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On transversely holomorphic flows II

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1 Introduction

This paper is a complement to the preceding one by Marco Brunella [2]. Our purpose is to complete the classification of transversely holomorphic flows on closed 3-manifolds, avoiding the "rationality" assumption made in [2]. The main tool is still the existence of a harmonic atlas established by M. Brunella and we shall only add some simple ideas coming from the theory of complex surfaces.

We follow the notations of [2]. As a matter of fact, our main result is the following:

Theorem 1.1 *Let $\mathscr{L}$ be an orientable transversely holomorphic foliation on a closed connected 3-manifold M. Assume that $H^2(M;\mathscr{O})\neq 0$ where $\mathscr{O}$ denotes the sheaf of germs of functions which are constant along the leaves and holomorphic in the transverse direction. Then, $\mathscr{L}$ is riemannian, i.e. there is a riemannian metric on the normal bundle which is invariant under holonomy.*

Theorem 1 in [2] gives a complete description of the situation on closed 3-manifolds for which $H^2(M;\mathscr{O})=0$. On the other hand, Y. Carrière obtained in [3] a classification of riemannian foliations in dimension 3. Therefore, the association of Theorem 1.1. and Brunella's result gives a classification: the only transversely holomorphic foliations on closed orientable connected 3-manifolds are examples 1) to 6) described in [2].

I would like to thank Yves Carrière: I had the pleasure to make with him the first attempts to classify these objects. As the reader will notice, the main ideas are contained in Brunella's paper; I wish to thank him for communicating these ideas to me.

2 General method

We fix a transversely holomorphic orientable foliation $\mathscr{L}$ on a connected closed 3-manifold M. We first give a very simple criterium which guarantees that the foliation is riemannian. We say that a differential form of degree 1, with complex values, is a *basic holomorphic* 1*-form* if it is locally the pullback of a holomorphic 1-form by the projection on a local leaf space.

Lemma 2.1 *If there exists a non trivial basic holomorphic.* 1*-form, then* $\mathscr{L}$ *is riemannian.*

Proof. Let ω be a basic holomorphic 1-form and assume first that ω has no singularity. We can define a hermitian metric g on the normal bundle (of complex dimension 1) in such a way that the length of a vector v is the modulus of $\omega(v)$. Since ω is basic, the same is true for g, i.e. $\mathscr{L}$ is riemannian.

In general, the singular locus of ω is transversely isolated, i.e. is a finite union of compact leaves $L_1, \dots, L_n$ of $\mathscr{L}$ (which are of course circles). Again, we can construct a hermitian metric g on the normal bundle but g vanishes along these leaves L_i. Consider the holonomy h_i of the leaf L_i; this is the germ of a holomorphic diffeomorphism in the neighborhood of a fixed point in a small transverse disc D_i. By choosing a suitable local coordinate in D_i, we can assume that the restriction of ω to D_i is $z^k dz$ for some integer $k > 0$ in the neighborhood of 0. The invariance condition of ω by h_i means that $h_i^k(z)h_i'(z) = z^k$ so that $h_i^{k+1}(z) - z^{k+1}$ is a constant. Evaluating at the origin, we see that this constant vanishes so that h_i is actually the germ of a rigid rotation of finite order. Therefore, $\mathscr{L}$ is riemannian in the neighborhood of L_i, i.e. we can find a saturated neighborhood of L_i in which $\mathscr{L}$ admits a transverse invariant (non degenerate) metric g_i. We can now multiply g_i by a bump function depending only on the modulus of z in order to obtain a transverse invariant "metric" g_i' for $\mathscr{L}$ which is non degenerate in the neighborhood of L_i but vanishes outside of some other tubular neighborhood of L_i. The sum of the g_i' and g is therefore everywhere non degenerate and is a transverse invariant metric. This shows that $\mathscr{L}$ is riemannian. □

Recall that M. Brunella proved the existence of a *harmonic atlas*. This means that there is a covering of M by a finite number of open sets U_j, whose intersections are connected and simply connected, equipped with diffeomorphisms $\Psi_j : U_j \to V_j = \Psi_j(U_j) \subset \mathbf{D} \times \mathbf{R}$ such that:

- In each U_j, the foliation $\mathscr{L}$ is the pull-back by Ψ_j of the foliation of $\mathbf{D} \times \mathbf{R}$ whose leaves are the lines $\{\bigstar\} \times \mathbf{R}$.
- Changes of coordinates $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ have the following form on their domain of definition $V_{ij} = \psi_j(U_i \cap U_j)$:

$$\psi_{ij} : (z,t) \in V_{ij} \mapsto (\phi_{ij}(z), t + h_{ij}(z)) \in V_{ji} \,,$$

where ϕ_{ij} is holomorphic and h_{ij} is *harmonic*.

Let H_{ij} be a holomorphic fonction whose real part is h_{ij}. Define:

$$\widehat{V}_j = V_j \times \mathbf{R} \subset \mathbf{D} \times \mathbf{R} \times \mathbf{R} \simeq \mathbf{D} \times \mathbf{C}$$

$$\widehat{V}_{ij} = V_{ij} \times \mathbf{R} \subset \mathbf{D} \times \mathbf{R} \times \mathbf{R} \simeq \mathbf{D} \times \mathbf{C}\,.$$

$$\Psi_{ij} : (z, t + i.s) \in \widehat{V}_{ij} \mapsto (\phi_{ij}(z), t + i.s + H_{ij}(z)) \in \widehat{V}_{ji}\,.$$

Unfortunately, the Ψ_{ij} do not necessarily define a cocycle, i.e. $\Psi_{ij} \circ \Psi_{jk}$ does not necessarily coincide with Ψ_{ik}. Therefore, we cannot in general define a complex surface X like M. Brunella in the "rational case". The main idea which will guide our discussion is that when one glues the open sets $\widehat{V}_j$ together using the Ψ_{ij}, one gets however some kind of "singular object" X which projects naturally onto M and which is not a complex manifold "only in the direction of $\partial/\partial s$". We shall not try to give a precise meaning to the previous sentence (in a suitable category...). We shall only recall that tensors on the "surface" X which are invariant under the translations along $\partial/\partial s$ can be defined with no ambiguity. In the next section, using this heuristic idea, we shall give the precise definitions of these tensors on X invariant by $\partial/\partial s$.

3 Some sheaves on *M*

We first define a fibre bundle $\mathcal{T}$ with fiber $\mathbf{C}^2$ on M. Consider the complex tangent bundle of each $\widehat{V}_j$. This is a holomorphic vector bundle with fiber $\mathbf{C}^2$ on which there is a natural action of the translations τ_σ (for $\sigma \in \mathbf{R}$) that we shall call *vertical*:

$$\tau_\sigma : (z, t + i.s) \in \widehat{V}_j \mapsto (z, t + i.(s + \sigma)) \in \widehat{V}_j\,.$$

The quotient of $\widehat{V}_j$ by the free action of these translations can be canonically identified with V_j so that we get a natural fiber bundle on V_j, with fiber $\mathbf{C}^2$. Since the Ψ_{ij} define a cocycle "up to vertical translations", we get therefore a fibre bundle on M (which is obtained from the V_j by gluing with the ψ_{ij}). This is the announced bundle $\mathcal{T}$.

We shall show that, although M is not a complex manifold, most properties of the cohomology of complex manifolds can be generalized to M. We follow the notations of [4] of which we quickly survey Sect. 15 and point out the modifications which are necessary in our situation.

We denote by $\mathbf{T}$ the dual bundle to $\mathcal{T}$ and by $\overline{\mathbf{T}}$ the conjugate bundle of $\mathbf{T}$. If p and q are two integers (smaller than or equal to 2), we consider the vector bundle $\Lambda^p(\mathbf{T}) \otimes \Lambda^q(\overline{\mathbf{T}})$, tensor product of exterior powers. Sections of this bundle are called *forms of type* (p,q) on M. Local sections define a sheaf denoted by $\mathcal{A}^{p,q}$.

Locally, a form ω of type (p,q) defined on V_j is identified with a form $\widehat{\omega}$ of type (p,q), in the usual sense, of the complex surface $\widehat{V}_j$, invariant under vertical translations. Since the decomposition $d = \partial + \overline{\partial}$ of the exterior differential is of course invariant under vertical translations, and since these operators commute with the biholomorphisms Ψ_{ij}, we get well defined operators, ∂ and $\overline{\partial}$

$$\partial : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q}$$
$$\overline{\partial} : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1} .$$

The sum of the $\mathscr{A}^{p,q}$ with $p+q=r$ is denoted by $\mathscr{A}^r$. The kernel of $\overline{\partial} : \mathscr{A}^{p,0} \to \mathscr{A}^{p,1}$ is the sheaf $\Omega(\Lambda^p(\mathbf{T}))$ of germs of *holomorphic p-forms* on M. Since a holomorphic function on $\widehat{V}_j$ invariant under vertical translations is in fact a function which depends only on the variable z and is holomorphic in this variable, the sheaf of germs of *holomorphic* 0-*forms* is identified with the sheaf $\mathcal{O}$ of functions on M which are constant along the leaves and which are transversely holomorphic.

If we consider each $\widehat{V}_j$ and a version of Dolbeault's theorem which is equivariant under the action of vertical translations, we get the following resolution analogous to the classical one:

$$0 \to \Omega(\Lambda^p(\mathbf{T})) \to \mathscr{A}^{p,0} \to \mathscr{A}^{p,1} \to \cdots \to \mathscr{A}^{p,q} \to \cdots .$$

Hence, Dolbeault's theorem holds in our context. More precisely, the q-th cohomology group $H^{p,q}(M)$ of M with values in the sheaf $\Omega(\Lambda^p(\mathbf{T}))$ can be identified with:

$$Z^{p,q}/\overline{\partial}(\mathscr{A}^{p,q-1}) ,$$

where, of course, $Z^{p,q}$ denotes the global forms of type (p,q) which are $\overline{\partial}$-closed. The dimension of $H^{p,q}(M)$ will be denoted by $h^{p,q}$ (we shall see that it is finite). Note that $H^2(M;\mathcal{O}) = H^{0,2}(M)$.

We now show how to extend Serre's duality and Hodge's theory in our context. The main point is to define a "fundamental class", i.e. to be able to "integrate" a $(2,2)$-form. So, let us consider such a form ω. In an open set V_j, this form corresponds to a $(2,2)$-form $\widehat{\omega}_j$ in the classical sense of $\widehat{V}_j$, invariant under vertical translations. The interior product of $\widehat{\omega}_j$ by the vector field $\partial/\partial s$ is a 3-form on $\widehat{V}_j$, basic for $\partial/\partial s$, i.e. which is a pull-back of some 3-form $\tilde{\omega}_j$ on V_j. Clearly, these 3-forms $\tilde{\omega}_j$ are compatible on the intersections of the V_j, i.e. they define a global 3-form $\tilde{\omega}$ on M. By convention, we define the integral of ω, denoted by $\int\omega$, as the integral of $\tilde{\omega}$ on M.

The simple (but crucial) observation is that Stokes' theorem can be extended with no difficulty:

Lemma 3.1 *If* α *is a form of type* $(2,1)$ *and if* $\omega = d\alpha (= \overline{\partial}\alpha)$, *when* $\int\omega$ *vanishes.*

Proof. We can assume that α has a compact support contained in some V_j. We consider the corresponding 3-form $\widehat{\alpha}_j$ in $\widehat{V}_j$, invariant under vertical translations, and whose differential is the form $\widehat{\omega}_j$. Since $\widehat{V}_j = V_j \times \mathbf{R}$, we can embed V_j in $\widehat{V}_j$ as $V_j \times \{s_0\}$. By definition, the integral of $d\alpha$ is equal to the integral of the interior product $i_{\partial/\partial s}\widehat{\omega}_j$ on $V_j \times \{t_0\}$ (which is indeed independent of s_0). As $\widehat{\alpha}_j$ is invariant under vertical translations:

$$i_{\partial/\partial s} d\widehat{\alpha}_j + d i_{\partial/\partial s}\widehat{\alpha}_j = 0 ,$$

so that $i_{\partial/\partial s} d\widehat{\alpha}_j$ is an exact form. On the other hand, the support of the restriction of $i_{\partial/\partial s} d\widehat{\alpha}_j$ à $V_j \times \{s_0\}$ is compact since α has a compact support contained in V_j. Hence, the lemma follows from usual Stokes' theorem. □

Therefore, the integration of the exterior product defines linear maps:

$$\imath : H^{p,q}(M) \otimes H^{2-p,2-q}(M) \to \mathbf{C} .$$

Let us now introduce some hermitian metric on $\mathbf{T}$. This enables us to define, as usual, anti-isomorphisms:

$$\sharp : \Lambda^p(\mathbf{T}) \otimes \Lambda^q(\overline{\mathbf{T}}) \ \to \ \Lambda^{2-p}(\mathbf{T}) \otimes \Lambda^{2-q}(\overline{\mathbf{T}})$$
$$\sharp : \mathscr{A}^{p,q} \ \to \ \mathscr{A}^{2-p,2-q}$$

and the operator:

$$\vartheta = -\sharp\,\overline{\partial}\,\sharp : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q-1} .$$

The operators ϑ and $\overline{\partial}$ are adjoint for the scalar product:

$$(\alpha, \beta) = \int (\alpha \wedge \sharp\beta)$$

because of 3.1 and for the same reason as in the classical case. The Laplace operator $\square = \vartheta\overline{\partial} + \overline{\partial}\vartheta$ is elliptic since, locally, it coincides with the usual Laplace operator acting on forms which are invariant under vertical translations.

Therefore, we get the finite dimensionality of cohomology groups $H^{p,q}$ and Hodge decomposition:

$$\mathscr{A}^{p,q} = \overline{\partial}\mathscr{A}^{p,q-1} \oplus \vartheta\mathscr{A}^{p,q+1} \oplus B^{p,q} ,$$

where $B^{p,q}$ denotes the space of $\square$-harmonic forms, i.e. the intersection of the kernels of ϑ and $\overline{\partial}$. According to Dolbeault's theorem, mentioned above, $B^{p,q}$ is identified with $H^{p,q}(M)$. In the same way we get Serre's duality, i.e. the isomorphism between $H^{p,q}(M)$ and $H^{2-p,2-q}(M)$.

4 Proof of the theorem

We can now prove the theorem. *We assume now that $\mathscr{L}$ is not riemannian* and we shall prove that $H^2(M; \mathscr{O}) = 0$.

According to Serre's duality, we know that $h^{0,2} = h^{2,0}$ and therefore it suffices to show that $h^{2,0} = 0$, i.e. that there is no non trivial holomorphic 2-form on M.

Let us start by observing that any holomorphic 1-form on M is closed. This is a well known fact for any holomorphic 1-form in the classical sense on a complex compact surface (see [1] page 115) and the proof only uses Stokes' theorem, for which we have proved the analogous version 3.1.

Let ω be a holomorphic 2-form on M and $\widehat{\omega}_j$ be the corresponding holomorphic 2-form on $\widehat{V}_j$. By contracting with the vertical holomorphic vector field in $\widehat{V}_j$, we get a holomorphic 1-form $\widehat{\alpha}_j$ in $\widehat{V}_j$. In other words, we construct a holomorphic 1-form α on M. By construction, $\partial/\partial s$ is in the kernel of $\widehat{\alpha}_j$. Since we observed that α is necessarily closed, the forms $\widehat{\alpha}_j$ are obtained by pull-back of some forms α_j in V_j which are basic for $\mathscr{L}$ (a closed form, vanishing on the leaves of a foliation, is a basic form). Hence, these forms α_j define a global basic holomorphic form for $\mathscr{L}$. Since we assumed that $\mathscr{L}$ is not riemannian, there is no non trivial form by 2.1. Hence ω vanishes.

References

1. Barth, W., Peters, C., Van de Ven A.: Compact complex surfaces, Springer, Berlin (1984)
2. Brunella, M.: On transversely holomorphic flows, I. Probab. Theory Relat. Fields, **106**, 265–279 (1996)
3. Carrière, Y.: Flots riemanniens, in Structures transverses des feuilletages, Astérisque **116**, 31–52 (1984)
4. Hirzebruch, E.: Topological methods in algebraic geometry, Grundlehren der mathematischen Wissenschaften **131**, Springer, Berlin (1978)