



# A Singular Mathematical Promenade

Reviewed by Rafe Jones



*A Singular Mathematical Promenade*

By Étienne Ghys

After fifteen seconds with Étienne Ghys's *A Singular Mathematical Promenade*, I knew that it was unlike any other book I had read. My first casual flip-through revealed a kaleidoscope of photos, drawings, and diagrams on nearly every page, and yet this visual brilliance occurred

under chapter headings containing mathematical phrases from the deep trenches of graduate-level mathematics, like "singular operads" and "resolution of singularities." I saw immediately the scale of the book's ambition: to appeal to almost everyone in the mathematical world from the undergraduate level up while still presenting a rich tapestry of advanced mathematical ideas. In the end, it comes remarkably close to achieving this lofty vision.

The journey begins with polynomials. Stuck in a particularly boring meeting, the author's colleague Maxim Kontsevich passes him a Parisian metro ticket with a scribble and the single word "impossible!" (see Figure 1). After some whispering, the author figures out that Kontsevich has shown that there cannot be four real polynomials satisfying  $P_1(x) < P_2(x) < P_3(x) < P_4(x)$  for small  $x < 0$  and  $P_2(x) < P_4(x) < P_1(x) < P_3(x)$  for small  $x > 0$ . Yet there are no such "forbidden permutations" for three polynomials. Surprised by

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this constraint on the relative positions of the graphs of four real polynomials, the author tries to understand all such constraints, then to place the problem in the more general context of singularities of real algebraic plane curves, that is, curves defined implicitly by equations of the form  $F(x,y) = 0$  for some real polynomial  $F$ . See Figure 2 for an example studied by Newton,  $F(x,y) = y^6 - 5xy^5 + x^3y^4 - 7x^2y^2 + 6x^3 + x^4$ .

The singularities of such curves have a rich structure, and if the promenade has a goal, it is to understand this structure. On the way, many mathematical vistas open, some that will be familiar to nearly all readers, and others that will be familiar to nearly none. The author warns us that we will get lost from time to time but gives these words of encouragement: "as d'Alembert said once, 'go on...and faith will catch up to you!' You may see every now and then a beautiful panorama emerging from the mist..." (p. 3). To illustrate, in a way, he directs us to the gorgeous painting *Wanderer Above the Sea of Fog*, by Caspar David Friedrich, reproduced at the very beginning of the text (see Figure 3).

Although its *sui generis* quality makes the book hard to describe precisely, one can easily say what it is not, as in this sentence from the back cover: "This is neither an elementary

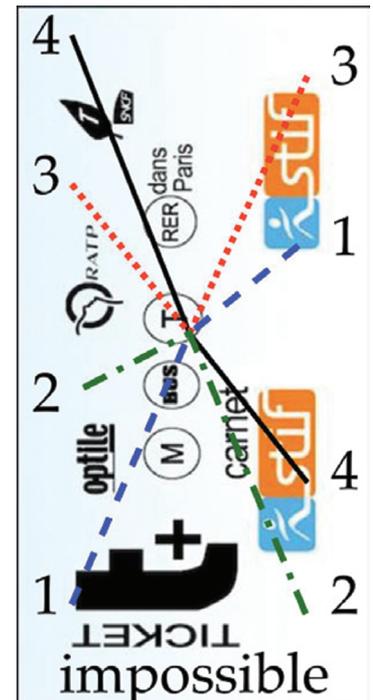
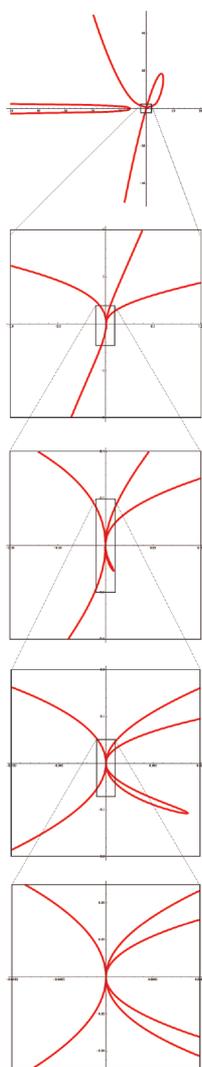


Figure 1. Kontsevich's theorem on a metro ticket (p. 1).



**Figure 2.** Zooming in on a singularity (p. 57).

introduction to the theory of singularities, nor a specialized treatise containing many new theorems." It is a series of mathematical-historical vignettes, each essentially self-contained but placed thoughtfully so as to resonate with those around it. The word "promenade" from the title is well chosen, as the book takes a roughly chronological path but freely explores side avenues when they arise. A highly useful "roadmap" (see Figure 4), superimposed on a sixteenth-century Chinese landscape drawing, gives a sense not only of the content of the book but of the progression and interconnectedness of the promenade.

Some chapters are detours, followed solely for their own interest (one example is a delightful chapter on divergent series on p. 87). Others emerge from the mist, only to have their relevance become clear later on, such as a sudden chapter on the Möbius band (p. 111). Still others add texture and richness to the mathematical ideas, such as a chapter on the local topology of singularities of complex algebraic curves (p. 157).

The roadmap shows two main strands, both leading towards a description of a combinatorial structure called a chord diagram, which can be used to classify the essential features of a singularity of a real algebraic plane curve. Starting from Kontsevich's observation about polynomials, four chapters discuss which permutations are allowed—they have already been studied by combinatorists under the

name "separable permutations"—and how to count them. An exercise from Donald Knuth's *The Art of Computer Programming* is the inspiration for one chapter, and the author traces the roots of the counting problem back 2000 years to Hipparchus.

The next strand generalizes the problem from intersections of polynomial graphs to singularities of real algebraic plane curves. Newton gave us the first significant results in this vein, and two marvelous chapters take us through parts of the original manuscript where he introduced what are now known as Newton's method and Newton polygons. Newton shows (without proof) that in a neighborhood of a singularity, a real algebraic plane curve may be decomposed into a finite number of branches, each of which is the "graph" of a formal power series with rational exponents. (You can see an illustration of this in Figure 2.) We then pay a visit to Gauss, who in his dissertation gave a topological



**Figure 3.** Caspar David Friedrich, *Wanderer Above the Sea of Fog*. The author proposes it as "a suggestion of the mathematical world" (p. ii).

proof of the fundamental theorem of algebra, relying on the assertion that an algebraic curve cannot stop or retrace itself—in other words, if it enters a disc, it must get out. But Gauss neglected to prove this! The fascinating history of the search for a proof of this claim occupies an additional chapter, and further chapters complete the proof of Newton's claims. This establishes that "a small circle around a singularity of a real algebraic plane curve intersects the curve in an even number of points and defines a chord diagram, i.e.,  $2n$  points cyclically ordered on a circle and grouped in pairs" (p. 6).

To go deeper into the topological structure of singularities, one must blow them up. This algebraic technique is explosive only metaphorically—it is "a kind of microscope enabling us to look deeply into the singularity" (p. 6). The process transforms a neighborhood of a singularity into a Möbius band, and repeated applications lead to generalizations called Möbius necklaces. Blowing up plays a crucial role in the crowning chapters on chord diagrams and in particular in a result that certain chord diagrams cannot arise from singularities of plane curves. There is an extended, though very interesting, detour into the topological structure of singularities of *complex* algebraic plane curves; these are the chapters on the bottom of the roadmap picture. Just before the chapters on chord diagrams, the thread of separable permutations returns, leading into the

theory of operads—a quite modern framework for understanding certain classes of operations like grafting trees or braids onto one another or perturbing polynomials with other polynomials.

Author Étienne Ghys, a CNRS senior researcher at the École Normale Supérieure de Lyon, lists on his website research interests in “Geometry, Topology, Dynamical Systems, mathematics in general.” After reading this book, one can only conclude the last phrase is not an overstatement. This sentence from the back cover aptly captures the dazzling mathematical breadth of his *Promenade*: “We play with a bit of algebra, topology, geometry, complex analysis, combinatorics, and computer science.”

Equally dazzling is the wealth of mathematical history woven into the exposition. Its breadth is breathtaking. This history comes not in sidebars but rather as an essential part of the mathematical narrative. For instance, the chapter on Newton’s method introduces its mathematical ideas by explicating several very clear reproductions of pages from Newton’s 1671 unpublished *De Methodis Serierum et Fluxionum*, as well as a few from a 1736 English translation (see Figure 5 for an example). We see reformulations of these originals into modern mathematical language, thereby connecting the old ideas to ones that most readers will have encountered in courses. These primary texts showcase both Newton’s brilliance and (perhaps) laziness—he eschewed formal proof and contented himself with working out examples as illustrations of his general method.

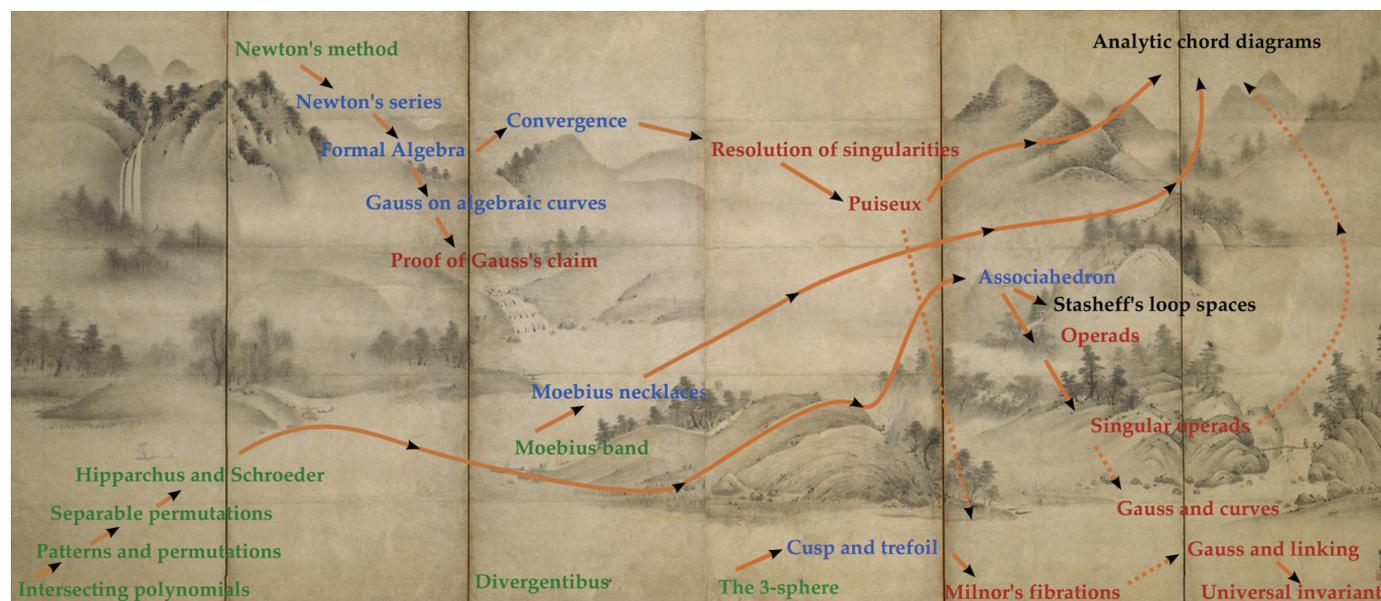
Two chapters later we learn that Cramer read Newton’s work and furnished complete proofs in his 1750 book *Introduction à l’Analyse des Lignes Courbes Algébriques*. However, Cramer was not pleased about this, writing in the preface: “It is regrettable that Mr. Newton was content to parade his discoveries without including proofs, and that

he preferred the pleasure of making himself admired rather than of instructing” (p. 67). Here I have translated the quotation into English; this is one of a handful of unfortunate places where interesting quotations are left untranslated from the French.

At other places history serves as a welcome digression. At the end of a reasonably demanding chapter on the Hopf fibration, there comes a marvelous discussion about whether the universe in Dante’s *Divine Comedy* is homeomorphic to a 3-sphere (p. 155). The prominent role of history makes this book more of a narrative than many other math books and creates a personal feel that makes the abstract ideas seem more immediate. Nearly every chapter has some element of history, personal anecdote, or visual embellishment. For example, in the chapter relating to Donald Knuth’s work, there is a sketch of Knuth (p. 17) and photos of old editions of *The Art of Computer Programming* (p. 19). Another example comes at the beginning of the chapter on Milnor’s fibration (p. 177): one finds a picture of Milnor, whose avuncular smile warms the page.

The book makes use of the margins of each page in a positively magical way. The margins are home to photos, stories, quotes, diagrams, examples, jokes, and references. They function almost as a second text, an internal voice of the author that pipes up over and around the primary public voice found in the main text. They are at their best when you most need them—when you are a bit confused by the main text or you need a moment to work out a test case to make sure you understand a theorem. Then you will find an explanatory aside or precisely the test case that you need to solidify your intuition.

There is a joy to this book that suffuses all of its aspects. There are jokes, quips, and winks in the text, in the images, and in the margins. There are flights of fancy, moments of



**Figure 4.** *Landscape of the Four Seasons (Eight Views of the Xiao and Xiang Rivers)*, by Soami, early 16<sup>th</sup> century (pp. 8–9). Colors give an idea of the difficulty of each chapter; in increasing order of difficulty: green, blue, red, and black.

$y^3 - 2y - 5 = 0$		+ 2, 10000000 - 0, 00544852 + 2, 09455148, &c. = y
$2 + p = y.$	+ $y^3$ - $2y$ - 5	+ 8 + 12p + 6p <sup>2</sup> + p <sup>3</sup> - 4 - 2p - 5
The Sum		- 1 + 10p + 6p <sup>2</sup> + p <sup>3</sup>
$0, 1 + q = p.$	+ $p^3$ + 6p <sup>2</sup> + 10p - 1	+ 0, 001 + 0, 03q + 0, 3q <sup>2</sup> + q <sup>3</sup> + 0, 06 + 1, 2 + 6, + 1, + 10, - 1,
The Sum		0, 061 + 11, 23 q + 6, 3 q <sup>2</sup> + q <sup>3</sup>
$- 0, 0054 + r = q.$	$q^3$ + 6, 3q <sup>2</sup> + 11, 23q + 0, 061	- 0, 000000157464 + 0, 00008748r - 0, 01621 <sup>2</sup> + r <sup>3</sup> + 0, 000183708 - 0, 06804 + 6, 3 - 0, 060642 + 11, 23 + 0, 061
The Sum		+ 0, 0005416 + 11, 162r
$- 0, 00004852 + s = r.$		

**Figure 5.** A page from Colson’s 1736 translation of Newton’s *De Methodis Serierum et Fluxionum*. In it, we see the application of what is now known as Newton’s method to approximate the root of  $x^3 - 2x - 5$  that is close to 2. For a striking image of the original in Newton’s handwriting, see p. 48 of the book.

high artistry, and sheer excitement leaping off the page. One example is the previously mentioned opening story of the book: of Kontsevich scrawling a theorem on a metro ticket during a boring meeting and passing it to the author. At another moment, the reader is ordered to go read Euler’s papers (p. 109). During a strenuous chapter on singularities of complex curves, the margin shows road signs indicating increasing grades. When defining an operad, the author writes that he will not “write down the formulas expressing these axioms since I would be unable myself to read the formulas that I wrote. I prefer to give first an example...” The book begins and ends with reproductions of lush, contemplative paintings by Caspar David Friedrich, the first the already mentioned *Wanderer Above the Sea of Fog* and the second a sunset painting called *Tree of Crows*. The book is alive with imagination, feeling, and humor.

While reading the book, I wondered at many junctures who the target audience is. The mathematical level fluctuates enormously over the course of the book, ranging from elementary and likely familiar (e.g., separable permutations, Newton’s method) to quite advanced and likely unfamiliar (e.g., the chapter on Stasheff’s loop spaces). Some chapters are complex enough that even mathematical veterans must frequently pause with pencil and paper to work through the arguments. The author addresses the question of audience directly in the preface, but in a way I found somewhat unsatisfying: “I wrote this ‘petit livre’ with one specific reader in mind: myself, when I was an undergraduate...” (p. 3). Yet the narrow audience implied by this statement sells the book short. Seasoned mathematicians

will enjoy its freewheeling yet mathematically substantial approach and are almost certain to learn new mathematics and new historical contexts for mathematics they already know. They will be dazzled by the lovely images and illustrations as much as a calculus student, and the latter’s curiosity may be piqued by the chapter on Newton’s method. If a reader enjoys mathematics and has a modicum of experience with it, she will find something to enjoy in this book. Few readers will be drawn into every last detail, but all will gain some appreciation of the beauty and interconnectedness of

the mathematical universe.

A PDF of the book is available for free on the author’s website, though it is certainly a joy to hold the physical volume, which qualifies as a work of art independent of the mathematical meaning of its contents. The pictures and diagrams appear in outstanding resolution, with crisp and vibrant colors, and the page width is pleasingly wider than a typical book—helping to give the margins and their magnificent contents their due. It makes a fabulous addition to a mathematical library.



Rafe Jones

**Credits**

Figures 1, 2, and 4 are courtesy of Étienne Ghys. Figure 3 is by Caspar David Friedrich [Public domain], via Wikimedia Commons. Figure 5 is from the John Adams Library (call number 2310187), Rare Books & Manuscripts Department, Boston Public Library. Author photo is by Laura Chihara.