MR1324140 (96b:57032) 57R30 (57M99)
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Topologie des feuilles génériques. (French) [Topology of generic leaves]
Ann. of Math. (2) **141** (1995), *no. 2,* 387–422.

FEATURED REVIEW.

J. Cantwell and L. Conlon proved in 1987 that given a noncompact surface, any closed 3-manifold can be given a 2-dimensional foliation having the surface as a leaf. In contrast, some of the main results of the paper under review show that for a given foliation of a compact 3-manifold, the topology of "most" leaves is constrained (measure-theoretic technicalities will be suppressed in stating most of the results below). For example, if there is no compact leaf then an uncountable number of the leaves are one of six surfaces: the plane, the cylinder, the 2-sphere minus a Cantor set, and these same three surfaces with infinitely many handles attached in such a way that they limit to every end of the surface. Much more generally: for any 2-dimensional lamination of a compact space, the union of the compact leaves and the leaves that are one of these six types is a Borel set of full "harmonic" measure.

Before proving these and other results, the author provides a number of illustrative examples. There are examples of laminations of compact 3-manifolds having each of these six surfaces as the generic leaf. Other examples illustrate measure-theoretic subtleties, and an example of a 2-dimensional foliation of a 5-manifold is given for which almost all leaves have two ends, but a countable number have more than two. The most elaborate example gives a 4-dimensional lamination of a noncompact space whose leaves are all dense and such that no two distinct leaves are diffeomorphic.

Although the latter example might appear to be discouraging for genericity theorems in higher dimensions, the author proves a striking positive result: if $\mathcal{F}$ is a lamination of a compact space M, then for almost all x in M, the leaf passing through x has either 0, 1, 2, or a Cantor set of ends. Of course, this is an analogue of H. Hopf's result that a regular covering of a compact polyhedron has either 0, 1, 2, or an infinite number of ends, and the six noncompact surfaces that can be generic leaves for a foliated 3-manifold are exactly the six surfaces that arise as noncompact regular coverings of compact orientable surfaces. The author pursues this analogy in a couple of ways. First, if the leaf through almost every point has two ends, then $\mathcal{F}$ is "a finite extension of a 1-dimensional foliation". That is, there exists a Borel function π from M to a compact Borel space X with a 1-dimensional foliation $\mathcal{G}$ such that for almost every leaf L of $\mathcal{F}$, the restriction of π is a continuous proper map to a leaf l of $\mathcal{G}$, taking the two ends of L to the two ends of l. Second, there is an analogue of Stallings' structure theorem for groups with infinitely many ends.

Underlying many of these results is the "fundamental proposition", which says that if M is a compact space containing a lamination and $B \subset M$ is a Borel set, then for almost all $x \in M$ the intersection of B with the leaf L_x passing through x is either empty or approaches every end of L_x. The proposition relies on the fact that L_x has bounded geometry, meaning that the injectivity radii at its points and the sectional curvatures are uniformly bounded.

This paper is extremely well written. The rich variety of examples and the author's motivating explanations clarify the results and proofs admirably.

Reviewed by *Darryl McCullough*