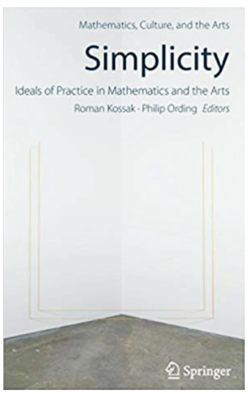
Simplicity: Ideals of Practice in Mathematics and the Arts

Reviewed by Douglas Norton



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Simplicity: Ideals of Practice in Mathematics and the Arts
Edited by Roman Kossak and Philip Ording

Simplify this fraction. Simplify the expression. Simplify your answer. We certainly present simplicity to our students as a desired goal, sometimes to the extent of conflating in significance the path to a solution and the form of the solution. On the research side of our mathematical lives, embedded in our own reference to a proof as "elegant" is the idea of a proof demonstrating some sort of simplicity. One hundred years after David Hilbert (Figure 1) presented his famous list of unsolved problems at the International Congress of Mathematicians in 1900 [1], historian of mathematics Rüdiger Thiele discovered another problem buried away in Hilbert's mathematical notebooks: "The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs" [2]. While Hilbert's list of problems inspired and challenged the mathematical community throughout the twentieth century, his 24th problem never appeared in the literature until this relatively recent discovery. Nevertheless, the ideas appear independently as formal threads in twentieth-century proof theory, model theory, and algorithmic information theory.

Figure 1. David Hilbert, c. 1900.

The book *Simplicity: Ideals of Practice in Mathematics and the Arts* addresses ideas of simplicity in mathematical proof in a general and philosophical way that requires no previous grounding in the specialty theories of the preceding paragraph while providing both subtle and fascinating insights into the questions raised. The volume presents selected lectures and additional contributions from a conference also titled Simplicity, held at the Graduate Center of the City University of New York in April of 2013. (See the conference poster in Figure 2.)

Why mathematics *and* the arts? Mathematical proportions proposed by the Greek sculptor Polykleitus in the fifth century BCE, perspective in Renaissance Italian painting, symmetry in Islamic tilings, geometry in the paintings of Piet Mondrian and the De Stijl school, and tessellations in M. C. Escher are all examples of specific mathematical tools utilized by artists. The past two decades have found both a broadening of the content and a widening of the appeal of the crossover between mathematics and the arts. The Bridges Organization works to "foster research, practice, and new interest in mathematical connections to art, music, architecture, education, and culture" through its annual Bridges Conferences [3]. *The Journal of Mathematics and the Arts* [4] is a peer-reviewed journal that focuses on connections between mathematics and the arts. An impressive juried exhibition of mathematical art has become a regular

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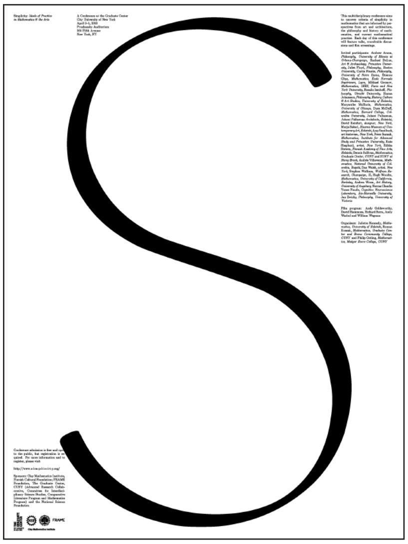

Figure 2. Poster for the conference Simplicity, held April 3–5, 2013, at the Graduate Center of the City University of New York.

feature at the Joint Mathematics Meetings [5]. Educators at all levels have begun to advocate for the inclusion of the arts in the push for science, technology, engineering, and mathematics education, with STEM evolving into STEAM [6].

Organizers of the Simplicity conference were Juliette Kennedy (University of Helsinki), Roman Kossak (the Graduate Center of CUNY), and Philip Ording (then at Medgar Evers College of CUNY, now at Sarah Lawrence College), all mathematicians with crossover interests in logic and philosophy, model theory, and mathematics and the arts, respectively. In their preface to the book, editors Kossak and Ording provide the following:

> That mathematicians attribute aesthetic qualities to theorems or proofs is well known. The question that interests us here is to what extent aesthetic sensibilities inform mathematical practice itself. When one looks at various aspects of mathematics from this perspective, it is hard not to notice analogies with other areas of creative endeavor—in particular, the arts.… [W]e find that a more profound connection between art and mathematics than any formal similarity is a similarity in method. For this reason the conference emphasized ideals of *practice* [pp. viii–ix].

They see, and the conference participants explore, simplicity as the essence of the similarity of method and the ideal of practice common to twentieth-century Western art and Hilbert's quest for consistency, efficiency, and rigor in proofs. The papers gathered in this volume present a fascinating peek at what the interactions among the mathematicians, artists, and philosophers gathered at the conference were like. Talks at the conference (as in Figure 3) were complemented by panel discussions across disciplinary boundaries; see Figure 4. The observations below are intended to follow a few threads that wend their way through the text rather than providing a sequential stroll through the papers in the collection.

Juliet Floyd, philosopher of mathematics and of language at Boston University, opens her piece "The Fluidity of Simplicity: Philosophy, Mathematics, Art" with the line: "Simplicity is not simple" [p. 155]. Is there a definition on which we can agree in the mathematical context? Is there one in the arts? Are they mirror images, funhouse mirror images, or completely unrelated? Andrés Villaveces, professor of mathematics at the National University of Colombia, Bogotá, observes in his piece "Simplicity via Complexity: Sandboxes, Reading Novalis":

> The *simplicity question*—the quest for the simplest proof or the simplest design, line, or resolution of architectural space or rhyme or melody…draws a tenuous but intriguing connection between mathematics and various other disciplines (architecture, physics, design, chemistry, music, etc.) [p. 192].

Let us first consider what the authors have to say in the mathematical arena.

Étienne Ghys, mathematician at the École Normale Supérieure in Lyon, presented the first address at the conference and the first paper in the collection, entitled "Inner Simplicity vs. Outer Simplicity." In these, he demonstrates why he was the inaugural recipient of the Clay Mathematics Institute Award for Dissemination of Mathematical

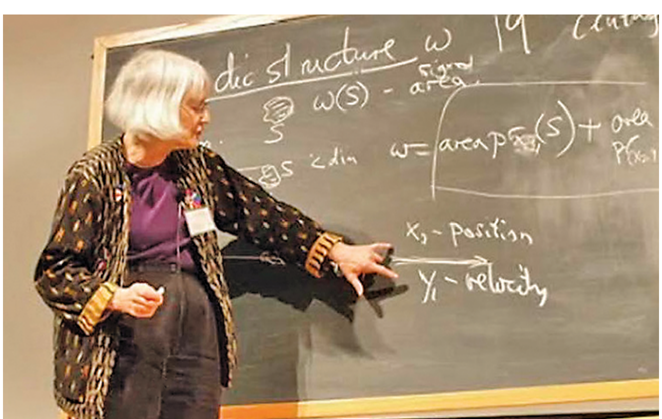

Figure 3. Professor Dusa McDuff of Barnard College, presenting at the Simplicity conference.

Figure 4. Panel discussion at the Simplicity conference. Panelists from left: Philip Ording, Amy Baker Sandback, Rachael DeLue, and Étienne Ghys.

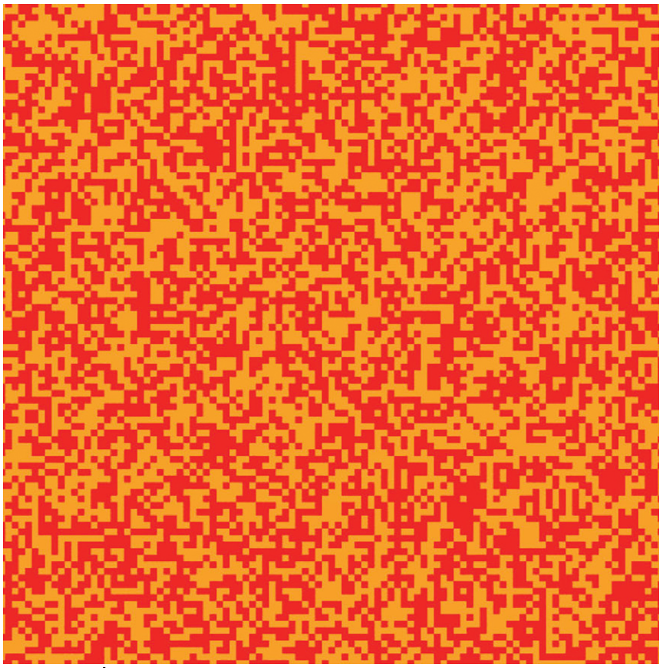

Figure 5. Étienne Ghys, "…my own art object.This is a totally random object."

Knowledge by posing questions, giving examples, and setting the tone for the conference and the collection. He sees a basic dichotomy in mathematics: "For me, mathematics is just about understanding. And understanding is a personal and private feeling. However, to appreciate and express this feeling, you need to communicate with others…" [p. 3]. For him, this dichotomy translates into two kinds of simplicity: inner simplicity, reflecting an ease of personal understanding that may nonetheless be difficult to communicate, and outer simplicity, in which something may be easy to express concisely but difficult to comprehend.

Because Ghys's outer simplicity relies on communication and description, he invokes Kolmogorov complexity as a measure of simplicity. In information theory, the Kolmogorov complexity of an object is the length of the shortest computer algorithm that produces the object as output. A more general usage would be that the complexity of an object is the length of the shortest description of the object. Ghys contrasts high and low Kolmogorov complexity through two pictures. A square with a random distribution of yellow and orange dots (Figure 5) would require a long sentence for a complete dot-by-dot description, while just a few short lines of code can generate the Mandelbrot set (Figure 6). This brief description renders the Mandelbrot set "simple" from the outer simplicity perspective, but Ghys finds this unsatisfactory; it is not simple in terms of inner simplicity.

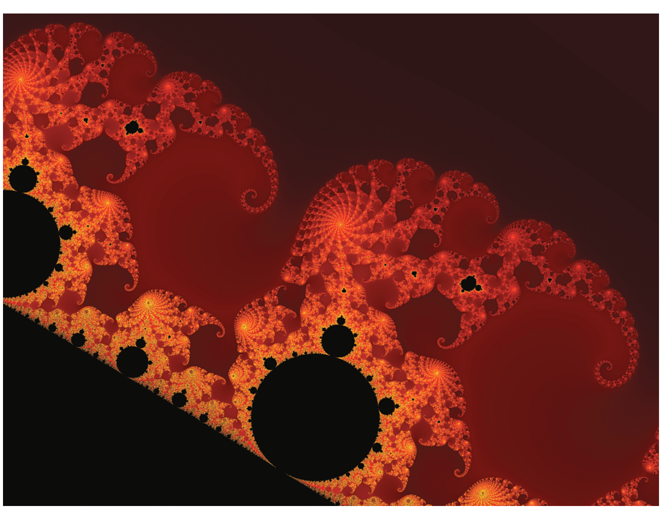

Figure 6. Closeup of the Mandelbrot set centered at (0.282, -0.01) at magnification 195.3125 times along the Y-axis using 5,000 iterations.

Ghys provides another example with proofs. He presents a single sentence from a number theory book by Jean-Pierre Serre that he recalls and describes as follows:

> I spent two days on this one sentence. It's only one sentence, but looking back at this sentence, I see now that it is just perfect. There is nothing to change in it; every single word, even the smallest, is important in its own way.… Serre's language is so efficient, so elegant, so simple. It is so simple that I don't understand it.… Everything, every single word is fundamental. Yet, from the Kolmogorov point of view, this is very simple.… Finally, at the end of the second day, all of a sudden, I grasped it and I was so happy that I could understand it. From Kolmogorov's point of view, it's simple, and yet for me—and, I imagine many students—it's not simple [p. 6].

He provides counterpoint to this with a delightful meander through networks, density, the Internet, and a theorem by Endre Szemerédi, with the following conclusion: "[I]t's an example of a theorem for which the published proof is complicated, but nevertheless I understand it. For me it's simple. I think I will never forget the proof because I understand it. And this is the exact opposite of the one-line by Jean-Pierre Serre, which was so short that it took me days to understand it" [p. 14].

Many of the views of mathematical simplicity expressed throughout the volume are a conflation of these two, always about a basic interaction with mathematical ideas and proofs. Dennis Sullivan, professor of mathematics and Einstein Chair holder at the Graduate Center of CUNY, reiterates this connection in his final essay of the collection, entitled "Simplicity Is the Point": "Understanding is more important to me than proofs.… So, proof and understanding are intimately tied, but understanding is, for me, the primary goal, and simplicity plays a role in that" [p. 269].

Marjorie Senechal addresses simplicity in science in her piece entitled "The Simplicity Postulate." Senechal is the Louise Wolff Kahn Professor Emerita in Mathematics and History of Science and Technology at Smith College, as well as editor-in-chief of *The Mathematical Intelligencer*.

What do we mean by simplicity in proofs? Senechal proposes *ease, unpretentious,* and *minimal* as words that come to mind [p. 79]. She adds *probable* to the list and tells the story of Dorothy Wrinch, Harold Jeffreys, and the Simplicity Postulate, a scientific version of the classic Occam's Razor. Occam's Razor is the principle that in solving a problem, the solution with the fewest assumptions—the simplest—tends to be the correct one. Wrinch and Jeffreys applied this reasoning to claim that scientists choose from among competing theories to explain a given physical phenomenon by selecting equations with smaller order, degrees, and magnitudes of coefficients. This Simplicity Postulate did not last long in scientific circles, and yet it had an impact on the wider acceptance of Bayesian statistics. (This story contains one of my favorite lines in a book full of interesting lines: Senechal says that Wrinch "crossed disciplinary boundaries easily, without glancing for oncoming trains" [p. 80].) While Senechal's discussion centers on selection between competing scientific theories rather than forms of mathematical proof, the linguistic distinctions are helpful. Reducing a proof to some sort of minimality may be quite difficult, sacrificing one type of simplicity for another, just as Serre's sentence in the Ghys paper may have simultaneously minimized length of description and ease of comprehension.

On the other end of the spectrum is the approach of proof theory, the branch of mathematical logic that studies proofs themselves as formal mathematical objects. One may define the simplicity of a proof in terms of minimizing some sort of measure: the number of symbols in some reduced version of a proof or the number of references to a class of topics. Andrew Arana, associate professor of philosophy at l'Université Paris 1 Panthéon-Sorbonne, addresses this approach in "On the Alleged Simplicity of Impure Proof." Arana calls a proof *pure* if it draws only on "what is 'close' or 'intrinsic' to that theorem" [p. 207]. By what metric do we define "closeness"? He mentions two: "A proof is *elementally close* to a theorem if the proof draws only on what is more elementary or simpler than the theorem.… A proof is *topically close* to a theorem if the proof draws only on what belongs to the content of the theorem, or what we have called the *topic* of the theorem" [p. 208]. These metrics may bring to mind some contrasting proofs in number theory. Zagier provides a one-sentence proof that every prime congruent to 1 mod 4 can be written as the sum of two squares [7]. The involution and fixed-point results required for the proof are sophisticated terms for the simple checking of algebraic identities, no less elementary than the ideas of prime numbers and squares of whole numbers. They are both simple and lie within the mathematical neighborhood of the topic of the theorem. By either metric, this proof may be simple in terms of length as well as in terms of structural content. Compare this with the Wiles proof of Fermat's Last Theorem: a theorem similarly simple to state, with a proof extremely long and distant from the statement by either of Arana's metrics.

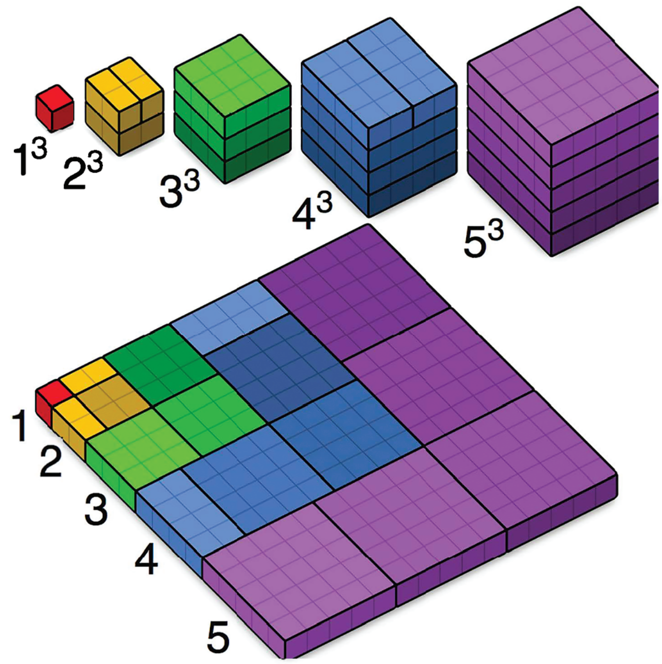

Figure 7. Proof without words of the Nicomachus theorem (squared triangular numbers).

In a similar vein, Rosalie Iemhoff, philosopher at Utrecht University in the area of proof theory, claims in "Remarks on Simple Proofs" that a proof should "not contain reasoning about geometric objects when the conclusion of the proof is a statement about the natural numbers" [p. 147]. This constraint, or Arana's "topically close" constraint above, would render many of the entries in the *Mathematics Magazine* series "Proofs Without Words" impure or not simple, as the proof of Nicomachus's theorem in Figure 7. On the other hand, such proofs certainly score high (for simplicity) on the Kolmogorov complexity version of the test of how many words it takes to carry out the proof!

Maryanthe Malliaris and Assaf Peretz approach the question from a different direction in "What Simplicity Is Not." Malliaris is a mathematician at the University of Chicago, and Peretz is in the interdisciplinary Group in Logic at Berkeley. Their list includes the following: simplicity is not outside existence, is not totally subjective, is not necessarily timeless, is not necessarily functional, is not necessary, does not equal perfection, is not necessarily shallow or shorter or a final minimum [pp. 51–8]. They present a helpful observation of simplicity as time-dependent. That a proof may become shorter or seem simpler as years go by may be the result of "re-orienting the field, restructuring certain basic paradigms and assumptions so that the theorem in question appears, to later students, as much closer to the source" [p. 55]. Concepts such as limits in calculus, complex numbers, and Gödel's incompleteness theorem, along with the proofs associated with them, have grown to be a part of our common lexicon. They were strange or difficult or not simple when introduced, but we've grown accustomed to their faces. Are they actually simpler, or do they just seem that way? This may be akin to something I tell my students in Advanced Calculus class when introducing a particular maneuver: "The first time you see this, it is a trick. The next time you see it, it is a technique. The next time, it is an old friend." It is not just the passage of time but the acquired familiarity that can provide a shift toward "simple."

Jan Zwicky, professor emerita at the University of Victoria, considers an appropriate opposite for simple in "The Experience of Meaning." Zwicky channels environmental philosopher Arne Næss to distinguish between complicated and complex. Unsurprisingly, Næss sees an ecosystem as complex—intricate, interrelated, and structured—while finding one's way through a huge unfamiliar city without a map is complicated—disorganized, even messy. "By definition, a complex thing cannot be simple in the sense of having no parts or divisions. It will have multiple aspects, and there are often many different relations between these aspects. But complexity is uncluttered. Everything fits" [p. 94]. This echoes the etymological lesson in the opening article by Ghys. The roots of the words "simple" and "complex" are both related to the French word *plier* for fold. Something simple has one fold; something complex has many folds. To explain something is to unfold it.

Zwicky continues by evoking the great twentieth-century mathematician Paul Erdős, who claimed that, as Ghys describes it, "somewhere in heaven there is THE BOOK and in it are some jewels, some wonderful proofs and we should work toward these beautiful proofs, simple proofs, elegant proofs" [p. 10]. The Erdős version of simplicity suggests a proof that utilizes just what it needs, with an economy of thought and a style of presentation. Martin Aigner and Günter Ziegler's book *Proofs from THE BOOK* [8] is now in its fifth edition, suggesting a continuing interest in this idea of the beauty and simplicity of proofs. Erdős inspired Aigner and Ziegler to write the book and assisted them as they began writing it. The volume begins with Euclid's proof of the infinitude of primes. Zwicky quotes Marjorie Senechal:

> Though no one has seen the book or ever will, all mathematicians know that Euclid's proof of the infinitude of primes is in it, and no mathematician doubts that computer-generated proofs, the kind that methodically check case after case, are not. The proofs in God's book are elegant. They surprise. In other words, they are light, quick, exact, and visible [p. 95].

Zwicky continues: "In other words, the proofs in God's book…are potent with meaning. They may be complex, but they are not complicated; there is no clutter" [p. 95]. And yet although Ghys finds most of the proofs in Aigner and Ziegler "wonderful," he finds that he does not remember them. They fail the "understanding means remembering" test of his inner simplicity.

As an aside, I find it interesting that there are six different proofs of the infinitude of primes to lead off the book about THE BOOK. I wonder if some are simpler than others by the various definitions of simplicity we see in this collection on *Simplicity*. On a related note, one of the editors of this volume, Philip Ording, has a new book entitled *99 Variations on a Proof* [9].

In his essay referenced above, Andrés Villaveces posits that complexity often provides a key step in the simplification process. There may even be "spiraling, back-and-forth movement between simplification and complexification" along the way [p. 191]. I confess that I first read the title as "Simplicity vs. Complexity," but the actual title makes more sense. Villaveces's example of "Simplicity via Complexity" is the combination of Gödel's completeness and incompleteness theorems. He proposes that one could interpret the incompleteness result as motivation to expand the terrain or "play in a larger sandbox," as he puts it, leading to the more satisfying result of completeness in a larger setting. I propose that the expansion of our number systems across the centuries, from natural numbers to fractions and negatives and irrationals and complex numbers, added a level of complexity at each stage that simplified by broadening the validity of the sentence "Every equation of this type has a solution." Here is a more quotidian example. In the midst of a big cleanup project around the house or at the office, I often refer to "the storm before the calm," flipping the usual phrase: sometimes you need to pull everything out and make a bigger mess in order to reconfigure in a better result. The same holds with proofs and complexity, I guess.

Villaveces sees the question of simplicity in proof as three problems:

> The notion itself of a *simplest* proof is the first and perhaps trickiest problem; the existence of such a simplest proof is a second, independent issue; finally, the question of *how to provide* such a simplest proof—provided it exists—is a third problem [p. 193].

These are all reasonable concerns. Is the idea of a simplest proof a valid one, and if so, what do we mean by it? Does a simplest proof of a given theorem exist, or is the process of simplification an asymptotic one? And then, of course, the practical question: if the first two questions are answered in the affirmative, how in the world do we do it? Iemhoff, the philosopher referenced above, adds the following: "Mathematics is the science par excellence that can be simple and complex at the same time.… In contrast with the use of the word in daily life, in mathematics, a simple argument does not necessarily mean that it is easy to find" [p. 145].

The authors address all of these questions in careful and multifaceted ways; this summary provides only brief peeks at their observations. What of the notion of simplicity in art, and how does it relate to that in mathematics?

Juliette Kennedy swaps her organizer hat for a presenter one; she explores the writings and philosophy of twentieth-century sculptor Fred Sandback in her essay "Kant, Co-Production, Actuality, and Pedestrian Space: Remarks on the Philosophical Writings of Fred Sandback." (A photo of one of Sandback's works appears on the cover of the book, reproduced in Figure 8.) Sandback's pieces are made with acrylic yarn stretched across walls and corners in what he termed "habitable drawings" [p. 39]. He describes some of his work:

> Around 1968, a friend and I coined the term "pedestrian space," which seemed to fit the work we were doing at the time.... Pedestrian space was literal, flat-footed, and everyday. The idea was to have the work right there along with everything else in the world, not up on a spatial pedestal [10].

In this sense, art can be something larger than ourselves as long as it successfully meets us where we are: personal and yet accessibly personal for each viewer in a different way. This compares with Arana's idea of *topical closeness* above. A simple proof in this sense uses mathematical material that is nearby, *pedestrian*, not as lacking in inspiration or excitement but as within walking distance. Sandback's pedestrian space mines the immediacy of nearby space for artistic expression; its topical closeness is an experiential closeness.

These ideas are consonant with the writings of the late twentieth-century American conceptual artist Sol LeWitt:

> Conceptual art is not necessarily logical.... Some ideas are logical in conception and illogical perceptually. The ideas need not be complex. Most ideas that are successful are ludicrously simple. Successful ideas generally have the appearance of simplicity because they seem inevitable [11].

His use of the term *inevitable* suggests that simplicity is related to accessibility. Accessibility may be a key to artistic expression as well as appreciation, just as it can be key to both understanding and communicating mathematics.

Art, then, can be a reduction, in the cooking sense: the process of thickening or intensifying the flavor of a liquid mixture such as a sauce by evaporation. In his essay "The Complexity of Simplicity: The Inner Structure of the Artistic Image," Finnish architect and former dean of the Helsinki University of Technology Juhani Pallasmaa quotes master twentieth-century Finnish architect Alvar Aalto: "Almost every formal assignment involves dozens often hundreds, sometimes thousands of conflicting elements that can be forced into functional harmony only by an act of will. This harmony cannot be achieved by any other means than art" [p. 18]. As Pallasmaa says, "Instead of analyzing and separating things, art is fundamentally engaged in merging and fusing opposites" [p. 18]. It is more than a merging of opposites: "The ultimate ideal of all art (and an impossibility, we must admit) is to fuse the complexity of human experiences into a singular image…" [p. 21], much as the William Blake poem calls us "[t]o see a World in a Grain of Sand" [12]. It is this simultaneous universality of meaning and the individual process of association and interpretation that provide the challenge and the evocative richness of art, through which "simplicity turns into labyrinthian complexity" [p. 22]. In a similar fashion, the mathematician evaporates out the unnecessary in a reduction process to turn what is simply a proof into a simple proof, even when the ideas involved are larger than the simplicity of the argument may suggest.

So what do the mathematical and artistic claims on simplicity have in common? The authors and editors make no attempt at tying it all together with a neat bow at the end. We end up where we began: "Simplicity is not simple." Perhaps it is the reduction of the broader to the essential. In mathematics, it is both the culling of the unnecessary and retaining focus on the core meaning of the concept at hand, ultimately toward comprehensibility and even memorability by the general reader. In art, it can be distancing from the subjectivity of the artist toward universality. As Pallasmaa quotes Balthus, twentieth-century Polish-French modern artist: "Great painting has to have a universal

Figure 8. Fred Sandback, *Untitled (Fourth of Ten Corner Constructions, Sculptural Study, Yellow Version)*, c. 1981/2007. Gold acrylic yarn 97 1/8 x 70 x 70 inches (246.7 x 177.8 x 177.8 cm).

meaning.... I want to give painting back its lost anonymity, because the more anonymous painting is, the more real it is" [p. 23]. In both art and mathematics, there is a certain irony in how the universality of a proof or an artistic piece confers personal meaning or individual relatability to the reader or the viewer or anyone experiencing the art, or the art in the proof, on their own terms.

Singers have made a subtle change to the words to the old Shaker song "Simple Gifts" over the years, replacing the article "the" by "a" in the first lines. In the original, the song begins:

> 'Tis the gift to be simple, 'tis the gift to be free,
> 'Tis the gift to come down, where we ought to be... [13].

Here, simplicity is seen not as one of many gifts but as THE gift, from Divine Providence, from the Muses, from the Keeper of THE BOOK. The contributors to this volume give us fascinating evidence that they have plumbed the depths of this simple and complex topic. Their sharing is a gift to us.

Simplicity is the first volume in a new Springer series on Mathematics, Culture, and the Arts. If this volume is any indication of the intellectual richness and payoff to come, I look forward to reading *Great Circles: The Transits of Mathematics and Poetry* and *Africa and Mathematics: From Colonial Findings Back to the Ishango Rods*, the next books in the series.

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