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Commutators and diffeomorphisms of surfaces. (English summary)
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Given a group G one defines the commutator length

$$\mathrm{comm}\colon G' \to \mathbb{N}$$

on the subgroup $G' \subset G$ generated by commutators as the minimal length of a chain of generators for $g \in G'$.

The aim of the paper is to show that groups of area-preserving diffeomorphisms of (compact) surfaces admit unbounded commutator length. The strategy used is to construct (homogeneous) quasi-morphisms $\varphi\colon G \to \mathbb{R}$. Quasi-morphisms are functions for which there is a constant $D_\varphi > 0$ with

$$|\varphi(ab) - \varphi(a) - \varphi(b)| \le D_\varphi.$$

Two quasi-morphisms are called equivalent if their difference is bounded. Now the equivalence class of every quasi-morphism φ contains a unique homogeneous quasi-morphism

$$\Phi(a) := \lim_{p\to\infty} \frac{1}{p}\varphi(a^p).$$

Using

$$\mathrm{comm}(g^p) \ge p\frac{\Phi(g)}{4D_\Phi}$$

one has unbounded commutator length if one can find a nontrivial homogeneous quasi-morphism. The main result is the following:

Theorem. For every closed, oriented surface Σ, there exist homogeneous quasi-morphisms $\Phi\colon \mathrm{Diff}^\infty_0(\Sigma, \mathrm{area}) \to \mathbb{R}$ which are nontrivial, even when restricted to the kernel of Calabi's homomorphism (for $\Sigma \ne \mathbb{S}^2$). Moreover the vector space of these homogeneous quasi-morphisms is infinite-dimensional.

After stating the theorem, the authors construct infinite families of independent quasi-morphisms for surfaces of all genera and the disc.

This paper is very well written, and might be of interest for anybody studying surface diffeomorphism groups.

Reviewed by *Martin J. Schmoll*

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