10

Infinite groups as geometric objects (after Gromov)

Etienne Ghys and Pierre de la Harpe

Many ideas presented here are due to M. Gromov, and our title above is precisely that of his I.C.M. address in Warsaw (Gromov, 1984). The notes of J. Cannon (Cannon, Chapter 11) discuss closely related topics.

There are many geometric problems which may be solved by algebraic tools. Indeed, much of algebraic topology is best motivated by geometric or topological problems (the introduction of Steenrod (1972) is highly recommended). But the point of this chapter is to show that the opposite move may also be fruitful: given an algebraic problem, translate it into geometry to make it 'visible'. The algebraic problem of concern below is to understand finitely generated groups. In this way, we hope to show how the geometric methods discussed in this volume may also be important for algebraists.

10.1 The Cayley graph of a group and the notion of quasi-isometry

Let Γ be a finitely generated group, with neutral element denoted by e.

Choose a finite set S of generators for Γ; for simplicity, we shall *always* assume that $e \notin S$ and that $S^{-1} = S$ (namely $s \in \Gamma$ is in S iff s^{-1} is in S). Define the *length* $\ell_S(\gamma)$ of any $\gamma \in \Gamma$ to be the smallest integer n such that there exists a sequence $(s_1, \ldots, s_n)$ of generators in

Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces. T. Bedford, M. Keane, C. Series (Eds). Oxford University Press, Oxford, 1991.

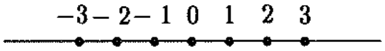

Fig.1

S for which $\gamma = s_1 s_2 \dots s_n$ and define the *distance* $d_S : \Gamma \times \Gamma \to \mathbb{R}_+$ by $d_S(\gamma_1, \gamma_2) = \ell_S(\gamma_1^{-1}\gamma_2)$. It is easy to check that d_S makes Γ a *metric space.*

Since d_S takes integral values, this metric space is discrete, and this may impede geometric understanding. One way out (logically useless but intuitively worthwhile) is to introduce the *Cayley graph* $G(\Gamma, S)$: this is a graph with vertex set Γ in which two vertices γ_1, γ_2 are the two ends of an edge if and only if $\gamma_1^{-1}\gamma_2 \in S$. (This gives a non-oriented graph, without any loop or multiple edge, which is infinite whenever Γ is infinite.) There is an obvious action from the left of Γ on this graph which is transitive on the set of vertices.

Each edge of $G(\Gamma, S)$ can be made a metric space isometric to the segment [0,1], in such a way that the left action of Γ produces isometries between the edges. One defines naturally the length of a path between two points (not necessarily two vertices) of the graph, and the distance between two points is defined to be the infimum of the appropriate path-lengths. In this way $G(\Gamma, S)$ is made a *metric space* which is *arc-connected*, and the natural inclusion $\Gamma \subset G(\Gamma, S)$ is an isometry.

Cayley diagrams of finite groups were introduced by Cayley (in 1878) and others, but the first use of infinite Cayley diagrams is due to Dehn (in 1910), for surface groups and for the group of the trefoil knot. This is why one says also 'Dehn Gruppenbild' for 'Cayley graph' (Cannon, Chapter 11).

Example 10.1 *If Γ is the infinite cyclic group $\mathbb{Z}$ and if $S = \{1, -1\}$, then $G(\Gamma, S)$ is isometric to the real line (Figure 1) and $\Gamma \subset G(\Gamma, S)$ is the usual inclusion $\mathbb{Z} \subset \mathbb{R}$.*

Example 10.2 *If $\Gamma = \mathbb{Z}^2$ and $S = \{\binom{1}{0}, \binom{-1}{0}, \binom{0}{1}, \binom{0}{-1}\}$, then $G(\Gamma, S)$ is isometric to the standard square grid $\{(x, y) \in \mathbb{R}^2 : \text{at least one of } x, y \text{ is in } \mathbb{Z}\}$ in the Euclidean plane.*

Similarly, for any $n \geq 3$, the group $\mathbb{Z}^n$ produces a graph which embeds in $\mathbb{R}^n$.

Example 10.3 *Let Γ be the (non-Abelian) free group F_2 on two generators a, b, and set $S = \{a, a^{-1}, b, b^{-1}\}$. Then $G(F_2, S)$ is a tree in which any vertex has four neighbours, (Figure 2).*

(The vertices shown on Figure 2 are those of a ball of radius 2.) There

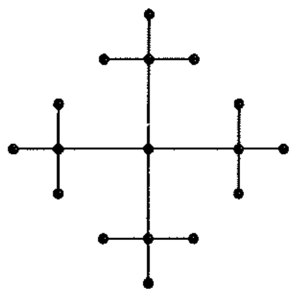

Fig.2

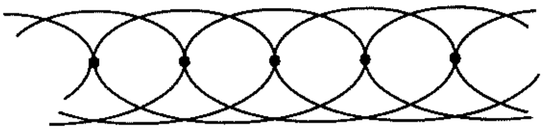

Fig.3

is no isometric embedding of this Cayley graph in a Euclidean plane (though amateurs of curiosities may enjoy Kuiper (1955)), but it is easy to draw $G(F_2, S)$ in the hyperbolic plane H^2 (see Section 1 in Series (Chapter 5)). It is fascinating to look at classical pictures, such as those reproduced in Magnus' book (Magnus, 1974); on Figure 19.a of that book, the graph dual to the tesselation is a copy of $G(F_2, S)$.

There is a similar example with the group F_k for each $k > 2$, producing a tree where each vertex has $2k$ neighbours and which may also be drawn in H^2.

Example 10.4 *If $\Gamma = \mathbb{Z}$ and if $S = \{2, -2, 3, -3\}$, then $G(\Gamma, S)$ is shown in Figure 3.*

This shows that $G(\Gamma, S)$ does indeed depend on both Γ and S. The last figure looks like that of Example 1 when looked at from far enough away. This motivates the following definition.

Definition 10.5 *Let (X, d) and (X', d') be two metric spaces. A map $f : X \to X'$ is a quasi-isometry if there exist constants $\lambda > 0, C \geq 0$ such that*

$$\frac{1}{\lambda}d(x,y) - C \leq d'(f(x), f(y)) \leq \lambda d(x,y) + C$$

for all $x, y \in X$. The spaces (X, d) and (X', d') are quasi-isometric if there exists a quasi-isometry $f : X \to X'$ and a constant $D \geq 0$ such that $d'(f(X), x') \leq D$ for all $x' \in X'$.

Observe that f is *not* necessarily continuous. For example, $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric, as one may check with the map $f : \mathbb{R} \to \mathbb{Z}$ which takes a number to its integral part. More generally, given a group Γ and a finite generating set S, the Cayley graph $G(\Gamma, S)$ defined above and the metric space (Γ, d_S) are quasi-isometric; the case of $\mathbb{R}$ and $\mathbb{Z}$ is just the particular case of Example 1.

Exercise 10.6

(i) *Two metric spaces (X, d) and (X', d') are quasi-isometric if and only if there exist two maps $f : X \to X'$ and $g : X' \to X$ as well as two constants $\lambda > 0, C \geq 0$ such that*

$$d'(f(x), f(y)) \leq \lambda d(x, y) + C$$

$$d(g(x'), g(y')) \leq \lambda d'(x', y') + C$$

$$d'(f(g(x')), x') \leq C$$

$$d(g(f(x)), x) \leq C$$

for all $x, y \in X$ and $x', y' \in X'$.

(ii) *Quasi-isometry is an equivalence relation between metric spaces.*

Proposition 10.7 *Let S and S' be two finite sets of generators of a group Γ, and let d, d' be the distances defined on Γ by S and S' respectively. Then (Γ, d) and (Γ, d') are quasi-isometric.*

Proof. Let f denote the identity transformation of Γ, viewed as a map from (Γ, d) to (Γ, d'). Set $\lambda_1 = \max\{d'(s, e) : s \in S\}$ and $\lambda_2 = \max\{d(s', e) : s' \in S'\}$. For all $x, y \in \Gamma$, it is easy to check (by induction on $d(x, y)$) that $d'(f(x), f(y)) \leq \lambda_1 d(x, y)$. Similarly $d(f^{-1}(x), f^{-1}(y)) \leq \lambda_2 d'(x, y)$. The proposition follows. □

Though (Γ, d_S) does depend on S and is not well defined as a metric space by the group Γ alone, it follows from Proposition 7 that (Γ, d_S) is *well defined up to quasi-isometry* by the group Γ alone. Thus any property invariant by quasi-isometry of the metric space (Γ, d_S) is a *property of the group* Γ.

In particular, given two finitely generated groups Γ and Γ', one may ask whether they are quasi-isometric or not. The question makes sense even though one has not chosen finite generating sets in Γ and Γ'.

10.2 Examples of quasi-isometric groups

The first example is obvious:

Example 10.8 *A metric space is quasi-isometric to a point if and only if its diameter is finite. In particular, the metric space $G(\Gamma, S)$ defined in Section 1 is quasi-isometric to a point if and only if the group Γ is finite. Otherwise said, the quasi-isometry class of the group $\{e\}$ is precisely the class of finite groups.*

Other examples will come after a proposition for which we introduce some vocabulary.

A metric space X is a *geodesic space* if, for every pair (x, y) of points of X, there exists an isometry $g : [0, d(x, y)] \to X$ such that $g(0) = x$ and $g(d(x, y)) = y$. (We do not ask that this isometry be unique.) Examples of geodesic spaces: Cayley graphs as above, and complete Riemannian manifolds (by the Hopf-Rinow Theorem).

A metric space is *proper* if all its closed balls are compact. A proper space is locally compact and complete (and the converse holds for geodesic spaces, see Theorem 1.10 in Gromov (1981a)).

A discrete group Γ acting by homeomorphisms on a locally compact topological space X is said to act *properly* if, for every compact subspace K of X, the set $\{\lambda \in \Gamma : \lambda K \cap K \neq \emptyset\}$ is finite. (There are equivalent definitions, e.g. in Bourbaki 1971, chap. III, §4); see in particular the remark on page 34. For an action of a discrete group, one says also 'discontinuously' instead of 'properly' : see Section 3 of Beardon (Chapter 1).

Proposition 10.9 *Let X be a metric space which is geodesic and proper. Let Γ be a group acting (say from the left) isometrically and properly on X. If $\Gamma \backslash X$ is compact, then Γ is a finitely generated group which is quasi-isometric to X.*

Proof. (See e.g. Lemma 2 in Milnor (1968); see also Cannon (Chapter 11)). Let $\pi : X \to \Gamma \backslash X$ be the canonical projection. The space $\Gamma \backslash X$ has a canonical metric defined by $d(p, q) = \inf\{d(x, y) : x \in \pi^{-1}(p) \text{ and } y \in \pi^{-1}(q)\}$. As $\Gamma \backslash X$ is compact, its diameter

$$R = \sup\{d(p, q) : p, q \in \Gamma \backslash X\}$$

is finite. Choose a base point $x_0 \in X$ and set

$$B = \{x \in X : d(x_0, x) \leq R\}.$$

Observe that $(\gamma B)_{\gamma \in \Gamma}$ is a covering of X. Set

$$S = \{\gamma \in \Gamma : \gamma \neq e \quad \text{and} \quad \gamma B \cap B \neq \emptyset\}.$$

Observe that $S^{-1} = S$, and that S is finite because the action is proper. Finally, set

$$r = \inf\{d(B, \gamma B) : \gamma \in \Gamma - (S \cup \{e\})\}.$$

As the infimum is achieved over a finite set (e.g. over those $\gamma \in \Gamma - (S \cup \{e\})$ such that $d(x_0, \gamma x_0) \geq 5R$) and as B is compact, one has $r > 0$.

We claim firstly that S generates Γ and that, for every $\gamma \in \Gamma$, one has $d_S(e, \gamma) \leq \frac{1}{r} d(x_0, \gamma x_0) + 1$.

Indeed, consider an element $\gamma \in \Gamma$. Let k be the smallest integer such that $d(x_0, \gamma x_0) < kr + R$. As X is geodesic, one may choose points $x_1, x_2, \ldots, x_{k+1} = \gamma x_0$ so that $d(x_0, x_1) \leq R$ and $d(x_i, x_{i+1}) < r$ for $i = 1, \ldots, k$. As $(\alpha B)_{\alpha \in \Gamma}$ is a covering of X, one may also choose $\gamma_1, \gamma_2, \ldots, \gamma_{k+1} \in \Gamma$ so that $x_i \in \gamma_i B$ for $i = 1, \ldots, k+1$, with moreover $\gamma_1 = e$ and $\gamma_{k+1} = \gamma$. Set $s_i = \gamma_i^{-1} \gamma_{i+1}$, so that $\gamma = s_1 s_2 \ldots s_k$. As we have on the one hand

$$d(\gamma_i^{-1} x_i, \gamma_i^{-1} x_{i+1}) < r$$

and on the other hand

$$\gamma_i^{-1} x_i \in B \qquad \gamma_i^{-1} x_{i+1} = s_i \gamma_{i+1}^{-1} x_{i+1} \in s_i B$$

it follows from the definitions of r and S that $s_i \in S \cup \{e\}$ for $i = 1, \ldots, k$, so that $d_S(e, \gamma) \leq k$. But $(k-1)r + R \leq d(x_0, \gamma x_0)$ by definition of k, so that

$$d_S(e, \gamma) \leq \frac{1}{r} d(x_0, \gamma x_0) + 1 - \frac{R}{r}.$$

We claim secondly that $d(x_0, \gamma x_0) \leq \lambda d_S(e, \gamma)$ for all $\gamma \in \Gamma$, where $\lambda = \sup\{d(x_0, s x_0) : s \in S\}$. This claim is straightforward, by induction on $d_S(e, \gamma)$.

Consider now the map

$$f : \begin{cases} \Gamma \longrightarrow X \\ \gamma \longmapsto \gamma x_0. \end{cases}$$

The first claim implies that

$$d_S(\gamma_1, \gamma_2) \leq \frac{1}{r} d(f(\gamma_1), f(\gamma_2)) + 1$$

for all $\gamma_1, \gamma_2 \in \Gamma$. The second claim implies that

$$d_S(f(\gamma_1), f(\gamma_2)) \leq \lambda d_S(\gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. One has finally

$$d(f(\Gamma), x) \leq R$$

for all $x \in X$ because $(\alpha B)_{\alpha \in \Gamma}$ is a covering of Γ. The proposition follows. □

Corollary 10.10

(i) *Let Γ be a finitely generated group and let Γ' be a subgroup of finite index in Γ. Then Γ' is finitely generated and quasi-isometric to Γ.*

(ii) *Let $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ be a short exact sequence of groups with Γ' finite and Γ, Γ'' finitely generated. Then Γ and Γ'' are quasi-isometric groups.*

(iii) *Let Γ be the fundamental group of a closed orientable surface of genus $g \geq 2$. Then Γ is quasi-isometric to the hyperbolic plane H^2.*

Proof.

(i) Choose a finite set S of generators of Γ and apply Proposition 9 to the natural action of Γ' on the Cayley graph $G(\Gamma, S)$.

(ii) Choose a finite set S'' of generators of Γ'' and consider the natural action of Γ on $G(\Gamma'', S'')$.

(iii) Choose a Riemannian metric of constant curvature -1 on the surface and consider the associated action of Γ on H^2. □

Observe that claim (iii) carries over to any cocompact Fuchsian group (see Beardon, Chapter 1).

Example 10.11 *For each integer $k \geq 2$, denote by F_k the free group on k generators. Then F_k and F_2 are quasi-isometric.*

To check this, it is enough to show that F_2 has a subgroup of finite index isomorphic to F_k. But consider the graph H which has one vertex of degree four and two edges (the figure eight), and a connected regular covering $\pi : G \to H$ with $k-1$ sheets. The connected graph G has $k-1$ vertices and $2(k-1)$ edges, thus its Euler characteristic is $1-k$ and its fundamental group is isomorphic to F_k. It follows that π induces an inclusion of F_k onto a subgroup of $F_2 = \pi_1(H)$ of index $k-1$. Figure 4 shows the situation for $k=3$.

Exercise 10.12 *Check that F_2 is quasi-isometric to the modular group $PSL_2(\mathbb{Z})$. Hint: the images in $PSL_2(\mathbb{Z})$ of the matrices $\binom{12}{01}$ and $\binom{10}{21}$ generate a subgroup of finite index which is isomorphic to F_2.*

From the point of view of quasi-isometries, the examples in this section show that finite groups are irrelevant. For this reason, it is

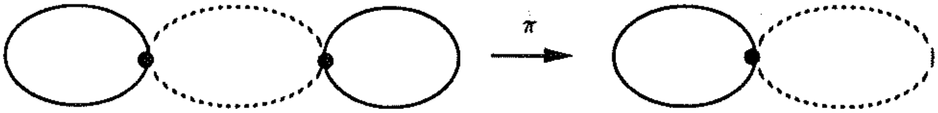

Fig.4

convenient to introduce the following terminology.

Let (P) be a property of discrete groups. A group Γ is said to *have virtually the property* (P) if Γ has a subgroup of finite index which has property (P). In particular, Γ is *virtually Abelian* if it has an Abelian subgroup of finite index (examples: the infinite dihedral group, or crystallographic groups à la Bieberbach (Buser, 1985)); *virtually nilpotent* if it has a nilpotent subgroup of finite index (examples: products of a nilpotent group and a finite group); *virtually free* if it has a free subgroup of finite index (examples: the modular group of Exercise 12, or indeed any Fuchsian group which is not cocompact). Observe however that some properties are not interesting for this 'virtualification'; for example, a group is obviously 'virtually finite' if and only if it is finite.

To conclude this section, let us mention without proof that there are pairs of quasi-isometric groups (Γ, Γ') for which there is not any group Γ_o which is isomorphic to a subgroup of finite index in both Γ and Γ' (the groups Γ and Γ' are not 'commensurable'). An example is given by $\Gamma = \pi_1(M)$ and $\Gamma' = \pi_1(M')$, where M and M' are 3-dimensional compact Riemannian manifolds of constant curvature -1 such that the ratio of the Riemannian volumes $\frac{\mathrm{vol}(M')}{\mathrm{vol}(M)}$ is irrational. For the existence of such pairs, see Thurston's notes (Thurston, 1978).

10.3 Growth, and examples of non-quasi-isometric groups

In order to give examples as in the title, we introduce the following growth invariant.

Let Γ be a group generated by a finite subset S (always with $e \notin S$ and $S^{-1} = S$, as in Section 1). For each integer $k \geq 0$, denote by $b_S(k)$ the cardinal of the finite ball $\{\gamma \in \Gamma : d(e, \gamma) \leq k\}$. Define the *degree of polynomial growth*

$$d(\Gamma) = \limsup_{k\to\infty} \frac{\log b_S(k)}{\log k} \in [0, \infty].$$

If T is another finite set generating Γ, it is easy to check that $b_T(k) \leq b_S(\lambda k)$ for all $k \geq 0$, with $\lambda = \max\{d_S(e,t) : t \in T\}$. It follows that $d(\Gamma)$ depends only on Γ and not on the choice of S.

Proposition 10.13 *With the above notation, one has the following properties of the degree of polynomial growth.*

(i) *If Γ is finite, $d(\Gamma) = 0$.*

(ii) *If Γ is infinite, $b_S(k+1) \geq b_S(k)+1$ for all $k \geq 0$, and $d(\Gamma) \geq 1$.*

(iii) *If Γ'' is a quotient of Γ, then $d(\Gamma'') \leq d(\Gamma)$.*

(iv) *If Γ' is a finitely generated subgroup of Γ, then $d(\Gamma') \leq d(\Gamma)$.*

(v) *If $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ is a short exact sequence of finitely generated groups, then $d(\Gamma) \geq d(\Gamma') + d(\Gamma'')$; if moreover Γ is the direct product $\Gamma' \times \Gamma''$, equality holds.*

These properties are straightforward to check. As a digression let us mention a few others:

If Γ' is a finitely generated subgroup of infinite index in Γ, then $d(\Gamma') \leq d(\Gamma) - 1$. This is the 'Splitting Lemma' of Gromov (1981b); see also Tits (1981).

If Γ is such that $d(\Gamma) < \infty$, then $d(\Gamma)$ is an integer. The only proof we know of this uses the full strength of Gromov (1981b).

In property (v) above, equality does not hold in general. Say first that a group Γ has *exponential growth* if, notations being as above, there exist constants $u > 0$ and $v > 1$ such that $b_S(k) \geq uv^k$ for all $k \geq 0$. Then, a semi-direct product of the form $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ associated to a matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in GL_2(\mathbb{Z})$$

has exponential growth if and only if α has an eigenvalue μ such that $|\mu| > 1$. In particular, one may have $d(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}) = \infty$.

Example 10.14 *For each integer $n \geq 1$, one has $d(\mathbb{Z}^n) = n$. For each integer $k \geq 2$, the non-Abelian free group F_k on k generators has exponential growth; in particular, $d(F_k) = \infty$.*

Proposition 10.15 *Two finitely generated groups which are quasi-isometric have the same degree of polynomial growth.*

Proof. Let Γ_j be a group generated by a finite set S_j, and let d_j be the associated distance on Γ_j $(j = 1, 2)$. Let us assume that there exists a quasi-isometry $f : \Gamma_1 \to \Gamma_2$ such that

$$\frac{1}{\lambda} d_1(x, y) - C \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + C \quad \text{for all} \quad x, y \in \Gamma_1$$

for some constants $\lambda > 0, C \geq 0$. We shall show that $d(\Gamma_2) \geq d(\Gamma_1)$, and the proposition will follow.

For each $k \geq 0$, denote by $B_j(k)$ the ball of radius k around e in Γ_j and by $b_j(k)$ its cardinal $(j = 1, 2)$. Set $D = C + d_2(f(e), e)$. If $\gamma \in B_1(k)$, then $d_2(f(e), f(\gamma)) \leq \lambda k + C$; this shows that $f(B_1(k)) \subset B_2(\lambda k + D)$. If $\gamma, \gamma' \in \Gamma_1$ are such that $f(\gamma) = f(\gamma')$, then $d_1(\gamma, \gamma') \leq \lambda C$; this shows that two points in Γ_1 with the same image by f are in some common ball of radius λC. Consequently

$$b_2(\lambda k + D) \geq \frac{b_1(k)}{b_1(\lambda C)} \quad \text{for all} \quad k \geq 0.$$

This implies clearly that $d(\Gamma_2) \geq d(\Gamma_1)$. □

Corollary 10.16

(i) *The free Abelian groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are quasi-isometric if and only if $m = n$.*

(ii) *A free Abelian group $\mathbb{Z}^n$ cannot be quasi-isometric to a non-Abelian free group F_k* $(k \geq 2)$.

Proof. See Example 14 and Proposition 15. □

In fact, the next section will hopefully convince the reader that two groups cannot be quasi-isometric unless they are, algebraically speaking, 'very similar'.

10.4 Geometric properties, and open problems

A property (P) of finitely generated groups is said to be a *geometric property* if, whenever Γ_1 and Γ_2 are quasi-isometric finitely generated groups, Γ_1 has property (P) if and only if Γ_2 has property (P).

It is remarkable that there is an abundance of geometric properties, so that quasi-isometry is a very interesting relation between groups : though it ignores finite details (Section 2), it preserves a lot of distinct properties. As a trivial example, finiteness is a geometric property (Example 8). Our first genuine example is a reformulation of Proposition 15:

Proposition 10.17 *The degree of polynomial growth is a geometric property.*

For the following examples, we give only sketches of arguments, or just references.

Proposition 10.18 *Being of finite presentation is a geometric property.*

Proof. Let G be a Cayley graph of the group Γ. Given an integer $k \geq 1$, denote by $L(k)$ the set of loops in G of the form $\varphi\psi\varphi^{-1}$, where φ is a path from e to some vertex x in G and where ψ is a loop starting at x of length at most k. Then Γ is finitely presented if and only if the following holds :

($\star$) $L(k)$ generates the fundamental group $\pi_1(G)$ for k large enough.

One may check that ($\star$) is a geometric property. □

Proposition 10.19 *To be virtually cyclic is a geometric property.*

The proof has two steps : given a finitely generated group Γ which is quasi-isometric to $\mathbb{Z}$, one shows firstly that Γ has an element γ of infinite order, and secondly that the quotient set $\Gamma/\gamma^{\mathbb{Z}}$ is finite. Though this proof can be made completely elementary (see Chapter 1 in Ghys and de la Harpe, 1990), details make the arguments longer than we would wish. Of course Proposition 19 follows also from rather easy growth estimates and from the spectacular result of Gromov (1981b); see also Tits (1981).

Theorem 10.20 *To be virtually nilpotent is a geometric property.*

Theorem 10.21 *To be virtually Abelian is a geometric property.*

Problem 10.22 *It is an open problem to find a proof of Theorem 21 which does not use the full strength of Theorem 20. For example, it would be nice to have a rather short proof of the following: a finitely generated group which is quasi-isometric to $\mathbb{Z}^2$ has a subgroup of finite index which is isomorphic to $\mathbb{Z}^2$.*

Theorem 10.23 *Amenability is a geometric property.*

Proof. See Følner (1955) or for a more recent proof pages 446–448 of Connes (1976); see also Chapter 6 in Gromov (1981a). □

Before stating other open problems, let us recall the following notions (see e.g. Kargapolov and Merzliakov, 1985). Let Γ be a group. A (finite) space *matriochka* in Γ is a nested sequence

$$(\star) \qquad \{e\} = \Gamma_o \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \Gamma$$

where each Γ_j is a subgroup of Γ. A matriochka is *subnormal* if each Γ_{j-1} is a normal subgroup of Γ_j; when this holds the quotient groups Γ_j/Γ_{j-1} are the *quotients* of the matriochka. The matriochka $(\star)$ is *normal* if each Γ_j is normal in Γ. It is *central* if it is normal and if $[\Gamma_j, \Gamma] \subset \Gamma_{j-1}$, namely if Γ_j/Γ_{j-1} lies in the centre of Γ/Γ_{j-1} for $j = 1, \ldots, n$. Subnormal matriochkas are also called 'composition sequences'.

A group is *nilpotent* if it has a central matriochka, *polycyclic* if it has a subnormal matriochka with cyclic quotients (finite or infinite), and *solvable* if it has a subnormal matriochka with Abelian quotients.

For example, a group of upper triangular matrices with ones on the diagonal is nilpotent. A finitely generated nilpotent group is polycyclic (see Theorem 17.2.2 in Kargapolov and Merzliakov, 1985). The semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ associated to the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ via powers of $\alpha = \binom{21}{11}$ is polycyclic, but is not nilpotent. A group which is either nilpotent or polycyclic is obviously solvable. A group which is finite, or which is more generally a subgroup of $GL_n(\mathbb{Z})$ for some integer n, is polycyclic if and only if it is solvable (see Theorem 21.2.1 in Kargapolov and Merzliakov, 1985). For any group Γ, let $D\Gamma$ denote the subgroup of Γ generated by the commutators $xyx^{-1}y^{-1}$, with $x, y \in \Gamma$; recall that F_2 denotes the free group on two generators; then $F_2/D(DF_2)$ is an example of a finitely generated solvable group which is not polycyclic (because its subgroup $DF_2/D(DF_2)$ is not finitely generated and in particular is not polycyclic, whereas any subgroup of a polycyclic group is again polycyclic).

Problem 10.24 *Is polycyclicity a geometric property ? Is solvability a geometric property ?*

These two questions constitute the beginning of a list that one may extend at will.

10.5 Hyperbolic groups

Let X be a geodesic metric space (the definition is just after Example 8). A *geodesic segment* between two points $x, y \in X$ is the image of an isometry $g : [0, d(x,y)] \to X$ such that $g(0) = x$ and $g(d(x,y)) = y$; such a segment is often denoted by $[x,y]$, though there are in general *several* segments between two given points. A *geodesic triangle* in X is a subset

$$\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$$

where the x_i's are points in X and where $[x_i, x_j]$ denotes some segment between x_i and x_j.

Definition 10.25 *The geodesic space X is hyperbolic if there is a constant $\delta \geq 0$ such that the following holds: for any geodesic triangle $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ in X one has $d(y, [x_1, x_2] \cup [x_2, x_3]) \leq \delta$ for all $y \in [x_1, x_3]$.*

See Cannon (Chapter 11) for more on this notion.

The following result is far from being obvious. See Gromov (1987) and Ghys and de la Harpe (1990).

Theorem 10.26 *Let X, X' be two geodesic metric spaces which are quasi-isometric. The spacr X is hyperbolic if and only if X' is hyperbolic.*

Definition 10.27 *A finitely generated group Γ is hyperbolic if the associated Cayley graphs $G(\Gamma, S)$ are hyperbolic.*

To check whether a group is hyperbolic or not, it is enough by Theorem 26 and Proposition 7 to consider *one* set of generators only.

For example, it is easy to check that the hyperbolic plane H^2 is hyperbolic in the sense of Definition 25. One shows by standard methods (comparison theorems) that any simply connected complete Riemannian manifold of sectional curvature bounded above by some constant $\kappa < 0$ is a hyperbolic space. It follows from Proposition 9 that the fundamental group of a closed Riemannian manifold of negative curvature is a hyperbolic group. Gromov indicates in Gromov (1987) many more examples of hyperbolic groups, and indeed claims that a finitely presented group 'is hyperbolic with probability one'.

Hyperbolic groups should be thought of as *negatively curved* groups. One of the main problems in extending the theory is that one does not have, so far, a good definition of groups of *non-positive curvature.*

Hyperbolicity for groups is a very fine notion: on one hand it covers so many groups, but on the other hand it allows remarkably precise results, of which the next theorem indicates just a sample.

Theorem 10.28 *Let Γ be a hyperbolic group. Then*

(i) *Γ is finitely presented.*

(ii) *Γ has finite cohomological dimension over the rationals, namely $H^k(\Gamma, \mathbb{Q}) = \{0\}$ for k large enough.*

(iii) *Any virtually solvable (or more generally any amenable) subgroup of Γ is virtually cyclic.*

(iv) *Let S be any finite set of generators of Γ, and denote by $\sigma(n)$ the cardinal of the sphere $\{\gamma \in \Gamma : d_S(e,\gamma) = n\}$. Then the formal power series $\sum_{n=0}^{\infty} \sigma(n)t^n$ is a rational function of the variable t.*

The rational function of (iv) does depend on S. How do its properties (zeros, poles,...) depend on S ? How do they relate to properties of Γ ? (See Parry 1988, and references therein.) Problems concerning this rational function are related to the material of Lalley (Chapter 8) and Pollicott (Chapter 6).

The theory of hyperbolic groups solves problems which were formulated long before this theory. Without entering any detail, let us quote two more results of Gromov.

Theorem 10.29 *Let Γ be the fundamental group of a compact Riemannian manifold of negative curvature. Then Γ has infinite quotient groups in which all elements have finite orders.*

Theorem 10.30 *There are countable groups which have property (T) of Kazhdan and which are not finitely presented.*

Theorem 29 gives natural constructions of finitely generated infinite torsion groups, and answers problems about amenable groups going back to von Neumann (1929). For previous answers, see Grigorchuk (1983) and Ol'shanskii (1980); see also Grigorchuk (1985). Theorem 30 answers a question from Kazhdan (1967).

10.6 Bibliography

Bourbaki, N. (1971). *Topologie générale*, Chaps. 1–4. Hermann. Paris.

Buser, P. (1985). A geometric proof of Bieberbach's theorems on crystallographic groups. *L'Enseignement Math.*, **31**, 137-145.

Følner, E. (1955). On groups with full Banach mean value. *Math. Scand.*, **3**, 243-254.

Connes, A. (1976). On the classification of von Neumann algebras and their automorphisms. In *Symposia Math. XX*, pp. 435-478.

Academic Press.

Ghys, E., and de la Harpe, P. (1990). *Sur les groupes hyperboliques d'après Mikhael Gromov.* Progress in Mathematics, **83.** Birkhäuser. Zurich.

Grigorchuk, R.I. (1983). On Milnor's problem of group growth. *Soviet Math. Dokl.*, **28-1**, 23-26.

Grigorchuk, R.I. (1985). Degrees of growth of finitely generated groups and the theory of invariant means. *Math. USSR Izvestiya*, **25-2**, 259-300.

Gromov, M. (1981a). *Structures métriques pour les variétés riemanniennes.* (rédigé par Lafontaine, J. et Pansu, P. Cedic/F. Nathan). Paris.

Gromov, M. (1981b). Groups of polynomial growth and expanding maps. *Publ. Math. I.H.E.S.*, **53**, 53-73.

Gromov, M. (1984). Infinite groups as geometric objects. *Proc. I.C.M. Warszawa*, **1**, 385-392.

Gromov, M. (1987). Hyperbolic groups. In *Essays in group theory*, (ed. S.M. Gersten), pp. 75-263. Springer. New York.

Kargapolov, M. and Merzliakov, I. (1985). *Eléments de la théorie des groupes.* Mir, Moscow.

Kazhdan, D. (1967). Connection of the dual space of a group with the structure of its closed subgroups. *Funct. Anal. and its Appl.*, **1**, 63-65.

Kuiper, N.H. (1955). On C^1 isometric imbedding, II. *Proc. Kon. Ned. Akad. v. Wetensch*, **A58**, 683-689.

Magnus, W. (1974). *Noneuclidean tesselations and their groups.* Academic Press. New York.

Milnor, J. (1968). A note on curvature and fundamental group. *J. Differential Geometry*, **2**, 1-7.

Von Neumann, J. (1929). Zur allgemeinen Theorie des Masses. *Fund. Math.*, **13**, 73-116.

Ol'shanskii, A. Ju. (1980). On the problem of the existence of an invariant mean on a group. *Russian Math. Surveys*, **35**, 180-181.

Parry, W. (1988). Counterexamples involving growth series and Euler characteristics. *Proc. Amer. Math. Soc.*, **102**, 49-51.

Steenrod, N.E. (1972). Cohomology operations, and obstructions to extending continuous functions. *Adv. in Math.*, **8**, 371-416.

Thurston, W.P. (1978). *The geometry and topology of 3-manifolds.* Unpublished lecture notes, Princeton University.

Tits J. (1981). Groupes à croissance polynomiale. In *Lecture Notes in Mathematics*, **901**, pp. 176-188, Springer. Berlin.

Etienne Ghys
Laboratoire de Mathématiques
Ecole Normale Supérieure de Lyon
46, Allée d'Italie
69364 Lyon cedex 07
France.

Pierre de la Harpe
Section de Mathématiques
Université de Genève
C.P. 240
1211 Genève 24
Switzerland