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Holomorphic Anosov systems.
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Ghys investigates the structure of holomorphic Anosov diffeomorphisms and flows. In the lowest-dimensional situations he obtains a classification up to holomorphic conjugacy: Theorem A considers a holomorphic Anosov diffeomorphism of a compact complex surface S and shows that S is a torus and the diffeomorphism is holomorphically conjugate to a linear automorphism. The underlying reasons for this fact are that one can adapt the traditional arguments showing that codimension-1 invariant foliations are C^1 to the complex case to conclude that here both stable and unstable foliations are C^1 and hence holomorphic. (In the real domain smoothness of these foliations is an extra assumption.) The proof is nevertheless not straightforward—in fact Section 4 of the paper is needed to adapt earlier arguments by Ghys for the real case to the absence of an ordering on $\mathbf{C}$. Theorem B classifies holomorphic Anosov flows on 3-manifolds. Complex Anosov flows are defined somewhat similarly to real ones, but with allowances made for a 2-dimensional flow direction. For reasons discussed in Section 6 it makes sense to define them as a multiplicative action of $\mathbf{C}^*$. The models are holomorphic suspensions and their "twisted" counterparts, the action by left translations of $\left(\begin{smallmatrix} T & 0 \\ 0 & 1/T \end{smallmatrix}\right)$ on compact quotients of $\mathrm{SL}(2,\mathbf{C})$ and "twisted" quotients (by a construction of Ghys [Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 2, 251–270; MR0911758 (89h:58153)]), and "kth roots" of $\mathbf{C}^*$-actions (assume the action restricted to kth roots of unity is free on M with projection π_k to the quotient and let ${}_k\varphi(T)(\pi_k(x)) := \pi_k(\varphi(T^{1/k})(x))$). Theorem C shows that, up to finite covers, any holomorphic Anosov flow on a compact complex 3-manifold is holomorphically conjugate to one of these examples.

In higher dimension similar results can be obtained for codimension-1 systems: According to Theorem B every transitive holomorphic Anosov diffeomorphism of a compact complex manifold with real 2-dimensional unstable foliation is topologically conjugate to a linear toral automorphism. (Note that Theorem A does not assume transitivity; this necessitates extra work there.)

These results should be compared to their real counterparts; Ghys refers to his previous papers [Invent. Math. **82** (1985), no. 3, 479–526; MR0811548 (87f:58084); Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 209–247; MR1162561 (93j:58111); Inst. Hautes Études Sci. Publ. Math. No. 78 (1993), 163–185 (1994); MR1259430 (95d:57009)].

The constructions in this paper also produce an interesting result about foliations per se, Theorem D (or Theorem 6.5): There is a holomorphic complex 1-dimensional foliation $\mathcal{F}$ on a compact complex 3-manifold M such that every leaf is dense, of polynomial growth, and conformally equivalent to $\mathbf{C}$, but there is no flat Hermitian metric on the tangent bundle of $\mathcal{F}$ which is continuous on M and smooth along leaves. This is in contrast to a theorem of Candel for conformally hyperbolic leaves which gives a Hermitian metric with constantly curved leaves.

Reviewed by *Boris Hasselblatt*