sigma_k$.. For $1 \leq i \leq k$, we have

$${\gamma_i}^{-1}x_i \in B, \quad {\gamma_i}^{-1}x_{i+1} = \sigma_i(\gamma_{i+1})^{-1}x_{i+1} \in B \ \text{ and } \ |\gamma_I^{-1}x_i - \gamma_i^{-1}x_{i+1}| < r$$

Hence, (*) forces $\sigma_i \in S\bigcup\{\epsilon\}$ for $i = 1, \ldots, k$. This shows that S generates Γ (since γ was arbitrary); more precisely, we have shown that $d_S(1,\gamma) \leq k$.

Now, $(k-1)r + R \leq |x_0 - \gamma x_0|$ (by minimality of k), and therefore

$$d_S(1,\gamma) \leq k \leq \frac{1}{r}|x_0 - \gamma x_o| + 1 - \frac{R}{r}$$

Inequality (2) can be verified inductively: if $\gamma = \sigma_1\sigma_2\ldots\sigma_n$ with

$$\begin{aligned} |x_0 - \gamma x_0| &= |x_0 - \sigma_1\sigma_2\ldots\sigma_n x_0| \\ &\leq |x_0 - \sigma_1 x_0| + |\sigma_1 x_0 - \sigma_2\ldots\sigma_n x_0| \\ &\leq \lambda + (n-1)\lambda \end{aligned}$$

Finally, (3) is obvious because $\Gamma B = X$ and $\mathrm{diam}(X) < \infty$. □

20. Corollary.

(1) *Let Γ_1 be a finite index subgroup of a finitely generated group Γ_2. Then Γ_1 and Γ_2 are quasi-isometric.*
(2) *If $1 \to \triangle \to \Gamma_1 \to \Gamma_2$ is an exact sequence of groups, and if $\triangle$ is finite, then Γ_2 and Γ_1 are quasi-isometric.*

Proof.

(1) Let $X = G(\Gamma_2, S)$ be the cayley graph of Γ_2 with respect to a finite system S of generators. Then X is locally compact and Γ_1 acts on X (properly and by isometries); morevoer $\Gamma_1\backslash X$ is compact.
(2) The homomorphism $\Gamma_1 \to \Gamma_2$ defines a proper action of Γ_1 on X. for which $\Gamma_1\backslash X$ is compact.

□

We can also state this corollary by saying that two finitely generated groups which are commensurable are quasi-isometric.

Applied to riemannian manifolds, proposition 19 implies the following result.

21. Corollary. *Let $\hat{M}$be a riemannian manifold, and let $\Gamma \subset rmIso(\hat{M})$ be a discrete group of isometries which acts properly. Then Γ is finitely generated and is quasi-isometric to $\hat{M}$.*

In particular, we have:

(1) The fundamental group of a compact riemannian manifold M is quasi-isometric to its universal cover $\hat{M}$.
(2) Two compact riemannian manifolds with commensurable funddamental groups have quasi-isometric universal covers.
(3) The fundamental group of a compact riemannian manifold with positive Ricci curvature is finite (in fact, it follows from Myer's theorem that the universal cover is compact. See [Gve], proposition 4.1).

Recall that a group Γ is called hyperbolic if it is finitely generated and there is a finite system $S \subset \Gamma$ of generators such that Γ, given the word metric induced by S, is a hyperbolic space. In anticipation of chapter 5 (where we prove that hyperbolicity is invariant under quasi-isometry for geodesic spaces), we obtain the following characterisation of hyperbolic groups.

22. Theorem. *Let Γ be a dicrete group; then the following conditions are equivalent:*

(1) *Γ is finitely generated and hyperbolic.*
(2) *There is a proper hyperbolic geodesic space X on which Γ acts properly by isometries and such that $diam(\Gamma\backslash X) < \infty$.*

Proof. If Γ satifies (1), we can take X to be the Cayley Graph of Γ defined by any finite system of generators. Conversely, if there is a space X which meets the hypothesis, then Γ is quasi-isometric to X by porposition 19, hence Γ is hyperbolic (by theorem 5.12). □

We will see, in the next chapter, that every hyperbolic group acts simplicially on a contractible, finite-dimensional polyhedron (the action is properly discontinuous and has compact quotient). For infinite groups, this is a non-trivial condition (for finite groups, one can take the polyhedron to be a point).

Theorem 22, with corollary 20, implies

23. Corollary.

(1) *If Γ_1 is a finite index subgroup of a hyperbolic group Γ_2, then Γ_1 is hyperbolic.*
(2) *If $1 \to \triangle \to \Gamma_1 \to \Gamma_2$ is an exact sequence of groups, and if $\triangle$ is finite and Γ_2 is hyperbolic, then Γ_1 is hyperbolic.*

Finally, proposition 19 and theorem 22 imply the promised result.

24. Theorem. *Let $(\hat{M}, g)$ be a complete, simply connected, riemannian manifold with curvature $K \leq k < 0$, and let $\Gamma \subset Iso(\hat{M})$ be a discrete group of isometries such that $M = \Gamma\backslash\hat{M}$ is compact. then Γ is finitely generated and hyperbolic.*

In particular, the fundamental group of a surface with negative Euler characteristic is a hyperbolic group. In dimension 3, there are many examples of negative curved compact manifolds. In dimensions greater than 3, it is non-trivial to construct such manifolds. Nevertheless, one sees [Bor], that the rank 1 symmetric spaces (which are riemannian manifolds whith curvature $-4 \leq K \leq 1$) possess cocompact discrete groups of isometries (which are, therefore, examples of hyperbolic groups).

4. Convexity and CAT

25. Definitions. A geodesic space X is called *geodesically convex* if there exists one and only one geodesic segment $[p, q]$ between any two points in X. It is called *metrically convex* if, given two affinely parametrised geodesics (i.e. parametrised proportionally to arc-length) $g : [0, 1] \to X$ and $h : [0, 1] \to X$, we have

$$|g(s) - h(s)| \leq (1 - s)|g(0) - h(0)| + s|g(1) - h(1)|$$

(i.e. the distance function is convex).

26. Example. A vector space is metrically convex.

Proof. Let $g(s) = sx + (1-s)y$ and $h(s) = sx' + (1-x)y'$ be two geodesics. We then have, by the triangle inequality

$$|g(s) - h(s)| = |s(x - x') + (1-s)(y - y')| \leq (1-s)|x - x'| + s|y - y'|$$

□

27. Proposition. *A geodesic metrically convex space is geodesically convex.*

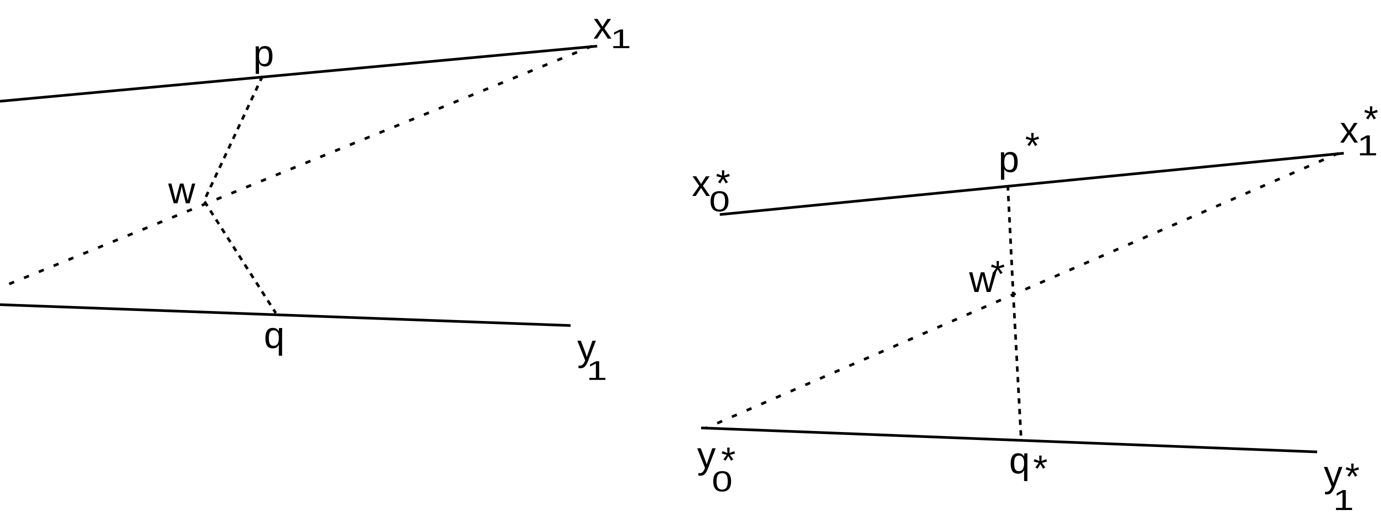
This proposition is obvious. However, the converse is false; for example, the open half-sphere is geodesically convex and not metrically convex.

28. Proposition. *A geodesic metrically convex space is contractible.*

Proof. We choose a base point and contract the points in the space along the unique geodesic joining them to the base point. The convexity of the distance function implies that this is continuous.

29. Proposition. *If X is a geodesic space satisfying CAT_k, $k \leq 0$; then X is metrically convex.*

Proof. Suppose $g : [0,1] \to X$ and $h : [0,1] \to X$ are two affinely parametrised geodesic segments. Let $t \in [0,1]$ and denote $x_0 = g(0), x_1 = g(1), y_0 = h(0), y_1 = h(1), p = g(t)$ and$q = h(t)$. Choose six points $x_0*, x_1*, y_0*, y_1*, p*$ and$q* \in H_k$ with the following properties: First, x_0*, x_1*, y_0* form a comparison triangle for x_0, x_1, y_0 and $p* \in [x_0*, x_1*]$ is the point corresponding to $p \in [x_0, x_1]$. Next, y_0*, y_1*, x_1* form a comparison triangle for y_0, y_1, x_1 and $q* \in [y_0*, y_1*]$ is the point corresponding to $q \in [y_0, y_1]$. Finally, the points x_0* and y_1* are on opposite sides of the line $x_1 * y_0*$ (i.e. $(x_o*, y_1*) \cap (x_1*, y_0*) = \emptyset$).



The segment $[p*, q*]$ intersects the diagonal $[x_1*, y_0*]$ in a point $w*$. We observe that $|p * - q *| = |p * - w *| + |w * - q *|$. we choose a segment $[x_1, y_0] \subset X$, and use

w to denote the point in $[x_1, y_0]$ such that $|w - x_1| = |w * - x_1 *|$. Since X satisfies CAT_k, we have, by criterion (T):

$$|p - w| \leq |p * - w *| \text{ and } |q - w| \leq |q * - w *|$$

whence we deduce:

$$|p - q| \leq |p - w| + |w - q| \leq |p * - w *| + |w * - q *| = |p * - q *|$$

On the other hand, for $k \leq 0$, H_k is metrically convex and therefore

$$|p * - q *| \leq (1 - t)|x_0 * - x_1 *| + t|y_0 * - y_1 *|$$

It follows that

$$\begin{aligned} |g(t) - h(t)| = |p - q| &\leq (1 - t)|x_0 - x_1| + t|y_0 - y_1| \\ &= (1 - t)|g(0) - g(1)| + t|h(0) - h(1)| \end{aligned}$$

□

In particular, if X is a geodesic space satifying CAT_k, for $k \leq 0$, then X is contractible.

30. Remark. The converse of proposition 29 is false; a metrically convex space is does not necessarily satisfy CAT_0.

Consider, for example, $\mathbb{R}^2$ given the norm $|(x, y)| = (|x|^p + |y|^p)^{\frac{1}{p}}$. We denote the associated metric space by L_p. L_p is a metrically convex space (since is is a normed vector space). We claim that L_p satisfies CAT_0 if and only if $p = 2$.

Suppose that $g(s) = (s, 0)$ and $h(t) = (0, t)$ (these are two geodesics in L_p originating at the point $(0, 0)$). The comparison angle with the Euclidean plane is given by the law of cosines

$$\cos(\alpha^0(s, t)) = \frac{s^2 + t^2 - |g(s) - h(t)|^2}{2st}$$

We show that the criteria (A_0) is not satisfied if $p \neq 2$. Set $s = 1$, and define $f(t)$ by

$$\frac{d}{dt}\cos(\alpha^0(s, t)) = \frac{1}{2t} f(t)$$

Then $f(t) = \frac{1}{2}[(t^2 - 1) - (1 - t^p)(1 + t^p)^{\frac{2}{p} - 1}]$— in particluar, $f(t)$ satisifes

(1) $f(t) \sim 0 \Leftrightarrow p = 2$;
(2) If $p \neq 2$, then $f(t) = 0 \Leftrightarrow t = 1$;
(3) $f(t) = -f(\frac{1}{t})$.

Hence, if $p \neq 2$, then f changes sign at $t = 1$. It follows that $\cos(\alpha^0(s, t))$ is not monotone at $(s, t) = (1, 1)$.

Metrically convex spaces were first studeied by Busemann [Bus].

Translated by WILLIAM GROSSO

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

FIRST PROPERTIES OF HYPERBOLIC GROUPS (CHAPTER 4)

Eliane Salem

Abstract. In this chapter, we prove a result due to I. Rips which implies that a hyperbolic group Γ has the following properties (see theorem 1.36):

(1) Γ is finitely presented.
(2) The number of conjugacy classes of torsion elements in Γ is finite.
(3) The cohomology groups $H^k(\Gamma, \mathbb{Q})$ are trivial for large enough k.

Here is the result:

1. Theorem (I. Rips). *Let Γ be a hyperbolic group. There is a contractible, locally finite, finite dimensional, simplicial complex P on which Γ acts simplically, faithfully, properly discontinuously, and so that the quotient $\Gamma\backslash P$ is compact.*

In the first section of this chapter, we associate the Rips Complex $P_n(\Gamma, S)$ to each finitely generated group Γ with a given generating set S (with $S = S^{-1}$ and $e \notin S$). It is a locally finite, finite dimensional, simplicial complex on which the group Γ acts simplically, faithfully, properly discontinuously and with compact quotient. In the second section of this chapter, we show that, if Γ is a hyperbolic group (for the word metric associated to the generating system S), then the simplicial complex $P_n(\Gamma, S)$ is contractible for n sufficiently large (Rips' Theorem). Finally, in the third section, we show that a hyperbolic group has the three properties stated at the beginning of this chapter.

The results in this chapter are stated in sections 1.7 and 2.2 of [Gr5]. The proofs given here are based on introductory lectures of E. Ghys.

1. Construction of the Rips Complex $P_n(\Gamma, S)$

Throughout this chapter, Γ denotes a finitely generated group and S a finite system of generators for Γ; we always assume (and we won't mention it again) that $S^{-1} = S$ and $e \notin S$. We denote the word metric on Γ associated to S by d. It is invariant under left translations: if $\gamma, \gamma', \gamma'' \in \Gamma$ then $d(\gamma\gamma', \gamma\gamma'') = d(\gamma', \gamma'')$.

2. Definition. Let n be an integer, $n \geq 1$. The Rips Complex $P_n(\Gamma, S)$ is the simplicial complex whose k-simplices are the $(k+1)$-tuples $(\gamma_0, \gamma_1, \ldots, \gamma_k)$ of pair-wise distinct elements of Γ such that $\max_{\{i,j\}} d(\gamma_i, \gamma_j) \leq n$. We give $P_n(\Gamma, S)$ the weak topology.

3. Remarks.

(1) The complex $P_n(\Gamma, S)$ is locally finite and finite dimensional. In fact, let b be the cardinality of the closed ball

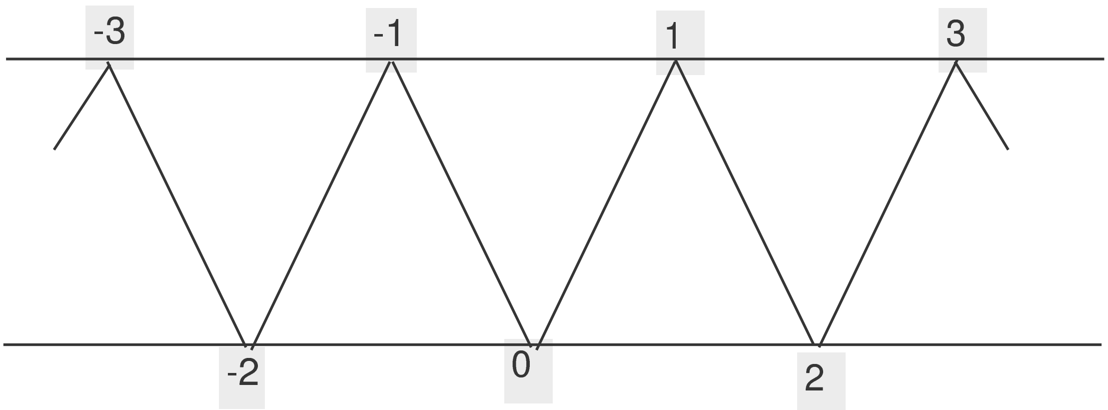
$$B(e, n) = \{\gamma \in \Gamma : d(e, \gamma) \leq n\}$$

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

Then $P_n(\Gamma, S)$ has dimension at most $b-1$ and every vertex of $P_n(\Gamma, S)$ belongs to exactly $b-1$ edges.
(2) The vertices of $P_n(\Gamma, S)$ are the elements of Γ. The 1-skeleton of $P_n(\Gamma, S)$ is the Cayley Graph of the group Γ for the system of generators $B(e,n)-\{e\}$.

4. Examples. If $\Gamma = \mathbb{Z}$ and $S = \{\pm 1\}$, the Rips Complex $P_2(\Gamma, S)$ is represented by the figure below



The group Γ acts simplically on $P_n(\Gamma, S)$ by left translation: if $\sigma = (\gamma_0, \gamma_1, \ldots, \gamma_k)$ is a k-simplex of $P_n(\Gamma, S)$ and if $\gamma \in \Gamma$, then $\gamma\sigma$ is the k-simplex $(\gamma\gamma_0, \gamma\gamma_1, \ldots, \gamma\gamma_k)$.

5. Proposition. *The action of Γ on $P_n(\Gamma, S)$ defined above possesses the following properties:*

(1) *The action is faithful.*
(2) *The stabiliser of any simplex is finite.*
(3) *The action is properly discontinuous.*
(4) *The space of orbits is compact.*
(5) *Furthermore, if Γ is torsion-free, the action is free.*

Proof.

(1) The action is free on the set of vertices of $P_n(\Gamma, S)$ and therefore faithful
(2) If $\gamma \in \Gamma$ leaves the k-simplex $(\gamma_0, \gamma_1, \ldots, \gamma_k)$ invariant, then γ permutes the $(k+1)$ elements $\gamma_0, \gamma_1, \ldots, \gamma_k \in \Gamma$. Hence, the stabiliser of any simplex in $P_n(\Gamma, S)$ is finite.
(3) If σ and σ' are two simplices of $P_n(\Gamma, S)$, the set

$$\{\gamma \in \Gamma : \gamma\sigma \cap \sigma \neq \emptyset\}$$

is finite. Since any compact set C in $P_n(\Gamma, S)$ can be covered by a finite number of simplices of $P_n(\Gamma, S)$, we deduce that the set

$$\{\gamma \in \Gamma : \gamma C \cap C \neq \emptyset\}$$

is finite.
(4) Let A be the finite union of all the simplies of $P_n(\Gamma, S)$ with $e \in \Gamma$ as one of their vertices. The set A is compact and the inclusion of A in $P_n(\Gamma, S)$

induces a homeomorphism of $\Gamma\backslash A$ onto $\Gamma\backslash P_n(\Gamma, S)$. We remark that, as a matter of fact, for every simplex σ of $P_n(\Gamma, S)$, there is an element $\gamma \in \Gamma$ such that $\gamma\sigma \in A$.

(5) The assertion follows from (ii)

□

6. Remark. In general, the quotient $\Gamma\backslash P_n(\Gamma, S)$ is not a simplicial complex. Let $P_n''(\Gamma, S)$ be the second barycentric subdivision of $P_n(\Gamma, S)$. The action of Γ on $P_n(\Gamma, S)$ by left translation induces a simplical action of Γ on $P_n''(\Gamma, S)$ which also has properties (i) to (v) of proposition 5. Suppose, moreover, that p is a vertex of $P_n''(\Gamma, S)$ and γ is an element of Γ such that $\gamma p \neq p$; then the stars of p and γp are disjoint. It follows that the quotient $\Gamma\backslash P_n''(\Gamma, S)$ is a simplicial complex, and that the projection

$$\pi : P_n''(\Gamma, S) \to \Gamma\backslash P_n''(\Gamma, S)$$

is simplicial (see [Bre], page 117).

7. Notation. The Cayley Graph $G(\Gamma, S)$ (defined in chapter 1) is identified with a subset of the 1-skeleton of $P_n(\Gamma, S)$. Thus, two vertices γ and γ' of $P_n(\Gamma, S)$ can be joined by a curve in $P_n(\Gamma, S)$ which is in $G(\Gamma, S)$. and which is a geodesic in the sense of numbers 1.25 and 1.26. We denote such a curve by $[\gamma, \gamma']$, although it is not, in general, uniquely defined by the endpoints γ, γ'. Every expression of $\gamma^{-1}\gamma'$ as a product $s_1 s_2 \dots s_l$ of a minimal number of generators corresponds to a geodesic segment $[\gamma, \gamma']$ which is the union of the 1-simplices.

$$(\gamma, \gamma s_1), (\gamma s_1, \gamma s_2), \dots, (\gamma s_1 s_2 \dots s_{l-1}, \gamma')$$

of $P_n(\Gamma, S)$; these simplices are also edges of length 1 in $G(\Gamma, S)$

2. Proof of Rips' Theorem

We consider, as in §1 a group Γ, a finite system of generators S, an integer $n \geq 1$ and the Rips complex $P_n(\Gamma, S)$. The goal of §2 is to finish the proof of theorem 1 by showing that, if Γ is hyperbolic, then $P_n(\Gamma, S)$ is contractible for large enough n.

Fix a number $\delta \geq 0$. We say that Γ is δ-hyperbolic for S if the space Γ satisfies the conditions of definition 2.3 when given the word metric defined by S.

8. Lemma. *We assume that Γ is δ-hyperbolic for S. We choose an integer $n \geq 4\delta + 2$ and a vertex γ_0 of $P_n(\Gamma, S)$.*

If γ is a vertex of $P_n(\Gamma, S)$ such that $d(\gamma_0, \gamma) > [\frac{n}{2}]$, there is a vertex γ' of $P_n(\Gamma, S)$ satisfying

(1) $d(\gamma_0, \gamma') = d(\gamma_0, \gamma) - d(\gamma, \gamma')$,

(2) $d(\gamma', \gamma) = [\frac{n}{2}]$,

(3) *For all vertices γ'' of $P_n(\Gamma, S)$, we have*

$$d(\gamma', \gamma'') \leq \max\{[\frac{n}{2}] + d(\gamma_0, \gamma'') - d(\gamma_0, \gamma); d(\gamma, \gamma'') - [\frac{n}{2}]\} + 2\delta$$

Proof. We choose a geodesic segment $[\gamma_0, \gamma]$ as in number 7 and take for γ' the vertex of this segment such that $d(\gamma', \gamma) = [\frac{n}{2}]$; we clearly have (i) and (ii).

For all $\gamma'' \in \Gamma$, we have, by definition of δ-hyperbolicity

$$d(\gamma', \gamma'') + d(\gamma_0, \gamma) \leq \max\{d(\gamma', \gamma) + d(\gamma_0, \gamma''), d(\gamma_0, \gamma') + d(\gamma'', \gamma)\} + 2\delta$$

Inequality (iii) now follows. □

9. Proposition. *We suppose that the group Γ is δ-hyperbolic for the system of generators S. We consider a finite simplicial complex K of vertices $\{p_0, p_1, \dots, p_k\}$, an integer $n \geq 4\delta + 2$ and a simplicial map $f : K \to P_n(\Gamma, S)$. Then there is a continuous homotopy $h : K \times [0,1] \to P_n(\Gamma, S)$ of f to a simplicial map $f' : K \to P_n(\Gamma, S)$ such that*

$$d(f'(p_0), f'(p_j)) \leq \frac{n}{2} \; \forall P_j \in K$$

Proof. Let p_i be a vertex of K such that

$$d(f(p_0), f(p_i)) = sup_{1 \leq j \leq k} d(f(p_0), f(p_j))$$

We may assume that $d(f(p_0), f(p_i)) > \frac{n}{2}$ (if not, there is nothing to show). By applying lemma 8 with $\gamma_0 = f(p_0)$ and $\gamma = f(p_i)$, we see that there is a vertex γ' of $P_n(\Gamma, S)$ such that

(a) $$d(f(p_0), \gamma') = d(f(p_0), f(p_i)) - [\frac{n}{2}]$$

(b)
For every vertex p_j of K, we have $d(\gamma', f(p_j)) \leq \max\{[\frac{n}{2}] + 2\delta; d(f(p_i), f(p_j)) - [\frac{n}{2}] + 2\delta\}$

Since $n \geq 4\delta + 2$, we have $[\frac{n}{2}] + 2\delta \leq n$ and $2\delta - [\frac{n}{2}] \leq 0$, and therefore

(c) for every vertex p_j of K, we have $d(\gamma', f(p_j)) \leq \max\{n; d(f(p_i), f(p_j))\}$

Let $h : K \times [0,1] \to P_n(\Gamma, S)$ be the map defined on the vertices of K by

$$h_1(p_j, t) = \begin{cases} f(p_j) & iff(p_j) \neq f(p_i) \\ t\gamma' + (1-t)f(p_i) & iff(p_j) = f(p_i) \end{cases}$$

and extended to the simplices of K (this is possible by inequality (c)) by

$$h_1(\Sigma_{j=0}{}^k \lambda_j p_j, t) = \Sigma_{j=0}{}^k \lambda_j h_1(p_j, t)$$

where $0 \leq \lambda_j \leq 1$ and $\Sigma_{j=0}^k \lambda_j = 1$. Then h_1 is a continuous homotopy of f to a simplicial map $f_1 : K \to P_n(\Gamma, S)$ satisfying:

(d) If p_j is a vertex of K such that $d(f(p_0), f(p_j)) \leq [\frac{n}{2}]$ then $f_1(p_j) = f(p_j)$

(e) We have $d(f_1(p_0), f_1(p_i)) = d(f(p_0), f(p_i)) - [\frac{n}{2}]$

If $sup_{1\leq j\leq k} d(f_1(p_0), f_1(p_j)) \leq n$, the proof is finished.
If not, we choose a vertex $p_{i'}$ of K such that

$$d(f_1(p_0), f_1(p_{i'})) = sup_{1\leq j\leq k} d(f_1(p_0), f_1(p_j))$$

and we construct, as above, a homotopy h_2 of f_1 to a map f_2 such that

$$d(f_2(p_0), f_2(p_{i'})) = d(f_1(p_0), f_1(p_{i'})) - [\frac{n}{2}]$$

By repeating this a finite number of times, we obtain a sequence $h_1, h_2, \dots$ whose composition is a homotopy with the stated properties. □

10. Rips' Theorem. *Suppose that Γ is δ-hyperbolic for the system of generators S. If $n \geq 4\delta + 2$, the Rips Complex $P_n(\Gamma, 2)$ is contractible.*

Proof. Let K be a finite, non-empty subcomplex of $P_n(\Gamma, S)$ with vertices $\{p_0, p_1, \dots, p_k\}$. It suffices to show that K retracts to a point in $P_n(\Gamma, S)$. But, proposition 9 shows that K retracts in $P_n(\Gamma, S)$ to a subcomplex K' whose vertices are contained in the ball

$$B = \{\gamma \in \Gamma : d(p_0, \gamma) \leq \frac{n}{2}\}$$

Since $\max\{d(\gamma', \gamma'') : \gamma', \gamma'' \in B\} \leq n$, the subcomplex K' is contained in a simplex of $P_n(\Gamma, S)$, and therefore can be retreacted into a point. □

Corollary. *We suppose that the group Γ is δ-hyperbolic for the system of generators S. If $n \geq 4\delta + 2$, every simplicial map of the circle S^1 into $P_n(\Gamma, S)$ extend to a a simplicial map from the disk D^2 into $P_n(\Gamma, S)$.*

We summarise proposition 5 and theorem 10 in the following manner:

12. Theorem. *Let Γ be a hyperbolic group. There is a finite dimensional, locally finite, contractible, simplicial complex P on which Γ acts faithfully, simplicially, and properly discontinuously , in such a way that*

(1) *The stabiliser of any simplex is finite.*
(2) *If p is a vertex of P and γ an element of Γ such that $p \neq \gamma p$, then the stars of p and γp are disjoint.*
(3) *The space of orbits $\Gamma\backslash P$ is a finite, simplicial complex and the projection $\pi : P \to \Gamma\backslash P$ is simplicial.*
(4) *Moreover, if Γ is torsion free, the action is free.*

Proof. Let S and n be as in theorem 10. One can take P to be the second barycentric subdivision of $P_n(\Gamma, S)$. □

3. Corollaries of Rip's Theorem

13. Proposition. *A hyperbolic group Γ possesses a finite number of conjugacy classes of torsion elements.*

Proof. Let P be a simplicial complex as per the conclusion of theorem 12. If $\gamma \in \Gamma$ is a torsion element ($\gamma^n = e, n \geq 2, \gamma^m \neq e$ for $0 < m < n$), then γ leaves a simplex

σ of P invariant. In fact, if this is not the case, the cyclic group $\mathbb{Z}/n\mathbb{Z}$ generated by γ acts freely on P and, therefore there is an $l \geq 0$ such that $H^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ for all $k > l$; this is absurd ([Bro], page 58). Further, if γ leaves the simplex γ invariant, then $\gamma' \gamma {\gamma'}^{-1}$ leaves the simplex $\gamma' \sigma$ invariant for all $\gamma' \in \Gamma$.

We use $S(\gamma)$ to denote the set $\{\sigma \in P : \gamma\sigma = \sigma\}$. Since $S(\gamma'\gamma{\gamma'}^{-1}) = \gamma' S(\gamma)$ for any $\gamma' \in \Gamma$, we can associate a non-empty subset $\pi(S(\gamma))$ of the finite set of simplices in $\Gamma \backslash P$ to every torsion element $\gamma \in \Gamma$.

If ν is another torsion element of Γ with $\pi(S(\gamma)) = \pi(S(\nu))$, then for every simplex σ of $S(\gamma)$, there is an element $\gamma' \in \Gamma$ such that $\gamma'\sigma \in S(\nu)$, hence such that ${\gamma'}^{-1}\nu\gamma'$ leaves σ invariant. Since the simplicial complex $\Gamma\backslash P$ is finite and since the stabiliser of any simplex of P is finite, we deduce that there are only a finite number of conjugacy classes of torsion elements in Γ. $\square$

For all integers $k \geq 0$, we use $H^k(\Gamma, \mathbb{Q})$ to denote the k'th cohomology group of Γ with coefficients in $\mathbb{Q}$ (Γ acts trivially on $\mathbb{Q}$). Recall that $H^k(\Gamma, \mathbb{Q}) = H^k(X, \mathbb{Q})$ if X is an Eilenberg-MacLane space $K(\Gamma, 1)$; see [Bro], chapter III.

14. Proposition. *If Γ is a hyperbolic group, there is an integer $l \geq 0$ such that $H^k(\Gamma, \mathbb{Q}) = 0$ for all $k > l$.*

Proof. Let P be a simplicial complex as per the conclusion of theorem 12.

First, we assume that Γ is torsion free, so that Γ acts freely on P and that $H^k(\Gamma, \mathbb{Q}) = H^k(\Gamma\backslash P, \mathbb{Q})$ for all $k \geq 0$. Then $H^k(\Gamma, \mathbb{Q}) = 0$ for all k greater than the dimension of $\Gamma\backslash P$ (which is the dimension of P).

If Γ is a group with torsion, we can invoke a standard spectral sequence argument. We can also proceed as follows. Let $B\Gamma$ be the classifying space of Γ and $E\Gamma$ its universal cover, which is contractible. We use $E\Gamma \times_\Gamma P$ to denote the quotient of $E\Gamma \times P$ by the natural action of Γ. The projection $E\Gamma \times P \to E\Gamma$ induces a map $E\Gamma \times_\Gamma P \to B\Gamma$ which is the fibration (with fibre P) associated to the covering $E\Gamma \to B\Gamma$ and to the action of Γ on P; since the fibre P is contractible, the fibration is a homotopy equivalence , and we have $H^k(\Gamma, \mathbb{Q}) = H^k(E\Gamma \times_\Gamma, \mathbb{Q})$ for all $k \geq 0$.

The projection $E\Gamma \times P \to P$ induces a map $f : E\Gamma \times_\Gamma P \to \Gamma\backslash P$ which is continuous and surjective. We denote the vertices of the finite simplicial complex $\Gamma\backslash P$ by $\{p_1, \ldots, p_m\}$. We choose a vertex $q_i \in P$ which projects to p_i and denote the isotropy group of q_i in Γ by Γ_i. Let V_i be the union of all the open simplices of P containing q_i in their closure (this is the star of q_i) and let U_i be the image of V_i in $\Gamma\backslash P$. Then V_i is invariant under Γ_i and we have $\gamma V_i \cap V_i = \emptyset$ for all $\gamma \in \Gamma - \Gamma_i$ (we are using property (2) of theorem 12 here). Furthermore,

$$U_i = \Gamma_i \backslash V_i \text{ and } f^{-1}(U_i) = E\Gamma \times_\Gamma \Gamma V_i = E\Gamma \times_\Gamma V_i$$

where ΓV_i denotes the orbit of V_i under the action of Γ.

Since Γ acts freely on $E\Gamma$, the space $E\Gamma/\Gamma_i$ is a $K(\Gamma_i, 1)$. the natural map $E\Gamma \times_\Gamma V_i \to E\Gamma/\Gamma_i$ is a fibration with contractible fibre V_i. Therefore, it is a homotopy equivalence and $H^*(f^{-1}(U_i), \mathbb{Q}) = H^*(\Gamma_i, \mathbb{Q})$. Γ_i is a finite group; therefore its rational cohomology is that of of a point (see [Bro], page 59) and it follows that $H^*(f^{-1}(U_i), \mathbb{Q})$ is the cohomology of a point for each $i = 1, \ldots, m$. Similarly, for all non-empty intersections $W = U_{i_1} \cap \ldots \cap U_{i_k}$ (with each $U_{i_j} \in \{U_1, \ldots, U_k\}$), the cohomology $H^*(f^{-1}(W), \mathbb{Q})$ is that of a point.

A proposition of Leray shows that the cohomology $H^*(E\Gamma \times_\Gamma P, \mathbb{Q})$ is the Čech cohomology of the complex $C^*(W, \mathbb{Q})$ where W is the cover $(f^{-1}(U_i))_{1\leq i \leq m}$ of $E\Gamma \times_\Gamma P$. (This proposition follows from a Mayer-Vietoris argument; see [Ler] and also see theorem 7.26 of [Rot]). It follows that $H^K(E\Gamma \times_\Gamma P, \mathbb{Q}) = 0$ for large enough k. □

15. Notation. Let Γ be a group generated by a finite set S of generators. We use $L(S)$ to denote the free group on S and $\pi : L(S) \to \Gamma$ to denote the canonical homomorphism. Each element $w \in L(S)$ possesses a unique reduced word $w = s_1 s_2 \cdots s_l$ with $s_\in S$ and $s_{i+1} \neq {s_i}^{-1}$; the integer l is the length of w.

Consider an integer $n \geq 1$ and the Rips Complex $P_n(\Gamma, S)$. If (γ_0, γ_1) is a 1-simplex in $P_n(\Gamma, S)$, we define $E(\gamma_0, \gamma_1)$ to be the subset of $L(S)$ consisting of the reduced words w such that $\pi(w) = \gamma_0 {\gamma_1}^{-1}$. We define $E(\gamma_0, \gamma_1)$ to be the empty word if $\gamma_0 = \gamma_1$. If $\sigma = (\gamma_0, \gamma_1, \gamma_2)$ is an oriented 2-simplex in $P_n(\Gamma, S)$, we define $E(\gamma_0, \partial\sigma)$ to be the set of words of the form $w_0 w_1 w_2$ where $w_0 \in E(\gamma_0, \gamma_1)$, $w_1 \in E(\gamma_1, \gamma_2)$, and $w_2 \in E(\gamma_2, \gamma_0)$.

Assume that we are given a simplicial map $F : D^2 \to P_n(\Gamma, S)$ where D^2 is a simplicial disk. We associate the subset $E(x_0, x_1)$ to each oriented 1-simplex (x_0, x_1) in D^2. If c is a path in D^2 which is the union of the 1-simplices (x_i, x_{i+1}), $0 \leq i \leq m-1$, we associate the set $E(x_0, c)$ of all the words of the form $w_0 w_1 w_2 \cdots w_m$ ($w_i \in E(x_i, x_{i+1})$) to c.

16. Definition of R_n. Let Γ be a group generated by a finite set S. For every integers $n \geq 1$, the ser R_n is the subset of the free group $L(S)$ consisting of
Words of the form ww'^{-1} with $w, w' \in E(e, \gamma)$ and
(e, γ) a 1-simplex of $P_n(\Gamma, S)$ and
the words in $E(e, \partial\sigma)$ where σ is a two-simplex of $P_n(\Gamma, S)$containing e.

Notice that the words in R_n all have length at most $3n$; in particular, R_n is finite.

17. Proposition. *Let Γ be δ-hyperbolic for a finite system of generators S and let $n \geq 4\delta + 2$. Let R_n be defined as above. Then $< S : R_n >$ is a presentation of Γ.*

Proof. Let $\pi : L(S) \to \Gamma$ be the canonical homomophism and let $w \in \operatorname{Ker}(\pi)$. We must show that w is the product of conjugates of elements in R_n.

Let $w = s_1 \cdots s_n$ be the reduced form of w in $L(S)$, and let S^1 be an oriented simplicial circle with l vertices $p_0 < p_1 < \ldots < p_l = p_0$. Associate to w the pointed simplicial map $f : (S^1, p_0) \to (P_n(\Gamma, S), e)$ which maps the 1-simplex (p_{i-1}, p_i) of S^1 to the 1 simplex $(s_1 \cdots s_{i-1}, s_1 \cdots s_i)$ of $P_n(\Gamma, S)$.

Since $n \geq 4\delta + 2$, the complex $P_n(\Gamma, S)$ is contractible. Therefore, there is an oriented simplicial 2-disk D^2 and a simplicial map $F : D^2 \to P_n(\Gamma, S)$ such that the boundary of D^2 is the is the circle S^1 (with vertices $p_0 < p_1 < \ldots < p_l = p_0$) and such that F extends f. We denote the oriented two simplices of D^2 by $\sigma_1, \sigma_2, \ldots, \sigma_N$.

Given $i \in \{1, \ldots, N\}$ and a vertex x of σ_i, we define the subset $E_F(x, \partial\sigma_i) \subset L(S)$ as follows. If $F(\sigma_i)$ is a 0-simplex, $E_F(x, \partial\sigma_i)$ is the empty word. If $F(\sigma_i)$ is a

1-simplex, $(F(x), F(x'))$, then $E_F(x, \partial\sigma_i)$ is the set of words of the form ww'^{-1} with $w, w' \in E(e, F(x)^{-1}F(x'))$. Finally, if $F(\sigma_i)$ is an oriented 2-simplex of $P_n(\Gamma, S)$, then $E_F(x, \partial\sigma_i)$ us the set

$$E(e, \partial(F(x)^{-1}F(\sigma_i)))$$

defined in number 15. In each case, $E_F(x, \partial\sigma_i)$ is a subset of R_n.

For $i \in \{1, \dots, N\}$, we consider the set A_i of pairs (x, c) where x is a vertex of σ_i and c is an oriented path from p_0 to x in the 1-skeleton of D^2. We use M_i to denote the words of the form $\alpha r \alpha^{-1}$, where $\alpha \in E_F(p_0, c)$ and $r \in E_F(x, \partial\sigma_i)$.

By the construction of f, we have $w \in E_F(p_0, S^1)$. Now, every word in $E_F(p_0, S^1)$ can be written as the prodcut of words $m_i \in M_i$ for $i = 1, \dots, N$, hence is a product of conjugates of words in R_n. $\square$

Translated by WILLIAM GROSSO

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

QUASI-ISOMETRIES AND QUASI-GEODESICS (CHAPTER 5)

Etienne Ghys and Pierre de la Harpe

Abstract. We have already indicated, in chapter 1, how it is sometimes necessary and useful to weaken the relation of isometry to that of quasi-isometry. Here, we similarly introduce the idea of a quasi-isometric map of one space into another; it is an idea which is well-suited to hyperbolic spaces. The quasi-geodesics of a metric space X are then the quasi-isometries (or their images) from the real line (or the rational numbers) into X. For technical reasons, we also introduce local quasi-geodesics in X: these are maps $\mathbb{R} \to X$ or $\mathbb{Z} \to X$ whose restrictions to appropriate intervals are quasi-isometries.

This chapter is dedicated to the proof of a fundamental approximation result for geodesic hyperbolic spaces: within a finite controllable distance from a quasi-geodesic (or even a local quasi-geodesic) there always is a genuine geodesic segment (§1 and §4, thereoms 11 and 22); when the space is also proper, this extends to rays (half-geodesics) and to minimising geodesics (§5, theorem 25). In particular, it follows that, for a finitely generated group, hyperbolicity, which is defined in terms of a finite system of generators, is independent of the system chosen (§2, corollary to theorem 12). Gromov gives a much different, and for now elliptical, method of proof (page 76 of [Gr5]) for this crucial point of the theory. Finally, we give a criterion for when a sequence of points defines a quasi-geodesic segment (§1 , theorem 16) which will be used in chapter 8.

1. Geodesic and Quasi-geodesic Segments.

In the first two definitions, the ideas involving L are not used much before §4.

1. Definition. Consider metric spaces X, X_0, a map $F : X_0 \to X$ and three numbers $\lambda \geq 1, c \geq 0, L > 0$. We say that F is an isometry if

$$|F(s) - F(t)| = |s - t|$$

for all $s, t \in X_0$ (this does not imply that F is surjective), and that F is a (λ, c, L) quasi-isometry if

$$\frac{1}{\lambda}|s-t| - c \leq |F(s) - F(t)| \leq \lambda|s-t| + c$$

for all $s, t \in X_0$ such that $|s - t| \leq L$, and that F is a (λ, c) quasi-isometry if the last inequalities are true for all $s, t \in X_0$.

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

2. Definition. Let X be a metric space and I in $\mathbb{Z}$ or $\mathbb{R}$. An isometry $g : I \to X$ is

(1) A *geodesic segment* in X if I is bounded (see definition 1.25).
(2) A *ray* or a *minimising ray* if I is semi-infinite.
(3) A *minimising geodesic* if I is $\mathbb{Z}$ or $\mathbb{R}$

A (λ, c) quasi-isometry $f : I \to X$ is

(1) A (λ, c) *quasi-geodesic segment* in X if I is bounded.
(2) A (λ, c) *quasi-ray* if I is semi-infinite.
(3) A (λ, c) *quasi-geodesic* if I is $\mathbb{Z}$ or $\mathbb{R}$

A (λ, c, L) local quasi-isometry $f : I \to X$ is

(1) A (λ, c, L) *local quasi-geodesic segment* in X if I is bounded.
(2) A (λ, c, L) *local quasi-ray* if I is semi-infinite.
(3) A (λ, c, L) *local quasi-geodesic* if I is $\mathbb{Z}$ or $\mathbb{R}$

3. Remarks.

(1) We also say quasi-isometry and local quasi-isometry , quasi-geodesic segment and local quasi-geodesic segment,..., if the constants λ, c, L are not important.
(2) It is easy to verify that two spaces X, X_0 are quasi-isometric in the sense of definition 1.7 if and only if there is a quasi-isometry $F : X_0 \to X$ such that $\sup_{x \in X} d(x, \ \mathrm{Im}(F)) < \infty$.
(3) Quasi-geodesic segments, quasi-rays, and quasi-geodesics are clearly invariant under quasi-isometry, in contrast to the ideas introduced by Gromov in §7.2 of [Gr5].
(4) As in the case of segments, we define *parametrised quasi-geodesic segments* and *geometric quasi-geodesic segments*, and we commit the same abuses (see definition 1.25).
(5) A quasi-isometry with domain $\mathbb{R}$ is not necessarily continuous; for example the map $\mathbb{R} \to \mathbb{Z}$ which maps a number to its whole part. A quasi-isometry can be locally "very irregular"; for example, there is an easy construction of a map $f : \mathbb{R} \to \mathbb{R}^2$ which, for every $i \in \mathbb{Z}$, maps the interval $[i, i+1]$ onto the square defined by the equations $i \leq x \leq i+1$ and $0 \leq y \leq 1$ (it is a variant of the Peano Curve).
(6) It might be less abusive to call the quasi-geodesics defined above "minimising quasi-geodesics". Although we do do not do this, in a manifold, a geodesic is classically defined by a local property, and is not necessarily a quasi-geodesic in the above sense.
(7) When $I = \{m, m+1, \ldots, n\}$ is a bounded interval in $\mathbb{Z}$, we often denote a map from I to X by $(x_i)_{m \leq i \leq n}$. We use $(x_i)_{i \geq 0}$ and $(x_i)_{i \in \mathbb{Z}}$ similarly.

4. Definitions. Let X be a metric space, Y and $\mathbb{Z}$ two non-empty subset of X, and H a positive real number.

We call the set $\{x \in X : d(x, Y) \leq H\}$ the *H-neighborhood* of Y in X and denote it by $\nu_H(Y)$.

We define the *Haussdorff distance* of Y and $\mathbb{Z}$, denoted by $\mathcal{H}(Y, \mathbb{Z})$, to be the number defined by

$$\inf\{H > 0 : Y \subset \nu_H(\mathbb{Z})\text{and}\mathbb{Z} \subset \nu_H(Y)\}$$

when this expression makes sense (and ∞ otherwise).

Furthermore, suppose A, B are two non-empty sets and $f : A \to X$, $G : B \to X$ are two maps. We denote the Haussdorff distance between the images $f(A)$, $g(B)$ by $\mathcal{H}(f, g)$ and call this the *Haussdorff distance between f and g.*

5. Remark. The Haussdorff distance is not a distance in the strict sense; First, because it can take on infinite values and, also $H(Y, \mathbb{Z}) = 0$ does not imply that $Y = \mathbb{Z}$. But, this is not important here.

The following result is the central theorem of this chapter. It reappears below in a different, reinforced, form as theorem 11, and one can think of theorems 12, 16 ,21 and 25 as corollaries or variants of it.

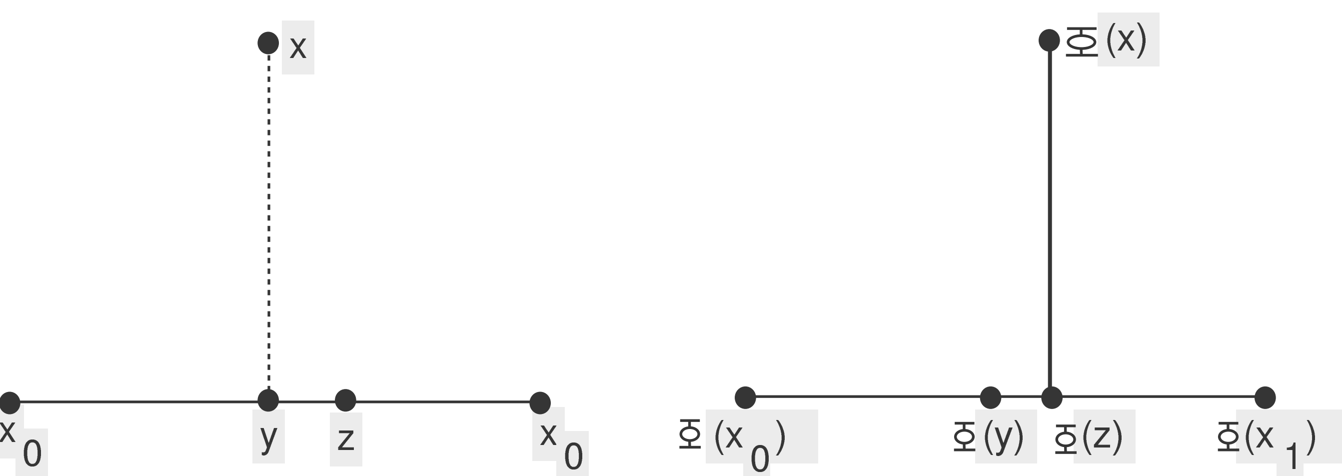
6. Theorem. *Given three real numbers $\delta \geq 0$, $\lambda \geq 1$, and $c > 0$, there is a constant $H = H(\delta, \lambda, c)$ with the following property.*

Let X be a geodesic δhyperbolic space, $I = [0, a]$ a bounded interval in $\mathbb{Z}$ or $\mathbb{R}$, and $f : I \to X$ a (λ, c) quasi-geodesic segment. Denote the interval $[0, |f(a) - f(0)|]$ in $\mathbb{R}$ by J and choose a geodesic segment $g : J \to X$ with origin $f(0)$ and endpoint $f(a)$. Then $Im(f) \subset \nu_H(Im(g))$.

7. Lemma. *Consider a metric space F, two points $x_0, x_n \in F$ and a geodesic segment $[x_0, x_n]$ which joins them. Also consider a constant $c' > 0$, a metric tree T and a map $\Phi : F \to T$ such that*

(1) *The restriction of Φ to $[x_0, x_n]$ is an isometry.*
(2) *We have $|u - v| - c' \leq |\Phi(u) - \Phi(v)| \leq |u - v|$ for all $u, v \in F$.*

Let x be a point in F. Choose $y \in [x_0, x_n]$ such that $|x - y| = d(x, [x_0, x_n])$. Let $z' \in T$ be the point in $[\Phi(x_0), \Phi(x_n)]$ with minimal distance from $\Phi(x)$ and let $z \in [x_0, x_n]$ be such that $\Phi(z) = z'$. Then $|y - z| \leq c'$.



Proof. We have $|x-y| \leq |x-z|$ and

$$|x-y| - c' \leq |\Phi(x) - \Phi(y)| \leq |x-y|$$
$$|x-z| - c' \leq |\Phi(x) - \Phi(z)| \leq |x-z|$$

Therefore,

$$|\Phi(y) - \Phi(z)| = |\Phi(y) - \Phi(x)| - |\Phi(x) - \Phi(z)| \leq |x-y| - |x-z| + c'$$

And it follows that $|y-z| \leq c'$. □

8. Lemma. *The reduction to the case $c = 0$. We use the notation of theorem 6 and consider a (λ, c) quasi-geodesic segment $f : I \to X$.*

Then there is a constant $\lambda' = \lambda'(\lambda, c)$, an interval I' of $\mathbb{Z}$ and a $(\lambda', 0)$ quasi-geodesic segment $f : I' \to X$ from $f(0)$ to $f(a)$ such that

$$Im(f) \subset \nu_H(\ Im(f')) \ \text{where} \ \ H = \max\{(2\lambda^2+1)c, \lambda^2 c + 2\lambda + c\}$$

Proof. First, assume that I is an interval $[0,a] \subset \mathbb{R}$. If $a \leq 4\lambda c$, every point $t \in [0,a]$ is within $2\lambda c$ of $\{0,a\}$, and the distance between $f(t)$ and $f(\{0,a\})$ never exceeds $(2\lambda^2+1)c$. Thus, we choose a geodesic segment f' from $f(0)$ to $f(a)$, and we have $\mathrm{Im}(f) \subset \nu_{(2\lambda^2+1)c}(\ \mathrm{Im}(f'))$.

We can therefore suppose that $a > 4\lambda c$. Choose an integer a' such that $\frac{a}{4\lambda c} < a' < \frac{a}{2\lambda c}$, and let I' denote the interval $\{0, \dots, a'\}$ in $\mathbb{Z}$. Set $t_i = \frac{i}{a'}a$ for all $i \in I'$ (so that $t_0 = 0$ and $t_{a'} = a$). We define $f' : I' \to X$ by $f'(i) = f(t_i)$. One can easily check that

$$\mathrm{Im}(f) \subset \nu_{2(\lambda^2+1)c}(\ \mathrm{Im}(f'))$$

Let $i, j \in I'$ be such that $i \neq j$. By the hypotheses on f, we have

$$\frac{1}{\lambda}|t_i - t_j| - c \leq |f(t_i) - f(t_j)| \leq \lambda|t_i - t_j| + c$$

hence

$$[\frac{1}{\lambda}\frac{a}{a'} - c]|i-j| \leq |f'(i) - f'(j)| \leq [\lambda\frac{a}{a'} + c]|i-j|$$

As $\frac{1}{\lambda}\frac{a}{a'} - c > c$ and $\lambda\frac{a}{a'} + c < 4\lambda^2 c + c$ (by choice of a'), we can set

$$\lambda' = \max\{\frac{1}{c}, \lambda^2 c\}$$

We have

$$\frac{1}{\lambda'}|i-j| \leq |f'(i) - f'(j)| \leq \lambda'|i-j|$$

for all $i, j \in I'$; therefore f' is a $(\lambda', 0)$ quasi-geodesic segment.

Next, assume that I is an interval $\{0, 1, \dots, a\}$ in $\mathbb{Z}$. We set $\lambda' = \max\{\lambda, \lambda^2 c + 2\lambda + c\}$. Let N be the integer such that $\lambda c + 1 \leq N < \lambda c + 2$, and let a' be the integral part of $\frac{a}{N}$. We denote the interval $\{0, \dots, a'\}$ by I' and define $f' : I' \to X$

by $f'(i) = f(Ni)$. for each $i \in I$, there is an $i' \in I'$ such that $|i - Ni'| \leq N$, hence such that

$$|f(i) - f'(i')| \leq \lambda N + c \leq \lambda^2 c + 2\lambda + c \leq lambda'$$

It follows that $\operatorname{Im}(f) \subset \nu_H(\operatorname{Im}(f'))$.

Let $i, j \in I'$ be such that $i \neq j$. Since $\frac{N}{\lambda} - c \geq \frac{1}{\lambda} \geq \frac{1}{\lambda'}$ and $\lambda N + c \leq \lambda'$, we have, by our hypotheses on f

$$\frac{1}{\lambda'}|i-j| \leq \frac{1}{\lambda}|Ni - Nj| - c \leq |f'(i) - f'(j)| \leq \lambda|Ni - Nj| + c \leq \lambda'|i-j|$$

and f' is a $(\lambda', 0)$ quasi-geodesic segment. □

9. Proof of Theorem 6. The preceding lemma shows that we may assume, without loss of generality, that I is an interval $\{0, \ldots, n\}$ of $\mathbb{Z}$ and that $f : I \to X$ is a $(\lambda, 0)$ quasi-geodesic segment. Set $x_i = f(i)$ for each $i \in I$. Choose a geodesic segment $[x_0, x_n]$ with endpoints x_0 and x_n. We need to show that $\{x_i\}_{0 \leq i \leq n}$ is contained within the H-neighborhood $\nu_H([x_0, x_n])$ for a constant H which only depends on δ, λ, and c. The argument which follows is an adaptation of that which one finds in the notes of Thurston (see proposition 5.9.2 of [Thu]).

First, we define H. Choose an integer $N \geq 1$ such that $\log_2(N+2) < \frac{N}{4\lambda\delta} - 2$. We use k to denote the integer satifying $k - 1 < \log_2(N+2) \leq k$ and set $c' = 2(k+1)\delta$ and $R = \frac{1}{2}\lambda N + c'$. Thus, we observe that

$$\begin{aligned}\frac{1}{\lambda} - 2\frac{c'}{N} &> \frac{4\delta}{N}(\log_2(N+2) + 2) - 2\frac{c'}{N} \\ &> \frac{2}{N}(2\delta(k+1) - c') = 0\end{aligned}$$

and, finally, we set

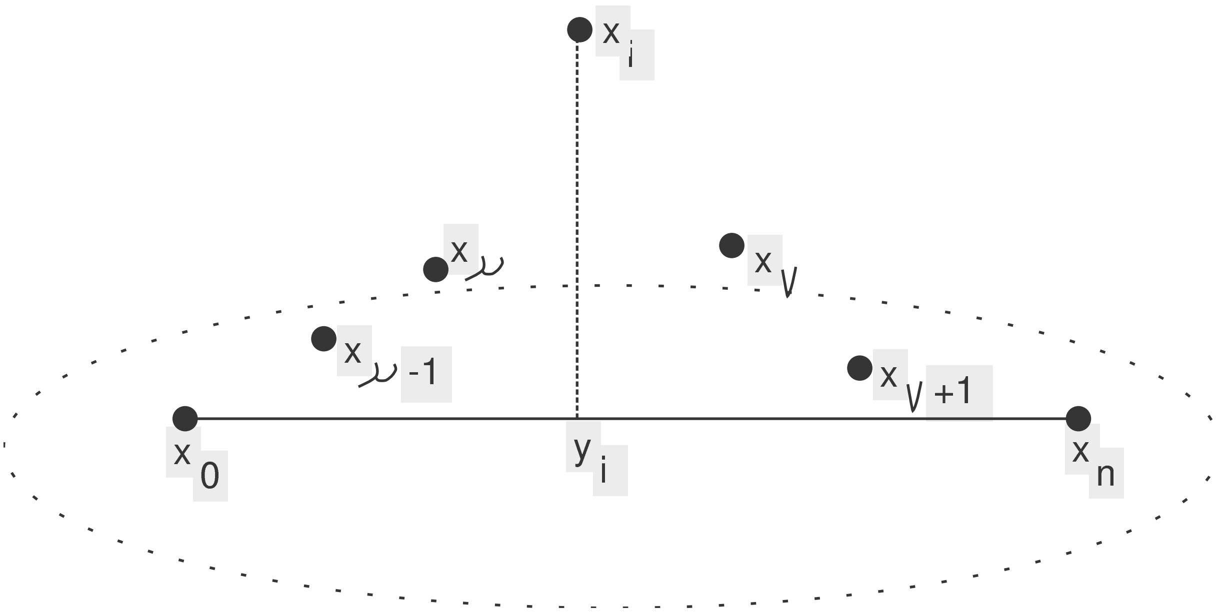
$$H = R + \lambda + 2\lambda\left(\frac{1}{\lambda} - 2\frac{c'}{N}\right)^{-1}(R + \lambda + c')$$

There is nothing to show if $\operatorname{Im}(f)$ is in the R neighborhood of the segment $[x_0, \ldots, x_n]$ and henceforth, we suppose that $\operatorname{Im}(f) \not\subset \nu_R$, where we write ν_R for $\nu_R([x_0, x_n])$.

Consider the integers $u, v \in \{1, 2, \ldots, n-1\}$ such that

$$\begin{gathered} u \leq v \\ x_{u-1} \in \nu_R \\ x_{\nu+1} \in \nu_R \\ \{x_u, x_{u+1}, \ldots, x_v\} \subset X - \nu_R \end{gathered}$$

We set $I_0 = \{u, \ldots, v\}$. For each $i \in I_0$, we choose $y_i \in [x_0, x_n]$ such that $|x_i - y_i| = d(x_i, [x_0, x_n])$.



Consider the integers $i, j \in I_0$ such that $0 \leq j - i \leq N$. We denote the set $[x_0, x_n] \bigcup \{x_i, \dots, x_j\}$ by F and choose x_0 to be the base point of F. Since $N + 2 < 2^k$, theorem 2.12.ii shows that there is a pointed real tree T and a map $\Phi : F \to T$ such that

(1) The restriction of Φ to $[x_0, x_n]$ is an isometry.
(2) We have $|p - q| - c' \leq |\Phi(p) - \Phi(q)| \leq |p - q|$ for all $p, q \in F$.

We have

$$d(\Phi(x_i), [\Phi(x_0), \Phi(x_n)]) \geq R - c'$$
$$d(\Phi(x_j), [\Phi(x_0), \Phi(x_n)]) \geq R - c'$$
$$|\Phi(x_i) - \Phi(x_j)| \leq |x_i - x_j| \leq \lambda N$$

Since $\lambda N = 2(R - c')$, the point in $[\Phi(x_0), \Phi(x_n)]$ a minimum distance from $\Phi(x_i)$ coincides with the point having minimum distance from $\Phi(x_j)$. It follows from lemma 7 that $|y_i - y_j| \leq 2c'$.

By successively applying the preceding argument to the pairs

$$(i, j) = (u, u + N), (u + N, u + 2N), \dots$$

We obtain

$$|x_u - x_v| \leq |x_u - y_u| + |y_u - y_{u+N}| + |y_{u+N} - y_{U+2N}| + \dots + |y_v - x_v|$$
$$\leq R + \lambda + (\frac{|u - v|}{N} + 1)2c' + \lambda + R$$

But, on the other hand, $\frac{1}{\lambda}|u - v| \leq |x_u - x_v|$ and it follows that

$$|u - v|(\frac{1}{\lambda} - 2\frac{c'}{N}) \leq 2(R + \lambda + c')$$

It follows that

$$|x_u - x_i| \leq \lambda(\frac{1}{\lambda} - 2\frac{c'}{N})^{-1} 2(R + \lambda + c')$$

for all $i \in \{u, u+1, \ldots, v\}$ so that $\{x_u, x_{u+1}, \ldots, x_v\}$ is in the ball centered at x_u with radius $H - R - \lambda$, and hence in $\nu_H[x_0, x_n]$.

The preceding statement is equivalent to : for all "connected components" of $\{x_u, x_{u+1}, \ldots, x_v\}$ in $\mathrm{Im}(f) \cap (X - \nu_R([x_0, x_n]))$ we have $\mathrm{Im}(f) \subset \nu_H([x_0, x_n])$. □

Theorem 11 uses the same idea as proposition 6.

10.Lemma. *Given three real numbers $\lambda \geq 1$,$c \geq 0$ and $H' \geq 0$, there is a constant $H = H(\lambda, c, H')$ with the following property.*

Let X be a metric space, $I = [p, q]$ and $J = [r, s]$ two intervals in $\mathbb{Z}$ or $\mathbb{R}$, with two maps $f : I \to X$ and $g : J \to X$. We assume that

1. *f is a (λ, c) quasi-geodesic segment.*
2. *g is a geodesic segment.*
3. *$|f(p) - g(r)| \leq H'$ and $|f(q) - g(s)| \leq H'$.*
4. *$Im(f) \subset \nu_{H'}(Im(g))$.*

then, $\mathcal{H}(f, f') \leq H$

Proof. The constant $H = 2H' + \lambda + c$ is suitable. In fact, consider a partition $p = t_0 < t_1 < \ldots < t_n = q$ of I such that $|t_i - t_{i-1}| \leq 1$ for all $i \in \{1, \ldots, n\}$. By hypothesis, for all $i \in \{0, \ldots, n\}$ there is a point $u_i \in J$ such that $|f(t_i) - g(u_i)| \leq H'$, and we can suppose that $u_0 = r$ and $u_n = s$. Since

$$|f(t_{i-1}) - f(t_i)| \leq \lambda + c$$

we also have

$$|g(u_{i-1}) - g(u_i)| \leq 2H' + \lambda + c$$

For all $u \in J$, there exists an $i \in \{0, \ldots, n\}$ such that

$$|u - u_i| = |g(u) - g(u_i)| \leq H' + \frac{1}{2}(\lambda + c)$$

It follows that $d(g(u),\ \mathrm{Im}(f)) \leq 2H' + \frac{1}{2}(\lambda + c)$ □

11. Theorem. *Given three real numbers $\delta \geq 0$, $\lambda \geq 1$, and $c \geq 0$, there exists a constant $H = H(\delta, \lambda, c)$ with the following property.*

Let X be a geodesic δ-hyperbolic space, $I = [a, b]$ a bounded interval in $\mathbb{Z}$ or $\mathbb{R}$, and $f : I \to X$ a (λ, c) quasi-geodesic segment. Let $J \subset \mathbb{R}$ be an interval of length $|f(a) - f(b)|$ and let $g : J \to X$ be a geodesic segement with origin $f(a)$ and endpoint $f(b)$. Then $\mathcal{H}(f, g) \leq H$.

Proof. This result follows from proposition 6 and lemma 10. □

2. Invariance of Hyperbolicity under Quasi-Isometry and Quasi-Invariance of the Gromov Product

The following result (which immediately implies theorem 1.29) is essential for the idea of a hyperbolic group to even make sense.

12. Theorem. *Consider two geodesic metric spaces X, Y and a quasi-isometry $F : X \to Y$. If Y is hyperbolic, then X is too.*

Proof. We use $\delta \geq 0$, $\lambda \geq 1$ and $c \geq 0$ to denote the constants such that Y is δ-hyperbolic and such that F is a (λ, c) quasi-isometry. Denote the constant $H(\delta, \lambda, c)$ of theorem 11 by H.

Let I be a bounded interval in $\mathbb{R}$ (or in $\mathbb{Z}$) and let $g : I \to X$ be a geodesic segment. Then $F \circ g : I \to Y$ is a (λ, c) quasi-geodesic segment. It follows that, if $(F \circ g)_0$ denotes a geodesic segment with the same endpoints as $F \circ g$, we have $\mathcal{H}(F \circ g, (F \circ g)_0) \leq H$.

Consider a geodesic triangle $\triangle$ in X and denote its three sides by $g_j : I_j \to X$ $(j = 1, 2, 3)$. We obtain,as above, a geodesic triangle $F\triangle_0$ in Y with edges $(F \circ g_j)_0 : J_j \to Y$. For each point y in the image of $(F \circ g_3)_0$, we have, by the hypothesis on Y and δ and by proposition 2.21

$$d(y,\ \mathrm{Im}((F \circ g_1)_0) \bigcup \mathrm{Im}((F \circ g_2)_0)) \leq 4\delta$$

For each point y' in the image of $F \circ g_3$, we therefore have

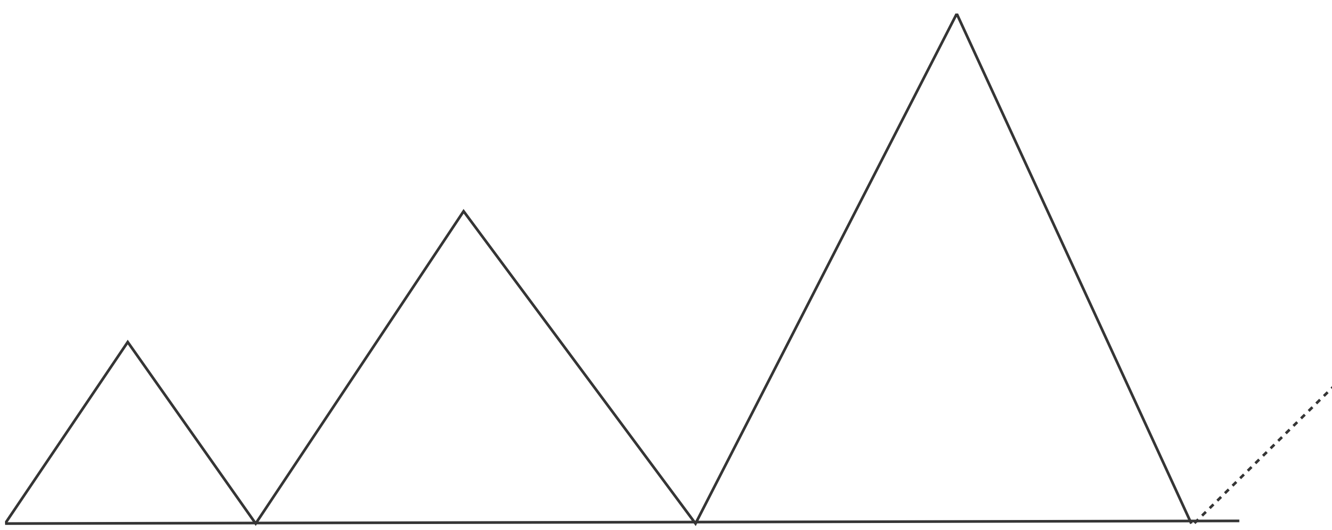
$$d(y',\ \mathrm{Im}(F \circ g_1) \bigcup \mathrm{Im}(F \circ g_2)) \leq 4\delta + 2H$$

It follows that , for each point x in the image of g_3, we have

$$d(x,\ \mathrm{Im}(g_1) \bigcup \mathrm{Im}(g_2)) \leq \lambda(4\delta + 2H + c)$$

By again applying proposition 2.21, we see that X is δ'-hyperbolic with $\delta' = 8\lambda(4\delta + 2H + c)$. $\square$

13. Remark (J. Heber). Theorem 12 is no longer correct if we omit the assumption that X is geodesic. Indeed, let X be the broken semi-infinite line in in the Euclidean plane $\mathbb{C}$ defined by $X = \bigcup_{n \geq 0} \phi^n(X_0)$, with $X_0 = [0, 1+i] \bigcup [1+i, 2]$ and $\phi : z \mapsto 2z + 2$. Let Y be the half-axis $\mathbb{R}^+$ and let $F : X \to Y$ be the vertical projection. We give X and Y the metrics induced by the usual metric on $\mathbb{C}$. The F is a $(\sqrt{2}, 0)$ quasi-isometric bijection and Y is 0-hyperbolic, but X is not hyperbolic.



14. Corollary. *Let Γ be a finitely generated group, S, T two finite systems of generators of Γ such that $S = S^{-1}$ and $T = T^{-1}$, and let $G(\Gamma, S)$, $G(\Gamma, T)$ be the associated Cayley Graphs as in §1.2. Then $G(\Gamma, S)$ is hyperbolic if and only if $G(\Gamma, T)$ is.*

Proof. This follows from theorem 12, because, as we saw in 1.9, the spaces $G(\Gamma, S)$ and $G(\Gamma, T)$ are quasi-isometric. □

So, the notion of hyperbolic group makes sense, as we had announced in definition 1.30

In view of chapter 7 (which concerns the boundary), it is also useful to study the effects of a quasi-isometry on Gromov Products. The next proposition is a contribution (among others) of M. Berger to our text.

15. Proposition. *Given three real numbers $\delta \geq 0$, $\lambda \geq 1$ and $c \geq 0$, there is a constant $A = A(\delta, \lambda, c)$ with the following property.*

Let X, Y be two geodesic δ-hyperbolic spaces and $F : X \to Y$ a (λ, c) quasi-isometry. Let w, x, y, z be points in X. Then

(1) $\frac{1}{\lambda}(x|y)_w - A \leq (F(x)|F(y))_{F(w)} \leq \lambda(x|y)_w + A$

(2) $\frac{1}{\lambda}|(x|y)_w - (y|z)_w| - A \leq |(F(x)|F(y))_{F(w)} - (F(y)|F(z))_{F(w)}|$

$$\leq \lambda|(x|y)_w - (y|z)_w| + A$$

Proof. Denote the constant $H(\delta, \lambda, c)$ of theorem 11 by H.

(1) We choose geodesic segments $[x, y]$ in X and $[F(x), F(y)]$ in Y. We have

$$(x|y)_w \leq d(w, [x, y]) \leq (x|y)_w + \delta$$
$$(F(x)|F(y))_{F(w)} \leq d(F(w), [F(x), F(y)]) \leq (F(x)|F(y))_{F(w)} + \delta$$

by lemma 2.17 and

$$\mathcal{H}(F([x, y]), [F(x), F(y)]) \leq H$$

by theorem 11. On the other hand,

$$\frac{1}{\lambda}d(w, [x, y]) - c \leq d(F(w), F([x, y])) \leq \lambda d(w, [x, y]) + c$$

by the hypothesis on F. We therefore have

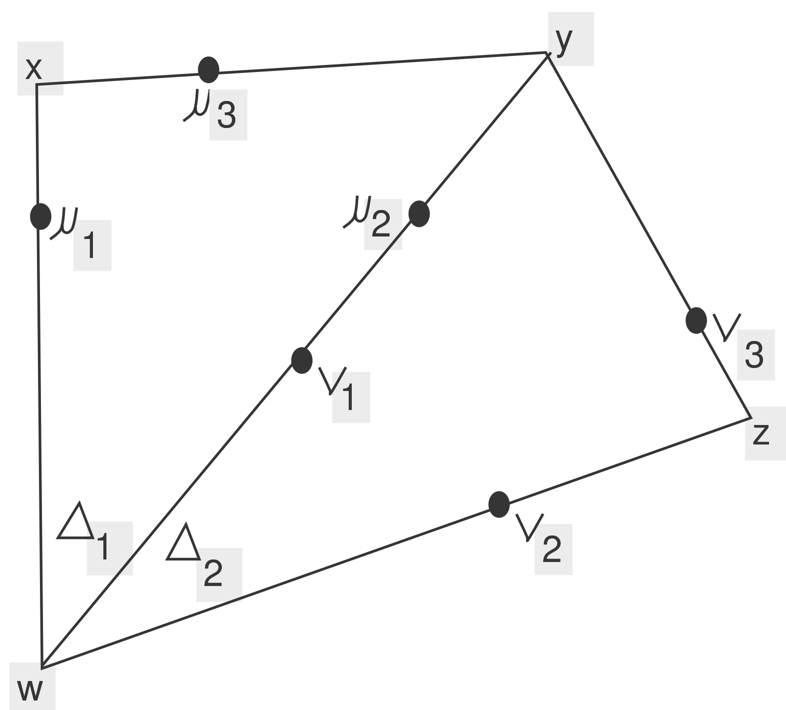
$$\begin{aligned}(F(x)|F(y))_{F(w)} &\leq d(F(w), [F(x), F(y)]) \\ &\leq d(F(w), F([x, y])) + H \\ &\leq \lambda d(w, [x, y]) + c + H \\ &\leq \lambda(x|y)_w + \lambda\delta + c + H\end{aligned}$$

and, similarly

$$(F(x)|F(y))_{F(w)} \geq \frac{1}{\lambda}(x|y)_w - \lambda\delta - c - H$$

Every constant $A \geq \lambda\delta + c + H$ is therefore suitable for 1.

(2) We choose geodesic segments $[w,x],[w,y][w,z],[x,y] and [y,z]$ in X. We use $\triangle_1$ to denote the geodesic triangle $[w,x]\bigcup[w,y]\bigcup[x,y]$ and $u_1 \in [w,x]$, $u_2 \in [w,y]$, and $u_3 \in [x,y]$ to denote its inscribed triple of points (the inverse image of the center of the corresponding tripod, see definition 2.18).


x
y
z
w
μ3
μ1
μ2
ν1
ν3
ν2
Δ1
Δ2

We use $\triangle_2$ to denote the geodesic triangle $[w,y]\bigcup[w,z]\bigcup[y,z]$ $v_1\in[w,y]$, $v_2\in[w,z]$, and $v_3\in[y,z]$ to denote the points of its inscribed triple. We have that

$$|w-u_1|=|w-u_2|=(x|y)_w$$
$$|w-v_1|=|w-v_2|=(y|z)_w$$

Similarly, we introduce a geodesic triangle $\triangle_1'$ in Y

$$\triangle_1'=[F(w),F(x)]\bigcup[F(w),F(y)]\bigcup[F(x),F(y)]$$

with inscribed triple u_1', u_2', u_3' and a geodesic triangle

$$\triangle_2'=[F(w),F(y)]\bigcup[F(w),F(z)]\bigcup[F(y),F(z)]$$

with inscribed triple v_1', v_2', v_3'. We also have

$$|F(w)-u_1'|=|F(w)-u_2'|=(F(x)|F(y))_{F(w)}$$
$$|F(w)-v_1'|=|F(w)-v_2'|=(F(y)|F(z))_{F(w)}$$

In view of theorem 11, we can also choose points

$$u_1''\in[F(w),F(x)]$$
$$u_2''\in[F(w),F(y)]$$
$$u_3''\in[F(x),F(y)]$$
$$v_1''\in[F(w),F(y)]$$
$$v_2''\in[F(w),F(z)]$$
$$v_3''\in[F(y),F(z)]$$

such that

$$|F(u_i)-u_i''|\le H \qquad i=1,2,3$$
$$|F(v_i)-v_i''|\le H \qquad i=1,2,3$$

The tails of $\triangle_1$ and $\triangle_2$ are less than 4δ (see definition 2.18 and proposition 2.21), and hence the diamters of the sets $\{F(u_1),F(u_2),F(u_3)\}$ and $\{F(v_1),F(v_2),F(v_3)\}$ are less than $4\delta\lambda+c$. It follows that the diamters of $\{u_1'',u_2'',u_3''\}$ and $\{v_1'',v_2'',v_3''\}$ are less than $4\delta\lambda+c+2H$. The argument used in the proof of lemma 2.20 shows that

$$|u_2''-u_2'|\le\frac{3}{2}(4\delta\lambda+c+2H)$$
$$|v_1''-v_1'|\le\frac{3}{2}(4\delta\lambda+c+2H)$$

If we set $B = \frac{3}{2}(4\delta\lambda + c + 2H) + H$ and $A = 2B + c$, we have, in light of the preceding equations,

$$|F(u_2) - u_2'| \leq B \qquad \text{and} \qquad |F(v_1) - F(v_1')| \leq B$$

hence

$$\begin{aligned} |(F(x)|F(y))_{F(w)} - (F(y)|F(z))_{F(w)}| &= ||F(w) - u_2'| - |F(w) - v_1'|| \\ &= |u_2' - v_1'| \\ &\leq |F(u_2) - F(v_1)| + 2B \\ &\leq \lambda|u_2 - v_1| + 2B + c \\ = \lambda|(x|y)_w - (y|z)_w| + A \end{aligned}$$

We get the lower bound in the same way

□

3. A Criterion For Quasi-Geodesics.

We consider a constant $\delta > 0$ and a geodesic hyperbolic space X which satisfies the Rips condition with constant δ (that is, $d(u, [x, y] \bigcup [z, x]) \leq \delta$ for every geodesic triangle $[x, y] \bigcup [y, z] \bigcup [z, x]$ and for every $u \in [y, z]$ — see definition 1.27). The criterion of this section is an approximation of result 7.2C in [Gr5]

16. Theorem. *We consider a constant $\kappa > 0$ and a sequence $(x_i)_{0 \leq i \leq n}$ of points in X such that*

$$|x_{i-1} - x_{i+1}| \geq max\{|x_{i-1} - x_i|, |x_i - x_{i+1}|\} + 18\delta + \kappa$$

for each $i \in \{1, \ldots, n-1\}$. Then $|x_0 - x_n| \geq \kappa n$.

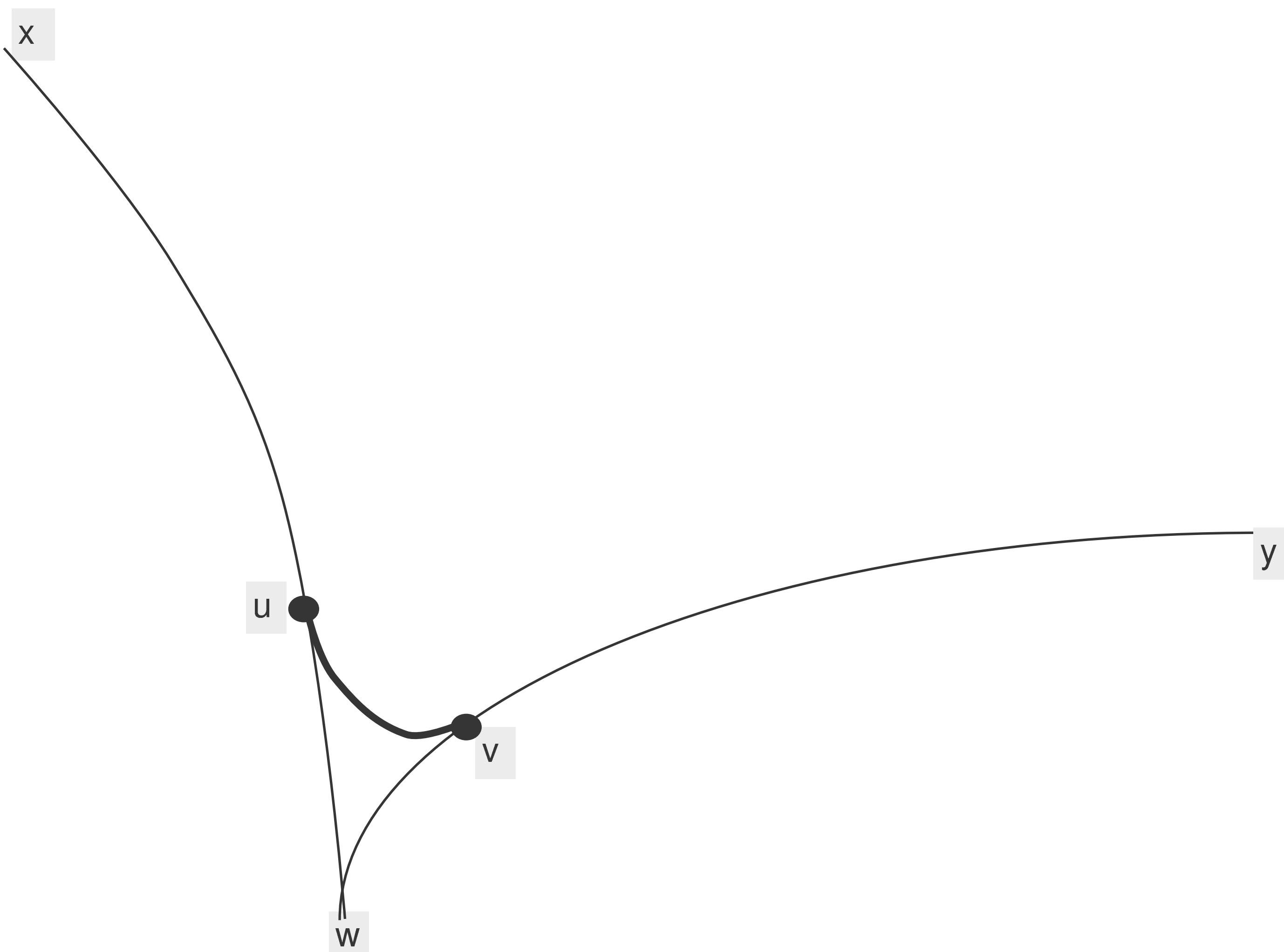
Suppose, furthermore, that L is a constant such that $|x_{i-1} - x_i| \leq L$ for all $i \in \{1, \ldots, n-1\}$. If we set $\lambda = max\{\frac{1}{\kappa}, L\}$, then $(x_i)_{0 \leq i \leq n}$ is a $(\lambda, 0)$ quasi-geodesic segment.

The second assertion follows easily from the first. For the proof of the first statement (and, in particular, in numbers 17 and 19), we adapt arguments of J. Cannon (see [Ca1] and [Ca2]).

17. Definition. *Let $[w, x]$ and $[w, y]$ be two geodesic segments with the same origin in X, and let $c > 0$ be a constant. We call the quantity*

$$\sup\{\max\{|u - w|, |v - w|\} \ : \ u \in [w, x]\ , v \in [w, y]\ \textit{and}\ \ |u - c| \leq c\}$$

the $c-$deviation of $[w, x]$ and $[w, y]$ and denote it by $dev_c([w, x], [w, y])$.


x
y
u
v
w

18. Remarks. We use the notation of definition 17, but write dev_c for $\mathrm{dev}_c([w,x],[w,y])$.

(1) We clearly have $c \leq \mathrm{dev}_c \leq max\{|x-w|,|y-w|\}$, and one easily verifies that $\mathrm{dev}_c \leq (x|y)_w + c$.
(2) Furthermore, suppose that $c \geq 2\delta$. We also have

$$|x-w| + |y-w| - 2(\mathrm{dev}_c + \delta) \leq |x-y| \leq |x-w| + |y-w|$$

In fact, choose a segment $[x,y]$ and let p be the point in $[x,y]$ farthest from x such that $d(p,[w,x]) \leq \delta$; then $d(p,[w,y]) \leq \delta$. Let $u \in [w,x]$ and $v \in [w,y]$ be points such that $|p-u| \leq \delta$ and $|p-v| \leq \delta$. Then $|u-v| \leq c$, hence $|u-w| \leq rmdev_c$ and $|v-w \leq \mathrm{dev}_c$. It follows that

$$|x-w| + |y-w| \leq |x-u| + |y-v| + 2\mathrm{dev}_c \leq |x-p| + |y-p| + 2\mathrm{dev}_c$$

and the assertion follows because $|x-y| = |x-p| + |p-y|$.

19. Lemma. *We consider a constant $\kappa > 0$ and a sequence of points $(x_i)_{0\leq i\leq n}$ in X. For each $i \in \{1,\dots,n\}$, we choose a geodesic segment $[x_{i-1},x_i]$ joining x_{i-1} and x_i; we set $dev_0 = 0$ and*

$$dev_i = dev_{3\delta}([x_{i-1},x_i],[x_i,x_{i+1}])$$

for each $i \in \{1,\dots,n\}$. If we have

$$|x_{i-1} - x_i| > dev_{i-1} + dev_i + 5\delta \qquad \text{for } i \in \{1,\dots,n-1\}$$
$$|x_{i-1} - x_i| \geq 2dev_{i-1} + 12\delta + \kappa \qquad \text{for } i \in \{1,\dots,n-1\}$$

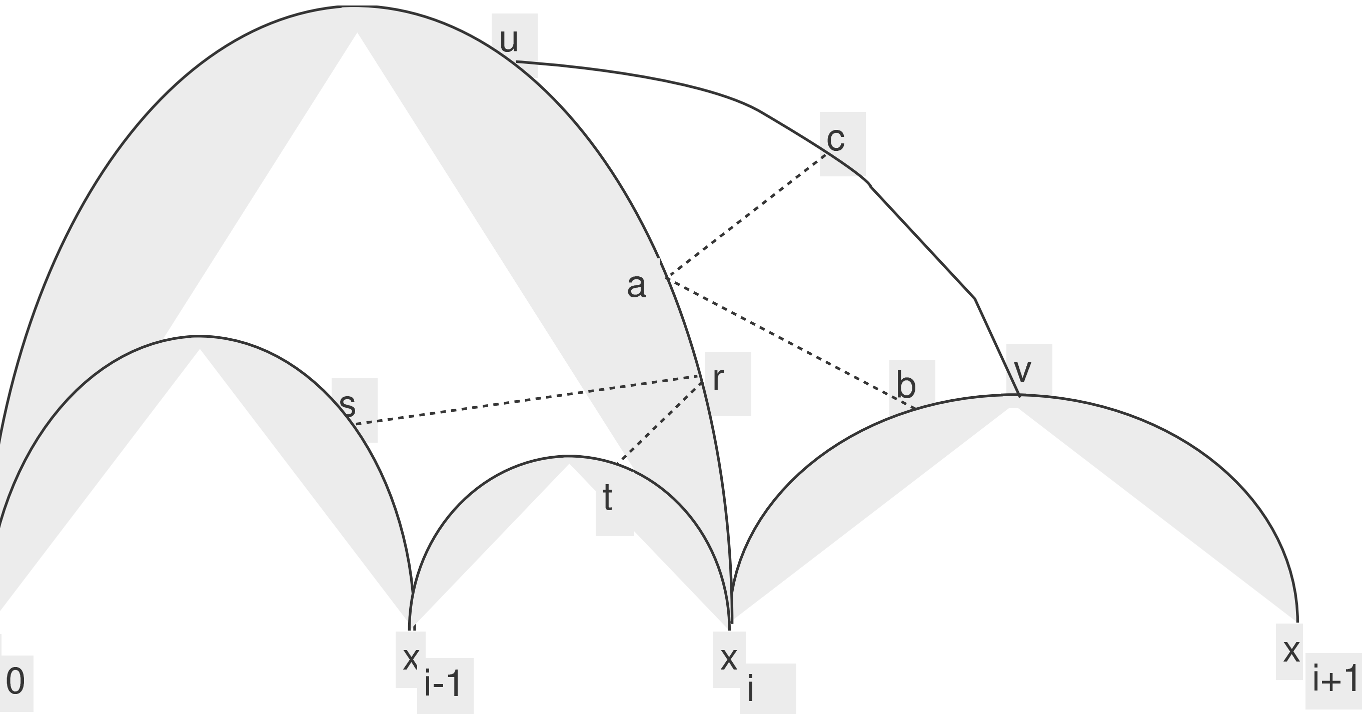
then

$$|x_0 - x_n| \geq \kappa n$$

Proof. for every $i \in \{2,\dots,n\}$, we choose a geodesic segment $[x_0,x_i]$. We first show that

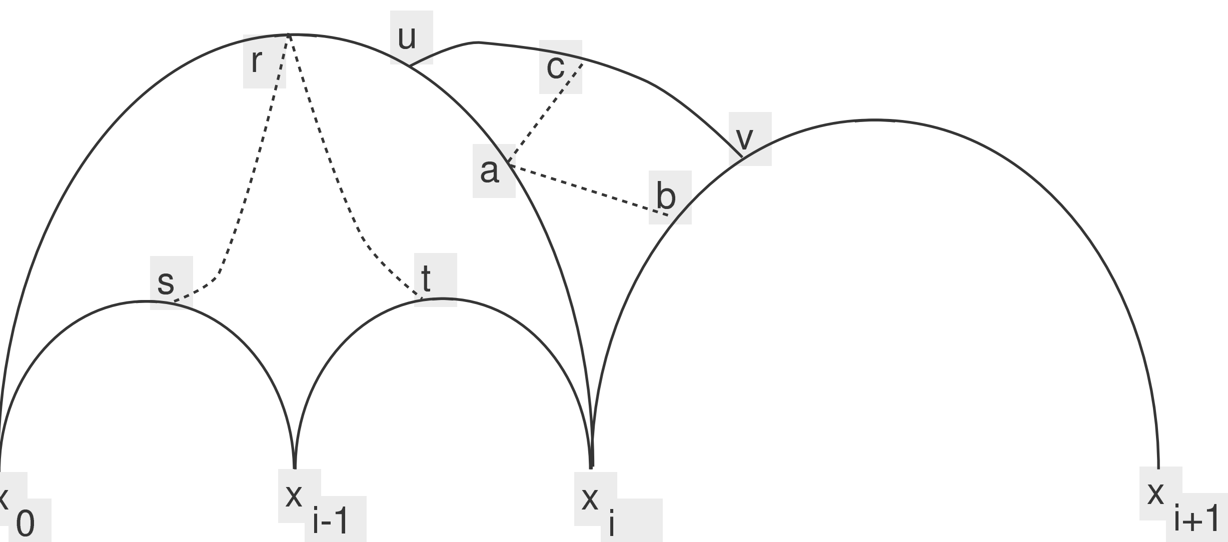
$$\mathrm{dev}_{3\delta}([x_0,x_i],[x_i,x_{i+1}]) \leq \mathrm{dev}_i + 5\delta$$

This assertion is trivial for $i=1$; we proceed by induction on i. Hence, suppose $i \geq 2$ and that the assertion is true up to $i-1$. Let $u \in [x_0,x_i]$ and $v \in [x_i, xi+1]$ be points such that $|u-v| \leq 3\delta$. we must show that $|u-x_i|$ and $|v-x_i|$ are less than $\mathrm{dev}_i + 5\delta$.



We choose a point $r \in [x_0, x_i]$ for which there is $s \in [x_0, x_{i-1}]$ and $t \in [x_{i-1}, x_i]$ such that $|r-s| \leq \delta$ and $|r-t| \leq \delta$. Similarly, we choose $a \in [u, x_i]$, $b \in [x_i, v]$ and $c \in [u, v]$ such that $|a-b| \leq \delta$ and $|a-c| \leq \delta$.

Suppose that $r \in [a, x_i]$. By considering a triangle with vertices x_i, a, b, we see that $d(r, [x_i, b]) \leq 2\delta$. It follows that $d(t, [x_i, b]) \leq 3\delta$ and hence that $|t - x_i| \leq \mathrm{dev}_i$. But, $|t-s| \leq 2\delta$, hence $|t - x_{i-1}| \leq \mathrm{dev}_{i-1} + 5\delta$ by the induction hypothesis. We therefore have $|x_{i-1} - x_i| \leq \mathrm{dev}_{i-1} + \mathrm{dev}_i + 5\delta$, which is contrary to this lemma's hypotheses.


u
r
c
v
a
b
s
t
x 0
x i-1
x i
x i+1

We can therefore suppose that $r \in [x_0, a]$. By considering a triangle with vertices x_i, r, t, we see that $d(a, [x_i, t]) \leq 2\delta$; it follows that $d(b, [t, x_i]) \leq 3\delta$ and hence that $|b - x_i| \leq \mathrm{dev}_i$. Since $|v-b| \leq |v-c|+|c-a|+|a-b| \leq 5\delta$, we have $v - x_i| \leq \mathrm{dev}_i + 5\delta$. Finally,

$$|u - x_i| \leq |u - c| + |c - a| + |a - b| + |b - x_i| \leq \mathrm{dev}_i + 5\delta$$

and we have finished the induction on i.

The assertion of the lemma if trivial if $n = 1$, and we now proceed to induct on n. We assume that $n \geq 2$ and $|x_0 - x_{n-1}| \geq \kappa(n-1)$. Remark 18.2 shows that

$$|x_0 - x_n| \geq |x_0 - x_{n-1}| + |x_{n-1} - x_n| - 2\mathrm{dev}_{3\delta}([x_0, x_{n-1}], [x_{n-1}, x_n]) - 2\delta$$

Now, the first part of this proof showed that

$$\begin{aligned} |x_{n-1} - x_n| - 2\mathrm{dev}_{3\delta}([x_0, x_{n-1}], [x_{n-1}, x_n]) - 2\delta &\geq |x_{n-1} - x_n| - 2\mathrm{dev}_{n-1} - 12\delta \\ &\geq \kappa \end{aligned}$$

Therefore, we have that $|x_0 - x_n| \geq \kappa n$. □

20. Proof of Theorem 16. It suffices to show that the hypothesis of lemma 19 are satisfied. We choose $i \in \{1, \ldots, n-1\}$ and two points $u \in [x_{i-1}, x_i], v \in [x_i, x_{i+1}]$ such that $|u - v| \leq 3\delta$. As in chapter 2, we now associate a metric tripod T and a map $f : [x_{i-1}, x_i] \bigcup [x_i, x_{i+1}] \to T$. Since f doesn't increase distances, $|f(u) - f(v)| \leq 3\delta$. Hence,

$$|u - x_i| = |f(u) - f(x_i)| \leq 3\delta + (x_{i-1}|x_{i+1})_{x_i}$$

and a similar formula for $|v - x_i|$. Also,

$$\mathrm{dev}_i \leq 3\delta + (x_{i-1}|x_{i+1})_{x_i}$$

First, it follows that, for $i \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned} \mathrm{dev}_{i-1} + \mathrm{dev}_i + 5\delta &\leq 11\delta + \frac{1}{2}\{|x_{i-2} - x_{i-1}| + |x_{i-1} - x_i| - |x_{i-2} - x_i| + |x_{i-1} - x_i| + |x_i - x_{i+1} - |x_{i-1} - x_{i+1}|\} \\ &\leq 11\delta + |x_{i-1} - x_i| + \frac{1}{2}\{|x_{i-2} - x_{i-1}| - |x_{i-2} - x_i| + |x_i - x_{i+1} - |x_{i-1} - x_{i+1}|\} \\ &\leq 11\delta + |x_{i-1} - x_i| - \frac{1}{2}\{|x_{i-2} - x_i| - |x_{i-2} - x_{i-1}|\} - \frac{1}{2}\{|x_{i-1} - x_{i+1}| - |x_i - x_{i+1}|\} \\ &\leq 11\delta + |x_{i-1} - x_i| - 18\delta - \kappa \leq |x_{i-1} - x_i| \end{aligned}$$

It also follows that, for $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} 2\mathrm{dev}_{i-1} + 12\delta + \kappa &\leq 18\delta + \kappa + |x_{i-1} - x_i| + |x_i - x - i + 1| - |x_{i-1} - x_i| \\ &\leq |x_{i-1} - x_i| \end{aligned}$$

Hence, we can apply lemma 19. □

4. Local Quasi-Geodesic Segments.

Using theorem 16, we are going to show the following extension of theorem 11

21. Theorem. *Given 3 real numbers $\delta \geq 0, \lambda \geq 1$ and $c \geq 0$, there exists a constant $H = H(\delta, \lambda, c)$ and $L = L(\delta, \lambda, c)$ with the following property:*

Let X be a geodesic δ-hyperbolic space, $I = [a, b]$ a bounded interval in $\mathbb{Z}$ or $\mathbb{R}$, and $f : I \to X$ a (λ, c, L) local quasi-geodesic segment. Let $J \subset \mathbb{R}$ be an interval of length $|f(a) - f(b)|$, and let $g : J \to X$ be a geodesic segment with origin $f(a)$ and endpoint $f(b)$.

Then $\mathcal{H}(f, g) \leq H$.

In the rest of this section, X denotes a geodesic δ-hyperbolic space. Here is, first, a special case of theorem 21 corresponding to $I \subset \mathbb{Z}$ and $c = 0$.

22. Lemma. *Given a real number $\Lambda \geq 1$, there is a constant $H_2 = H_2(\delta, \Lambda)$ and $L_2 = L_2(\delta, \Lambda)$ with the following property: Consider an integer $n \geq 1$, a $(\Lambda, 0, L_2)$ local quasi-geodesic segment $(y_i)_{0 \leq i \leq n}$ and a geodesic segment $[y_0, y_n]$ joining y_0 to y_n. Then*

$$\mathcal{H}(\{y_i\}_{0 \leq i \leq n}, [y_0, y_n]) \leq H_2$$

Proof. We denote the constant $H(\delta, \Lambda, 0)$ of theorem 11 by H_1 and choose an integer N such that

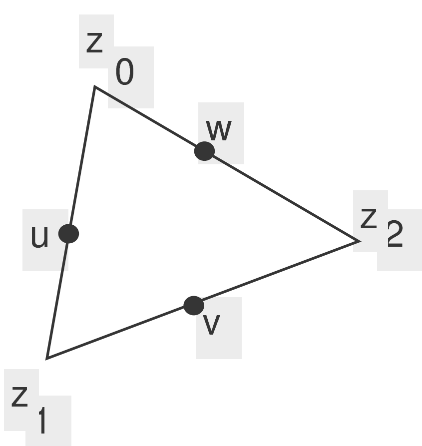
$$N \geq 2\Lambda(18\delta + 1) \qquad \text{and} \qquad N \geq 4\Lambda(H_1 + 2\Lambda^2 H_1 + 4\Lambda^2\delta)$$

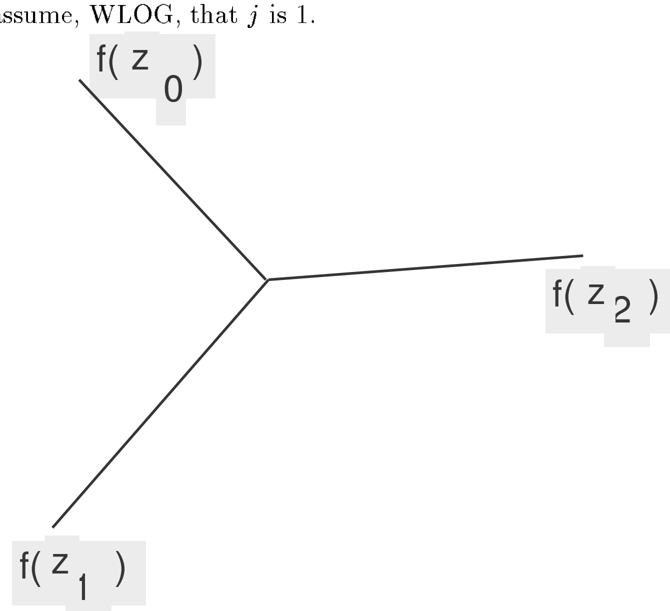
and set $L_2 = 2N$. The constant H_2 will be given precisely below so that $H_2 \leq 2\Lambda N$.

Let p be the whole part of $\frac{n}{N}$. We define a sequence $(z_j)_{0 \leq j \leq p}$ by $z_j = y_{Nj}$. If $n < 2N$, then $|y_i - y_0| \leq \Lambda 2N$ for each $i \in \{0, 1, \ldots, n\}$ and the lemma is clearly true. We may therefore suppose that $p \geq 2$. We show that

$$(*) \qquad |z_{j-1} - z_{j+1}| \leq \max\{|z_{j-1} - z_j|, |z_j - z_{j+1}|\} + 18\delta + 1$$

for all $j \in \{1, \ldots, p-1\}$. We may assume, WLOG, that j is 1.





Choose geodesic segments $[z_0, z_1]$, $[z_1, z_2]$ and $[z_0, z_2]$. This defines a geodesic triangle $\triangle$ and we associate a metric tripod T and a map $f : \triangle \to T$ to $\triangle$ as in chapter 2. We denote the inverse images of the center of T by $u \in [z_0, z_1]$, $v \in [z_1, z_2]$, and $w \in [z_0, z_2]$. Recall that $|u - v| \leq 4\delta$ (proposition 2.21). Theorem 11 shows that there are integers i, j such that $0 \leq i \leq N \leq j \leq 2N$ and such that

$$|y_i - u| \leq H_1 \quad \text{and} \quad |y_j - v| \leq H_1$$

Since $|i - j| \leq 2N = L_2$, we have $\frac{1}{\Lambda}|i - j| \leq |y_i - y_j| \leq 2H_1 + 4\delta$ and hence that $|i - N| \leq \Lambda(2H_1 + 4\delta)$. Therefore

$$|u - z_1| \leq |u - y_i| + |y_i z_1| \leq H_1 + \Lambda^2(2H_1 + 4\delta)$$

On the other hand, $|z_0 - z_1| \geq \frac{1}{\Lambda}N$. One hypothesis on N implies that we have $|u - z_1| \leq \frac{1}{4}|z_0 - z_1|$. Similarly, $|u - z_1| = |v - z_1| \leq \frac{1}{4}|z_1 - z_2|$. We can now compute

$$\begin{aligned}
|z_0 - z_2| &= |z_0 - w| + |w - z_2| = |z_0 - z_1| + |z_1 - z_2| - |u - z_1| - |v - z_1| \\
&\geq |z_0 - z_1| + |z_1 - z_2| - \frac{1}{2}\min\{|z_0 - z_1|, |z_1 - z_2|\} \\
&= \max\{|z_0 - z_1|, |z_1 - z_2|\} + \frac{1}{2}\min\{|z_o - z_1|, |z_1 - z_2|\} \\
&\geq \max\{|z_0 - z_1|, |z_1 - z_2|\} + \frac{N}{2\Lambda}
\end{aligned}$$

The other hypothesis on N shows that the inequality (*) is therefore true.

Since $(y_i)_{0 \leq i \leq n}$ is a $(\Lambda, 0, L_2)$ quasi-geodesic segment, we have that $|z_{j-1} - z_j| \leq \Lambda N$ for all $j \in \{1, \ldots, p\}$. Therefore, inequality (*) and theorem 16 show that $(z_j)_{0 \leq j \leq p}$ is a $(\lambda N, 0)$ quasi-geodesic segment.

We let H_1' denote the constant $H(\delta, \Lambda N, 0)$ of theorem 11 and and choose a segment $[y_0, y_{Np}]$ joining $y + 0 = z_0$ to $y_{Np} = z_p$. then $d(z_j, [y_0, y_{Np}]) \leq H_1'$ for all $j \in \{0, \ldots, p\}$. Each point of $[y_0, y_{Np}]$ is within 4δ of $[y_0, y_n] \bigcup [y_n, y_{Np}]$ (by proposition 2.21), and hence within $4\delta + \Lambda N$ of $[y_o 0, y_n]$. so, $d(z_j, [y_0, y_n]) \leq H_1' + 4\delta + \Lambda N$. Since N and H_1' don't depend on δ or Λ, lemma 10 shows that there is a constant H_2', independent of δ and Λ, such that

$$\mathcal{H}(\{z_j\}_{0 \leq j \leq p}, [y_0, y_n]) \leq H_2'$$

But

$$\mathcal{H}(\{z_j\}_{0 \leq j \leq p}, \{y_i\}_{0 \leq i \leq n}) \leq \Lambda N$$

It therefore suffices to set $H_2 = H_2' + 2\Lambda N$. $\square$

23. Lemma. *Given constants $\lambda \geq 1$, $c \geq 0$, and $L_2 \geq 1$, there are constants $H = H(\lambda, c)$, $\Lambda = \Lambda(\lambda, c)$ and $L = L(\lambda, c, L_2)$ with the property below.*

Let X be a metric space and $(x_i)_{0 \leq i \leq a}$ be (λ, cL) local quasi-geodesic segment in X. Then there is a $(\Lambda, 0, L_2)$ local quasi-geodesic segment $(y_i)_{0 \leq j \leq b}$ such that

$$y_0 = x_0 \quad |y_b - x_a| \leq H \quad \text{and} \quad \mathcal{H}(\{x_i\}_{0 \leq i \leq a}, \{y_j\}_{0 \leq j \leq b}) \leq H$$

Proof. Set $H = \lambda^2 c + 2\lambda + c$, $\Lambda = \max\{\lambda, H\}$, and $L = L_2(\lambda c + 2)$.

Let N be the integer such that $\lambda c + 1 \leq N \leq \lambda c + 2$ and let b be the whole part of $\frac{a}{N}$; we set $y_j = x_{Ni}$ for all $j \in \{0, \dots, b\}$.

For all $i \in \{0, \dots, a\}$, there exists a $jin\{0, \dots, b\}$ such that $|i - Nj| \leq N$. Since $N \leq L$, we have $|x_i - y_j| = |x_i - x_{Nj}| \leq \lambda N + c$. Hence, we have $\mathcal{H}(\{x_i\}_{0 \leq i \leq a}, \{y_j\}_{0 \leq j \leq b}) \leq H$ and, similarly, $|y_b - x_a| \leq H$.

Let $i, j \subset n\{0, \dots, b\}$ be such that $0 \leq |i - j| \leq L_2$. Since $NL_2 \leq L$, we have $|Ni - Nj| \leq L$ and

$$\left(\frac{N}{\lambda} - c\right)|i-j| \leq \frac{1}{\lambda}|Ni - Nj| - c \leq |y_i - y_j| \leq \lambda|Ni - Nj| + c \leq (\lambda N + c)|i - j|$$

Now $\frac{N}{\lambda} - c \geq \frac{\lambda c + 1}{\lambda} - c \geq \frac{1}{\Lambda}$ and $\lambda N + c \leq \lambda^2 c + 2\lambda + c \leq \Lambda$; in this way, we see that $(y_j)_{0 \leq j \leq b}$ is a $(\Lambda, 0, L_2)$ local quasi-geodesic segment. □

24. Proof of Theorem 21. We consider the statement of theorem 21 in which I is an interval in $\mathbb{Z}$, and let δ, λ, c be as given in the theorem.

We use $\Lambda = \Lambda(\delta, c)$ to denote the constant of lemma 23, and then set $H_2 = H_2(\delta, \Lambda)$ and $L_2 = L_2(\delta, \Lambda)$ to be the constants from lemma 22. Finally, we set $H' = H(\lambda, c)$ and $L = L(\lambda, c, L_2)$ to be the constants from lemma 23.

If $(x_i)_{0 \leq i \leq a}$ is a (λ, c, L) local quasi-geodesic segmen, we obtain, as in lemma 23, a $(\Lambda, 0L_2)$ local quasi-geodesic segment $(y_j)_{0 \leq j \leq b}$ such that $y_0 = x_0$, $|x_a - y_b| \leq H'$ and $\mathcal{H}(\{x_i\}_{0 \leq i \leq a}, \{y_j\}_{0 \leq j \leq b}) \leq H'$. Lemma 22 shows that

$$\mathcal{H}(\{y_j\}_{0 \leq j \leq b}, [y_0, y_b]) \leq H_2$$

But, on the other hand, $\mathcal{H}([x_0, x_a], [y_0, y_b]) \leq H' + 4\delta$. It follows that

$$\mathcal{H}(\{x_i\}_{0 \leq i \leq a}, [x_0, x_a]) \leq 2H' + H_2 + 4\delta$$

and it therefore suffices to set $H = 2H' + H_2 + 4\delta$.

We leave the details of the statement when $I \subset \mathbb{R}$ to the reader. □

5. Local Quasi-rays and Local Quasi-geodesics in Proper Spaces

Recall that a metric space is proper if every bounded ball is compact (definition 3.17). For these spaces, we can, via a theorem of Ascoli, extend theorems 11 and 21 to rays and minimising geodesics. More precisely,

25. Theorem. *Given three real numbers $\delta \geq 0$, $\lambda \geq 0$, and $c \geq 0$, there exist constants $H = H(\delta, \lambda, , c)$ and $L = L(\delta, \lambda, c)$ with the property below.*

Let X be a proper, geodesic, δ-hyperbolic metric space.

(i) Let f be a map from $\mathbb{R}_+$ or $\mathbb{N}$ to X which is a (λ, cL) local quasi-ray. There exists a minimising ray $g : \mathbb{R}_+ \to X$ such that $g(0) = f(0)$ and $\mathcal{H}(f, g) \leq H$.

(ii) Let f be a map from $\mathbb{R}$ or $\mathbb{Z}$ to X which is a (lambda, c, L) local quasi-geodesic. There exists a minimising geodesic $g : \mathbb{R} \to X$ such that $\mathcal{H}(f, g) \leq H$.

Proof. Let H' and L be the constants given by theorem 21 (where they are called H and L). We will show that one can set $H = 2H' + 1$.

Let f be as in assertions (i) and (ii). For al integers $k \geq 1$, we consider the set ε_k of all geodesic segments originating at $f(0)$ and with endpoints in the ball $B_k = \{x \in X : |x - f(0)| = k\}$; we give ε_k the topology of uniform convergence (for this to make sense, we consider all geodesic segments in ε_k with length $l \leq k$ as maps defined on $[0, k]$ which are constant on $[l, k]$). Since the ball B_k is compact (X is poper), Ascoli's theorem shows that ε_k is compact.

For each geodesic segment $h : J \to X$, where $J = [a, b] \subset \mathbb{R}_+$ and $h(0) = f(0)$, we set $J^{(K)} = J \bigcup [o, k]$ and we denote the restriction of h to $J^{(k)}$ by $H^{(k)}$. Hence, we have $h^{(k)} \in \varepsilon_k$.

(i) Consider a (λ, cL) local quasi-ray f. For each integer $n \geq 1$, set $a_n = |f(n) - f(0)|$ and $J_n = [0, a_n] \subset \mathbb{R}_+$. Denote the restriction of f to $[0, n]$ by f_n. We choose a geodesic segment $h_n : J_n \to X$ with endpoints $f(0)$ and $f(n)$ such that $\mathcal{H}(f_n, h_n) \leq H'$.

Let $k \geq 1$. Since ε_k is compact, there is a subsequence $(h_{k,m})_{m \geq 1}$ such that the sequence $(h^{(k)}{}_{k,m})_{m \geq 1}$ is uniformly convergent. Cantor's diagonalization argument provides us with a subsequence $(g_m)_{m \geq 1}$ of $(h_n)_{n \geq 1}$ witht he following properties: first, for all $k \geq 1$, the sequence $(g^{(k)}{}_m)_{m \geq 1}$ converges uniformly to a segment $g^(k) : [0, k] \to B_k$; second, $g^{(l)}$ extends $g^{(k)}$ if $l \geq k$. These geodesic segments $g^{(k)}$ define a ray $g : \mathbb{R}_+ \to X$ originating at $f(0)$.

It remains to estimate $\mathcal{H}(f, g)$.

Let $x \in \operatorname{Im}(f)$. Let n be a large enough integer so that $x \in \operatorname{Im}(f_n)$. We can choose $y_n \in \operatorname{Im}(h_n)$ such that $|x - y_n| \leq H'$. The sequence $(y_n)_{n \gg 1}$ possesses a subsequence which converges to a point $y \in \operatorname{Im}(g)$, and we still have $|x - y| \leq H'$. Hence, the image of f is in the H' neighborhood of the image of g.

On the other hand, let $y \in \operatorname{Im}(g)$. For all m sufficientlky large, we choose a point $y_m \in \operatorname{Im}(g_m)$ in such a way that the sequence $(y_m)_{m \gg 1}$ converges to y, and we also choose a point $x_m \in \operatorname{Im}(f)$ such that $|y_m - x_m| \leq H'$. If m is sufficiently large, we therefore have $|y - x_m| \leq H' + 1$ and sot he image of g is in the $(H' + 1)$ neighborhood of that of f.

(ii) Consider a (λ, c, L) local quasi-geodesic f. For each integer $n \geq 1$, we choose a geodesic segment $[x_{-n}, x_n]$ with endpoints $x_{-n} = f(-n)$ and $x_n = f(n)$. It follows from theorem 21 that there exists $y_n \in [x_{-n}, x_n]$ such that $|y_n - f(0)| \leq H'$. Let $y \in X$ be the limit of a convergent subsequence $(y_{n_j})_{j \geq 1}$ of $(y_n)_{n \geq 1}$; moreover, we have $|y - f(0)| \leq H'$.

Furthermore, by freely passing to a subsequence, we show as above in (i) that the segments $[x_{-n_j}, x_{n_j}]$ coverge to a geodesic $g : \mathbb{R} \to X$ passing through y, and that $\mathcal{H}(f, g) \leq 2H' + 1$. □

Translated by WILLIAM GROSSO

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CA 94720

THE BOUNDARY OF A TREE (CHAPTER 6)

Etienne Ghys and Pierre de la Harpe

Abstract. This chapter is a look at an example preparing us for the boundary of a hyperbolic space. We define several metrics on the boundary of a tree, then we examine how the isometries and quasi-isometries of the tree act on its boundary. The last section is dedicated to definitions suitable for a "conformal" vocabulary.

1. Distances on the boundary

Let X be a locally finite metric tree whose edges are all of length 1 (see chapter 2, §1). We also assume that each vertex of X possesses at least 3 neighbors (in particular X is infinite).

Recall (definition 5.2) that the *rays* in X are the minimizing semi-infinite geodesics in X. We say that two rays are *equivalent* if they have the same end, this means that they contain a common subray. (the rays can also be geometric rays, this means they are subsets of X; see definition 1.25 and remark 5.3.4

1. Definition. The boundary ∂X of X is the set of equivalence classes of rays of X.

We choose a vertex $w \in X$ as a *base point*. Each element a of ∂X is represented by a unique ray $[w, a)$ starting at w and going to a. We define the Gromov Product on ∂X by:

$$(a|b) = \lim(x|y)$$

where x (respectively y) is a point on the ray going to a (respectively b) and where the limit is defined for large enough distance $|x-w|$ and $|y-w|$. For all real number $\epsilon > 0$, we can then define

$$d_\epsilon(a,b) = e^{-\epsilon(a|b)}$$

We immediately have:

$$\begin{aligned}
&(a|b) = +\infty \quad \& \quad d_\epsilon(a,b) = 0 \Leftrightarrow a = b \\
&(b|a) = (a|b) \quad \& \quad d_\epsilon(a,b) = d_\epsilon(b,a) \\
&(a|c) = \min\{(a|b),(b|c)\} \quad \& \quad d_\epsilon(a,c) = \max\{d_\epsilon(a,b), d_\epsilon(b,c)\}
\end{aligned}$$

for all $a, b, c \in \partial X$. Thus, d_ϵ is an *ultrametric distance* on ∂X. We use d for d_1 and note that $d_e(a,b) = d(a,b)^\epsilon$. In particular, the distances d_ϵ are pairwise

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

Typeset by $\mathcal{AMS}$-TEX

topologically equivalent and therefore define a topology on ∂X. Furthermore, this topology does not depend on the choice of w. We may also verfiy that ∂X can be identified, with its topology, with the space of ends of X. We will not emphasize this aspect, which is very dependent on the fact that X is a tree. The boundary of a hyperbolic space (defined studied in the next chapter) is, in general, different from the space of ends (see proposition 7.17).

2. Proposition.

(1) *The pair* $=(X, \partial X)$ *is a visibility space.*
(2) *The space* ∂X *is compact and perfect (hence uncountable).*
(3) *Suppose that there exists an integer* $v \geq 2$ *such that each vertex of* X *has at least* $v+1$ *neighbors. Every distance* d_ϵ *makes* ∂X *into a metric space with finite Haussdorff dimension.*

Proof and Comments.

(1) The assertion says that, for all $a, b \in \partial X$ such that $a \neq b$, there is a minimising geodesic in X with endpoints a and b (the geodesic is unique here, but this is not part of the definition of visibility).
(2) The space ∂X is homeomorphic to the projective limit $\varprojlim F_n$ where F_n is the set of points distance n from the base point, given the discrete topology, and with $F_{n+1} \to F_n$ maps $x \in F_{n+1}$ to $[w, x] \bigcap F_n$ (exercise: modify assertion (2) for the case where X possesses vertices with 2 neighbors).
(3) For the definition of Haussdorff dimension, see, for example, [Fa1]. The proof of (3) is an easy exercise. If is equally easy to construct an example of a a tree X which doesn't meet the condition on v and for which the Haussdorff dimension of $(\partial X, d_\epsilon)$ is infinite for all ϵ.

□

We choose another vertex w' in X as the base point. We use $(a|b)'$ to denote the Gromov Product and $d_\epsilon{}'(a, b)$ to denote the distances on ∂X which are defined as above., but this time relative to w'. In view of the arbitrary choice of base point, we need to know how the distances d_ϵ and $d_\epsilon{}'$ compare.

A rough comparison shows that, if $n = |w - w'|$, we have, successively:

$$(x|y) - n \leq (x|y)' \leq (x|y) + n$$
$$(a|b) - n \leq (a|b)' \leq (a|b) + n$$
$$e^{-\epsilon n} d_\epsilon(a, b) \leq d_\epsilon{}'(a, b) \leq e^{\epsilon n} d_\epsilon(a, b) \forall a, b \in \partial X$$

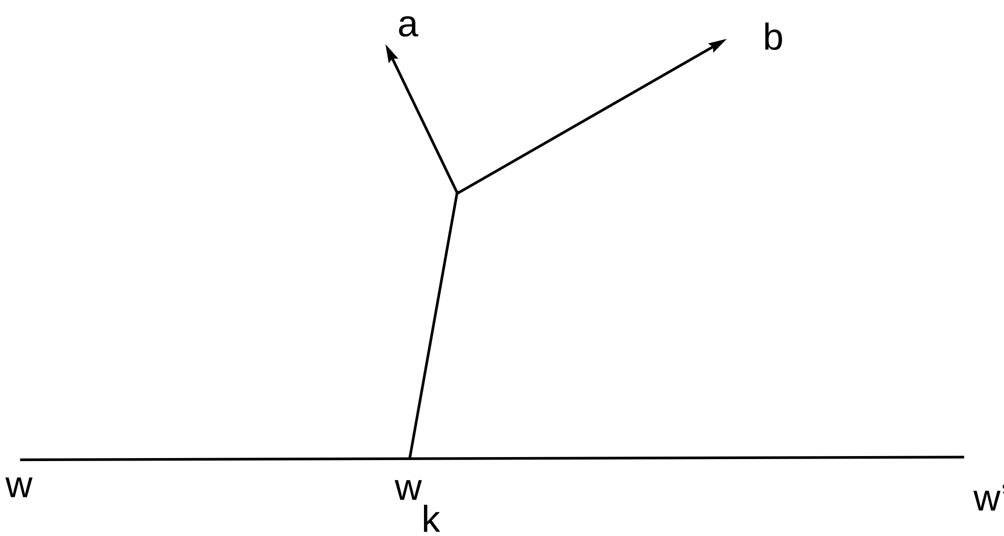
It follows that the *Lipschitz equivalence class* (defined in section 4) of d_ϵ does not depend on the choice of basepoint.

For a finer way to compare metrics, we use vertices $w_o = w, w_1, \ldots, w_n = w'$ of X between w and w'. For $k = 0, 1, \ldots n$, we set

$$\mathfrak{U}_k = \{a \in \partial X | [w, a) \bigcap [w, w') = [w, w_k]\}$$

Then, the $\mathfrak{U}_k$ are an open cover of ∂X. Let $a, b \in \mathfrak{U}_k$. We have

$$(a|b)' = (a|b) - k + (n-k)$$
$$d_\epsilon{}'(a,b) = e^{\epsilon(2k-n)} d_\epsilon$$



In particular, in $\mathfrak{U}_k$, the balls (not their rays) are the same whether we use d_ϵ or $d_\epsilon{}'$. In short, in the vocabulary of §4 :

3. Proposition. *With the above notation, the distances d_ϵ and $d_\epsilon{}'$ on ∂X are in the same Lipschitz class and are conformally equivalent. Two distances d_ϵ and $d_{\epsilon'}{}'$, associated to two basepoints w and w' and two numbers ϵ and ϵ' are in the same Hölder class and are conformally equivalent.*

Proof. For the second assertion, we observe that $d_{\epsilon'}{}' = (d_{\epsilon'})^{\epsilon'/\epsilon}$. □

2. The Action on the boundary induced by Isometries and Quasi-isometries.

We consider, as in section 1, a tree X, a vertex $w \in X$, a number $\epsilon > 0$ and a corresponding distance $d = d_\epsilon$. Let G be the group of isometries of X. Every isometry $\gamma \in G$ acts naturally on the boundary ∂X. If we set $w' = \gamma^{-1}(w)$ and if we use d' to denote the distance relative to w' and ϵ, we have $d(\gamma a, \gamma b) = d'(a,b)$ for all $a, b \in \partial X$. Hence, it follows from proposition 3 that we have:

4. Corollary. *Every isometry of the tree X induces a lipschitz, conformal homeomorphism on the boundary ∂X.*

Recall the ideas of *quasi-isometry* between tow metric spaces X and Y, of *quasi-geodesic* $f : \mathbb{R} \to X$ and of *quasi-ray* $f : \mathbb{R}_+ \to X$ as defined in the beginning of chapter 5. Henceforward, we consider X to be a metric tree.

5. Lemma.

(1) *Let $f : \mathbb{R} \to X$ be a quasi-geodesic. There exists a unique geodesic $f_0 : \mathbb{R} \to X$ sucht hat the images of f and f_0 are a finite Haussdorff distance apart.*
(2) *Suppose $f : \mathbb{R}_+ \to X$ is a quasi-ray. There exists a unique ray $f_0 : \mathbb{R}_+ \to X$ such that $f_0(0) = f(0)$ and such that the images of f and f_0 are a finite Haussdorff distance apart.*
(3) *Let γ" $X \to X'$ be a quasi-isometry and let f, f' be two rays in X which define the same point in ∂X. then the geodesic rays $(\gamma f)_0$ and $(\gamma f')_0$ define the same point in $\partial X'$.*

Proof. These statements are just easy special cases of the results shown in chapter 5.

6. Proposition. *A quiasi-isometry from a tree to itself induces a hölder, quasi-conformal homeomorphism on the boundary*

Proof. Choose a base point w in the tree X and a number $\epsilon > 0$, along with a distance $d = d_\epsilon$ on the boundary ∂X. We show that $\gamma : \partial X \to \partial X$ is quasi-conformal.

Choose a point $a \in \partial X$ represented by a ray $[w, a)$ and a point $x \in X$ on this ray. Set $p = |w - x|$. In ∂X, the sphere $S_{a,p}$ centered at a with radius $e^{-\epsilon p}$ consists of the points b such that $(a|b) = p$; this means such that $[w, b) \cap [w, a) = [w, x]$.

Set $w' = \gamma w$; we use $(|)'$ to denote the Gromov Product (and $d' = d_\epsilon{}'$ the distance) corresponding to w'. The point γa is represented by the ray $[w, \gamma a)$. Denote the projection of γx to this ray by x' and set $p' = |x' - w'|$. For all $b \in S_{a,p}$, the two rays $[w, a)$, $[w, b)$ and the geodesic (a, b) pass through x. Hence, their images under γ are two quasi-rays and one quasi-geodesic passing through γx. If x'_b designates the unique intersection point of $[w', \gamma a) \cap [w', \gamma b) \cap (\gamma a, \gamma b)$, there is a constant k such that $|x'_b - \gamma x| \leq k$ and $|x' - \gamma x| \leq k$ and such that k does not depend on γ, w, a, or b. Since $(\gamma a|\gamma b)' = |w' - x'_b|$, we have

$$p' - 2k \leq (\gamma a|\gamma b)' \leq p' + 2k$$

$$e^{-\epsilon(p'+2k)} \leq d'(\gamma a, \gamma b) \leq e^{-\epsilon(p'-2k)}$$

and it follows that

$$\frac{\sup\{d'(\gamma a, \gamma b) : d(a, b) = e^{-\epsilon p}\}}{\inf\{d'(\gamma a, \gamma b) : d(a, b) = e^{-\epsilon p}\}} \leq e^{\epsilon 4k}$$

And, finally, it follows from proposition 3 that we also have

$$\limsup_{p \to \infty} \frac{\sup\{d(\gamma a, \gamma b) : d(a, b) = e^{-\epsilon p}\}}{\inf\{d(\gamma a, \gamma b) : d(a, b) = e^{-\epsilon p}\}} < \infty$$

The constants in question don't depend on a ; it follows that $\gamma : \partial X \to \partial X$ is quasi-conformal. □

For example, if X is a Cayley Graph of a free group F on a finite set of generators, all automorphisms γ of F define a quasi-isometry of X, hence a quasi-conformal map on ∂X. However, the constant of quasi-conformity is *not* uniformly bounded when γ ranges over the group of automorphisms of F.

7. Exercise. Let F be the free group on two generators s, t and let γ be the automorphism mapping s to s and t to st. Show that the homeomorphism defined by γ on the boundary ∂X of the corresponding graph is not conformal.

Hint. Let $a \in \partial X$ be the point defined by the sequence s^∞, let $b \in \partial X$ be the sequence defined by $s^n t$ and let $b' \in \partial X$ be the sequence defined by $s^n t^{-1}$. Then:

$$d(\gamma a, \gamma b)/d(\gamma a, \gamma b') = e^{-\epsilon}$$

8. Exercise. We are given a partition $A \bigsqcup B$ of the edges in a tree X and two real numbers $\alpha > 0$ and $\beta > 0$. We use X' to denote the metric tree associated to X by making each edge in A have length α and each edge in B have length β. Let $f : X \to X'$ be the map which induces the indentity on the underlying set; it is a $(\mu, 0)$ quasi-isometry, where $\mu = \max(\alpha, \beta, \alpha^{-1}, \beta^{-1})$ (see definition 5.1). Verify that f induces a map $\partial X \to \partial X'$. We identify ∂X with $\partial X'$ using f. Show that the distances d_ϵ and d'_ϵ on the boundary, defined using X and X' are in the same Hölder equivalence class (and are, in fact, quasi-conformal).

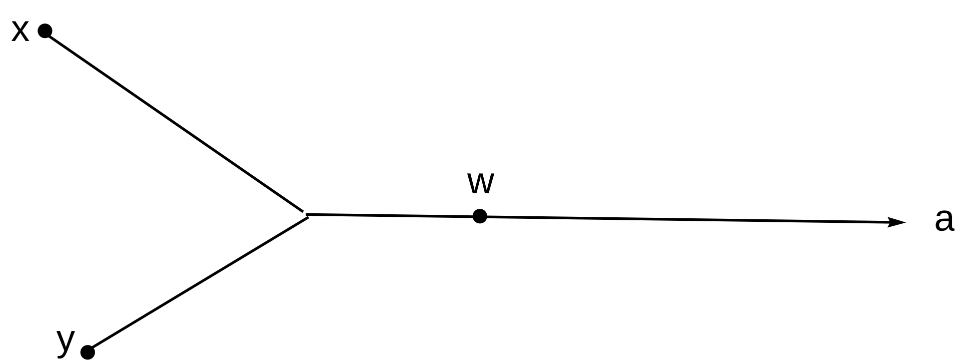
3. The Classification of Isometries of a Tree

In this section, we classify the isometries of a tree as either elliptic or hyperbolic. In preparation for our approach, we describe the horospheres of a tree, as well as the distances associated to suitable subsets of the boundary.

Let X be a metric tree and a a point on its boundary ∂X. For $x, y \in X$, the distances $|x - a|$ and $|y - a|$ aren't defined; however, their difference $|y - a| - |x - a|$ is well defined. More precisely, we define

$$\beta_a(x, y) = |x - w| - |y - w| \in \mathbb{R}$$

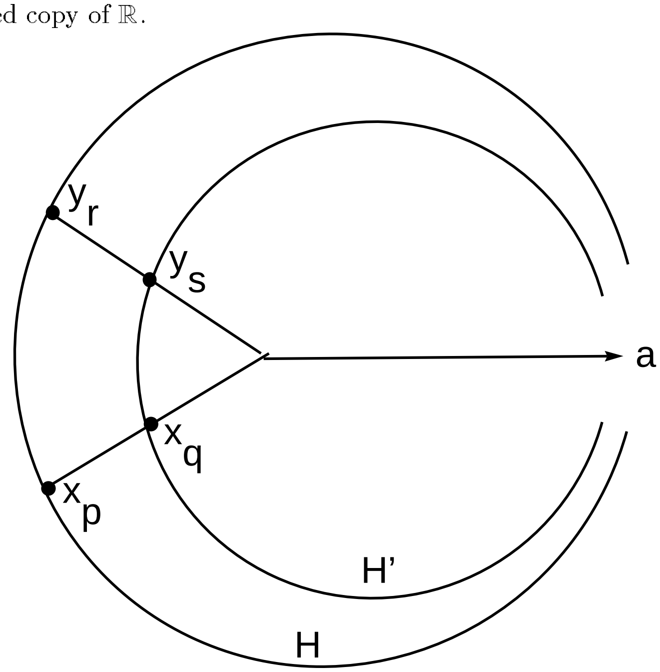
where w is a point in the intersetciob of the rays $[x, a)$ and $[y, a)$– we remark that the number $\beta_a(x, y)$ doesn't depend on the choice of w.



9. Definition. The *horosphere* centered at $a \in \partial X$ is the set $\{y \in X"\beta_a(x, y) = 0\}$.

such a horosphere intersects a geodesic heading towards a in a unique point. More precisely, let $(x_t)_{t\in\mathbb{R}}$ and $(y_t)_{t\in\mathbb{R}}$ be two geodesics such that $a = \lim_{t\to\infty} x_t =$

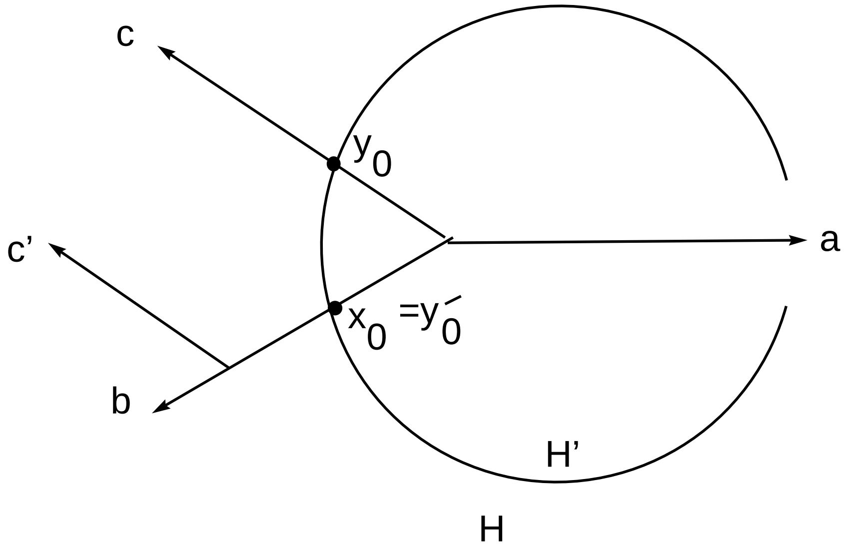
$\lim_{t\to\infty} y_t$, let H and H' be two horospheres centered at a, and let p, q, r, s be numbers defined by $x_p, y_r \in H$, $x_q, y_s \in H'$. then $q - p = s - r$' we set $\triangle(H, H') = q - p \in \mathbb{R}$. Then the set of horospheres centered at a is naturally parametrised as an oriented copy of $\mathbb{R}$.



If we are given a horosphere H centered at a and a real number $\epsilon > 0$, we can define, as follows, a distance $d_{H,\epsilon}$ on $\partial X - \{a\}$. Let b, c be two points in $\partial X - \{a\}$; we parametrise the geodesic $(x_t)_{t\in\diagdown} = (a, b)$ and the geodesic $(y_t)_{t\in\diagdown} = (a, c)$ so that $x_0, y_0 \in H$ and $\lim_{t\to\infty} x_t = \lim_{t\to\infty} y_t = a$. We then set

$$(b|c)_H = \lim_{t\to\infty} [(b|c)_{x_t} - t]$$

$$d_{H,\epsilon}(b, c) = e^{-\epsilon(b|c)_h}$$



On the other hand, let d be the restriction of one of the distances defined on ∂X in §1 to $\partial X - \{a\}$ (d is defined relative to a positive number and a chosen basepoint of X). Then

10. Proposition. *With the above notation:*

(1) *The distances d and $d_{H,\epsilon}$ define the same topology on $\partial X - \{a\}$.*
(2) *Moreover, they're conformally equivalent.*
(3) *The distance $d_{H,\epsilon}$ makes $\partial X - \{a\}$ a complete metric space.*
(4) *Let H' be another horosphere centered at a, and let $\triangle(H, H')$ be as above– then $d_{H',\epsilon} = e^{-\epsilon\triangle(H,H')} d_{H,\epsilon}(b, c)$ for all $b, c \in \partial X - \{a\}$.*

Proof. We consider a point $b \neq a \in \partial X$, the geodesic g between a and b and the point $x = g \bigcap H$ (we can think of x as being "near a"). Then $(b|c)_x = (b|c)_H$ for every $c \in \partial X$ close enough to b, where $(b|c)_x$ is defined as in the beginning of §1. Hence, claims (1) and (2) follow from proposition 3.. The proof of claim (3) is left to the reader (however, see proposition 8.7) and claim (4) follows because $(b|c)_{H'} = (b|c)_H + \triangle(H, H')$. □

Let a be a point on the boundary ∂X. We use G_a to denote the isotropy group of a in the group G of isometries of X. Each isometry $\gamma \in G_a$ premutes the horospheres centered at a. More precisely, if $g = (x_t)_{t\in\mathbb{R}}$ is a geodesic such that $\lim x_t = a$ and if $(H_t)_{t\in\mathbb{R}}$ ia the parametrisation associated to the horospheres centered at a (with $x_t = g \bigcap H_t$), there exists a real number $\lambda(\gamma)$ such that γ maps H_t to $H_{T+\lambda(\gamma)}$ for all $t \in \mathbb{R}$. The *force* of γ relative to a is the positive number $\Phi_a(\gamma) = e^{\lambda(\gamma)}$.

11. Proposition.

(1) *the map $G_a \to (\mathbb{R}*, +)$ defined by the force, $\gamma \mapsto \Phi_a(\gamma)$ is a group homomorphism.*
(2) *Every isometry $\gamma \in G_a$ with $\Phi_a(\gamma) = 1$ possesses a fixed point in X.*
(3) *In particular, every element γ in the commutator subgroup G'_a of G_a acts by isometries on $\partial X - \{a\}$ and possesses a fixed point in X.*

Proof of (2). Let $b \in \partial X - \{a\}$ and suppose x is a point in $(a, b) \bigcap (a, \gamma b)$. If $\Phi_a(\gamma) = 1$, it is easy to show that γ fixes x. □

12. Proposition. *We consider a number $\epsilon > 0$, a point $a \in \partial X$, the family $(H_t)_{t \in \mathbb{R}}$ of horospheres centered at a and we write H for H_0. Let $\gamma \in G_a$ be an isometry of X fixing a and let $\Phi_a(\gamma)$ be the force of γ relative to a. Then:*

$$d_{H,\epsilon}(\gamma b, \gamma c) = \Phi_a(\gamma)^\epsilon d_{H,\epsilon}(b, c)$$

for all $b, c \in \partial X - \{a\}$. In particular:

If $\Phi_a(\gamma) < 1$ (respectively $\Phi_a(\gamma) > 1$) then γ is a contraction (resp. expansion) of the metric space $(\partial X - \{a\}, d_{H,\epsilon})$ possessing a unique fixed point $a' \in \partial X - \{a\}$.

If $\Phi_a(\gamma) = 1$, then γ is an isometry of $(\partial X - \{a\}, d_{H,\epsilon})$.

Proof. Since

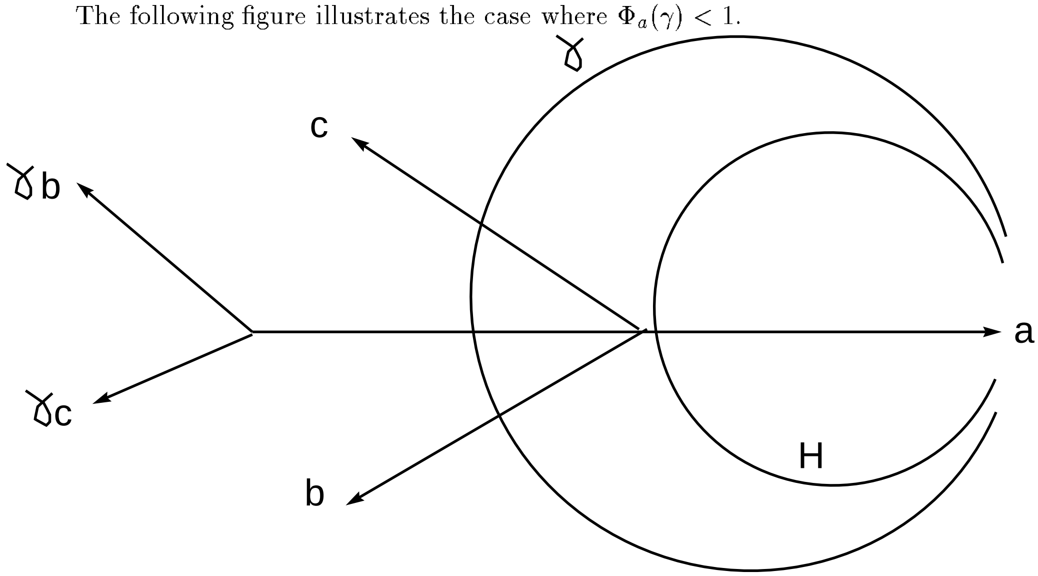
$$(\gamma b|\gamma c)_H + \triangle(H, \gamma H) = (\gamma b|\gamma c)_{\gamma H} = (b|c)_H$$

and since $\triangle(H, \gamma H) = \ln \Phi_a(\gamma)$, we obtain

$$d_{H,\epsilon}(\gamma b, \gamma c) = \Phi_a(\gamma)^\epsilon d_{H,\epsilon}(b, c)$$

for all $b, c \in \partial X - \{a\}$. □

The following figure illustrates the case where $\Phi_a(\gamma) < 1$.

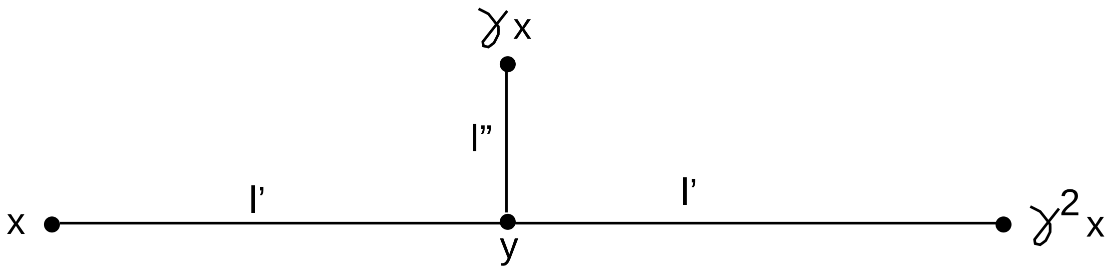


13. Definition. We say that an isometry γ of the tree X is elliptic if it possesses a fixed point in X, and hyperbolic otherwise.

14. Proposition. *Let γ be a hyperbolic isometry of a tree X. Then γ possesses exactly two fixed points a and a' in ∂X. Furthermore, we can assume the notation is such that a and a' possess the following property:*

For each neighborhood $\mathfrak{U}$ *of* a *and each neighborhood* $\mathfrak{U}'$ *of* a' *in* ∂X*, there exists an integer* $k > 0$ *such that* $\gamma^n(\partial X \setminus \mathfrak{U}') \subset \mathfrak{U}$ *and* $\gamma^{-n}(\partial X \setminus \mathfrak{U}) \subset \mathfrak{U}'$ *for all integers* $n \geq k$*; we say that* a *is the target of* γ *and* a' *is its source.*

Proof. Let $x \in X$. The smallest sub-tree of X containing $x, \gamma x, \gamma^2 x$ is a tripod– we denote its center by y and set $l' = |x - y| = |\gamma^2 x - y|$ and $l'' = |\gamma x - y|$. Then $L' > l''$ (if not, then there is a point of $[y, \gamma x]$ fixed by γ); we set $l = l' - l''$. The tree generated by $(\gamma^n x)_{n \in \mathbb{Z}}$ is a quasi-geodesic in X invariant under γ.



In particluar, γ possesses an invariant geodesic, and hence also possesses two fixed points in ∂X. Proposition 14 then results from proposition 12. See also I.6.4 of [Ser]. □

4. Several Types of Homeomorphisms of Metric Spaces

Let (X, d) be a metric space and let $\Phi : X \to X$ be a homeomorphism. For the rest of this chapter, we assume that X is perfect; in this case, for every $x \in X$, there exists a descending sequence $(\epsilon_j)_{j \geq 1}$ of real numbers tending towards zero such that all the spheres $\{y \in X : d(y, x) = \epsilon_j\}$ is non-empty. To simplify our notation, we write the limits taken using these spheres with $\epsilon \to 0$ rather than $j \to \infty$.

15. Definition. The *conformal dilation* of Φ at a point $x \in X$ for the distance d is the "number"

$$H_\Phi{}^d(x) = \limsup_{\epsilon \to 0} \frac{\sup\{d(\Phi(x), \Phi(y)) : y \in X \text{ and } d(y, x) = \epsilon\}}{\inf\{d(\Phi(x), \Phi(y)) : y \in X \text{ and } d(y, x) = \epsilon\}} \in [1, +\infty)$$

We say that a homeomorphism Φ is
conformal if $H_\Phi{}^d(x) = 1$ for all $x \in X$.
quasi-conformal if $\sup_{x \in X} H_\Phi{}^d(x) \infty$.
K-quasi-conformal if $H_\Phi{}^d(x) < K$ for all $x \in X$ (where K is a given constant).

16. Examples.

We say that the homeomorphism Φ is *lipschitz* for d if there exists a constant $L \geq 1$ such that

$$\frac{1}{L} d(x, y) \leq d(\Phi(x), \Phi(y)) \leq L d(x, y)$$

for all $x, y \in X$. Any such homeomorphism is L^2-quasi-conformal.
Every C^1 diffeomorphism of the circle or line is conformal.

We say that the homeomorphism Φ is *Hölder* for d if there exists constants $\alpha > 0$ and $c > 0$ such that:

$$\frac{1}{c} d(x,y)^{1/\alpha} \leq d(\Phi(x), \Phi(y)) \leq c d(x,y)^{\alpha}$$

for all $x, y \in X$. In general, such homeomorphisms are not quasi-conformal, as is shown by the example defined on the Euclidean line by

$$\phi(x) = x^{\alpha} if x \geq 0$$

$$\phi(x) = x if x \leq 0$$

with $0 \leq \alpha \leq 1$.

In many familiar metric spaces, it is a strong condition for a homeomorphism to be conformal or even quasi-conformal. Consider, for example, the sphere S^n with dimension ≥ 2, given the usual angular distance. Then, a quasi-conformal homeomorphism is differentiable almost everywhere. If Φ is conformal, it is a Möbius transformation(see [Ahl], [LeV], and [Ric]).

On the other hand, the notions of conformality and quasi-conformality lose much of their interest in "general" metric spaces, as is shown in the following proposition. It follows that, when one proves that a transformation of the boundary ∂X of a hyperbolic metric space X is (quasi)conformal for a metric d_ϵ, it is important to recall certain properties of $(X, \partial X)$, among which is finite haussdorff dimension.

17. Proposition. *Let (X, d) be a metric space. There exists a distance d' on X, topologically equivalent to d, such that every hölder homeomorphism of (X, d) is a conformal homeomorphism of (X, d').*

18. Exercise. Choose a space (X, d) and verify that the space (X, d') obtained by the following construction has infinte haussdorff dimension

Before the proof of proposition 17, we recall the following facts. Leyt $f : \mathbb{R}_+ \to \mathbb{R}_+$. We say that f is *concave* if

$$f((1-\alpha)s + \alpha t) \geq (1-\alpha) f(s) + \alpha f(t)$$

for all $s, t \in \mathbb{R}_+$ and $\alpha \in [0,1]$. When f is C^2, it is well known that f is concave if and only if $f'' \leq 0$.

19. Exercise. Show that a concave function f such that $f(0) = 0$ satisfies $f(s + t) \leq f(s) + f(t)$ for every $s, t\mathbb{R}_+$.

Let (x, d) be a metric space and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, concave function with $f(0) = 0$. It follows from the preceding exercise that the map

$$d_f = f \circ d : X \times X \to \mathbb{R}_+$$

is a distance function on X which is topologically equivalent to d.

20. Lemma. *For all $0 < t < e^{-1}$, we set $g(t) = \frac{1}{\log(-\log t)}$. Then :*

(1) $\lim_{t \to 0} g(t) = 0$.
(2) *There exists $t_0 > 0$ such that g is increasing and concave in $[0, t_o)$ and such that $g(t_0) > 0$.*
(3) *Let c, β be constants with $c > 0$ and $\beta > 0$. Then*

$$\lim_{t \to 0} \frac{g(ct^\beta}{g(\frac{1}{c}t} = 1$$

Proof. Elementary analysis. □

Proof of Proposition 17. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, increasing and concave function such that:

$$f(0) = 0$$
$$f(t) = g(t) \text{ if } 0 < t < t_0 \text{ with } g \text{ and } t_0 \text{ as above.}$$
$$f \text{ is linear if } t > t_0$$

the reader can easily verify, by utilising conclusion (3) of the preceding lemma, that the metric $d' = d_f$ has the properties required by proposition 17. □

21. Definitions. All the preceding extends naturally to the case of a homeomorphism Φ from one metric space to another. Suppose, in particular, that X is a space given two distances d and d' which are topologically equivalent. We say that d and d' are in the same quasi-conformal class or conformal class (or lipschitz or Hölder ...) if the identity map on X is of the appropriate type.

Translated by WILLIAM GROSSO

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

THE BOUNDARY OF A HYPERBOLIC SPACE (CHAPTER 7)

ETIENNE GHYS AND PIERRE DE LA HARPE

ABSTRACT. There are many ways to define the boundary ∂X of a hyperbolic space X. In §1, we define it in three ways which are equivalent if X is geodesic and proper. In §2, we define a topology on ∂X which makes it into a compact space. However, ∂X is more than just a topological space— it can be given a natural metric and a conformal structure. This metric is defined in §3. We dedicate §4 to the action on the boundary of isometries and quasi-isometries of X and we attempt to describe some examples of boundaries in §5.

1. THE BOUNDARY AS A SET

We consider a given number $\delta \geq 0$ and a δ-hyperbolic metric space X with a base point W; we write the Gromov Product of x and y relative to w as $(x|y)$. We assume that X is a geodesic space (defintion 1.25) and *proper* (all closed balls are compact, definition 3.17). Recall (definition 5.2) that a *quasi-ray* in X is a quasi-isometric map $f : \mathbb{R}_+ \to X$.

1. Definition. Two quasi-rays in X are *equivalent* if their images are a finite Haussdorff distance from each other.

2. Proposition. *Let g and H be two rays in a geodesic δ-hyperbolic space X. The following properties are equivalent.*

(1) *The rays g and h are equivalent, by which we mean $\mathcal{H}(g,h) < \infty$.*
(2) *We have $\sup_{t\geq 0} |h(t) - g(t)| < \infty$.*
(3) *There exists $t_1 \geq 0$ such that, for all $t \geq t_1$, there exists s_t such that $|h(t) - g(s_t)| \leq 8\delta$.*
(4) *There exists $u \in \mathbb{R}$ and $t_0 \geq \max\{0, u\}$ such that $|h(t) - g(t-u)| \leq 16\delta$ for all $t \geq t_0$.*

Proof (see [Pau]).

First, we show that (1)⇒(2). We define $H = (g,h)$ and choose $s_0 \geq 0$ such that $|h(0) - g(s_0)| \leq H$. For all $t > 0$, there exists s_t such that $|h(t) - g(s_t)| \leq H$. We have

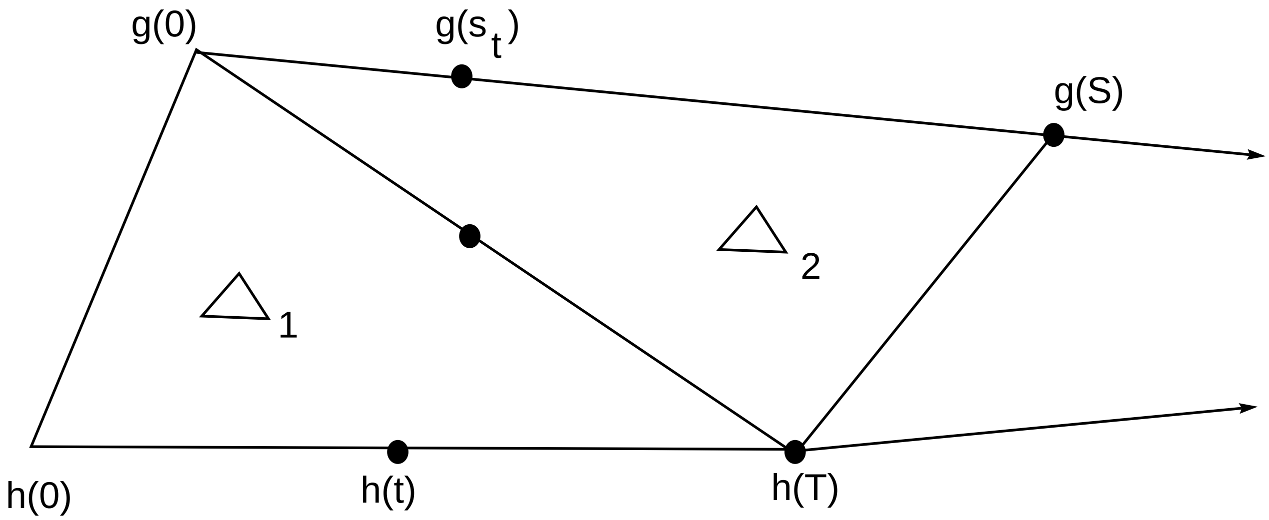
$$|s_t - s_0| = |g(s_t) - g(s_0)| \leq |h(t) - h(0)| + 2H = t + 2H$$
$$|s_t - s_0| \geq |h(t) - h(0)| - 2H = t - 2H$$

1991 *Mathematics Subject Classification.* Primary 20F05, 20F14.
Key words and phrases. Hyperbolic Groups.

Hence

$$|s_t - t| \le s_o + 2H \text{ and } |h(t) - g(t)| \le s_0 + 3H$$

We now show (2)⇒(3). We write $D = \sup_{t\ge 0} |h(t) - g(t)|$ and set $t_1 = d + 4\delta + 1$. Suppose $t, T \in \mathbb{R}_+$ such that $t \ge t_1$ and $T \ge t + D + 8\delta + 1$ and let $S \ge 0$ be such that $|h(T) - g(S)| \le D$. We complete the geodesic segment $[h(0), h(T)]$ to a geodesic triangle $\triangle_1$ with vertices $h(0), h(T), g(0)$ and then comlete the segments $[g(0), g(S)]$ and $[g(0), h(T)]$ to a geodesic triangle $\triangle_2$.



We apply proposition 2.21 first to $\triangle_1$ and then to $\triangle_2$.

If there exists $y \in [g(0), h(0)]$ such that $|h(t) - y| \le 4\delta$, we have

$$t = |h(t) - h(0)| \le |h(t) - y| + |g(0) - h(0)| \le D + 4\delta$$

contradicting the choice of t. Hence there exists $x \in [g(0), h(T)]$ such that $|x - h(t)| \le 4\delta$. If there exists $z \in [g(s), h(T)]$ such that $|z - x| \le 4\delta$, we have

$$T - t = |h(T) - h(t)| \le |h(T) - g(S)| + |z - x| + |x - h(t)| \le D + 8\delta$$

contradicting the choice of T. Hence there exists $s_t \in [0, S]$ such that $|g(s_t) - x| \le 4\delta$. It follows that $|g(s_t) - h(t)| \le 7\delta$.

We show (3)⇒(4). We write s_1 for s_{t_1} and set $u = t_1 - s_1$. We have $|h(t_1) - g(t_1 - u)| \le 8\delta$. Because $\lim_{t\to\infty} |h(t) - g(t_1 - u)| = \infty$, there exists $T_1 \ge 1$ such that $t \ge T_1$ implies $s_t \ge s_1$. We set $t_0 = \max\{t_1, T_1, s_1\}$. If $t \ge t_0$

$$\begin{aligned} |h(t) - g(t-u)| &\le |h(t) - g(s_t)| + |g_t - (t-u)| \\ &\le |(s_t - s_1) - (t - t_1)| + 8\delta \\ &= ||g(s_t) - g(s_1)| - |h(t) - h(t_1)|| + 8\delta \\ &\le 24\delta \end{aligned}$$

If there exists $\epsilon > 0$ and $t > t_0$ such that $|h(t) - g(t-u)| \geq 16\delta + \epsilon$, we have, for all $T \geq t$ (see propositions 2.21 and 2.25)

$$|h(t_1) - g(t_1 - u)| + \epsilon \leq |h(t) - g(t-u)| - 8\delta$$
$$\leq \frac{t}{T}|h(T) - g(T-u)| + (1 - \frac{t}{T})|h(t_1) - g(t_1 - u)|$$

and it follows that $\lim_{T\to\infty} |h(T) - g(T-u)| = \infty$, which contradicts the previous calculation. Hence $|h(t) - g(t-u)| \leq 16\delta$ for all $t \geq t_0$.

The implication (4)⇒(1) is trivial. □

3. Corollary. *Let X be a geodesic, δ-hyperbolic space.*

(1) *If g and h are two equivalent rays with the same origin, then*

$$\sup_{t\geq 0} |h(t) - g(t)| \leq 8\delta$$

(2) *Suppose g and h are two minimizing geodesics in X. If $\mathcal{H}(g,h) < \infty$, then there exists $u \in \mathbb{R}$ such that $\sup_{t\in\mathbb{R}} |h(t) - g(t-u)| \leq 16\delta$.*

Proof.

(1) If there exists $\epsilon > 0$ and $t > 0$ such that $|h(t) - g(t)| \geq 8\delta + \epsilon$, we have, as in the proof of (3)⇒(4) above, $\lim_{t\to\infty} |h(t) - g(t)| = \infty$, which is absurd.

(2) In view of the proposition, we may assume, WLOG, that $|h(0) - g(0)| \leq 8\delta$. The same argument then shows that $|h(t) - g(t)| \leq 16\delta$ for all $t \in \mathbb{R}$.

□

There are many ways to define the boundary of X. The *first model* for the boundary is the set $\partial_q X$ of of equivalence classes of quasi-rays in X. If a is the equivalence class of a quasi-ray f, we say that f tends to a and write $f \to a$ or $f(t) \to a$.

The *second model* is the set $\partial_{r,w} X$ of equivalence classes of rays originating at a base point w; we also write $\partial_r X$ if there is no ambiguity about the base point. We denote the image of a ray originating at w and tending to a by $[w, a)$— there are, in general, many rays of this type, even if the notation doesn't reflect it! Nevertheless, the proof of the preceding corollary shows that if g and h are two rays originating at w and tending towards a, then $\sup_{t\geq 0} |g(t) - h(t)| \leq 8\delta$.

A sequence $(x_i)_{i\geq 1}$ of points in X *tends to* ∞ if $\lim_{i,j\to\infty}(x_i|x_j) = \infty$. The definition doesn;t depend on the choice of w. Two sequences $(x_i)_{i\geq 1}$ and $(y_j)_{j\geq 1}$ which tend to ∞ are *equivalent* if $\lim_{i,j\to\infty}(x_i|y_j) = \infty$. The *third model* of the boundary of X is the set $\partial_s(X)$ of equivalence classes of sequences tending to ∞. If a is the equivalence class of the sequence $(x_i)_{i\geq 1}$, we write $x_i \to a$ (as an exercise, show that if $X = \mathbb{R}^2$, then the relation defined between sequences tending to ∞ is not an equivalence relation).

4. Proposition. *Using the above notation, there are natural bijections between the sets $\partial_q X$, $\partial_r X$ and $\partial_s X$.*

Proof. First, we know that every quasi-ray is a finite Haussdorff distance from a ray (§5.5), and so $\partial_q X$ and $\partial_r X$ are the "same" set. It remains to show that $\partial_r X$ and $\partial_s X$ are identifiable.

We can associate a sequence $(x_i)_{i\geq 1}$ to every map $g : \mathbb{R}_+ \to X$ by defining $x_i = g(i)$. We verify, on one hand, that $(x_i)_{i\geq 1}$ tends to ∞ if g is a ray and, on the other hand, that two equivalent rays g and h are associated to two equivalent sequences. This is a natural map $\partial_r X \to \partial_s X$.

Conversely, let $(x_i)_{i\geq 1}$ be a sequence tending to ∞ (tending to a point $x \in \partial_s X$). For every $i \geq 1$, we choose a geodesic segment $[x_0, x_i]$ between x_o and x_i. Using Arscoli's theorem, we show, as in theorem 5.25, that there is a subsequence $(y_i)_{i\geq 1}$ of $(x_i)_{i\geq 1}$ such that the segments $[x_0, y_i]$ converge to a ray g. This is a map $\partial_s X \to \partial_r X$ (it is easy check that it is well defined) and it is the inverse to the map $\partial_r X \to \partial_s X$ defined above. □

5. Definition. From now on, we identify the three models described above to a single set and denote it by $\partial X-$ we call this the *boundary* of X

Each model has its advantages. For example, to show that a quasi-isometry of X induces a map on ∂X, we use the model $\partial_q X$. $\partial_r X$ is well suited for the proof, below, that ∂X is compact. And, $\partial_s X$ makes sense even if X is neither geodesic nor proper.

There are other constructions of the boundary of a metric space (X, d). Let $\mu : X \to \mathbb{R}_+^*$ be a continuous function and let d_μ be the distance defined by $d_\mu(x, y) = \inf \int_C d_\mu$, where the infinum is taken over the curves C between x and y and where $\int_C d_\mu$ denotes the μ-length of such a curve. The μ-boundary of X is $\overline{X_\mu} - \overline{X}$, where $\overline{X_\mu}$ is the completion of (X, d_μ) and $\overline{X}$ that of (X, d). When μ is well-chosen, we recover the boundary ∂X of definition 5 (corollary 7.2M of [Gr5], and [Flo]).

There is also a construction of the boundary which uses Busemann functions (§7.5 of [Gr5]).

6. Proposition. *Suppose that X is geodesic and proper. The pair $(X, \partial X)$ is a visibility space.*

Proof. (See the proof of proposition 6.2 for the definition of visibility space). Let $a, b \in \partial X$ with $a \neq b$. We choose rays g, h originating at the base point w and tending, respectively, to a and b. Let $f : \mathbb{R} \to X$ be the map defined by $f(-t) = g(t)$ and $f(t) = h(t)$ for all $t \geq 0$. By approximating the image of f by a tree, we see that f is a quasi-geodesic. The proposition then follows from the existence of a geodesic a finite Haussdorff distance from f (theorem 5.25) □

7. Exercise. The boundary of a hyperbolic space X is clearly empty if X has finite diameter. Show that the converse is true if X is geodesic and proper. In the case of non-proper spaces, examine the 0-hyperbolic (but not locally compact) space obtained by giving the "tree distance" to the union of the segments $[0, ne^{i\frac{\pi}{n}}]$ (for $n \geq 1$) in the complex plain.

2. The Boundary As a Topological Space

As in §1, the space X is assumed δ-hyperbolic, geodesic, proper, and given a base point w. We think of the boundary ∂X as given by the sequences model, and we plan to introduce a topology on ∂X which makes it compact.

The Gromov Product of two points in ∂X is defined by

$$(a|b) = \sup \liminf_{i.j\to\infty}(x_i|y_j)$$

where the supremum is taken over all sequences $(x_i)_{i\geq 1}$ tending towards a and $(y_j)_{j\geq 1}$ tending towards b.

7. Remark. Let $a, b \in \partial X$. For all sequences $x_i \to a$ and $y_j \to b$, we have:

$$(a|b) - 2\delta \leq \liminf(x_i|y_j) \leq (a|b)$$

Proof. Suppose $\gamma > 0$. There exist sequences x_i^γ and y_j^γ tending towards a and b such that

$$(a|b) - \gamma \leq \liminf(x_i^\gamma|y_j^\gamma) \leq (a|b)$$

But

$$(x_i|y_j) \geq \min\{(x_i|x_i^\gamma), (x_i^\gamma|y_j^\gamma), (y_j^\gamma|y_j)\} - 2\delta$$

Two of the terms inside the braces tend towards infinityand one is bounded if $a \neq b$. Consequently,

$$\liminf(x_i|y_j) \geq \liminf(x_i^\gamma|y_j^\gamma) - 2\delta \geq (a|b) - \gamma - 2\delta$$

Since this is true for all $\gamma > 0$, we have the desired inequality. □

For every rational number $r > 0$, define:

$$V_r = \{(a, b) \in \partial X \times \partial X : (a|b) \geq r\}$$

For a given $r \in \mathbb{Q}_+^*$, we choose $s \in \mathbb{Q}_+^*$ such that $s \geq r + 6\delta$. The definition of hyperbolicity, and the preceding remark, show that, if $(a, b) \in V_s$ and $(b, c) \in V_s$, then $(a, c) \in V_r$. It follows that the family $(V_r)_{r\in Q_+^*}$ is a fundamental system of open sets for a uniformly separating structure on ∂X (see [BTG], chap II, §1) which is metrizable (id, chapter 9, §4, prop. 2). from now on, we give ∂X the corresponding topology.

9. Proposition. *The topological space ∂X is compact.*

Proof. Since ∂X is metrizable, it suffices to show that ∂X is sequentially compact.

Let a_n be a sequence of elemeents in ∂X. For each n, choose a ray g_n originating at the origin and tending towards a_n. We know that the sequence $x_i^n = g_n(i)$ tends towards a_n. Since X is proper, we may suppose (by extracting a sub-sequence) that g_n converge uniformly on each compact set to a ray g originating at w. Let $x_i = g(i)$ and let a be the point of ∂X corresponding to $(x_i)_{i\geq 1}$ (or, since it comes to to the same thing, corresponding to the ray g). We are going to show that the sequence a_n (or the sequence which we have extracted) converges to a.

Let R be an arbitrarily large integer. There exists $n_0(R)$ such that, for all $n \geq n_0(R)$, we have

$$|x_R^n - x_R| \leq 1$$

We evaluate $(x_i^n|x_j)$ for all $i, j \geq R$. We have

$$(x_i^n|x_j) = \frac{1}{2}(|x_i^n| + |x_j| - |x_i^N - x_j|)$$
$$= \frac{1}{2}(i + j - |x_i^n - x_j|)$$

Now

$$|x_i^n - x_j| \leq |x_i^n - x_R^n| + |x_R^n - x_R| + |x_r - x_j|$$
$$\leq i - R + 1 + j - R$$

Consequently,

$$(x_i^n|x_j) \geq R - \frac{1}{2}$$

By using remark 8, we find that

$$(a_n|a) \geq R - \frac{1}{2} - 2\delta \quad \text{for all } n \geq n_0(R)$$

This shows that the sequence a_n converges to a in ∂X. □

3. The boundary as a Metric Space

Throughout, we keep the same hypotheses on the δ-hyperbolic space X; it is proper, geodesic and has a basepoint w. We have seen in §2 that the boundary ∂X is metrizable. The goal of this section is to explicitly construct a family of metrics on ∂X whose conformal properties will be studied in §4.

We suppose that we are given a real number $\epsilon > 0$ and we set, for all $a, b \in \partial X$:

$$\varrho_\epsilon(a, b) = \exp(-\epsilon(a|b))$$

The Gromov Product on ∂X has the following properties:

$$(a|b) = (b|a) \text{ for all } a, b \in \partial X,$$
$$(a|b) = \infty \text{ if and only if } a = b,$$
$$(a|c) \geq \min\{(a|b), (b|c)\} - \delta \text{ for all } a, b, c \in \partial X.$$

It follows that ϱ_ϵ satisfies

(1) $\varrho_\epsilon(a, b) = \varrho_\epsilon(b, a)$
(2) $\varrho_\epsilon(a, b) = 0$ if and only if $a = b$
(3) $\varrho_\epsilon(a, c) \leq (1 + \epsilon')\max\{\varrho_\epsilon(a, b), \varrho_\epsilon(b, c)\}$ for all $a, b, c \in \partial X$ with $\epsilon' = \exp(\epsilon\delta) - 1$.

The "quasi-ultrametric" inequality (3) does not guarantee that ϱ_ϵ satisfies the triangle inequality… Hence, we will slightly modify ϱ_ϵ to define a distance.

A *chain* between two points a and b in ∂X is a finite sequence $a = a_0, a_1, \ldots, a_n = b$ of points in ∂X; we denote the set of chains between a and b by $C_{a,b}$. We define

$$\varrho_\epsilon(a_0, a_1, \ldots, a_n) = \sum_{i=1}^{n} \varrho_\epsilon(a_{i-1}, a_i)$$
$$d_\epsilon(a, b) = \inf\{\varrho_\epsilon(c) : c \in C_{a,b}\}$$

10. Proposition. *If $\epsilon' \leq \sqrt{2}-1$, d_ϵ is a distance on ∂X and we have*

$$(1-2\epsilon')\varrho_\epsilon(a,b) \leq d_\epsilon(a,b) \leq \varrho_\epsilon(a,b)$$

for all $a, b \in \partial X$. In particular, d_ϵ is compatible with the topology introduced in §2.

Proof. The proof is an adaptation of ([BTG], chapter 9, §1, n° 4). The only non-trivial point is to show the inequality

$$(1-2\epsilon')\varrho_\epsilon(a,b) \leq d_\epsilon(a,b)$$

This follows from the following assertion: for all integers $n \geq 1$ and for every chain $a = a_0, a_1, \dots, a_n = b$ of length n between a and b, we have

$$(1-2\epsilon')\varrho_\epsilon(a,b) \leq \sum_{i=1}^{n} \varrho_\epsilon(a_{i-1}, a_i)$$

We show this assertion by induction on n. If $n = 1$, there is nothing to show, and we henceforth assume that $n \geq 2$.

Set $R = \sum_{i=1}^{n} \varrho_\epsilon(a_{i-1}, a_i)$. The inequality is true if $R \geq 1 - 2\epsilon'$ since $\varrho_\epsilon(a,b) < 1$. Suppose that $R < 1 - 2\epsilon'$. Let p be the largest of the indices q such that $\sum_{i=1}^{q} \varrho_\epsilon(a_{i-1}, a_i) \leq \frac{1}{2}R$. We then have

$$\sum_{i=1}^{p} \varrho_\epsilon(a_{i-1}, a_i) \leq \frac{1}{2}R \text{ and } \sum_{i=p+2}^{n} \varrho_\epsilon(a_{i-1}, a_i) \leq \frac{1}{2}R$$

By the induction hypothesis, we have

$$\varrho_\epsilon(a, a_p) \leq \frac{R}{2(1-2\epsilon')} \text{ and } \varrho_\epsilon(a_{p+1}, b) \leq \frac{R}{2(1-2\epsilon')}$$

On the other hand, $\varrho_\epsilon(a_p, a_{p+1}) \leq R$. Hence

$$\begin{aligned}\varrho_\epsilon &\leq (1+\epsilon')^2 \max\{\varrho_\epsilon(a, a_p), \varrho_\epsilon(a_{p+2}, b)\} \\ &\leq \frac{(1+\epsilon')^2}{1-2\epsilon'} \max\{\frac{1}{2}, 1-2\epsilon'\}R\end{aligned}$$

Since $(1'_\epsilon)^2(1-2\epsilon') \leq 1$ for all $\epsilon' \geq 0$, and since $\frac{(1+\epsilon')^2}{2} \leq 1$ if $\epsilon' \leq \sqrt{2}-1$, we obtain

$$\varrho_\epsilon(a,b) \leq \frac{1}{1-2\epsilon'} \sum_{i=1}^{n} \varrho_\epsilon(a_{i-1} a_i)$$

The argument is valid for every $n \geq 2$ and we have finished the proof. □

11. Proposition. *If X is quasi-isometric to the Cayley Graph of a hyperbolic group, each of the distances d_ϵ makes ∂X into a metric space with finite haussdorff dimension.*

Proof. Suppose, more generally, that there exists constants $k > 0$ and $C > 0$ with with the following property: for all integers $n \geq 1$, there exists a finite family $(x_i^{(n)})_{i \in I_n}$ of points in the sphere $\mathbb{S}(n) = \{x \in X : |x - w| = n\}$, with cardinality of $I_n \leq Ce^{kn}$, such that the balls $B_i^{(n)}$ of radius 1 centered at the points $x_i^{(n)}$ cover $\mathbb{S}(n)$. For $i \in I_n$, define

$$U_i^{(n)} = \{a \in \partial X : \text{ there exists a ray } [w, a) \text{ which intersects } B_i^{(n)}\}$$

let $a, b \in U_i^{(n)}$. By approximating the set $[w, a) \cup [w, b)$ by a tree, we see that $(a|b) \geq n - C_1$ for a suitable constant C_1. It follows that $\varrho_\epsilon(a, b) \leq \exp(-\epsilon(n - C_1))$ and the d_ϵ-diameter of $U_i^{(n)}$ is less than $C_2 \exp(-\epsilon n)$, where C_2 is another convenient constant. If s is a number such that $k - \epsilon s < 0$, we then have

$$\lim_{n\to\infty} \sum_{i \in I_n} (\text{diam } (U_i^{(n)}))^s \leq \lim_{n\to\infty} Ce^{kn} C_2 e^{-s\epsilon n} = 0$$

It follows that the d_ϵ Hausdorff dimension is less that s. □

12. A Coding Problem. Let X be a geodesic δ-hyperbolic space. Show there exists a metric tree T and a surjective continuous map $\partial T \to \partial X$. Furthermore, suppose that X is a connected metric graph whose edges are all of length 1, and there exists a integer v such that all the vertices of X have at most $v + 1$ neighbors. Show that we can obtain a map $\partial T \to X$ with finite fibres (there are hints in n° 7.6 of [GR5]).

4. The Action of Isometries and Quasi-Isometries on the Boundary of a Hyperbolic Space

Let X be a geodesic, proper hyperbolic space with base point w. Suppose ϵ is a sufficiently small real number (as in proposition 10) and let d_ϵ be the corresponding distance on the boundary ∂X. In this section, we assume that ∂X is perfect, as is the case if X is the Cayley Graph of a non-elementary hyperbolic group (in the sense of number 15, below). We write as if, for all $a \in \partial X$ and $r > 0$, the sphere $\{b \in \partial X : d_\epsilon(a, b) = r\}$ is never empty (see the beginning of §6.4).

Given a homeomorphism ψ of ∂X and a point $a \in \partial X$, recall that the conformal dilatation of ψ at a for the distance d_ϵ is given by (definition 6.15)

$$H_\psi{}^\epsilon = \limsup_{r\to 0} \frac{\sup\{d_\epsilon(\psi(a), \psi(b)) : b \in \partial X \text{ and } d_\epsilon(a, b) = r\}}{\inf\{d_\epsilon(\psi(a), \psi(b)) : b \in \partial X \text{ and } d_\epsilon(a, b) = r\}}$$

Furthermore, we introduce

$$\tilde{H}_\psi^\epsilon = \limsup_{r\to 0} \frac{\sup\{\varrho_\epsilon(\psi(a), \psi(b)) : b \in \partial X \text{ and } \varrho_\epsilon(a, b) = r\}}{\inf\{\varrho_\epsilon(\psi(a), \psi(b)) : b \in \partial X \text{ and } \varrho_\epsilon(a, b) = r\}}$$

Also recall that ψ is K-quasi-conformal for d_ϵ if $H_\psi{}^\epsilon(a) \leq K$ for all $a \in \partial X$.

13. Proposition. *There exists a constant K_ϵ such that any isometry Φ of X induces a lipschitz, K_ϵ-quasi-conformal homeomorphism $\partial\Phi$ of ∂X. Moreover, K_ϵ tends towards 1 as ϵ tends towards 0.*

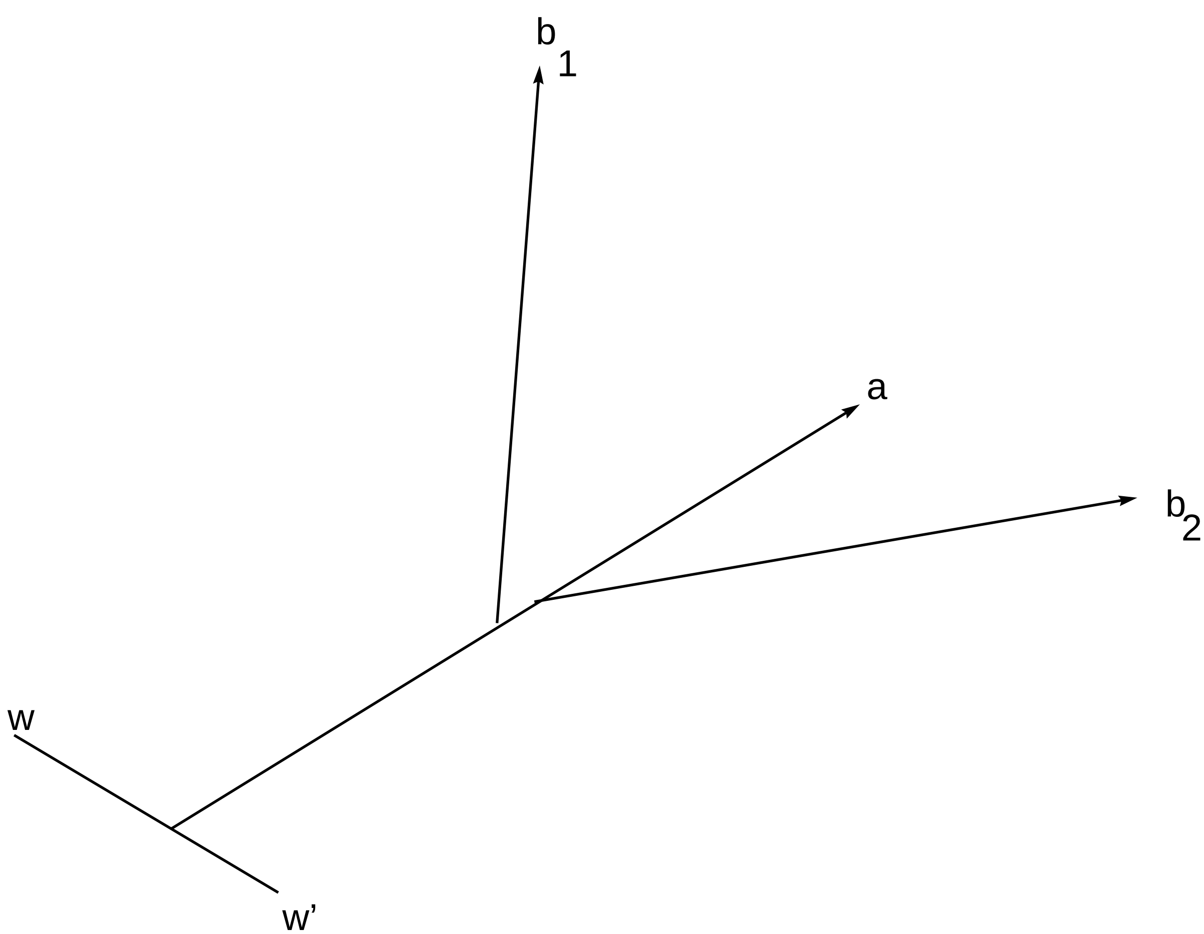
Proof. (Compare to corollary 6.4) We check, as in chapter 6, that $\partial\Phi$ is lipschitz.

Define $w' = \Phi^{-1}(w)$ and $R = |w - w'|$; we denote the Gromov Product of $a, b \in \partial X$ relative to w' by $(a|b)'$ and similarly define ϱ'_ϵ and d'_ϵ. Let a, b_1, b_2 be three points in ∂X. When X is a tree, the inequalities

$$(a|b_1) > R \text{ and } (a|b_2) > R$$

imply

$$(a|b_1) - (a|b_1)' = (a|b_2) - (a|b_2)'$$



In the general case, we choose a segement $[w, w']$ and rayons $[w, a)$, $[w, b_1)$, and $[w, b_2)$. As in theorem 2.12, we can approximate

$$[w, a) \cup [w, b_1) \cup [w, b_2) \cup [w, w']$$

by a tree. It follows that there is a constant C (depending only on δ) with the following poperty: If

$$(a|b_1) > R + C \text{ and } (a|b_2) > R + C$$

then

$$|[(a|b_1) - (a|b_1)'] - [(a|b_2) - (a|b_2)']| \leq C$$

One can deduce that

$$\exp(-\epsilon C)\frac{\varrho_\epsilon(a,b_1)}{\varrho_\epsilon(a,b_2)} \leq \frac{\varrho'_\epsilon(a,b_1)}{\varrho'_\epsilon(a,b_2)} \leq \exp(\epsilon C)\frac{\varrho_\epsilon(a,b_1)}{\varrho_\epsilon(a,b_2)}$$

It follows that

$$\tilde{H}^\epsilon_{\partial\Phi}(a) \leq \exp(2\epsilon C)$$

and

$$H_{\partial\Phi}{}^\epsilon(a) \leq \exp(2\epsilon C)(1-2\epsilon)^{-4}$$

by proposition 10. □

14. Proposition. *Let X, Y be two proper, geodesic δ-hyperbolic spaces with perfect boundaries $\partial X, \partial Y$, and let $\Phi : C \to Y$ be a quasi-isometry.*

Then Φ induces a map $\partial X \to \partial Y$, hereafter denoted by Φ, which is hölder and quasi-conformal.

Further, if there exists a quasi-isometry $\psi : Y \to X$ such that

$$\sup_{x\in X} |\Psi\Phi(x) - x| < \infty \text{ and } \sup_{y\in Y} |\Phi\Psi(y) - y| < \infty$$

(for example, if $\Phi : X \to Y$ is a quasi-isometric homeomorphism), then $\Phi : \partial X \to \partial Y$ is a homeomorphism.

Proof. The map $\Phi : \partial X \to \partial Y$ is hölder by property (1) of proposition 5.15, and quasi-conformal by property (2) (also see proposition 6.6).

Moreover, if there exists a map $\Psi : Y \to X$ with the stated properties, the the compositions of $\Phi : X \to Y$ and $\Psi : Y \to X$ are the identity maps of ∂X and ∂Y. □

5. Some Examples

Let Γ be a hyperbolic group. Proposition 5.15 shows that the boundary $\partial\Gamma$ is well defined up to quasi-isometry; two finite systems of generators S, S' of Γ define quasi-isometries between their Cayley Graphs G, G' and hence their boundaries are quasi-conformally equivalent. We use $\partial\Gamma$ to denote the boundary of any of these graphs G.

15. Proposition - definition.. *Let Γ be a hyperbolic group.*

1. *If Γ is finite, its boundary is empty.*
2. *If Γ contains an finite index index cyclic subgroup, its boundary has precisely two elements.*
3. *In all other cases, the boundary of Γ has at least 3 elements (and we show in corollary 18 that the boundary is therefore infinite).*

In cases (1) and (2), we say that the group Γ *is elementary.*

Proof. Assertion (1) is obvious. Hence, we assume that Γ is infinite with a given system S of generators. For all integers $n \geq 1$, we choose an element γ_n with length $2n$ (with respect to S); let γ_n' be the midpoint of a geodesic segment joining e and γ_n. We have

$$|e - \gamma_n'| = |\gamma_n' - \gamma_n| = n \text{ and } (\gamma_n | e)_{\gamma_n'} = 0$$

We set

$$\beta_n' = {\gamma_n'}^{-1} \text{ and } \beta_n = {\gamma_n'}^{-1}\gamma_n$$

Which implies

$$|\beta_n' - e| = |\beta_n - e| \text{ and } (\beta_n | \beta_n')_e = 0$$

Let a (respectively a') be a point in $\partial\Gamma$ which is the limit of a subsequence of $(\beta_n)_{n\geq 1}$ (respectively $(\beta_n')_{n\geq 1}$). It is clear that $a \neq a'$ and hence that $\partial\Gamma$ contains at least two points.

Now, suppose that $\partial\Gamma$ contains exactly two points a and b. It remains to show that Γ contains a finite index subgroup isomorphic to $\mathbb{Z}$. Let $g : \mathbb{Z} \to \Gamma$ be a geodesic connecting a and b. Hence, using corollary 3.2, every element of Γ lies on a geodesic whose haussdorf distance from g is at most 16δ. It follows that every point of γ is within 16δ of a point in the image of g and hence Γ is quasi-isometric to $\mathbb{Z}$. The assertion now follows from exercise 1.16. □

Recall that $\partial\Gamma$ is a Cantor set if Γ is a finitely generated, non-abelian free group (chapter 6).

We examinne the case of the fundamental group of a closed surface M with genus at least 2. Let g be a negatively curved riemannian metric on M and let Y be the universal cover of M.

For all $y \in Y$, let ${\mathbb{S}^1}_{g,y}$ denote the "visual circle" of tangent vectors at y in Y with length 1. The map $h_{g,y} : {\mathbb{S}^1}_{g,y} \to \partial Y$ which maps a vector η to the class of rays beginning at y with tangent η is a homeomorphism. An interesting exercise is to show that $h_{g,y}$ is K_ϵ-quasi-conformal if we give ${\mathbb{S}^1}_{g,y}$ the angular metric and ∂Y the metric d_ϵ which we constructed earlier. Moreover, $\lim_{\epsilon\to 0} K_\epsilon = 1$.

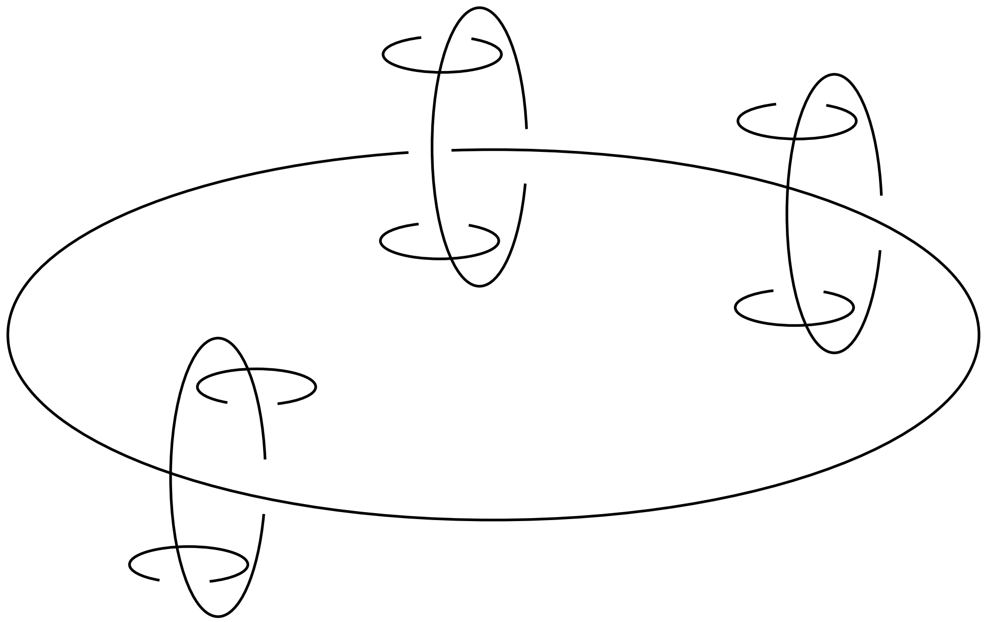
If we compare the homeomorphisms associated to points y, y', we obtain a homeomorphism, $h_{g,y'} \circ h_{g,y'}^{-1}$, of circles which is C^1 (this is not true for all negatively curved surfaces). If we compare the homeomorphisms associated to y and y' with different metrics, $h_{g',y'} \circ h_{g,y}^{-1}$ is a K-quasi-conformal homeomorphism of circles with K depending on g and g'.

Since the fundamental group Γ of M is quasi-isometric to Y, we see that $\partial\Gamma$ is a circle which is well-defined up to quasi-conformality.

Let Γ_1 and Γ_2 be fundamental groups of two closed surfaces M_1 and M_2 which have genus at least two. Let Γ be the free product $\Gamma_1 * \Gamma_2$. Let M be the polyhedron obtained by attaching M_1 and M_2 at a common point c. Choose a base point w in M. Every point in $\partial\Gamma$ can be represented bya geodesic in M originating at w (in this particular case, we use geodesic to mean a *locally* isometric map $\mathbb{R}^+ \to M$; this is the classical, differential-geometric meaning of the word "geodesic").

For concreteness, we will assume that the base point w in in M_1 and is distinct from the attaching point c. We identify the circle $\mathbb{S}^1$ with the set of unit tangent vectors at w in M. Every point of $\mathbb{S}^1$ defines a ray in M_1 and these rays contribute a copy T_1 of the circle to $\partial\Gamma$.

Let η be a point in $\mathbb{S}^1$, let g_η be the geodesic ray in M_1 corresponding to η, and let $t_0 > 0$ be such that $g_\eta(t_0) = c$— the points η for which there exists at least one such t_0 form a countable dense subset of $\mathbb{S}^1$. There exists a family g_ζ of rays in M which agree with g_η for all $t \leq t_0$ and which are in M_2 for $t \geq t_0$; the parameter ζ describes the unit circle in M_2 at c. This family contributes a "second generation" circle $T_{2,\eta}$ to $\partial\Gamma$. It is useful to think of $T_{2,\eta}$ as a circle around the point $\eta \in T_1$ and the corresponding rays as small where t_0 is large.



Iif g_η goes through c many times, there are many disjoint circles surrounding η.

In each circle $T_{2,\eta}$, there is a countable dense set of points corresponding to those rays which might bifurcate into M_1 (at c) and ....

In the final reckoning, $\partial\Gamma$ is the projective limit of the spaces $T_{(j)}$ relative to the surjective maps $\Phi_j : T_{(j+1)} \to T_{(j)}$. The map Φ_j has a countably infinite collection of fibres which are homeomorphic to circles (for all $r > 0$ there are infinitely many of these circles with rays longer than r), and each of the other fibres deforms to a point. For example, $T_{(1)} = T_1$ and $T_{(2)}$ is the countable union of the $T_{2,\eta}$ with T_1.

16. Exercise. Let $\Gamma_1\Gamma_2, M_1, M_2$ be as above, and let c_j be a simple closed geodesic in M_j $(j = 1, 2)$. We assume that c_1 and c_2 are the same length and consider the polyhedron M obtained by isometrically glueing the image of $c_1$in M_2 to the image of c_2 in M_2; we use c to denote the resulting curve in M. The fundamental group Γ of M is an amalgamated product $\Gamma_1 *_{\mathbb{Z}} \Gamma_2$. We use a base point $w \in (M_1 - c) \subset M$.

Let η be a point in the circle $\mathbb{S}^1$ of unit tangent vectors at w in M. If the geodesic ray g_η in M_1 corresponding to η doesn't intersect c then η defines a point in $\partial\Gamma$. The other possibility corresponds to rays which may bifurcate in M, hence to intervals in the first circle $T_{(1)}$ which cause doubling in $\partial\Gamma$. The exercise is to find a precise definition of $\partial\Gamma = \operatorname{proj\,lim} T_{(j)}$, where $T_{(j)}$ corresponds to rays in M

crossing $j-1$ times between M_1 and M_2.

The next set of ideas allow us to compare the idea of *ends* of a locally compact space with the boundary which we have introduced. Since the results are not essential to the rest of the book, we have just given sketches of the proofs. For the theory of ends, see [ScW].

The following statement was told to us by F. Paulin; see also [Pav].

17. Proposition. *Let X be a proper, hyperbolic, geodesic metric space. Let ∂X be the boundary of X and bt(X) the space of ends of X. There exists a continuous surjection from ∂X to btX whose fibres are the connected components of ∂X.*

Sketch of the Proof. Let $g:\mathbb{R}^+\to X$ be a geodesic ray. Since the ray leaves every compact subset of X, it defines an end. On can easily check that this end depends only on the point on the boundary of X defined by g and that the resulting map $\pi:\partial X\to \mathrm{bt}(X)$ is continuous. Let $(x_n)_{n\geq 1}$ be a sequence of points of X which define an end ε of X. If a is a limit of a subsequence of $(x_n)_{n\geq 1}$ in ∂X, we have that $\pi(a)=\varepsilon$. Hence, π is surjective. Since $\mathrm{bt}(X)$ is totally disconnected, the connected components of ∂X are contained in the fibres of π.

It remains to show that the fibres of π are connected. Suppose that there exists a partition of a fibre $\pi^{-1}\varepsilon$ into two closed, non-empty sets F_1 and F_2; we choose two open, disjoint sets Ω_1 and Ω_2 containing F_1 and F_2 (respectively). We choose a base point $w\in X$ and denote the union of of rays beginning at w and ending at a point in Ω_j by $\hat{\Omega}_j$ $(j=1,2)$. If B_R denotes the ball of radius R centered at w then, for R large enough, $\hat{\Omega}_1\setminus B_R$ and $\hat{\Omega}_2\setminus B_R$ are disjoint and are a positive distance apart.

Let g_1 (respectively g_2) be a ray originating at w and ending at a point in F_1 (respectively F_2). By our hypothesis, the sequences $(g_1(n))_{n\geq 1}$ and $(g_2(n))_{n\geq 1}$ define the same end. Hence, for all $N\geq 1$, there exists an integer $n_N >\geq 1$ and a path $l_N:[0,1]\to X$ connecting $g_1(n_N)\in\hat{\Omega}_1$ with $g_2(n_N)\in\hat{\Omega}_2$ which stays outside the ball B_N. Since, for N large enough, $\hat{\Omega}_1\setminus B_R$ and $\hat{\Omega}_2\setminus B_R$ are a positive distance apart, there exists $t_N\in[0,1]$ such that $x_N=l_N(t_N)$ is not in $\hat{\Omega}_1$ or $\hat{\Omega}_2$. Choose a geodesic segment $[w,x_N]$ from w to x_N. Since the lengths of these segments tends to ∞, there is a subsequence of $([w,x_N])_{N\geq 1}$ which converges to a ray g originating at w. It is obvious that g defines the same end ε as g_1 and g_2, but it is not in F_1 or F_2. Since this is absurd, it follows that $\pi^{-1}(\varepsilon)$ is connected. □

18. Corollary. *If the bounary of a hyperbolic group Γ has at least three distinct points, then it is infinite (and, in fact, uncountable).*

Proof. If $\partial\Gamma$ is finite, then proposition 17 shows that $\partial\Gamma=\mathrm{bt}(\Gamma)$. But, we know that the space of ends of a group is a set with $0,1$ or 2 points or is a Cantor set. The corollary follows. □

19. Theorem. *Let Γ be a group which is quasi-isometric to a free group on a finite set of generators. then Γ contains a finite-index, free subgroup.*

Sketch of Proof. Since Γ is quasi-isometric to a free group, Γ is hyperbolic and its boundary is homeomorphic to a Contor set. It follows from corollary 18 that Γ has infinitely many ends.

We first assume that Γ is torsion-free and show that, in this case, Γ is a free group.

Stallings' theoreom, along with Grŭsko's theorem,imply that Γ can be written as a free product $\Gamma_1 * \Gamma_2 * \ldots \Gamma_n$ where each Γ_i is a finitely presented group with 1 or 2 ends. For each Γ_i, $i \in \{1, \ldots, n\}$, we choose a finite set S_i of generators. the the union $S = \cup_{i=1}^{i=n} S_i$ is a system of generators for Γ and the immersions $(\Gamma_i, d_{S_i}) \to (\Gamma, d_S)$ are isometries. It follows that the Γ_i are hyperbolic and that their boundaries (mapped into $\partial\Gamma$) are totally discontinuous; in this way, the boundary of Γ_i can be idenitifed with the space of ends of Γ_i for $i \in \{1, \ldots, n\}$. Hence, the boundary of Γ_i has 1 or 2 elements and Γ_i is elementary (in the sense of number 15); since Γ_i is torsion-free, Γ_i is infinite cyclic and Γ is isomorphic to the free group $\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ (n-factors).

When Γ has torsion, the idea of the proof is the same, but one must overcome some technical difficulties.

To start, since Γ is hyperbolic, Γ is finitely presented. A result of Dunwoody [Dun] shows that Γ is accessible. In other words (see lemma 7.1 of [ScW]), Γ is isomorphic to the fundamental group of a graph of groups where the edge groups are finite and where the vertex groups have at most one end. This is analogous to the decomposition of Γ as a free product of Γ_i and represents an "iterated free product with amalgamation / HNN extension". We proceed as in the preceding case by showing that the vertex groups map isometrically into Γ (for a good choice of generators) and showing that the vertex groups cannot have 1 end and are therefore finite. We finish by observing that the fundamental group of a graph of groups with has finite vertex groups contains a finite-index, free subgroup. For more details, see [ScW] and [Ser]. □

Translated by WILLIAM GROSSO

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720