

On quasi-homomorphism rigidity for lattices in simple algebraic groups

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Abstract

Property (TTT) was introduced by Ozawa as a strengthening of Kazhdan's property (T) and Burger and Monod's property (TT) . In this paper, we improve Ozawa's result by showing that any simple algebraic group of rank ≥ 2 over a local field has property (TTT) . We also show that lattices in a second countable locally compact group inherit property (TTT) . Finally, we study to what extent Lie groups with infinite center fail to have properties (TT) and (TTT) .

1 Introduction

Let G be a locally compact second countable group. It is well-known that by the Delorme-Guichardet theorem, G has Kazhdan's property (T) if and only if every affine isometric action on a Hilbert space has a fixed point (see [BdlHV08, Ch. 2]). This result can also be interpreted in terms of cocycles.

Definition 1.1. Let G be a locally compact group and H a Hilbert space. We say that a Borel locally bounded (i.e. bounded on compact subsets) map $b : G \rightarrow H$ with a Borel map $\pi : G \rightarrow \mathcal{U}(H)$ is a

- cocycle if π is a representation and $\forall g, h \in G, b(gh) = b(g) + \pi(g)b(h)$;
- quasi-cocycle if π is a representation and $\sup_{g, h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$;
- wq-cocycle if $\sup_{g, h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$.

It is known that G has property (T) if and only if every cocycle on G is bounded. In [BM99], Burger and Monod introduced a strengthening of property (T) : G has property (TT) if every quasi-cocycle is bounded. In this article, we study a stronger property introduced by Ozawa ([Oza11]).

Definition 1.2. Let G be a locally compact group and A a subgroup of G . The pair (G, A) has relative property (TTT) if any wq-cocycle on G is bounded on A .

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If G has property (TT) , then all quasimorphisms $G \rightarrow \mathbb{R}$, that is to say maps $\varphi : G \rightarrow \mathbb{R}$ such that $\{\varphi(gh)(\varphi(g)\varphi(h))^{-1} | g, h \in G\}$ is relatively compact, are bounded. Property (TTT) allows one to study such questions for quasi-homomorphisms, when the target group is no longer $\mathbb{R}$ (see [Oza11, Thm. A]).

Ozawa showed that for any local field $\mathbb{K}$, the group $SL_3(\mathbb{K})$ has property (TTT) as well as all its lattices ([Oza11, Thm. B]). However, it was not clear to him whether property (TTT) passes to lattices that are not cocompact. We show that this is true.

Theorem A. *Let G be a locally compact second countable group and Γ a lattice in G . Then G has property (TTT) if and only if Γ has property (TTT) .*

Our main result is an extension of the result on SL_n to higher rank simple algebraic groups.

Theorem B. *Let G be a connected simple algebraic group over a local field $\mathbb{K}$ with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property (TTT) .*

We follow the same idea as the classical proof of property (T) for these groups: we reduce the proof to the cases of the classical groups SL_3 and Sp_4 . As mentioned before, it is already known that SL_3 has property (TTT) . We show that for any local field $\mathbb{K}$, $Sp_4(\mathbb{K})$ has property (TTT) in Theorem 3.1.

Finally, Theorem B applies to higher rank simple Lie groups with finite center. But when G has infinite center, it is well-known that G has an unbounded quasimorphism $\phi : G \rightarrow \mathbb{R}$ (see [BG92, Prop. 6]). In particular, G does not have property (TT) nor (TTT) . However, we show in Proposition 5.2 that the unbounded wq-cocycles of G are completely controlled by the unbounded wq-cocycles of its center.

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2 Related properties

2.1 Positive definite kernels and completely bounded norm

Let G be a locally compact second countable group endowed with a left Haar measure. A function $\theta \in L^\infty(G \times G)$ is a positive definite kernel if for any $\xi \in L^1(G)$, $\int \theta(x, y) \xi(x) \overline{\xi(y)} dx dy \geq 0$. Equivalently, θ is a positive definite kernel if and only if there exists a separable Hilbert space H and a measurable map $F : G \rightarrow H$ such that $\theta(x, y) = \langle F(x), F(y) \rangle$ almost everywhere (see [BO08, Appendix D]). If θ is continuous, F can be taken to be continuous, and equality holds everywhere. We say that θ is normalized if $\theta(x, x) = 1$ for any $x \in G$.

The following inequality will be useful throughout the paper. Let θ be a normalized positive definite kernel on G . Let $x, y, z \in G$. We have

$$\begin{aligned}
|\theta(x, z) - \theta(y, z)| &= |\langle F(x) - F(y), F(z) \rangle| \\
&\leq \|F(x) - F(y)\| \\
&\leq \left(\|F(x)\|^2 + \|F(y)\|^2 - \langle F(x), F(y) \rangle - \langle F(y), F(x) \rangle \right)^{1/2} \\
&\leq \left(2 - \theta(x, y) - \overline{\theta(x, y)} \right)^{1/2} \\
&\leq \sqrt{2} |\theta(x, y) - 1|^{1/2}.
\end{aligned}$$

Let $\theta \in L^\infty(G \times G)$. We define the cb-norm of θ by

$$\|\theta\|_{cb} = \inf \{ \|P\| \|Q\| : P, Q \in L^\infty(G; H), \theta(x, y) = \langle P(x), Q(y) \rangle \}.$$

2.2 Property (T_P) and (T_Q)

Let G be a locally compact second countable group and A a subgroup of G .

Definition 2.1 ([Oza11]). The pair (G, A) has relative property (T_P) if $\forall \varepsilon > 0, \exists \delta > 0$ and $K \subset G$ compact such that for any $\theta : G \times G \rightarrow \mathbb{C}$ Borel normalized positive definite kernel satisfying

$$\sup_{g \in G} \|\theta(g \cdot, g \cdot) - \theta\|_{cb} < \delta \quad (2.1)$$

and

$$\sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta \quad (2.2)$$

then

$$\sup_{x, y \in A} |\theta(x, y) - 1| < \varepsilon. \quad (2.3)$$

Remark 2.2. As noticed by Ozawa ([Oza11, Section 3]), it is enough to consider only continuous kernels instead of Borel kernels. Furthermore, the hypothesis (2.2) can be weakened to

$$\sup_{x \in K} |\theta(x, 1) - 1| < \delta. \quad (2.4)$$

Indeed, if θ verifies (2.1) and (2.4), then for any $g, h \in G$ with $g^{-1}h \in K$, there is $x \in K$ such that $h = gx$ so

$$\begin{aligned}
|\theta(g, h) - 1| &= |\theta(g, gx) - 1| \\
&\leq |\theta(g, gx) - \theta(1, x)| + |\theta(1, x) - 1| \\
&< 2\delta
\end{aligned}$$

so θ verifies (2.2) up to replacing δ by 2δ .

Definition 2.3. The pair (G, A) has relative property (T_Q) if $\forall \varepsilon > 0, \exists \delta > 0$ and $K \subset G$ compact such that for any Borel map $\pi : G \rightarrow \mathcal{U}(H)$ and every unit vector $\xi \in H$ satisfying

$$\sup_{g, h \in G} \|\pi(gh)\xi - \pi(g)\pi(h)\xi\| < \delta \quad (2.5)$$

and

$$\sup_{g \in K} \|\pi(g)\xi - \xi\| < \delta \quad (2.6)$$

then

$$\sup_{x \in A} \|\pi(x)\xi - \xi\| < \varepsilon. \quad (2.7)$$

2.3 Measurable factorisation

Let X be a σ -finite measure space such that $L^2(X)$ is a separable Hilbert space, for example X a locally compact second countable group. Then $L^1(X)$ is also separable. If E is a separable Banach space, a function $\phi : X \rightarrow E^*$ is w^* -measurable if $x \mapsto \langle \phi(x), v \rangle$ is measurable for any $v \in E$. Since E is separable, let (x_n) be a dense sequence in the unit sphere of E . Then

$$\|\phi(\cdot)\| = \sup_{n \in \mathbb{N}} |\phi(\cdot)(x_n)|$$

is measurable, as the supremum of measurable functions. Thus, we can define $L_\sigma^p(X; E^*)$ as the space of w^* -measurable functions $\phi : X \rightarrow E^*$ such that

$$\|\phi\|_p = \|\|\phi(\cdot)\|\|_p < +\infty$$

(see [DU77] for more details). By Pettis measurability theorem ([DU77, Ch. II, Thm. 2]), if E^* is separable and $\phi : X \rightarrow E^*$ is such that $x \mapsto u(\phi(x))$ is measurable for any $u \in E^{**}$, then ϕ is Bochner measurable. This implies that when E is a separable reflexive Banach space, the space $L_\sigma^p(X; E^*)$ coincides with the space $L^p(X; E^*)$ of (Bochner) measurable functions. This holds more generally when E^* has the Radon-Nikodym property (see [DU77, Ch. IV]).

Let E, F be two Banach spaces. We denote $E \hat{\otimes} F$ the completion of the algebraic tensor product $E \otimes F$ for the projective tensor norm (see [DU77, Ch. VIII]). When E, F are separable, this is a separable Banach space. By [DU77, Ch. VIII.2, Coro. 2], there is an isometric isomorphism

$$(E \hat{\otimes} F)^* \simeq B(E, F^*) \quad (2.8)$$

and $\phi : E \hat{\otimes} F \rightarrow \mathbb{C}$ corresponds to the unique bounded operator $u : E \rightarrow F^*$ such that $\forall x, y \in E \otimes F, \phi(x \otimes y) = u(x)(y)$. Thus, we can define the spaces $L_\sigma^\infty(X; B(E, F^*))$ when E, F are separable Banach spaces.

Let E be a Banach space. By [DU77, Ch. VIII.1, Ex.10], the natural embedding $L^1(X) \otimes E \rightarrow L^1(X; E)$ extends to an isometric isomorphism

$$L^1(X) \hat{\otimes} E \simeq L^1(X; E). \quad (2.9)$$

Furthermore, if E is separable, the map

$$\begin{array}{ccc} L^\infty_\sigma(X; E^*) & \rightarrow & L^1(X; E)^* \\ \xi & \mapsto & u \mapsto \int_X [\xi(x)](u(x)) dx \end{array} \quad (2.10)$$

is an isometric isomorphism (see [Coi17, Thm. 1.16] or [Pis16, Prop. 2.20, 2.26 and Thm. 2.29]).

Let H be a separable Hilbert space and $\overline{H}$ be the complex conjugate of H . There is a canonical isometric isomorphism $H^* \simeq \overline{H}$ by Riesz representation theorem, and a canonical identification $H \simeq H^{**}$. Combining (2.9) and (2.10) with $E = H^*$, a function in $L^\infty(X; H) = L^\infty_\sigma(X; H)$ corresponds to a functional ϕ on $L^1(X) \hat{\otimes} \overline{H}$, which is defined on simple tensors $u \otimes \overline{y} \in L^1(X) \otimes \overline{H}$ by

$$\phi(u \otimes \overline{y}) = \int_X u(x) \langle \xi(x), y \rangle dx.$$

Thus, by (2.8), the map

$$T: \begin{array}{ccc} L^\infty(X; H) & \rightarrow & B(L^1(X), H) \\ \xi & \mapsto & u \mapsto \int_X u(x) \xi(x) dx \end{array} \quad (2.11)$$

is an isometric isomorphism.

Let E, F be two separable Banach spaces. The above properties give isometric isomorphisms

$$\begin{aligned} L^\infty_\sigma(X; B(E, F^*)) &\simeq L^\infty_\sigma(X; (E \hat{\otimes} F)^*) && \text{by (2.8)} \\ &\simeq L^1(X; E \hat{\otimes} F)^* && \text{by (2.10)} \\ &\simeq (L^1(X) \hat{\otimes} (E \hat{\otimes} F))^* && \text{by (2.9)} \\ &\simeq (E \hat{\otimes} L^1(X) \hat{\otimes} F)^* && \\ &\simeq (E \hat{\otimes} L^1(X; F))^* && \text{by (2.9)} \\ &\simeq B(E, L^1(X, F)^*) && \text{by (2.8)} \\ &\simeq B(E, L^\infty_\sigma(X; F^*)) && \text{by (2.10)} \end{aligned}$$

and following the path of isomorphisms shows that

$$\begin{array}{ccc} L^\infty_\sigma(X; B(E, F^*)) & \rightarrow & B(E, L^\infty_\sigma(X; F^*)) \\ \xi & \mapsto & u \mapsto \xi(\cdot)(u) \end{array} \quad (2.12)$$

is an isometric isomorphism.

Let

$$\Gamma_2(L^1(X), L^\infty(X)) = \left\{ T \in B(L^1(X), L^\infty(X)) \left| \begin{array}{l} T = SR \text{ where } R \in B(L^1(X), H), \\ S \in B(H, L^\infty(X)) \text{ for some} \\ \text{separable Hilbert space } H \end{array} \right. \right\}$$

with norm $\gamma(T) = \inf \|S\| \|R\|$. Let $z \in L^1(X) \otimes L^1(X)$. We define

$$\|z\|_* = \inf \left(\sum \|u_i\|^2 \right)^{1/2} \left(\sum \|v_i\|^2 \right)^{1/2}$$

where the infimum runs over all finite families $(u_i), (v_i)$ such that for $\xi, \eta \in (L^1(X))^*$,

$$|(\xi \otimes \eta)(z)| \leq \left(\sum |\xi(u_i)|^2 \right)^{1/2} \left(\sum |\eta(v_i)|^2 \right)^{1/2}.$$

Then, $\|\cdot\|_*$ is a norm on $L^1(X) \otimes L^1(X)$. By [Pis86, Thm. 2.8 and Coro. 2.9], there is an isometric isomorphism

$$\Gamma_2(L^1(X), L^\infty(X)) \simeq (L^1(X) \otimes_* L^1(X))^* \quad (2.13)$$

where $L^1(X) \otimes_* L^1(X)$ is the completion of the tensor product $L^1(X) \otimes L^1(X)$ for the norm $\|\cdot\|_*$. Thus, this space has a separable predual and we can consider the spaces $L^\infty_\sigma(Y; \Gamma_2(L^1(X), L^\infty(X)))$.

If $\varphi \in L^\infty(X \times X)$, we can define $r_\varphi \in B(L^1(X), L^\infty(X))$ by

$$r_\varphi(f)(s) = \int_X f(t) \varphi(t, s) dt.$$

By [Spr04, Thm. 3.3], φ is a Schur multiplier if and only if $r_\varphi \in \Gamma_2(L^1(X), L^\infty(X))$, and in that case, $\|\varphi\|_{cb} = \gamma(r_\varphi)$.

Let $\phi \in L^\infty(X \times X \times X)$ and denote $\phi_x = \phi(\cdot, x, \cdot)$. Such a map defines an operator $\tilde{\phi} \in L^\infty_\sigma(X; B(L^1(X), L^\infty(X)))$ by

$$\tilde{\phi}(x)(u) = \int_X \phi(t, x, \cdot) u(t) dt = r_{\phi_x}(u).$$

If $f \in L^1(X)$, let $\bar{f}: x \mapsto \overline{f(x)}$. Then $f \mapsto \bar{f}$ is an antilinear involution of $L^1(X)$.

Proposition 2.4. *Let G be a locally compact second countable group. Let $\theta \in L^\infty(G \times G)$ be a positive definite kernel on G such that for any $g \in G$, $\|g\theta - \theta\|_{cb} \leq \delta$. Denote $\phi(x, g, y) = \theta(gx, gy) - \theta(x, y)$. Then there exists a separable Hilbert space H and two functions $a, b \in L^\infty_\sigma(G; B(L^1(G), H))$ such that for almost every $g \in G$ and for every $u, v \in L^1(G)$,*

$$[\tilde{\phi}(g)(u)](v) = \langle a(g)(u), b(g)(\bar{v}) \rangle$$

with $\|a\|_\infty \|b\|_\infty \leq \delta$.

Proof. Since ϕ_g is a Schur multiplier for any $g \in G$, we have

$$\tilde{\phi} \in L^\infty_\sigma(G; \Gamma_2(L^1(G), L^\infty(G)))$$

with $\|\tilde{\phi}\|_{\infty, \Gamma_2} = \sup_{g \in G} \gamma(r_{\psi_g}) \leq \delta$.

By [CLMS21, Thm 5.1], there are two maps $a \in L^\infty_\sigma(G; B(L^1(G), H))$ and $b' \in L^\infty_\sigma(G; B(L^1(G), H^*))$ such that for almost every $g \in G$ and for every $u, v \in L^1(G)$,

$$[\tilde{\phi}(g)(u)](v) = [b'(g)(v)](a(g)(u)).$$

Then by composition of each $b'(g)$ with the antilinear identification $H^* \simeq \overline{H} \rightarrow H$, and the antilinear involution $L^1(G) \rightarrow L^1(G)$, $v \rightarrow \overline{v}$, we obtain $b \in L^\infty_\sigma(G; B(L^1(G), H))$ such that for almost every $g \in G$ and for every $u, v \in L^1(G)$,

$$[\tilde{\phi}(g)(u)](v) = \langle a(g)(u), b(g)(\overline{v}) \rangle. \quad \square$$

Lemma 2.5. *Let H be a separable Hilbert space, X, Y measured spaces such that $L^2(X), L^2(Y)$ are separable and Y is complete. Let $\alpha, \beta \in L^\infty_\sigma(Y; B(L^1(X), H))$ be two maps such that for almost every $y \in Y$ and every $u, v \in L^1(X)$,*

$$\langle \alpha(y)(u), \alpha(y)(v) \rangle = \langle \beta(y)(u), \beta(y)(v) \rangle. \quad (2.14)$$

Then there exists a map $\pi : Y \rightarrow \mathcal{U}(H \oplus \ell^2(\mathbb{N}))$ which is measurable when the group $\mathcal{U}(H \oplus \ell^2(\mathbb{N}))$ is endowed with the Borel σ -algebra coming from the strong operator topology, such that for almost all $y \in Y$, for all $u \in L^1(X)$, $U_y(\alpha(y)(u)) = \beta(y)(u)$.

Proof. First, by (2.12) with $E = L^1(X)$, $F = H^* \simeq \overline{H}$, the map

$$\begin{array}{ccc} L^\infty_\sigma(Y; B(L^1(X), H)) & \rightarrow & B(L^1(X), L^\infty_\sigma(Y; H)) \\ \alpha & \mapsto & u \mapsto \alpha(\cdot)(u) \end{array}$$

is an isometric isomorphism. Furthermore, $L^\infty_\sigma(Y; H) = L^\infty(Y; H)$ since H is a separable Hilbert space. Thus, for $u \in L^1(X)$, the maps $y \mapsto \alpha(y)(u)$ and $y \mapsto \beta(y)(u)$ are measurable.

Set $H' = H \oplus \ell^2(\mathbb{N})$. Since $L^1(X)$ is separable, we can consider $(u_n)_{n \in \mathbb{N}}$ a dense sequence in $L^1(X)$. Denote Y' a conull set in Y such that (2.14) holds for all $y \in Y'$.

If $y \in Y$, define $H_y = \overline{\alpha(y)(L^1(X))}$, then the sequence $(\alpha(y)(u_n))_{n \in \mathbb{N}}$ is dense in H_y . We apply the Gram-Schmidt process to this family: set $a_0(y) = \alpha(y)(u_0)$ which is measurable. If we have constructed $a_0(y), \dots, a_{n-1}(y)$ such that

$$\text{span}(a_0(y), \dots, a_{n-1}(y)) = \text{span}(\alpha(y)(u_0), \dots, \alpha(y)(u_{n-1}))$$

and each a_k is measurable, we set

$$a_n(y) = \alpha(y)(u_n) - \sum_{k < n, a_k(y) \neq 0} \frac{\langle a_k(y), \alpha(y)(u_n) \rangle}{\|a_k(y)\|^2} a_k(y).$$

Recursively, this give a family of vectors $(a_n(y))_{n \in \mathbb{N}}$ which for each y contains an orthogonal basis of H_y and some zero vectors. Since $\{y | a_n(y) \neq 0\}$ is measurable, replacing $a_n(y)$ by $a_n(y)/\|a_n(y)\|$ on this set still gives a measurable function, and now $(a_n(y))$ contains an orthonormal basis and some zero vectors for each $y \in Y$.

With the same process, we construct for each $y \in Y$ a family $(b_n(y))_{n \in \mathbb{N}}$ containing an orthonormal basis of $K_y = \overline{\beta(y)(L^1(X))}$ and some zero vectors such that for each $n \in \mathbb{N}$, $y \mapsto b_n(y)$ is measurable.

The crucial point is that using the hypothesis (2.14), for any $y \in Y'$ we have

$$a_n(y) = 0 \iff b_n(y) = 0 \quad (2.15)$$

and

$$a_n(y) = \sum_{k=0}^n \lambda_k(y) \alpha(y)(u_k) \iff b_n(y) = \sum_{k=0}^n \lambda_k(y) \beta(y)(u_k). \quad (2.16)$$

Now, consider an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H and an orthonormal basis $(f_n)_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$. Since $\ell^2(\mathbb{N})$ has uncountable dimension, there exists $(f'_n)_{n \in \mathbb{N}}$ such that $(f_n) \cup (f'_n)$ is linearly independent. Let $(g_n) = (e_n + f'_n) \cup (f_n)$. This is a total family in $H' = H \oplus \ell^2(\mathbb{N})$. Let

$$c_n(y) = P_{H_y^\perp}(g_n) = g_n - \sum \langle a_n(y), g_n \rangle a_n(y)$$

and

$$d_n(y) = P_{K_y^\perp}(g_n) = g_n - \sum \langle b_n(y), g_n \rangle b_n(y).$$

As limits of measurable functions, c_n, d_n are measurable since Y is complete. The family $(c_n(y))_{n \in \mathbb{N}}$ is total in $H_y^\perp$, and linearly independent. Indeed, if there is a relation $\sum_{i=1}^n \lambda_i c_i(y) = 0$, then $\sum_{i=1}^n \lambda_i g_i \in H_y$, but $(\text{span}(g_n)_{n \in \mathbb{N}}) \cap H = \{0\}$ by construction, so $\lambda_i = 0$ for any $1 \leq i \leq n$.

Similarly, the family $(d_n(y))$ is total in $K_y^\perp$ and linearly independent. Thus, applying the Gram-Schmidt process produces $(a'_n(y))_{n \in \mathbb{N}}$ and $(b'_n(y))_{n \in \mathbb{N}}$, which are also measurable functions and an orthonormal basis of $H_y^\perp, K_y^\perp$ respectively.

For $y \in Y'$, we have two orthonormal bases of $H' = H \oplus \ell^2(\mathbb{N})$. Thus, there is a unique unitary map U_y sending $a_n(y)$ to $b_n(y)$ and $a'_n(y)$ to $b'_n(y)$, using (2.15) to ensure that U_y is well-defined on the zero vectors in $(a_n(y))_{n \in \mathbb{N}}$. On $Y \setminus Y'$, we set $U_y = \text{Id}$.

Using (2.16) we show recursively that for any $y \in Y'$, $n \in \mathbb{N}$,

$$U_y(\alpha(y)(u_n)) = \beta(y)(u_n).$$

Thus, by density of (u_n) and continuity of U_y , we get that for any $u \in L^1(X)$,

$$U_y(\alpha(y)(u)) = \beta(y)(u).$$

Let $\xi \in H'$. Then for $y \in Y'$,

$$\xi = \sum_{n \geq 0} (\langle a_n(y), \xi \rangle a_n(y) + \langle a'_n(y), \xi \rangle a'_n(y))$$

so

$$U_y \xi = \sum_{n \geq 0} (\langle a_n(y), \xi \rangle b_n(y) + \langle a'_n(y), \xi \rangle b'_n(y)).$$

Again since Y is complete, $y \mapsto U_y \xi$ is measurable as a pointwise limit almost everywhere of measurable functions.

Since this is true for any $\xi \in H'$ and since H' is separable, this implies that $y \mapsto U_y$ is measurable for the strong operator topology on $\mathcal{U}(H')$. $\square$

2.4 Relation between properties

Ozawa showed the following implications between these strengthenings of property (T) ([Oza11, Thm. 1]).

Theorem 2.6.

$$\text{rel. property}(T_P) \implies \text{rel. property}(TTT) \implies \text{rel. property}(T_Q).$$

We aim to show that these properties are all equivalent.

Theorem 2.7. *Let G be a second countable locally compact group and A a subgroup of G . If (G, A) has relative property (T_Q) , then it has relative property (T_P) .*

Proof. Let $\varepsilon > 0$ and θ be a continuous positive definite normalized kernel verifying (2.1) and (2.2) for some δ, K to be determined later. By definition, there exists a separable Hilbert space H and a continuous map $\xi : G \rightarrow H$ such that $\forall g, h \in G$, $\theta(g, h) = \langle \xi(g), \xi(h) \rangle$ and $\forall g \in G$, $\|\xi(g)\| = 1$.

By Proposition 2.4, there exists a separable Hilbert space H' and two functions $a, b \in L^\infty_\sigma(G; B(L^1(G), H'))$ such that for almost every $g \in G$ and for all $u, v \in L^1(G)$,

$$\int_{G \times G} (\theta(gx, gy) - \theta(x, y)) u(x) \overline{v(y)} dx dy = \langle a(g)(u), b(g)(v) \rangle$$

with $\|a\|_\infty \|b\|_\infty < \delta$. Up to multiplying a, b by some constant, we can actually assume that $\|a\|_\infty < \sqrt{\delta}$ and $\|b\|_\infty < \sqrt{\delta}$.

With the notation of (2.11), we also get

$$\begin{aligned} \int_{G \times G} (\theta(gx, gy) - \theta(x, y)) u(x) \overline{v(y)} dx dy \\ = \langle T(g^{-1}\xi)(u), T(g^{-1}\xi)(v) \rangle - \langle T(\xi)(u), T(\xi)(v) \rangle. \end{aligned}$$

But then, setting $\tilde{a}(g)(u) = \frac{a(g)(u) + b(g)(u)}{2}$ and $\tilde{b}(g)(u) = \frac{a(g)(u) - b(g)(u)}{2}$, we also have $\|\tilde{a}\|_\infty < \sqrt{\delta}$ and $\|\tilde{b}\|_\infty < \sqrt{\delta}$. In the space $H \oplus H'$, we have for almost every $g \in G$ and any $u, v \in L^1(G)$, we get

$$\begin{aligned} \langle (T(\xi)(u), \tilde{a}(g)(u)), (T(\xi)(v), \tilde{a}(g)(v)) \rangle \\ = \langle (T(g^{-1}\xi)(u), \tilde{b}(g)(u)), (T(g^{-1}\xi)(v), \tilde{b}(g)(v)) \rangle. \end{aligned}$$

We apply Lemma 2.5 to $X = Y = G$ and

$$\alpha(g)(u) = (T(\xi)(u), \tilde{a}(g)(u)), \quad \beta(g)(u) = (T(g^{-1}\xi)(u), \tilde{b}(g)(u)),$$

to get a map $\pi : G \rightarrow \mathcal{U}(H \oplus H' \oplus \ell^2(\mathbb{N}))$ which is measurable for the completion of the Borel σ -algebra on G , and such that for almost every $g \in G$ and every $u \in L^1(G)$,

$$\pi(g)(T(\xi)(u), \tilde{a}(g)(u)) = (T(g^{-1}\xi)(u), \tilde{b}(g)(u)). \quad (2.17)$$

Then,

$$\begin{aligned}
\|T(g^{-1}\xi)(u) - \pi(g)T(\xi)(u)\| &\leq \|\tilde{b}(g)(u)\| + \|(T(g^{-1}\xi)(u), \tilde{b}(g)(u)) - \pi(g)T(\xi)(u)\| \\
&\leq \sqrt{\delta}\|u\| + \|\pi(g)(T(\xi)(u), \tilde{a}(g)(u)) - \pi(g)T(\xi)(u)\| \\
&\leq \sqrt{\delta}\|u\| + \|\tilde{a}(g)(u)\| \\
&\leq 2\sqrt{\delta}\|u\|.
\end{aligned}$$

But since

$$\pi(g)T(\xi)(u) = \pi(g) \int_G u(x)\xi(x)dx = \int_G u(g)\pi(g)(\xi(x))dx = T(\pi(g) \circ \xi)(u),$$

we get that for almost every g ,

$$\|T(g^{-1}\xi - \pi(g) \circ \xi)(u)\| \leq 2\sqrt{\delta}\|u\|$$

thus

$$\|T(g^{-1}\xi - \pi(g) \circ \xi)\|_{B(L^1(X), H)} \leq 2\sqrt{\delta}.$$

Since T is an isometry and ξ is continuous, for almost every $g \in G$ and for all $x \in G$,

$$\|\xi(gx) - \pi(g)\xi(x)\| \leq 2\sqrt{\delta}. \quad (2.18)$$

We want to change π so that (2.17) holds everywhere and π is a Borel map. We proceed as in [Oza11]. Let M be a Borel subset of G of measure zero such that (2.17) holds for all $g \in G \setminus M$. There exists also a Borel subset N of measure zero such that π is Borel $G \setminus N$. By regularity of the Haar measure, there exists a G_δ set of measure zero $N' = \bigcap_n U_n$ such that $M \cup N \subset N'$. Let K be any compact neighborhood of G and consider the map multiplication map $m : (G \setminus N') \times (K \setminus N') \rightarrow G$. Since N' has zero measure and K positive measure, m is surjective. Furthermore, for any $g \in G$,

$$\begin{aligned}
m^{-1}(\{g\}) &= \{(gk^{-1}, k) | k \in K \setminus N', xk^{-1} \in G \setminus N'\} \\
&= \bigcup_{p, q \in \mathbb{N}} \left\{ (gk^{-1}, k) | k \in K \cap U_p^c \right\} \cap ((G \setminus U_n) \times K)
\end{aligned}$$

so $m^{-1}(\{g\})$ is σ -compact. Thus, applying the Lusin-Novikov uniformization theorem ([Kec12, Thm.35.46]), there exists a Borel section $s : G \rightarrow (G \setminus N') \times (K \setminus N')$ of m . Then $t = p_K \circ s : G \rightarrow K$ is a Borel map such that $\forall g \in G, gt_g^{-1}, t_g \in G \setminus N'$.

Set $\tilde{\pi}(g) = \pi(gt_g^{-1})\pi(t_g)$, this is a Borel map and $\forall g \in G$,

$$\begin{aligned}
\|\xi(gx) - \tilde{\pi}(g)\xi(x)\| &\leq \|\xi(gx) - \pi(gt_g^{-1})\xi(t_gx)\| \\
&\quad + \|\pi(gt_g^{-1})\xi(t_gx) - \pi(gt_g^{-1})\pi(t_g)\xi(x)\| \\
&\leq 2\sqrt{\delta} + \|\xi(t_gx) - \pi(t_g)\xi(x)\| \\
&\leq 4\sqrt{\delta}
\end{aligned}$$

since (2.18) holds for t_g and gt_g^{-1} .

Let $\xi = \xi(e)$. Let us show that (2.5) and (2.6) hold for the pair $(\tilde{\pi}, \xi)$, in order to apply relative property (T_Q) .

By hypothesis (2.2), we have for any $g \in G, x \in K$,

$$|\theta(g, gx) - 1| < \delta \iff |\langle \xi(g), \xi(gx) \rangle - 1| < \delta.$$

Thus,

$$\begin{aligned} \|\xi(gx) - \xi(g)\|^2 &= \|\xi(gx)\|^2 + \|\xi(g)\|^2 - \langle \xi(gx), \xi(g) \rangle - \langle \xi(g), \xi(gx) \rangle \\ &\leq |1 - \langle \xi(gx), \xi(g) \rangle| + |1 - \langle \xi(g), \xi(gx) \rangle| \\ &\leq 2|\langle \xi(g), \xi(gx) \rangle - 1| \\ &\leq 2\delta. \end{aligned}$$

Hence, for any $g \in G, x \in K$, we have

$$\|\xi(gx) - \xi(g)\| < \sqrt{2\delta}. \quad (2.19)$$

Then, if $x \in K$, we have

$$\|\tilde{\pi}(x)\xi - \xi\| \leq \|\tilde{\pi}(x)\xi(e) - \xi(x)\| + \|\xi(x) - \xi(e)\| \leq (4 + \sqrt{2})\sqrt{\delta} = \delta'$$

by (2.18) and (2.19).

Let $g, h \in G$. We have that

$$\begin{aligned} \|\tilde{\pi}(gh)\xi - \tilde{\pi}(g)\tilde{\pi}(h)\xi\| &\leq \|\tilde{\pi}(gh)\xi(e) - \xi(gh)\| + \|\xi(gh) - \tilde{\pi}(g)\xi(h)\| \\ &\quad + \|\tilde{\pi}(g)\xi(h) - \tilde{\pi}(g)\tilde{\pi}(h)\xi(e)\| \\ &< 4\sqrt{\delta} + 4\sqrt{\delta} + \|\xi(h) - \tilde{\pi}(h)\xi(e)\| \\ &< 12\sqrt{\delta} = \delta''. \end{aligned}$$

Now since (G, A) has relative property (T_Q) , choosing K associated to ε in (T_Q) and δ small enough so that δ', δ'' are associated to ε , we get that (2.5) and (2.6) hold for $(\tilde{\pi}, \xi)$. Then we have by relative (T_Q) (2.7) that for any $x \in A$,

$$\|\tilde{\pi}(x)\xi - \xi\| = \|\tilde{\pi}(x)\xi(e) - \xi(e)\| < \varepsilon.$$

Let $x, y \in A$. We have that

$$\begin{aligned} |\theta(x, y) - 1|^2 &= |\langle \xi(x), \xi(y) \rangle - 1|^2 \\ &= (1 - \langle \xi(x), \xi(y) \rangle)(1 - \overline{\langle \xi(x), \xi(y) \rangle}) \\ &= 1 - \langle \xi(x), \xi(y) \rangle - \overline{\langle \xi(x), \xi(y) \rangle} + |\langle \xi(x), \xi(y) \rangle|^2 \\ &\leq 2 - \langle \xi(x), \xi(y) \rangle - \overline{\langle \xi(x), \xi(y) \rangle} \\ &= \|\xi(x) - \xi(y)\|^2 \end{aligned}$$

thus

$$\begin{aligned}
|\theta(x, y) - 1| &\leq \|\xi(x) - \xi(y)\| \\
&\leq \|\xi(x) - \tilde{\pi}(x)\xi(e)\| + \|\tilde{\pi}(x)\xi(e) - \xi(e)\| \\
&\quad + \|\xi(e) - \tilde{\pi}(y)\xi(e)\| + \|\tilde{\pi}(y)\xi(e) - \xi(y)\| \\
&\leq 4\sqrt{\delta} + \varepsilon + \varepsilon + 4\sqrt{\delta} = \varepsilon'
\end{aligned}$$

by relative (T_Q) and by (2.18).

Hence, we showed (2.3) for ε' , so (G, A) has relative property (T_P) . $\square$

It was shown in [Oza11] that both (T_P) and (T_Q) passes to lattices, but as noticed in the introduction, it was not clear whether (TTT) passes to non cocompact lattices. The equivalence of these three properties immediately implies Theorem A.

Corollary 2.8. *Let G be a locally compact group and Γ a lattice in G . Then G has (TTT) if and only if Γ has (TTT) .*

3 The symplectic group $Sp_4(\mathbb{K})$

Let $\mathbb{K}$ be a local field. We consider the symplectic group

$$Sp_4(\mathbb{K}) = \{g \in GL_4(\mathbb{K}) \mid {}^t g J g = J\}$$

where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Let also $S^{2*}(\mathbb{K}^2)$ be the vector space of symmetric bilinear form on $\mathbb{K}^2$ which can be identified with the space of symmetric matrices in $M_2(\mathbb{K})$. Then the group $SL_2(\mathbb{K})$ acts on $S^{2*}(\mathbb{K}^2)$ by $g.B = gB^t g$.

Consider the subgroup

$$G_2 = \left\{ g_A = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \mid A \in SL_2(\mathbb{K}) \right\} \simeq SL_2(\mathbb{K})$$

as well as the two subgroups

$$N_2^+ = \left\{ X_B^+ = \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \mid B \in M_2(\mathbb{K}), {}^t B = B \right\}$$

and

$$N_2^- = \left\{ X_B^- = \begin{pmatrix} I_2 & 0 \\ B & I_2 \end{pmatrix} \mid B \in M_2(\mathbb{K}), {}^t B = B \right\}.$$

Then the maps

$$\begin{aligned}
\iota_1 : \quad & SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2) & \rightarrow & Sp_4(\mathbb{K}) \\
& (A, B) & \mapsto & X_B^+ g_A
\end{aligned}$$

and

$$\iota_2 : \begin{array}{ccc} SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2) & \rightarrow & Sp_4(\mathbb{K}) \\ (A, B) & \mapsto & X_B^- g^t A^{-1} \end{array}$$

define two group embeddings of $SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2)$ with N_2^+, N_2^- as images of $S^{2*}(\mathbb{K}^2)$.

It is known that the pair $(SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$ has relative property (T) (see [BdlHV08, Coro. 1.5.2]) thus by [Oza11, Prop. 3], it has relative property (T_P) .

Theorem 3.1. *Let $\mathbb{K}$ be a local field. The group $Sp_4(\mathbb{K})$ has property (T_P) .*

We first need a Mautner type lemma adapted to the context of "almost invariance" instead of the usual invariance.

Lemma 3.2. *Let G be a locally compact group, $\theta : G \times G \rightarrow \mathbb{C}$ a normalized positive definite kernel such that $\sup_{g \in G} \|g.\theta - \theta\|_{cb} < \varepsilon$. Let $x, y \in G$ be such that*

$$|\theta(y^{-1}xy, 1) - 1| < \varepsilon \quad \text{and} \quad |\theta(y, 1) - 1| < \varepsilon.$$

Then

$$|\theta(x, 1) - 1| < 2\varepsilon + 4\varepsilon^{1/2}.$$

Proof. First, note that for any $g \in G$,

$$|\theta(gy, g) - 1| \leq |\theta(gy, g) - \theta(y, 1)| + |\theta(y, 1) - 1| < 2\varepsilon.$$

We have

$$\begin{aligned} |\theta(x, 1) - 1| &\leq |\theta(x, 1) - \theta(y^{-1}xy, 1)| + |\theta(y^{-1}xy, 1) - 1| \\ &< |\theta(x, 1) - \theta(y^{-1}x, y^{-1})| + |\theta(y^{-1}x, y^{-1}) - \theta(y^{-1}xy, 1)| + \varepsilon \\ &< 2\varepsilon + |\theta(y^{-1}x, y^{-1}) - \theta(y^{-1}x, 1)| + |\theta(y^{-1}x, 1) - \theta(y^{-1}xy, 1)| \\ &< 2\varepsilon + \sqrt{2}|\theta(1, y^{-1}) - 1|^{1/2} + \sqrt{2}|\theta(y^{-1}xy, y^{-1}x) - 1|^{1/2} \\ &< 2\varepsilon + 2\sqrt{2}(2\varepsilon)^{1/2}. \end{aligned}$$

□

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\varepsilon > 0$ and (K_0, δ) associated to ε in property (T_P) for the pair $(SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$. We may assume $\delta < \varepsilon$. Consider ι_1 and ι_2 the embeddings of $SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2)$ into $G = Sp_4(\mathbb{K})$, and set $K = \iota_1(K_0) \cup \iota_2(K_0)$.

Let θ be a normalized positive definite kernel on $Sp_4(\mathbb{K})$, that we may assume continuous (by a Remark in [Oza11, Section 3]), such that

$$\sup_{g \in G} \|g.\theta - \theta\|_{cb} < \delta$$

and

$$\sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta.$$

Then by relative property (T_P) for $(SL_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$, we get that

$$\sup_{s \in N_2^+ \cup N_2^-} |\theta(s, 1) - 1|.$$

Consider the subgroup

$$H = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid ad - bc = 1 \right\} \simeq SL_2(\mathbb{K})$$

and its two subgroups $N^+ = \{g \in H \mid a = d = 1, c = 0\}$, $N^- = \{g \in H \mid a = d = 1, b = 0\}$. Since $N^+ \cup N^- \subset N_2^+ \cup N_2^-$, for any $s \in N^+ \cup N^-$, $|\theta(s, 1) - 1| < \varepsilon$.

If $g \in H, s \in N^+ \cup N^-$, we have

$$\begin{aligned} |\theta(g, 1) - 1| &\leq 2|\theta(g, s) - 1| \\ &\leq 2(|\theta(g, s) - \theta(s, 1)| + |\theta(s, 1) - 1|) \\ &< 2(\delta + \varepsilon) < 4\varepsilon. \end{aligned}$$

But every element g in H can be written as a product of at most 3 elements of $N^+ \cup N^-$ (these corresponds to the transvections in $SL_2(\mathbb{K})$). Thus, we get that for any $g \in H$,

$$|\theta(g, 1) - 1| \leq 4\varepsilon^{1/2} + \varepsilon = \varepsilon'.$$

For any $\lambda \in \mathbb{K}$, the matrix $d_\lambda = \text{Diag}(\lambda, 1, \lambda^{-1}, 1)$ is an element of H . For $x \in \mathbb{K}$, consider the matrices

$$a(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \quad \text{and} \quad a'(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $x \in \mathbb{K}$ fixed. If $\lambda^{-1} \rightarrow 0$, we have $d_\lambda^{-1} a(x) d_\lambda \rightarrow 1$. In particular, by continuity of θ , there is λ such that $|\theta(d_\lambda^{-1} a(x) d_\lambda, 1) - 1| < \varepsilon'$. Thus, by Lemma 3.2, we have

$$|\theta(a(x), 1) - 1| < 2\varepsilon' + 4\sqrt{\varepsilon'} = \varepsilon''.$$

Similarly, if $\lambda \rightarrow 0$, we have $d_\lambda^{-1} a'(x) d_\lambda \rightarrow 1$ and thus $|\theta(a'(x), 1) - 1| < \varepsilon''$.

Finally, there is some integer ℓ such that any element $g \in G$ is a product of at most ℓ elements of $a(\mathbb{K}) \cup a'(\mathbb{K}) \cup N_2^+ \cup N_2^-$ (see [Neu03]).

Thus for any $g \in G$,

$$|\theta(g, 1) - 1| \leq 2\ell\sqrt{\varepsilon''}$$

which shows that G has (T_P) . □

4 Algebraic groups over local fields

We now know that $SL_3(\mathbb{K})$ and $Sp_4(\mathbb{K})$ have property (TTT) . As in the classical proof of property (T) , we want to use this fact to show that any almost $\mathbb{K}$ -simple algebraic group of rank at least 2 has (TTT) , where $\mathbb{K}$ is a local field. Before that, we need to show that (TTT) is stable under some operations.

If G_1, G_2 are locally compact group, a quasi-homomorphism is a Borel map $\varphi : G_1 \rightarrow G_2$ which is regular (i.e. the image of a compact subset of G_1 is relatively compact) and such that $\{\varphi(gh)^{-1}\varphi(g)\varphi(h) | g, h \in G_1\}$ is a relatively compact subset of G_2 .

Proposition 4.1. *Let G_1, G_2 be two locally compact groups. Let $\varphi : G_1 \rightarrow G_2$ be a surjective quasi-homomorphism. If G_1 has (TTT) , then G_2 has (TTT) .*

Proof. Let b be Borel wq-cocycle on G_2 . Then since φ is a quasi-homomorphism, $b \circ \varphi$ is a wq-cocycle on G_1 , hence bounded by (TTT) . Since φ is surjective, b is bounded. $\square$

Proposition 4.2. *Let G be a second countable locally compact group, $N \triangleleft G$ a closed normal subgroup. If N and G/N have (TTT) , then G has (TTT) .*

Proof. Let b be a wq-cocycle on G , and let

$$D = \sup_{g, h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$$

be its defect. Then $b|_N$ is a wq-cocycle on N , hence bounded by C by property (TTT) . By [Mac52, Lemma 1.1], there exists a Borel section $\sigma : G/N \rightarrow G$ which is regular, meaning that the image of any compact subset of G/N is relatively compact in G . Denote $n_g = g^{-1}\sigma(gN)$. Set $\tilde{b} = b \circ \sigma$ and $\tilde{\pi} = \pi \circ \sigma$. Then $\tilde{b}$ is a wq-cocycle on G/N associated to $\tilde{\pi}$. Indeed, if $g, h \in G$,

$$\begin{aligned} & \|\tilde{b}(ghN) - \tilde{b}(gN) - \tilde{\pi}(gN)\tilde{b}(hN)\| \\ &= \|b(ghn_{gh}) - b(gn_g) - \pi(gn_g)b(hn_h)\| \\ &\leq \|b(ghn_{gh}) - b(gn_g h n_h)\| + D \\ &\leq \|b(ghn_{gh}) - b(ghn')\| + D \\ &\leq \|b(ghn_{gh}) - b(gh) - \pi(gh)b(n_{gh})\| + \|b(gh) - b(ghn')\| + \|b(n_{gh})\| + D \\ &\leq \|b(ghn') - b(gh) - \pi(gh)b(n')\| + \|b(n')\| + 2D + C \\ &\leq 3D + 2C, \end{aligned}$$

using that N is a normal subgroup, thus $h^{-1}n_g h \in N$. Since G/N has property (TTT) , $\tilde{b}$ is bounded by C' . Thus, for any $g \in G$,

$$\|b(g)\| \leq \|b(gn_g) - b(g) - \pi(g)b(n_g)\| + \|b(gn_g)\| + \|b(n_g)\| \leq D + C + C'$$

so b is bounded on G , and thus G has property (TTT) . $\square$

In [Oza11, Thm. 6], Ozawa showed that a lattice in a group with property (TTT) inherits property (TTT) . In fact, his proof also shows the following results.

Theorem 4.3. [Oza11, Thm. 6] *Let H be a closed subgroup of G locally compact second countable such that there exists a finite Borel measure on G/H invariant under the action of G . If G has property (TTT) , then H has property (TTT) .*

We will now turn to algebraic groups. By algebraic group, we will always mean an affine algebraic group realised as an algebraic subgroup of GL_n . We will use the notations of [Mar91, Ch. I], where more details can be found.

Lemma 4.4. *Let $\mathbb{K}$ be a local field, G a connected semisimple $\mathbb{K}$ -group and $\tilde{G}$ its simply connected cover (in the algebraic sense). Then $G(\mathbb{K})$ has (TTT) if and only if $\tilde{G}(\mathbb{K})$ has (TTT) .*

Proof. Let $\pi : \tilde{G} \rightarrow G$ be a central $\mathbb{K}$ -isogeny. Then by [Mar91, Ch. I, Thm. 2.3.4], $\pi(\tilde{G}(\mathbb{K}))$ is a closed normal subgroup of $G(\mathbb{K})$ such that $G(\mathbb{K})/\pi(\tilde{G}(\mathbb{K}))$ is compact (thus has (TTT) as well as a finite Borel measure invariant by $G(\mathbb{K})$). By Proposition 4.2, $\pi(\tilde{G}(\mathbb{K}))$ has (TTT) implies $G(\mathbb{K})$ has (TTT) . Conversely, by Theorem 4.3, if $G(\mathbb{K})$ has (TTT) , so does $\pi(\tilde{G}(\mathbb{K}))$.

Furthermore, $\tilde{G}(\mathbb{K})/(\ker \pi)(\mathbb{K}) \rightarrow \pi(\tilde{G}(\mathbb{K}))$ is a homeomorphism. Thus, since the subgroup $(\ker \pi)(\mathbb{K})$ is finite hence has (TTT) , by Propositions 4.1 and 4.2, $\tilde{G}(\mathbb{K})$ has (TTT) if and only if $\pi(\tilde{G}(\mathbb{K}))$ has (TTT) . $\square$

In [dC09], Yves de Cornulier studied lengths on algebraic groups and showed the following result. A semigroup length on G is a map $\ell : G \rightarrow \mathbb{R}_+$ which is locally bounded and such that $\forall x, y \in G, \ell(xy) \leq \ell(x) + \ell(y)$.

Theorem 4.5. [dC09, Thm. 1.4] *Let G be an almost $\mathbb{K}$ -simple algebraic group over a local field $\mathbb{K}$, then every semigroup length on $G(\mathbb{K})$ is bounded or proper.*

To prove Theorem B, we will show using (TTT) on SL_3 and Sp_4 that a certain length is not proper, thus is bounded.

Theorem 4.6. *Let $\mathbb{K}$ be a local field, G a connected almost $\mathbb{K}$ -simple $\mathbb{K}$ -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property (TTT) .*

Proof. By [Mar91, Ch.I, Prop. 1.6.2], G contains an almost $\mathbb{K}$ -simple $\mathbb{K}$ -subgroup H whose (algebraic) simply connected cover is SL_3 or Sp_4 . Thus, by [Oza11, Thm.5], Theorem 3.1 and Lemma 4.4, $H(\mathbb{K})$ has property (TTT) .

Let b be a wq-cocycle on $G(\mathbb{K})$. Let $C = \sup_{g, h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$. Then $b|_{H(\mathbb{K})}$ is a wq-cocycle on $H(\mathbb{K})$ hence is bounded by (TTT) .

Consider the function $\ell : g \mapsto \|b(g)\| + C$. We have that $\ell(gh) \leq \ell(g) + \ell(h)$. Furthermore, ℓ is locally bounded since by definition b is. Then, by [dC09, Thm. 1.4], ℓ is either proper or bounded. But b is bounded on $H(\mathbb{K})$ which is not relatively compact, thus b is bounded. $\square$

Remark 4.7. Let G be a connected simple Lie group with finite center of rank at least 2. Then G is locally isomorphic to the group of $\mathbb{R}$ -point of an almost- $\mathbb{R}$ -simple algebraic group, thus has (TTT) .

Corollary 4.8. *Let $\mathbb{K}$ be a local field, G a connected almost $\mathbb{K}$ -simple $\mathbb{K}$ -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Let Γ be a lattice in $G(\mathbb{K})$. Then Γ has (TTT) .*

Proof. This is a direct consequence of the theorem and the fact that (TTT) passes to lattices. $\square$

Let $\varphi : G \rightarrow G'$ be a quasi-homomorphism. As noticed by Ozawa in [Oza11], if b is a wq-cocycle on G' , then $b \circ \varphi$ is a wq-cocycle on G . Hence, if G has property (TTT) and there exists b a proper wq-cocycle on G' (i.e. such that $\{g \mid \|b(g)\| \leq n\}$ is relatively compact in G' for any $n \in \mathbb{N}^*$), then any quasi-homomorphism $G \rightarrow G'$ has a relatively compact image.

Corollary 4.9. *Let Γ be a lattice in an higher rank almost $\mathbb{K}$ -simple algebraic group. Then any quasi-homomorphism $\Gamma \rightarrow G'$ where G' admits a proper wq-cocycle has relatively compact image.*

This applies in particular when G' has Haagerup property, or when G' is hyperbolic. Thus, it gives another proof of [FK16, Coro. 4.3].

5 Simple Lie groups with infinite center

In the previous section, we showed that any connected simple Lie group with finite center of rank at least 2 has (TTT) . We say that a quasi-homomorphism $\Phi : G \rightarrow \mathbb{R}$ is homogeneous if for any $g \in G, n \in \mathbb{N}, \Phi(g^n) = n\Phi(g)$. In that case, if g, h commute, then $\Phi(gh) = \Phi(g) + \Phi(h)$. Let G be a connected simple Lie group with infinite center $Z(G)$ and rank at least 2. Then by [BG92, Prop. 6], the space of homogeneous quasi-morphism is one dimensional. In particular, a nonzero element of this space is a wq-cocycle (and even a quasi-cocycle) which is unbounded, thus G does not have property (TTT) (and (TT) as well).

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Let A, K be the analytic subgroups of G with Lie algebras $\mathfrak{a}, \mathfrak{k}$ respectively. Then $G = KAK$ as in the finite center case. However, note that K is not compact. Indeed, $Z(G) \subset K$ is an infinite discrete subgroup, but $K/Z(G)$ is compact.

The following lemma is due to Yves de Cornulier and Mikael de la Salle in an unpublished note. We here reproduce their proof.

Lemma 5.1. *Let G be a connected simple Lie group with infinite center. There exists a Borel regular section $s : G/Z(G) \rightarrow G$ such that*

$$S = \{s(ghZ(G))(s(gZ(G))s(hZ(G)))^{-1} \mid g, h \in G\}$$

is finite and $s(\exp_{\text{Ad}(G)}(X)) = \exp_G(X)$ for any $X \in \mathfrak{a}$.

Proof. Let Φ be a Barge-Ghys morphism, normalized by $\Phi(Z(G)) \subset \mathbb{Z}$. Since Φ is homogeneous, we can define $s(gZ(G))$ by g if $\Phi(g) \in [-\frac{1}{2}, \frac{1}{2}]$. Since Φ is a quasimorphism, there is $C > 0$ such that $|\Phi(gh) - \Phi(g) - \Phi(h)| \leq C$. But we have

$$\begin{aligned} |\Phi(s(xy)s(y)^{-1}s(x)^{-1})| &\leq 2C + |\Phi(s(xy))| + |\Phi(s(y)^{-1})| + |\Phi(s(x)^{-1})| \\ &\leq 2C + \frac{3}{2} \end{aligned}$$

bounded independently of $x, y \in G/Z(G)$. Since $\{s(xy)s(y)^{-1}s(x)^{-1}\} \subset Z(G)$, it is finite. $\square$

Note that $s(1) = 1$. We want to study wq-cocycles on G , up to bounded functions. Let H be a Hilbert space and $\pi : G \rightarrow \mathcal{U}(H)$ be fixed. Let

$$Z_w(G, \pi) = \{b : G \rightarrow H \mid b \text{ wq-cocycle for } \pi\}$$

and $B_w(G, \pi)$ the subspace of bounded Borel functions. We want to understand the space $H_w(G, \pi) = Z_w(G, \pi) / B_w(G, \pi)$.

Let $i : Z(G) \rightarrow G$ denote the inclusion. Right-composition by i induces a map

$$i_* : H_w(G, \pi) \rightarrow H_w(Z(G), \pi).$$

Denote $z_g = gs(gZ(G))^{-1} \in Z(G)$.

Proposition 5.2. *The map i_* is injective and*

$$i_*(H_w(G, \pi)) = \left\{ [b] \mid \sup_{g \in G, z \in Z(G)} \|\pi(g)b(z) - \pi(z_g)b(z)\| < +\infty \right\}.$$

Proof. Let b be a wq-cocycle with defect D such that $i_*[b] = 0$. Then $b \circ i$ is bounded. The map $b \circ s$ is also a wq-cocycle on $G/Z(G)$. Indeed, since s is Borel regular, $b \circ s$ is Borel and locally bounded. Furthermore, if $g, h \in G/Z(G)$, then

$$\begin{aligned} \|b(s(gh)) - b(s(g)) - \pi(s(g))b(s(h))\| &\leq \|b(s(gh)) - b(s(g)s(h))\| + D \\ &\leq \|b((s(g)s(h))^{-1}s(gh))\| + 2D \end{aligned}$$

which is bounded in g, h since S is finite. But then since $G/Z(G)$ has finite center, by Theorem 4.6, $b \circ s$ is bounded.

For any $g \in G$, denote $z_g = gs(gZ(G))^{-1} \in Z(G)$. Thus

$$\|b(g)\| \leq D + \|b(s(gZ(G)))\| + \|b(z_g)\|$$

so b is bounded and $[b] = 0$.

Let $\tilde{b}$ be a wq-cocycle on $Z(G)$ with defect D . Suppose that there exists a wq-cocycle b on G with defect D' such that $i_*[b] = [\tilde{b}]$. Then, for any $g \in G, z \in Z(G)$,

using that z, z_g commute with G ,

$$\begin{aligned}
\|\pi(g)b(z) - \pi(z_g)b(z)\| &\leq \|\pi(g)b(z) + b(z_g s(gZ(G))) - b(gz) \\
&\quad - b(z_g s(gZ(G))) + b(z_g) + \pi(z_g)b(s(gZ(G))) \\
&\quad + b(gz) - b(z_g) - \pi(z_g)b(zs(gZ(G))) \\
&\quad + \pi(z_g)b(zs(gZ(G))) - \pi(z_g)b(z) - \pi(z_g)\pi(z)b(s(gZ(G))) \\
&\quad - \pi(z_g)b(s(gZ(G))) + \pi(z_g)\pi(z)b(s(gZ(G)))\| \\
&\leq 4D' + 2\|b(s(gZ(G)))\|.
\end{aligned}$$

But since $b \circ s$ is bounded, we get that

$$\sup_{g \in G, z \in Z(G)} \|\pi(g)b(z) - \pi(z_g)b(z)\| < +\infty.$$

Now since $b|_{Z(G)} - \tilde{b}$ is bounded by assumption, we get the necessary condition

$$\sup_{g \in G, z \in Z(G)} \|\pi(g)\tilde{b}(z) - \pi(z_g)\tilde{b}(z)\| < +\infty \quad (5.1)$$

Finally, we show that condition (5.1) is sufficient. So let $\tilde{b}$ be a wq-cocycle such that (5.1) holds and let C be the supremum. Define $b(g) = \tilde{b}(gs(gZ(G))^{-1}) = \tilde{b}(z_g)$ which is Borel and locally bounded. Then b is a wq-cocycle. Indeed, if $g, h \in G$, then

$$\begin{aligned}
\|b(gh) - b(g) - \pi(g)b(h)\| &= \|\tilde{b}(z_{gh}) - \tilde{b}(z_g) - \pi(g)\tilde{b}(z_h)\| \\
&\leq \|\tilde{b}(z_{gh}) - \tilde{b}(z_g) - \pi(z_g)\tilde{b}(z_h)\| + \|\pi(g)\tilde{b}(z) - \pi(z_g)\tilde{b}(z)\| \\
&\leq \|\tilde{b}(z_{gh}) - \tilde{b}(z_g z_h)\| + D + C \\
&\leq \|\tilde{b}((z_g z_h)^{-1} z_{gh})\| + 2D + C.
\end{aligned}$$

But $(s(gZ(G))s(hZ(G)))^{-1}s(ghZ(G)) = (z_g z_h)^{-1} z_{gh}$ so since S is finite,

$$\sup_{g, h \in G} \|\tilde{b}((z_g z_h)^{-1} z_{gh})\| < +\infty.$$

Finally, for any $z \in Z(G)$, $b(z) = \tilde{b}(zs(1)^{-1}) = \tilde{b}(z)$ so that $i_*[b] = [\tilde{b}]$. $\square$

Remark 5.3. In particular, any wq-cocycle on $Z(G)$ associated with $\pi : Z(G) \rightarrow \mathcal{U}(H)$ induces a wq-cocycle on G , associated to $\pi' : G \rightarrow \mathcal{U}(H)$ defined by

$$\pi'(g) = \pi(gs(gZ(G))^{-1}) = \pi(z_g).$$

Furthermore, any wq-cocycle on G is bounded on A , since $A \subset s(G/Z(G))$.

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