

REGULARITY OF K -FINITE MATRIX COEFFICIENTS OF SEMISIMPLE LIE GROUPS

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ABSTRACT. We consider a semisimple Lie group G with finite center and K a maximal compact subgroup of G . We study the regularity of K -finite matrix coefficients of unitary representations of G . More precisely, we find the optimal value $\kappa(G)$ such that all such coefficients are $\kappa(G)$ -Hölder continuous. The proof relies on analysis of spherical functions of the symmetric Gelfand pair (G, K) , using stationary phase estimates from Duistermaat, Kolk and Varadarajan. If U is a compact form of G , then (U, K) is a compact symmetric pair. Developing similar tools in this context, we study the regularity of K -finite coefficients of unitary representations of U , improving on previous results obtained by the author.

1. INTRODUCTION

The aim of this article is to investigate the regularity of matrix coefficients of unitary representations of a (compact or non-compact) semisimple Lie group G with finite center. The asymptotic behavior of matrix coefficients of unitary representations of semisimple Lie groups is well understood (see [HT12, Ch. V]), but far less is known about the local behavior and regularity of such coefficients. Of course, by definition, all these functions are continuous, but there is no reason to expect a better regularity.

If K is a maximal compact subgroup of G , it is known from the work of Harish-Chandra ([HC53]) that matrix coefficients associated with K -finite vectors of irreducible unitary representations of G (and more generally, admissible representations) are smooth. Thus, a natural space of matrix coefficients to consider is the space of K -finite matrix coefficients. Every unitary representation of G decomposes as a direct integral of irreducible representations, but since Harish-Chandra's estimates depend on the representations, they do not provide any estimates for arbitrary representations.

Recently, Vincent Lafforgue ([Laf08]) pointed out a surprising fact on local regularity: matrix coefficients of $SO(3)$ that are $SO(2)$ -bi-invariant are all $\frac{1}{2}$ -Hölder continuous in a neighborhood of any point (outside of some singular points). This result has strong consequences for the analysis of $SL(3, \mathbb{R})$ and is a key ingredient in the proof of strong property (T) . In broad generality, one may embed $SO(3)$ (not necessarily as a subgroup) inside $SL(3, \mathbb{R})$ in several ways. Thus, any coefficient of a (not necessarily unitary) representation of $SL(3, \mathbb{R})$ inherits the $\frac{1}{2}$ -Hölder

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regularity in each direction corresponding to an embedding. For a large class of representations with controlled growth, the local behavior can be promoted to asymptotic decay by combinatorics in the Weyl chamber of $SL(3, \mathbb{R})$. This was further improved in [HdL13] to cover the case of $Sp(4, \mathbb{R})$ (and thus by classification, all higher rank simple Lie groups). Again, Haagerup and de Laat make use of a compact subgroup of $Sp(4, \mathbb{R})$ where some regularity result holds for a class of matrix coefficients.

These phenomena of regularity have already led to several advances in various domains. In dynamical systems, strong property (T) plays an important role in the resolution of several cases of Zimmer's conjecture on actions of lattices on manifolds ([BFH22]). In harmonic analysis, the results were also extended to explore the behavior of Fourier multipliers, which was much more mysterious than that of coefficients of representations ([PRdlS22]). In operator algebras, the method permitted to obtain examples of groups without the approximation property (AP) of Haagerup and Kraus ([LdlS11]). It also shows that non-commutative L^p spaces associated with some lattices in higher rank groups do not have the (CBAP) for some values of p ([LdlS11, dL13, dLdlS18]). The values of p where the theorem holds depend on the group, and come from the regularity result at the level of the compact subgroup.

A deep open question of Connes is the following: are the von Neumann algebras of $SL_n(\mathbb{Z})$ and $SL_m(\mathbb{Z})$ non-isomorphic when $n \neq m$? The aforementioned result [dLdlS18, Thm. 1.1] is directly related to this question, as it hints at a different behavior of the critical parameter p for $SL_n(\mathbb{Z})$ as n varies, which comes directly from the value of the regularity for the $SO(n)$ -bi-invariant coefficients of $SO(n+1)$ as n varies. However, currently, all those results have been obtained through ad hoc computations in particular cases. In order to better understand the limits of these methods, it is then of fundamental importance to completely solve the question of the optimal regularity of matrix coefficients for the compact semisimple groups.

Haagerup and de Laat ([HdL13]) noticed that this result can be interpreted in terms of Gelfand pairs (G, K) . In this setting, any K -bi-invariant coefficient of a unitary representation of G decomposes as a direct integral of positive definite spherical functions of the pair (G, K) (see Section 2.2). In [Dum24], we explored in a more systematic way the phenomena for pairs (U, K) such that U/K is a symmetric space of compact type. In particular, we found the optimal regularity of K -finite coefficients whenever U/K is of rank 1 or isomorphic to a Lie group ([Dum24, Thm. A and B]). These results were again the fruit of a particular expression of spherical functions. Nonetheless, we still conjectured a result in the general case, and we established several partial results supporting this claim.

In this paper, we aim to provide an (almost) complete answer in the general case, and thus to answer this conjecture. The novelty in our approach is that we first consider a different setting in which we study *non-compact* Gelfand pairs. Indeed, if G is a semisimple Lie group with finite center and K a maximal compact subgroup of G , then (G, K) is a Gelfand pair. The quotient space G/K also has a natural structure of Riemannian symmetric space (of non-compact type). In this framework, a lot is known about the spherical functions of the pair ([Hel79], [Hel00]).

The key point is that every such symmetric space G/K has a simply connected dual of compact type, which is of the form U/K , where U is a compact semisimple Lie group containing K as a subgroup. The pair (U, K) is a Gelfand pair, as studied

in [Dum24]. Since both G and U can be realized as subgroups of the same complex Lie group $G_{\mathbb{C}}$, we will be able to relate the behavior on U to that on G through complex analysis.

The first goal of the paper is to find the optimal value $(r, \delta) \in \mathbb{N} \times [0, 1]$ such that any K -finite matrix coefficient of G is in the Hölder space $C^{(r, \delta)}(G_r)$ (see Section 2.1 for the precise definition of these Hölder spaces). Here, G_r is the dense open subset of regular points of G (see Proposition 3.1 and afterwards for the definition). We point out that the restriction to regular points is essential, as our results fail outside of G_r (see Remark 4.7). However, this is not a serious limitation in applications, in particular, this set was sufficient in applications such as in [Laf08].

Given a Cartan involution of $\mathfrak{g}$, we have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, we can consider the associated root system $\Sigma \subset \mathfrak{a}^*$ and a choice of positive roots Σ^+ (see Section 2.3 for more details, in particular for definitions of those objects). For $\lambda \in \mathfrak{a}^*$, define

$$n(\lambda) = \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha)$$

and set

$$\kappa(G) = \inf_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{n(\lambda)}{2}.$$

Note that this value, which at this point may seem arbitrary, will arise geometrically later as the minimal codimension of a certain critical manifold

Our first main result provides a complete answer to the question of regularity in the non-compact setting (Theorem 4.6 in the text).

Theorem A. *Let G be a semisimple Lie group with finite center and K a maximal compact subgroup of G . Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of G is in $C^{(r, \delta)}(G_r)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant coefficient of a unitary representation of G which is not in $C^{(r, \delta')}(G_r)$.*

As explained above, the idea is to first prove this result for K -bi-invariant coefficients only. In that case, we use the theory of decomposition of a representation into a direct integral to reduce this problem to a question of boundedness of positive definite spherical functions in Hölder spaces (Lemmas 2.6 and 2.7). In the setting of semisimple Lie groups, these spherical functions have a well-known expression as an oscillatory integral over the maximal compact subgroup K ([Hel00, Ch. IV]). The key input is then the use of the stationary phase approximation to obtain estimates on these integrals, using the results from [DKV83]. The last step is to prove that the regularity obtained for K -bi-invariant coefficients can be enhanced to all K -finite coefficients. The arguments for this step follow the same idea as in [Dum24, Section 5] for compact semisimple Lie groups, but some steps are more involved due to the lack of a rich finite-dimensional representation theory (Section 3.2).

If $G_{\mathbb{C}}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and U the analytic subgroup corresponding to the subalgebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, it is known that U is a compact simply connected semisimple Lie group and $K \subset U$. As explained above, (U, K) is a Gelfand pair and U/K is a symmetric space of compact type. Our second goal in

this paper is to find the optimal value $(r, \delta) \in \mathbb{N} \times [0, 1]$ such that any K -finite matrix coefficient of G is in the Hölder space $C^{(r, \delta)}(U_r)$. As already mentioned, the dual version of Theorem A was obtained in special cases, and we conjectured that the optimal regularity of K -finite coefficients on the subset of regular points U_r should also be $\kappa(G)$ (see [Dum24, Conjecture 1.3] or Conjecture 5.9 for the statement). With the tools of stationary phase approximation developed for the non-compact case and some complex analysis, we generalize these results and obtain a complete answer, but in an open subset smaller than U_r (Theorems 5.8 and 5.15 in Section 5).

Theorem B. *Let G be a semisimple Lie group with finite center and (U, K) defined as above. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then, there exists an open subset $V \subset U_r$ such that any K -finite matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(V)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient which is not in $C^{(r, \delta')}(U_r)$.*

This result provides a definitive answer to the question in the context of symmetric pairs, originally motivated by compact groups. In light of the application discussed earlier, restricting to the open subset V is of little consequence, as none of the results require the use of the entire set U_r .

The idea is to use analytic continuation to obtain an expression of spherical functions as an oscillatory integral (Lemma 5.4), which allows us to use the stationary phase approximation. Since the phase function is now complex-valued, the proof is more involved. Theorem B cannot be extended to all of U_r , because we only have such an oscillatory integral in a neighborhood of the identity. However, this still shows that the conjecture ([Dum24, Conjecture 1.3]) is true at least in some open subset. Furthermore, given some other partial results obtained in [Dum24], we also get the full conjecture for several new families of groups (Corollary 5.10).

We point out that in this statement, the compact group U must be simply connected. However, we can obtain from this theorem a general result with no assumptions on the fundamental group of U (Theorem 5.18).

Some of the ideas in the compact case appeared in [Cle88]. In this paper, Clerc actually gives an expression as an oscillatory integral for any point of U . However, the domain of integration lacks compactness. Thus, he could only get estimates for some specific subfamily of positive definite functions, which is not enough for our purpose. On the contrary, we obtain uniform bounds on the Hölder norms of all spherical functions of symmetric Gelfand pairs of compact type.

Organisation of the paper. Section 2 contains preliminaries on Gelfand pairs and spherical functions, especially in the context of semisimple Lie groups. In Section 3, we reduce the main question from K -finite coefficients to K -bi-invariant (Theorem 3.7) and we explain how to work at the level of the Lie algebra of G . Section 4 is devoted to the proof of Theorem A in the case of K -bi-invariant coefficients. Finally, in Section 5, we study the same questions for compact symmetric pairs and prove Theorem B.

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2. PRELIMINARIES

2.1. Hölder spaces.

Definition 2.1. Let (X, d) be a metric space and U an open subset of X , $(E, \|\cdot\|)$ a normed vector space, $\alpha \in]0, 1]$. A function $f : U \rightarrow E$ is α -Hölder continuous if for any compact subset K of U , there is $C_K > 0$ such that

$$\forall x, y \in K, \|f(x) - f(y)\| \leq C_K d(x, y)^\alpha.$$

If X is also a normed vector space and $r \in \mathbb{N}$, we say that the map f belongs to $C^{(r, \alpha)}(U, E)$ if $f \in C^r(U, E)$ and the r -th differential $D^r f$ is α -Hölder continuous as a map from U to the vector space of multilinear r -forms. We extend to $\alpha = 0$ by $C^{(r, 0)}(U, E) = C^r(U, E)$.

For K a compact subset of U and $f \in C^{(r, \alpha)}(U, E)$, define

$$\|f\|_{C^{(r, \alpha)}(K, E)} = \max \left\{ \max_{k \leq r} \sup_{x \in K} \|D^k f(x)\|, \sup_{x, y \in K, x \neq y} \frac{\|D^r f(x) - D^r f(y)\|}{d(x, y)^\alpha} \right\}.$$

The family of semi-norms $\|\cdot\|_{C^{(r, \alpha)}(K, E)}$ for K a compact subset of U makes the space $C^{(r, \alpha)}(U, E)$ into a Fréchet space.

Finally, if (X, d) is a Riemannian manifold, we say that $f \in C^{(r, \alpha)}(U, E)$ if for any chart (φ, V) of U , $f \circ \varphi^{-1} \in C^{(r, \alpha)}(\varphi(V), E)$.

Remark 2.2. If U is locally compact, a function $f : U \rightarrow E$ is α -Hölder continuous if and only if for any $x \in U$, there exists a neighbourhood U_x of x and a constant $C_x > 0$ such that for any $y, z \in U_x$, $\|f(y) - f(z)\| \leq C_x d(y, z)^\alpha$.

We will denote $C^{(r, \alpha)}(U, \mathbb{C})$ by $C^{(r, \alpha)}(U)$.

The following lemma will be useful throughout the article and can be found in [Dum24, Lemma 2.1]

Lemma 2.3. *Let (X, d) and (Y, d') be two Riemannian manifolds and U, V open subsets of X, Y respectively. Let $\alpha > 0$ and $r \in \mathbb{N}$. Let $\varphi : U \rightarrow V$ be a function of class C^∞ . Then $\varphi_* : f \mapsto f \circ \varphi$ maps $C^{(r, \alpha)}(V)$ to $C^{(r, \alpha)}(U)$ and is continuous.*

2.2. Gelfand pairs.

Definition 2.4. Let G be a locally compact topological group with a left Haar measure dg and K a compact subgroup with normalized Haar measure dk . The pair (G, K) is a Gelfand pair if the algebra of continuous K -bi-invariant functions on G with compact support is commutative for the convolution.

A spherical function of (G, K) is a continuous K -bi-invariant non-zero function on G such that for all $x, y \in G$,

$$\int_K \varphi(xky) dk = \varphi(x)\varphi(y).$$

A standard result (see [vD09, Coro. 6.3.3]) gives a link between spherical functions of (G, K) and unitary representations of G .

Proposition 2.5. *If (G, K) is a Gelfand pair, then for any irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$, the subspace $\mathcal{H}^K$ of K -invariant vectors is of dimension at most 1.*

The positive definite spherical functions of G are exactly the matrix coefficients $g \mapsto \langle \pi(g)v, v \rangle$ where π is an irreducible unitary representation of G and v a K -invariant unit vector.

If G is compact, any spherical function is positive definite.

More details on Gelfand pairs can be found in [vD09, Ch. 5,6,7].

Given a Gelfand pair (G, K) , it is natural to study spherical functions in order to get results on K -bi-invariant matrix coefficients of unitary representations. Indeed, any matrix coefficient of a unitary representation decomposes into an integral of spherical functions - an infinite sum if G is compact. Then studying the boundedness of positive definite spherical functions in some Hölder spaces is enough to obtain regularity for all K -bi-invariant matrix coefficients of unitary representations. More precisely, the optimal regularity of such coefficients is exactly the optimal uniform regularity of spherical functions. The proof of the following two lemmas can be found in [Dum24, Section 2.2].

Lemma 2.6. *Let (G, K) be a Gelfand pair with G second countable. Let φ be a K -bi-invariant matrix coefficient of a unitary representation π on a Hilbert space $\mathcal{H}$. Then, there exists a standard Borel space X and a σ -finite measure μ on X such that*

$$\varphi = \int_X c_x \varphi_x d\mu(x)$$

where φ_x is a positive definite spherical function of (G, K) for any $x \in X$ and $c \in L^1(X, \mu)$.

Let $\widehat{G}_K$ be the unitary spherical dual of (G, K) , that is to say the set of all isomorphism classes of irreducible unitary representations of G with a nonzero K -invariant vector. It follows immediately from Proposition 2.5 that there is a bijection between the family of positive definite spherical functions of the Gelfand pair (G, K) and $\widehat{G}_K$. For $\lambda \in \widehat{G}_K$, we will thus denote φ_λ the associated (positive definite) spherical function.

Lemma 2.7. *Let (G, K) be a Gelfand pair with G a Lie group endowed with a Riemannian metric d and U any open subset of G . Let $(\varphi_\lambda)_{\lambda \in \widehat{G}_K}$ be the family of positive definite spherical functions of (G, K) . Then $(\varphi_\lambda)_{\lambda \in \widehat{G}_K}$ is bounded in $C^{(r, \delta)}(U)$ if and only if any K -bi-invariant matrix coefficient of a unitary representation of G is in $C^{(r, \delta)}(U)$.*

2.3. Spherical functions of semisimple Lie groups. Let G be a connected real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Note that all results will apply to complex semisimple Lie groups, by viewing them naturally as real Lie groups. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ in ± 1 -eigenspaces of θ . Then $K = \exp \mathfrak{k}$ is a maximal compact subgroup of G . Since all maximal compact subgroups of G are conjugated, whenever we say “a maximal compact subgroup of G ” in the sequel, we always assume that it is defined by a Cartan involution. Consider $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. The rank of G is $\text{rank } G = \dim \mathfrak{a} = \ell$. For $\alpha \in \mathfrak{a}^*$, define $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}, [H, X] = \alpha(H)X\}$ the root space associated to α . Let $m(\alpha) = \dim(\mathfrak{g}^\alpha)$ and $\Sigma = \{\alpha \neq 0 \mid m(\alpha) \geq 1\}$ be the set of roots. We say that Σ is the restricted root system of G . Let $\mathfrak{m} = \mathfrak{g}^0 \cap \mathfrak{k}$.

Then the Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha.$$

The Killing form of $\mathfrak{g}$ induces an inner product on $\mathfrak{a}$, denoted $\langle \cdot, \cdot \rangle$. Then for $\lambda \in \mathfrak{a}^*$, there is a unique $H_\lambda \in \mathfrak{a}$ such that for any $H \in \mathfrak{a}$, $\lambda(H) = \langle H_\lambda, H \rangle$. We use the isomorphism $\lambda \mapsto H_\lambda$ to define an inner product on $\mathfrak{a}^*$ by

$$\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle.$$

Let W be the Weyl group of the root system Σ , which is the subgroup of $O(\mathfrak{a}^*)$ generated by the reflections $s_\alpha : x \mapsto x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$. The group W also acts on $\mathfrak{a}$ by $wH_\lambda = H_{w\lambda}$. By [Var13, Thm. 4.3.24], this action can be extended to automorphisms of the Lie algebra $\mathfrak{g}$. In particular, we get that $\mathfrak{g}^{w\alpha} = w(\mathfrak{g}^\alpha)$ and so $m(w\alpha) = m(\alpha)$. The hyperplanes $\{\alpha(H) = 0\}$ divide $\mathfrak{a}$ into $|W|$ connected components. We choose one, which we denote $\mathfrak{a}^+$ and call the positive Weyl chamber, and we define the positive roots $\Sigma^+ = \{\alpha \in \Sigma \mid \forall H \in \mathfrak{a}^+, \alpha(H) > 0\}$. Then $\Sigma = \Sigma^+ \cup (-\Sigma^+)$. We say that $\alpha \in \Sigma^+$ is simple if it cannot be decomposed as $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Sigma^+$. Let Δ be the set of simple roots. Then Δ is a basis of $\mathfrak{a}^*$ and we can write $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Given $\alpha \in \Sigma^+$, $\alpha = \sum_{i=1}^\ell n_i(\alpha) \alpha_i$ with $n_i(\alpha) \in \mathbb{N}$. Furthermore, the group W is generated by the reflections $\{s_\alpha\}_{\alpha \in \Delta}$ ([Bou07, Ch. VI, Thm. 2]). For any $\alpha \in \Delta$, the reflection s_α permutes the positive roots that are not proportional to α ([Bou07, Ch. VI, Prop. 17]).

For $\lambda \in \mathfrak{a}^*$, define

$$(2.1) \quad n(\lambda) = \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha)$$

and set

$$(2.2) \quad \kappa(G) = \inf_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{n(\lambda)}{2}.$$

We can express κ in a more computable way. First, notice that n is invariant under the Weyl group. Since every orbit under the Weyl group meets $\mathfrak{a}^+$, $\kappa(G)$ is also the infimum over $\lambda \neq 0$ such that $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Sigma^+$. For such a λ , if $\alpha = \beta + \gamma$ with $\alpha, \beta, \gamma \in \Sigma^+$, then $\langle \alpha, \lambda \rangle = 0$ implies $\langle \beta, \lambda \rangle = 0 = \langle \gamma, \lambda \rangle$. Thus, if $n(\lambda)$ is minimal, there is a unique $\alpha_i \in \Delta$ such that $\langle \alpha_i, \lambda \rangle \neq 0$. Therefore, we get

$$(2.3) \quad \kappa(G) = \frac{1}{2} \min_{1 \leq i \leq \ell} \sum_{\substack{\alpha \in \Sigma^+ \\ n_i(\alpha) \geq 1}} m(\alpha).$$

The values of κ were computed for simple groups in [Dum24, Section 4.3, Tab. 3], in which $\kappa(G)$ is denoted $r(M)$, where M is the compact dual of the symmetric space G/K . For the sake of completeness, we reproduce this table in Appendix A with the non-compact group G as a label. Note that $\kappa(G)$ can also be interpreted geometrically as half of the minimum dimension of G/P where P is a parabolic subgroup of G .

Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Denote $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$ and $N = \exp \mathfrak{n}$. The Iwasawa decomposition says that

$$(2.4) \quad \begin{array}{ccc} K \times A \times N & \rightarrow & G \\ (k, a, n) & \mapsto & kan \end{array}$$

is a diffeomorphism ([Kna02, Thm. 6.46]). Let $H : G \mapsto \mathfrak{a}$ be the Iwasawa projection, that is, the unique map such that $\forall g \in G, g \in K \exp H(g)N$. Then H is smooth.

The pair (G, K) is a Gelfand pair. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m(\alpha)\alpha$. Let $\mathfrak{a}_{\mathbb{C}}^*$ be the space of $\mathbb{R}$ -linear forms on $\mathfrak{a}_{\mathbb{C}}$. Then the spherical functions of (G, K) are

$$(2.5) \quad \varphi_\lambda : g \mapsto \int_K e^{(i\lambda - \rho)(H(gk))} dk$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, where dk is the Haar measure on K with total mass 1 ([Hel00, Ch. IV, Thm. 4.3]).

Since we want to study unitary matrix coefficients, we only need to consider positive definite spherical functions. In particular, we want to determine for which λ the function φ_λ is positive definite. A complete answer is only known in a few special cases, see for example [Kos69, FJK79] in rank 1. In general, it is still an open problem to completely classify such λ , which is related to understanding the unitary dual of G . However, some partial results are known. The following proposition can be found in [Hel00, Section IV, Thm. 8.1 and B.9.(i)] and will be sufficient for our purposes.

Proposition 2.8. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, then:*

- φ_λ is bounded if and only if $\text{Im } \lambda \in C = \text{Conv}(W\rho)$,
- If φ_λ positive definite, then φ_λ bounded by $\varphi(e) = 1$,
- If $\text{Im } \lambda = 0$, then φ_λ is positive definite.

3. K -FINITE MATRIX COEFFICIENTS

3.1. K -bi-invariant functions on Lie groups vs. Lie algebras. We collect in this subsection some results on decompositions of semisimple groups that we need to study K -finite coefficients. Given the previous notations, a semisimple Lie group with finite center has a KAK decomposition. This means that the K -bi-invariant functions can be studied as functions on A . The following proposition is [Kna02, Thm. 7.39].

Proposition 3.1 (KAK decomposition). *For any $g \in G$, there exists $k_1, k_2 \in K$ and a unique $a \in \overline{A^+}$ such that $g = k_1 a k_2^{-1}$. Furthermore, if $a \in A^+$, k_1 is unique up to multiplication on the right by an element of $M = Z_K(A)$.*

Denote $G_r = KA^+K$. Then G_r is a dense open subset of G which we call the set of regular elements. Let also $P : G \rightarrow \mathfrak{a}^+$ be the map such that for any $g \in G$, $g \in K \exp P(g)K$, which is well-defined by the above proposition. Note that by definition, $P(G_r) = \mathfrak{a}^+$.

Let φ be a K -bi-invariant function on G . Let also $\psi = \varphi \circ \exp|_{\overline{\mathfrak{a}^+}}$. Then $\varphi = \psi \circ P$ by K -bi-invariance. Hence, we can study ψ and recover information on φ using P . Since we want to study regularity of functions, we will use Lemma 2.3. However, P is not smooth on G , but we will show that it is smooth on G_r . We will then only obtain results on the regular points G_r .

Lemma 3.2. *The map*

$$q : \begin{array}{ccc} K \times K \times \mathfrak{a}^+ & \rightarrow & G_r \\ (k_1, k_2, H) & \mapsto & k_1 \exp(H) k_2^{-1} \end{array}$$

is a submersion.

Proof. If $g \in G$, denote L_g and R_g the translations by g on the left and right respectively. Let $m : G \times G \rightarrow G$ be the multiplication map, its differential at (a, b) is

$$T_{(a,b)}m : \begin{array}{ccc} T_a G \times T_b G & \rightarrow & T_{ab} G \\ (X_a, X_b) & \mapsto & T_a R_b(X_a) + T_b L_a(X_b) \end{array}.$$

We can identify $T_g G$ with $\mathfrak{g}$ by the isomorphism $T_e L_g$. Under this identification, we have $\forall g, h \in G$, $T_h L_g = \text{Id}$ and $T_h R_g = \text{Ad}(g^{-1})$, so that the tangent map becomes $T_{(a,b)}m(X_a, X_b) = \text{Ad}(b^{-1})(X_a) + X_b$. Furthermore, if $k \in K$, since $L_k(K) = K$, $T_k K \subset T_k G$ is identified with $\mathfrak{k} \subset \mathfrak{g}$. Thus by the chain rule we have

$$T_{(k_1, k_2, H)}q : \begin{array}{ccc} \mathfrak{k} \times \mathfrak{k} \times \mathfrak{a} & \rightarrow & \mathfrak{g} \\ (X_1, X_2, Y) & \mapsto & \text{Ad}(k_2)(\text{Ad}(\exp(-H))(X_1) + T_H \exp(Y)) - X_2 \end{array}.$$

We know that $\text{Ad}(k)$ is an isomorphism of $\mathfrak{g}$ and an isomorphism of $\mathfrak{k}$ in restriction. Furthermore, $T_H \exp : \mathfrak{a} \mapsto \mathfrak{a}$ is also an isomorphism. Thus, the map $T_{(k_1, k_2, H)}q$ is surjective if and only if $u = \text{Ad}(k_2^{-1}) \circ T_{(k_1, k_2, H)}q \circ (\text{Id}, \text{Ad}(k_2), (T_H \exp)^{-1})$ is surjective. We have

$$u(X_1, X_2, Y) = \text{Ad}(\exp(-H))(X_1) - X_2 + Y.$$

For $\alpha \in \Sigma^+$, let $\mathfrak{k}^\alpha = \mathfrak{k} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$ and $\mathfrak{p}^\alpha = \mathfrak{p} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$. From [Loo69, Ch. VI, Prop. 1.4], we get

$$\begin{aligned} \mathfrak{k} &= \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{k}^\alpha = \mathfrak{m} \oplus \mathfrak{l}, \\ \mathfrak{p} &= \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}^\alpha = \mathfrak{a} \oplus \mathfrak{b}. \end{aligned}$$

We also get that for $\alpha \in \Sigma^+$, there exists $Z_{\alpha,1}, \dots, Z_{\alpha,m(\alpha)}$ basis of $\mathfrak{g}^\alpha$, such that setting $Z_{\alpha,i}^+ = Z_{\alpha,i} + \theta(Z_{\alpha,i})$ and $Z_{\alpha,i}^- = Z_{\alpha,i} - \theta(Z_{\alpha,i})$, $\{Z_{\alpha,i}^+\}$ is a basis of $\mathfrak{k}_\alpha$ and $\{Z_{\alpha,i}^-\}$ is a basis of $\mathfrak{p}_\alpha$.

Let also $H_1, \dots, H_\ell$ be a basis of $\mathfrak{a}$ and $Y_1, \dots, Y_r$ a basis of $\mathfrak{m}$. Then for $H \in \mathfrak{a}$, we have $[H, Y_i] = 0$, $[H, Z_{\alpha,i}^+] = \alpha(H)Z_{\alpha,i}^-$ and $[H, Z_{\alpha,i}^-] = \alpha(H)Z_{\alpha,i}^+$. Thus, we see that

- $u(0, 0, H_i) = H_i$,
- $u(Y_i, 0, 0) = e^{-\text{ad}(H)}(Y_i) = Y_i$,
- $u(0, Y_i, 0) = -Y_i$,
- $u(Z_{\alpha,i}^+, 0, 0) = e^{-\text{ad}(H)}(Z_{\alpha,i}^+) = \cosh(\alpha(H))Z_{\alpha,i}^+ - \sinh(\alpha(H))Z_{\alpha,i}^-$,
- $u(0, Z_{\alpha,i}^+, 0) = -Z_{\alpha,i}^+$.

Since $H \in \mathfrak{a}^+$, $\alpha(H) \neq 0$ for any $\alpha \in \Sigma^+$ and u is indeed surjective. $\square$

Proposition 3.3. *The map $P : G \rightarrow \overline{\mathfrak{a}^+}$ is smooth on G_r . Furthermore, for each $g \in G_r$, there exists a neighbourhood U_g of g in G_r and a choice of $g \mapsto k_i(g)$ such that k_i is smooth on U_g , $i = 1, 2$ and for any $g \in U_g$, $g = k_1(g) \exp(P(g)) k_2(g)^{-1}$.*

Proof. The proof is identical to [Dum24, Prop. 5.8] using the previous proposition. $\square$

Corollary 3.4. *Let φ be a K -bi-invariant function on G , then $\varphi \in C^{(r,\delta)}(G_r)$ if and only if $\varphi \circ \exp \in C^{(r,\delta)}(\mathfrak{a}^+)$.*

Furthermore, if $(\varphi_i)_{i \in I}$ is a family of K -bi-invariant function on G , then (φ_i) is bounded in $C^{(r,\delta)}(G_r)$ if and only if $(\varphi_i \circ \exp)$ is bounded in $C^{(r,\delta)}(\mathfrak{a}^+)$.

Proof. This is a direct consequence of Lemma 2.3 and Proposition 3.3. $\square$

3.2. K -finite matrix coefficients. In this section, we want to show that the optimal regularity of all K -bi-invariant matrix coefficients of unitary representations of G coincides with the optimal regularity of all K -finite matrix coefficients of unitary representations. The ideas are very similar to [Dum24, Section 5] where the same result is shown for compact symmetric pairs. However, this proof relied heavily on [dLMdlS16, Lemma 2.2], which only works in the compact case as it extensively uses finite dimensional representations. Thus, the first step of our proof is to show a similar lemma in a general setting.

Definition 3.5. Let π be a unitary representation of G on $\mathcal{H}$ and (ρ, V) a representation of K . We say that $\xi \in \mathcal{H}$ is

- K -finite if $\text{span}(\pi(K)\xi)$ is finite-dimensional,
- of K -type V if $\text{span}(\pi(K)\xi) \simeq V$ as a representation of K .

Note that this definition of K -type V is not standard.

Let G be any Lie group and K a compact subgroup of G . If π is a unitary representation of G on a Hilbert space $\mathcal{H}$, we say that $\xi \in \mathcal{H}$ is a smooth vector if $g \mapsto \pi(g)\xi$ is a smooth map from G to $\mathcal{H}$. Clearly, matrix coefficients associated to smooth vector are smooth. Consider the Gårding subspace of π , denoted $\mathcal{H}_\infty$, which is the subspace of $\mathcal{H}$ spanned by all vectors of the form

$$\pi(f)\xi = \int_G f(g)\pi(g)\xi dg$$

for $\xi \in \mathcal{H}, f \in C_c^\infty(G)$, where dg is a left-invariant Haar measure. Then it is well-known that $\mathcal{H}_\infty$ is dense in $\mathcal{H}$ and every vector of $\mathcal{H}_\infty$ is smooth ([Kna01, Prop. 3.14, Thm. 3.15]). In fact, it is a deep result of Dixmier and Malliavin ([DM78]) that the Gårding subspace coincides with the space of smooth vectors, but we do not need this. Note also that $\mathcal{H}_\infty$ is stable under $\pi(\mu)$ for any μ complex measure on G with compact support.

Let $U = K \times K$. Let (ρ, E) be a finite-dimensional unitary representation of U . For $g \in G$, let U_g be the stabilizer of g under the action of U by left-right multiplication, i.e.

$$U_g = \{(k_1, k_2) \in U \mid k_1 g k_2^{-1} = g\}.$$

Let $E_g = E^{U_g}$ the subspace of vectors fixed by U_g , and $P_g : E \rightarrow E_g$ the orthogonal projection.

Lemma 3.6. *For any $g_0 \in G$, there exists a smooth function $\psi : G \rightarrow B(E)$ such that*

- (1) $\forall u \in U, g \in G, \psi(u.g) = \psi(g) \circ \rho(u)^{-1}$,
- (2) $\forall v_1, v_2 \in E, g \mapsto \langle \psi(g)v_1, v_2 \rangle$ is a matrix coefficient of a unitary representation of G ,
- (3) $\psi(g_0) = P_{g_0}$.

Proof. Let F be the set of all functions $\phi : G \rightarrow B(V)$ such that for any $v_1, v_2 \in E$, the map $g \mapsto \langle \phi(g)v_1, v_2 \rangle$ is a matrix coefficients of a unitary representation of G with vectors in the Gårding subspace of that representation. Then such a map F is smooth and verifies (2).

If $\phi \in F$, define $\psi(g) = \int_U \phi(u.g)\rho(u)du$ where du is the Haar probability measure of U . Since U is compact and ϕ smooth, ϕ is bounded on the compact orbit $U.g$ so the integral makes sense. Then a simple change of variables shows that ψ verifies (1). Let $e_1, \dots, e_d$ be an orthonormal basis of E . Let π_{ij} be unitary representations of G and ξ_{ij}, η_{ij} vectors in the Gårding subspace of π_{ij} such that for all $1 \leq i, j \leq d$ and $g \in G$,

$$\langle \phi(g)e_i, e_j \rangle = \langle \pi_{ij}(g)\xi_{ij}, \eta_{ij} \rangle.$$

Let $u = (k_1, k_2) \in U$, then $\rho(u) = \rho(k_1, 1_K)\rho(1_K, k_2)$. We define functions $K \rightarrow \mathbb{C}$ such that

$$\rho(1_K, k)e_i = \sum_{j=1}^d \lambda_{ij}(k)e_j$$

and

$$\rho(k, 1_K)e_i = \sum_{j=1}^d \mu_{ij}(k)e_j.$$

Then

$$\begin{aligned} & \langle \psi(g)e_i, e_j \rangle \\ &= \int_U \langle \phi(u.g)\rho(u)e_i, e_j \rangle du \\ &= \int_{K \times K} \sum_{p,q=1}^d \mu_{pq}(k_1)\lambda_{ip}(k_2)\langle \phi(k_1 g k_2^{-1})e_q, e_j \rangle dk_1 dk_2 \\ &= \int_{K \times K} \sum_{p,q=1}^d \mu_{pq}(k_1)\lambda_{ip}(k_2)\langle \pi_{qj}(k_1 g k_2^{-1})\xi_{qj}, \eta_{qj} \rangle dk_1 dk_2 \\ &= \sum_{p,q=1}^d \int_{K \times K} \langle \pi_{qj}(g) (\lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj}), \overline{\mu_{pq}(k_1)}\pi_{qj}(k_1^{-1})\eta_{qj} \rangle dk_1 dk_2 \\ &= \sum_{p,q=1}^d \left\langle \pi_{qj}(g) \left(\int_K \lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj} dk_2 \right), \left(\int_K \overline{\mu_{pq}(k_1)}\pi_{qj}(k_1^{-1})\eta_{qj} dk_1 \right) \right\rangle. \end{aligned}$$

Since ξ_{qj} is in the Gårding subspace of π_{qj} , the same is true for the vectors $\int_K \lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj} dk_2$ and $\int_K \overline{\mu_{pq}(k_1)}\pi_{qj}(k_1^{-1})\eta_{qj} dk_1$. Thus, $\langle \psi(g)e_i, e_j \rangle$ is a matrix coefficient of $\bigoplus_{q=1}^d \pi_{qj}^{\oplus d}$ whose vectors are in the Gårding subspace. By linearity, this is true for $v_1, v_2 \in E$. Thus, we showed that if $\phi \in F$, then $\psi \in F$, hence ψ is smooth and verifies (2).

It remains to show that there exists $\phi \in F$ such that $\psi(g_0) = P_{g_0}$. Notice that if $u \in U_{g_0}$, then

$$\psi(g_0) = \psi(u.g_0) = \psi(g_0) \circ \rho(u)^{-1}.$$

Thus $E_{g_0}^\perp = \sum_{u \in U_{g_0}} \text{Im}(\rho(u) - \text{Id}) \subset \ker \psi(g_0)$. Thus, condition (3) is a "maximal rank" condition.

First, let us find $\phi \in F$ such that $\text{rank } \psi(g_0) = \dim E_{g_0}$. Consider $O \simeq U/U_{g_0}$ the U -orbit of g_0 in G . Let s be a measurable section, that is to say $s : O \mapsto U$ such that $s(u.g_0) \in uU_{g_0}$. Let $h : O \mapsto B(E)$ be the map $x \mapsto \rho(s(x))^{-1}$. Then $\int_U h(u.g_0)\rho(u) du = \int_U \rho(s(u.g_0)^{-1}u) du$ is the identity on E_{g_0} , and by the above discussion 0 on $E_{g_0}^\perp$. Thus $\int_U h(u.g_0)\rho(u) du = P_{g_0}$. Let μ be the image of the Haar measure on O by the map $p : u \mapsto u.g_0$. Then $h \in L^1(O; B(E), \mu)$. By density of continuous function, there are continuous maps $f : O \rightarrow B(E)$ arbitrarily close to h in $\|\cdot\|_1$. But then,

$$\begin{aligned} \left\| \int_U f(u.g_0)\rho(u)du - \int_U h(u.g_0)\rho(u)du \right\| &\leq \int_U \|f(u.g_0) - h(u.g_0)\| du \\ &\leq \int_U \|(f - h) \circ p\| du \\ &\leq \int_O \|f - h\| d\mu \\ &\leq \|f - h\|_1. \end{aligned}$$

So we can take f close enough so that $\text{rank } \int_U f(u.g_0)\rho(u)du = \text{rank } P_{g_0}$. Then since O is closed in G normal, by Tietze extension theorem, we can extend f to a continuous map $G \mapsto B(E)$, that we still denote f .

Let L be a compact subset of G containing g_0 and $\varepsilon > 0$. Let $f_{ij} : g \mapsto \langle f(g)e_i, e_j \rangle$. By the Gelfand-Raikov theorem ([Fol16, Thm. 3.34], the (irreducible) unitary representations of G separate points. An application of Stone-Weierstrass theorem shows that matrix coefficients of unitary representations (restricted to L) are dense in the space of continuous functions on G . Thus, for any i, j , there exists a unitary representation π_{ij} of G and a matrix coefficient ϕ_{ij} of π_{ij} such that

$$\sup_{g \in L} |f_{ij}(g) - \phi_{ij}(g)| \leq \varepsilon.$$

Then by density of the Gårding subspace, we may find a coefficient $\tilde{\phi}_{ij}$ of π_{ij} whose vectors are in the Gårding subspace of π_{ij} and such that

$$\sup_{g \in L} |\phi_{ij}(g) - \tilde{\phi}_{ij}(g)| \leq \varepsilon.$$

Then, define $\phi : G \rightarrow B(E)$ by $\phi(g)e_i = \sum_{j=1}^d \tilde{\phi}_{ij}(g)e_j$, by construction $\phi \in F$. So we can find ϕ in F arbitrarily close to f on any compact subset containing g_0 , in particular on the orbit O . Thus, for ε small enough, $\int_U \phi(u.g_0)\rho(u)du$ is of rank $\dim E_{g_0}$.

Finally, we get $\phi \in F$ such that $\psi(g_0)$ has rank $\dim E_{g_0}$ and is zero on $E_{g_0}^\perp$. Thus there is $A \in B(E)$ such that $A\psi(g_0) = P_{g_0}$. Replace ϕ by $A\phi$ and we get the result. $\square$

We now return to our setting and finally prove the reduction to K -bi-invariant matrix coefficients.

Theorem 3.7. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G . Assume that any K -bi-invariant matrix coefficient of unitary representations of G belongs to $C^{(r,\delta)}(G_r)$. Let π be a unitary representation of G and φ a K -finite matrix coefficient of π . Then $\varphi \in C^{(r,\delta)}(G_r)$.*

Proof. Let $\mathcal{H}$ be the representation space and $\xi, \eta \in \mathcal{H}$ K -finite vectors such that $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$.

We first assume that $\xi, \eta \in \mathcal{H}$ are of K -type V, W respectively, for V, W irreducible representations of K . Denote $V_\xi = \text{span}(\pi(K)\xi)$. Then there is an isomorphism $i_\xi : V \rightarrow V_\xi \subset \mathcal{H}$, denote $\xi_0 = i_\xi^{-1}(\xi)$. Similarly, define V_η, i_η and η_0 . Then the map

$$(3.1) \quad f : \begin{array}{ccc} B(\mathcal{H}) & \rightarrow & L(V, W^*) \simeq V^* \otimes W^* \\ A & \mapsto & i_\eta^* A i_\xi \end{array}$$

is $K \times K$ equivariant.

For the associated matrix coefficient, we have $\varphi(g) = \langle \pi(g)\xi, \eta \rangle = \langle f(\pi(g))\xi_0, \eta_0 \rangle$.

Now denote (ρ, V_ρ) the irreducible representation of $U = K \times K$ on $V^* \otimes W^*$. The U -equivariance of f means that for any $(k, k') \in U$ and $A \in B(\mathcal{H})$, we have

$$(3.2) \quad f(\pi(k)A\pi(k')^{-1}) = \rho(k, k')(f(A)).$$

Furthermore, there are $v_1, \dots, v_n \in V_\rho$ and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$(3.3) \quad f(A) = \sum_{i=1}^n \langle A\xi_i, \eta_i \rangle v_i.$$

Let $g_0 \in G_r$ and ψ given by Lemma 3.6 for the representation (ρ, V_ρ) . Let $\tilde{f} : g \mapsto \psi(g)(f(\pi(g)))$. By (3.2) and (1) of Lemma 3.6, we have

$$(3.4) \quad \tilde{f}(u.g) = \psi(u.g)(f(\pi(u.g))) = \psi(g)\rho(u)^{-1}\rho(u)(f(\pi(g))) = \tilde{f}(g)$$

so $\tilde{f}$ is a K -bi-invariant map.

Let $(e_1, \dots, e_d)$ be an orthonormal basis of V_ρ , by (2) of Lemma 3.6 there are $(\pi_{ij}, \mathcal{H}_{ij})$ unitary representations of G and $a_{ij}, b_{ij} \in \mathcal{H}_{ij}$ such that

$$\langle \psi(g)v_i, e_j \rangle = \langle \pi_{ij}(g)a_{ij}, b_{ij} \rangle$$

so $\psi(g)v_i = \sum_{j=1}^d \langle \pi_{ij}(g)a_{ij}, b_{ij} \rangle e_j$ and finally with (3.3),

$$(3.5) \quad \tilde{f}(g) = \sum_{i=1}^n \sum_{j=1}^d \langle (\pi_{ij} \otimes \pi)(g)(a_{ij} \otimes \xi_i), b_{ij} \otimes \eta_i \rangle e_j.$$

Hence, $\tilde{f}$ is a sum of K -bi-invariant matrix coefficients of unitary representations of G , so by the hypothesis, $\tilde{f} \in C^{(r, \delta)}(G_r)$.

By Lemma 3.1, if $a \in A^+ = \exp \mathfrak{a}^+$, we have $U_a = \Delta(M)$. Thus, $V_a = V_0$ is independent of $a \in A^+$. If $g = (k_1, k_2).a = k_1 a k_2^{-1}$, we have $(k, k') \in U_g$ if and only if $(k_1^{-1} k k_1, k_2^{-1} k' k_2) \in \Delta(M)$ and so $V_g = \rho(k_1, k_2)V_0$.

Let $g_0 = k_0 a_0 k_0'^{-1}$ and $V_1 = V_{g_0}$. Since $\psi(g_0) = P_{g_0}$, there is an orthonormal basis adapted to V_1 such that

$$\psi(g_0) = \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, since ψ is smooth, there is A_{g_0} neighbourhood of g_0 such that

$$\psi(g) = \begin{pmatrix} A(g) & * \\ * & * \end{pmatrix}$$

with $g \mapsto A(g)$ smooth, $A(g)$ invertible for any $g \in A_{g_0}$. Up to restricting A_{g_0} , by Proposition 3.3, we have $g = k_1(g) \exp(P(g)) k_2(g)^{-1} = k_1(g) a(g) k_2(g)^{-1}$ with k_1, k_2 smooth on A_{g_0} .

By (3.4), for any $g \in A_{g_0}$, we have

$$\tilde{f}(g) = \tilde{f}(a(g)) = \tilde{f}(k_0 a(g) k_0'^{-1}).$$

But then $f(\pi(k_0 a(g) k_0'^{-1})) \in V_{k_0 a(g) k_0'^{-1}} = \rho(k_0, k_0') V_0 = V_1$. Set

$$\Phi(g) = \rho(k_1(g) k_0^{-1}, k_2(g) k_0'^{-1}) \begin{pmatrix} A(k_0 a(g) k_0'^{-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

it is a smooth map on A_{g_0} because A is smooth invertible, k_1, k_2 are smooth and ρ is a finite-dimensional representation of U thus smooth. Since $f(\pi(k_0 a(g) k_0'^{-1})) \in V_1$, we have

$$\begin{aligned} \Phi(g)(\tilde{f}(g)) &= \Phi(g)(\tilde{f}(\pi(k_0 a(g) k_0'^{-1}))) \\ &= \Phi(g) \psi(g)(f(\pi(k_0 a(g) k_0'^{-1}))) \\ &= \rho(k_1(g) k_0^{-1}, k_2(g) k_0'^{-1})(f(\pi(k_0 a(g) k_0'^{-1}))) \\ &= f(\pi(k_1(g) a(g) k_2(g)^{-1})) \\ &= f(\pi(g)) \end{aligned}$$

Now let $B : B(V) \times V \rightarrow V$ be the bilinear map sending (u, v) to $u(v)$. We showed that on A_{g_0} , $f \circ \pi = B \circ (\Phi, \tilde{f})$. Since Φ is smooth on A_{g_0} and $\tilde{f} \in C^{(r, \delta)}(G_r)$, we get by Leibniz formula that $f \circ g \in C^{(r, \delta)}(A_{g_0})$.

So for any $g_0 \in G_r$, there exists a neighbourhood A_{g_0} such that $f \circ \pi \in C^{(r, \delta)}(A_{g_0})$. Thus, $f \circ \pi \in C^{(r, \delta)}(G_r)$ and since $\varphi(g) = \langle f(\pi(g)) \xi_0, \eta_0 \rangle$, $\varphi \in C^{(r, \delta)}(G_r)$.

Finally in the general case, if ξ, η are K -finite, V_ξ, V_η are finite-dimensional representations of K , so they decompose into a finite number of irreducible representations. Thus, φ is a finite sum of matrix coefficients of the previous case, so $\varphi \in C^{(r, \delta)}(G_r)$. $\square$

Remark 3.8. We point out that this proof heavily relies on the nice behaviour of the KAK decomposition of G , which was also the case in [Dum24]. In the more recent paper [Dum26], a general setting was developed to obtain the reduction from K -finite coefficients to K -bi-invariant coefficients under some assumptions on a Gelfand pair, which is the existence of such a well-behaved KAK decomposition.

4. BOUNDEDNESS OF POSITIVE DEFINITE SPHERICAL FUNCTIONS

In this section, we study the spherical functions of the pair (G, K) and their Hölder norms to obtain Theorem A. We will use the estimates obtained by Duistermaat, Kolk and Varadarajan in [DKV83] using the method of stationary phase. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, we denote by $\psi_\lambda = \varphi_\lambda \circ \exp|_{\mathfrak{a}}$ the spherical functions on the Lie algebra.

Theorem 4.1. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Let C be a bounded subset of $\mathfrak{a}^*$. Then the family of spherical functions ψ_λ of (G, K) with $\text{Im } \lambda \in C$ is bounded in $C^{(r, \delta)}(\mathfrak{a}^+)$.*

Remark 4.2. By Proposition 2.8, this implies that the family of bounded spherical functions, thus the subfamily of positive definite spherical functions which we are interested in, is bounded in $C^{(r,\delta)}(\mathfrak{a}^+)$.

Proof. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we write $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$ and for $t \in \mathbb{R}$, $\lambda_t = t\xi + i\eta$. Denote $f(\xi, \eta, t, Y, k) = e^{(i\lambda_t - \rho)(H(\exp(Y)k))} \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^* \times \mathbb{R} \times \mathfrak{a} \times K)$. We denote D the operator which differentiates a function with respect to the variable $Y \in \mathfrak{a}$. Let $s \in \mathbb{N}$. Then for any $Y \in \mathfrak{a}$, $X = (X_1, \dots, X_s) \in \mathfrak{a}^s$,

$$(4.1) \quad D^s \psi_{\lambda_t}(Y)(X) = \int_K D^s f(\xi, \eta, t, Y, k)(X) dk.$$

By induction on s , there is a polynomial $P \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^* \times \mathfrak{a} \times K \times \mathfrak{a}^s)[t]$ of degree s such that

$$(4.2) \quad D^s f(\xi, \eta, t, Y, k)(X) = P(t) e^{it\xi(H(\exp(Y)k))}.$$

For $0 \leq j \leq s$, let $g_j(\xi, \eta, Y, X) \in C^\infty(K)$ be defined by

$$(4.3) \quad g_j = \frac{1}{j!} \frac{d^j}{dt^j} \left((D^s f) e^{-it\xi(H(\exp(Y)k))} \right) \Big|_{t=0}.$$

Then we have

$$(4.4) \quad D^s \psi_{\lambda_t}(Y)(X) = \sum_{j=0}^s t^j \int_K e^{it\xi(H(\exp(Y)k))} g_j(\xi, \eta, Y, X)(k) dk.$$

Let S be the unit sphere of $\mathfrak{a}^*$ and L a compact subset of $\mathfrak{a}^+$. Up to replacing L by its convex hull, which is still a compact subset of $\mathfrak{a}^+$ by Carathéodory's theorem and the convexity of $\mathfrak{a}^+$, we can assume that L is convex. We consider $C^\infty(K)$ endowed with the topology given by the family of seminorms $p_i : g \mapsto \sup_{k \in K} \|D^i g(k)\|$.

For $(Y, \xi) \in L \times S$, by [DKV83, Prop. 9.2], there exists a neighbourhood $V_{Y,\xi}$ of Y, ξ in $\mathfrak{a} \times \mathfrak{a}^*$ and a continuous seminorm $v_{Y,\xi}$ on $C^\infty(K)$ such that for any $(Y', \xi') \in V_{Y,\xi}$, $g \in C^\infty(K)$, $t \geq 1$,

$$(4.5) \quad \left| \int_K e^{it\xi'(H(\exp(Y')k))} g(k) dk \right| \leq v_{Y,\xi}(g) t^{-n(\xi)/2} \leq v_{Y,\xi}(g) t^{-\kappa(G)}.$$

Now write $\mathfrak{a}^* \times C = ([0, 1]S \times C) \cup (\mathbb{R}_{\geq 1}S \times C)$.

For any $(\xi, \eta) \in S \times C$, $0 \leq t \leq 1$, $Y \in L$, then (4.4) implies

$$(4.6) \quad \begin{aligned} \|D^s \psi_{\lambda_t}(Y)\| &= \sup_{\|X_i\|=1} |D^s \psi_{\lambda_t}(Y)(X)| \\ &\leq \sup_{\|X_i\|=1} \sum_{j=0}^s t^j \int_K \|g_j(\xi, \eta, Y, X)\|_\infty dk \\ &\leq \sup_{\substack{\|X_i\|=1 \\ Y \in L, \xi \in S, \eta \in C, k \in K}} \sum_{j=0}^s |g_j(\xi, \eta, Y, X)(k)| \\ &\leq C_{L,s,C} \end{aligned}$$

where $C_{L,s,C} > 0$ is independent from $Y \in L$, $\xi \in S$, $\eta \in C$, using that g_j is smooth hence bounded on compact subsets.

On the other hand, write $L \times S = \bigcup_{(Y,\xi) \in L \times S} V_{Y,\xi}$. Since $L \times S$ is compact, there exists a finite subcover $U_{Y_1,\xi_1}, \dots, U_{Y_n,\xi_n}$. Then for any $(\xi, \eta) \in S \times C$, $t \geq 1$, $Y \in L$,

$$\begin{aligned}
\|D^s \psi_{\lambda_t}(Y)\| &= \sup_{\|X_i\|=1} |D^s \psi_{\lambda_t}(Y)(X)| \\
&\leq \sup_{\|X_i\|=1} \sum_{j=0}^s t^j \left| \int_K e^{it\xi(H(\exp(Y)k))} g_j(\xi, \eta, Y, X)(k) dk \right| \\
(4.7) \quad &\leq \sum_{j=0}^s t^j \max_{1 \leq i \leq n} \sup_{\substack{\|X_i\|=1 \\ Y \in L, \xi \in S, \eta \in C}} v_{Y_i, \xi_i}(g_j(\xi, \eta, Y, X)) t^{-\kappa(G)} \\
&\leq D_{L,s,C} t^{s-\kappa(G)}
\end{aligned}$$

where $D_{L,s,C} > 0$ is independent from $Y \in L$, $\xi \in S$, $\eta \in C$. Again, the last line comes from the fact that g_j is smooth hence any of its differentials is bounded on compact subsets, and the topology on $C^\infty(K)$ is given by the seminorm p_i , thus if v is a continuous seminorms, there is i such that $v(g) \leq C p_i(g) = C \sup_{k \in K} \|D^i g(k)\|$.

Thus combining (4.6) and (4.7), for any λ such that $\text{Im } \lambda \in C$, $H \in L$, $s \leq r$,

$$(4.8) \quad \|D^s \psi_\lambda(Y)\| \leq \max(C_{L,s,C}, D_{L,s,C}) = M_{L,s}.$$

Thus the differentials of the family of spherical functions are bounded independently on λ such that $\text{Im } \lambda \in C$ up to order r . If $r = \kappa(G)$, the proof is complete.

Otherwise, $\kappa(G) - r = \frac{1}{2}$. Then using (4.7) for $s = r$ and $s = r + 1$, we show that for any $x, y \in L$, $\xi \in S, \eta \in C$, $t \geq 1$, we have on the one hand

$$(4.9) \quad \|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq \|D^r \psi_{\lambda_t}(x)\| + \|D^r \psi_{\lambda_t}(y)\| \leq 2D_{L,r,C} t^{-1/2},$$

with $\lambda_t = t\xi + i\eta$, and on the other hand, since L is convex, we get by the mean value theorem that

$$(4.10) \quad \|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq \left(\sup_{Y \in L} \|D^{r+1} \psi_{\lambda_t}(Y)\| \right) \|x - y\| \leq D_{L,r+1,C} t^{1/2} \|x - y\|.$$

Thus, combining (4.9) and (4.10) yields

$$(4.11) \quad \|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq (2D_{L,r,C} D_{L,r+1,C})^{1/2} \|x - y\|^{1/2}.$$

Hence, setting $M_L = \max\left((2D_{L,r,C} D_{L,r+1,C})^{1/2}, C_{L,r+1,C}(\text{diam } L)^{1/2}\right)$, we have that for any λ such that $\text{Im } \lambda \in C$, $x, y \in L$,

$$(4.12) \quad \|D^r \psi_\lambda(x) - D^r \psi_\lambda(y)\| \leq M_L \|x - y\|^{\kappa(G)-r}. \quad \square$$

Remark 4.3. If $\text{Im } \lambda$ is allowed to be unbounded, then the functions are not even bounded in $C(\mathfrak{a}^+)$.

We will now show that this result is optimal. The bounds used in the previous theorem are not sharp in general, but for a subfamily where $n(\xi) = \frac{\kappa(G)}{2}$, they are. Thus, we will show that this particular subfamily is already unbounded in Hölder spaces of higher regularity.

Theorem 4.4. *We keep the notations of Theorem 4.1. For any $\delta' > \delta$, the family of positive definite spherical functions of (G, K) is not bounded in $C^{(r,\delta')}(\mathfrak{a}^+)$.*

Before this, we first prove a lemma showing that complex exponentials are not bounded in Hölder spaces. We will reduce the problem for spherical functions of (G, K) to such functions.

Lemma 4.5. *Let E be a finite-dimensional real vector space, U an open subset of E such that $0 \in \overline{U}$. Let $u_1, \dots, u_n \in E^*$ distinct and non-zero, and $f_1, \dots, f_n : E \rightarrow \mathbb{C}$ continuous functions such that for any U' open subset of U , there is $x \in U'$ such that $\sum |f_j(x)| \neq 0$. Then there exist $C > 0$, $d > 0$, $x \in U$, and an open set V with $0 \in \overline{V}$ such that for all $y = x + h$, $h \in V$, $m \in \mathbb{N}$ and $N \geq \frac{d}{\|h\|}$,*

$$\frac{1}{N} \sum_{t=m}^{m+N-1} \left| \sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right|^2 \geq C.$$

Proof. Up to multiplying f_j by $x \mapsto e^{imu_j(x)}$, we can assume that $m = 0$ as long as the constants we find depend only on $|f_j|$ and not f_j . We have that

$$\begin{aligned} & \left| \sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right|^2 \\ &= \left(\sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right) \left(\sum_{k=1}^n \left(\overline{f_k(x)} e^{-itu_k(x)} - \overline{f_k(y)} e^{-itu_k(y)} \right) \right) \\ &= \sum_{j,k=1}^n \left(f_j(x) \overline{f_k(x)} e^{it(u_j(x) - u_k(x))} - f_j(x) \overline{f_k(y)} e^{it(u_j(x) - u_k(y))} \right. \\ & \quad \left. - f_j(y) \overline{f_k(x)} e^{it(u_j(y) - u_k(x))} + f_j(y) \overline{f_k(y)} e^{it(u_j(y) - u_k(y))} \right). \end{aligned}$$

Note that if $z \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\left| \sum_{t=0}^{N-1} e^{itz} \right| \leq \frac{1}{|\sin(z/2)|}$. The set $H = \bigcup_{j \neq k} \ker(u_j - u_k)$ is a finite union of hyperplane, thus $U' = U \cap (E \setminus H)$ is open and non-empty with 0 in its closure.

Let x in U' with $\|x\| \leq \min_{j,k} \frac{\pi}{\|u_j\| + \|u_k\|}$ and such that there is j_0 with $f_{j_0}(x) \neq 0$. Then there is a neighbourhood V_x of x in U' and $\varepsilon > 0$ such that for $y \in V_x$ and $j \neq k$, $\varepsilon \leq \left| \frac{u_j(y) - u_k(y)}{2} \right| \leq \pi - \varepsilon$ and $\varepsilon \leq \left| \frac{u_j(x) - u_k(y)}{2} \right| \leq \pi - \varepsilon$.

Then, let $V_0 = V_x - x$, let $h_0 \in V_0 \setminus \bigcup_i \ker u_i$. Let $\eta = \frac{1}{2\|h_0\|} \min_j |u_j(h_0)| > 0$. Let

$$V = V_0 \setminus \{h \in E \mid \forall 1 \leq j \leq n, |u_j(h)| \leq \eta \|h\|\}.$$

Then V is an open subset of U , containing h_0 thus such that $0 \in \overline{V}$. For any $h \in V$, by definition we have $\|h\| < \frac{1}{\eta} |u_j(h)|$ for any $1 \leq j \leq n$.

Hence we get for any $y = x + h, h \in V, N \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right|^2 \geq \\ \sum_{j=1}^n \left(|f_j(x)|^2 + |f_j(y)|^2 - \frac{|f_j(x)f_j(y)|}{N \left| \sin \frac{u_j(x) - u_j(y)}{2} \right|} \right) \\ - \sum_{j \neq k} \left(\frac{|f_j(x)f_k(x)|}{N \left| \sin \frac{u_j(x) - u_k(x)}{2} \right|} + \frac{|f_j(y)f_k(y)|}{N \left| \sin \frac{u_j(y) - u_k(y)}{2} \right|} + 2 \frac{|f_j(x)f_k(y)|}{N \left| \sin \frac{u_j(x) - u_k(y)}{2} \right|} \right). \end{aligned}$$

Now, for each of the terms with $j \neq k$, the assumptions on x, y ensures that the arguments in $\sin$ are bounded away from $0, \pi$. Furthermore, up to restricting V to a bounded set if necessary, the functions f_i are bounded. Thus, there is N_0 such that for $N > N_0$, we get

$$\frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right|^2 \geq \frac{|f_{j_0}(x)|^2}{2} - \sum_{j=1}^n \frac{|f_j(x)f_j(x+h)|}{N \left| \sin \frac{u_j(h)}{2} \right|}.$$

Up to restricting again V , we may assume that $h \in V$ is small enough so that $|u_j(h)| \leq \pi$ for any $h \in V, 1 \leq j \leq n$. Then, since the function $x \mapsto \frac{x}{\sin x}$ is bounded on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can define for any $1 \leq j \leq n$

$$d_j = \sup_{h \in V} \frac{4|f_j(x)f_j(x+h)|}{|f_{j_0}(x)|^2} \frac{|u_j(h)|}{\left| \sin \frac{u_j(h)}{2} \right|}$$

which is finite.

For any $N \geq \frac{d_j}{|u_j(h)|}$, by definition of d_j we obtain

$$\frac{|f_j(x)f_j(x+h)|}{N \left| \sin \frac{u_j(h)}{2} \right|} \leq \frac{|f_{j_0}(x)|^2}{4}.$$

Thus, for $d > (\max d_j)/\eta$, then for any $y = x + h, h \in V$ and $N > \max \left(N_0, \frac{d}{\|h\|} \right)$,

$$\frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n \left(f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \right|^2 \geq \frac{|f_{j_0}(x)|^2}{4}. \quad \square$$

Proof of Theorem 4.4. Consider $\lambda \in \mathfrak{a}^*$ such that $n(\lambda) = 2\kappa(G)$, and such that $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Sigma^+$. By Proposition 2.8, $\varphi_{t\lambda}$ is positive definite for any $t \in \mathbb{R}$. As in (4.4), for any $Y \in \mathfrak{a}, t \geq 1, X \in \mathfrak{a}^r$,

$$(4.13) \quad D^r \psi_{t\lambda}(Y)(X) = \sum_{j=0}^r t^j \int_K e^{it\lambda(H(\exp(Y)k))} g_j(Y, X)(k) dk.$$

Let $I_j(Y, X, t) = \int_K e^{it\lambda(H(\exp(Y)k))} g_j(Y, X)(k) dk$. Let W_λ denote the stabilizer of λ under the action of the Weyl group W , and K_Y, K_λ be the centralizers of Y, λ in K . Let also

$$\Sigma^+(\lambda) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \lambda \rangle \neq 0\}$$

and

$$\sigma_w = - \sum_{\alpha \in \Sigma^+(\lambda) \neq 0} m(\alpha) \operatorname{sgn}(\langle \alpha, \lambda \rangle)(w\alpha)(Y)).$$

Let $d_0 k$ denote the Riemannian measure on K induced by the bi-invariant metric defined by the Killing form on $\mathfrak{k}$. Let $\operatorname{Vol}(K) = \int_K d_0 k$. We also denote by $d_0 k$ the induced Riemannian measure on the submanifold $K_a w K_\lambda$ - the measure coming from the restriction of the Riemannian metric of K to a Riemannian metric on the submanifold. For $w \in W$, $g \in C^\infty(K)$, $Y \in \mathfrak{a}^+$, set

(4.14)

$$c_{w,a}(g) = e^{i\frac{\pi}{4}\sigma_w} \prod_{\alpha \in \Sigma^+(\lambda)} \left| \frac{\langle \alpha, \lambda \rangle}{4\pi} \left(1 - e^{-2(w\alpha)(Y)}\right)^{-\frac{m(\alpha)}{2}} \right| \frac{1}{\operatorname{Vol}(K)} \int_{K_a w K_\lambda} g(k) d_0 k.$$

Then, by [DKV83, Thm. 9.1], for any $Y \in \mathfrak{a}^+$, there is a neighbourhood U_Y of Y in $\mathfrak{a}^+$ and $D(Y) > 0$ such that for any $0 \leq j \leq r$, $t \geq 1$, $Y' \in U_Y$ and X with $\|X_i\| = 1$ for all i ,

$$(4.15) \quad \left| I_j(Y', X, t) - \sum_{W/W_\lambda} e^{it(w\lambda)(Y')} t^{-\kappa(G)} c_{w,Y'}(g_j(Y', X)) \right| \leq D(Y) t^{-\kappa(G)-1}.$$

We use that g_j is smooth in all variables, hence bounded on compacts and that the bound is uniform in the parameter Y' of the phase function.

In particular, combining (4.13) with (4.15) for $0 \leq j < r$, for any Y there is a neighbourhood V_Y of Y and a constant $C(Y)$ such that for any $t \geq 1$, $Y' \in V_Y$ and X with $\|X_i\| = 1$,

$$(4.16) \quad |D^r \psi_{t\lambda}(Y')(X) - t^r I_r(Y', X, t)| \leq C(Y) t^{-1}.$$

For X fixed with $\|X_i\| = 1$, let $S_t(x) = \sum_{W/W_\lambda} e^{it(w\lambda)(x)} c_{w,x}(g_r(x, X))$. Combining (4.15) and (4.16), if $t \geq 1$, and $x, y \in U_Y \cap V_Y$,

(4.17)

$$\begin{aligned} t^{-\delta} |S_t(x) - S_t(y)| &\leq t^r |t^{-\kappa(G)} S_t(x) - I_r(x, X, t)| + t^r |I_r(x, X, t) - I_r(y, X, t)| \\ &\quad + t^r |I_r(y, X, t) - t^{-\kappa(G)} S_t(y)| \\ &\leq 2D(Y) t^{-\delta-1} + |t^r I_r(x, X, t) - D^r \psi_{t\lambda}(x)(X)| \\ &\quad + |D^r \psi_{t\lambda}(x)(X) - D^r \psi_{t\lambda}(y)(X)| \\ &\quad + |D^r \psi_{t\lambda}(y)(X) - t^r I_r(y, X, t)| \\ &\leq 2D(Y) t^{-\delta-1} + 2C(Y) t^{-1} + |D^r \psi_{t\lambda}(x)(X) - D^r \psi_{t\lambda}(y)(X)| \\ &\leq \|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\| + 2(C(Y) + D(Y)) t^{-1}. \end{aligned}$$

Now the functions $c_{w,x}(g_r(x, X))$ are all zero at x if and only if $g_r(x, X) = 0$ almost everywhere on $\bigcup K_a w K_\lambda$. Let $f_k(x) = H(\exp(x)k)$, then

$$g_r(x, X)(k) = e^{-\rho(H(\exp(x)k))} \prod_{i=1}^r \lambda(Df_k(x)(X_i)).$$

If $P_{\mathfrak{a}}$ denote the orthogonal projection onto $\mathfrak{a}$, by [DKV83, Section 5] we have

$$Df_k(x)(X_i) = P_{\mathfrak{a}}(\operatorname{Ad}(t(\exp(x)k)^{-1})(X_i))$$

where $t(g) = a(g)n(g)$ in the Iwasawa decomposition (see (2.4)). The function $k \mapsto f_k(x)$ is left K_x -invariant, and by [DKV83, Prop. 5.6], $k \mapsto \lambda(f_k(x))$ is right

K_λ -invariant as $H_\lambda \in \overline{\mathfrak{a}^+}$. Thus, $g_r(x, X)$ is constant on $K_x w K_\lambda$ for any w . Thus given any open subset of $\mathfrak{a}^+$, we can choose x, X such that $g_r(x, X)(e) \neq 0$. Thus the hypotheses of Lemma 4.5 hold for the family of functions $f_w : x \mapsto c_{w,x}(g_r(x, X))$, for $U = \mathfrak{a}^+$. Let C, d, x, V be given by Lemma 4.5, $W_x = x + V$ such that for any $y \in W_x$, $m \in \mathbb{N}$, $N \geq \frac{d}{\|x-y\|}$,

$$(4.18) \quad \sum_{t=m}^{m+N-1} |S_t(x) - S_t(y)|^2 \geq CN.$$

From now on, we choose $Y = x$ given above. Let $M = 4(C(x) + D(x))^2$, we get from (4.17) that for any $t \geq 1$, $y \in U_x \cap V_x \cap W_x$,

$$(4.19) \quad \frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \leq \|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\|^2 + Mt^{-2}$$

Assume now that the family of positive definite spherical functions of (G, K) is bounded in $C^{(r, \delta')}(\mathfrak{a}^+)$ for $\delta' > \delta$. In particular, up to reducing $U_x \cap V_x \cap W_x$ to a bounded subset of diameter L if necessary, there is $D > 0$ such that for any $y \in U_x \cap V_x \cap W_x$ and $t \geq 1$,

$$(4.20) \quad \|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\| \leq D\|x - y\|^{\delta'}$$

For y fixed, set m, N such that

$$(4.21) \quad \frac{1}{\|x - y\|^{\delta'}} \leq m \leq \frac{1}{\|x - y\|^{\delta'}} + 1$$

and

$$(4.22) \quad \frac{d}{\|x - y\|} \leq N \leq \frac{d}{\|x - y\|} + 1.$$

Combining (4.18), (4.19) and (4.20) gives

$$(4.23) \quad \begin{aligned} \frac{CN}{2(m+N)^{2\delta}} &\leq \sum_{t=m}^{m+N-1} \frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \\ &\leq \sum_{t=m}^{m+N-1} (\|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\|^2 + Mt^{-2}) \\ &\leq ND^2\|x - y\|^{2\delta'} + \frac{MN}{m^2} \end{aligned}$$

thus

$$(4.24) \quad \frac{C}{2(m+N)^{2\delta}} \leq D^2\|x - y\|^{2\delta'} + \frac{M}{m^2} \leq (D^2 + M)\|x - y\|^{2\delta'}$$

by (4.21). But by (4.21) and (4.22), we have

$$(4.25) \quad \begin{aligned} m + N &\leq \frac{d}{\|x - y\|} + 1 + \frac{1}{\|x - y\|^{\delta'}} + 1 \\ &\leq \frac{1}{\|x - y\|} (d + 2\|x - y\| + \|x - y\|^{1-\delta'}) \\ &\leq \frac{1}{\|x - y\|} (d + 2L + L^{1-\delta'}) \end{aligned}$$

hence (4.24) becomes

$$(4.26) \quad \frac{C}{2(d+2L+L^{1-\delta'})^{2\delta}} \|x-y\|^{2\delta} \leq (D^2+M) \|x-y\|^{2\delta'}.$$

Since (4.26) holds for any $y \in U_x \cap V_x \cap W_x$ with the constant involved independent from y and $\delta' > \delta$, we get a contradiction as y goes to x (which is possible because $0 \in \bar{V}$ hence $x \in \overline{U_x \cap V_x \cap W_x}$). $\square$

Theorem 4.6 (Theorem A in the introduction). *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of G is in $C^{(r,\delta)}(G_r)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of G that is not in $C^{(r,\delta')}(G_r)$.*

Proof. It follows from Lemma 2.7, Corollary 3.4 and Theorems 3.7, 4.1 and 4.4. $\square$

Remark 4.7. If $a = \exp(Y) \in G \setminus G_r$, the behaviour around a is more complicated. Indeed by [DKV83], the critical set of the phase functions becomes $\bigcup_W K_a w K_\lambda$. If $\Sigma_w(\lambda, a) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \lambda \rangle (w\alpha)(Y) \neq 0\}$, then the codimension of the critical set is $n_w = \sum_{\alpha \in \Sigma_w(\lambda, a)} m(\alpha)$. In particular, n_w can be smaller than $2\kappa(G)$ so that the regularity of matrix coefficients around a is strictly worse than on G_r .

5. COMPACT SEMISIMPLE GROUPS

5.1. An upper bound on regularity. We first recall some notations of Section 2.3 and introduce new ones (more details in [Cle88, Hel79]). If $\mathfrak{g}$ is a semisimple real Lie algebra, we introduced a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into eigenspaces of a Cartan involution θ . Let $G_{\mathbb{C}}$ be the simply connected Lie group whose Lie algebra is the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Let G, K be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to the subalgebras $\mathfrak{g}, \mathfrak{k}$. Consider $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and U the corresponding analytic subgroup of $G_{\mathbb{C}}$. Then U is a maximal compact subgroup of $G_{\mathbb{C}}$ and is simply connected. Consider also $K_{\mathbb{C}}$ the analytic subgroup corresponding to $\mathfrak{k}_{\mathbb{C}}$. Finally, recall that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Let $A, A_{\mathbb{C}}, N, N_{\mathbb{C}}$ be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{a}, \mathfrak{a}_{\mathbb{C}}, \mathfrak{n}, \mathfrak{n}_{\mathbb{C}}$.

The involution θ extends to $\mathfrak{g}_{\mathbb{C}}$ and thus induces an involution of $G_{\mathbb{C}}$, also denoted θ . The subgroup K is the subgroup of fixed points of θ in U , so (U, K) is a symmetric Gelfand pair and the symmetric space $M = U/K$ is the compact dual of G/K . Since the Killing form of $\mathfrak{g}_{\mathbb{C}}$ restricted to $\mathfrak{g} \times \mathfrak{g}$ coincides with the Killing form of $\mathfrak{g}$, we continue to denote $\langle \cdot, \cdot \rangle$ both on $\mathfrak{g}$ and its complexification. It must be noted that $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$ -bilinear on $\mathfrak{g}_{\mathbb{C}}$ and not sesquilinear, and thus not a scalar product.

Such pairs (U, K) were studied in [Dum24]. Note that interesting pairs also arise when U is not simply connected - for example, $(SO(3), SO(2))$ -, we will lift this restriction in Section 5.3. The optimal regularity of K -finite coefficients of U was found in some specific cases and a conjecture was given in the general case. In what follows, we will extend the results using methods similar to what we did above in the non-compact setting.

Let Q be the connected component of $\mathfrak{a}_r = \{H \in \mathfrak{a} \mid \forall \alpha \in \Sigma, \alpha(H) \notin \pi\mathbb{Z}\}$ contained in $\mathfrak{a}^+$ and whose closure contains 0. Then, there is a KAK decomposition in the group U ([Dum24, Prop. 5.8]).

Proposition 5.1. *For any $u \in U$, there exists a decomposition*

$$u = k_1(u) \exp(iP(u)) k_2(u)^{-1}$$

where $P(u) \in \overline{Q}$ is unique and $k_1(u), k_2(u) \in K$. The map $u \mapsto P(u)$ is smooth on the set $U_r = K \exp(iQ)K$. Furthermore, for each $u \in U_r$, there exists a neighbourhood V_u of u in U_r and a choice of $u \mapsto k_i(u)$ such that k_i is smooth on V_u , $i = 1, 2$.

The set U_r is a dense open subset of U and we call it the set of regular points. This set will play the same role as G_r in the non-compact case.

As in the previous section for the non-compact setting, we want to study the spherical functions of the pair (U, K) . Let $\widehat{U}_K$ denote the set of classes of irreducible finite-dimensional representations of U with a non-zero K -invariant vector, which we call the spherical dual of U . Then $\widehat{U}_K$ (and thus spherical functions of (U, K)) are parameterized by a subset of $\mathfrak{a}_{\mathbb{C}}^*$. Note that since U is compact, any spherical function is positive definite ([vD09, Thm. 6.5.1]).

Theorem 5.2 (Cartan-Helgason). *Let $\Lambda = \{\mu \in \mathfrak{a}^* \mid \forall \alpha \in \Sigma^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}\}$. Then the map which sends a representation to its highest weight is a bijection from $\widehat{U}_K$ onto Λ .*

There exists $\mu_1, \dots, \mu_\ell \in \mathfrak{a}^*$ such that

$$\frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, 2\alpha_j \notin \Sigma \\ 2 & \text{if } i = j, 2\alpha_j \in \Sigma \end{cases}$$

The elements (μ_i) are called spherical fundamental weights and $\Lambda = \{\sum m_i \mu_i, m_i \in \mathbb{N}\}$ (see [Vre76]).

Let $\mu \in \Lambda$ and let π_μ be an irreducible finite-dimensional representation of U with highest weight μ . Let e_K be a unit K -invariant vector. Then $\psi_\mu : u \mapsto \langle \pi_\mu(u)e_K, e_K \rangle$ is a spherical function of (U, K) . Since π_μ is a finite-dimensional representation, it is smooth and its differential induces a representation of $\mathfrak{u}$, which extends to $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ and is itself the differential of a representation of the simply connected group $G_{\mathbb{C}}$. Thus, π_μ extends to a representation of $G_{\mathbb{C}}$, so ψ_μ is defined on all of $G_{\mathbb{C}}$. Given the notations of Section 2.3, $\psi_\mu|_G = \varphi_{-i(\mu+\rho)}$.

Thus on G , we know that ψ_μ has an integral representation. But since the Iwasawa decomposition does not extend to $G_{\mathbb{C}}$, the integral does not have meaning outside of G . However, even if $K_{\mathbb{C}} \times A_{\mathbb{C}} \times N_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ is not a diffeomorphism, it is a diffeomorphism in a neighbourhood of the identity $e \in G_{\mathbb{C}}$ so we can still work there. The following lemma is [Cle76, Lemme 1].

Lemma 5.3. *There exists a neighbourhood V of e in $G_{\mathbb{C}}$ which is invariant by conjugation by K and analytic maps $\kappa : V \rightarrow K_{\mathbb{C}}$, $n : V \rightarrow N_{\mathbb{C}}$ and $H : V \rightarrow \mathfrak{a}_{\mathbb{C}}$ such that*

- (1) $H(e) = 0$,
- (2) $\forall g \in V, g = \kappa(g) \exp H(g) n(g)$.

The map H coincides with the Iwasawa projection on $G \cap V$. Since $H(k^{-1}gk) = H(gk)$ for any $g \in G, k \in K$, we can extend the expression of ψ_μ on $G \cap V$ to all of V by analytic continuation ([Cle76, Lemme 3]).

Lemma 5.4. *Let $\mu \in \Lambda$. For any $g \in V$, $\psi_\mu(g) = \int_K e^{\mu(H(k^{-1}gk))} dk$.*

Unlike the non-compact case, the phase function is now complex-valued. However, for any $u \in U \cap V$, $\operatorname{Re} \mu(H(k^{-1}uk)) \leq 0$ ([Cle88, Coro. 2.4]) which is the condition to apply the method of stationary phase (see [Hör83, Section 7.7]).

We will now state the version of the stationary phase approximation on manifolds that we will use. The proof follows [Cha74], but with a complex-valued phase function, hence requiring some modifications.

Theorem 5.5. *Let (Z, g) be a compact Riemannian manifold of dimension d , dz its volume measure, and U an open subset of $\mathbb{R}^n$. Let $f \in C^\infty(Z)$ and $\phi \in C^\infty(Z \times U)$ be complex-valued functions. Let W_a be the set of critical points of $\phi_a : z \mapsto \phi(z, a)$ for $a \in U$ and assume that $W_a = W$ for any a . Assume also that W is finite, and for any $w \in W$, the Hessian of ϕ_a at w is non-degenerate. Furthermore, suppose that $\operatorname{Re} \phi \leq 0$, with equality at critical points $w \in W$. Set*

$$I(f, a, t) = \int_Z e^{t\phi(z, a)} f(z) dz$$

and fix $a_0 \in U$. Then there exists a semi-norm ν on $C^\infty(Z)$ and an open neighbourhood $U' \subset U$ of a_0 such that for any $t \geq 1$, $a \in U'$, $f \in C^\infty(Z)$,

$$\left| I(f, a, t) - \sum_{w \in W} e^{t\phi(w, a)} t^{-d/2} f(w) \left(\frac{(2\pi)^d}{\det(-\operatorname{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq \nu(f) t^{-\frac{d}{2}-1}$$

where the square root is taken as the branch of the square root which is deformed to 1 under the homotopy $(1-s)(-\operatorname{Hess}_{\phi_a}(w)) + s \operatorname{Id}$.

Proof. We can cover Z by a finite number of chart open subsets Z_j , $j \in J$ which contains at most one element of W . Up to shrinking some sets, we may assume as well that $Z_j \cap W = \overline{Z_j} \cap W$. In particular, whenever $Z_j \cap W = \emptyset$, the differential $D\phi_a$ is nonzero on the compact set $\overline{Z_j}$ thus $\|D\phi_a\|$ is bounded below on Z_j . Using a partition of unity subordinated to this open cover, we can write

$$(5.1) \quad I(f, a, t) = \sum_{j \in J} I_j(f, a, t)$$

where

$$(5.2) \quad I_j(f, a, t) = \int_{Z_j} e^{t\phi(z, a)} f_j(z) dz$$

and $\sum f_j(z) = f(z)$ for any $z \in Z$. In the chart Z_j , consider the local coordinates given by $H_j : Z_j \rightarrow \mathbb{R}^d$. Let $G = \det(g_{i,j})$ where $g_{i,j}(z) = g_z(\partial_i, \partial_j)$. Then by definition of the volume measure on Z , we have

$$(5.3) \quad I_j(f, a, t) = \int_{\mathbb{R}^d} e^{\phi(H_j^{-1}(x), a)} f_j(H_j^{-1}(x)) \sqrt{G(H_j^{-1}(x))} dx.$$

For each $j \in J$, we will bound $I_j(f, a, t)$. We need to distinguish between two cases, depending on the size of $Z_j \cap W$.

Case 1: if $Z_j \cap W = \emptyset$, by [Hör83, Thm. 7.7.1], for any $n > 0$, there exists $U_j^{(n)} \subset U$ neighbourhood of a_0 and $C_n, C'_n > 0$ such that for any $t \geq 1$, $a \in U_j^{(n)}$,

$$|I_j(f, a, t)| \leq \frac{C_n}{t^n} \sum_{|k| \leq n} \sup \|D^k(f_j \sqrt{G})(x)\| \leq \frac{C'_n}{t^n} \sum_{|k| \leq n} \sup \|D^k(f)(x)\|.$$

We used Leibniz formula to replace $f_j\sqrt{G}$ by f , up to changing the constant C_n to some C' taking into account norms of differentials of G and of the partition of unity. In particular, let $n_0 = \lceil \frac{d}{2} \rceil + 1$, $U_j = U_j^{(n_0)}$ and $v_j(f) = C'_{n_0} \sum_{|k| \leq n_0} \sup \|D^k(f)(x)\|$. Then for any $t \geq 1$, $a \in U_j$,

$$(5.4) \quad |I_j(f, a, t)| \leq v_j(f)t^{-\frac{d}{2}-1}$$

Case 2: if $Z_j \cap W = \{w\}$, up to a translation, we may assume that the local coordinates are such that $H_j(w) = 0$. By [Hör83, Thm. 7.7.5], there exists an open neighbourhood $U_j \subset U$ of a_0 such that for any $t \geq 1$, $a \in U_j$,

$$(5.5) \quad \left| I_j(f, a, t) - e^{t\phi_a(w)} t^{-d/2} f_j(w) \sqrt{G(w)} \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a \circ H_j^{-1}}(0))} \right)^{\frac{1}{2}} \right| \leq v_j(f)t^{-\frac{d}{2}-1}$$

where the square root is taken as in the statement of the theorem. Again, the semi-norm v_j should be applied to $f_j\sqrt{G}$ instead of f , but since it is defined as a differential operator, by Leibniz formula the inequality remains true with f up to changing the semi-norm.

For $u, v \in T_0\mathbb{R}^d$, at the critical point we have that

$$\text{Hess}_{\phi_a \circ H_j^{-1}}(0)(u, v) = \text{Hess}_{\phi_a}(w)(DH_j^{-1}(0)u, DH_j^{-1}(0)v).$$

Fix an orthonormal basis of $T_w Z$ with respect to the inner product g_w and consider the canonical basis $\frac{\partial}{\partial x_i}|_0$ of $T_0\mathbb{R}^d$, then

$$\det(-\text{Hess}_{\phi_a \circ H_j^{-1}}(0)) = \det(-\text{Hess}_{\phi_a}(w)) (\det DH_j^{-1}(0))^2.$$

But since $\partial_i|_w = DH_j^{-1}(0) \left(\frac{\partial}{\partial x_i} |_0 \right)$, we also have that $(g_{ij}(w))_{1 \leq i, j \leq n} = {}^t A A$ where A is the matrix of $DH_j^{-1}(0)$ in the previous bases, thus $G(w) = \det(DH_j^{-1}(0))^2$. Thus, (5.5) becomes

$$(5.6) \quad \left| I_j(f, a, t) - e^{t\phi_a(w)} t^{-d/2} f_j(w) \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq v_j(f)t^{-\frac{d}{2}-1}.$$

We now combine (5.1) and (5.4), (5.6) by triangular inequality. Setting $U' = \bigcap_{j \in J} U_j$, there is a semi-norm ν on $C^\infty(Z)$ such that for any $f \in C^\infty(Z)$, $a \in U'$, and $t \geq 1$,

$$\left| I(f, a, t) - \sum_{w \in W} e^{t\phi(w, a)} t^{-d/2} f(w) \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq \nu(f)t^{-\frac{d}{2}-1}$$

which concludes the proof. $\square$

We now compute the critical points and the Hessian of the phase function to make use of the previous theorem. We will use arguments of analytic continuation to compute the critical manifold from the results of [DKV83] on G , which we used in Section 4.

For $a \in V$, consider the phase function

$$F_{a, \mu} : \begin{array}{ccc} K & \rightarrow & \mathbb{C} \\ k & \mapsto & \mu(H(k^{-1}ak)) \end{array} ,$$

which we can define thanks to Lemma 5.3. For any k , the map $a \mapsto F_{a,\mu}(k)$ is an analytic continuation of the phase function studied in [DKV83, Section 4]. Denote $H_\mu \in \mathfrak{a}$ the unique vector such that for any $H \in \mathfrak{a}_\mathbb{C}$, $\mu(H) = \langle H, H_\mu \rangle$. Recall that K_μ is the centraliser in K of H_μ . For $\mu \in \Lambda$, $H_\mu \in \mathfrak{a}^+$. Then, by [DKV83, Proposition 5.6] and analytic continuation, the map $F_{a,\mu}$ is right K_μ -invariant.

For $x \in G$, let

$$\theta_x : \begin{array}{ccc} K & \rightarrow & K \\ k & \mapsto & \kappa(xk) \end{array}.$$

By uniqueness of the Iwasawa decomposition on G , it is clear that for any $x \in G$ and $k \in K$, $\kappa(xk) = k\kappa(k^{-1}xk)$. Now for $x \in V$, since we can extend κ analytically on V by Lemma 5.3 and V is K -invariant, the element $k\kappa(k^{-1}xk) \in K_\mathbb{C}$ is well-defined and the formula

$$\theta_x : \begin{array}{ccc} K & \rightarrow & K_\mathbb{C} \\ k & \mapsto & k\kappa(k^{-1}xk) \end{array}$$

extends θ on $V \times K$, and $x \mapsto \theta_x(k)$ is analytic on V for each k . Let $a \in V \cap U_r$ such that $a = \exp(iY)$ with $Y \in Q$.

Lemma 5.6. *The set of critical points of $F_{a,\mu}$ is*

$$\mathcal{C}_\mu = \bigcup_{w \in W} k_w K_\mu$$

where $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and k_w is a representative of $w \in W$.

Proof. We identify $T_k K$ with $\mathfrak{k}$ under the isomorphism $T_e L_k$. Then by [DKV83, Lemma 5.1 and Corollary 5.2], for any $x \in G$, the tangent map of $F_{x,\mu}$ at k is

$$(5.7) \quad T_k F_{x,\mu} : Z \mapsto \langle Z, \text{Ad}(n(k^{-1}xk)^{-1})(H_\mu) \rangle.$$

Again by Lemma 5.3, for $k \in K$ and $Z \in \mathfrak{k}$ fixed, the right-hand side of (5.7) extends to a well-defined analytic function of x on V . The map $x \mapsto F_{x,\mu}(k)$ is also analytic on V for $k \in K$ fixed. For $Z \in \mathfrak{k}$,

$$T_k F_{x,\mu}(Z) = \lim_{t \rightarrow 0} \frac{F_{x,\mu}(k \exp(tZ)) - F_{x,\mu}(k)}{t}$$

and the convergence is uniform for x in a compact set. Thus, as a uniform limit of analytic functions, $T_k F_{x,\mu}(Z)$ is analytic in $x \in V$. Hence, both sides of (5.7) extends analytically to functions of $x \in V$ which coincides on $G \cap V$, and so by uniqueness of analytic continuation, (5.7) holds for any $x \in V$.

Similar arguments of analytic continuation will be used several times in what follows. We will not give full details as the proofs are identical.

By [DKV83, Prop. 5.4], any $k \in \mathcal{C}_\mu$ is a critical point of $F_{a,\mu}$ for $a \in A$. For $\mu \in \Lambda$, $k \in \mathcal{C}_\mu$, $Z \in \mathfrak{k}$ fixed, the map $a \mapsto T_k F_{a,\mu}(Z)$ is analytic in $a \in V \cap A_\mathbb{C}$ and zero on A . Thus, by analytic continuation it is identically 0 and k is a critical point of $F_{a,\mu}$.

Conversely, let k be a critical point of $F_{a,\mu}$. Then $\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) \in \mathfrak{g}_\mathbb{C}$ is orthogonal to $\mathfrak{k}$ so $\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) \in \mathfrak{p}_\mathbb{C}$. Since $H_\mu \in \mathfrak{a} \subset \mathfrak{p}_\mathbb{C}$, we get that

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{p}_\mathbb{C}.$$

Furthermore, for any $X \in \mathfrak{n}_\mathbb{C}$, $\text{Ad}(\exp(X))(H_\mu) = e^{\text{ad}(X)}(H_\mu) = H_\mu \pmod{\mathfrak{n}_\mathbb{C}}$. Since $\exp(\mathfrak{n}_\mathbb{C})$ generates $N_\mathbb{C}$, we get $\text{Ad}(n)(H_\mu) = H_\mu \pmod{\mathfrak{n}_\mathbb{C}}$ for any $n \in N_\mathbb{C}$. Thus,

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{n}_\mathbb{C}$$

so

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{p}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C} = \{0\}.$$

So we get

$$(5.8) \quad \text{Ad}(n(k^{-1}ak))(H_\mu) = H_\mu.$$

From [DKV83, Lemma 1.1 and Lemma 5.9], we also have for $a' = \exp(Y') \in A$, $k \in K$ and $Z \in \mathfrak{k}$ that

$$T_k F_{a',\mu}(Z) = -\langle [Y', \text{Ad}(\theta_{a'}(k))(H_\mu)], \tilde{Z} \rangle$$

where $\tilde{Z} = (\sinh \text{ad}(Y') / \text{ad}(Y')) \circ \text{Ad}(k)(Z)$. By analytic continuation, we extend this expression to V . In particular for $a = \exp(iY)$, we get

$$T_k F_{a,\mu}(Z) = -i \langle [Y, \text{Ad}(\theta_a(k))(H_\mu)], \tilde{Z} \rangle$$

where $\tilde{Z} = (\sin \text{ad}(Y) / \text{ad}(Y)) \circ \text{Ad}(k)(Z)$.

We claim that the map $Z \mapsto \tilde{Z}$ is an isomorphism of $\mathfrak{k}$. Since $\text{Ad}(k)$ is an isomorphism of $\mathfrak{k}$, it suffices to show that

$$T = (\sin \text{ad}(Y) / \text{ad}(Y)) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} \text{ad}(Y)^{2n}$$

is an isomorphism of $\mathfrak{k}$. Consider the basis of $\mathfrak{k}$ used in the proof of Lemma 3.2. Let $Y_1, \dots, Y_r$ be a basis of $\mathfrak{m} = \mathfrak{k}^\mathfrak{a}$. For $\alpha \in \Sigma^+$, let $Z_{\alpha,1}, \dots, Z_{\alpha,m(\alpha)}$ be a basis of $\mathfrak{g}^\alpha$. Let $Z_{\alpha,i} = Z_{\alpha,i} + \theta(Z_{\alpha,i}) \in \mathfrak{k}$. Then $(Y_i)_{1 \leq i \leq r} \cup (Z_{\alpha,i}^+)_{\alpha \in \Sigma^+, 1 \leq i \leq m(\alpha)}$ is a basis of $\mathfrak{k}$. In this basis, T is a diagonal operator and its eigenvalues are nonzero. Indeed, for any $1 \leq i \leq r$, we have $T(Y_i) = Y_i$ - since Y_i commutes with $\mathfrak{a}$. For any $\alpha \in \Sigma^+$ and $1 \leq i \leq m(\alpha)$, we have $\text{ad}(Y)(Z_{\alpha,i}) = \alpha(Y)Z_{\alpha,i}$ and $\text{ad}(Y)\theta(Z_{\alpha,i}) = -\alpha(Y)\theta(Z_{\alpha,i})$. Thus, $\text{ad}(Y)^2(Z_{\alpha,i}^+) = \alpha(Y)^2 Z_{\alpha,i}^+$. Hence,

$$T(Z_{\alpha,i}^+) = \sum_{n \in \mathbb{N}} \frac{(-1)^n \alpha(Y)^{2n}}{(2n+1)!} Z_{\alpha,i}^+ = \frac{\sin(\alpha(Y))}{\alpha(Y)} Z_{\alpha,i}.$$

Since we assumed $Y \in Q$, $\alpha(Y) \notin \pi\mathbb{Z}$ for any $\alpha \in \Sigma^+$ and thus the claim is proved and T is an isomorphism.

Hence if k a critical point of $F_{a,\mu}$, we get that $[Y, \text{Ad}(\theta_a(k))(H_\mu)] \in \mathfrak{g}_\mathbb{C}$ is orthogonal to $\mathfrak{k}$ thus $[Y, \text{Ad}(\theta_a(k))(H_\mu)] \in \mathfrak{p}_\mathbb{C}$. But since $\theta_a(k) \in K_\mathbb{C}$, we also have $\text{Ad}(\theta_a(k))(H_\mu) \in \mathfrak{p}_\mathbb{C}$ thus

$$(5.9) \quad [Y, \text{Ad}(\theta_a(k))(H_\mu)] \in [\mathfrak{p}_\mathbb{C}, \mathfrak{p}_\mathbb{C}] \cap \mathfrak{p}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \cap \mathfrak{p}_\mathbb{C} = \{0\}.$$

Since $k^{-1}ak \in V$, by definition of κ, H, n , we can decompose

$$ak = k\kappa(k^{-1}ak) \exp(H(k^{-1}ak))n(k^{-1}ak) = \theta_a(k) \exp(H(k^{-1}ak))n(k^{-1}ak).$$

Thus combining (5.8) and (5.9), we get that

$$\begin{aligned} [Y, \text{Ad}(ak)(H_\mu)] &= [Y, \text{Ad}(\theta_a(k)) \circ \text{Ad}(\exp(H(k^{-1}ak))) \circ \text{Ad}(n(k^{-1}ak))(H_\mu)] \\ &= [Y, \text{Ad}(\theta_a(k))(H_\mu)] \\ &= 0 \end{aligned}$$

Since $\text{Ad}(a)$ is an automorphism of Lie algebra and $A_\mathbb{C}$ is abelian, we get

$$[\text{Ad}(a^{-1})(Y), \text{Ad}(k)(H_\mu)] = [Y, \text{Ad}(k)(H_\mu)] = 0.$$

Now since $k \in K$ and Y is regular, we conclude with [DKV83, Prop. 1.2] that $k \in \mathcal{C}_\mu$. $\square$

We now see $F_{a,\mu}$ as a function defined on K/K_μ , whose distinct critical points are $k_w K_\mu$ for $w \in W/W_\mu$ - thus, there are only finitely many such points. For $\alpha \in \Sigma^+$, let $\mathfrak{k}^\alpha = \mathfrak{k} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$. Denote $\Sigma^+(\mu) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \mu \rangle \neq 0\}$ and

$$\mathfrak{l}_\mu = \bigoplus_{\alpha \in \Sigma^+(\mu)} \mathfrak{k}^\alpha.$$

Then $\dim K/K_\mu = \dim(\mathfrak{l}_\mu) = \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) = n(\mu)$. Let $F_\alpha : \mathfrak{l}_\mu \rightarrow \mathfrak{k}^\alpha$ be the orthogonal projection.

Let $w \in W/W_\mu$. The value at a critical point of the phase is $F_{a,\mu}(k_w K_\mu) = i\mu(w^{-1}Y) = i(w\mu)(Y)$. By analytic continuation and [DKV83, Prop. 6.5], the Hessian at $k_w K_\mu$ is given by

$$\text{Hess}_{F_{a,\mu}}(k_w K_\mu)(Y, Z) = -\langle Y, L_{a,\mu,w}(Z) \rangle$$

where

$$(5.10) \quad L_{a,\mu,w} = -\frac{1}{2} \sum_{\alpha \in \Sigma^+(\mu)} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) F_\alpha$$

and $-\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{k}$. Finally, to apply the method of stationary phase, we need to compute the square root of $\det(-L_{a,\mu,w})^{-1}$ which is continuously deformed to 1 by the homotopy $s\text{Id} + (1-s)(-L_{a,\mu,w})$.

If $w\alpha \in \Sigma^+$, we have

$$\frac{1}{2} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) = i \langle \alpha, \mu \rangle e^{-i(w\alpha)(Y)} \sin((w\alpha)(Y))$$

and $0 < (w\alpha)(Y) < \pi$ so we take $e^{i\frac{\pi}{4}} \langle \alpha, \mu \rangle^{\frac{1}{2}} e^{-i\frac{(w\alpha)(Y)}{2}} \sin((w\alpha)(Y))^{\frac{1}{2}}$ as a square root.

If $w\alpha \in -\Sigma^+$, we have

$$\frac{1}{2} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) = -i \langle \alpha, \mu \rangle e^{-i(w\alpha)(Y)} |\sin((w\alpha)(Y))|$$

since $-\pi < (w\alpha)(Y) < 0$, so we take $e^{-i\frac{\pi}{4}} \langle \alpha, \mu \rangle^{\frac{1}{2}} e^{-i\frac{(w\alpha)(Y)}{2}} |\sin((w\alpha)(Y))|^{\frac{1}{2}}$ as a square root.

Let $\sigma_w = \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) \text{sgn}((w\alpha)(Y))$. We get

$$(5.11) \quad \det(-L_{a,\mu,w})^{-\frac{1}{2}} = e^{-i\sigma_w \frac{\pi}{4}} e^{i(w\rho_\mu)(Y)} \prod_{\alpha \in \Sigma^+(\mu)} \langle \alpha, \mu \rangle^{-\frac{m(\alpha)}{2}} |\sin((w\alpha)(Y))|^{-\frac{m(\alpha)}{2}}$$

where $\rho_\mu = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) \alpha$.

Theorem 5.7. *For $\mu \in \Lambda$, let $\Psi_\mu = \psi_\mu \circ \exp|_{iQ}$. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then for any $\delta' > \delta$, the family $(\Psi_\mu)_{\mu \in \Lambda}$ is not bounded in $C^{(r,\delta')}(iQ)$.*

The proof of this result is a variation of the proof of Theorem 4.4 using the expression of spherical functions of (U, K) from Lemma 5.4 and the method of stationary phase for complex-valued phase functions.

Proof. By (2.3), we see that $\kappa(G)$ is attained on a fundamental spherical weight, which is an element of Λ . Thus, we can choose $\mu \in \Lambda$ be such that $n(\mu) = 2\kappa(G)$. Let V be as above given by Lemma 5.3 and V' an open subset of Q such that $a = \exp(iY) \in V$ for any $Y \in V'$. Let $d(kK_\mu)$ be the image of the Haar measure on K on K/K_μ . Then for any $n \in \mathbb{N}$ and $Y \in V'$, $\Psi_{n\mu}(iY) =$

$\int_{K/K_\mu} e^{n\mu(H(k^{-1}\exp(iY)k))} d(kK_\mu)$. As in the proof of Theorem 4.4, for any $X \in \mathfrak{a}^r$ there are functions $g_j(Y, X) \in C^\infty(K/K_\mu)$ which do not depend on $n \in \mathbb{N}$ such that

$$D^r \Psi_{n\mu}(iY)(X) = \sum_{j=0}^r n^j \int_{K/K_\mu} e^{nF_{a,\mu}(k)} g_j(Y, X)(k) d(kK_\mu).$$

Let $I_j(Y, X, n) = \int_{K/K_\mu} e^{nF_{a,\mu}(k)} g_j(Y, X)(k) d(kK_\mu)$ for $1 \leq j \leq r$. We want to evaluate this integral as n goes to infinity. Let $d_0(kK_\mu)$ be the volume measure on K/K_μ associated with the (invariant) Riemannian metric induced by the restriction of the inner product $-\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$. Denote $\text{Vol}(K/K_\mu) = \int_{K/K_\mu} d_0(K/K_\mu)$. By uniqueness of the invariant measure on K/K_μ , we have $d(kK_\mu) = \frac{1}{\text{Vol}(K/K_\mu)} d_0(kK_\mu)$. If $w \in W/W_\mu$, $g \in C^\infty(K/K_\mu)$, let

$$c_{w,Y}(g) = \frac{(2\pi)^{n(\mu)/2}}{\text{Vol}(K/K_\mu)} \det(-L_{\exp(iY),\mu,w})^{-\frac{1}{2}} g(k_w K_\mu).$$

By the method of stationary phase (Theorem 5.5) and the computations on the phase functions $F_{a,\mu}$, there is a neighbourhood U_Y of Y and a constant $D(Y) \geq 0$ such that for any $1 \leq j \leq r$, for any $n \in \mathbb{N}^*$, $Y' \in U_Y$ in V' and $X \in \mathfrak{a}^r$ with $\|X_i\| = 1$ for all i ,

$$(5.12) \quad \left| I_j(Y', X, n) - \sum_{w \in W/W_\mu} e^{in(w\mu)(Y')} n^{-\frac{n(\mu)}{2}} c_{w,Y'}(g_j(Y', X)) \right| \leq D(Y) n^{-\frac{n(\mu)}{2}-1}.$$

We used that $g_j : V' \times \mathfrak{a}^r \times K \rightarrow \mathbb{C}$ is smooth, so that $\nu(g_j(Y', X))$ is bounded when Y', X stay in compact sets. Using the previous inequality in the cases $1 \leq j < r$, for any $Y \in V'$, there is a neighbourhood V_Y of Y and a constant $C(Y) > 0$ such that for any $n \geq 1$, $Y' \in V_Y$ and X with $\|X_i\| = 1$ for all $1 \leq i \leq r$,

$$(5.13) \quad |D^r \Psi_{n\mu}(iY')(X) - n^r I_r(Y', X, n)| \leq C(Y) n^{-1}.$$

Let $S_n(x) = \sum_{w \in W/W_\mu} e^{in(w\mu)(x)} c_{w,x}(g_r(x, X))$. Since $n(\mu) = 2\kappa(G)$ by choice of μ , combining (5.12) with $j = r$ and (5.13) yields that for any $n \in \mathbb{N}^*$ and $x, y \in U_Y \cap V_Y$,

$$(5.14) \quad \begin{aligned} n^{-\delta} |S_n(x) - S_n(y)| &\leq n^r |n^{-\kappa(G)} S_n(x) - I_r(x, X, n)| + n^r |I_r(x, X, n) - I_r(y, X, n)| \\ &\quad + n^r |I_r(y, X, n) - n^{-\kappa(G)} S_n(y)| \\ &\leq 2D(Y) n^{r-\kappa(G)-1} + |n^r I_r(x, H, n) - D^r \Psi_{n\lambda}(x)(X)| \\ &\quad + |D^r \Psi_{n\lambda}(x)(X) - D^r \Psi_{n\lambda}(y)(X)| \\ &\quad + |D^r \Psi_{n\lambda}(y)(X) - n^r I_r(y, X, n)| \\ &\leq 2D(Y) n^{-\delta-1} + 2C(Y) n^{-1} + |D^r \Psi_{n\lambda}(x)(X) - D^r \psi_{n\lambda}(y)(X)| \\ &\leq \|D^r \Psi_{n\lambda}(x) - D^r \Psi_{n\lambda}(y)\| + 2(C(Y) + D(Y)) n^{-1}. \end{aligned}$$

The hypotheses of Lemma 4.5 hold for the family of functions $x \mapsto c_{w,x}(g_r(x, X))$, for $U = V'$. Let C, d, x, V be given by Lemma 4.5, $W_x = x + V$ such that for any

$$y \in W_x, m \in \mathbb{N}, N \geq \frac{d}{\|x-y\|},$$

$$(5.15) \quad \sum_{t=m}^{m+N-1} |S_n(x) - S_n(y)|^2 \geq CN.$$

The end of the proof follows as in Theorem 4.4. $\square$

Theorem 5.8 (Theorem B, Part 1). *Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of a unitary representation of U which is not in $C^{(r, \delta')}(U_r)$.*

Proof. By Theorem 5.7, the family of spherical functions viewed on the Lie algebra is not bounded in $C^{(r, \delta')}(Q)$. By Proposition 5.1 and Lemma 2.3, it follows that the family of spherical functions of (U, K) is not bounded in $C^{(r, \delta')}(U_r)$. Thus, the result follows from Lemma 2.7. $\square$

In [Dum24], we made a conjecture on the expected optimal regularity of coefficients of (U, K) , which should be $\kappa(G)$.

Conjecture 5.9. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Any K -finite matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(U_r)$ and this regularity is optimal.

By [Dum24, Thm. 5.2], it suffices to look at K -bi-invariant matrix coefficients. This conjecture has been proven in rank 1 and when U/K is a Lie isomorphic to a Lie group (which is equivalent to G being a complex Lie group), see [Dum24, Thm. A and B]. Theorem 5.8 gives a partial result towards this conjecture : the optimal regularity cannot be greater than $\kappa(G)$. Furthermore, the conjecture is also shown in some new specific cases.

Corollary 5.10. *The conjecture is true for*

- $(U, K) = (SU(p+q), S(U(p) \times U(q)))$ with $q \geq p \geq 2$, which corresponds to $G = SU(p, q)$;
- $(U, K) = (SO(8), U(4))$, which corresponds to $G = SO^*(8)$;
- $(U, K) = (SO(10), U(5))$, which corresponds to $G = SO^*(10)$;
- $(U, K) = (Sp(n), Sp(2) \times Sp(n-2))$ with $n \geq 4$, which corresponds to $G = Sp(2, n-2)$.

Proof. This is a consequence of Theorem 5.8 and [Dum24, Thm. 4.9 and 4.18]. $\square$

5.2. Towards a lower bound. We now know that the optimal regularity of K -finite matrix coefficient of U on U_r is at most $\kappa(G)$, with equality in many cases. We will conclude this paper by showing that equality holds in all cases, but only in some open subset of U_r .

In Theorem 5.7, we used the method of stationary phase to obtain a sharp estimate of the spherical functions, but only on a subfamily. In order to obtain a lower bound on regularity, we need estimates for all $\mu \in \Lambda$. Clearly, we know that we cannot obtain sharp estimates of this kind. Indeed, as μ varies, the critical submanifold of the phase function $F_{a, \mu}$ varies in dimension.

Thus, we cannot treat μ as a parameter and use stationary phase approximation to obtain uniform estimate when μ is close to some μ_0 . However, in a local chart, we can see K in coordinates (x, y) such that K_μ is given by $\{y = 0\}$, and treat x, a, μ as parameters, as done in [DKV83, Prop. 9.2] and used in Theorem 4.1.

This leads to brutal estimates, as we forget what happens in the x coordinate, but sufficient for our purposes.

A second issue arises in the statement of stationary phase used before (Theorem 5.5). When choosing a parameter $\mu_0 \in \Lambda$, we need to consider a neighbourhood of μ_0 in $\mathfrak{a}^*$. However, a necessary hypothesis is that the real part of the phase is non-positive, which is only true on Λ , thus there may not exist a neighbourhood of μ_0 such that this is true.

We begin our proof with another statement of the stationary phase approximation which is well-suited to holomorphic phase functions - instead of simply C^∞ as in Theorem 5.5.

Theorem 5.11 ([Sjö82, Thm 2.8 and Rem. 2.10], [BJ20, Prop. 1.3]). *Let $U \times V$ be an open subset of $\mathbb{C}^n \times \mathbb{C}^k$ and Γ a k -dimensional real submanifold of V with boundary. Let $\Phi, u : U \times V \rightarrow \mathbb{C}$ be holomorphic functions. For $x \in U$, denote $\Phi_x : y \mapsto \Phi(x, y)$. Let $x_0 \in U$. Assume that there exists a unique critical point y_0 of Φ_{x_0} in Γ , which is non-degenerate and in the interior of Γ and that $\Phi(x_0, y_0) = 0$. Furthermore, assume that $\text{Im } \Phi(x_0, y) \geq 0$ for any $y \in \Gamma$, and $\text{Im } \Phi(x_0, y) > 0$ for any $y \in \partial\Gamma$.*

Then, there exists an open neighbourhood $U' \subset U$ of x_0 such that for any $x \in U'$, the map Φ_x has a unique critical point $y_c(x)$ close to y_0 and $x \mapsto y_c(x)$ is holomorphic. Furthermore, there exists $C > 0$ such that for any $x \in U'$, $t \geq 1$,

$$\left| e^{-it\Phi(x, y_c(x))} \int_{\Gamma} e^{it\Phi(x, y)} u(x, y) dy \right| \leq C \|u\|_{\infty, U' \times \Gamma} t^{-k/2}.$$

If we want to estimate the integral, we need to understand the term $e^{-it\Phi(x, y_c(x))}$. It is clear that this term is bounded below as $t \rightarrow +\infty$ except when $\text{Im } \Phi(x, y_c(x)) < 0$. But since $y_c(x)$ is not necessarily real, even for x real, it is *a priori* not enough to understand $\text{Im } \Phi(x, y)$ on real points y .

Lemma 5.12 ([BJ20, Lem. 1.16]). *Let Φ be as in Theorem 5.11 and $y_c : U' \rightarrow V$ given by the theorem. Assume that $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\text{Im Hess}_{\Phi_{x_0}}(y_0)$ is positive. Then there exists an open neighbourhood $U'' \subset U'$ of x_0 such that for any $x \in U'' \cap \mathbb{R}^n$ with $\text{Im } \Phi(x, y) \geq 0$ for all $y \in V \cap \mathbb{R}^k$, then*

$$\text{Im } \Phi(x, y_c(x)) \geq 0.$$

Remark 5.13. This lemma is stated with the hypothesis that $\text{Im } \Phi(x, y) \geq 0$ for **all** x, y real - and thus of course, $\text{Im } \Phi(x, y_c(x)) \geq 0$ for all x real. However it is clear in the proof that we can remove this assumption by adding positivity of the Hessian matrix, and we get the result only for x real where the assumption holds, as stated above. This is exactly what will allow us to take care of parameters μ_0 on the boundary of $\tilde{\Lambda}$.

We now come back to the Lie group setting, and we will use all the notations introduced in Section 5.1. Notice that since the subgroup K is compact, there exists a relatively compact open subset $\tilde{K}$ of $K_{\mathbb{C}}$ such that $K \subset \tilde{K}$. By compactness, up to shrinking the open subset V given by Lemma 5.3, we may assume it is invariant by conjugation by elements of $\tilde{K}$. Thus, the element $F_{g, \mu}(k) = \mu(H(k^{-1}gk)) \in \mathbb{C}$ is well-defined for any $g \in V$, $k \in \tilde{K}$, $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ and is holomorphic in all variables.

Recall also that there are fundamental weights $\mu_1, \dots, \mu_\ell \in \mathfrak{a}^*$ such that

$$\Lambda = \left\{ \sum_{i=1}^{\ell} m_i \mu_i \mid m_i \in \mathbb{N} \right\}$$

and that for any $u \in U \cap V$, $\mu \in \Lambda$,

$$\operatorname{Re} \mu(H(u)) \leq 0.$$

Thus, defining the cone

$$\tilde{\Lambda} = \left\{ \sum_{i=1}^{\ell} t_i \mu_i \mid t_i \in \mathbb{R}_+ \right\},$$

it is clear that for any $u \in U \cap V$, $\mu \in \tilde{\Lambda}$,

$$\operatorname{Re} \mu(H(u)) \leq 0.$$

Furthermore, the function $\psi_\mu : g \mapsto \int_K e^{\mu(H(k^{-1}gk))} dk$ is also well-defined for any $\mu \in \mathfrak{a}_\mathbb{C}^*$.

Theorem 5.14. *For $\mu \in \mathfrak{a}_\mathbb{C}^*$, let $\Psi_\mu = \psi_\mu \circ \exp|_{iQ}$. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Let $Q_0 = \{Y \in Q \mid \exp(iY) \in V\}$. Then the family $(\Psi_\mu)_{\mu \in \Lambda}$ is bounded in $C^{(r,\delta)}(iQ_0)$.*

Proof. First consider $\tilde{Q} = \{Y \in \mathfrak{a}_\mathbb{C} \mid \exp(Y) \in V\}$. Let $\mu \in \mathfrak{a}_\mathbb{C}^*$ and $Y \in \tilde{Q}$. Then the phase function $F_{\exp Y, \mu} : k \mapsto \mu(H(k^{-1} \exp(Y)k))$ is well-defined on $\tilde{K}$ and holomorphic in all variables. Let $t \in \mathbb{R}$, consider $\Psi_{t\mu}(Y) = \int_K e^{tF_{a,\mu}(k)} dk$. Let $s \in \mathbb{N}$ and $X \in \mathfrak{a}_\mathbb{C}^*$. For $0 \leq j \leq s$, we can consider the functions

$$g_j(\mu, Y, X) : k \in \tilde{K} \mapsto \frac{1}{j!} \frac{d^j}{dt^j} \left(\left(D^s(e^{tF_{a,\mu}(k)})(Y)(X) \right) e^{-tF_{a,\mu}(k)} \right) \Big|_{t=0}$$

where D is the differential operator with respect to the variable Y . The functions g_j are holomorphic in all variables. As in Theorem 4.1, we get that

$$(5.16) \quad D^s \Psi_{t\mu}(Y)(X) = \sum_{j=0}^s t^j \int_K e^{tF_{a,\mu}(k)} g_j(\mu, Y, X)(k) dk.$$

Let $S = \{\mu \in \tilde{\Lambda} \mid \|\mu\| = \min \|\mu_i\|\}$ which is a compact set. For now, fix $\mu_0 \in S$, $Y_0 \in Q_0$, $a_0 = \exp(iY_0)$. Let $\mathcal{C} = \mathcal{C}_{\mu_0}$ be the critical set of F_{a_0, μ_0} computed in Lemma 5.6. The critical set μ varies with μ , but it depends on K_μ hence on the set of roots orthogonal to μ . We can choose a neighbourhood V_{μ_0} of μ_0 in $\mathfrak{a}_\mathbb{C}^*$ such that for any $\mu \in V_{\mu_0} \cap \mathfrak{a}$, $\mathcal{C}_\mu \subset \mathcal{C}$.

Consider a tubular neighbourhood of $\mathcal{C}$, that is to say a vector bundle $\pi : E \rightarrow \mathcal{C}$ together with a map $J : E \rightarrow K$ such that if 0_E denote the zero section of the vector bundle, $J(0_E(x)) = x$ and J is a diffeomorphism from an open neighbourhood Ω of $0_E(\mathcal{C})$ to an open neighbourhood of $\mathcal{C}$ (we refer to [Lee03] for more details on the construction of such a bundle). For any $k_0 \in \mathcal{C}$, consider an open neighbourhood U_{1,k_0} which is a local trivialization of the bundle. Then $U_{2,k_0} = J(\Omega \cap \pi^{-1}(U_{1,k_0}))$ is a neighbourhood of k_0 in K . Notice that if $x \in U_{2,k} \cap U_{2,k'}$, then $\pi(J^{-1}(x)) \in U_{1,k} \cap U_{1,k'}$. Let V_{k_0} be a neighbourhood of k_0 in $\tilde{K}$ containing U_{2,k_0} , which we may assume up to reducing U_{2,k_0} to be the domain of an analytic chart

$$H_{k_0} : \begin{array}{ccc} V_{k_0} & \rightarrow & \mathbb{C}^{\dim(\mathcal{C})} \times \mathbb{C}^{n(\mu_0)} \\ k & \mapsto & (z_1, z_2) \end{array}$$

defined in such a way that $H_{k_0}(k_0) = (0, 0)$, $H_{k_0}(U_{2,k_0}) = H_{k_0}(V_{k_0} \cap K) = H(V_{k_0}) \cap (\mathbb{R}^{\dim(\mathcal{C})} \times \mathbb{R}^{n(\mu_0)})$ and $\mathcal{C}$ is given is those local coordinates by $H_{k_0}(U_{2,k_0}) \cap \{z_2 = 0\}$. Indeed, by the computations in Lemma 5.6, there exists $w \in W$ such that $k_0 \subset wK_{\mu_0}$, and we may assume that $U_{1,k_0} \cap \mathcal{C} \subset wK_{\mu}$. Then set $O = \exp_K^{-1}(w^{-1}U_{2,k_0})$, $O_{\mathbb{C}} = O + iO$ open subset of $\mathfrak{k}_{\mathbb{C}}$ and $V_{k_0} = w \exp_{K_{\mathbb{C}}}(O_{\mathbb{C}})$ - up to shrinking at each step so that the exponential map is a diffeomorphism in the neighbourhoods considered. Then if we consider a decomposition $\mathfrak{k} = \mathfrak{k}_{\mu} \oplus E$, the chart $H_{k_0}(k) = \exp_{K_{\mathbb{C}}}^{-1}(w^{-1}k)$ satisfies the requirements.

Let $x = (Y, \mu, z_1) \in V_1 = \tilde{Q} \times V_{\mu_0} \times \text{pr}_1(H_{k_0}(V_{k_0}))$, $y = z_2 \in V_2 = \text{pr}_2(H_{k_0}(V_{k_0}))$. We will treat x as a parameter and apply the stationary phase only in the y coordinate. Let $x_0 = (Y_0, \mu_0, 0)$ and $\Phi(x, y) = -i(F_{\exp(iY), \mu}(H_{k_0}^{-1}(z_1, z_2)) - F_{a_0, \mu_0}(k_0))$. Then $\Phi_{x_0} : y \mapsto \Phi(x_0, y)$ has a unique critical point $y_0 = 0$ in V_2 , which is non-degenerate. Since $\mu_0 \in \tilde{\Lambda}$ and $\text{Re } F_{a_0, \mu_0}(k_0) = 0$, we also have $\text{Im } \Phi(x_0, y) \geq 0$ for any $y \in V_2$. Furthermore, by (5.10), the imaginary part of the Hessian matrix of Φ_{x_0} at y_0 is positive definite. By Taylor's formula, this implies that $\text{Im } \Phi(x_0, y) > 0$ for y real and close to y_0 . Thus, we can chose a small ball $\Gamma_{k_0} \subset V_2 \cap \mathbb{R}^{n(\mu_0)}$ centered around y_0 such that $\text{Im } \Phi(x_0, y) > 0$ for $y \in \partial \Gamma_{k_0}$. Thus, we can apply Theorem 5.11 and Lemma 5.12. Thus, there is an open neighbourhood $V'_1 \subset V_1$ of x_0 , which we may assume to be of the form $V'_{Y_0}(k_0) \times V'_{\mu_0}(k_0) \times V'_0(k_0)$, such that for any $x \in V'_1$, the map Φ_x has a unique critical point $y_c(x)$. Up to shrinking, we may assume that $V'_0(k_0)$ is a ball centered at 0, whose closure is contained in V_{k_0} . For $x = (Y, \mu, z_1)$ real and such that $\mu \in \tilde{\Lambda}$, Lemma 5.12 ensures that $\text{Im } \phi(x, y_c(x)) \geq 0$. Finally, there is a constant $C_{Y_0, \mu_0, k_0} > 0$ such that for any holomorphic map g on $V_1 \times V_2$, for any $t \geq 1$ and $x = (Y, \mu, z_1) \in V'_1$ real with $\mu \in \tilde{\Lambda}$,

$$(5.17) \quad \left| \int_{\Gamma_{k_0}} e^{it\phi(x,y)} g(x,y) dx \right| \leq C_{Y_0, \mu_0, k_0} \|g\|_{\infty} t^{-n(\mu_0)/2} \leq C_{Y_0, \mu_0, k_0} \|g\|_{\infty} t^{-\kappa(G)}.$$

For the last inequality, we used that $\kappa(G) \leq \frac{n(\mu)}{2}$ for any $\mu \in \mathfrak{a}^*$ by definition.

Let $\Omega_{k_0} = H_{k_0}^{-1}((V'_0(k_0) \cap \mathbb{R}^{\dim \mathcal{C}}) \times \{0\}) = H_{k_0}^{-1}(V'_0(k_0) \times \mathring{\Gamma}_{k_0}) \cap \mathcal{C}$. Then Ω_{k_0} is an open neighbourhood of k_0 in $\mathcal{C}$, contained in U_{1,k_0} . We will construct an "almost disjoint" cover of $\mathcal{C}$. Since $\mathcal{C} = \bigcup_{k \in \mathcal{C}} \Omega_k$ is compact, we can extract a finite subcover $\Omega_{k_1}, \dots, \Omega_{k_p}$. Set $V_1 = \Omega_{k_1}$ and inductively define $V_j = \Omega_{k_j} \cap (\mathcal{C} \setminus \bigcup_{i < j} \overline{V_i})$. This construction yields p disjoint open subsets V_j of $\mathcal{C}$ such that $\mathcal{C} = \bigcup_{j=1}^p \overline{V_j} = \bigcup_{j=1}^p V_j \cup \bigcup_{j=1}^p \partial V_j$. We have that

$$\begin{aligned} \partial V_j &\subset \partial \Omega_{k_j} \cup \partial \left(\mathcal{C} \setminus \bigcup_{i < j} \overline{V_i} \right) \\ &= \partial \Omega_{k_j} \cup \partial \bigcup_{i < j} \overline{V_i} \\ &\subset \partial \Omega_{k_j} \cup \bigcup_{i < j} \partial \overline{V_i} \\ &\subset \partial \Omega_{k_j} \cup \bigcup_{i < j} \partial V_i. \end{aligned}$$

Since $\partial V_1 = \partial \Omega_1$, recursively we obtain

$$\partial V_j \subset \bigcup_{i \leq j} \partial \Omega_{k_j}.$$

But $\partial \Omega_{k_j}$ is contained in the embedded image of a $\dim \mathcal{C} - 1$ dimensional sphere, thus it has Riemannian volume 0. This implies that each ∂V_j has volume 0.

Let $N_j = H_{k_j}^{-1}(H_{k_j}(\overline{V_j}) \times \Gamma_{k_j}) \subset U_{2,k_j}$ and $D = \bigcup_{j=1}^p N_j$. Since $\overline{V_j}$ are disjoint up to negligible set, and contained in U_{1,k_j} local trivialization of the tubular neighbourhood, then N_j are also disjoint up to negligible sets (for the volume measure on K). Indeed, $N_i \cap N_j \subset J(\Omega \cap \pi^{-1}(\partial V_j))$. Note that $\mathcal{C} \subset \mathring{D}$ and set $K' = K \setminus \mathring{D}$. Then $K' \cap D = \partial D \subset \bigcup_j H_{k_j}^{-1}(H_{k_j}(\partial \overline{V_j}) \times \Gamma_{k_j}) \cup H_{k_j}^{-1}(H_{k_j}(\overline{V_j}) \times \partial \Gamma_{k_j})$ which has volume 0 again.

Denote $A_{Y_0} = Q \cap \bigcap_{j=1}^p V'_{Y_0}(k_j)$ and $S_{\mu_0} = \bigcap_{j=1}^p V'_{\mu_0}(k_j)$, by finiteness of the intersection, these are still open neighbourhoods of Y_0, μ_0 respectively, and we may assume, up to shrinking if necessary, that they are both bounded. Up to reducing once again A_{Y_0} to a ball, it will be convenient to assume it is convex.

By assumption, for $a = \exp(iY)$ with $Y \in Q$ and μ real in S_{μ_0} , $\mathcal{C}_{\mu_0} \subset \mathcal{C}$ so $F_{a,\mu}$ has no critical point in K' since $\mathcal{C} \cap K' = \emptyset$. By compactness of K' and since with these parameters, $\operatorname{Re} F_{a,\mu}(k) \leq 0$ for any $k \in K$, this implies that $\|T_k F_{\exp(iY),\mu}\|^2 - \operatorname{Re} F_{a,\mu}(k)$ is bounded below by $\delta > 0$ for $k \in K'$, $Y \in A_{Y_0}$ and $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$. Thus, by [Hör83, Thm. 7.7.1], for any $n \in \mathbb{N}$, there exists a semi-norm ν_{n,Y_0,μ_0} on $C^\infty(K')$ such that for any $g \in C^\infty(K')$, $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$ and $t \geq 1$,

$$(5.18) \quad \left| \int_{K'} e^{tF_{\exp(iY),\mu}(k)} g(k) dk \right| \leq \nu_{n,Y_0,\mu_0}(g) t^{-n}.$$

Now, as in Theorem 5.7, let $d_0 k$ denote the volume measure on K associated to the invariant Riemannian metric h induced by the inner product $-\langle \cdot, \cdot \rangle$ and $\operatorname{Vol}(K) = \int_K d_0 k$. Then the Haar measure dk is $\frac{1}{\operatorname{Vol}(K)} d_0 k$. Let also $G(k) = \det(h_k(\partial_i, \partial_j))$. Denote also

$$\tilde{g}_{j,m}(\mu, Y, X)(z_1, z_2) = g_j(\mu, Y, X)(k) \sqrt{G(k)}$$

where $(z_1, z_2) = H_{k_m}(k)$. Since by construction, the set of points counted several times is negligible, we can write that for any $1 \leq j \leq s$,

$$\begin{aligned} & \left| \int_K e^{tF_{\exp(iY),\mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \leq \left| \int_{K'} e^{tF_{\exp(iY),\mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \quad + \sum_{m=1}^p \left| \int_{N_m} e^{tF_{\exp(iY),\mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \leq \left| \int_K e^{tF_{\exp(iY),\mu}(k)} g_j(\mu, Y, X)(k) u_0(k) dk \right| \\ & \quad + \frac{1}{\operatorname{Vol}(K)} \sum_{m=1}^p \left| \int_{H_{k_m}(\overline{V_m})} \int_{\Gamma_{k_m}} e^{it\phi((Y,\mu,z_1),z_2)} \tilde{g}_{j,m}(\mu, Y, X)(z_1, z_2) dz_2 dz_1 \right| \end{aligned}$$

We can now apply (5.17), and (5.18) with $n = \lceil \kappa(G) \rceil$ to get that for any $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$, $X \in \mathfrak{a}^s$ and $t \geq 1$,

$$\left| \int_K e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \leq \nu_{\lceil \kappa(G) \rceil, Y_0, \mu_0}(g_j(\mu, Y, X)u_0)t^{-\lceil \kappa(G) \rceil} \\ + \sum_{m=1}^p \frac{C_{Y_0, \mu_0, k_i} \text{Vol } H_{k_m}(\overline{V_m})}{\text{Vol}(K)} \|\tilde{g}_{j,m}(\mu, Y, X)\|_{\infty} t^{-\kappa(G)}$$

Now since the functions g_j are smooth in all variables, they are bounded in $C^\infty(K)$ when μ, Y, X remain bounded, thus there exists a constant $D_{j, Y_0, \mu_0} > 0$ such that for any $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$, $X \in \mathfrak{a}^s$ with $\|X_i\| = 1$ and any $t \geq 1$,

$$(5.19) \quad \left| \int_K e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \leq D_{j, Y_0, \mu_0} t^{-\kappa(G)}.$$

Since S is compact and $S \subset \bigcup_{\mu_0 \in S} S_{\mu_0}$, we can extract a finite cover $S_{\mu_1}, \dots, S_{\mu_q}$. Set

$$M_{Y_0, s} = \max_{1 \leq k \leq q} \sum_{j=1}^s D_{j, Y_0, \mu_k}.$$

Combining (5.19) with (5.16), for any $Y \in A_{Y_0}$, $\mu \in S$ and any $t \geq 1$,

$$(5.20) \quad \|D^s \Psi_{t\mu}(Y)\| = \sup_{\|X_i\|=1} |D^s \Psi_{t\mu}(Y)(X)| \leq M_{Y_0, s} t^{s-\kappa(G)}.$$

Note that $\Lambda \setminus \{0\} \subset \mathbb{R}_{\geq 1} S$, so the previous equation immediately implies that for any $\mu \in \Lambda \setminus \{0\}$ and $s \leq r$,

$$(5.21) \quad \|D^s \Psi_{t\mu}(Y)\| \leq M_{Y_0, s}.$$

Thus if $\kappa(G)$ is an integer, the proof is complete.

Otherwise, $\delta = \kappa(G) - r = \frac{1}{2}$. Then applying (5.20) with $s = r$ and triangular inequality gives that for any $Y, Y' \in A_{Y_0}$, $\mu \in S$, $t \geq 1$,

$$\|D^r \Psi_{t\mu}(Y) - D^r \Psi_{t\mu}(Y')\| \leq 2M_{Y_0, r} t^{-1/2}.$$

On the other hand, using (5.20) with $s = r + 1$ and the mean value theorem gives that for any $Y, Y' \in A_{Y_0}$, $\mu \in S$, $t \geq 1$,

$$\|D^r \Psi_{t\mu}(Y) - D^r \Psi_{t\mu}(Y')\| \leq \sup_{Z \in A_{Y_0}} \|D^{r+1} \Psi_{t\mu}(Z)\| \|Y - Y'\| \leq M_{Y_0, r+1} t^{1/2} \|Y - Y'\|.$$

Combining both estimates, we get that for any $Y, Y' \in A_{Y_0}$, $\mu \in \Lambda \setminus \{0\}$,

$$(5.22) \quad \|D^r \Psi_{t\mu}(Y) - D^r \Psi_{t\mu}(Y')\| \leq (2M_{Y_0, r} M_{Y_0, r+1})^{1/2} \|Y - Y'\|$$

which completes the proof when $\kappa(G)$ is not an integer. $\square$

Theorem 5.15 (Theorem B, Part 2). *Consider (U, K) as before. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Let $Q_0 = \{Y \in Q \mid \exp(iY) \in V\}$ and $U_0 = K \exp(iQ_0)K \subset U_r$. Then any K -finite matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(U_0)$.*

Proof. By Theorem 5.14, Proposition 5.1 and Lemma 2.3, the family of spherical functions of (U, K) is bounded in $C^{(r, \delta)}(U_0)$. By Lemma 2.7, it follows that any K -bi-invariant matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(U_0)$. Finally, [Dum24, Thm. 5.2] allows to extend this regularity to K -finite coefficients. $\square$

Remark 5.16. Combining Theorems 5.8 and 5.15, we get that Conjecture 5.9 is true for any compact symmetric pair (U, K) , but only in some open subset U_0 and not all of U_r .

The same proof cannot extend this regularity to all of U_r . It is clear that for any $g \in K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ - in particular for any $a \in U_r \cap A_{\mathbb{C}}$ - we can consider an analytic extension of H in a neighbourhood of g . However, since g is not a fixed point of the action by conjugation it cannot be chosen K -invariant as in Lemma 5.3, thus we cannot get the integral expression of spherical functions around g to work with.

However Clerc gave in [Cle88] a multivalued analytic extension of H to all of $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$, as well as an integral formula of spherical functions of (U, K) very similar to Lemma 5.4. But the domain of integration in the expression of $\varphi_{\mu}(g)$ is now an open subset K_g of K . Thus, the lack of compactness does not allow for a similar proof.

When μ is regular - meaning that $\langle \mu, \alpha \rangle \neq 0$ for any root α - Clerc managed to reduce this integral to a compact subset and get precise estimate of spherical functions. However, this is not sufficient for our purposes, and this does not work when μ is not regular.

5.3. Non simply connected groups. Until that point, we assumed U to be simply connected. We will now prove that we can drop this assumption and extend our results to any symmetric pair (U, K) of compact type.

First, it is pointed out in [Dum24, Lemma 2.9] that the spherical functions of (U, K) depend only on the symmetric space of compact type $M = U/K$. Thus, as soon as M is simply connected, our results extend to the pair (U, K) . However, when M is no longer simply connected, the situation becomes more complicated: the pair (U, K) may have fewer spherical functions.

Let U be a connected semisimple compact Lie group, let σ be an involutive automorphism of U and K a subgroup such that $(U^{\sigma})_0 \subset K \subset U^{\sigma}$. Let $M = U/K$ be the associated symmetric space. Let $\tilde{U}$ be the universal cover of U , σ induces an automorphism of $\tilde{U}$ and the subgroup $\tilde{K} = \tilde{U}^{\sigma}$ is connected (by [Bor98, Ch.V, Thm 3.3]). Then $\tilde{M} = \tilde{U}/\tilde{K}$ is simply connected and is the universal cover of M . If $p: \tilde{U} \rightarrow U$ is the covering map, then $F = \ker p$ is a subgroup of the finite group $Z(\tilde{U})$ and $p(\tilde{K}) = K_0$.

Recall that σ induces the decomposition $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Then choose $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{u}$ containing $i\mathfrak{a}$, Φ the root system of $\mathfrak{h}_{\mathbb{C}}$ and Φ a choice of positive roots, such that $\Sigma^+ = \{\phi|_{\mathfrak{a}} \mid \phi \in \Phi^+, \phi|_{\mathfrak{a}} \neq 0\}$.

We recall some facts on the representation theory of compact Lie groups (see [Kna02, Ch. V]). We say that $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ is a dominant element if for any $\alpha \in \Phi^+$, $\langle \mu, \alpha \rangle \geq 0$. We say that μ is (algebraically) integral if

$$\forall \alpha \in \Sigma_{\mathfrak{h}}, 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

We say that μ is analytically integral if for any $H \in \mathfrak{h}$ such that $\exp(H) = 1$, $\mu(H) \in 2\pi i\mathbb{Z}$. Let $P(\mathfrak{u})$ be the set of dominant integral element and $P(U)$ the set of dominant analytically integral elements. Note that $P(U) \subset P(\mathfrak{u})$ and that $P(\tilde{U}) = P(\mathfrak{u})$ ([Kna02, Thm. 5.107]). Finally, define a partial order on dominant integral element: λ is higher than μ if $\lambda - \mu$ is a combination of roots in Φ^+ with coefficients in $\mathbb{N}$.

By the theorem of the highest weight ([Kna02, Thm. 5.110]), there is a one-one correspondence between the set $\hat{U}$ of finite-dimensional irreducible representations of U (up to isomorphism) and $P(U)$. More precisely, if π is a finite-dimensional representation of $\mathfrak{u}$ on V , then V is the direct sum of its weight space, i.e. $V = \bigoplus V^{(\lambda)}$ where

$$V^{(\lambda)} = \{v \in V \mid \forall H \in \mathfrak{h}, \pi(H)v = \lambda(H)v\}.$$

If π is a finite-dimensional representation of U , we still denote the representation of $\mathfrak{u}$ induced by its differential π . A highest weight of π is a weight which is higher than every other weight appearing in the decomposition of V . Then the theorem of the highest weight states that

- (1) If (π, V) is an irreducible representation of U , there exists a unique highest weight λ , and $\lambda \in P(U)$;
- (2) If two representations have the same highest weight, they are isomorphic;
- (3) For any $\lambda \in P(U)$, there exists an irreducible representation (π_λ, V_λ) with highest weight λ , and the weight space of λ is one-dimensional.

Of course, this theorem applies to $\tilde{U}$ with the set of dominant integral elements $P(\mathfrak{u})$ (equivalently, to the Lie algebra $\mathfrak{u}$).

If $\lambda \in P(U)$, denote v_λ a generator of the weight space of λ . If $\lambda, \mu \in P(U)$, consider the tensor representation $V = V_\lambda \otimes V_\mu$. Then the decomposition into irreducible representations of V is

$$V = \bigoplus V_\nu^{\oplus m_\nu}$$

with $m_{\lambda+\mu} = 1$, and the weight space of $\lambda + \mu$ is generated by $v_\lambda \otimes v_\mu$.

Finally, recall that by the Cartan-Helgason theorem (Theorem 5.2), if we set

$$\Lambda = \left\{ \mu \in i\mathfrak{a}^* \mid \forall \alpha \in \Sigma_{\mathfrak{a}}^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \right\},$$

then the map which sends a representation to its highest weight is a bijection from $\hat{U}_{\tilde{K}}$ to Λ , where $\hat{U}_{\tilde{K}}$ is the subset of classes of irreducible representations of $\tilde{U}$ with a nonzero $\tilde{K}$ -invariant vector.

We now want to understand the set Λ_M of highest weight of representations of U with a nonzero K -invariant vector.

Proposition 5.17. *Let $\Lambda_M = \{\lambda \in P(U) \mid V_\lambda^K \neq \{0\}\}$. Then there exists $N \in \mathbb{N}$ such that*

$$N\Lambda \subset \Lambda_M \subset \Lambda.$$

Proof. One inclusion is easy: if $\lambda \in \Lambda_M$, there exists an irreducible representation $\pi_\lambda \in \hat{U}_K$ of highest weight λ . Let ξ_λ be a generator of the space of K -invariant vector. Consider the composition $\tilde{\pi}_\lambda = \pi_\lambda \circ p$, then $\tilde{\pi}_\lambda \in \hat{\tilde{U}}$ and ξ_λ is $p^{-1}(K)$ -invariant. Since $\tilde{K} \subset p^{-1}(K_0) \subset p^{-1}(K)$, $\lambda \in \Lambda$.

For the other inclusion, we first claim that there is $N_0 \in \mathbb{N}$ such that $N_0 P(\mathfrak{u}) \subset P(U)$. Indeed, if $\lambda \in P(\mathfrak{u})$, there is an irreducible representation $\pi_\lambda \in \hat{\tilde{U}}$ on V_λ . Let v_λ be a nonzero vector of weight λ . Let $F = \ker p \subset Z(\tilde{U})$. Then $\lambda \in P(U)$ if and only if π_λ descends to U , i.e. $\pi_\lambda|_F = \text{Id}$. Now since F is central in $\tilde{U}$ and π_λ is irreducible, by Schur's lemma F acts by scalar multiplication. Thus, there exists a morphism $\tau : F \rightarrow S^1$ such that $\pi_\lambda(x) = \tau_\lambda(x) \text{Id}$ for any $x \in F$. Let $n \in \mathbb{N}$,

consider $v_{n\lambda} = v_\lambda^{\otimes n} \in V_{n\lambda}$ seen as a subrepresentation of $V_\lambda^{\otimes n}$. Then on the one hand, for any $x \in F$, $\pi_{n\lambda}(x)v_{n\lambda} = \tau_{n\lambda}(x)v_{n\lambda}$. On the other hand, $\pi_{n\lambda}$ acts as the tensor representation so

$$\pi_{n\lambda}(x)v_{n\lambda} = (\pi_\lambda(x)v_\lambda)^{\otimes n} = (\tau_\lambda(x))^n v_\lambda^{\otimes n} = \tau_\lambda(x^n)v_{n\lambda}.$$

Thus, we obtain that for any $\lambda \in P(\mathfrak{u})$, $x \in F$, $n \in \mathbb{N}$, $\tau_{n\lambda}(x) = \tau_\lambda(x^n)$. Thus, let $N_0 = |F|$ the order of the finite group F , then $\tau_{N_0\lambda} = 1$ for any $\lambda \in P(\mathfrak{u})$, so $\pi_{N_0\lambda}$ induces an irreducible representation of U . Thus,

$$(5.23) \quad N_0 P(\mathfrak{u}) \subset P(U).$$

Let $\lambda \in \Lambda \cap P(U)$. Then there exists an irreducible representation (π_λ, V_λ) of U such that $\pi_\lambda = \tilde{\pi}_\lambda \circ p$ is an irreducible representation of $\tilde{U}$, and a nonzero $\tilde{K}$ -invariant vector ξ_λ . By the proof of the Cartan-Helgason theorem ([Wol07, Thm. 11.4.10]), the generator ξ_λ can be chosen as $\Pi(v_\lambda)$ where Π is the projection on the space of K_0 -invariant vectors. In particular, this implies that $\langle \xi_\lambda, v_\lambda \rangle \neq 0$. Clearly since $p(\tilde{K}) = K_0$, ξ_λ is K_0 -invariant. We want to understand when ξ_λ is actually invariant under the whole subgroup K and not only its connected component.

Since (U, K_0) is a Gelfand pair, the subspace $V_\lambda^{K_0}$ is one-dimensional, generated by ξ_λ (Proposition 2.5). The connected component K_0 is a normal subgroup of K , so we get that for any $k \in K$, $\pi_\lambda(k)\xi_\lambda$ is also a K_0 -invariant vector, thus a multiple of ξ_λ . Thus K acts on $V_\lambda^{K_0}$ by scalar multiplication. Since K_0 acts trivially, this induces a morphism

$$\chi_\lambda : K/K_0 \rightarrow S^1$$

where K/K_0 is a finite group. Then ξ_λ is K -invariant if and only if this character χ_λ is trivial. We will use a similar strategy as in the case of τ to prove that χ is multiplicative. Let $\lambda, \mu \in \Lambda \cap P(U)$. We claim that

$$\chi_{\lambda+\mu} = \chi_\lambda \chi_\mu.$$

As explained before, consider $V_{\lambda+\mu}$ as a subrepresentation of $W = V_\lambda \otimes V_\mu$.

Let $\xi = \xi_\lambda \otimes \xi_\mu$. Then for any $k \in K$, we have that

$$(5.24) \quad (\pi_\lambda \otimes \pi_\mu)(k)\xi = \chi_\lambda(kK_0)\xi_\lambda \otimes \chi_\mu(kK_0)\xi_\mu = \chi_\lambda(kK_0)\chi_\mu(kK_0)\xi.$$

However ξ need not be in $V_{\lambda+\mu}$, so we cannot argue directly as above. As a U -representation,

$$W = \bigoplus_\nu V_\nu^{\oplus m_\nu}$$

and thus, the space of K_0 -invariant vector is

$$W^{K_0} = \bigoplus_\nu \bigoplus_{i=1}^{m_\nu} V_\nu^{K_0}.$$

On each summand $V_\nu^{K_0}$, either $V_\nu^{K_0}$ is trivial, or it is one-dimensional generated by ξ_μ and K/K_0 acts by multiplication by χ_ν . Since $\xi \in W^{K_0}$, there exists a unique decomposition

$$\xi = \sum_\nu \sum_{i=1}^{m_\nu} a_{\nu,i} \xi_{\nu i}.$$

Then for any $k \in K$,

$$(5.25) \quad (\pi_\lambda \otimes \pi_\mu)(k)\xi = \sum_\nu \sum_{i=1}^{m_\nu} a_{\nu,i} \pi_\nu(k)\xi_{\nu i} = \sum_\nu \sum_{i=1}^{m_\nu} a_{\nu,i} \chi_\nu(kK_0) \xi_{\nu i}.$$

By uniqueness of the decomposition of ξ , we obtain combining (5.24) and (5.25) that for any ν such that there is i with $a_{\nu,i} \neq 0$, then

$$\chi_\nu = \chi_\lambda \chi_\mu.$$

It now suffices to show that $a_{\lambda+\mu,1} \neq 0$ - recall that $m_{\lambda+\mu} = 1$. But

$$\langle \xi, v_\lambda \otimes v_\mu \rangle = \langle \xi_\lambda, v_\lambda \rangle \langle \xi_\mu, v_\mu \rangle \neq 0$$

so ξ has a nonzero $V_{\lambda+\mu}$ -component. Thus, we obtain $\chi_{\lambda+\mu} = \chi_\lambda \chi_\mu$ so for any $n \in \mathbb{N}$, $\lambda \in \Lambda \cap P(U)$, $\chi_{n\lambda} = \chi_\lambda^n$.

Let $N_1 = |K/K_0|$ be the order of the finite group K/K_0 , then $\chi_{N_1\lambda} = 1$ for all $\lambda \in \Lambda \cap P(U)$ thus $\pi_{N_1\lambda}$ has a nonzero K -invariant vector. Thus,

$$(5.26) \quad N_1(\Lambda \cap P(U)) \subset \Lambda_M.$$

Set $N = N_0 N_1$, combining (5.23) and (5.26), we obtain

$$N\Lambda = N(\Lambda \cap P(\mathfrak{u})) \subset N_1(\Lambda \cap N_0 P(\mathfrak{u})) \subset N_1(\Lambda \cap P(U)) \subset \Lambda_M$$

which completes the proof. $\square$

Theorem 5.18. *Let (U, K) be a symmetric pair of compact type, let $\tilde{M} = \tilde{U}/\tilde{K}$ be the universal cover of U/K with $\tilde{U}$ the universal cover of U . Then $(\tilde{U}, \tilde{K})$ is the compact dual of the pair $(G, \tilde{K})$ for some semisimple Lie group with finite center G . Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Then*

- (i) *there exists an open set V in U such that any K -finite matrix coefficient of U is in $C^{(r,\delta)}(V)$;*
- (ii) *for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of a unitary representation of U which is not in $C^{(r,\delta')}(p(\tilde{U}_r))$.*

Furthermore, the regularity in (i) can be improved to all of $p(\tilde{U}_r)$ whenever $(\tilde{U}, \tilde{K})$ is a pair of rank 1, of the form $(\tilde{K} \times \tilde{K}, \tilde{K})$ or one of the pairs of Corollary 5.10.

Proof. (i). This is the easy part: let $p : \tilde{U} \rightarrow U$ be the covering map. If φ is a K -finite coefficient of a representation π , then $\varphi \circ p$ is a $\tilde{K}$ -finite coefficient of $\tilde{U}$. Thus by Theorem 5.15, there exists an open set U_0 such that $\varphi \circ p \in C^{(r,\delta)}(U_0)$. Since p is a local diffeomorphism, for any $x \in V = p(U_0)$, we can find a neighborhood $U_x \subset V$, and an open $\tilde{U}_x \subset U_0$ such that $p : \tilde{U}_x \rightarrow U_x$ is a diffeomorphism. Thus by Lemma 2.3, $\varphi \in C^{(r,\delta)}(U_x)$. Thus $\varphi \in C^{(r,\delta)}(V)$.

To obtain the improvement in specific cases, we use the results of [Dum24].

(ii). If $\delta' > \delta$, we know from Theorem 5.7 that the family $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$ of spherical functions of $(\tilde{U}, \tilde{K})$ is not bounded in $C^{(r,\delta)}(U_r)$. Let N be given by Proposition 5.17 so that $N\Lambda \subset \Lambda_{U/K}$. Then it is easy to see from the proof of Theorem 5.7 that the subfamily $(\tilde{\varphi}_\lambda)_{\lambda \in N\Lambda}$ is already not bounded in $C^{(r,\delta)}(U_r)$. But since $N\Lambda \subset \Lambda_{U/K}$, $\tilde{\varphi}_\lambda = \varphi_\lambda \circ p$ with φ_λ a spherical function of (U, K) . Since p is smooth, by Lemma 2.3, $(\varphi_\lambda)_{\lambda \in N\Lambda}$ is not bounded in $C^{(r,\delta)}(p(U_r))$. We then conclude with Lemma 2.7 as in the proof of Theorem 5.8. $\square$

STATEMENTS

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability. No datasets were generated or analyzed during the current study.

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APPENDIX A. COMPUTATION OF $\kappa(G)$

If G is semisimple with finite center, then $G/Z(G) = \prod_{i=1}^n G_i$ with G_i simple. Then, $\kappa(G) = \min_{i \text{ s.t. } G_i \text{ not compact}} \kappa(G_i)$. Indeed, the restricted root system of G is the direct sum of the restricted root system of the G_i 's, and the restricted root system of a compact group is trivial (because it is its own maximal compact subgroup). Thus, it suffices to compute κ for non-compact simple Lie groups.

Table 1 deals with complex simple Lie groups (viewed as real Lie groups) and their compact real form U . In this case, all multiplicities are 2. In Table 2, M is the compact symmetric space dual to G/K , that is to say U/K where U is a compact real form of G . The computation uses the classification and multiplicities for such symmetric spaces given in [Loo69, Ch. VII].

TABLE 1. Values of $\kappa(G)$ for complex simple Lie groups.

G	U	$\dim G$	$\text{rank } G$	$\kappa(G)$
$SL(n, \mathbb{C})$	$SU(n), n \geq 2$	$2(n^2 - 1)$	$n - 1$	$n - 1$
$SO(2n + 1, \mathbb{C})$	$SO(2n + 1), n \geq 1$	$2n(2n + 1)$	n	$2n - 1$
$Sp(2n, \mathbb{C})$	$Sp(n), n \geq 1$	$2n(2n + 1)$	n	$2n - 1$
$SO(2n, \mathbb{C})$	$SO(2n), n \geq 2$	$2n(2n - 1)$	n	1 if $n = 2$ 3 if $n = 3$ $2n - 2$ else
$(G_2)_{\mathbb{C}}$	G_2	28	2	5
$(F_4)_{\mathbb{C}}$	F_4	104	4	15
$(E_6)_{\mathbb{C}}$	E_6	156	6	16
$(E_7)_{\mathbb{C}}$	E_7	266	7	27
$(E_8)_{\mathbb{C}}$	E_8	496	8	57

TABLE 2. Values of $\kappa(G)$ for real simple Lie groups.

M		G	$\text{rank } G$	$\kappa(G)$
AI	$SU(n)/SO(n), n \geq 2$	$SL(n, \mathbb{R})$	$n - 1$	$\frac{n-1}{2}$
AII	$SU(2n)/Sp(n), n \geq 2$	$SU^*(2n)$	$n - 1$	$2(n - 1)$
$AIII$	$SU(p+q)/S(U(p) \times U(q)), p+q \geq 3$	$SU(p, q)$	$\min(p, q)$	2 if $p = q = 2$ $p + q - \frac{3}{2}$ else
BDI	$SO(p+q)/SO(p) \times SO(q), p+q \geq 3$	$SO_0(p, q)$	$\min(p, q)$	$\frac{1}{2}$ if $p = q = 2$ $\frac{3}{2}$ if $p = q = 3$ $\frac{p+q}{2} - 1$ else
CI	$Sp(n)/U(n), n \geq 1$	$Sp(2n, \mathbb{R})$	n	$n - \frac{1}{2}$
CII	$Sp(p+q)/Sp(p) \times Sp(q), p+q \geq 2$	$Sp(p, q)$	$\min(p, q)$	5 if $p = q = 2$ $2(p+q) - \frac{5}{2}$ else
$DIII$	$SO(4n)/U(2n), n \geq 1$	$SO^*(4n)$	n	$n(n - \frac{1}{2})$ if $n \leq 3$ $4n - \frac{7}{2}$ if $n > 3$
	$SO(4n+2)/U(2n+1), n \geq 1$	$SO^*(4n+2)$	n	$4n - \frac{3}{2}$
EI		$E_{6(6)}$	6	8
EII		$E_{6(2)}$	4	$\frac{21}{2}$
$EIII$		$E_{6(-14)}$	2	$\frac{21}{2}$
EIV		$E_{6(-26)}$	2	8
EV		$E_{7(7)}$	7	$\frac{27}{2}$
EVI		$E_{7(-5)}$	4	$\frac{33}{2}$
$EVII$		$E_{7(-24)}$	3	$\frac{27}{2}$
$EVIII$		$E_{8(8)}$	8	$\frac{57}{2}$
EIX		$E_{8(-24)}$	4	$\frac{57}{2}$
FI		$F_{4(4)}$	4	$\frac{15}{2}$
FII		$F_{4(-20)}$	1	$\frac{15}{2}$
G		$G_{2(2)}$	2	$\frac{5}{2}$