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LOCALLY FREE HOLOMORPHIC ACTIONS OF THE COMPLEX AFFINE GROUP

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ABSTRACT

In this paper we classify holomorphic volume preserving locally free actions of the complex affine group on compact complex 3-manifolds. The proof is a mixture of complex analytic methods and of classical hyperbolic theory for dynamical systems. Some examples come from automorphisms of 2-dimensional complex tori that we describe them explicitly.

0. Introduction

Let us denote by Aff the affine group of the complex line, i.e. the group of transformations $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. The purpose of this paper is to classify locally free, holomorphic, and volume-preserving actions of Aff on compact complex 3-manifolds. There are several motivations for such a study. The first is that there are interesting examples obtained by arithmetical constructions and we

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believe that these examples deserve a careful analysis, especially from the dynamical point of view. A second motivation is that the corresponding problem in the real domain, i.e. actions of the real affine group on real 3-manifolds, has already been studied in [Gh1], leading to rigidity results. As a general principle, real and complex one dimensional dynamics should be very similar so that one could expect a good understanding of holomorphic codimension one actions of complex Lie groups.

This paper is independent of [Gh3] but is closely related to it. In [Gh3], the first author studies holomorphic Anosov flows on complex 3-manifolds and shows that these flows have to preserve some volume form. In the present paper we make the assumption that the action under consideration is volume-preserving but *we know of no example where this condition is not satisfied.*

In section 1, we describe examples of actions of Aff on 3-manifolds. In section 2, we prove our main theorem according to which any locally free, holomorphic, volume-preserving action of Aff is conjugate to one of the examples described in section 1. Finally, the appendix contains a description of automorphisms of 2-dimensional complex tori, basically known to algebraic geometers but which might be useful to others.

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1. Construction of examples

We shall in fact study actions of the covering spaces of Aff, so we will introduce some notation for them. If k is a positive integer, we denote by Aff_k the group of pairs (a,b) with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ with multiplication:

$$(a,b)(a',b') = (aa', a^k b' + b).$$

Of course Aff_1 coincides with Aff and the map $(a,b) \mapsto (a^k, b)$ shows that Aff_k is the k-fold cover of Aff. The center of Aff_k is cyclic of order k and consists of elements of the form $(a,0)$ with $a^k = 1$. Similarly, we introduce the universal cover, Aff_∞ of Aff. This can be seen as the group of pairs (α, b) with $\alpha \in \mathbb{C}$ and $b \in \mathbb{C}$ and multiplication given by $(\alpha,b)(\alpha',b') = (\alpha+\alpha', exp(\alpha)b' + b)$.

The center of Aff_∞ is infinite cyclic, generated by $(2i\pi, 0)$.

The subgroup of Aff_k (respectively Aff_∞) which consists of elements of the form $(a,0)$ (respectively $(\alpha,0)$) will be called the *a-subgroup* (respectively *α-subgroup*). Similarly the *b-subgroup* of Aff_k (respectively Aff_∞) consists of elements of the form $(1,b)$ (respectively $(0,b)$). This is the first commutator subgroup of Aff_k (respectively Aff_∞). It is important to note that these groups Aff_k and Aff_∞ are solvable.

Recall that an action of a Lie group is called locally free if all stabilizers are discrete.

1.1. Homogeneous spaces of $\mathrm{SL}(2,\mathbb{C})$

First note that there is an embedding of Aff_2 in $\mathrm{SL}(2,\mathbb{C})$ given by:

$$(a,b) \in \mathrm{Aff}_2 \mapsto \begin{pmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

If Γ is a discrete co-compact subgroup of $\mathrm{SL}(2,\mathbb{C})$, we get a locally free action of Aff_2 on the quotient manifold $M = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ by considering right translations on left cosets. This action is obviously volume preserving since the Haar measure of $\mathrm{SL}(2,\mathbb{C})$ is bi-invariant and yields an invariant volume form on M.

Similarly, we have an embedding of Aff in $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$, given by:

$$(a,b) \in \mathrm{Aff} \mapsto \pm\begin{pmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}).$$

The same construction as before gives examples of actions of Aff on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{C})$.

Before we proceed describing the dynamics of these actions, let us show that there are no similar constructions for $k \neq 1, 2$. First notice that $\mathrm{SL}(2,\mathbb{C})$ is simply connected with center $\{\pm I\}$ so that $\mathrm{PSL}(2,\mathbb{C})$ is the only connected Lie group locally isomorphic but not isomorphic to $\mathrm{SL}(2,\mathbb{C})$. A nontrivial element in $\mathrm{PSL}(2,\mathbb{C})$ has an abelian centralizer and this remark enables us to analyze homomorphisms from Aff_k to $\mathrm{PSL}(2,\mathbb{C})$ by considering the center of Aff_k. The reader will easily prove the following statements:

(1) Any locally injective homomorphism from Aff to $\mathrm{PSL}(2,\mathbb{C})$ is conjugate to the one described above.
(2) Any locally injective homomorphism from Aff_k, $k \in \mathbb{N} \cup \{\infty\}$, to $\mathrm{PSL}(2,\mathbb{C})$ factors through its quotient Aff.
(3) There is a locally injective homomorphism from Aff_k to $\mathrm{SL}(2,\mathbb{C})$ if and only if k is even (or ∞). Any such homomorphism factors through Aff_2 and is conjugate to the one described above.

We now give a short *description of the dynamics* of the examples on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{C})$. Recall that $\mathrm{PSL}(2,\mathbb{C})$ is the group of orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$ and that it acts freely and transitively on the positive orthonormal frames in the tangent bundle of $\mathbb{H}^3$ (see, for instance [Th]). If Γ is a discrete co-compact subgroup of $\mathrm{PSL}(2,\mathbb{C})$, one can consider the quotient $V = \Gamma\backslash\mathbb{H}^3$. Due to the possible existence of elements of finite order in Γ, this V might not be a manifold but is instead a so-called "hyperbolic orbifold". In any case, Γ contains a torsion-free subgroup of finite index so that, up to finite covers, we can restrict ourselves to the case where V is a manifold. The quotient $M = \Gamma\backslash\mathrm{PSL}(2,\mathbb{C})$ is therefore the orthonormal frame bundle of the Riemannian manifold V. The structure of the $SO(3)$-principal bundle of the projection $M \to V$

is given by right translations by $PU(2,\mathbb{C}) \simeq SO(3)$. Also note that the unit tangent bundle, $T_1(V)$, of V fits into a sequence of bundles $M \to T_1(V) \to V$.

Orbits of the a-subgroup in M project to geodesics in V. The geodesic flow of V, acting on $T_1(V)$ is a quotient of the dynamics of the a-subgroup acting on M. This action is ergodic, mixing, and with positive entropy. Recall that there is exactly one closed geodesic in V for each conjugacy class of elements of Γ of infinite order. Any element γ of Γ, considered as an element of $\mathrm{PSL}(2,\mathbb{C})$, has two eigenvalues ω_1 and ω_2 (defined up to sign) whose product is 1. The modulus of both ω_1 and ω_2 is different from 1 if γ is of infinite order; it is related to the length of the corresponding closed geodesic. Correspondingly, we obtain a compact orbit for the a-subgroup in $M = \Gamma\backslash \mathrm{PSL}(2,\mathbb{C})$; it is an elliptic curve, quotient of $\mathbb{C}^*$ by the multiplicative group generated by ω_1^2. The union of these compact orbits is dense in M. We were informed by Karl Oeljenklaus that it is unknown whether the complex manifold M could contain compact holomorphic curves besides these elliptic curves.

As for the orbits of the b-subgroup they project in V into horospherical orbits. It is well known that they are dense in V. All orbits of the b-subgroup are dense in M, the action being uniquely ergodic with zero entropy. One can consult [Gh2] for a survey, more general statements, and a bibliography on this kind of dynamical systems.

Observe that the case of $\Gamma\backslash \mathrm{SL}(2,\mathbb{C})$ is very similar to that of $\Gamma\backslash \mathrm{PSL}(2,\mathbb{C})$: one just has to replace the orthonormal frame bundle by the spin bundle. The corresponding dynamical systems have the same properties as stated before

1.2. Holomorphic suspensions

Let Λ be a lattice in $\mathbb{C}^2$, i.e. a discrete co-compact subgroup (hence isomorphic to $\mathbb{Z}^4$). Let Φ be a complex linear automorphism of $\mathbb{C}^2$ that preserves Λ and induces an automorphism of Λ. Assume that Φ has an eigenvalue ω of modulus different from 1. Of course for a generic Λ there is no such linear map but we give in the appendix a list of all possibilities, some being remarkably interesting.

Denote by $T_\Lambda = \mathbb{C}^2/\Lambda$ the corresponding complex torus and by $\bar{\Phi} : T_\Lambda \to T_\Lambda$ the holomorphic diffeomorphism induced by Φ. Let $Y \in \mathbb{C}^2$ be an eigenvector such that $\Phi(Y) = \omega Y$. Choose a positive integer k. On $\mathbb{C}^2 \times \mathbb{C}^*$ one has the following right action of Aff_k:

$$((x,z),(a,b)) \in \mathbb{C}^2 \times \mathbb{C}^* \times \mathrm{Aff}_k \mapsto (x + bz^kY, za) \in \mathbb{C}^2 \times \mathbb{C}^*.$$

This action commutes with the action of Λ on $\mathbb{C}^2 \times \mathbb{C}^*$ by translations on the first factor and therefore yields an action of Aff_k on $T_\Lambda \times \mathbb{C}^*$, which obviously commutes with the following diffeomorphism F of $T_\Lambda \times \mathbb{C}^*$:

$$F(\bar{x}, z) = (\bar{\Phi}(\bar{x}), \omega^{1/k}z).$$

where $\omega^{1/k}$ denotes some fixed k-th root of ω. Since $|\omega| \neq 1$, the group generated by F acts freely and properly on $T_\Lambda \times \mathbb{C}^*$ and the quotient M is a compact complex manifold equipped with a holomorphic locally free action of Aff_k. The manifold M fibres over an elliptic curve with fibres isomorphic to the torus T_Λ. We call this kind of example a *holomorphic suspension*.

Note that since $\bar{\Phi}$ is a diffeomorphism of T_Λ the determinant of Φ has to be of modulus 1. Hence the volume form $\sqrt{-1}|z|^{-2}dz \wedge d\bar{z} \wedge dx_1 \wedge d\overline{x_1} \wedge dx_2 \wedge d\overline{x_2}$ of $\mathbb{C}^2 \times \mathbb{C}^*$ descends to M and is invariant under the action of Aff_k.

As is well known, $\bar{\Phi}$ is an example of Anosov diffeomorphism of a torus. It is ergodic, mixing, with positive entropy. Periodic points of $\bar{\Phi}$ are rational points (i.e. torsion elements) in T_Λ. For the dynamics of the a-subgroup of Aff_k on M one has analogous properties: ergodicity and density of compact orbits. However, the action is not mixing (i.e. the real flow corresponding to real values of a is not mixing) since we have a holomorphic projection of M onto the elliptic curve $E = \mathbb{C}^*/\{\omega^{n/k}\}$ which exhibits the nonmixing transitive action of $\mathbb{C}^*$ on E as a quotient of the action of the a-subgroup on M.

It is very easy to describe the dynamics of the b-subgroup. Any orbit is dense in one of the tori which are the fibres of the projection of M onto E.

The manifold M does not always admit a nonzero holomorphic 3-form. Indeed, such a form lifts to $\mathbb{C}^2 \times \mathbb{C}^*$ as $u(x_1, x_2, z)dx_1 \wedge dx_2 \wedge \frac{dz}{z}$ and the invariance under translations by Λ shows that u has to be a function $u(z)$ of z alone. The invariance condition becomes: $u(\omega^{1/k}z) = \rho u(z)$ where ρ is the determinant of Φ. Writing the Laurent expansion of u, one sees that this is only possible if u is a constant multiple of z^n where $n \in \mathbb{Z}$ is such that $\rho = \omega^{n/k}$. We have seen that $|\rho| = 1$ and, by assumption, $|\omega| \neq 1$ so that the only possibility is $n = 0$ and $\rho = 1$. In the appendix we study the cases where $\rho = 1$ and we show that this is not always the case. Hence, there are 3-manifolds with no nontrivial holomorphic 3-forms but which admit a locally free action of Aff. In particular, these manifolds M are not homogeneous spaces of a complex 3-dimensional Lie group so that the corresponding actions are quite different from the actions we described on $\Gamma\backslash \mathrm{SL}(2, \mathbb{C})$.

As a matter of fact, it is not difficult to check that if $\rho \neq 1$ the full group of biholomorphisms of M coincides with Aff_k up to finite index.

If on the contrary, $\rho = 1$, then M is a homogeneous space. Indeed in this case one has a nonzero vector Z in $\mathbb{C}^2$ such that $\Phi(Z) = \omega^{-1}Z$ and one can consider the 3-dimensional Lie group G_k which is a semi-direct product $\mathbb{C}^2 \ltimes \mathbb{C}^*$ given by the multiplication: $(b_1, b_2; a)(b_1', b_2'; a') = (b_1 + a^k b_1', b_2 + a^{-k} b_2'; aa')$. Note that G_k contains Aff_k and that G_k acts on the right on $\mathbb{C}^2 \times \mathbb{C}^*$ by right translations on itself. This action descends to a transitive action of G_k on M, making M a homogeneous space and extending the action of Aff_k.

If ρ is not a root of unity, M is not even a finite quotient of a homogeneous space, the so-called "infra-homogeneous" spaces.

Finally, we indicate shortly how to modify these suspension examples by taking finite quotients. Consider a character $\chi : \Lambda \to \mathbb{C}^*$ which is invariant by Φ. We can consider the quotient of $\mathbb{C}^2 \times \mathbb{C}^*$ by the following action of Λ:

$$(\lambda, (x, z)) \in \Lambda \times \mathbb{C}^2 \times \mathbb{C}^* \mapsto (x + \lambda, \chi(\lambda)z) \in \mathbb{C}^2 \times \mathbb{C}^*.$$

The quotient $T_\Lambda \rtimes_\chi \mathbb{C}^*$ is a $\mathbb{C}^*$-bundle over T_Λ which is nontrivial if χ is nontrivial. The diffeomorphism $(x, z) \in \mathbb{C}^2 \times \mathbb{C}^* \mapsto (\Phi(x), \omega^{1/kz})$ still descends to a diffeomorphism of $T_\Lambda \rtimes_\chi \mathbb{C}^*$ which yields a compact 3-manifold M admitting an action of Aff_k provided $\chi(\lambda)^k = 1$ for any $\lambda \in \Lambda$. We shall call this kind of example a *twisted suspension*.

Observe that χ is trivial on a finite-index sublattice of Λ so that these twisted suspensions are in fact finite quotients of the previous "untwisted" examples.

2. The classification

In this section we prove the following theorem:

Theorem. *Any holomorphic action of* Aff_∞*, which is locally free and volume-preserving on a compact complex 3-manifold factors through* Aff_k *for some* (finite) integer k and is holomorphically conjugate to one of the examples described in section 1.

Let us begin by some general comments and notations. Any locally free action of a (connected) Lie group lifts to an action of its universal cover (the action might not be faithful). This is why we restrict ourselves, with no loss of generality, to actions of Aff_∞. Suppose we have such a holomorphic right action of Aff_∞ on a compact complex 3-manifold M preserving some volume form v (which is *a priori* a 6-form of class C^0). Since we have parametrized Aff_∞ by pairs (α, b) with $\alpha \in \mathbb{C}$ and $b \in \mathbb{C}$, the α-subgroup and b-subgroup generate two complex flows that we denote by g^α and h^b (for"geodesic" and "horocyclic"). The induced holomorphic vector fields on M are denoted by X and Y respectively. They satisfy the relation $[X, Y] = Y$, defining the Lie algebra of Aff_∞. The assumption that the action is locally free means that X and Y are linearly independent at each point of M. Since X and Y correspond to left invariant vector fields on Aff_∞, they transform under right-translations by the adjoint representation. One verifies easily that:

$$(g^\alpha)_* Y = \exp(\alpha) Y \qquad (g^\alpha)_* X = X.$$

We shall equip M with a Hermitian metric, denoted by $\| \ \|$ such that X and Y are orthogonal, of norm 1, and whose volume form is precisely v.

Lemma. *Let $\mathcal{B}$ be the Banach space of continuous vector fields on M equipped with the sup-norm. The spectrum of the linear operator induced in $\mathcal{B}$*

by the differential of the flow g^α $(\alpha \in \mathbb{C})$ is contained in the union of $\{1, \exp(\alpha)\}$ and the circle of radius $|\exp(-\alpha)|$.

Proof. Let $\mathcal{B}_t \subset \mathcal{B}$ be the closed subspace of vector fields which are tangent to $\mathbb{C}X \oplus \mathbb{C}Y$, *i.e.*, to the orbits of Aff_∞. According to the previous formulae, the spectrum of $(g_1^\alpha)_*$ acting on $\mathcal{B}_t$ is precisely $\{1, \exp(\alpha)\}$. Let $\mathcal{B}_n = \mathcal{B}/\mathcal{B}_t$ be the quotient Banach space (of "normal" vector fields), with the quotient norm. Since the Hermitian metric $\|\ \|$ defines the invariant volume form v and that the volume along the orbits of Aff_∞ is multiplied by $|\exp(\alpha)|^2$ by g^α, we deduce that g^α contracts by $|\exp(\alpha)|^2$ the transversal volume. It follows that the action of $(g^\alpha)_*$ on $\mathcal{B}_n$ contracts the norm exactly by $|\exp(\alpha)|$ and this proves the lemma. □

Corollary. *There is a unique continuous field of complex 1-dimensional tangent lines* E^s *in* M *which is transversal to* $\mathbb{C}X \oplus \mathbb{C}Y$ *and invariant under* $(g^\alpha)_*$ $(\alpha \in \mathbb{C})$. This E^s is uniquely integrable and defines a continuous foliation $\mathcal{F}^s$ of M by holomorphic curves.

Proof. This is a simple application of the theory of stable manifolds as explained, for instance, in [HPS]. We recall the main results of the theory. Let g be a C^∞-diffeomorphism of a compact manifold M such that the spectrum of the linear operator g_* acting on $\mathcal{B}$ does not intersect some circle $\{|z| = r\}$. Then there is a unique splitting of the tangent bundle of M as a sum of two continuous sub-bundles E_1 and E_2 which are invariant by g_* and have the following property. The spectrum of g_* acting on continuous sections of E_1 (resp. E_2) lies inside (resp. outside) the disc $\{|z| < r\}$. Moreover if $r > 1$ the sub-bundle E_2 is integrable and defines a continuous foliation with C^∞ leaves.

We can apply these results to g^α where α has a negative real part and $1 < r < |\exp(-\alpha)|$. Of course the bundle E_1 coincides with $\mathbb{C}X \oplus \mathbb{C}Y$ and E_2 is a 2-dimensional real sub-bundle. By uniqueness and the fact that all g^α commute, E_2 is invariant under all $(g^\alpha)_*$ (even if $\Re(\alpha) = 0$). Again by uniqueness and since g^α is holomorphic, E_2 has to be invariant by multiplication by $\sqrt{-1}$ so that it is actually a complex line sub-bundle of the tangent bundle. In particular the leaves of the induced foliation are holomorphic curves. We denote E_2 by E^s since it contains vectors which are stable under $(g^\alpha)_*$ when the real part of α goes to $+\infty$. □

Note that we did not (yet) claim that E^s is a holomorphic sub-bundle.

Remark. We used the existence of an invariant volume in the previous lemma to get some contraction in the direction transversal to the orbits of Aff_∞. This is the only place where we need this invariant volume so that the main theorem would hold true under the weaker assumption that the spectrum of $(g^\alpha)_*$ acting on $\mathcal{B}_n = \mathcal{B}/\mathcal{B}_t$ lies inside the unit disc for $\Re(\alpha) > 0$. Note also that it would suffice to ask for a volume form invariant under the flow g^α (but it is easy to verify that such a form is also necessarily invariant under the horocyclic flow h^b).

Corollary. *Any volume-preserving locally free holomorphic action of* Aff_∞ *factors through one of its quotients* Aff_k.

Proof. We shall establish that the real flow g^{it} ($t \in \mathbb{R}$) must be a periodic flow. This will imply that the action of Aff_∞ under consideration is not faithful. Its kernel must be a discrete nontrivial normal subgroup of Aff_∞ necessarily contained in the infinite center of Aff_∞ and that implies the corollary.

Choose the hermitian norm $\|\ \|$ such that the line field E^s is orthogonal to $\mathbb{C}X \oplus \mathbb{C}Y$. It follows from the invariance of E^s under $(g^\alpha)_*$ and our previous calculations that g^{it} is an isometry of $\|\ \|$. The closure C of this 1-parameter subgroup in the compact open topology is therefore compact by Ascoli's theorem. Of course, C is connected and Abelian. Moreover, it is a closed subgroup of the full biholomorphism group of M which is a complex Lie group. It follows that C is a torus $T^l = \mathbb{R}^l/\mathbb{Z}^l$ for some $l \geq 1$ and we want to show that $l = 1$. The Lie algebra of T^l gives rise to vector fields V on M which are obviously invariant by the elements of the flow g^α. By our previous description of the action of $(g^\alpha)_*$ on vector fields on M it follows that V must be tangent to $\mathbb{C}X$. Being invariant by $(g^\alpha)_*$, V actually is a (complex) multiple of X. In other words, the compact group C is contained in the complex 1-parameter group g^α ($\alpha \in \mathbb{C}$). We know that g^α is an isometry only if $\Re(\alpha) = 0$ so that we have proven that C coincides with the real 1-parameter flow g^{it} which must therefore be periodic. This proves the corollary □

In the terminology of [Gh3] we have established that g^α is a holomorphic Anosov flow. Of course, we could use the results of [Gh3] but we won't do it because that would not simplify much of the following discussion. However we shall borrow the following lemma from [Gh3] but we present here a more general proof.

Lemma. *The plane field* $E^s_0 = E^s \oplus \mathbb{C}X$ (of complex dimension 2) is a holomorphic plane field.

Proof. By pushing the leaves of E^s along the flow g^α one sees that E^s_0 is integrable and generates a foliation $\mathcal{F}^s_0$, of class C^0 with leaves of complex codimension 1. We want to show that this foliation is actually holomorphic.

Let us begin by a general remark concerning this kind of foliation. In suitable local holomorphic coordinates (x_1, x_2, x_3) in $\mathbb{C}^3$ (where $|x_1| < 1$, $|x_2| < 1$, $|x_3| < 1$) the leaves have equations of the form: $x_3 = F(x_1, x_2, c)$, $F(0, 0, c) = c$ where c is a constant depending on the leaf. Since the leaves are holomorphic and don't intersect two by two, for fixed x_1, x_2 we know that $F(x_1, x_2, c)$ is an injective function of c and, fixing c, the function $F(x_1, x_2, c)$ is holomorphic in x_1, x_2. This is the definition of a so-called holomorphic motion (see for instance [Do] for an interesting discussion). In particular, it follows that $F(x_1, x_2, c)$ is a quasi-conformal homeomorphism in the variable c and that the quasi-conformal distortion on a small neighborhood of 0 goes to 1 as (x_1, x_2) goes to $(0, 0)$.

Hence we have shown that a *codimension one foliation on a compact complex manifold, with holomorphic leaves, is necessarily transversely quasi-conformal.*

Let us apply this to our situation. Let x and y be two nearby points in the same leaf of $\mathcal{F}_0^s$. Through x and y we can consider small pieces of orbits of the flow h^b, which can be considered as small holomorphic curves U_x and U_y transversal to $\mathcal{F}_0^s$ at x and y. Consider the holonomy of a small path connecting x and y in their leaf of $\mathcal{F}_0^s$. As we have seen, this is a quasi-conformal homeomorphism $\gamma_{x,y}$ from a neighborhood of x in U_x to a neighborhood of y in U_y sending x to y. We have to show that these homeomorphisms are holomorphic. If x, y, z are three nearby points in the same leaf, we obviously have $\gamma_{x,z} = \gamma_{y,z} \circ \gamma_{x,y}$. Note that if $y = g^\alpha(x)$ for some small α, the holonomy $\gamma_{x,y}$ is the restriction of g^α to U_x and is therefore holomorphic. We can therefore assume that x and y are two nearby points in the same leaf of $\mathcal{F}^s$. By the compactness of M and the property of holomorphic motions mentioned above, for every $\epsilon > 0$ there is $r > 0$ such that if the distance between x and y is less than r then $\gamma_{x,y}$ is $(1+\epsilon)$-quasi-conformal in a neighborhood of x. Of course $g^\alpha \gamma_{x,y} g^{-\alpha} = \gamma_{g^\alpha(x), g^\alpha(y)}$ so that taking α with a sufficiently big real part, $g^\alpha(x)$ and $g^\alpha(y)$ are at a distance less than r in the same leaf of $\mathcal{F}^s$ and $\gamma_{g^\alpha(x), g^\alpha(y)}$ is $(1+\epsilon)$-quasi-conformal. Since the quasi-conformal distortion is invariant under holomorphic conjugation, we get that $\gamma_{x,y}$ is $(1+\epsilon)$-quasi-conformal in a neighborhood of x. Since this is true for every $\epsilon > 0$, we get that $\gamma_{x,y}$ is 1-quasi-conformal, i.e. holomorphic. The lemma is established. □

Lemma. *The 1-dimensional foliation $\mathcal{F}^s$ is holomorphic.*

Proof. Let η be the holomorphic 1-form which vanishes on $E_0^s = E^s \oplus \mathbb{C}X$ and is equal to 1 on the vector field Y. Since η is integrable, there is a unique holomorphic 1-form ξ such that:

$$d\eta = \eta \wedge \xi, \qquad \xi(Y) = 0.$$

Evaluating on (X, Y), we find:

$$\begin{aligned} d\eta(X,Y) &= X\eta(Y) - Y\eta(X) - \eta([X,Y]) \\ &= \eta(X)\xi(Y) - \eta(Y)\xi(X) \\ &= -\xi(X) = -1, \end{aligned}$$

so that $\xi(X) = 1$.

Of course η satisfies $(g^\alpha)^*\eta = \exp(\alpha)\eta$ so that ξ is invariant under $(g^\alpha)^*$. Therefore the kernel of ξ is a holomorphic plane field transverse to $\mathbb{C}X$ which is invariant under $(g^\alpha)_*$; it must coincide with $E^s \oplus \mathbb{C}Y$. It follows that E^s is the intersection of $E^s \oplus \mathbb{C}X = \ker\eta$ and $E^s \oplus \mathbb{C}Y = \ker\xi$ which are both holomorphic plane fields. Hence E^s is a holomorphic line field. □

Corollary. *There are only two possibilities:*

I- *$E^s \oplus \mathbb{C}Y$ is integrable.*

II- *there is a holomorphic vector field Z, everywhere independent of X and Y, such that* $[X,Z]=-Z$; $[Y,Z]=X$ (and $[X,Y]=Y$).

Proof. Consider the holomorphic 1-form ξ vanishing on $E^s \oplus \mathbb{C}Y$ and such that $\xi(X)=1$. We have seen that ξ is invariant under $(g^\alpha)^*$ so that the holomorphic 3-form $\xi \wedge d\xi$ is also invariant under $(g^\alpha)^*$. We distinguish several cases:

I- $\xi \wedge d\xi$ vanishes everywhere, so that $E^s \oplus \mathbb{C}Y$ is integrable.

II- The form $\xi \wedge\ d\xi$ is not identically zero.

In case **II**, we consider the zero set Σ of $\xi \wedge\ d\xi$ and we shall show that Σ must be empty. The space of holomorphic 3-forms on M is a finite dimensional space equipped with a natural positive definite Hermitian form given by integration. The group $\mathrm{Aff_k}$ acts on this space by isometries and by complex automorphisms. Since a complex Lie group of $\mathrm{GL}(n,\mathbb{C})$ which is relatively compact is necessarily trivial, we see that $\mathrm{Aff_k}$ must act trivially on holomorphic 3-forms. In particular $\xi \wedge d\xi$ is invariant under $\mathrm{Aff_k}$ and Σ is a 2-dimensional analytic set invariant under the action. Since all the orbits of $\mathrm{Aff_k}$ are 2-dimensional, the singular set of Σ, being of dimension 0 or 1, must be empty. Hence Σ is a finite union of orbits of $\mathrm{Aff_k}$. This shows that Σ is empty since the action of $\mathrm{Aff_k}$ has no compact orbits (discrete subgroups of $\mathrm{Aff_k}$ are Abelian and not co-compact).

Let Z be the holomorphic vector field tangent to E^s such that $\xi \wedge d\xi(X,Y,Z)=1$. Then $(g^\alpha)_*Z = \exp(-\alpha)Z$ so that $[X,Z]=-Z$. The Jacobi identity then yields $[[Y,Z],X]=0$. We have already noticed that a vector field commuting with X must be a constant multiple of X so that multiplying Z by a suitable constant we have, as required:

$$[X,Z]=-Z, \quad [Y,Z]=X, \quad [X,Y]=Y.$$

This proves the Corollary. □

Of course, in case II, the vector fields X, Y, Z generate a Lie algebra which is isomorphic to the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ so that we get a locally free action of $\mathrm{SL}(2,\mathbb{C})$ on M. This case corresponds to the examples described in section 1.1. Therefore, in order to prove the theorem we are left with case I.

Lemma. *In case I, the line field E^s is invariant under the action of* $\mathrm{Aff_k}$.

Proof. Recall that we have holomorphic forms η and ξ such that:

$$\begin{aligned} \mathbb{C}X \oplus E^s &= \ker \eta, \qquad \eta(Y)=1 \\ \mathbb{C}Y \oplus E^s &= \ker \xi, \qquad \xi(X)=1 \\ d\eta &= \eta \wedge \xi. \end{aligned}$$

Choose a locally defined holomorphic vector field Z tangent to E^s. We have:

$$\begin{aligned} d\eta(Y,Z) &= Y\eta(Z) - Z\eta(Y) - \eta([Y,Z]) = -\eta([Y,Z]) \\ &= \eta(Y)\xi(Z) - \eta(Z)\xi(Y) = 0. \end{aligned}$$

Hence $[Y, Z]$ lies in the plane $\mathbb{C}X \oplus E^s$. If we assume, as in case I, that $\mathbb{C}Y \oplus E^s$ integrable, $[Y, Z]$ lies in $\mathbb{C}Y \oplus E^s$ so that both conditions imply that $[Y, Z]$ is parallel to Z, *i.e.*, the flow $(h^b)_*$ preserves E^s. Since we already know that $(g^\alpha)_*$ preserves E^s we deduce that E^s is indeed invariant under the action of the full group Aff_k. □

We are ready to reconstruct M.

Lemma. *In case I, there is a regular covering $\widehat{M}$ of M which is biholomorphic to $\mathbb{C} \times \mathrm{Aff}_k$ in such a way that:*

(1) *the action of Aff_k lifts to $\widehat{M}$ to the action by right translations on the second factor.*
(2) *let Γ be the Galois group of this cover. There are two homomorphisms $\sigma : \Gamma \to \mathrm{Aff}$ and $\tau : \Gamma \to \mathrm{Aff}_k$ such that the action of $\gamma \in \Gamma$ on $\widehat{M}$ is given by:*
$$\gamma(x, g) = (\sigma(\gamma)(x), \tau(\gamma)(g)).$$

Proof. Let L be a leaf of $\mathcal{F}^s$ and recall the classical argument from Anosov theory that shows that L is homeomorphic to a plane. Fix a point x in L and $\epsilon > 0$ sufficiently small such that all balls of radii ϵ in leaves of $\mathcal{F}^s$ are homeomorphic to discs in the plane. Then L can be considered as the increasing union of the images by g^{-n} ($n \in \mathbb{N}$) of the balls of radii ϵ in the leaf of $\mathcal{F}^s$ passing through $g^n(x)$. This shows that L is homeomorphic to a plane and we shall see in a moment that L is actually isomorphic to $\mathbb{C}$, as a Riemann surface.

Using the action of Aff_∞ (lifted from Aff_k and non-faithful), we get a holomorphic map:
$$\pi : L \times \mathrm{Aff}_\infty \to M.$$
We claim that this is the universal cover of M. Of course, π is a local diffeomorphism and it is enough to show that paths in M can be uniquely lifted to $L \times \mathrm{Aff}_\infty$. If x is a point in L, the restriction of π to $\{x\} \times \mathrm{Aff}_\infty$ is a covering onto the orbit of x in M. Hence any path contained in an orbit can be lifted. By the previous lemma, if g is in Aff_∞, the image of L by g is a leaf of $\mathcal{F}^s$. Therefore any path contained in a leaf of $\mathcal{F}^s$ can be lifted. Using the connectivity of M (that we tacitly assumed), any path in M can be arbitrarily approximated by a path made of a succession of paths alternatively tangent to $\mathcal{F}^s$ and to the orbits of Aff_∞. This shows that π is indeed the universal cover of M.

Of course, the lifted action of Aff_∞ is by right translations on the second factor. As for the action of the fundamental group, it must respect the splitting and commutes with right Aff_∞ translations. Hence we have two homomorphisms $\bar{\sigma}$ and $\bar{\tau}$ from the fundamental group of M to $\mathrm{Aut}(L)$ and Aff_∞ such that the action has the form:
$$\gamma \cdot (x, g) = (\bar{\sigma}(\gamma)(x), \bar{\tau}(\gamma)(g)).$$
We know that L is isomorphic to $\mathbb{C}$ or to the Poincaré disc. The latter case is impossible since $\mathrm{Aut}(L)$ would be a group of isometries (of the Poincaré metric) so

that the action of Aff_k that we study on M would respect a Riemannian metric, transversely to its orbits which is a contradiction with the normal contraction of g^α. Therefore, we can identify L with $\mathbb{C}$ and its automorphism group $\mathrm{Aut}(\mathbb{C})$ with Aff.

The action of Aff_∞ on M is not faithful since $(2i\pi k, 0)$ acts trivially. In other words, there is an element γ_0 in the fundamental group of M such that $\bar{\sigma}(\gamma_0) = (1,0)$ and $\bar{\tau}(\gamma_0) = (2i\pi k, 0)$. Such an element is central in the fundamental group and the quotient of $\mathbb{C} \times \mathrm{Aff}_\infty$ by this central subgroup $\{\gamma_0^n\}$ is $\mathbb{C} \times \mathrm{Aff}_k$. We get the regular covering space $\widehat{M}$ whose existence is claimed by the lemma. The Galois group Γ is the quotient of the fundamental group by $\{\gamma_0^n\}$ and σ and τ are induced by $\bar{\sigma}$ and $\bar{\tau}$. The lemma is proved. □

We now analyze the two homomorphisms:

$$\sigma : \gamma \in \Gamma \mapsto (a_1(\gamma), b_1(\gamma)) \in \mathrm{Aff}$$
$$\tau : \gamma \in \Gamma \mapsto (a_2(\gamma), b_2(\gamma)) \in \mathrm{Aff}_k .$$

Let Γ_1 and Δ be respectively the kernel and image of the homomorphism:

$$\gamma \in \Gamma \mapsto (a_1(\gamma), a_2(\gamma)^k) \in \mathbb{C}^* \times \mathbb{C}^*.$$

Finally, let Λ be the image of the homomorphism:

$$\gamma \in \Gamma_1 \mapsto (b_1(\gamma), b_2(\gamma)) \in \mathbb{C}^2.$$

Lemma. *Λ is a lattice in $\mathbb{C}^2$ and Δ contains an infinite cyclic subgroup of finite index.*

Proof. If $\gamma \in \Gamma_1$, its action on $\mathbb{C} \times \mathrm{Aff}_k$ has the form

$$(z; x, y) \mapsto (z + b_1(\gamma); a_2(\gamma)x, y + b_2(\gamma)).$$

Of course the action of Γ_1 on $\mathbb{C} \times \mathrm{Aff}_k$ is discrete so that Λ is a discrete subgroup of $\mathbb{C}^2$. Note that Γ_1 contains the first commutator group of Γ so that if Λ were trivial, Γ would be Abelian and would be isomorphic to a discrete subgroup of $\mathbb{C}^* \times \mathbb{C}^*$. This is impossible since the quotient of $\mathbb{C} \times \mathrm{Aff}_k$ by Γ is compact. Hence Λ is a non trivial discrete subgroup of $\mathbb{C}^2$.

The group Δ acts by diagonal matrices in $\mathbb{C}^2$ preserving Λ. Moreover, Λ is not contained in $\mathbb{C} \times \{0\}$ or $\{0\} \times \mathbb{C}$ since, otherwise, there would be a codimension one subspace of $\mathbb{C} \times \mathrm{Aff}_k$ which is invariant by Γ, contradicting the compactness of the quotient. Therefore, it is a discrete subgroup of $\mathbb{C}^* \times \mathbb{C}^*$ which cannot contain an element of the form (ω_1, ω_2) with $|\omega_1| < 1$ and $|\omega_2| < 1$. It follows that Δ is a discrete subgroup of a group isomorphic to $\mathbb{R} \times S^1 \times S^1$. Hence Δ either contains an infinite cyclic subgroup of finite index or is finite. The latter case is not possible since otherwise the quotient of $\mathbb{C} \times \mathrm{Aff}_k$ by Γ would not be compact.

It remains to show that Λ is a lattice, *i.e.*, that $\mathbb{C}^2/\Lambda$ is compact. We know that M is the quotient of $\mathbb{C}^2/\Lambda \times \mathbb{C}^*$ by the action of $\Lambda/\Lambda_1 = \Delta$. Since M is compact and Δ contains an infinite cyclic subgroup of finite index, it follows that $\mathbb{C}^2/\Lambda$ is compact, as claimed by the lemma. □

The end of the proof of the theorem is now easy. Choose a generator Φ of a direct summand of the torsion part of Δ. The action of Φ on $\mathbb{C}^2/\Lambda \times \mathbb{C}^*$ is exactly as in the suspension case, described in section 1.2. The quotient of $C^2/\Lambda \times \mathbb{C}^*$ by Φ is a finite cover of M. To get the exact structure of M, one has to take into account the torsion subgroup of Δ which gives rise to the twisted suspension. □

Appendix
Automorphisms of 2-dimensional complex tori

Let us introduce some notations. If Λ is a lattice in $\mathbb{C}^n$ $(n \geq 1)$, we define:

$$\begin{aligned}
\mathrm{End}(\Lambda) &= \{f \in M(n,\mathbb{C}) \mid f(\Lambda) \subset \Lambda\} \\
\mathrm{Aut}(\Lambda) &= \{f \in M(n,\mathbb{C}) \mid f(\Lambda) = \Lambda\} \\
\mathrm{End}(\Lambda \otimes \mathbb{Q}) &= \{f \in M(n,\mathbb{C}) \mid f(\Lambda \otimes \mathbb{Q}) \subset \Lambda \otimes \mathbb{Q}\}.
\end{aligned}$$

If Λ_1 and Λ_2 are two lattices, they are *isomorphic* if there is $f \in \mathrm{GL}(n,\mathbb{C})$ such that $f(\Lambda_1) = \Lambda_2$ and *isogenous* if there is $f \in \mathrm{GL}(n,\mathbb{C})$ such that $f(\Lambda_1)$ is a sublattice (hence of finite index) in Λ_2. If Λ_1 and Λ_2 are isogenous, $\mathrm{Aut}(\Lambda_1)$ and $\mathrm{Aut}(\Lambda_2)$ are commensurable, *i.e.*, contain subgroups of finite index which are conjugate.

The purpose of this appendix is to list explicitly all lattices Λ in $\mathbb{C}^2$ for which $\mathrm{Aut}(\Lambda)$ is infinite and to give a description of the possible groups $\mathrm{Aut}(\Lambda)$. All this discussion will be made up to isogeny and commensurability, in order to avoid lengthy lists.

Most of this appendix can be extracted from classical books, like for instance [Mu], but we believe that this explicit description might be useful.

First recall the elementary facts concerning the case $n = 1$.

For a generic $\Lambda \subset \mathbb{C}$, one has $\mathrm{Aut}(\Lambda) = \{\pm id\}$ and $\mathrm{End}(\Lambda) \simeq \mathbb{Z}$

$\mathrm{Aut}(\Lambda)$ is always finite; it is nontrivial only if Λ is isomorphic to the lattice $\mathbb{Z}[\sqrt{-1}]$ of Gaussian integers or the lattice $\mathbb{Z}[\sqrt[3]{-1}]$ of Eisenstein integers.

$\mathrm{End}(\Lambda)$ is not isomorphic to $\mathbb{Z}$ if and only if Λ is isogenous with the lattice of integers in some imaginary quadratic field.

We shall now describe five families of lattices in $\mathbb{C}^2$ whose automorphism group is infinite and then sketch the proof of the fact that this is the complete list up to isogeny. We order the examples by decreasing "size" of $\mathrm{Aut}(\Lambda)$.

Example 1. Let p and q be two positive integers. Consider the quaternion algebra $H_{p,q}$ over $\mathbb{Q}$ generated by i, j, k with:

$$\begin{array}{lll} i^2 = p & j^2 = -q & k^2 = pq \\ ij = k = -ji & jk = qi = -kj & ki = -pj = -ik. \end{array}$$

As is well known, $H_{p,q} \otimes \mathbb{R}$ is isomorphic to the algebra $M(2,\mathbb{R})$ via the embedding:

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i \to \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix},$$
$$j \to \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \qquad k \to \begin{pmatrix} 0 & \sqrt{p} \\ q\sqrt{p} & 0 \end{pmatrix}.$$

The norm of an element $x_0 + x_1 i + x_2 j + x_3 k$ of $H_{p,q} \otimes \mathbb{R}$ is $x_0^2 - px_1^2 + qx_2^2 - pqx_3^2$ and the group of unitary quaternions in $H_{p,q} \otimes \mathbb{R}$ is isomorphic with $\mathrm{SL}(2,\mathbb{R})$.

Right multiplication by $j/\sqrt{q}$ gives a complex structure on $M(2,\mathbb{R})$ which can therefore be considered as $\mathbb{C}^2$. Left multiplication by elements of $H_{p,q} \otimes \mathbb{R}$ act complex linearly on $\mathbb{C}^2$. Let Λ be the lattice of quaternions of $H_{p,q}$ with integral coordinates, considered as a lattice in $\mathbb{C}^2$. Let $G_{p,q}$ be the group of integral quaternions of norm 1; it acts on $\mathbb{C}^2$ by complex linear mappings, preserving the lattice Λ.

The groups $G_{p,q}$ are examples of arithmetical Fuchsian groups (see for instance [Ka]). They are always discrete subgroups of finite co-volume in $\mathrm{SL}(2,\mathbb{R})$ and are co-compact if the equation $-px_1^2 + qx_2^2 - pqx_3^2 = 0$ has no nontrivial integral solution (x_1, x_2, x_3). In any case, these groups $G_{p,q}$ are quite big in the sense that, for instance, they contain nonabelian free subgroups.

Example 2. Let Λ_0 be a lattice in $\mathbb{C}$ and consider the lattice $\Lambda = \Lambda_0 \times \Lambda_0$ in $\mathbb{C}^2$. The automorphism group of Λ is $\mathrm{GL}(2, \mathrm{End}(\Lambda_0))$. It contains at least $\mathrm{GL}(2,\mathbb{Z})$ and, therefore, a free nonabelian group.

Example 3. Let K be a degree 4 number field which is totally imaginary, *i.e.*, admitting two embeddings i_2, i_2 in $\mathbb{C}$ which are distinct and nonconjugate. Let $i = (i_1, i_2) : K \to \mathbb{C} \times \mathbb{C}$. Let $\mathcal{O} \subset K$ be the ring of integers of K and $\Lambda = i(\mathcal{O})$; it is a lattice in $\mathbb{C}^2$ (see for instance [BS]). By Dirichlet's theorem, the group of units of $\mathcal{O}$ has the form $Z \times F$ where F is the finite group of roots of unity contained in K. Any unit u acts in $\mathbb{C}^2$ by the diagonal matrix $\begin{pmatrix} i_1(u) & 0 \\ 0 & i_2(u) \end{pmatrix}$ preserving the lattice Λ. Hence this construction produces examples of lattices whose automorphism group contains an infinite subgroup of finite index.

Note that the four conjugates of $i_1(u)$ are $i_1(u)$, $\overline{i_1(u)}$, $i_2(u)$, $\overline{i_2(u)}$. We have noticed that since the lattice Λ is invariant, one has $|i_1(u) i_2(u)| = 1$. If $|i_1(u)|$ and $|i_2(u)|$ were equal to 1, the four conjugates would be of modulus 1 and u would be of finite order according to Kronecker's theorem. Therefore, if u is a unit of infinite order, the corresponding automorphism of Λ has no eigenvalue of modulus 1 and can therefore be used in the construction of a holomorphic suspension.

We discuss now the question mentioned in section 1.2 concerning the product $\rho = i_1(u)i_2(u)$ which is the determinant of the linear operator of $\mathbb{C}^2$ preserving Λ. We want to describe in which cases it is equal to 1. Consider the characteristic polynomial of the unit u (considered as an integral 4×4 matrix acting on Λ); it has the form:

$$P(T) = T^4 + n_3T^3 + n_2T^2 + n_1T \pm 1 \quad \text{where } n_i \in \mathbb{Z}.$$

Conversely, any such polynomial, if irreducible over $\mathbb{Q}$, defines a quartic field $\mathbb{Q}[T]/P(T)$ and a unit $u = T$ in its ring of integers. The condition expressing that this field is purely imaginary is expressed by finitely many polynomial inequalities in the n_i's. Of course, the four roots of this polynomial P are the four conjugates of $i_1(u)$ that we denote, for better symmetry, by $\omega_1, \omega_2, \omega_3, \omega_4$. We have to decide if one of the products $\omega_i\omega_j$ $(i \neq j)$ is equal to 1. Apply the usual method for solving quartic equations, *i.e.*, let us define:

$$\Omega_1 = \omega_1\omega_2 + \omega_3\omega_4 \quad \Omega_2 = \omega_1\omega_3 + \omega_2\omega_4 \quad \Omega_3 = \omega_1\omega_4 + \omega_2\omega_3.$$

Symmetric polynomials of the Ω_i's can be explicitly expressed in the n_i's so that the Ω_i's satisfy a cubic equation (Galois resolvent). If one uses the fact that $\omega_1\omega_2\omega_3\omega_4 = \pm 1$, one finds:

$$\Omega^3 - n_2\Omega^2 + (n_1n_3 \mp 4)\Omega + (\pm n_3 \pm 2n_2 - n_1^2) = 0.$$

A product $\omega_i\omega_j$ $(i \neq j)$ is equal to 1 if and only if the previous cubic has 2 (or 0) as a solution. We therefore find the necessary and sufficient condition in the n_i's so that $\rho = 1$.

In particular, we find many examples where $\rho = 1$ and many examples where $\rho \neq 1$ (one has also to check that the condition is compatible with the polynomial inequalities mentioned above but this is an easy exercise). Note also that in any case ρ is of degree 1, 2, 3, 4 or 6 over $\mathbb{Q}$ so that, if ρ is a root of unity, its order is a priori bounded (by 18) and the sum $\rho + 1/\rho = \Omega$ can take only finitely many explicit values. In other words, we find analogous conditions in the n_i's for ρ to be a root of unity. Therefore most holomorphic suspensions do not lead to infra-homogeneous manifolds (see section 1.2).

Observe finally that if $\rho = 1$, the previous cubic equation has to split over $\mathbb{Q}$ so that Ω_1 and Ω_2 belong to a (real) quadratic field. Hence in this case, K is a purely imaginary quadratic extension of a real quadratic field. In this case the complex torus $\mathbb{C}^2/\Lambda$ is algebraic, but we shall not focus here on the algebraicity of our examples (see [Mu]).

Example 4. Let K be a real quadratic field, $\mathcal{O}$ its ring of integers, and u a unit in $\mathcal{O}$ of infinite order. The action of u on $\mathcal{O}$ gives rise to a matrix U in $\mathrm{GL}(2, \mathbb{Z})$ which is diagonalizable over the reals. The 4×4 matrix $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ preserves

$\mathbb{Z}^4$ and has two invariant subspaces in $\mathbb{R}^4$, both of dimension 2, on which it is a homothety. Choose any complex structure on each one of these subspaces so that $\mathbb{R}^4$ is now identified with $\mathbb{C}^2$ in such a way that the matrix acts complex linearly and preserves the lattice $\Lambda = \mathbb{Z}^4 \subset \mathbb{R}^4 \simeq \mathbb{C}^2$. Once again, we found lattices with infinite automorphisms groups. Note that, unlike examples 3, this construction produces uncountably many examples since we have a lot of freedom for the choices of complex structures on the 2-dimensional invariant subspaces.

Example 5. We now come to a very degenerate case. Let $\Lambda_0 \subset \mathbb{C}$ be a lattice and let $\Lambda \subset \mathbb{C}^2 \simeq \mathbb{R}^4$ be the image of $\Lambda_0 \times \Lambda_0$ by a linear map whose matrix has the form $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$ where A is a 2×2 matrix. For a generic choice of A, the line $\{0\} \times \mathbb{C}$ is the only complex line in $\mathbb{C}^2$ that intersects Λ on a lattice (*i.e.*, isomorphic to $\mathbb{Z}^2$). Automorphisms of Λ have the form $\begin{pmatrix} \omega_1 & b \\ 0 & \omega_2 \end{pmatrix}$ where ω_1 and ω_2 are in the finite group $\mathrm{Aut}(\Lambda_0)$ and b is in the ring $\mathrm{End}(\Lambda_0)$. In particular, $\mathrm{Aut}(\Lambda)$ contains an Abelian group of finite index, consisting only in unipotent elements. Hence these examples cannot be used in the holomorphic suspension construction of section 1.2.

Proposition. *Up to isogeny, all lattices in $\mathbb{C}^2$ whose automorphism group is infinite are described in examples 1 to 5.*

Proof. We give a sketch of the proof, leaving details to the reader. Let Λ be a lattice in $\mathbb{C}^2$ such that $\mathrm{Aut}(\Lambda)$ is infinite. We distinguish several cases:

A) There are at least two complex lines in $\mathbb{C}^2$ intersecting Λ on a lattice. In this case, Λ is isogenous to a product of two lattices of $\mathbb{C}$. These two lattices must be isogenous since $\mathrm{Aut}(\Lambda)$ is supposed to be infinite; we are in the example 2 case.

B) There is a unique complex line in $\mathbb{C}^2$ that intersects Λ on a lattice Λ_0. It is not difficult to see that Λ/Λ_0 must be isogenous to Λ_0 and that we are in the case of example 5.

C) There is no complex line intersecting Λ on a lattice. In this case $D = \mathrm{End}(\Lambda \otimes \mathbb{Q})$ is a division algebra since the image and the kernel of an element of $\mathrm{End}(\Lambda \otimes \mathbb{Q})$ are complex subspaces of $\mathbb{C}^2$ which are rational with respect to the lattice Λ, *i.e.*, the kernel and image are either trivial or $\mathbb{C}^2$. Of course $\Lambda \otimes \mathbb{Q} \simeq \mathbb{Q}^4$ appears as a vector space over D so that one has:

$$4 = \dim_D(\Lambda \otimes \mathbb{Q}) \dim_{\mathbb{Q}} D.$$

Let K be the center of D so that K is a number field and:

$$\dim_{\mathbb{Q}} D = \dim_{\mathbb{Q}} K \dim_K D.$$

As is well known $\dim_K D$ is a square so that one has three possibilities:

C-1) $K = \mathbb{Q}$, $\dim_{\mathbb{Q}} D = 4$ and $\Lambda \otimes \mathbb{Q}$ is 1-dimensional over D. By Frobenius' theorem, D is a quaternion algebra over $\mathbb{Q}$. We are in example 1.

C-2) $D = K$, $\dim_{\mathbb{Q}} K = 4$ and $\Lambda \otimes \mathbb{Q}$ is 1-dimensional over K. In this case, Λ is commensurable with the ring of integers of K and $\mathrm{Aut}(\Lambda)$ with the group of units. It follows from the fact that K has to act by complex linear maps that K is purely imaginary. This is example 3.

C-3) $D = K$, K is a quadratic field and $\Lambda \otimes \mathbb{Q}$ is a 2-dimensional K-vector space. This is example 4.

This finishes the sketch of the proof of the proposition. □

We have described $\mathrm{Aut}(\Lambda)$ up to commensurability but it would not be difficult to give a complete description. We shall not do it, in order to avoid long lists, and we only mention an interesting example of a finite subgroup of $\mathrm{Aut}(\Lambda)$.

Let Λ be the Hurwitz lattice in $\mathbb{C}^2 \simeq \mathbb{R}^4$ consisting of points (x_1, x_2, x_3, x_4) whose coordinates are all integers or all half an odd integer. Then $\mathrm{Aut}(\Lambda)$ contains a subgroup with 96 elements. Actually, the complex torus $\mathbb{C}^2/\Lambda$ is the Jacobian of the algebraic curve which is the 2-fold cover over the Riemann sphere, branched over the 6 vertices of a regular octahedron. This Riemann surface has the biggest automorphism group among genus-2 Riemann surfaces; it contains 48 elements.

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