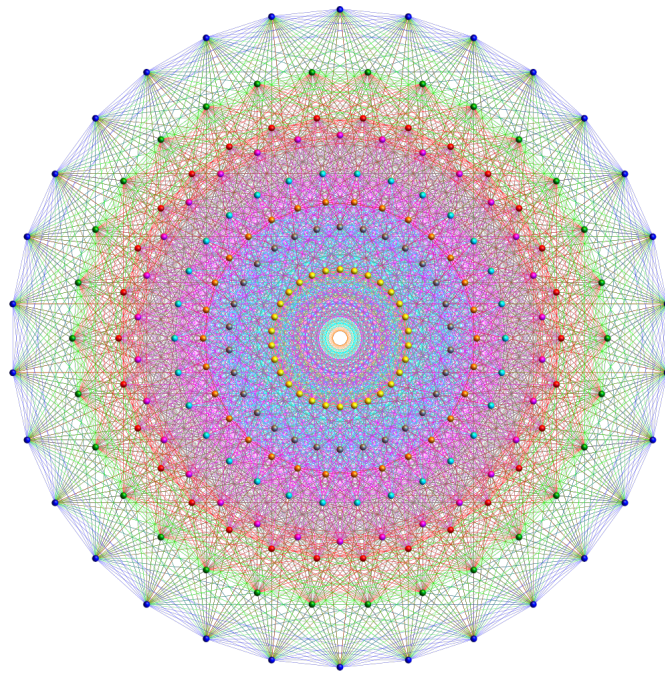


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Quelques propriétés de rigidité des groupes semi-simples :

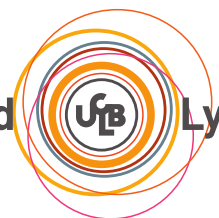
régularité de coefficients matriciels et rigidité de quasi-homomorphismes



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Thèse de doctorat

Université Claude Bernard



Lyon 1

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Quelques propriétés de rigidité des groupes semi-simples : régularité de coefficients matriciels et rigidité de quasi-homomorphismes

Thèse de doctorat

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Introduction (version française)

Cette thèse est consacrée à l'étude de plusieurs propriétés de rigidité des groupes de Lie semi-simples. Le manuscrit est divisé en deux parties indépendantes. La première partie porte sur l'étude de résultats de régularité pour les coefficients matriciels des représentations unitaires des groupes de Lie semi-simples (ainsi que d'autres groupes étroitement liés). Ces questions de régularité trouvent leur origine dans les travaux de Vincent Lafforgue sur la propriété (T) renforcée. Dans la seconde partie de la thèse, nous considérons un renforcement complètement différent de la propriété (T) : nous étudions la propriété (TTT) , introduite par Narutaka Ozawa pour comprendre les quasi-homomorphismes. Contrairement à la propriété (T) , la propriété (TTT) étudie des objets qui ne sont plus nécessairement des représentations de groupe. Nous considérons également des versions de cette propriété dans le cadre des espaces de Banach.

Régularité de coefficients matriciels

Dans la première partie de cette thèse, nous étudions la régularité et le comportement local des coefficients matriciels K -bi-invariants et K -finis des représentations unitaires de certains groupes de Lie G avec K un sous-groupe compact. Plus précisément, nous cherchons à déterminer la plus grande valeur de $\alpha \geq 0$ telle que tout coefficient matriciel K -fini d'une représentation unitaire de G soit localement α -Hölderien. Puisque les morphismes de groupes continus entre groupes de Lie sont lisses, si π est une représentation unitaire de dimension finie de G , alors ses coefficients sont lisses. En particulier, par le théorème de Peter-Weyl, tout coefficient matriciel d'une représentation unitaire irréductible d'un groupe de Lie compact G est lisse.

Dans le cadre non compact, si G est semi-simple et si K est un sous-groupe compact maximal de G , il est connu, grâce aux travaux de Harish-Chandra ([HC53]), que les coefficients matriciels associés aux vecteurs K -finis des représentations unitaires irréductibles de G (et plus généralement des représentations admissibles) sont de classe C^∞ . Les représentations unitaires de G se décomposent en intégrales directes de représentations irréductibles, mais comme les estimées dépendent des représentations, cela ne fournit pas d'estimées générales pour des représentations arbitraires. Un autre résultat important est le fait que, bien que tous les vecteurs ne soient pas lisses dans une représentation arbitraire, l'ensemble des vecteurs lisses y est dense ([Kna01, Ch.3]).

L'exemple des groupes semi-simples appartient à une classe plus large de paires (G, K) où le sous-groupe K est tel que (G, K) est une paire de Gelfand. Dans ce cas, il existe une correspondance bijective entre les fonctions sphériques définies positives de (G, K) , qui apparaissent comme les caractères de l'algèbre de convolution (abélienne) des fonctions K -bi-invariantes continues à support compact, et les représentations unitaires irréductibles possédant des vecteurs K -invariants non nuls. Dans ce cadre, tout coefficient matriciel

K -bi-invariant d'une représentation unitaire de G se décompose comme une intégrale directe de fonctions sphériques définies positives (Section 1.2). Ainsi, établir des estimées uniformes sur la famille de ces fonctions sphériques permet d'obtenir des estimées sur tout coefficient matriciel K -bi-invariant d'une représentation unitaire de G . De plus, comme nous le verrons en Section 1.3, nous pourrions étendre cette méthode aux coefficients K -finis sous certaines hypothèses supplémentaires sur (G, K) .

Cette approche a été utilisée par Lafforgue pour montrer que les coefficients $SO(2)$ -bi-invariants de $SO(3)$ sont $\frac{1}{2}$ -Hölder continus en dehors des points singuliers, un ingrédient clé dans sa preuve de la propriété (T) renforcée pour $SL(3, \mathbb{R})$ ([Laf08]). Grâce à ce résultat, il a obtenu des estimées asymptotiques des coefficients de représentations bien plus générales que les représentations unitaires sur des espaces de Hilbert (voir aussi [dLS22] pour un article de survol). Ce résultat a ensuite été amélioré dans [dLdLS15] pour tous les groupes de Lie simples de rang supérieur sur une large classe d'espaces de Banach. Cette idée a permis de résoudre divers problèmes en algèbre d'opérateurs avec différentes paires, mais aussi dans d'autres domaines. Les mêmes techniques s'étendent à l'étude des multiplicateurs de Fourier ([PRdLS22]). Dans [LdLS11], [HdL13] et [HdL16], elles sont utilisées pour montrer que les groupes de Lie simples de rang supérieur ne possèdent pas la propriété d'approximation. Ces résultats impliquent également des conclusions sur l'absence de plongements grossiers pour certaines familles d'expansions ([dLdLS18]). En systèmes dynamiques, la propriété (T) forte a également joué un rôle important dans le programme de Zimmer ([BFH22], [BFH20], [BFH21], [Bro22], [Fis22]).

Le résultat de Lafforgue sur $(SO(3), SO(2))$ repose sur l'analyse harmonique de la paire de Gelfand, comme nous l'avons expliqué ci-dessus – bien que sans employer cette terminologie – et sur le fait que les fonctions sphériques dans ce cas sont simplement les polynômes de Legendre, ce qui rend l'analyse plus accessible. Dans cette thèse, nous souhaitons étudier plus systématiquement cette régularité locale pour les paires de Gelfand symétriques, c'est-à-dire lorsque G/K est un espace symétrique. Un espace symétrique riemannien est une variété riemannienne munie d'une structure supplémentaire. Du point de vue de la théorie de Lie, il peut être défini comme un espace homogène $M = G/K$ où G est un groupe de Lie connexe et K un sous-groupe compact ouvert du groupe des points fixes d'un automorphisme involutif de G . Dans ce cas, (G, K) est une paire de Gelfand. Lorsque M est simplement connexe, il se décompose en un produit d'un espace euclidien, d'un espace de type compact et d'un espace de type non compact (cette terminologie sera précisée dans la Section 1.4). De manière informelle, les espaces de type non compact correspondent aux espaces homogènes G/K où G est un groupe de Lie semi-simple de centre fini et K un sous-groupe de Lie compact maximal. Les espaces de type compact sont de la forme U/K où U est un groupe de Lie semi-simple compact. Dans ce cas, U peut être vu comme une forme réelle compacte d'un groupe semi-simple, ce qui établit une dualité entre espaces symétriques. Nous étudierons ces deux types d'espaces ainsi que certaines paires euclidiennes liées aux groupes semi-simples (voir Section 1.7).

Dans ce cadre, de nombreux résultats existent sur les fonctions sphériques de la paire (G, K) ([Hel79], [Hel00]). Ces fonctions sont paramétrées par un sous-ensemble Λ de $\mathfrak{g}_{\mathbb{C}}^*$, où \mathfrak{g} est l'algèbre de Lie de G . Nous appelons $\lambda \in \Lambda$ le paramètre spectral (voir Section 1.5). Les asymptotiques des fonctions sphériques en la variable du groupe ont été largement étudiées, notamment dans les travaux de Harish-Chandra, mais leur comportement local est beaucoup moins bien compris. Pour obtenir des informations sur ce comportement local, l'outil principal consiste à examiner le comportement asymptotique lorsque le paramètre spectral tend vers l'infini, tandis que la variable du groupe reste dans un compact. Une remarque importante est que les estimées souhaitées échouent au voisinage de l'identité de G , ainsi qu'en tout autre point singulier (voir la Définition 1.22 et la Proposition 3.8 pour

la définition des points singuliers et réguliers). Ainsi, nous n'obtiendrons la régularité des coefficients matriciels que sur l'ouvert dense des points réguliers.

Définition 1. Soit (X, d) un espace métrique et U un sous-ensemble ouvert de X , $(E, \|\cdot\|)$ un espace vectoriel normé, $\alpha \in]0, 1]$. Une fonction $f : U \rightarrow E$ est dite α -Hölder si, pour tout sous-ensemble compact K de U , il existe une constante $C_K > 0$ telle que $\forall x, y \in K$, $\|f(x) - f(y)\| \leq C_K d(x, y)^\alpha$.

Si X est aussi un espace vectoriel normé et $r \in \mathbb{N}$, on dit que l'application f appartient à $C^{(r, \alpha)}(U, E)$ si $f \in C^r(U, E)$ et si la r -ième différentielle $D^r f$ est α -Hölder en tant qu'application de U dans l'espace vectoriel des formes r -linéaires. On étend cette définition à $\alpha = 0$ en posant $C^{(r, 0)}(U, E) = C^r(U, E)$.

Pour K un sous-ensemble compact de U et $f \in C^{(r, \alpha)}(U, E)$, on définit

$$\|f\|_{C^{(r, \alpha)}(K, E)} = \max \left\{ \max_{k \leq r} \sup_{x \in K} \|D^k f(x)\|, \sup_{x, y \in K, x \neq y} \frac{\|D^r f(x) - D^r f(y)\|}{d(x, y)^\alpha} \right\}.$$

La famille de semi-normes $\|\cdot\|_{C^{(r, \alpha)}(K, E)}$ pour K un sous-ensemble compact de U munit l'espace $C^{(r, \alpha)}(U, E)$ d'une structure d'espace de Fréchet.

Enfin, si (X, d) est une variété riemannienne, on dit que $f \in C^{(r, \alpha)}(U, E)$ si, pour toute carte (φ, V) de U , on a $f \circ \varphi^{-1} \in C^{(r, \alpha)}(\varphi(V), E)$.

Soit G un groupe de Lie et K un sous-groupe compact. Nous cherchons à déterminer (r, α) tel que pour tout coefficient matriciel unitaire K -fini φ de G , on ait $\varphi \in C^{(r, \alpha)}(G_r)$, où G_r est l'ensemble ouvert dense des points réguliers (voir encore la Définition 1.22 et la Proposition 3.8). Si l'on suppose que (G, K) est une paire de Gelfand, l'outil principal sera l'étude de la bornitude de la famille des fonctions sphériques dans les espaces de Hölder. Si de plus (G, K) est une paire symétrique, alors G/K est un espace symétrique et il existe des résultats dans la littérature sur les fonctions sphériques.

Nous commençons au Chapitre 2 par l'étude des paires où G est lui-même un groupe compact. Notre premier résultat concerne les paires symétriques compactes de rang 1 (voir la Section 1.5 pour la notion de rang).

Théorème A. *Soit (G, K) une paire symétrique compacte de rang 1 et G_r l'ensemble ouvert dense des points réguliers. Soit*

$$\alpha = \frac{\dim G/K - 1}{2}.$$

Alors tout coefficient matriciel unitaire K -fini φ de G appartient à $C^{([\alpha], \alpha - [\alpha])}(G_r)$. De plus, cette régularité est optimale, c'est-à-dire que pour tout $(r, \delta) > ([\alpha], \alpha - [\alpha])$ dans l'ordre lexicographique, il existe un coefficient matriciel unitaire K -fini (et même K -bi-invariant) de G qui n'appartient pas à $C^{(r, \delta)}(G_r)$.

Nous prouvons également, dans le Corollaire 2.13, un résultat de régularité pour les multiplicateurs de Herz-Schur de $S_p(L^2(G))$ pour tout p .

Nous nous tournons ensuite vers les paires symétriques compactes de rang supérieur. Parmi elles, une classe simple est donnée par les paires $(G \times G, G)$ où G est un groupe de Lie compact semi-simple. L'espace symétrique associé à ces paires est isomorphe au groupe de Lie G . Étant donné Φ un système de racines pour G , Φ^+ un choix de racines positives et $\Delta = \alpha_1, \dots, \alpha_\ell$ une base, on peut écrire $\alpha = \sum_{i=1}^{\ell} n_i(\alpha) \alpha_i$ for $\alpha \in \Phi^+$ (voir la Section 1.5 pour plus de détails sur les systèmes de racines). Notre second résultat principal donne alors la régularité pour ces paires.

Théorème B. Soit $\gamma = \min_{1 \leq i \leq \ell} |\{\alpha \in \Phi^+ \mid n_i(\alpha) \geq 1\}|$ et G_r l'ensemble des points réguliers de $G \times G$. Alors tout coefficient matriciel G -fini d'une représentation unitaire de $G \times G$ appartient à $C^{(\gamma,0)}(G_r)$ et cette régularité est optimale.

Étant donné tous ces résultats, nous formulons une conjecture sur la régularité optimale dans le cas général. Soit (G, K) une paire symétrique compacte, il existe une décomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ de l'algèbre de Lie. Soit \mathfrak{a} un sous-espace abélien maximal de \mathfrak{p} et Σ le système de racines. Soit $\Lambda = \left\{ \mu \in i\mathfrak{a}^* \mid \forall \alpha \in \Sigma^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \right\}$. Pour $\alpha \in \Sigma$, notons $m(\alpha)$ la multiplicité de la racine.

Conjecture C. Soit G_r l'ensemble des points réguliers de G . Posons

$$\kappa = \inf_{\mu \in \Lambda \setminus \{0\}} \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} \frac{m(\alpha)}{2}.$$

Alors tout coefficient matriciel K -fini d'une représentation unitaire de G appartient à $C^{(\lfloor \kappa \rfloor, \kappa - \lfloor \kappa \rfloor)}(G_r)$ et cette régularité est optimale.

Dans le chapitre 3, nous nous intéressons aux paires non compactes issues d'un groupe de Lie semi-simple de centre fini G avec un sous-groupe compact maximal K . Une involution de Cartan permet de décomposer l'algèbre de Lie de G en $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Si \mathfrak{a} est un sous-espace abélien maximal de \mathfrak{p} , nous pouvons considérer le système de racines associé $\Sigma \subset \mathfrak{a}^*$ et un choix de racines positives Σ^+ (voir Section 1.6 pour plus de détails, notamment sur les définitions de ces objets).

Pour $\lambda \in \mathfrak{a}^*$, définissons

$$n(\lambda) = \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha)$$

et posons

$$\kappa(G) = \inf_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{n(\lambda)}{2}.$$

Nous étudions d'abord le groupe de déplacements de Cartan $H = \mathfrak{p} \rtimes K$ associé à G (voir Section 1.7). Notre premier résultat dans ce chapitre est le suivant :

Théorème D. Soit G un groupe de Lie semi-simple connexe de centre fini et $H = \mathfrak{p} \rtimes K$ son groupe de déplacements de Cartan. Soit $r = \lfloor \kappa(G) \rfloor$ et $\delta = \kappa(G) - r$. Alors tout coefficient matriciel K -fini d'une représentation unitaire de H appartient à $C^{(r,\delta)}(H_r)$. De plus, ce résultat est optimal au sens où, pour tout $\delta' > \delta$, il existe un coefficient K -fini (même K -bi-invariant) qui n'appartient pas à $C^{(r,\delta')}(H_r)$.

Nous poursuivons ce chapitre avec l'étude de la paire (G, K) . Contrairement au cas compact, nous apportons une réponse complète à la question de régularité des coefficients K -finis de G .

Théorème E. Soit G un groupe de Lie semi-simple de centre fini et K un sous-groupe compact maximal de G . Soit $r = \lfloor \kappa(G) \rfloor$ et $\delta = \kappa(G) - r$. Alors tout coefficient matriciel K -fini d'une représentation unitaire de G appartient à $C^{(r,\delta)}(G_r)$. De plus, pour tout $\delta' > \delta$, il existe un coefficient K -bi-invariant d'une représentation unitaire de G qui n'appartient pas à $C^{(r,\delta')}(G_r)$.

Les deux théorèmes sont établis en suivant une approche similaire. L'idée est d'abord de démontrer ce résultat pour les coefficients K -bi-invariants uniquement. Dans ce cas, nous utilisons la théorie de la décomposition des représentations en intégrales directes pour ramener le problème à une question de bornitude des fonctions sphériques définies positives dans des espaces de Hölder (Lemmes 1.9 et 1.10). Dans le cadre des groupes de mouvement de Cartan et des groupes de Lie semi-simples, ces fonctions sphériques s'expriment sous forme d'intégrales oscillantes sur le sous-groupe compact maximal K ([Hel00, Ch. IV], [Gin67]). L'ingrédient clé est alors l'utilisation de la méthode de la phase stationnaire pour obtenir des estimées sur ces intégrales, en exploitant les résultats de [DKV83]. La dernière étape est de montrer que la régularité obtenue pour les coefficients K -bi-invariants peut être étendue à tous les coefficients K -finis. Cette démonstration repose sur la décomposition KAK et un résultat général (Théorème 1.15) affirmant que, lorsqu'une telle décomposition existe avec certaines propriétés de régularité, les coefficients K -finis partagent la même régularité que les coefficients K -bi-invariants.

Dans le chapitre 4, nous revenons aux groupes compacts et apportons une réponse partielle à la Conjecture C. Pour cela, nous exploitons une dualité entre paires compactes et non compactes. Si $G_{\mathbb{C}}$ est le groupe de Lie simplement connexe d'algèbre de Lie $\mathfrak{g}_{\mathbb{C}}$, et U le sous-groupe analytique correspondant à la sous-algèbre $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, il est connu que U est un groupe compact simplement connexe semi-simple et que $K \subset U$. Alors (U, K) est une paire de Gelfand et U/K est un espace symétrique compact simplement connexe, qui est le dual compact de G/K . Dans cette notation, κ tel que défini dans la Conjecture C coïncide avec $\kappa(G)$. Ainsi, les Théorèmes A et B sont des versions duales du Théorème E dans certains cas particuliers, notamment lorsque U/K est de rang 1 ou lorsque U/K est un groupe de Lie. Grâce aux outils de phase stationnaire développés pour le cas non compact et à des arguments d'analyse complexe, nous généralisons ces résultats et prouvons le théorème suivant.

Théorème F. *Soit G un groupe de Lie semi-simple à centre fini et (U, K) défini comme ci-dessus. Soient $r = \lfloor \kappa(G) \rfloor$ et $\delta = \kappa(G) - r$. Alors, il existe un ouvert $V \subset U_r$ tel que tout coefficient matriciel K -fini d'une représentation unitaire de U appartient à $C^{(r, \delta)}(V)$. De plus, pour tout $\delta' > \delta$, il existe un coefficient K -bi-invariant qui n'appartient pas à $C^{(r, \delta')}(U_r)$.*

Comme expliqué ci-dessus, l'idée est d'utiliser le prolongement analytique pour obtenir une expression des fonctions sphériques sous forme d'une intégrale oscillante (Lemme 4.4), ce qui permet d'appliquer la méthode de la phase stationnaire. Étant donné que la fonction de phase est désormais à valeurs complexes, la démonstration est plus délicate. Ce résultat montre que la conjecture est vraie au moins sur un certain ouvert. De plus, nous établissons également la conjecture complète pour plusieurs nouvelles familles de groupes (Corollaire 4.10).

Propriété (TTT) et quasi-homomorphismes

Les deux derniers chapitres sont indépendants du reste de la thèse. Dans le Chapitre 5, nous étudions un renforcement de la propriété (T) introduit par Ozawa ([Oza11]). D'après le théorème de Delorme-Guichardet, un groupe G localement compact et σ -compact a la propriété (T) si et seulement si toute action par isométries affines sur un espace de Hilbert admet un point fixe (cette dernière propriété étant également connue sous le nom de propriété FH). Cela équivaut au fait que tout cocycle $b : G \rightarrow H$ (c'est-à-dire la partie de translation d'une telle action) est borné. Un wq-cocycle est une application

mesurable localement bornée $b : G \rightarrow H$ tel qu'il existe une application $\pi : G \rightarrow \mathcal{U}(H)$ qui est mesurable et qui satisfait

$$\text{def}(b) = \sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty.$$

Un groupe G a la propriété (TTT) si tous ses wq-cocycles sont bornés.

Tout cocycle b est un wq-cocycle avec π une représentation unitaire (la partie linéaire de l'action) et $\text{def}(b) = 0$. Ainsi, la propriété (TTT) implique clairement la propriété (T) . Cependant, la propriété (TTT) est en général beaucoup plus forte. Par exemple, il est bien connu qu'il existe de nombreux groupes hyperboliques ayant la propriété (T) (par exemple, les groupes aléatoires de Gromov pour certaines densités, voir [Oll05] ; ou encore les réseaux dans $Sp(n, 1)$). Toutefois, tout groupe hyperbolique admet un wq-cocycle propre ([Min01, Thm. 10]) donc n'a pas la propriété (TTT) .

La propriété (TTT) est particulièrement adaptée à l'étude des quasi-homomorphismes. Si G, G' sont des groupes localement compacts, un quasi-homomorphisme est une application $\varphi : G \rightarrow G'$ telle que

$$\left\{ \varphi(gh)^{-1} \varphi(g) \varphi(h) \mid g, h \in G \right\}$$

est un sous-ensemble relativement compact de G' . Si G a la propriété (TTT) et que G' admet un wq-cocycle propre (par exemple, si G' est hyperbolique comme expliqué plus haut), alors l'image d'un quasi-homomorphisme (continu) φ est relativement compacte.

L'objectif de ce chapitre est d'étudier la propriété (TTT) pour les réseaux dans les groupes de Lie simples de rang supérieur.

Nous montrons d'abord que les réseaux héritent de la propriété (TTT) .

Théorème G. *Soit G un groupe localement compact à base dénombrable, et soit Γ un réseau dans G . Alors G a la propriété (TTT) si et seulement si Γ a la propriété (TTT) .*

Ozawa avait déjà montré que les réseaux *cocompacts* héritent de la propriété (TTT) , et qu'une propriété plus forte, qu'il a appelée (TP) , passe aux réseaux généraux. Ce que nous prouvons ici, c'est que cette autre propriété est équivalente à (TTT) (nous renvoyons à la Section 5.2 pour la définition de cette propriété, et nous mentionnons simplement que (TP) est une version quasifiée de la propriété (T) de la même manière que (TTT) est une version quasifiée de la propriété (FH)).

Notre principal résultat dans ce chapitre est l'établissement de la propriété (TTT) pour les groupes algébriques simples de rang supérieur.

Théorème H. *Soit G un groupe algébrique simple connexe sur un corps local \mathbb{K} avec $\text{rank}_{\mathbb{K}} G \geq 2$. Alors $G(\mathbb{K})$ a la propriété (TTT) .*

Ce théorème avait déjà été démontré par Ozawa dans le cas de SL_n . Dans la Section 5.3, nous prouvons le Théorème H pour le cas de Sp_4 . Nous utilisons ensuite la classification des groupes simples, comme dans la preuve standard de la propriété (T) , pour obtenir le théorème. Un argument clé permettant de simplifier la démonstration est l'utilisation d'une dichotomie sur les longueurs dans les groupes de Lie algébriques simples due à Yves de Cornulier : si ℓ est une longueur (c'est-à-dire une application localement bornée et sous-additive $\ell : G \rightarrow \mathbb{R}_+$), alors ℓ est soit bornée, soit propre.

Nous considérons également le cas des groupes de Lie simples de rang supérieur de centre infini. Il est facile de montrer que ces groupes ne peuvent pas avoir la propriété (TTT) . En effet, ils admettent un wq-cocycle propre (en fait, un quasi-homomorphisme

$\phi : G \rightarrow \mathbb{R}$ appelé quasi-morphisme de Barge-Ghys, voir [BG92, Prop. 6]). Cependant, en utilisant le Théorème H, nous montrons que les wq-cocycles peuvent être contrôlés en fonction des wq-cocycles du groupe \mathbb{Z} . En particulier, cela étend l'unicité (à multiplication par un scalaire près) du quasi-morphisme de Barge-Ghys mentionné ci-dessus.

Jusqu'à présent, nous nous sommes restreints aux applications à valeurs dans un espace de Hilbert. Dans le Chapitre 6, nous étendons la propriété (TTT) au cadre des espaces de Banach.

Dans [BFGM07], deux analogues des propriétés (T) et FH ont été introduits.

Définition 2. Soit G un groupe topologique et E un espace de Banach.

Le groupe G a la propriété (F_E) si toute action $\alpha : G \curvearrowright E$ par isométries affines admet un point fixe.

Le groupe G a la propriété (T_E) si pour toute représentation isométrique continue $\pi : G \rightarrow O(E)$, la représentation induite $\pi' : G \rightarrow O(E/E^{\pi(G)})$ n'admet pas presque des vecteurs invariants.

Cependant, dans ce contexte plus général, le théorème de Delorme-Guichardet ne s'applique pas : la seule implication restante est que la propriété (F_E) implique la propriété (T_E) lorsque G est localement compact à base dénombrable.

Nous soulignons également que, en général, la propriété (F_E) n'est pas équivalente à la bornitude des cocycles à valeurs dans E . Cependant, lorsque E est super-réflexif (c'est-à-dire admet une norme équivalente uniformément convexe), l'équivalence est vraie ([BFGM07, Lemma 2.14]). Nous ne détaillons pas ici la définition de ces espaces (le lecteur pourra se référer à la Section 6.2 pour plus d'informations), mais les propriétés géométriques des espaces super-réflexifs en font une bonne classe d'espaces de Banach pour l'étude de ces propriétés. En particulier, de nombreux aspects de la géométrie hilbertienne peuvent être reformulés en termes de géométrie uniformément convexe. Ainsi, il a été conjecturé que les groupes algébriques simples de rang supérieur sur un corps local devraient avoir la propriété (F_E) pour tout espace super-réflexif. Pour les corps locaux non archimédiens, cela a été démontré par Lafforgue et Liao ([Laf08, Lia14]). Pour faire le lien avec la première partie de la thèse, soulignons qu'ils montrent en fait la propriété (T) renforcée pour ces groupes, ce qui est bien plus fort. Dans le cas réel, la conjecture a également été prouvée par de Laat et de la Salle ([dLdLS23]).

Inspirés par ces résultats, nous introduisons des versions banachiques de la propriété (TTT) .

Définition 3. Soit G un groupe localement compact. Soit E un espace de Banach. On dit qu'une application borélienne localement bornée (c'est-à-dire bornée sur les sous-ensembles compacts) $b : G \rightarrow E$ est un wq-cocycle s'il existe une application borélienne $\rho : G \rightarrow O(E)$ telle que

$$\sup_{g,h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| < +\infty.$$

Le groupe G a la propriété (FFF_E) si tout wq-cocycle $b : G \rightarrow E$ est borné.

Nous introduisons également dans la Section 6.3 une version de cette propriété traitant des vecteurs "presque invariants".

Définition 4. Soit G un groupe localement compact. Soit E un espace de Banach. Le groupe G a la propriété $(T_{Q,E})$ si pour tout $\varepsilon > 0$ et pour toute application borélienne

$\rho : G \rightarrow O(E)$, il existe $\delta > 0$ et un sous-ensemble compact $K \subset G$ tels que pour tout vecteur unitaire $\xi \in E$ vérifiant

$$\sup_{g,h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

et

$$\sup_{g \in K} \|\rho(g)\xi - \xi\| < \delta$$

alors

$$\sup_{x \in G} \|\rho(x)\xi - \xi\| < \varepsilon.$$

Si \mathcal{E} est une classe d'espaces de Banach, G possède la propriété $(T_{Q,\mathcal{E}})$ si G possède $(T_{Q,E})$ pour tout $E \in \mathcal{E}$.

Lorsque \mathcal{E} est une classe d'espaces super-réflexifs stable par somme directe d'un nombre infini de copies d'un espace (par exemple, la classe des espaces L^p , la classe $[\mathcal{H}]$ des espaces admettant une norme hilbertienne équivalente, ou simplement la classe de tous les espaces super-réflexifs), nous avons les implications suivantes entre les différentes propriétés que nous définissons :

$$\begin{array}{ccc} (FFF_{\mathcal{E}}) & \implies & (T_{Q,\mathcal{E}}) \\ \Downarrow & & \Downarrow \\ (F_{\mathcal{E}}) & \implies & (T_{\mathcal{E}}) \end{array}$$

Les implications horizontales sont proches du théorème de Delorme-Guichardet ([BFGM07, Thm. 1.3],6.20), tandis que les implications verticales sont démontrées dans la Proposition 6.19.

Nous commençons par étudier le comportement de ces propriétés vis-à-vis des réseaux. Les arguments de [Oza11, Thm. 6] montrent, *mutatis mutandis*, que si Γ est un réseau dans G , alors Γ hérite de la propriété $(T_{Q,E})$. Cependant, la situation est plus complexe dans le cas de la propriété (FFF_E) : *a priori*, nous ne pouvons obtenir un tel résultat que pour les réseaux cocompacts. En effet, dans le cadre hilbertien, notre démonstration du Théorème G reposait sur un théorème de type Delorme-Guichardet qui utilisait fortement la structure hilbertienne. Nous pouvons cependant étendre ce résultat à une classe bien plus large de réseaux en considérant des conditions d'intégrabilité comme dans [BFGM07, Section 8]. Soit G un groupe localement compact à base dénombrable et Γ un réseau dans G . Soit $\sigma : G/\Gamma \rightarrow G$ une section mesurable de la projection canonique et définissons $\beta : G \times G/\Gamma \rightarrow \Gamma$ par

$$\beta(g, x) = \sigma(x)^{-1}g\sigma(g^{-1}x).$$

Soit $p > 0$. Le réseau Γ est p -intégrable si Γ est de type fini et s'il existe un ensemble générateur fini S et une section borélienne $\sigma : G/\Gamma \rightarrow G$ telle que la frontière du domaine fondamental $\mathcal{D} = \sigma(G/\Gamma)$ soit de mesure nulle et que pour tout $g \in G$,

$$\int_{G/\Gamma} \ell_S(\beta(g, x))^p dx < +\infty,$$

où ℓ_S est la longueur associée à S . Cette condition d'intégrabilité est notamment vérifiée par tout réseau irréductible dans un groupe algébrique semi-simple de rang supérieur, pour tout $1 \leq p < +\infty$.

Théorème I. *Soit G un groupe localement compact à base dénombrable. Soit Γ un réseau dans G . Soit B un espace de Banach et $E = L^p(G/\Gamma; B)$.*

1. *Si Γ a la propriété (FFF_B) , alors G a la propriété (FFF_B) .*
2. *Si G a la propriété (FFF_E) et que Γ est p -intégrable, alors Γ a la propriété (FFF_B) .*

L'essentiel de la démonstration repose sur une induction classique d'une application $\Gamma \rightarrow B$ en une application $G \rightarrow E$, mais il faut prendre des précautions supplémentaires pour garantir que l'application ainsi obtenue est bien définie, car nous ne manipulons pas des morphismes de groupes.

Dans la Section 6.4, nous étudions le cas des espaces L^p . Les principaux résultats sont que (T_{Q,L^p}) est équivalent à la propriété (TTT) pour tout $1 < p < +\infty$, tandis que la propriété (FFF_{L^p}) est équivalente à (TTT) pour tout $1 < p \leq 2$. Les idées utilisées sont similaires à celles de [BFGM07] dans le cadre de la propriété (F_{L^p}) .

Enfin, nous considérons le cas de la classe $[\mathcal{H}]$ des espaces de Banach admettant une norme hilbertienne équivalente dans la Section 6.5. Le premier résultat concerne la propriété $(T_{Q,[\mathcal{H}]})$ dans le cas d'un produit semi-direct d'un groupe abélien, inspiré des caractérisations bien connues de la propriété (T) dans ce contexte ([Sha99, Ioa10, CT11, Oza11]).

Théorème J. *Soit $G = H \rtimes A$ le produit semi-direct d'un groupe abélien localement compact à base dénombrable A par un groupe localement compact et à base dénombrable H . Supposons que la seule moyenne H -invariante sur \hat{A} est δ_0 , où 0 est l'élément neutre du groupe dual \hat{A} de A . Alors la paire (G, A) a la propriété $(T_{Q,[\mathcal{H}]})$ relative.*

Munis de ce résultat, nous pouvons obtenir la propriété $(T_{Q,[\mathcal{H}]})$ pour les groupes algébriques simples de rang supérieur et leurs réseaux, ce qui constitue notre dernier résultat principal.

Théorème K. *Soit G un groupe algébrique simple connexe sur un corps local \mathbb{K} tel que $\text{rank}_{\mathbb{K}} G \geq 2$. Alors $G(\mathbb{K})$ possède la propriété $(T_{Q,[\mathcal{H}]})$. Soit Γ un réseau de $G(\mathbb{K})$, alors Γ possède également la propriété $(T_{Q,[\mathcal{H}]})$.*

La démonstration suit le même schéma que celui utilisé dans le Chapitre 5, avec quelques adaptations pour traiter les vecteurs presque invariants au lieu des wq-cocycles.

Introduction (English version)

This thesis is devoted to the study of several rigidity properties of semisimple Lie groups. The manuscript is divided into two independent parts. This first part is concerned with the study of regularity results for matrix coefficients of unitary representations of semisimple Lie groups (as well as other closely related groups). These questions of regularity take their origins from the work of Vincent Lafforgue on strong property (T) . In the second part of the thesis, we consider an entirely different strengthening of property (T) : we study property (TTT) , introduced by Narutaka Ozawa to understand quasi-homomorphisms. Unlike property (T) , property (TTT) deals with objects that may no longer be representations of the group. We also consider Banach versions of this property.

Regularity of matrix coefficients

In the first part of this thesis, we investigate the regularity and local behavior of K -bi-invariant and K -finite matrix coefficients of unitary representations of some Lie groups G with compact subgroup K . More precisely, we want to find the greatest $\alpha \geq 0$ such that any K -finite matrix coefficients of a unitary representation of G is locally α -Hölder continuous. Since continuous group morphisms of Lie groups are smooth, if π is a finite-dimensional unitary representation of G , then its coefficients are smooth. In particular, by the Peter-Weyl theorem, every matrix coefficient of an irreducible unitary representation of a compact Lie group G is smooth. In the non-compact setting, if G is semisimple and K is a maximal compact subgroup of G , it is known by the work of Harish-Chandra ([HC53]) that matrix coefficients associated to K -finite vectors of irreducible unitary representations of G (and more generally admissible representations) are smooth. Unitary representations of G decompose as direct integrals of irreducible representations, but since estimates depend on the representations, it does not provide any estimates for arbitrary representations. Another result to point out is the fact that although not every vector is smooth, for an arbitrary representation, the subspace of smooth vectors is dense ([Kna01, Ch.3])

The example of semisimple groups is part of a larger class of pairs where the subgroup K is such that (G, K) is a Gelfand pair. In that case, there is a 1-1 correspondence between positive-definite spherical functions of (G, K) , which arise as characters of the (abelian) convolution algebra of K -bi-invariant continuous functions with compact support, and irreducible unitary representations with non-zero K -invariant vectors. In this setting, any K -bi-invariant matrix coefficient of a unitary representation of G decomposes as a direct integral of positive-definite spherical functions (Section 1.2). Then, proving estimates of positive-definite spherical functions that are uniform on the whole family will produce estimates on any K -bi-invariant matrix coefficient of a unitary representation of

G . Furthermore, as we will see in Section 1.3, we may actually extend this to all K -finite coefficients - under some additional hypotheses on (G, K) .

This idea was used by Lafforgue to show that $SO(2)$ -bi-invariant coefficients of $SO(3)$ are $\frac{1}{2}$ -Hölder continuous outside of singular points, which was a key ingredient in the proof of his strengthening of property (T) for $SL(3, \mathbb{R})$ ([Laf08]). With this result, he showed asymptotic estimates of coefficients of representations far more general than unitary representations on Hilbert spaces (see also [dlS22] for a survey). This was further improved in [dLdlS15] for all higher rank simple Lie groups on a large class of Banach spaces. Various problems in operator algebra were solved using this idea with different pairs, but also in other fields. The same ideas extend to the study of Fourier multipliers ([PRdlS22]). In [LdlS11], [HdL13] and [HdL16], it is used to show that higher rank simple Lie groups do not have the approximation property. It also implies results on non-coarse embeddability of families of expanders ([dLdlS18]). In dynamical systems, strong property (T) was also an important part in the Zimmer's program ([BFH22],[BFH20],[BFH21],[Bro22],[Fis22]).

Lafforgue's result on $(SO(3), SO(2))$ relied on the harmonic analysis of the Gelfand pair, in the way we explained above - although not with the terminology introduced -, and used the fact that spherical functions are simply Legendre polynomials in this case, which makes the analysis quite manageable. In this thesis, we want to study more systematically this local regularity for symmetric Gelfand pairs, that is when G/K is a symmetric space. A Riemannian symmetric space is a Riemannian manifold M with some additional structure. Lie-theoretically, it can be defined as a homogeneous space $M = G/K$ where G is a connected Lie group and K a compact open subgroup of the group of fixed points of an involutive automorphism of G . In this case, (G, K) is a Gelfand pair. When M is simply connected, it is a product of a Euclidean space, a space of compact type and a space of non-compact type (this terminology will be explained in Section 1.4). Informally, spaces of non-compact type correspond to G/K where G is a semisimple Lie group with finite center and K is a maximal compact Lie group. Spaces of compact type are of the form U/K where U is compact semisimple. In that case, U can be seen as a compact real form of some semisimple group G , thus there is a duality between symmetric spaces. We will study both of these types, as well as some Euclidean pairs related to semisimple groups (see Section 1.7).

In this framework, there exist a lot of results on the spherical functions of the pair ([Hel79],[Hel00]). These functions are parameterized by a subset Λ of $\mathfrak{g}_\mathbb{C}^*$, where \mathfrak{g} is the Lie algebra of G . We call $\lambda \in \Lambda$ the spectral parameter (see Section 1.5). The asymptotics of spherical functions in the group variable have been studied a lot, in particular in the work of Harish-Chandra, but much less is known for local behavior. To get information on local behavior, the main tool is to study the asymptotic behavior as the spectral parameter goes to infinity, while the group variable remains in a compact subset. An important remark is that the estimates we want to show fail at the identity of G , and in fact at all other singular points (see Definition 1.22 and Proposition 3.8 for the definition of singular and regular points). Thus, we will only obtain regularity of matrix coefficients on the dense open subset of regular points.

Definition 0.1. Let (X, d) be a metric space and U open subset of X , $(E, \|\cdot\|)$ a normed vector space, $\alpha \in]0, 1]$. A function $f : U \rightarrow E$ is α -Hölder if for any compact subset K of U , there exists a constant $C_K > 0$ such that $\forall x, y \in K, \|f(x) - f(y)\| \leq C_K d(x, y)^\alpha$. If X is also a normed vector space and $r \in \mathbb{N}$, we say that the map f belongs to $C^{(r, \alpha)}(U, E)$ if $f \in C^r(U, E)$ and the r -th differential $D^r f$ is α -Hölder as a map from U to the vector space of multilinear r -forms. We extend to $\alpha = 0$ by $C^{(r, 0)}(U, E) = C^r(U, E)$.

For K a compact subset of U and $f \in C^{(r,\alpha)}(U, E)$, define

$$\|f\|_{C^{(r,\alpha)}(K,E)} = \max \left\{ \max_{k \leq r} \sup_{x \in K} \|D^k f(x)\|, \sup_{x,y \in K, x \neq y} \frac{\|D^r f(x) - D^r f(y)\|}{d(x,y)^\alpha} \right\}.$$

The family of semi-norms $\|\cdot\|_{C^{(r,\alpha)}(K,E)}$ for K a compact subset of U makes the space $C^{(r,\alpha)}(U, E)$ into a Fréchet space.

Finally if (X, d) is a Riemannian manifold, we say that $f \in C^{(r,\alpha)}(U, E)$ if for any chart (φ, V) of U , $f \circ \varphi^{-1} \in C^{(r,\alpha)}(\varphi(V), E)$.

Let G be a Lie group and K a compact subgroup. We want to find (r, α) such that for any K -finite unitary matrix coefficient φ of G , $\varphi \in C^{(r,\alpha)}(G_r)$, where G_r is the dense open subset of regular points (again, see Definition 1.22 and Proposition 3.8). If we assume that (G, K) is a Gelfand pair, the main tool for this will be to study the boundedness of the family of spherical functions in Hölder spaces. If furthermore (G, K) is a symmetric pair, then G/K is a symmetric space and there are results on the spherical functions in the literature.

We begin in Chapter 2 by studying pairs where G is itself a compact group. Our first result involves compact symmetric pairs of rank 1 (see Section 1.5 for the notion of rank).

Theorem A. *Let (G, K) be a compact symmetric pair of rank 1 and G_r the dense open subset of regular points. Let*

$$\alpha = \frac{\dim G/K - 1}{2}.$$

Then any K -finite unitary matrix coefficient φ of G is in $C^{([\alpha], \alpha - [\alpha])}(G_r)$. Furthermore, this regularity is optimal in the sense that for any $(r, \delta) > ([\alpha], \alpha - [\alpha])$ in lexicographic order, there exists a K -finite (and even K -bi-invariant) unitary matrix coefficient of G which is not in $C^{(r,\delta)}(G_r)$.

We also prove in Corollary 2.13 a regularity result for Herz-Schur multipliers of $S_p(L^2(G))$ for any p .

We then turn to higher rank compact symmetric pairs. Among them, a simple class is given by the pairs $(G \times G, G)$ with G a compact semisimple Lie group. The symmetric space associated to these pairs is isomorphic to the Lie group G . Given Φ a root system for G , Φ^+ a choice of positive roots and $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a basis, we can write $\alpha = \sum_{i=1}^\ell n_i(\alpha)\alpha_i$ for $\alpha \in \Phi^+$ (see Section 1.5 for more details about root systems). Then our second main result gives the regularity for these pairs.

Theorem B. *Let $\gamma = \min_{1 \leq i \leq \ell} \{|\alpha \in \Phi^+ \mid n_i(\alpha) \geq 1\}|$ and G_r the subset of regular points of $G \times G$. Then any G -finite matrix coefficient φ of a unitary representation of $G \times G$ is in $C^{(\gamma,0)}(G_r)$ and this regularity is optimal.*

Given all these results, we make a conjecture on the optimal regularity in the general case. Given (G, K) a compact symmetric pair, there is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra. Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} and Σ the root system. Let $\Lambda = \left\{ \mu \in \mathfrak{a}^* \mid \forall \alpha \in \Sigma^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \right\}$. For $\alpha \in \Sigma$, let $m(\alpha)$ be the multiplicity of the root.

Conjecture C. *Let G_r be the subset of regular points of G . Let*

$$\kappa = \inf_{\mu \in \Lambda \setminus \{0\}} \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} \frac{m(\alpha)}{2}.$$

Then any K -finite matrix coefficient φ of a unitary representation of G is in $C^{(\lfloor \kappa \rfloor, \kappa - \lfloor \kappa \rfloor)}(G_r)$ and this regularity is optimal.

In Chapter 3, we turn to non-compact pairs coming from a semisimple Lie group with finite center G with maximal compact subgroup K . Given a Cartan involution of \mathfrak{g} , we have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} , we can consider the associated root system $\Sigma \subset \mathfrak{a}^*$ and a choice of positive roots Σ^+ (see Section 1.6 for more details, in particular for definitions of those objects). For $\lambda \in \mathfrak{a}^*$, define

$$n(\lambda) = \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha)$$

and set

$$\kappa(G) = \inf_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{n(\lambda)}{2}.$$

We first study the Cartan motion group $H = \mathfrak{p} \rtimes K$ associated to G (see Section 1.7). Our first result in this chapter is the following:

Theorem D. *Let G be a connected semisimple Lie group with finite center and $H = \mathfrak{p} \rtimes K$ its Cartan motion group. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of H is in $C^{(r, \delta)}(H_r)$. Furthermore, this result is optimal in the sense that for any $\delta' > \delta$, there exists a K -finite (even K -bi-invariant) coefficient which is not in $C^{(r, \delta')}(H_r)$.*

We carry on this chapter with the study of the pair (G, K) itself. Unlike the compact case, we give a complete answer to the question of regularity of K -finite coefficients of G .

Theorem E. *Let G be a semisimple Lie group with finite center and K a maximal compact subgroup of G . Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of G is in $C^{(r, \delta)}(G_r)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant coefficient of a unitary representation of G which is not in $C^{(r, \delta')}(G_r)$.*

Both theorems are established using a similar approach. The idea is to first prove this result for K -bi-invariant coefficients only. In that case, we use the theory of decomposition of a representation into a direct integral to reduce this problem to a question of boundedness of positive-definite spherical functions in Hölder spaces (Lemmas 1.9 and 1.10). In the setting of Cartan motion groups and semisimple Lie groups, these spherical functions have a well-known expression as an oscillatory integral over the maximal compact subgroup K ([Hel00, Ch. IV], [Gin67]). The key input is then the use of the stationary phase approximation to obtain estimates on these integrals, using the results from [DKV83]. The last step is to prove that the regularity obtained for K -bi-invariant coefficients can be enhanced to all K -finite coefficients. There are two steps: first, we prove that these pairs admit a KAK decomposition with some smoothness properties (Sections 3.1.1 and 3.2.1). Then, we use a general result (Theorem 1.15) which states that whenever a pair has such a KAK decomposition, K -finite coefficients share the regularity of K -bi-invariant coefficients (Section 1.3).

In Chapter 4, we come back to compact groups and give a partial answer to Conjecture C. For this, we notice that there is a duality between compact and non-compact pairs. If $G_{\mathbb{C}}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and U the analytic subgroup corresponding to the subalgebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, it is known that U is a compact simply connected semisimple Lie group and $K \subset U$. Then (U, K) is a Gelfand pair and U/K is a

symmetric space of compact type, which is the simply connected compact dual of G/K . In those notations, κ as defined in Conjecture C coincides with $\kappa(G)$. Thus, Theorems A and B are dual versions of Theorem E in special cases, namely when U/K is of rank 1 and when U/K is a Lie group. With the tools of stationary phase approximation developed for the noncompact case and some complex analysis, we generalize these results. More precisely, we prove the following theorem.

Theorem F. *Let G be a semisimple Lie group with finite center and (U, K) defined as above. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then, there exists an open subset $V \subset U_r$ such that any K -finite matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(V)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient which is not in $C^{(r, \delta')}(U_r)$.*

As explained above, the idea is to use analytic continuation to obtain an expression of spherical functions as an oscillatory integral (Lemma 4.4) which allow to use the stationary phase approximation. Since the phase function is now complex-valued, the proof is more involved. This result shows that the conjecture is true at least in some open subset. Furthermore, we also get the full conjecture for several new families of groups (Corollary 4.10).

Property (TTT) and quasi-homomorphisms

The last two chapters are independent from the rest of the thesis. In Chapter 5, we study a strengthening of property (T) introduced by Ozawa ([Oza11]). By the Delorme-Guichardet theorem, a σ -compact locally compact group G has property (T) if and only if every action by affine isometries on a Hilbert space has a fixed point (also known as property FH). This is equivalent to the fact that every cocycle $b : G \rightarrow H$ (i.e. the translation part of such an action) is bounded. A wq-cocycle is a measurable locally bounded map $b : G \rightarrow H$ along with a map $\pi : G \rightarrow \mathcal{U}(H)$ which is measurable and such that

$$\text{def}(b) = \sup_{g, h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty.$$

A group G has property (TTT) if all wq-cocycles are bounded.

Every cocycle b is a wq-cocycle with π a unitary representation (the linear part of the action) and $\text{def}(b) = 0$. Thus, property (TTT) clearly implies property (T) . However, property (TTT) is in general far stronger. For example it is well-known that there are many hyperbolic groups with property (T) (for example, Gromov random group for certain densities, see [Oll05]; or lattices in $Sp(n, 1)$). However, every hyperbolic group admits a proper wq-cocycle ([Min01, Thm. 10]).

Property (TTT) is well-suited for the study of quasi-homomorphisms. If G, G' are locally compact group, a quasi-homomorphism is a map $\varphi : G \rightarrow G'$ such that

$$\left\{ \varphi(gh)^{-1} \varphi(g) \varphi(h) \mid g, h \in G \right\}$$

is a relatively compact subset of G' . If G has property (TTT) and G' admits a proper wq-cocycle (for example, G' hyperbolic as explained above), then the image of a (continuous) quasi-homomorphism φ is relatively compact.

The aim of this chapter is to study property (TTT) for lattices in higher rank simple Lie groups.

We first show that lattices inherit property (TTT) .

Theorem G. *Let G be a locally compact second countable group and Γ a lattice in G . Then G has property (TTT) if and only if Γ has property (TTT) .*

It was known by Ozawa that *cocompact* lattices inherit property (TTT) , and that a stronger property, which he called (T_P) , passes to general lattices. What we prove is that this other property is equivalent to (TTT) (we defer the definition of this related property to Section 5.2, and simply mention that (T_P) is a quasification of property (T) in the same way that (TTT) is a quasification of property FH).

Our main result in this chapter is property (TTT) for higher rank simple algebraic groups.

Theorem H. *Let G be a connected simple algebraic group over a local field \mathbb{K} with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property (TTT) .*

This theorem was already proven by Ozawa in the case of SL_n . In Section 5.3, we prove Theorem H in the case of Sp_4 . We then make use of the classification of simple groups, as in the standard proof of property (T) , to obtain the theorem. An important argument to simplify the proof is the use of a dichotomy for lengths in simple algebraic Lie groups due to Yves de Cornulier: if ℓ is a length (i.e. a locally bounded subadditive map $\ell : G \rightarrow \mathbb{R}_+$), then ℓ is either bounded or proper.

We also consider the case of higher rank simple Lie groups with infinite center. It is easy to show that these groups cannot have property (TTT) . Indeed, they admit a proper wq-cocycle (in fact, a quasi-homomorphism $\phi : G \rightarrow \mathbb{R}$ called a Barge-Ghys quasimorphism, see [BG92, Prop. 6]). However, making use of Theorem H, we show that wq-cocycles can be controlled in terms of the wq-cocycles of the group \mathbb{Z} . In particular, this extends the uniqueness (up to scalar multiplication) of the Barge-Ghys quasimorphism mentioned above.

Up until that point, we restricted ourselves to Hilbert valued maps. In Chapter 6, we extend property (TTT) to the realm of Banach spaces.

In [BFGM07], two analogues of properties (T) and FH were introduced.

Definition 0.2. Let G be a topological group and E a Banach space.

The group G has property (F_E) if for any action $\alpha : G \curvearrowright E$ by affine isometries, α has a fixed point.

The group G has property (T_E) if for any continuous isometric representation $\pi : G \rightarrow O(E)$, the induced representation $\pi' : G \rightarrow O(E/E^{\pi(G)})$ does not almost have invariant vectors.

However in this more general context, the Delorme-Guichardet theorem does not hold: the only remaining implication is that property (F_E) implies property (T_E) when G is locally compact second countable.

We also point out that in general, property (F_E) is not equivalent to the boundedness of cocycles with values in E . However whenever E is super-reflexive (i.e. has an equivalent uniformly convex norm), the equivalence holds ([BFGM07, Lemma 2.14]). We will not elaborate on the definition of those spaces in this introduction (the reader can look at Section 6.2 for more information), but the geometric properties of super-reflexive spaces make them a good class of Banach spaces to consider in the study of those properties. In particular, many aspects of Hilbertian geometry can be rephrased in terms of uniformly convex geometry and thus it was conjectured that higher rank simple algebraic groups over local field should have property (F_E) for any super-reflexive space. For non-Archimedean local fields, this was solved by the affirmative by Lafforgue and Liao in ([Laf08, Lia14]). As a link with the first part of this thesis, we point out that they proved a much stronger

result, namely Lafforgue's strong property (T) . In the real case, the conjecture was also proven to be true by de Laat and de la Salle ([dLdS23]).

Inspired by these results, we introduce Banach versions of property (TTT) .

Definition 0.3. Let G be a locally compact group. Let E be a Banach space. We say that a Borel locally bounded (i.e. bounded on compact subsets) map $b : G \mapsto E$ along with a Borel map $\rho : G \mapsto O(E)$ is a wq-cocycle if

$$\sup_{g,h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| < +\infty.$$

The group G has property (FFF_E) if every wq-cocycle $b : G \rightarrow E$ is bounded.

We also introduce in Section 6.3 a version of this property dealing with "almost invariant" vectors.

Definition 0.4. Let G be a locally compact group. Let E be a Banach space. The group G has property $(T_{Q,E})$ if for all $\varepsilon > 0$ and for any Borel map $\rho : G \mapsto O(E)$, there exist $\delta > 0$ and a compact subset $K \subset G$ such that for every unit vector $\xi \in E$ verifying

$$\sup_{g,h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{g \in K} \|\rho(g)\xi - \xi\| < \delta$$

then

$$\sup_{x \in G} \|\rho(x)\xi - \xi\| < \varepsilon.$$

If \mathcal{E} is a class of Banach spaces, G has $(T_{Q,\mathcal{E}})$ if G has $(T_{Q,E})$ for any $E \in \mathcal{E}$.

When \mathcal{E} is a class of super-reflexive spaces stable under direct sum of a infinite number of copies of a space (for example, the class of L^p spaces, the class $[\mathcal{H}]$ of spaces with an equivalent Hilbert norm, or simply the class of all super-reflexive spaces), we have the following implications between the various properties we define.

$$\begin{array}{ccc} (FFF_{\mathcal{E}}) & \implies & (T_{Q,\mathcal{E}}) \\ \Downarrow & & \Downarrow \\ (F_{\mathcal{E}}) & \implies & (T_{\mathcal{E}}) \end{array}$$

The horizontal implications are close to the Delorme-Guichardet theorem ([BFGM07, Thm. 1.3],6.20) while the vertical implications are proven in Proposition 6.19.

We begin by studying the behavior of these properties with respect to lattices. The arguments of [Oza11, Thm. 6] shows, *mutatis mutandis*, that if Γ is a lattice in G , Γ inherits property $(T_{Q,E})$. However, the situation is more complicated in the case of property (FFF_E) : we can only *a priori* get such a result for cocompact lattices. Indeed, in the Hilbert setting, our proof of Theorem G relied on a Delorme-Guichardet like theorem, which itself used a lot of the Hilbertian structure. We can however extend to a much larger class of lattices by considering integrability conditions as in [BFGM07, Section 8].

Let G be locally compact second countable group and Γ lattice in G . Let $\sigma : G/\Gamma \rightarrow G$ be a Borel section of the canonical projection and define $\beta : G \times G/\Gamma \rightarrow \Gamma$ by

$$\beta(g, x) = \sigma(x)^{-1}g\sigma(g^{-1}x).$$

Let $p > 0$. The lattice Γ is p -integrable if Γ is finitely generated and for some (equivalently, for any) finite generating set S , there is a Borel section $\sigma : G/\Gamma \rightarrow G$ such that the boundary of the fundamental domain $\mathcal{D} = \sigma(G/\Gamma)$ has zero measure and for any $g \in G$,

$$\int_{G/\Gamma} \ell_S(\beta(g, x))^p dx < +\infty$$

where ℓ_S is the word length associated to S .

This integrability condition is in particular verified by any irreducible lattice in higher rank semisimple algebraic groups, for any $1 \leq p < +\infty$.

Theorem I. *Let G be a locally compact second countable group. Let Γ be a lattice in G . Let B be a Banach space and $E = L^p(G/\Gamma; B)$. Then*

1. *If Γ has property (FFF_B) , G has property (FFF_B) .*
2. *If G has property (FFF_E) and Γ is p -integrable, then Γ has property (FFF_B) .*

The heart of the proof is a classical induction of a map $\Gamma \rightarrow B$ to a map $G \rightarrow E$, however we need to take extra caution to make sure that the map is well-defined since we are not dealing with objects that are group morphisms.

In Section 6.4, we consider the case of L^p spaces. The main results are that (T_{Q, L^p}) is equivalent to property (TTT) for any $1 < p < +\infty$, while property (FFF_{L^p}) is equivalent to property (TTT) for any $1 < p \leq 2$. The ideas are very similar to what is done in [BFGM07] in the context of property (F_{L^p}) .

Finally, we consider the case of the class $[\mathcal{H}]$ of Banach spaces with a compatible Hilbert norm in Section 6.5. The first result is a statement on property $(T_{Q, [\mathcal{H}]})$ in the case of a semi-direct product on an abelian group, inspired by well-known characterizations of property (T) in this context ([Sha99, Ioa10, CT11, Oza11]).

Theorem J. *Let $G = H \ltimes A$ be the semi-direct product of a locally compact second countable abelian group A by a locally compact second countable group H . Assume that the only H -invariant mean on \hat{A} is δ_0 where 0 is the neutral element in the Pontryagin dual \hat{A} of A . Then the pair (G, A) has relative property $(T_{Q, [\mathcal{H}]})$.*

Armed with this result, we can obtain property $(T_{Q, [\mathcal{H}]})$ for higher rank simple algebraic groups and their lattices, which is our last main result.

Theorem K. *Let \mathbb{K} be a local field, G a connected almost \mathbb{K} -simple \mathbb{K} -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$. Let Γ be a lattice in $G(\mathbb{K})$, then Γ has property $(T_{Q, [\mathcal{H}]})$.*

The proof follows the same pattern used in Chapter 5, with some adaptations to handle almost invariant vectors instead of wq-cocycles.

Part I

Regularity of matrix coefficients

Chapter 1

Spherical functions of semisimple Lie groups and matrix coefficients

In this first chapter, we present some preliminary results that will be used throughout the thesis. First, we recall some basic results on Hölder spaces. In Section 1.2, we explain how we can link regularity questions for K -bi-invariant coefficients to harmonic analysis on Gelfand pairs and the study of spherical functions. In Section 1.3, we show that under some additional hypotheses on the pair (G, K) , our original question on the regularity of K -finite coefficients reduces to the smaller and more manageable subspace of K -bi-invariant coefficients. Thus, combining these two reductions, once we find Gelfand pairs verifying this hypothesis, we can restrict ourselves to the study of the family of spherical functions. We then explain which Gelfand pairs we will consider in the subsequent chapters. In Section 1.4, we relate semisimple Lie groups to Gelfand pairs, through the theory of symmetric spaces. Finally, we present some preliminary results on the associated spherical functions in each of the following cases: compact symmetric pairs (Section 1.5), semisimple Lie groups (Section 1.6) and Cartan motion groups (Section 1.7). Most of these results can be found in the preliminary sections of [Dum24c, Dum24b, Dum25]. The study of K -finite coefficients is presented in [Dum25, Section 2.3] but was already achieved in [Dum24c, Section 5] and [Dum24b, Section 3.2] in a less general setting.

1.1 Hölder spaces

We recall here standard results on Hölder spaces.

Definition 1.1. Let (X, d) be a metric space and U open subset of X , $(E, \|\cdot\|)$ a normed vector space, $\alpha \in]0, 1]$. A function $f : U \rightarrow E$ is α -Hölder if for any compact subset K of U , there is $C_K > 0$ such that $\forall x, y \in K$, $\|f(x) - f(y)\| \leq C_K d(x, y)^\alpha$.

If X is also a normed vector space and $r \in \mathbb{N}$, we say that the map f belongs to $C^{(r, \alpha)}(U, E)$ if $f \in C^r(U, E)$ and the r -th differential $D^r f$ is α -Hölder as a map from U to the vector space of multilinear r -forms. We extend to $\alpha = 0$ by $C^{(r, 0)}(U, E) = C^r(U, E)$.

For K a compact subset of U and $f \in C^{(r, \alpha)}(U, E)$, define

$$\|f\|_{C^{(r, \alpha)}(K, E)} = \max \left\{ \max_{k \leq r} \sup_{x \in K} \|D^k f(x)\|, \sup_{x, y \in K, x \neq y} \frac{\|D^r f(x) - D^r f(y)\|}{d(x, y)^\alpha} \right\}.$$

The family of semi-norms $\|\cdot\|_{C^{(r, \alpha)}(K, E)}$ for K a compact subset of U makes the space $C^{(r, \alpha)}(U, E)$ into a Fréchet space.

Finally if (X, d) is a Riemannian manifold, we say that $f \in C^{(r,\alpha)}(U, E)$ if for any chart (φ, V) of U , $f \circ \varphi^{-1} \in C^{(r,\alpha)}(\varphi(V), E)$.

We will denote $C^{(r,\alpha)}(U, \mathbb{C})$ by $C^{(r,\alpha)}(U)$.

Remark 1.2. If U is locally compact, a function $f : U \rightarrow E$ is α -Hölder continuous if and only if for any $x \in U$, there exists a neighborhood U_x of x and a constant $C_x > 0$ such that for any $y, z \in U_x$, $\|f(y) - f(z)\| \leq C_x d(y, z)^\alpha$. Thus our definition is that of local Hölder continuity.

To see that this definition for manifolds makes sense, we should check that

- (i) the definition of $C^{(r,\alpha)}(U)$ for U open subset of X normed vector space agrees with the definition when X is seen as a manifold;
- (ii) the definition of $C^{(0,\alpha)}(U)$ for U open subset of a Riemannian manifold (X, d) agrees with the definition of $C^\alpha(U)$ when X is seen as a metric space.

These facts are a consequence of the following lemma, which is also a generalization of [PRdlS22, Proposition 4.1].

Lemma 1.3. *Let $(X, d), (Y, d')$ be metric spaces and U, V open subsets of X, Y respectively. Let E be a normed vector space and $\alpha \in [0, 1], r \in \mathbb{N}$.*

- (i) *If (X, d) is a Riemannian manifold and $f : U \rightarrow E$, then $f \in C^{(0,\alpha)}(U, E)$ if and only if f is α -Hölder (i.e. in the metric space sense).*
- (ii) *If X, Y are normed vector spaces and $\varphi : U \rightarrow V$ is a smooth function, then $\varphi_* : f \mapsto f \circ \varphi$ maps $C^{(r,\alpha)}(V, E)$ to $C^{(r,\alpha)}(U, E)$. Furthermore, if (f_n) is bounded in $C^{(r,\alpha)}(V, E)$, then $(f_n \circ \varphi)$ is bounded in $C^{(r,\alpha)}(U, E)$.*
- (iii) *If X, Y are Riemannian manifolds, the same conclusion as in (ii) holds with the Riemannian definition of $C^{(r,\alpha)}$.*

Proof. (i). Assume that $f \in C^{(0,\alpha)}(U, E)$. Let $x \in U$, there exists a chart (ψ, U_x) around x such that by assumption $f \circ \psi^{-1}$ is α -Hölder continuous. By [Lee19, Thm. 6.17], for any $x \in X$, there exists ε_0 such that for any $\varepsilon \leq \varepsilon_0$, the geodesic ball $B(x, \varepsilon)$ is geodesically convex, so we may assume U_x geodesically convex. Up to reducing U_x , we may assume that there exists $C_x > 0$ for any $y_0, z_0 \in \psi(U_x)$,

$$\|f(\psi^{-1}(y_0)) - f(\psi^{-1}(z_0))\| \leq C_x \|y_0 - z_0\|^\alpha$$

Let $y, z \in U_x$, then

$$\|f(y) - f(z)\| \leq C \|\psi(y) - \psi(z)\|^\alpha \leq C' d(y, z)^\alpha.$$

We used the mean value theorem to get the last inequality because U_x is geodesically convex and ψ smooth. Thus, f is α -Hölder.

The other direction is nearly identical, using that ψ^{-1} is smooth.

(ii). Let $f \in C^{(r,\alpha)}(V)$. Then $f \circ \varphi \in C^r(U)$. For any $x \in X$, consider a ball U_x centered at x such that $L = \overline{U_x}$ is compact and a subset of U .

The map $D^r(f \circ \varphi)(x)$ is a sum of terms of the form

$$D^i f(\varphi(x)) \circ (D^{j_1} \varphi(x), \dots, D^{j_i} \varphi(x))$$

with $i \leq r$ and $\sum j_i = r$. Take $y, z \in U_x$, then for the term $i = r$, we have

$$\begin{aligned}
& \|D^r f(\varphi(y)) \circ (D\varphi(y), \dots, D\varphi(y)) - D^r f(\varphi(z)) \circ (D\varphi(z), \dots, D\varphi(z))\| \\
& \leq \| [D^r f(\varphi(y)) - D^r f(\varphi(z))] \circ (D\varphi(y), \dots, D\varphi(y)) \| \\
& \quad + \sum_{k=1}^r \|D^r f(\varphi(z)) \circ (D\varphi(y), \dots, D\varphi(y) - D\varphi(z), \dots, D\varphi(z))\| \\
& \leq \|f\|_{C^{(r,\alpha)}(\varphi(L))} \|\varphi\|_{C^1(L)}^r d(\varphi(y), \varphi(z))^\alpha \\
& \quad + \sum_{k=1}^r \|f\|_{C^{(r,\alpha)}(\varphi(L))} \|\varphi\|_{C^1(L)}^{r-1} \|D\varphi(y) - D\varphi(z)\| \\
& \leq \|f\|_{C^{(r,\alpha)}(\varphi(L))} \|\varphi\|_{C^1(L)}^{r+\alpha} d(y, z)^\alpha \\
& \quad + \sum_{k=1}^r \|f\|_{C^{(r,\alpha)}(\varphi(L))} \|\varphi\|_{C^1(L)}^{r-1} \|\varphi\|_{C^2(L)} d(y, z)
\end{aligned}$$

We can use the mean value theorem to get the last inequality because U_x is convex. There is a constant $C(U_x, f, \varphi) > 0$ such that any term with $i < r$ is bounded in a similar way by $C(U_x, f, \varphi)d(y, z)$. The diameter of U_x is bounded so there is $C > 0$ such that $d(y, z) \leq Cd(y, z)^\alpha$. Thus, $f \circ \varphi \in C^{(r,\alpha)}(U)$.

Finally, φ_* is a linear map between Fréchet spaces. If $(f_n, f_n \circ \varphi)$ converges to (f, g) in $C^{(r,\alpha)}(V) \times C^{(r,\alpha)}(U)$, then f_n converge uniformly to f on compact subset and $f_n \circ \varphi$ converge uniformly to g on compact subset. Thus, $g = f \circ \varphi$ and the graph of φ_* is closed. By the closed graph theorem, φ_* is continuous. Thus, the image of a bounded subset is bounded.

(iii). This is an immediate consequence of the definition for manifolds □

Using Leibniz rule, we can prove the following lemma.

Lemma 1.4. *Let U be an open subset of a normed vector space E and $g : U \rightarrow \mathbb{C}$ a smooth function. If $(f_i)_{i \in I}$ is bounded in $C^{(r,\alpha)}(U)$, then (gf_i) is bounded in $C^{(r,\alpha)}(U)$.*

Proof. It is clear that if $f \in C^{(r,\alpha)}(U)$, gf is r times differentiable and for $x \in U$, $H = (H_1, \dots, H_r) \in E^n$,

$$D^r(gf)(x)(H) = \sum_{j \in J} D^{k_j} g(x)(H_{p_1(j)}, \dots, H_{p_{k_j}(j)}) D^{r-k_j} f(x)(H_{p_{k_j+1}(j)}, \dots, H_{p_r(i)})$$

where $k_j \leq r$ for each $j \in J$ and J finite. Thus, we see that $gf \in C^{(r,\alpha)}(U)$.

Furthermore, $m : f \mapsto gf$ is a well-defined linear map between Fréchet spaces. If (f_n, gf_n) converges uniformly to (f, h) in $C^{(r,\alpha)}(U) \times C^{(r,\alpha)}(U)$, then f_n converges uniformly to f on compact subsets of U and gf_n converges uniformly on compact subsets to h , and to gf . Thus $gf = h$, so the graph of m is closed. By the closed graph theorem, m is continuous so the image of a bounded subset is bounded. □

Given a family of functions which are eigenvalues of a map T into $B(\mathcal{H})$, the regularity of T can be linked to the boundedness of the eigenvalues in Hölder spaces.

Lemma 1.5. *Let U be an open subset of \mathbb{R}^d and \mathcal{H} a Hilbert space. Consider a map $T : U \mapsto B(\mathcal{H})$ such that there is an orthonormal basis (e_n) of \mathcal{H} and a family of maps $f_n : U \mapsto \mathbb{R}$, such that for any $X \in U$, $T(X)$ is diagonal in the basis (e_n) with eigenvalues $(f_n(X))$. If (f_n) is bounded in $C^{(r,\alpha)}(U)$ for some $\alpha > 0$, then T lies in $C^{(r,\alpha)}(U, B(\mathcal{H}))$.*

Proof. If T is C^k , then for any $X \in U$, $H_1, \dots, H_k \in \mathbb{R}^d$, we must have $f_n \in C^k$ for any $n \in \mathbb{N}$ and

$$D^k T(X)(H_1, \dots, H_k) e_n = D^k f_n(X)(H_1, \dots, H_k) e_n.$$

We prove the result by induction on r . If $S \in B(\mathcal{H})$, let $\|S\|_\infty = \sup_{\|H\|=1} \|S(H)\|$ be its operator norm. If $r = 0$, we have for $X, Y \in L$ compact subset of U ,

$$\|T(X) - T(Y)\|_\infty = \sup_n |f_n(X) - f_n(Y)| \leq C_L \|X - Y\|^\alpha$$

so T is α -Hölder.

Assume the result is true for $r - 1$. Since $C^{(r,\alpha)}(U, B(\mathcal{H})) \subset C^{(r-1,1)}(U, B(\mathcal{H}))$, we obtain by the induction hypothesis that $T \in C^{r-1}(U, B(\mathcal{H}))$. Let A be the r -linear map such that

$$A(H_1, \dots, H_r) e_n = D^r f_n(X)(H_1, \dots, H_r) e_n.$$

Note that since $(D^r f_n)$ is bounded in n on compacts, $A(H_1, \dots, H_r) \in B(\mathcal{H})$.

Denote $H = (H_1, \dots, H_{r-1})$ and $H^+ = (H_1, \dots, H_r)$. We must show that

$$\lim_{H_r \rightarrow 0} \frac{\|D^{r-1} T(X + H_r)(H) - D^{r-1} T(X)(H) - A(H^+)\|_\infty}{\|H_r\|_1} = 0$$

uniformly for H_1, \dots, H_{r-1} in bounded sets. We have

$$\begin{aligned} & \|D^{r-1} T(X + H_r)(H) - D^{r-1} T(X)(H) - A(H^+)\|_\infty \\ &= \sup_n |D^{r-1} f_n(X + H_r)(H) - D^{r-1} f_n(X)(H) - D^r f_n(X)(H^+)|. \end{aligned}$$

Assume that H_r is small enough, so that $B(X, \|H\|_\infty) \subset U$. Let

$$g_n : \begin{array}{ccc} [0, 1] & \rightarrow & \mathbb{R} \\ t & \mapsto & D^{r-1} f_n(X + tH_r)(H_1, \dots, H_{r-1}) \end{array}.$$

Then

$$g_n(1) - g_n(0) = g_n'(t) = D^r f_n(X + tH_r)(H_1, \dots, H_r)$$

for some $t \in]0, 1[$.

Then

$$\begin{aligned} & |D^{r-1} f_n(X + H_r)(H) - D^{r-1} f_n(X)(H) - D^r f_n(X)(H^+)| \\ &= |D^r f_n(X + tH_r)(H^+) - D^r f_n(X)(H^+)| \\ &\leq \|D^r f_n(X + tH_r) - D^r f_n(X)\| \prod_{i=1}^r \|H_i\| \\ &\leq \|X + tH_r - X\|^\alpha \prod_{i=1}^r \|H_i\| \\ &\leq \|H_r\|^\alpha \prod_{i=1}^r \|H_i\| \end{aligned}$$

so since $\alpha > 0$, we get what we want.

So we have that $D^r T(X)$ exists for any $X \in U$, and thus for $X, Y \in L$ compact subset of U ,

$$\|D^r T(X) - D^r T(Y)\| = \sup_n \|D^r f_n(X) - D^r f_n(Y)\| \leq C_L \|X - Y\|^\alpha$$

because $D^r f_n$ are uniformly α -Hölder on L . □

1.2 Gelfand pairs

Definition 1.6. Let G be a locally compact topological group with a left Haar measure dg and K a compact subgroup with normalized Haar measure dk . The pair (G, K) is a Gelfand pair if the algebra of continuous K -bi-invariant functions on G with compact support is commutative for the convolution.

A spherical function of (G, K) is a continuous K -bi-invariant non-zero function on G such that for all $x, y \in G$,

$$\int_K \varphi(xky) dk = \varphi(x)\varphi(y).$$

A standard result (see [vD09, Coro. 6.3.3]) gives a link between spherical functions of (G, K) and unitary representations of G .

Proposition 1.7. *If (G, K) is a Gelfand pair, then for any irreducible unitary representation π of G on a Hilbert space \mathcal{H} , the subspace \mathcal{H}^K of K -invariant vectors is of dimension at most 1.*

The positive-definite spherical functions of G are exactly the matrix coefficients $g \mapsto \langle \pi(g)v, v \rangle$ with π an irreducible unitary representation of G and v a K -invariant unit vector.

If G is compact, any spherical function is positive-definite.

Remark 1.8. If we assume that G, K are Lie groups, positive-definite spherical functions have a geometric interpretation. Let $D(G/K)$ be the algebra of differential operators on G/K invariant by the action of G by translation on G/K . Then $\varphi : G/K \rightarrow \mathbb{C}$ is a positive-definite spherical function if and only if $\varphi(K) = 1$, φ is invariant by the action of K and φ is an eigenvalue of all operators of $D(G/K)$.

When G/K is a compact symmetric space of rank 1 (a sphere or a projective space, see Section 2.1), then $D(G/K)$ is generated by the Laplacian. Hence in that case, positive-definite spherical functions are normalized K -invariant eigenvalues of the Laplacian.

More details on Gelfand pairs can be found in [vD09, Ch. 5,6,7].

Given a Gelfand pair (G, K) , it is natural to study spherical functions in order to get results on K -bi-invariant matrix coefficients of unitary representations. Indeed, any matrix coefficient of a unitary representation decomposes into an integral of spherical functions - an infinite sum if G is compact.

Lemma 1.9. *Let (G, K) be a Gelfand pair with G second countable. Let φ be a K -bi-invariant matrix coefficient of a unitary representation π on a Hilbert space \mathcal{H} . Then, there exists a standard Borel space X and a σ -finite measure μ on X such that*

$$\varphi = \int_X c_x \varphi_x d\mu(x)$$

where φ_x is a positive-definite spherical function of (G, K) for any $x \in X$ and $c \in L^1(X, \mu)$.

Proof. If $\varphi(g) = \langle \pi(g)u, v \rangle$, we can replace \mathcal{H} by $\overline{\text{span}(\pi(G)u, \pi(G)v)}$ which is a G -invariant separable subspace, since G is second-countable hence separable. Thus, we can assume that \mathcal{H} is separable.

Then, by [Kir76, Section 8.4], there exists (X, μ) and an isometry $U : \mathcal{H}_X \rightarrow \mathcal{H}$ where \mathcal{H}_X is the direct integral of the collection of Hilbert spaces $(\mathcal{H}_x)_{x \in X}$, such that $\forall g \in G$, $\pi(g) = U \circ \tilde{\pi}(g) \circ U^{-1}$, where $(\tilde{\pi}(g)\xi)_x = \pi_x(g)\xi_x$ and (π_x, \mathcal{H}_x) is an irreducible unitary representation of G for any $x \in X$.

Let P denote the projection on the space of K -invariant vectors in \mathcal{H} . Since φ is K -bi-invariant, we have

$$\varphi(g) = \int_K \int_K \varphi(kgk') dk dk' = \langle \pi(g)Pu, Pv \rangle.$$

Thus, we can assume that u, v are K -invariant, so $\xi = U^{-1}u, \eta = U^{-1}v$ are also K -invariant.

So for μ almost every x , ξ_x, η_x are K -invariant. Now, if π_x is such that 0 is the only K -invariant vector, $\langle \pi_x(g)\xi_x, \eta_x \rangle_{\mathcal{H}_x} = 0$ for every $g \in G$. On the other hand, assume π_x has non-zero K -invariant vectors. Then we know that the space of K -invariant vector is one-dimensional and that there is $c_x \in \mathbb{C}$ such that $\langle \pi_x(g)\xi_x, \eta_x \rangle_{\mathcal{H}_x} = c_x \varphi_x$, where φ_x is the spherical function associated to π_x (and so is positive-definite).

Setting $c_x = 0$ if π_x has no non-zero K -invariant vectors and φ_x the constant spherical function, we have

$$\varphi(g) = \langle \tilde{\pi}(g)\xi, \eta \rangle_{\mathcal{H}_X} = \int_X \langle \pi_x(g)\xi_x, \eta_x \rangle_{\mathcal{H}_x} d\mu(x) = \int_X c_x \varphi_x(g) d\mu(x).$$

Since $|c_x| \leq \|\xi_x\|_{\mathcal{H}_x} \|\eta_x\|_{\mathcal{H}_x}$, the function $x \mapsto c_x$ is in $L^1(X, \mu)$. \square

Lemma 1.10. *Let (G, K) be a Gelfand pair with G a Lie group endowed with a Riemannian metric d and U any open subset of G . Let $(\varphi_\lambda)_{\lambda \in \Lambda}$ be the family of positive-definite spherical functions of (G, K) . Then $(\varphi_\lambda)_{\lambda \in \Lambda}$ is bounded in $C^{(r, \delta)}(U)$ if and only if any K -bi-invariant matrix coefficient of a unitary representation of G is in $C^{(r, \delta)}(U)$.*

Proof. Assume that $(\varphi_\lambda)_{\lambda \in \Lambda}$ is bounded in $C^{(r, \delta)}(U)$. Let φ be a K -bi-invariant matrix coefficient of a unitary representation of G . Since G is a Lie group, G is second countable, thus by Lemma 1.9, there exists a σ -finite measured space (X, μ) such that

$$\varphi = \int_X c_x \varphi_{\lambda_x} d\mu(x)$$

with $c \in L^1(X, \mu)$ and $\lambda_x \in \Lambda$ for any $x \in X$.

Let $g_0 \in U$ and L a compact neighborhood of g_0 . Then there exists $C_L > 0$ such that $\|D^k \varphi_{\lambda_x}(g)\| \leq C_L$ for any $x \in X, g \in L, k \leq r$. Thus, $(x, g) \mapsto D^k(c_x \varphi_{\lambda_x})(g)$ is bounded by the integrable function $x \mapsto C_L c_x$. Hence φ is r times differentiable in a neighborhood of g_0 , and

$$D^k \varphi(g) = \int_X c_x D^k \varphi_{\lambda_x}(g) d\mu(x)$$

for any $k \leq r$. Since this hold for any $g_0 \in U$, $\varphi \in C^r(U)$. Finally, let L be any compact subset of U . There exists $D_L > 0$ such that for any $g, h \in L$ and $x \in X$,

$$\|D^r \varphi_{\lambda_x}(g) - D^r \varphi_{\lambda_x}(h)\| \leq D_L d(g, h)^\delta.$$

Thus,

$$\|D^r \varphi(g) - D^r \varphi(h)\| \leq \int_X |c_x| \|D^r \varphi_{\lambda_x}(g) - D^r \varphi_{\lambda_x}(h)\| d\mu(x) \leq D_L \|c\|_1 d(g, h)^\delta.$$

Hence, we showed that $\varphi \in C^{(r, \delta)}(U)$.

For the other direction, assume that any K -bi-invariant matrix coefficient of a unitary representation of G is in $C^{(r, \delta)}(U)$. Let E be the space of K -bi-invariant matrix coefficient of unitary representations, endowed with the norm

$$\|\varphi\| = \inf \{ \|\xi\| \|\eta\| \mid \exists \pi \text{ such that } \forall g \in G, \varphi(g) = \langle \pi(g)\xi, \eta \rangle \}.$$

Then E is a Banach space, and $\|\varphi\| \geq \|\varphi\|_\infty$. Consider $f : E \mapsto C^{(r,\delta)}(U)$ the linear map sending φ to its restriction to U . By the assumption on regularity, f is well-defined. Let \mathcal{G}_f be the graph of f in $E \times C^{(r,\delta)}(U)$. We claim that it is closed. Indeed, if $(\varphi_n, \varphi_n) \rightarrow (\varphi, \psi)$, then in particular, φ_n converges to φ uniformly on compact subsets of G , since $\|\varphi_n - \varphi\|_\infty \leq \|\varphi_n - \varphi\|$. On the other hand, by definition of the seminorms on $C^{(r,\delta)}(U)$, φ_n converges uniformly on compact subset of U to ψ . Thus $\psi = \varphi|_U$. Now, since E is a Banach space and $C^{(r,\delta)}(U)$ a Fréchet space, by the closed graph theorem, f is continuous. Finally, since the family of positive-definite spherical functions is in the unit ball of E , its image in $C^{(r,\delta)}(U)$ is bounded. \square

This result shows that studying boundedness of positive-definite spherical functions is enough to obtain regularity for all K -bi-invariant matrix coefficients of unitary representations and even that the optimal regularity of such coefficients is exactly the optimal regularity of spherical functions.

1.3 From K -bi-invariant to K -finite coefficients

Let G be a Lie group and K a compact subgroup of G - note that we do not assume that (G, K) is a Gelfand pair in this section.

Definition 1.11. Let π be a unitary representation of G on \mathcal{H} and (ρ, V) a representation of K . We say that $\xi \in \mathcal{H}$ is

- K -finite if $\text{span}(\pi(K)\xi)$ is finite-dimensional,
- of K -type V if $\text{span}(\pi(K)\xi) \simeq V$ as a representation of K . A matrix coefficient $\varphi = \langle \pi(\cdot)\xi, \eta \rangle$ is K -finite if ξ, η are K -finite.

Note that this definition of K -type V is not standard.

The aim of this subsection is to prove that a regularity result on the space of all K -bi-invariant coefficients of G implies that the same regularity holds for all K -finite coefficients of G . Thus, combining this result and Section 1.2, the original question on the regularity of K -finite coefficients of a Gelfand pair (G, K) is completely reduced to understanding spherical functions.

We point out that this reduction from K -finite to K -bi-invariant was done in [Dum24c, Section 5] for compact symmetric pairs, and then extended in [Dum24b, Section 3.2] for non-compact semisimple Lie groups. We here give a unified proof which holds under some mild assumptions on (G, K) . This proof was published in [Dum25].

The first step is a lemma which first appeared in [dLMdlS16, Lemma 2.2] in the case of compact groups, which will allow to smoothly transform a K -finite coefficient into invariant coefficients. The original proof relied on the fact that compact group have a rich finite-dimensional representation theory, and that these representations are smooth. In general, we cannot use finite-dimensional unitary representations, however it remains true that G has a rich set of smooth coefficients.

If π is a unitary representation of G on a Hilbert space H , we say that $\xi \in H$ is a smooth vector if $g \mapsto \pi(g)\xi$ is a smooth map from G to H . Clearly, matrix coefficients associated to smooth vectors are smooth. Consider the Gårding subspace of π , denoted H_∞ , which is the subspace of H spanned by all vectors of the form

$$\pi(f)\xi = \int_G f(g)\pi(g)\xi dg$$

for $\xi \in H, f \in C_c^\infty(G)$, where dg is a left-invariant Haar measure. Then it is well-known that H_∞ is dense in H and every vector of H_∞ is smooth ([Kna01, Prop. 3.14, Thm. 3.15]). In fact, it is a deep result of Dixmier and Malliavin ([DM78]) that the Gårding subspace coincides with the space of smooth vectors, but we do not need this.

Let $U = K \times K$. Let (ρ, E) be a finite-dimensional unitary representation of U . For $g \in G$, let U_g be the stabilizer of g under the action of U by left-right multiplication, i.e.

$$U_g = \{(k_1, k_2) \in U \mid k_1 g k_2^{-1} = g\}.$$

Let $E_g = E^{U_g}$ the subspace of vectors fixed by U_g , and $P_g : E \rightarrow E_g$ the orthogonal projection.

Lemma 1.12. *For any $g_0 \in G$, there exists a smooth function $\psi : G \rightarrow B(E)$ such that*

1. $\forall u \in U, g \in G, \psi(u.g) = \psi(g) \circ \rho(u)^{-1}$,
2. $\forall v_1, v_2 \in E, g \mapsto \langle \psi(g)v_1, v_2 \rangle$ is a matrix coefficient of a unitary representation of G ,
3. $\psi(g_0) = P_{g_0}$.

Proof. Let F be the set of functions $\phi : G \rightarrow B(V)$ such that for any $v_1, v_2 \in E$, the map $g \mapsto \langle \psi(g)v_1, v_2 \rangle$ is a matrix coefficients of a unitary representation of G with vectors in the Gårding subspace of that representation. Then such a map F is smooth and verifies (2).

If $\phi \in F$, define $\psi(g) = \int_U \phi(u.g)\rho(u) du$ where du is the Haar probability measure of U . Since U is compact and ϕ smooth, ϕ is bounded on the compact orbit $U.g$ so the integral makes sense. Then a simple change of variables show that ψ verifies (1). Let e_1, \dots, e_d be an orthonormal basis of E . Let π_{ij} be unitary representations of G and ξ_{ij}, η_{ij} vectors in the Gårding subspace of π_{ij} such that for all $1 \leq i, j \leq d$ and $g \in G$,

$$\langle \phi(g)e_i, e_j \rangle = \langle \pi_{ij}(g)\xi_{ij}, \eta_{ij} \rangle.$$

Let $u = (k_1, k_2) \in U$, then $\rho(u) = \rho(k_1, 1_K)\rho(1_K, k_2)$. We define functions $K \rightarrow \mathbb{C}$ such that

$$\rho(1_K, k)e_i = \sum_{j=1}^d \lambda_{ij}(k)e_j$$

and

$$\rho(k, 1_K)e_i = \sum_{j=1}^d \mu_{ij}(k)e_j.$$

Then

$$\begin{aligned} \langle \psi(g)e_i, e_j \rangle &= \int_U \langle \phi(u.g)\rho(u)e_i, e_j \rangle du \\ &= \int_{K \times K} \sum_{p,q=1}^d \mu_{pq}(k_1)\lambda_{ip}(k_2) \langle \phi(k_1 g k_2^{-1})e_q, e_j \rangle dk_1 dk_2 \\ &= \int_{K \times K} \sum_{p,q=1}^d \mu_{pq}(k_1)\lambda_{ip}(k_2) \langle \pi_{qj}(k_1 g k_2^{-1})\xi_{qj}, \eta_{qj} \rangle dk_1 dk_2 \\ &= \sum_{p,q=1}^d \int_{K \times K} \left\langle \pi_{qj}(g) \left(\lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj} \right), \overline{\mu_{pq}(k_1)}\pi_{qj}(k_1^{-1})\eta_{qj} \right\rangle dk_1 dk_2 \\ &= \sum_{p,q=1}^d \left\langle \pi_{qj}(g) \left(\int_K \lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj} dk_2 \right), \left(\int_K \overline{\mu_{pq}(k_1)}\pi_{qj}(k_1^{-1})\eta_{qj} dk_1 \right) \right\rangle. \end{aligned}$$

Since ξ_{qj} is in the Gårding subspace of π_{qj} , it is a finite linear combination of vectors of the form

$$\pi_{qj}(f)\xi = \int_G f(g)\pi_{qj}(g)\xi dg.$$

Then by left invariance of dg ,

$$\begin{aligned} \int_K \lambda_{ip}(k)\pi_{qj}(k^{-1})\pi_{qj}(f)\xi dk &= \int_K \int_G \lambda_{ip}(k)f(g)\pi_{qj}(k^{-1}g)\xi dg dk \\ &= \int_K \int_G \lambda_{ip}(k)f(kg)\pi_{qj}(g)\xi dg dk \\ &= \int_G \left(\int_K \lambda_{ip}(k)f(kg) dk \right) \pi_{qj}(g)\xi dg \end{aligned}$$

so $\int_K \lambda_{ip}(k)\pi_{qj}(k^{-1})\pi_{qj}(f)\xi dk$ is in the Gårding subspace and by linearity, so is the vector $\int_K \lambda_{ip}(k_2)\pi_{qj}(k_2^{-1})\xi_{qj} dk_2$. Similarly, $\int_K \mu_{pq}(k_1)\pi_{qj}(k_1^{-1})\eta_{qj} dk_1$ is in the Gårding subspace of π_{qj} . Thus, $\langle \psi(g)e_i, e_j \rangle$ is a matrix coefficient of $\bigoplus_{q=1}^d \pi_{qj}^{\oplus d}$ whose vectors are in the Gårding subspace. By linearity, this is true for $v_1, v_2 \in E$. Thus, we showed that if $\phi \in F$, then $\psi \in F$, hence ψ is smooth and verifies (2).

It remains to show that there exists $\phi \in F$ such that $\psi(g_0) = P_{g_0}$. Notice that if $u \in U_{g_0}$, then

$$\psi(g_0) = \psi(u.g_0) = \psi(g_0) \circ \rho(u)^{-1}.$$

Thus $E_{g_0}^\perp = \sum_{u \in U_{g_0}} \text{Im}(\rho(u) - \text{Id}) \subset \ker \psi(g_0)$. Thus, condition (3) is a "maximal rank" condition.

First, let us find $\phi \in F$ such that $\text{rank } \psi(g_0) = \dim E_{g_0}$. Consider $O \simeq U/U_{g_0}$ the U -orbit of g_0 in G . Let s be a measurable section, that is to say $s : O \mapsto U$ such that $s(u.g_0) \in uU_{g_0}$. Let $\phi : O \mapsto B(E)$ be the map $x \mapsto \rho(s(x))^{-1}$. Then $\psi : x \mapsto \int_U \rho(s(u.x))^{-1}u du$ is such that $\psi(g_0)$ is the identity on E_{g_0} , and by the above discussion 0 on $E_{g_0}^\perp$. Thus $\psi(g_0) = P_{g_0}$. Let μ be the image of the Haar measure on O by the map $p : u \mapsto u.g_0$. Then $\phi \in L^1(O; B(E), \mu)$. By density of continuous function, there are continuous maps $f : O \rightarrow B(E)$ arbitrarily close to ϕ in $\|\cdot\|_1$. But then,

$$\begin{aligned} \left\| \int_U f(u.g_0)\rho(u) du - \int_U \phi(u.g_0)\rho(u) du \right\| &\leq \int_U \|f(u.g_0) - \phi(u.g_0)\| du \\ &\leq \int_U \|(f - \phi) \circ p\| du \\ &\leq \int_O \|f - \phi\| d\mu \\ &\leq \|f - \phi\|_1. \end{aligned}$$

So we can take f close enough so that $\text{rank } \int_U f(u.g_0)\rho(u) du = \text{rank } P_{g_0}$. Then since O is closed in G normal topological space, by Tietze extension theorem, we can extend f to a continuous map $\phi : G \mapsto B(E)$.

Let L be a compact subset of G containing g_0 and $\varepsilon > 0$. Let $\phi_{ij} : g \mapsto \langle \phi(g)e_i, e_j \rangle$. By the Gelfand-Raikov's theorem, there exists π_{ij} a unitary representation of G and a matrix coefficient φ_{ij} of π_{ij} such that

$$\sup_{g \in L} |\phi_{ij}(g) - \varphi_{ij}(g)| \leq \varepsilon.$$

Then by density of the Gårding subspace, we may find a coefficient $\tilde{\varphi}_{ij}$ of π_{ij} whose vectors are in the Gårding subspace of π_{ij} and such that

$$\sup_{g \in L} |\varphi_{ij}(g) - \tilde{\varphi}_{ij}(g)| \leq \varepsilon.$$

Then, define $\varphi : G \rightarrow B(E)$ by $\varphi(g)e_i = \sum_{j=1}^d \tilde{\varphi}_{ij}(g)e_j$, by construction $\varphi \in F$. So we can find φ in F arbitrarily close to ϕ on any compact subset containing g_0 , in particular on the orbit O . Thus, for ε small enough, $\int_U \varphi(u.g_0)\rho(u) du$ is of rank $\dim E_{g_0}$.

Finally, we get $\phi \in F$ such that $\psi(g_0)$ has rank $\dim E_{g_0}$ and is zero on $E_{g_0}^\perp$. Thus there is $A \in B(E)$ such that $A\psi(g_0) = P_{g_0}$. Replace ϕ by $A\phi$ and we get the result. \square

With this lemma in hand, we can now prove regularity for K -invariant coefficients. However, although Lemma 1.12 holds without assumption, we need to assume that the pair (G, K) has some sort of a KAK decomposition to carry on - in the spirit of the well-known KAK decomposition of semisimple Lie groups.

Definition 1.13. Let G be a Lie group and K a compact subgroup of G . Let G_r be an open subset of G which is invariant by multiplication on the left and right by elements of K . We say that (G_r, K) has a well-behaved KAK decomposition if there exists a submanifold A_r of G and a subgroup M of K such that:

- for any $g \in G_r$, there exists a unique $a(g) \in A_r$ such that $g \in Ka(g)K$;
- the map $a : G_r \rightarrow A_r$ is smooth;
- the stabilizer of $a \in A_r$ under the left-right multiplication action of K is $\Delta(M) = \{(k, k) \mid k \in M\}$ and thus independent of a ;
- for any $x \in G_r$, there exists a neighborhood O_x of x and a choice of $g \mapsto k_i(g)$ defined on O_x , $i = 1, 2$ such that k_i is smooth and for any $g \in O_x$, $g = k_1(g)a(g)k_2(g)^{-1}$

The existence of a KAK decomposition of a semisimple Lie group is well-know, and we will see in Section 3.2.1 that it is well-behaved on a specific dense open subset. A similar result holds for the other pairs that we will study.

Let π be a unitary representation of G on \mathcal{H} and $\xi, \eta \in \mathcal{H}$ of K -type V, W respectively, for V, W irreducible representations of K . Denote $V_\xi = \text{span}(\pi(K)\xi)$. Then there is an isomorphism $i_\xi : V \rightarrow V_\xi \subset \mathcal{H}$, denote $\xi_0 = i_\xi^{-1}(\xi)$. Similarly, define V_η, i_η and η_0 . Then the map

$$f : \begin{array}{ccc} B(\mathcal{H}) & \rightarrow & L(V, W^*) \simeq V^* \otimes W^* \\ A & \mapsto & i_\eta^* A i_\xi \end{array} \quad (1.1)$$

is $K \times K$ equivariant.

For the associated matrix coefficient, we have $\varphi(g) = \langle \pi(g)\xi, \eta \rangle = \langle f(\pi(g))\xi_0, \eta_0 \rangle$. Indeed if $(e_i), (f_i)$ are orthonormal basis of V, W^* respectively and (e_i^*) the dual basis of (e_i) , then $(e_i^* \otimes f_j)$ is a basis of E and one can check that the coefficient of $f(A)$ in this basis are $\langle A i_\xi e_i, i_\eta f_j \rangle$.

Now denote (ρ, E) the irreducible representation of $U = K \times K$ on $V^* \otimes W^*$. For any $(k, k') \in U$ and $A \in B(\mathcal{H})$, using the U -equivariance of f , we have

$$f(\pi(k)A\pi(k')^{-1}) = \rho(k, k')(f(A)). \quad (1.2)$$

Furthermore, there are $v_1, \dots, v_n \in E$ and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$f(A) = \sum_{i=1}^n \langle A \xi_i, \eta_i \rangle v_i. \quad (1.3)$$

Proposition 1.14. Let G_r be an open subset of G such that (G_r, K) has a well-behaved KAK -decomposition. Assume that any K -bi-invariant matrix coefficient φ of a unitary representation of G is in $C^{(r, \delta)}(G_r)$, then the map $f \circ \pi$ is in $C^{(r, \delta)}(G_r, E)$.

Proof. Let $g_0 \in G_r$ and ψ given by Lemma 1.12 for the representation (ρ, E) . Consider the map $\tilde{f} : g \mapsto \psi(g)(f(\pi(g)))$. By (1.2) and (1) of Lemma 1.12, we have

$$\tilde{f}(u.g) = \psi(u.g)(f(\pi(u.g))) = \psi(g)\rho(u)^{-1}\rho(u)(f(\pi(g))) = \tilde{f}(g) \quad (1.4)$$

so \tilde{f} is a K -bi-invariant map.

Let (e_1, \dots, e_d) be an orthonormal basis of V_ρ , by (2) of Lemma 1.12 there are $(\pi_{ij}, \mathcal{H}_{ij})$ unitary representations of G and $a_{ij}, b_{ij} \in \mathcal{H}_{ij}$ such that

$$\langle \psi(g)v_i, e_j \rangle = \langle \pi_{ij}(g)a_{ij}, b_{ij} \rangle$$

so $\psi(g)v_i = \sum_{j=1}^d \langle \pi_{ij}(g)a_{ij}, b_{ij} \rangle e_j$ and finally with (1.3),

$$\tilde{f}(g) = \sum_{i=1}^n \sum_{j=1}^d \langle (\pi_{ij} \otimes \pi)(g)(a_{ij} \otimes \xi_i), b_{ij} \otimes \eta_i \rangle e_j. \quad (1.5)$$

Hence, \tilde{f} is a sum of K -bi-invariant matrix coefficients of unitary representations of G , so by the hypothesis, $\tilde{f} \in C^{(r,\delta)}(G_r, E)$.

Since (G_r, K) has a well-behaved KAK decomposition, consider A_r, M from Definition 1.13. Then for any $a \in A_r$, $U_a = \Delta(M)$. Thus, $E_a = E^{U_a}$ is independent of $a \in A_r$, and we denote it by E_0 . If $g = (k_1, k_2).a = k_1 a k_2^{-1}$, we have $(k, k') \in U_g$ if and only if $(k_1^{-1} k k_1, k_2^{-1} k' k_2) \in \Delta(M)$ and so $E_g = \rho(k_1, k_2)E_0$.

Let $g_0 = k_0 a_0 k_0'^{-1}$ and $E_1 = E_{g_0}$. Since $\psi(g_0) = P_{g_0}$, there is an orthonormal basis adapted to E_1 such that

$$\psi(g_0) = \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, since ψ is smooth, there is A_{g_0} neighborhood of g_0 such that

$$\psi(g) = \begin{pmatrix} A(g) & * \\ * & * \end{pmatrix}$$

with $g \mapsto A(g)$ smooth, $A(g)$ invertible for any $g \in A_{g_0}$. Up to restricting A_{g_0} , by assumption on (G_r, K) , we have $g = k_1(g)a(g)k_2(g)^{-1}$ with k_1, k_2 smooth on A_{g_0} .

By (1.4), for any $g \in A_{g_0}$, we have

$$\tilde{f}(g) = \tilde{f}(a(g)) = \tilde{f}(k_0 a(g) k_0'^{-1}).$$

But then $f(\pi(k_0 a(g) k_0'^{-1})) \in E_{k_0 a(g) k_0'^{-1}} = \rho(k_0, k_0')E_0 = E_1$. Set

$$\Phi(g) = \rho(k_1(g)k_0^{-1}, k_2(g)k_0'^{-1}) \begin{pmatrix} A(k_0 a(g) k_0'^{-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

it is a smooth map on A_{g_0} because A is smooth invertible, k_1, k_2 are smooth and ρ is a finite-dimensional representation of U thus smooth. Since $f(\pi(k_0 a(g) k_0'^{-1})) \in E_1$, we have

$$\begin{aligned} \Phi(g)(\tilde{f}(g)) &= \Phi(g)(\tilde{f}(\pi(k_0 a(g) k_0'^{-1}))) \\ &= \Phi(g)\psi(g)(f(\pi(k_0 a(g) k_0'^{-1}))) \\ &= \rho(k_1(g)k_0^{-1}, k_2(g)k_0'^{-1})(f(\pi(k_0 a(g) k_0'^{-1}))) \\ &= f(\pi(k_1(g)a(g)k_2(g)^{-1})) \\ &= f(\pi(g)) \end{aligned}$$

Now let $B : B(V) \times V \rightarrow V$ be the bilinear map sending (u, v) to $u(v)$. We showed that on A_{g_0} , $f \circ \pi = B \circ (\Phi, \tilde{f})$. Since Φ is smooth on A_{g_0} and $\tilde{f} \in C^{(r,\delta)}(G_r, E)$, we get by Leibniz formula that $f \circ g \in C^{(r,\delta)}(A_{g_0}, E)$.

So for any $g_0 \in G_r$, there exists a neighborhood A_{g_0} such that $f \circ \pi \in C^{(r,\delta)}(A_{g_0}, E)$. Thus, $f \circ \pi \in C^{(r,\delta)}(G_r, E)$. \square

Theorem 1.15. *Let G be a Lie group, G_r be an open subset of G and K a compact subgroup of G such that (G_r, K) has a well-behaved KAK -decomposition. The optimal regularity of K -bi-invariant matrix coefficient of unitary representations of G on G_r is equal to the optimal regularity of K -finite matrix coefficients of unitary representations of G on G_r .*

Proof. One inequality is trivial since K -bi-invariant coefficients are K -finite.

For the other inequality, let (r, δ) such that any K -bi-invariant matrix coefficient of unitary representations G is in $C^{(r,\delta)}(G_r)$. Let $\varphi : g \mapsto \langle \pi(g)\xi, \eta \rangle$ be a K -finite matrix coefficient of a unitary representation.

If ξ, η are of K -type V, W respectively, with V, W irreducible representations of K , we showed that $\varphi(g) = \langle \pi(g)\xi, \eta \rangle = \langle f(\pi(g))\xi_0, \eta_0 \rangle$ and in Proposition 1.14 that $f \circ \pi \in C^{(r,\delta)}(G_r)$, thus $\varphi \in C^{(r,\delta)}(G_r)$.

For the general case, if ξ, η are K -finite, V_ξ, V_η are finite-dimensional representations of K , so they decompose into a finite number of irreducible representations. Thus, φ is a finite sum of matrix coefficients of the previous case, so $\varphi \in C^{(r,\delta)}(G_r)$. \square

1.4 Symmetric spaces

Definition 1.16. A symmetric space is a smooth manifold M together with a smooth map $\mu : M \times M \rightarrow M$, denoted $\mu(x, y) = x \cdot y$, which verifies the following properties:

1. $\forall x \in M, x \cdot x = x$;
2. $\forall x, y \in M, x \cdot (x \cdot y) = y$;
3. $\forall x, y, z \in M, x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$;
4. $\forall x \in M$, there is a neighborhood $U \subset M$ of x such that if $y \in U, x \cdot y = y \Rightarrow x = y$.

Let G be a connected Lie group G and σ an involutive automorphism of G . Let G^σ denote the subgroup of fixed points of σ and $(G^\sigma)_0$ its identity component. For a subgroup K of G such that $(G^\sigma)_0 \subset K \subset G^\sigma$, the quotient space $M = G/K$ can be given a structure of symmetric space with $xK \cdot yK = x\sigma(x)^{-1}\sigma(y)K$ ([Loo69a]). Using a characterization of Gelfand pairs ([vd09, Prop. 6.1.3]), we see that (G, K) is a Gelfand pair if and only if K is compact. We call such pairs symmetric Gelfand pairs.

It turns out that all symmetric spaces arise in this way (see [Loo69a, Ch. II, Thm. 3.1]). Given M a connected symmetric space and $o \in M$, there is a canonical connected Lie group $G(M)$ associated to it, called the group of displacements. The group $G(M)$ is a subgroup of $\text{Aut}(M)$, the group of automorphisms of symmetric spaces of M , thus acts on M . If $K(M)$ is the isotropy subgroup of o , we have $M \simeq G(M)/K(M)$. Furthermore, $K(M)$ is compact if and only if the symmetric space is Riemannian ([Loo69a, Ch. IV, Prop. 1.7]).

Since there is a classification of Riemannian symmetric spaces ([Loo69b, Ch. VII]), it seems natural to study the Gelfand pairs associated to such spaces. We say that a Riemannian symmetric space M is

- euclidean if its sectional curvature is identically zero,
- of compact type if its sectional curvature is positive and not identically zero,
- of non-compact type if its sectional curvature is negative and not identically zero.

If M is a simply connected symmetric space, then there are M_0 euclidean, M_+ of non-compact type and M_- of compact type such that $M = M_0 \times M_+ \times M_-$ ([Loo69a, Ch. IV, Coro. 1]). In this thesis, we will study symmetric Gelfand pairs. However, a question arises: if two pairs represent the same symmetric space, are their spherical functions the same?

We say that a symmetric space M is semisimple if $G(M)$ is a semisimple Lie group.

Lemma 1.17. *Let (G, K) be a symmetric pair and $M = G/K$ the associated symmetric space. If M is semisimple, then there is a bijection between spherical functions of (G, K) and spherical functions of $(G(M), K(M))$, such that the image of φ induces the same function as φ on M .*

Proof. Let $\tau : G \rightarrow \text{Aut}(M)$ be the group morphism defined by $\tau(g) : xK \mapsto gxK$. Then $\ker \tau = \bigcap_{g \in G} gKg^{-1}$. Let $G_\sigma = \{x\sigma^{-1}(x) \mid x \in G\}$. By [Loo69a, Ch. II, Thm. 1.3], $G(M) = \langle \tau(G_\sigma) \rangle$ is a subgroup of $\tau(G) \simeq G/\ker \tau$.

By [Loo69a, Ch. IV, Prop. 1.4], since M is a semisimple symmetric space, we have that $G(M) = (\text{Aut}(M))_0$. So we have $G(M) = (\text{Aut}(M))_0 < \tau(G) < \text{Aut}(M)$ and $\tau(G)$ is connected because G is, so $G(M) \simeq G/\ker \tau$, and $K(M) \simeq K/\ker \tau$.

Let π be an irreducible unitary representation of $G(M)$ with a $K(M)$ -invariant vector ξ . By composition with the isomorphism and projection, it induces an irreducible unitary representation of G , with ξ which is a K -invariant vector.

Conversely, let π be an irreducible unitary representation of G on V with a K -invariant vector ξ . By irreducibility, $\text{span}(\pi(G)\xi) = V$. Let $x \in \ker \tau$. If $g \in G$, there is $k \in K$ such that $x = gkg^{-1}$. Hence,

$$\pi(x)\pi(g)\xi = \pi(g)\pi(k)\xi = \pi(g)\xi.$$

So for any $g \in G$, $\pi(g)\xi$ is $\pi(x)$ -invariant. By density of the vector space generated by these vectors, $\pi(x) = \text{Id}_V$. So $\ker \tau \subset \ker \pi$, thus π induces an irreducible representation of the quotient $G(M)$, with a $K(M)$ -invariant vector ξ . \square

This results says that in the case of a semisimple space, the spherical functions depend essentially only on the symmetric space. If M is of compact type or non-compact type, then M is semisimple. In fact ([Loo69a, Ch. IV, Thm. 3.5]), M is of compact type if and only if M is compact and semisimple, if and only if its universal cover is compact, if and only if $G(M)$ is compact and semisimple. When M is of non-compact type, M is always simply connected and $K(M)$ is a maximal compact subgroup of $G(M)$.

On the other hand, for a euclidean symmetric space, there is a variety of pairs representing the space which exhibit different behaviors.

Let $n \in \mathbb{N}$, and consider $M = \mathbb{R}^n$. Then $G(M) = \mathbb{R}^n$ and $\text{Aut}(M) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$. Let K be a compact subgroup of $GL_n(\mathbb{R})$ and $H = \mathbb{R}^n \rtimes K$, then (H, K) is a symmetric Gelfand pair (with the involution $\sigma(x, g) = (-x, g)$ for any $(x, g) \in H$). Clearly, when K is the trivial group - which corresponds to the canonical pair - there exists coefficients of unitary representations of \mathbb{R}^n that are continuous, but not Hölder continuous (for any exponent). However, it is interesting to ask what happens when K is non-trivial. For example, the spherical functions of $(\mathbb{R}^n \rtimes SO(n), SO(n))$ can be expressed in terms of Bessel functions ([Vil68]).

We will restrict ourselves to symmetric pairs associated to simply connected symmetric spaces of compact type or non-compact type. Thus by the previous discussion, we can study only one pair for each symmetric space (for example, the canonical pair). We will also consider some specific flat symmetric pairs related to semisimple groups (see Section 1.7).

1.5 Spherical functions of compact symmetric pairs

In this section, let M be a compact connected simply connected symmetric space and (G, K) the associated canonical compact symmetric Gelfand pair. Since G is compact, any spherical function is positive-definite and thus corresponds to an irreducible unitary representation of G with a non-zero K -invariant vector.

We know that the finite-dimensional irreducible representations of G are classified by the highest weights μ . Let \mathfrak{g} be the Lie algebra of G and \mathfrak{k} the Lie algebra of K . Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where \mathfrak{k} (resp. \mathfrak{p}) is the eigenspace of $+1$ (resp. -1) of σ . The space \mathfrak{p} is also the Lie triple system of M (see [Loo69a], Ch. II, Prop 2.3), that is to say the tangent space of M at K endowed with some additional structure. Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . If $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$, let $m(\alpha) = \dim\{X \in \mathfrak{g}_{\mathbb{C}} \mid \forall H \in \mathfrak{a}_{\mathbb{C}}, [H, X] = \alpha(H)X\}$. Let $\Sigma_{\mathfrak{a}} = \{\alpha \in \mathfrak{a}_{\mathbb{C}}^* \setminus \{0\} \mid m(\alpha) > 0\}$ be the root system of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Let $\Sigma_{\mathfrak{a}}^+$ a choice of positive roots. Let $\mathfrak{k}^{\mathfrak{a}} = \{x \in \mathfrak{k} \mid [x, \mathfrak{a}] = 0\}$ and \mathfrak{t} a Cartan subalgebra of $\mathfrak{k}^{\mathfrak{a}}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra of \mathfrak{g} . Consider $\Sigma_{\mathfrak{h}}$ the root system of $\mathfrak{h}_{\mathbb{C}}$ and $\Sigma_{\mathfrak{h}}^+$ a choice of positive roots, such that $\Sigma_{\mathfrak{a}}^+ = \{\phi|_{\mathfrak{a}} \mid \phi \in \Sigma_{\mathfrak{h}}^+, \phi|_{\mathfrak{a}} \neq 0\}$. Also, recall that by definition, $\text{rank } G = \dim \mathfrak{h}$ and $\text{rank } M = \dim \mathfrak{a}$.

We say that $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ is a dominant element if for any $\alpha \in \Sigma_{\mathfrak{h}}^+$, $\langle \mu, \alpha \rangle \geq 0$. We say that μ is analytically integral if for any $H \in \mathfrak{h}$ such that $\exp(H) = 1$, $\mu(H) \in 2\pi i\mathbb{Z}$. By the theorem of the highest weight ([Kna02, Thm 5.110]), there is a one-one correspondence between finite-dimensional irreducible representations of G (up to isomorphism) and dominant analytically integral elements. Let μ be a dominant analytically integral element in $\mathfrak{h}_{\mathbb{C}}^*$ and (π_{μ}, V_{μ}) a representative of the associated isomorphism class of finite-dimensional irreducible representations of G . Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_{\mathfrak{h}}^+} \alpha.$$

We know by the Weyl formula ([Hal03, Thm. 10.18]) that

$$d_{\mu} = \dim V_{\mu} = \frac{\prod_{\alpha \in \Sigma_{\mathfrak{h}}^+} \langle \alpha, \mu + \rho \rangle}{\prod_{\alpha \in \Sigma_{\mathfrak{h}}^+} \langle \alpha, \rho \rangle}.$$

The first question we want to ask is, given μ a dominant analytically integral element, what are the conditions for π_{μ} to have a non-zero K -invariant vector. In this context, the answer is given by the Cartan-Helgason theorem ([Hel00, Ch. V, Thm. 4.1]). Let \hat{G}_K denote the set of classes of irreducible finite-dimensional representations with a non-zero K -invariant vector.

Theorem 1.18 (Cartan-Helgason). *Let $\Lambda = \{\mu \in i\mathfrak{a}^* \mid \forall \alpha \in \Sigma_{\mathfrak{a}}^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}\}$. Then the map which sends a representation to its highest weight is a bijection from \hat{G}_K to Λ .*

Remark 1.19. In [Hel00], the result is stated for G simply connected. However, let \tilde{G} be the universal cover of G and $p : \tilde{G} \rightarrow G$ the covering map. Then $\ker(p) \subset Z(\tilde{G})$. Let

$\tilde{K} = p^{-1}(K)$. We have that $\tilde{G}/\tilde{K} \simeq M$. Since \tilde{G} is connected and M is simply connected, \tilde{K} is connected by the long exact sequence of homotopy groups. The group \tilde{K} contains \tilde{G}^σ which is connected (by [Bor98, Ch.V, Thm 3.3]) and has the same Lie algebra, thus $\tilde{K} = \tilde{G}^\sigma$.

This means that (\tilde{G}, \tilde{K}) is another symmetric pair for the symmetric space G/K , so by Lemma 1.17, there is a bijection between \hat{G}_K and $\hat{G}_{\tilde{K}}$. Finally, Λ depends only on the Lie algebra, and so by the theorem and the previous bijection, $\hat{G}_K \rightarrow \Lambda$ is a bijection too.

Also, note that $\ker p \subset \tilde{G}^\sigma$ since its elements act trivially on M . Hence,

$$\ker p \subset Z(\tilde{G})^\sigma = \{g \in Z(\tilde{G}) \mid \sigma(g) = g\} = \tilde{K} \cap Z(\tilde{G}).$$

For example, if $\text{rank } M = \text{rank } G$, $\mathfrak{a} = \mathfrak{h}$ so Λ is twice the set of dominant analytically integral elements. In general, let ℓ be the rank of M . Then the choice of $\Sigma_{\mathfrak{a}}^+$ gives a choice of a basis of the root system $\{\alpha_1, \dots, \alpha_\ell\}$. By [Vre76, Thm. 2.1], there are fundamental weights $\mu_i, 1 \leq i \leq \ell$ such that $\Lambda = \{\sum m_i \mu_i, m_i \in \mathbb{N}\} \simeq \mathbb{N}^\ell$. These fundamental weights verify $\langle \mu_i, \alpha_j \rangle = 0$ if $i \neq j$.

Let $\mathfrak{a}_r = \{H \in \mathfrak{a} \mid \forall \lambda \in \Sigma_{\mathfrak{a}}, \lambda(H) \notin i\pi\mathbb{Z}\}$ and Q the connected component of \mathfrak{a}_r contained in the positive Weyl chamber $\mathcal{C} = \{H \in \mathfrak{a} \mid \forall \lambda \in \Sigma_{\mathfrak{a}}^+, -i\lambda(H) > 0\}$ and such that $0 \in \overline{Q}$. Then in [Cle88, Prop. 3.2], we have the following result:

Proposition 1.20. *For any $g \in G$, there exists $k_1, k_2 \in K$ and a unique vector $H \in \overline{Q}$ such that $g = k_1 \exp(H) k_2^{-1}$.*

Remark 1.21. Again, this result is given for G simply connected. If G is not simply connected, note first that Q depends only on the Lie algebra. Let \tilde{G} be the universal cover of G and $p : \tilde{G} \rightarrow G$ the covering map. For $g \in G$, there is $\tilde{g} \in \tilde{G}$ such that $g = p(\tilde{g})$. The decomposition gives $k_1, k_2 \in \tilde{K}$ and $H \in Q$ such that

$$\tilde{g} = k_1 \exp_{\tilde{G}}(H) k_2^{-1},$$

so

$$g = p(k_1) p(\exp_{\tilde{G}}(H)) p(k_2)^{-1} = p(k_1) \exp_G(H) p(k_2)^{-1}$$

gives the decomposition for G . Furthermore, if there are H, H' such that $\exp_G(H) = \exp_G(H')$, then $\exp_{\tilde{G}}(H) = \exp_{\tilde{G}}(H') k, k \in \ker p \subset \tilde{K}$. By uniqueness for simply connected groups, $H = H'$.

Definition 1.22. Let $G_r = K \exp(Q) K$, then G_r is a dense open subset of G . We say that a point $g \in G_r$ is regular and $g \in G \setminus G_r$ is singular.

Now, let $(\varphi_\mu)_{\mu \in \Lambda}$ be the family of spherical functions of (G, K) . Consider the projection map

$$\text{pr} : \begin{array}{ccc} G & \rightarrow & K \backslash G / K \\ g & \mapsto & KgK \end{array} .$$

The functions φ_μ are K -bi-invariant, thus the value of $\varphi_\mu(g)$ depends only on the double coset $\text{pr}(g) = KgK$. Let $\psi_\mu = \varphi_\mu \circ \exp|_Q$. The family (ψ_μ) is a family of functions defined on an open subset of \mathbb{R}^ℓ . We are interested in the regularity of these functions. Let

$$r_{\text{opt}}(M) = \sup \left\{ (r, \alpha) \mid (\psi_\mu) \text{ bounded in } C^{(r, \alpha)}(Q) \right\}.$$

By Lemma 1.10, this is also the supremum of (r, α) such that every K -bi-invariant matrix coefficient of unitary representations of G is in $C^{(r, \alpha)}(Q)$.

Remark 1.23. Since the spherical functions can be defined on $K \backslash G / K$, we can see them as functions on \overline{Q} by the previous proposition without losing information on the function. However, we restrict to Q because on the singular points, the behavior cannot be controlled. Furthermore, Proposition 1.20 will be refined in Proposition 2.43, which will allow to recover the regularity obtained on the Lie algebra at the level of the group itself.

Let M_1, M_2 be two simply connected symmetric spaces of compact type and let (G_1, K_1) and (G_2, K_2) be the associated canonical Gelfand pairs. Then $M = M_1 \times M_2$ is a simply connected symmetric space of compact type, whose canonical Gelfand pair is (G, K) with $G = G_1 \times G_2$ and $K = K_1 \times K_2$. Let π be an irreducible representation of G . The irreducible representations of G are $\pi = \pi_1 \otimes \pi_2$ where π_i is an irreducible representation of G_i , and π has a non-zero K -invariant vector if and only if π_i has a non-zero K_i -invariant vector, $i = 1, 2$. In that case, we know that the space of K_i -invariant vectors is one-dimensional, let e_{K_i} be a unitary generator. Then, $e_K = e_{K_1} \otimes e_{K_2}$ is non-zero, unitary, K -invariant and generates the one-dimensional space of K -invariant vectors of π . Thus, if φ_i is the spherical function associated to π_i , and φ associated to π , we get

$$\varphi(g_1, g_2) = \varphi_1(g_1)\varphi_2(g_2).$$

Similarly, at the level of the Lie algebra, we have $Q = Q_1 \times Q_2$ so

$$\psi(H_1, H_2) = \psi_1(H_1)\psi_2(H_2).$$

Denote E_i vector space such that $Q_i \subset E_i$, and on $E_1 \times E_2$, we consider the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$. We write $\Lambda, \Lambda_1, \Lambda_2$ the sets of highest weights of representations with invariant vectors for M, M_1, M_2 . Since the constant function 1 is a spherical function of any pair, we can take $\psi_2 = 1$ and we get that for any $\mu \in \Lambda_1$, the function $(g_1, g_2) \mapsto \psi_\mu(g_1)$ is a spherical function of (G, K) .

Let $(r, \alpha) < r_{opt}(M)$. Then $(\psi_\mu)_{\mu \in \Lambda_1} \subset (\psi_\mu)_{\mu \in \Lambda}$ is bounded in $C^{(r, \alpha)}(Q_1)$, thus we get the inequality $r_{opt}(M_1) \geq r_{opt}(M)$. Symmetrically, $r_{opt}(M_2) \geq r_{opt}(M)$.

Conversely, let $(r, \alpha) \leq \min(r_{opt}(M_1), r_{opt}(M_2))$. Let L be a compact subset of Q . There are L_1, L_2 compact subsets of Q_1, Q_2 such that $L \subset L_1 \times L_2$. For $k \leq r$, let

$$C_{L_i, k} = \sup_{\mu \in \Lambda_i} \sup_{x \in L_i} \|D^k \psi_\mu(x)\|$$

and let

$$C_{L_i} = \sup_{\mu \in \Lambda_i} \sup_{x \in L_i} \frac{\|D^r f(x) - D^r f(y)\|}{\|x - y\|^\alpha}.$$

These are finite numbers since by assumptions, $(\psi_\mu)_{\mu \in \Lambda_i}$ is bounded in $C^{(r, \alpha)}(Q_i)$, $i = 1, 2$. We have $\Lambda \simeq \Lambda_1 \times \Lambda_2$. Consider $(\mu_1, \mu_2) \in \Lambda$, and

$$\psi_{(\mu_1, \mu_2)} : (x_1, x_2) \mapsto \psi_{\mu_1}(x_1)\psi_{\mu_2}(x_2).$$

Then clearly, $\psi_{(\mu_1, \mu_2)}$ is r times differentiable and we have

$$\begin{aligned} D^r \psi_{(\mu_1, \mu_2)}(x_1, x_2)((H_1, K_1), \dots, (H_r, K_r)) = \\ \sum_{i \in I} D^{k_i} \psi_{\mu_1}(x_1)(H_{j_1(i)}, \dots, H_{j_{k_i}(i)}) D^{r-k_i} \psi_{\mu_2}(x_2)(K_{j_{k_i+1}(i)}, \dots, K_{j_r(i)}) \end{aligned}$$

where $k_i \leq r$ for each $i \in I$ and I finite.

Thus, for all $(x_1, x_2), (y_1, y_2) \in L$, we have

$$\begin{aligned} \|D^r \psi_{(\mu_1, \mu_2)}(x_1, x_2) - D^r \psi_{(\mu_1, \mu_2)}(y_1, y_2)\| &\leq \\ &\sum_{i \in I} \|D^{k_i} \psi_{\mu_1}(x_1) D^{r-k_i} \psi_{\mu_2}(x_2) - D^{k_i} \psi_{\mu_1}(y_1) D^{r-k_i} \psi_{\mu_2}(y_2)\|. \end{aligned}$$

So we have

$$\begin{aligned} &\|D^k \psi_{\mu_1}(x_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(y_2)\| \\ &\leq \|D^k \psi_{\mu_1}(x_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(x_2) \\ &\quad + D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(y_2)\| \\ &\leq \|D^k \psi_{\mu_1}(x_1) - D^k \psi_{\mu_1}(y_1)\| \|D^{r-k} \psi_{\mu_2}(x_2)\| \\ &\quad + \|D^k \psi_{\mu_1}(y_1)\| \|D^{r-k} \psi_{\mu_1}(x_2) - D^{r-k} \psi_{\mu_1}(y_2)\| \end{aligned}$$

If $k \neq 0, r$, this gives

$$\begin{aligned} &\|D^k \psi_{\mu_1}(x_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(y_2)\| \\ &\leq C_{L_1, k+1} C_{L_2, r-k} \|x_1 - y_1\| + C_{L_1, k} C_{L_2, r-k+1} \|x_2 - y_2\|, \end{aligned}$$

if $k = 0$,

$$\begin{aligned} &\|D^k \psi_{\mu_1}(x_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(y_2)\| \\ &\leq C_{L_1, 1} C_{L_2, r} \|x_1 - y_1\| + C_{L_1, 0} C_{L_2} \|x_2 - y_2\|^\alpha \end{aligned}$$

and if $k = r$,

$$\begin{aligned} &\|D^k \psi_{\mu_1}(x_1) D^{r-k} \psi_{\mu_2}(x_2) - D^k \psi_{\mu_1}(y_1) D^{r-k} \psi_{\mu_2}(y_2)\| \\ &\leq C_{L_1} C_{L_2, 0} \|x_1 - y_1\|^\alpha + C_{L_1, r} C_{L_2, 1} \|x_2 - y_2\|. \end{aligned}$$

But since $\alpha \in [0, 1]$, there is $C_i > 0$ such that for all $x, y \in L_i$,

$$\|x - y\| \leq C_i \|x - y\|^\alpha$$

and so there is a constant $C_L > 0$ which does not depend on $(\mu_1, \mu_2) \in \Lambda$, such that

$$\begin{aligned} \|D^r \psi_{(\mu_1, \mu_2)}(x_1, x_2) - D^r \psi_{(\mu_1, \mu_2)}(y_1, y_2)\| &\leq C_L \max(\|x_1 - y_1\|^\alpha, \|x_2 - y_2\|^\alpha) \\ &= C_L \|(x_1, x_2) - (y_1, y_2)\|^\alpha. \end{aligned}$$

And thus, we showed that $(\psi_\mu)_{\mu \in \Lambda}$ is bounded in $C^{(r, \alpha)}(Q)$, so that

$$r_{opt}(M) \geq \min(r_{opt}(M_1), r_{opt}(M_2)).$$

By induction, we get the following :

Proposition 1.24. *Let M_i , $1 \leq i \leq n$ be a simply connected symmetric space of compact type, and $M = \prod_{i=1}^n M_i$. Then*

$$r_{opt}(M) = \min(r_{opt}(M_i)).$$

This result tells us that we can study only the irreducible simply connected symmetric spaces of compact type. By the classification in [Loo69b, Ch. VII], such a space M is either

1. one of the spaces in [Loo69b, Ch. VII, Table 4 and Table 8],
2. $(G \times G)/\Delta(G)$, where G is a simply connected compact simple Lie group and $\Delta(G)$ the diagonal subgroup of $G \times G$.

The first case contains the symmetric spaces of rank 1 for which we will solve the question in Section 2.1. In Section 2.2.1, we will study $(G \times G)/\Delta(G)$ (even for reducible and not simply connected). We also give some partial results for some other higher rank symmetric spaces in Section 2.2.2. Finally, we will improve these results in the general case in Chapter 4 using methods from the non-compact case.

1.6 Spherical functions of semisimple Lie groups

Let G be a connected real semisimple Lie group with finite center and \mathfrak{g} its Lie algebra. Note that all results will apply to complex semisimple Lie groups, by viewing them naturally as real Lie groups. Let B be the Killing form of \mathfrak{g} , defined by $B(X, Y) = \text{Tr}(\text{ad } X \circ \text{ad } Y)$. Let θ be a Cartan involution of \mathfrak{g} , that is to say an involutive Lie algebra automorphism such that

$$(X, Y) \mapsto -B(X, \theta Y)$$

is positive definite. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of \mathfrak{g} in ± 1 -eigenspaces of θ . Then $K = \exp \mathfrak{k}$ is a maximal compact subgroup of G . Since all maximal compact subgroups of G are conjugated, whenever we say "a maximal compact subgroup of G " in the sequel, we may always assume that it is defined by a given Cartan involution. The group K is connected and $K = G^\theta$. Then (G, K) is a symmetric pair of non-compact type, and all such pairs are of this form. Consider \mathfrak{a} a maximal abelian subspace of \mathfrak{p} . The rank of G is $\text{rank } G = \dim \mathfrak{a} = \ell$. For $\alpha \in \mathfrak{a}^*$, define $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}, [H, X] = \alpha(H)X\}$ the root space associated to α . Let $m(\alpha) = \dim(\mathfrak{g}_\alpha)$ and $\Sigma = \{\alpha \neq 0 \mid m(\alpha) \geq 1\}$ be the set of roots. We say that Σ is the restricted root system of G . Let $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k} = \mathfrak{k}^\mathfrak{a}$. Then the Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

The Killing form of \mathfrak{g} induces an inner product on \mathfrak{a} , denoted $\langle \cdot, \cdot \rangle$. Then for $\lambda \in \mathfrak{a}^*$, there is a unique $H_\lambda \in \mathfrak{a}$ such that for any $H \in \mathfrak{a}$, $\lambda(H) = \langle H_\lambda, H \rangle$. We use the isomorphism $\lambda \mapsto H_\lambda$ to define an inner product on \mathfrak{a}^* by

$$\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle.$$

Let W be the Weyl group of the root system Σ , which is the subgroup of $O(\mathfrak{a}^*)$ generated by the reflections $s_\alpha : x \mapsto x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$. The group W also acts on \mathfrak{a} by $wH_\lambda = H_{w\lambda}$ and is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ ([Kna02, Thm. 6.57]). By [Var13, Thm. 4.3.24], this action can be extended to automorphisms of the Lie algebra \mathfrak{g} . In particular, we get that $\mathfrak{g}_{w\alpha} = w(\mathfrak{g}_\alpha)$ and so $m(w\alpha) = m(\alpha)$. The hyperplanes $\{\alpha(H) = 0\}$ divide \mathfrak{a} into $|W|$ connected components. We choose one, which we denote \mathfrak{a}^+ and call the positive Weyl chamber, and we define the positive roots $\Sigma^+ = \{\alpha \in \Sigma \mid \forall H \in \mathfrak{a}^+, \alpha(H) > 0\}$. Then $\Sigma = \Sigma^+ \cup (-\Sigma^+)$. We say that $\alpha \in \Sigma^+$ is simple if it cannot be decomposed as $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Sigma^+$. Let Δ be the set of simple roots. Then Δ is a basis of \mathfrak{a}^* and we can write $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Given $\alpha \in \Sigma^+$, $\alpha = \sum_{i=1}^\ell n_i(\alpha) \alpha_i$ with $n_i(\alpha) \in \mathbb{N}$. Furthermore, the group W is generated by the reflections $\{s_\alpha\}_{\alpha \in \Delta}$ ([Bou07, Ch. VI, Thm. 2]). For any $\alpha \in \Delta$, the reflection s_α permutes the positive roots that are not proportional to α ([Bou07, Ch. VI, Prop. 17]).

For $\lambda \in \mathfrak{a}^*$, define

$$n(\lambda) = \sum_{\substack{\alpha \in \Sigma^+ \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha) \quad (1.6)$$

and set

$$\kappa(G) = \inf_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{n(\lambda)}{2}. \quad (1.7)$$

We can express κ in a more computable way. First, notice that n is invariant under the Weyl group. It suffices to prove this on generators s_γ , $\gamma \in \Delta$. Then

$$\begin{aligned} n(s_\gamma \lambda) &= \sum_{\substack{\alpha \in \Sigma^+ \setminus \{\gamma, 2\gamma\} \\ \langle \alpha, s_\gamma \lambda \rangle \neq 0}} m(\alpha) + (1 - \delta_{\langle \gamma, s_\gamma \lambda, 0 \rangle})(m(\gamma) + m(2\gamma)) \\ &= \sum_{\substack{\alpha \in \Sigma^+ \setminus \{\gamma, 2\gamma\} \\ \langle s_\gamma \alpha, \lambda \rangle \neq 0}} m(\alpha) + (1 - \delta_{-\langle \gamma, \lambda, 0 \rangle})(m(\gamma) + m(2\gamma)) \\ &= \sum_{\substack{\alpha \in \Sigma^+ \setminus \{\gamma, 2\gamma\} \\ \langle \alpha, \lambda \rangle \neq 0}} m(s_\gamma \alpha) + (1 - \delta_{\langle \gamma, \lambda, 0 \rangle})(m(\gamma) + m(2\gamma)) \\ &= \sum_{\substack{\alpha \in \Sigma^+ \setminus \{\gamma, 2\gamma\} \\ \langle \alpha, \lambda \rangle \neq 0}} m(\alpha) + (1 - \delta_{\langle \gamma, \lambda, 0 \rangle})(m(\gamma) + m(2\gamma)) \\ &= n(\lambda) \end{aligned}$$

using the fact that s_γ is a permutation of $\Sigma^+ \setminus \{\gamma, 2\gamma\}$ and the invariance of multiplicities under the Weyl group. Notice that $m(2\gamma)$ can be zero if $2\gamma \notin \Sigma$. Since every orbit under the Weyl group meets \mathfrak{a}^+ , $\kappa(G)$ is also the infimum over $\lambda \neq 0$ such that $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Sigma^+$. For such a λ , if $\alpha = \beta + \gamma$ with $\alpha, \beta, \gamma \in \Sigma^+$, then $\langle \alpha, \lambda \rangle = 0$ implies $\langle \beta, \lambda \rangle = 0 = \langle \gamma, \lambda \rangle$. Thus, if $n(\lambda)$ is minimal, there is a unique $\alpha_i \in \Delta$ such that $\langle \alpha_i, \lambda \rangle \neq 0$. Therefore, we get

$$\kappa(G) = \frac{1}{2} \min_{1 \leq i \leq \ell} \sum_{\substack{\alpha \in \Sigma^+ \\ n_i(\alpha) \geq 1}} m(\alpha). \quad (1.8)$$

The values of κ can be computed for simple groups and found in the tables in Appendix A.

Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Denote $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$ and $N = \exp \mathfrak{n}$. The Iwasawa decomposition says that

$$\begin{aligned} K \times A \times N &\rightarrow G \\ (k, a, n) &\mapsto kan \end{aligned} \quad (1.9)$$

is a diffeomorphism ([Kna02, Thm. 6.46]). Let $H : G \mapsto \mathfrak{a}$ be the Iwasawa projection, that is to say the unique map such that $\forall g \in G, g \in K \exp H(g)N$. Then H is smooth.

The pair (G, K) is a Gelfand pair. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m(\alpha)\alpha$. Let $\mathfrak{a}_{\mathbb{C}}^*$ be the space of linear forms on $\mathfrak{a}_{\mathbb{C}}$. Then the spherical functions of (G, K) are

$$\varphi_\lambda : g \mapsto \int_K e^{(i\lambda - \rho)(H(gk))} dk \quad (1.10)$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, where dk is the Haar measure on K with total mass 1 ([Hel00, Ch. IV, Thm. 4.3]).

Since we want to study unitary matrix coefficients, we only want to consider positive definite spherical functions. In particular, we want to know where λ is located when φ_λ is

positive-definite. A complete answer is only known in a few special cases, see for example [Kos69, FJK79] in rank 1. In general, it is still an open problem to completely classify such λ . However, some partial results are known. The following proposition is found in [Hel00, Section IV, Thm. 8.1 and B.9.(i)] and will be sufficient for our purposes.

Proposition 1.25. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, then:*

- φ_λ is bounded if and only if $\text{Im } \lambda \in C = \text{Conv}(W\rho)$,
- If φ_λ is positive-definite, then φ_λ is bounded by $\varphi(e) = 1$,
- If $\text{Im } \lambda = 0$, then φ_λ is positive-definite.

1.7 Cartan motion groups

If G is a semisimple Lie group with finite center, there is another interesting symmetric pair that we can construct. Again, consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and K a maximal compact subgroup of G . Then K acts on \mathfrak{p} by the adjoint representation. Thus, we can define the semi-direct product $H = \mathfrak{p} \rtimes K$, called the Cartan motion group associated to G . The pair (H, K) is a (flat) symmetric Gelfand pair and H/K is isomorphic to the euclidean symmetric space \mathfrak{p} .

Proposition 1.26 ([Gin67]). *The positive-definite spherical functions of (H, K) are*

$$\varphi_\lambda : X \mapsto \int_K e^{i\lambda(\text{Ad}(k)X)} dk$$

for $\lambda \in \mathfrak{a}^*$.

The Killing form induces a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} and its subspace \mathfrak{a} . Then for $\lambda \in \mathfrak{a}^*$, there exists $H_\lambda \in \mathfrak{a}$ such that $\lambda(X) = \langle X, H_\lambda \rangle$. Furthermore, for any $k \in K$, we have

$$\langle \text{Ad}(k)X, Y \rangle = \langle X, \text{Ad}(k^{-1})Y \rangle.$$

Since K is compact, we may use the change of variable $k \leftrightarrow k^{-1}$ to obtain the following expression for the spherical functions:

$$\varphi_\lambda(X) = \int_K e^{i\langle X, \text{Ad}(k)H_\lambda \rangle} dk \tag{1.11}$$

Remark 1.27. When $G = SO_0(n, 1)$, $K = SO(n)$ acts on $\mathfrak{p} \simeq \mathbb{R}^n$ by its usual action. Thus, the Cartan motion group associated to G is the Euclidean motion group. In this case, the spherical functions are Bessel functions ([Vil68, Ch. XI]).

The integral formula in Proposition 1.26 is a generalization of the integral formula for Bessel functions.

1.8 Duality

There is a duality between symmetric spaces of non-compact type and simply connected symmetric spaces of compact type (see [Loo69a, IV.1.3] for details on this correspondence at the level of symmetric spaces). We will explain the correspondence in terms of symmetric pairs, i.e. at the level of Lie groups.

If \mathfrak{g} is a semisimple real Lie algebra, we gave a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into eigenspaces of a Cartan involution θ . Let $G_{\mathbb{C}}$ be the simply connected Lie group whose Lie algebra is

the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} . Let G, K be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to the subalgebras $\mathfrak{g}, \mathfrak{k}$. Consider $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and U the corresponding analytic subgroup of $G_{\mathbb{C}}$. Then U is a maximal compact subgroup of $G_{\mathbb{C}}$ and is simply connected. The pair (G, K) is a symmetric pair of non-compact type, as in Section 1.6 and the pair (U, K) is its dual of compact type, as in Section 1.5.

Then, every object defined for the pair (U, K) in Section 1.5 can be defined in terms of the non-compact dual G . In particular, there is a connection between spherical functions. Let μ be the highest weight of an irreducible representation π_{μ} of U with a non-zero K -invariant unit vector e_{μ} (from Theorem 1.18). Then π_{μ} is a finite-dimensional representation of U . In particular, it descends to a representation of the Lie algebra \mathfrak{u} and its complexification $\mathfrak{g}_{\mathbb{C}}$. Since $G_{\mathbb{C}}$ is simply connected, π_{μ} induces a representation of $G_{\mathbb{C}}$. Thus, the spherical function $\psi_{\mu} : g \mapsto \langle \pi_{\mu}(g)e_{\mu}, e_{\mu} \rangle$ is defined on $G_{\mathbb{C}}$ and so $\psi_{\mu}|_G$ is a spherical function of the pair (G, K) . In the notations of (1.10), it turns out that

$$\psi_{\mu}|_G = \varphi_{-i(\mu+\rho)}.$$

Of course, $\psi_{\mu}|_G$ is usually not positive-definite since G has no non-trivial finite-dimensional unitary representations, but it establishes a link between spherical functions of (U, K) and (G, K) . As we will see later in the thesis, the pair (G, K) is easier to study, thus we will make use of this link to study (U, K) .

In Chapter 2, we will restrict to compact pairs and will use only the notations from Section 1.5. In Chapter 3, we will use notations from Sections 1.6 and 1.7, and study non-compact pairs. Only in Chapter 4 will we study compact pairs in relation with their non-compact dual and will make use of duality. To avoid overwhelming the reader with different notations, we will wait until this chapter to provide more details on the use of duality.

Chapter 2

Regularity of K -finite matrix coefficients of compact Gelfand pairs

In this chapter, we begin the study of optimal regularity of K -finite matrix coefficients in the case of symmetric pairs of compact type. The results presented here were published in [Dum24c]. We first restrict ourselves to working at the level of the Lie algebra. In Section 2.1, we give a complete answer in the case of a symmetric pair of rank 1. There are well-known descriptions of the spherical functions in this case involving Jacobi polynomials (Section 2.1.1). The main result is Theorem 2.5, where we use analysis of these polynomials to show that the family of spherical functions is bounded in some Hölder space. From this, we deduce Theorem A in the case of K -bi-invariant matrix coefficients of unitary representations. This generalizes results used in [Laf08, HdL13, dLdlS15, dLdlS18] for some specific pairs. In Section 2.1.3, using the aforementioned results, we prove in Corollary 2.13 regularity results for any K -bi-invariant Herz-Schur multipliers of $S_p(L^2(G))$, which improve on [PRdlS22]. We then show in Section 2.1.4 that our results are optimal.

In Section 2.2, we try to extend the results to higher rank. For the case $(G \times G, G)$, we can give the optimal regularity (Theorem 2.21 and Theorem 2.23), which proves Theorem B for K -bi-invariant coefficients. For this, we rely on the description of spherical functions of these pairs with characters of G and use the Weyl character formula. The techniques involved in the optimality are very similar to the rank 1 case, but require the study of the root system associated to G . We then show, for some of the remaining pairs, results of regularity in Section 2.2.2. We study these specific pairs because once again their spherical functions can be described with Jacobi polynomials. These different cases allow us to formulate a conjecture in the general case which will be partially solved in Chapter 4.

Finally, in Section 2.3, we prove that there is a well-behaved KAK decomposition for G . On the one hand, we use this result to recover regularity at the level of the Lie group instead of the Lie algebra. On the other hand, we can apply Section 1.3 to extend the regularity results to any K -finite unitary matrix coefficient, which completes the proof of Theorem A and B.

2.1 Symmetric pairs of rank 1

2.1.1 Spherical functions of symmetric pairs of rank 1

A classification of symmetric spaces of compact type can be found in [Loo69b, Ch. VII] and from this classification, we extract the canonical compact Gelfand pairs associated to compact symmetric spaces of rank 1. Table 2.1 lists these symmetric spaces with (G, K) the canonical pair associated to M , the dimension of M and two real parameters α, β used later to describe spherical functions.

| M | G | K | $\dim M$ | α | β |
|-------------------------------|---------|-------------------------|----------|-----------------|-----------------|
| $S^{k-1}, k \geq 3$ | $SO(k)$ | $SO(k-1)$ | $k-1$ | $\frac{k-3}{2}$ | $\frac{k-3}{2}$ |
| $\mathbb{R}P^{k-1}, k \geq 3$ | $SO(k)$ | $S(O(1) \times O(k-1))$ | $k-1$ | $\frac{k-3}{2}$ | $-\frac{1}{2}$ |
| $\mathbb{C}P^{k-1}, k \geq 2$ | $SU(k)$ | $S(U(1) \times U(k-1))$ | $2(k-1)$ | $k-2$ | 0 |
| $\mathbb{H}P^{k-1}, k \geq 2$ | $Sp(k)$ | $Sp(1) \times Sp(k-1)$ | $4(k-1)$ | $2k-3$ | 1 |
| $\mathbb{O}P^2$ | F_4 | $Spin(9)$ | 16 | 7 | 3 |

Table 2.1: Compact symmetric pairs of rank 1

Remark 2.1. We can see that $\alpha = \frac{\dim G/K}{2} - 1$.

The spherical functions of these pairs are well-known and can be found in [Hel00, Ch. V, Theorem 4.5], while the dimension of the associated representation are found in [CW75, Theorem 2.4, 3.2, 4.2, 5.2 and 6.2]. They can be expressed in terms of Jacobi polynomials.

Definition 2.2 (Jacobi polynomials). Let $\alpha > -1, \beta > -1$, the Jacobi polynomials of parameters (α, β) are defined as the unique polynomials $(P_n^{(\alpha, \beta)})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, $P_n^{(\alpha, \beta)}$ is of degree n , for all $m \neq n$,

$$\int_{-1}^1 P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx = 0$$

and for all $n \in \mathbb{N}$,

$$P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}.$$

Since G/K is of rank 1, we have either $\Sigma_{\mathfrak{a}}^+ = \{\alpha\}$ or $\Sigma_{\mathfrak{a}}^+ = \{\alpha, 2\alpha\}$. In both cases, $\mathfrak{a} \simeq \mathbb{R}$ by $H \mapsto -i\alpha(H)$. By this identification, we have $\mathcal{Q} \simeq]0, \pi[$. If φ is a spherical function of (G, K) , denote $\psi = \varphi \circ \exp|_{\overline{\mathcal{Q}}}$.

Theorem 2.3. *If (G, K) is a compact symmetric pair of rank 1, then its spherical functions are the functions φ_n defined at the level of the Lie algebra by*

$$\psi_n : \theta \in \overline{\mathcal{Q}} \mapsto \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} P_n^{(\alpha, \beta)}(\cos \theta).$$

Furthermore, the dimension m_n of the representation associated to φ_n is a polynomial in n of degree $(\dim G/K) - 1$.

Remark 2.4. To get the function φ_n itself, we need to understand the projection $P : G \mapsto \overline{Q}$ since $\varphi_n = \psi_n \circ P$ by K -bi-invariance.

For $(SO(n), SO(n-1))$, we have $P(g) = \arccos(g_{1,1})$. For the other non-exceptional rank 1 pairs, we have $P(g) = \arccos(2|g_{1,1}|^2 - 1)$. These functions are smooth on the subset of regular points.

However, we delay this study at the level of the group until Section 2.3 where we will explain a general result. By convention, "the family of spherical functions" will refer for now to the functions $(\psi_n)_{n \in \mathbb{N}}$.

2.1.2 Regularity of matrix coefficients

Theorem 2.5. *Let (G, K) be one of the symmetric pairs of rank 1. Let*

$$\alpha_\infty = \frac{\dim G/K - 1}{2}.$$

Then the family of spherical functions of the Gelfand pair (G, K) is bounded in the Hölder space $C^{(\lfloor \alpha_\infty \rfloor, \alpha_\infty - \lfloor \alpha_\infty \rfloor)}(Q)$.

Given that the spherical functions of the Gelfand pairs we are interested in are all Jacobi polynomials, of parameters (α, β) fixed by the pair, we can derive the theorem from the following result:

Theorem 2.6. *Let $\alpha \geq 0, \beta > -1$ be two reals. Then the family*

$$\left(\frac{\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 1)} P_n^{(\alpha, \beta)} \right)_{n \in \mathbb{N}}$$

is bounded in $C^{(\lfloor \alpha + \frac{1}{2} \rfloor, \alpha + \frac{1}{2} - \lfloor \alpha + \frac{1}{2} \rfloor)}([-1, 1])$.

Proof of Theorem 2.5 using Theorem 2.6. According to Section 2.1.1 and up to a reparameterization by Lemma 1.3, the spherical functions of (G, K) are normalized Jacobi polynomials of parameters (α, β) with

$$\alpha = \frac{\dim G/K}{2} - 1.$$

Since

$$\alpha + \frac{1}{2} = \frac{\dim G/K - 1}{2} + \frac{1}{2} = \alpha_\infty,$$

we get the result by Theorem 2.6. □

The proof of Theorem 2.6 relies on two ingredients that can be found in [Sze39, (4.21.7) and (8.21.10)].

Proposition 2.7. *For all $\alpha, \beta \geq 0$, and for all $n \geq k$, we have*

$$\frac{d^k}{dx^k} P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(\alpha + \beta + n + 1 + k)}{2^k \Gamma(\alpha + \beta + n + 1)} P_{n-k}^{(\alpha+k, \beta+k)}(x).$$

Proposition 2.8 (Darboux's formula). *We have*

$$P_n^{(\alpha, \beta)}(\cos \theta) = n^{-\frac{1}{2}} k(\theta) \cos(N\theta + \gamma) + O(n^{-\frac{3}{2}})$$

where

$$\begin{cases} k(\theta) = \pi^{-\frac{1}{2}} \sin^{-\alpha-\frac{1}{2}} \frac{\theta}{2} \cos^{-\beta-\frac{1}{2}} \frac{\theta}{2}, \\ N = n + \frac{1}{2}(\alpha + \beta + 1), \\ \gamma = -\frac{\pi}{2} \left(\alpha + \frac{1}{2} \right), \end{cases}$$

and $O(n^{-\frac{3}{2}})$ is uniform for θ in the compact $[\varepsilon, \pi - \varepsilon]$, for any $\varepsilon > 0$.

Proof of Theorem 2.6. Let $\varphi_n(x) = \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} P_n^{(\alpha,\beta)}(x)$. Set L a compact subset of $] -1, 1[$. Up to replacing L by its convex hull we can assume that L is an interval. There exists $\varepsilon > 0$ such that if $\cos \theta \in L$ with $\theta \in [0, \pi]$, then $\theta \in [\varepsilon, \pi - \varepsilon]$. Thus by Proposition 2.8, because the function $\theta \mapsto k(\theta)$ of Darboux's formula is bounded on compacts in $]0, \pi[$, there exists $C_{L,\alpha,\beta} > 0$ such that for all $x \in L$ and $n \in \mathbb{N}^*$,

$$|P_n^{(\alpha,\beta)}(x)| \leq C_{L,\alpha,\beta} n^{-\frac{1}{2}}. \quad (2.1)$$

Thus, by Proposition 2.7, we have for all $x \in L$ and $n > k$,

$$\left| \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) \right| \leq \frac{\Gamma(\alpha + \beta + n + 1 + k)}{2^k \Gamma(\alpha + \beta + n + 1)} C_{L,\alpha+k,\beta+k} (n - k)^{-\frac{1}{2}}. \quad (2.2)$$

For any $k \in \mathbb{N}$, and $n > k$, we have $n - k \geq \frac{n}{k+1}$. Using that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, we get that

$$\left| \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) \right| \leq \frac{\sqrt{k+1} \prod_{j=1}^k (\alpha + \beta + n + j)}{2^k} C_{L,\alpha+k,\beta+k} n^{-\frac{1}{2}}, \quad (2.3)$$

so

$$\begin{aligned} |\varphi_n^{(k)}(x)| &= \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} \left| \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) \right| \\ &\leq \frac{C_{L,\alpha+k,\beta+k} \sqrt{k+1} \Gamma(\alpha+1)}{2^k} \frac{\Gamma(n+1) \prod_{j=1}^k (\alpha+\beta+n+j)}{\Gamma(n+\alpha+1)} n^{-\frac{1}{2}}. \end{aligned} \quad (2.4)$$

Now, by the Stirling approximation for the Gamma function (see [Gor94, Thm. 5]), there are $C_1, C_2 > 0$ such that if $f(x) = \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}$, then for $x > 0$,

$$C_1 f(x) \leq \Gamma(x+1) \leq C_2 f(x).$$

Thus for all $n > 0$,

$$\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \leq \frac{C_2}{C_1} \frac{n^{n+\frac{1}{2}} e^{-n}}{(n+\alpha)^{n+\alpha+\frac{1}{2}} e^{-(n+\alpha)}} \leq \frac{C_2 e^\alpha}{C_1} (n+\alpha)^{-\alpha} \leq \frac{C_2 e^\alpha}{C_1} n^{-\alpha} \quad (2.5)$$

since $\alpha > 0$. Furthermore, for each $j \in \llbracket 1, k \rrbracket$ and $n > 0$, we have

$$(\alpha + \beta + n + j) \leq (\alpha + \beta + k + n) \leq (\alpha + \beta + k + 1)n$$

thus

$$\prod_{j=1}^k (\alpha + \beta + n + j) \leq (\alpha + \beta + k + 1)^k n^k. \quad (2.6)$$

Setting

$$\tilde{C}_{L,\alpha,\beta,k} = \frac{C_{L,\alpha+k,\beta+k} \sqrt{k+1} \Gamma(\alpha+1)}{2^k} \frac{C_2 e^\alpha}{C_1} (\alpha + \beta + k + 1)^k$$

and inserting (2.5) and (2.6) into (2.4), we finally get that for all $n > k$ and $x \in L$,

$$|\varphi_n^{(k)}(x)| \leq \tilde{C}_{L,\alpha,\beta,k} n^{k-\alpha-\frac{1}{2}}. \quad (2.7)$$

From this inequality, we see that the derivatives of the family of spherical functions are bounded in n up to order $\lfloor \alpha + \frac{1}{2} \rfloor$. If $\alpha + \frac{1}{2} \in \mathbb{Z}$, this shows that $(\varphi_n)_{n \in \mathbb{N}}$ is bounded in $C^{(\alpha+\frac{1}{2},0)}$. If $\alpha + \frac{1}{2} \notin \mathbb{Z}$, we must now verify the Hölder part.

There are two cases to consider.

Case 1: assume that $\alpha - \lfloor \alpha \rfloor < \frac{1}{2}$, which is equivalent to $\lfloor \alpha + \frac{1}{2} \rfloor = \lfloor \alpha \rfloor$. Set

$$\lambda = \alpha - \lfloor \alpha \rfloor + \frac{1}{2} = \alpha + \frac{1}{2} - \lfloor \alpha + \frac{1}{2} \rfloor.$$

We have $\lambda \in [\frac{1}{2}, 1[$. Then by (2.7) applied to $k = \lfloor \alpha \rfloor$, we get for $x \in L$ that

$$|\varphi_n^{(\lfloor \alpha \rfloor)}(x)| \leq \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor} n^{-\lambda},$$

so that for $x, y \in L$,

$$|\varphi_n^{(\lfloor \alpha \rfloor)}(x) - \varphi_n^{(\lfloor \alpha \rfloor)}(y)| \leq |\varphi_n^{(\lfloor \alpha \rfloor)}(x)| + |\varphi_n^{(\lfloor \alpha \rfloor)}(y)| \leq 2\tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor} n^{-\lambda}. \quad (2.8)$$

Applying (2.7) to $k = \lfloor \alpha \rfloor + 1$, we get for $x \in L$ that

$$|\varphi_n^{(\lfloor \alpha \rfloor + 1)}(x)| \leq \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 1} n^{1-\lambda},$$

so that for $x, y \in L$,

$$|\varphi_n^{(\lfloor \alpha \rfloor)}(x) - \varphi_n^{(\lfloor \alpha \rfloor)}(y)| \leq \sup_{t \in K} |\varphi_n^{(\lfloor \alpha \rfloor + 1)}(t)| |x - y| \leq \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 1} n^{1-\lambda} |x - y|. \quad (2.9)$$

Set $M = (2\tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor})^{1-\lambda} \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 1}^\lambda$. Combining (2.8) and (2.9) we have

$$|\varphi_n^{(\lfloor \alpha \rfloor)}(x) - \varphi_n^{(\lfloor \alpha \rfloor)}(y)| \leq M(n^{-\lambda})^{1-\lambda} (n^{1-\lambda} |x - y|)^\lambda = M |x - y|^\lambda,$$

which is the result we wanted.

Case 2: assume that $\alpha - \lfloor \alpha \rfloor \geq \frac{1}{2}$, which is equivalent to $\lfloor \alpha + \frac{1}{2} \rfloor = \lfloor \alpha \rfloor + 1$. Set

$$\lambda = \alpha - \lfloor \alpha \rfloor - \frac{1}{2} = \alpha + \frac{1}{2} - \lfloor \alpha + \frac{1}{2} \rfloor.$$

We have $\lambda \in [0, \frac{1}{2}]$. Similarly to the first case, we apply (2.7) to $k = \lfloor \alpha \rfloor + 1$ and $k = \lfloor \alpha \rfloor + 2$ to get that for all $x, y \in L$,

$$|\varphi_n^{(\lfloor \alpha \rfloor + 1)}(x) - \varphi_n^{(\lfloor \alpha \rfloor + 1)}(y)| \leq 2\tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 1} n^{-\lambda} \quad (2.10)$$

and

$$|\varphi_n^{(\lfloor \alpha \rfloor + 2)}(x) - \varphi_n^{(\lfloor \alpha \rfloor + 2)}(y)| \leq \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 2} n^{1-\lambda} |x - y|, \quad (2.11)$$

which combines as in the first case, giving

$$|\varphi_n^{(\lfloor \alpha \rfloor)}(x) - \varphi_n^{(\lfloor \alpha \rfloor)}(y)| \leq M'(n^{-\lambda})^{1-\lambda} (n^{1-\lambda} |x - y|)^\lambda \leq M' |x - y|^\lambda,$$

with $M' = (2\tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 1})^{1-\lambda} \tilde{C}_{L,\alpha,\beta,\lfloor \alpha \rfloor + 2}^\lambda$, which is the result we wanted. \square

Corollary 2.9. *Let (G, K) be a compact symmetric Gelfand pair of rank one. Let φ be a K -bi-invariant matrix coefficient of a unitary representation of G . Then $\varphi \circ \exp \in C^{(\lfloor \alpha_\infty \rfloor, \alpha_\infty - \lfloor \alpha_\infty \rfloor)}(Q)$.*

Proof. It follows directly from Lemma 1.10. \square

2.1.3 Schatten norm

Given a Hilbert space H , $1 \leq p \leq +\infty$ and T an operator on H , the Schatten p -norm of T is

$$\|T\|_{S_p} = \text{Tr}(|T|^p)^{1/p}$$

defined by functional calculus. If $p = +\infty$, this is the operator norm. Then $S_p(H)$ is the space of operators T such that $\|T\|_{S_p} < +\infty$.

Let (G, K) be a pair as in Section 2.1.1. Define $\tilde{T}_g = \int_{K \times K} \lambda(kgk') dk dk'$ where dk is the normalized Haar measure on K and λ the regular representation of G on $L^2(G)$. Since the map \tilde{T} is K -bi-invariant, the function $T = \tilde{T} \circ \exp : \bar{Q} \mapsto B(L^2(G))$ determines \tilde{T} . We want to investigate the regularity of T as map into $S_p(L^2(G))$. Let $p > 2 + \frac{2}{\dim G/K-1}$ and set

$$\alpha_p = \alpha_\infty - \frac{\dim G/K}{p} = \frac{\dim G/K - 1}{2} - \frac{\dim G/K}{p}$$

(so $\alpha_p > 0$).

Proposition 2.10. *The map $\delta \mapsto T_\delta$ belongs to*

$$C^{([\alpha_\infty]-1, \alpha_\infty - [\alpha_\infty] + 1)}(Q, S_\infty(L^2(G))).$$

Proof. There is an orthonormal basis of $L^2(G)$ such that for any $\delta \in Q$, T_δ is diagonal with eigenvalues $\psi_n(\delta)$ of multiplicity m_n the dimension of the representation associated to ψ_n .

By Theorem 2.5, we know that φ_n is bounded in $C^{([\alpha_\infty], \alpha_\infty - [\alpha_\infty])}(Q)$, thus by Lemma 1.5, we get the result. \square

Remark 2.11. If $\alpha = 0$ in Lemma 1.5, we cannot show that T is C^r . The derivative will exist in a weak sense, but we cannot show convergence in norm.

In particular, if $\alpha_\infty \in \mathbb{Z}$, we do not get that $\partial^{\alpha_\infty} T$ exists. In fact, we will see in the next subsection that the result in Proposition 2.10 is optimal.

The following theorem is a generalization of a result from [PRdLS22] on the group $SO(n)$.

Theorem 2.12. *Let $2 + \frac{2}{\dim G/K-1} < p < +\infty$. The map $\delta \mapsto T_\delta$ belongs to the Hölder space $C^{([\alpha], \alpha - [\alpha])}(Q, S_p(L^2(G)))$ where*

$$\alpha = \begin{cases} \alpha_p & \text{if } \alpha_p \notin \mathbb{Z} \\ \alpha_p - \varepsilon & \text{if } \alpha_p \in \mathbb{Z} \end{cases}$$

with $\varepsilon > 0$ arbitrarily small.

Proof. Let

$$\varphi_n(x) = \frac{\Gamma(u+1)\Gamma(n+1)}{\Gamma(n+u+1)} P_n^{(u,v)}(x) \quad (2.12)$$

be the spherical functions of (G, K) viewed on $\cos(Q) =]-1, 1[$ with $u = \frac{\dim G/K}{2} - 1$. By Lemma 1.3, we can assume that T is defined on $] - 1, 1[$ and is such that there is an orthogonal basis such that T_δ is diagonal with eigenvalues $\varphi_n(\delta)$ of multiplicity m_n , the dimension of the associated representation (see Section 2.1.1). Notice that there is a constant $C > 0$ which depends only on (G, K) such that for all $n \in \mathbb{N}$,

$$m_n \leq C(n+1)^{\dim G/K-1}. \quad (2.13)$$

Let $r < \alpha_\infty$ and $\delta \in]-1, 1[$. By Lemma 1.5, we have that $\partial^r T$ exists and

$$\|\partial^r T_\delta\|_{S_p} = \left(\sum_{n \geq 0} m_n |\varphi_n^{(r)}(\delta)|^p \right)^{1/p}.$$

Thus, using the inequality (2.7) from the proof of Theorem 2.6, we have that there is a constant $C(G, r, \delta) > 0$ such that

$$\begin{aligned} m_n |\varphi_n^{(r)}(x)|^p &\leq C(G, r, \delta) (n+1)^{\dim G/K - 1 + p(r - \frac{\dim G/K - 1}{2})} \\ &= C(G, r, \delta) (n+1)^{p(r - (\frac{\dim G/K - 1}{2} - \frac{\dim G/K}{p})) - 1}. \end{aligned}$$

Thus the sum converges as soon as $r < \frac{\dim G/K - 1}{2} - \frac{\dim G/K}{p} = \alpha_p$, and we get $\partial^r T_\delta \in S_p(L^2(G))$ for these r . This happens when $r = \lfloor \alpha \rfloor$.

In that case, for δ, δ' in some compact I of $] -1, 1[$, which we can assume to be an interval up to taking its convex hull, we have

$$\|\partial^{\lfloor \alpha \rfloor} T_\delta - \partial^{\lfloor \alpha \rfloor} T_{\delta'}\|_{S_p}^p = \sum_{n=0}^{\infty} m_n |\varphi_n^{(\lfloor \alpha \rfloor)}(\delta) - \varphi_n^{(\lfloor \alpha \rfloor)}(\delta')|^p.$$

We bound again m_n by $C(n+1)^{\dim G/K - 1}$. For the term $|\varphi_n^{(\lfloor \alpha \rfloor)}(\delta) - \varphi_n^{(\lfloor \alpha \rfloor)}(\delta')|$, we have two different bounds. By the expression of φ_n in (2.12) and using (2.7), on the one hand, there is $C_1 > 0$ such that

$$\begin{aligned} |\varphi_n^{(\lfloor \alpha \rfloor)}(\delta) - \varphi_n^{(\lfloor \alpha \rfloor)}(\delta')| &\leq |\varphi_n^{(\lfloor \alpha \rfloor)}(\delta)| + |\varphi_n^{(\lfloor \alpha \rfloor)}(\delta')| \\ &\leq C_1 (n+1)^{\lfloor \alpha \rfloor - u - \frac{1}{2}} \\ &\leq C_1 (n+1)^{\lfloor \alpha \rfloor - \frac{\dim G/K - 1}{2}}, \end{aligned}$$

which we use as soon as $n|\delta - \delta'| > 1$. On the other hand, there is $C_2 > 0$ such that

$$\begin{aligned} |\varphi_n^{(\lfloor \alpha \rfloor)}(\delta) - \varphi_n^{(\lfloor \alpha \rfloor)}(\delta')| &\leq \sup_{t \in I} |\varphi_n^{(\lfloor \alpha \rfloor + 1)}(t)| |\delta - \delta'| \\ &\leq C_2 (n+1)^{\lfloor \alpha \rfloor + 1 - \frac{\dim G/K - 1}{2}} |\delta - \delta'|, \end{aligned}$$

and we use this inequality as soon as $n|\delta - \delta'| \leq 1$. So we have, since

$$\dim G/K - 1 + p \left(\lfloor \alpha \rfloor - \frac{\dim G/K - 1}{2} \right) = p(\lfloor \alpha \rfloor - \alpha_p) - 1,$$

that

$$\begin{aligned} \|\partial^{\lfloor \alpha \rfloor} T_\delta - \partial^{\lfloor \alpha \rfloor} T_{\delta'}\|_{S_p}^p &\leq CC_1^p \sum_{n > \frac{1}{|\delta - \delta'|}} (n+1)^{p(\lfloor \alpha \rfloor - \alpha_p) - 1} \\ &\quad + CC_2^p \sum_{n \leq \frac{1}{|\delta - \delta'|}} (n+1)^{p(\lfloor \alpha \rfloor + 1 - \alpha_p) - 1} |\delta - \delta'|^p. \end{aligned}$$

There are now two cases to consider.

Case 1: if $\alpha_p \notin \mathbb{Z}$, we have $\alpha = \alpha_p$. Let $n_0 = \lfloor \frac{1}{|\delta - \delta'|} \rfloor + 1$. We get

$$\begin{aligned} \|\partial^{\lfloor \alpha \rfloor} T_\delta - \partial^{\lfloor \alpha \rfloor} T_{\delta'}\|_{S_p}^p &\leq CC_1^p \sum_{n=n_0}^{+\infty} \frac{1}{(n+1)^{1+p(\alpha - \lfloor \alpha \rfloor)}} \\ &\quad + CC_2^p |\delta - \delta'|^p \sum_{n=0}^{n_0-1} \frac{1}{(n+1)^{1+p(\alpha - \lfloor \alpha \rfloor - 1)}}. \end{aligned}$$

There are again two subcases to consider. First, assume that $1 + p(\alpha - [\alpha] - 1) \geq 0$, then

$$\begin{aligned}
\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p}^p &\leq CC_1^p \sum_{n=n_0}^{+\infty} \frac{1}{(n+1)^{1+p(\alpha-[\alpha])}} + CC_2^p |\delta - \delta'|^p \sum_{n=0}^{n_0-1} \frac{1}{(n+1)^{1+p(\alpha-[\alpha]-1)}} \\
&\leq CC_1^p \int_{n_0}^{\infty} \frac{1}{x^{1+p(\alpha-[\alpha])}} dx + CC_2^p |\delta - \delta'|^p \int_0^{n_0} \frac{1}{x^{1+p(\alpha-[\alpha]-1)}} dx \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])n_0^{p(\alpha-[\alpha])}} + \frac{CC_2^p |\delta - \delta'|^p}{-p(\alpha-[\alpha]-1)n_0^{p([\alpha]+1-\alpha)}} \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])} |\delta - \delta'|^{p(\alpha-[\alpha])} + \frac{CC_2^p |\delta - \delta'|^p}{p([\alpha]+1-\alpha)} \left(\frac{1+|\delta-\delta'|}{|\delta-\delta'|} \right)^{p([\alpha]+1-\alpha)} \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])} |\delta - \delta'|^{p(\alpha-[\alpha])} + \frac{CC_2^p (1+\pi)^{p([\alpha]+1-\alpha)}}{p([\alpha]+1-\alpha)} (|\delta - \delta'|)^{p(\alpha-[\alpha])} \\
&\leq \tilde{C}^p |\delta - \delta'|^{p(\alpha-[\alpha])}
\end{aligned}$$

where $\tilde{C} > 0$ is a constant which do not depend on $\delta, \delta' \in I$. Thus, we finally have

$$\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p} \leq \tilde{C} |\delta - \delta'|^{\alpha-[\alpha]}.$$

In the other subcase, we have $1 + p(\alpha - [\alpha] - 1) \leq 0$, thus

$$\begin{aligned}
\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p}^p &\leq CC_1^p \sum_{n=n_0}^{+\infty} \frac{1}{(n+1)^{1+p(\alpha-[\alpha])}} + CC_2^p |\delta - \delta'|^p \sum_{n=0}^{n_0-1} (n+1)^{-1+p([\alpha]+1-\alpha)} \\
&\leq CC_1^p \int_{n_0}^{\infty} \frac{1}{x^{1+p(\alpha-[\alpha])}} dx + CC_2^p |\delta - \delta'|^p \int_1^{n_0+1} x^{-1+p([\alpha]+1-\alpha)} dx \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])n_0^{p(\alpha-[\alpha])}} + \frac{CC_2^p |\delta - \delta'|^p}{-p(\alpha-[\alpha]-1)} \left((n_0+1)^{p([\alpha]+1-\alpha)} - 1 \right) \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])} |\delta - \delta'|^{p(\alpha-[\alpha])} + \frac{CC_2^p |\delta - \delta'|^p}{p([\alpha]+1-\alpha)} \left(\frac{1+2|\delta-\delta'|}{|\delta-\delta'|} \right)^{p([\alpha]+1-\alpha)} \\
&\leq \frac{CC_1^p}{p(\alpha-[\alpha])} |\delta - \delta'|^{p(\alpha-[\alpha])} + \frac{CC_2^p (1+2\pi)^{p([\alpha]+1-\alpha)}}{p([\alpha]+1-\alpha)} (|\delta - \delta'|)^{p(\alpha-[\alpha])} \\
&\leq \tilde{C}'^p |\delta - \delta'|^{p(\alpha-[\alpha])}
\end{aligned}$$

where $\tilde{C}' > 0$ does not depend on $\delta, \delta' \in I$. Again, we finally get

$$\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p} \leq \tilde{C}' |\delta - \delta'|^{\alpha-[\alpha]}.$$

Case 2: if $\alpha_p \in \mathbb{Z}$, let $0 < \varepsilon < 1$. We have $[\alpha] = [\alpha_p - \varepsilon] = \alpha_p - 1$. Let $n_0 = \left\lfloor \frac{1}{|\delta-\delta'|} \right\rfloor + 1$. We get

$$\begin{aligned}
\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p}^p &\leq CC_1^p \sum_{n=n_0}^{+\infty} \frac{1}{(n+1)^{1+p}} + CC_2^p |\delta - \delta'|^p \sum_{n=0}^{n_0-1} \frac{1}{n+1} \\
&\leq CC_1^p \int_{n_0}^{\infty} \frac{1}{x^{1+p}} dx + CC_2^p |\delta - \delta'|^p + CC_2^p |\delta - \delta'|^p \int_1^{n_0-1} \frac{1}{x} dx \\
&\leq \frac{CC_1^p}{pn_0^p} + CC_2^p |\delta - \delta'|^p + CC_2^p |\delta - \delta'|^p \ln(n_0 - 1) \\
&\leq \frac{CC_1^p}{p} |\delta - \delta'|^p + CC_2^p |\delta - \delta'|^p + CC_2^p |\delta - \delta'|^p \ln \left(\frac{1}{|\delta - \delta'|} \right) \\
&\leq D^p |\delta - \delta'|^p |\ln |\delta - \delta'||
\end{aligned}$$

where $D > 0$ does not depend on $\delta, \delta' \in I$. But $x^\varepsilon |\ln x|^{1/p} \xrightarrow{x \rightarrow 0} 0$ so there is a constant $C_\varepsilon > 0$ such that $|\ln |\delta - \delta'||^{1/p} \leq C_\varepsilon |\delta - \delta'|^{-\varepsilon}$ and thus,

$$\|\partial^{[\alpha]}T_\delta - \partial^{[\alpha]}T_{\delta'}\|_{S_p} \leq DC_\varepsilon |\delta - \delta'|^{1-\varepsilon} = DC_\varepsilon |\delta - \delta'|^{\alpha-[\alpha]}.$$

□

For $2 + \frac{2}{\dim G/K-1} < p \leq +\infty$, we now denote (r_p, δ_p) the regularity of $\delta \mapsto T_\delta$ as a map with values in $S_p(L^2(G))$ obtained in Proposition 2.10 and Theorem 2.12. We can use these two results to study the regularity of S_p -multipliers and give a generalization of [PRdlS22, Prop. 4.2]. Let $\varphi : G \rightarrow \mathbb{C}$ be a bounded measurable map. We can consider the map

$$S_\varphi : \begin{array}{ccc} S_2(L^2(G)) & \rightarrow & S_2(L^2(G)) \\ (a_{g,h})_{g,h \in G} & \mapsto & (\varphi(gh^{-1})a_{g,h})_{g,h \in G} \end{array} .$$

If $S_\varphi : S_p(L^2(G)) \cap S_2(L^2(G)) \mapsto S_p(L^2(G))$ is bounded, by density we can extend it to $S_p(L^2(G))$ and if it remains bounded, we say that φ is an S_p -multiplier.

Corollary 2.13. *Let $1 < p \leq +\infty$. Let φ be a K -bi-invariant S_p -multiplier of G and $\psi = \varphi \circ \exp$. Then $\psi \in C^{(r_p, \delta_p)}(Q)$.*

Proof. If $\delta \in Q$, we have $S_\varphi(T_\delta) = \psi(\delta)T_\delta$ by [PRdlS22, Remark 4.5]. Thus, if $\mathbf{1}$ denote the constant function, which belongs to $L^2(G)$, we have

$$\psi(\delta) = \langle S_\varphi(T_\delta)\mathbf{1}, \mathbf{1} \rangle.$$

So ψ is at least as regular as T is. □

2.1.4 Optimality of the results

In this section, we will show that the results obtained before are optimal. We keep the notations of Theorem 2.12, where we view the functions on $] - 1, 1[$ instead of Q (using Lemma 1.3).

Theorem 2.14. *Let $1 < p < +\infty$. For $\delta \in] - 1, 1[$, the operator $\partial^r T_\delta$ does not belong to $S_p(L^2(G))$ as soon as $r \geq \alpha_p$. Furthermore, if*

$$r = \begin{cases} \lfloor \alpha_p \rfloor & \text{if } \alpha_p \notin \mathbb{Z} \\ \lfloor \alpha_p \rfloor - 1 & \text{if } \alpha_p \in \mathbb{Z} \end{cases}$$

then for each compact interval I of $] - 1, 1[$, there exists a constant $C_{I,p}$ such that for all $x, y \in I$,

$$\|\partial^r T_x - \partial^r T_y\|_{S_p} \geq C_{I,p}|x - y|^{\alpha_p - r}.$$

Lemma 2.15. *Let $\alpha, \beta \geq 0$ two fixed reals. For any $\theta \in]0, \pi[$, there exists $C > 0$ and an integer N_0 such that for $N \geq N_0$, we have*

$$\left(\sum_{n=N+1}^{2N} n |P_n^{(\alpha, \beta)}(\cos \theta)|^2 \right)^{1/2} \geq C\sqrt{N}.$$

Proof. Denote

$$S = \left(\sum_{n=N+1}^{2N} n |P_n^{(\alpha, \beta)}(\cos \theta)|^2 \right)^{1/2}.$$

By Darboux's formula (Proposition 2.8), there is a constant M such that for all $n \in \mathbb{N}^*$,

$$|P_n^{(\alpha, \beta)}(\cos \theta) - n^{-\frac{1}{2}}k(\theta) \cos(N\theta + \gamma)| \leq Mn^{-\frac{3}{2}}.$$

By triangle inequality, we thus have

$$|k(\theta) \cos(N\theta + \gamma)| \leq \sqrt{n}|P_n^{(\alpha, \beta)}(\cos \theta)| + Mn^{-1}$$

so

$$\frac{1}{2}|k(\theta) \cos(N\theta + \gamma)|^2 \leq n|P_n^{(\alpha, \beta)}(\cos \theta)|^2 + M^2 n^{-2}.$$

Thus, we get

$$\begin{aligned} \sum_{n=N+1}^{2N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma)|^2 &\leq S^2 + M^2 \sum_{n=N+1}^{2N} n^{-2} \\ &\leq S^2 + M^2 \sum_{n=N+1}^{+\infty} n^{-2} \\ &\leq S^2 + M^2 N^{-1}. \end{aligned}$$

Taking square root, since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we get

$$S \geq \left(\sum_{n=N+1}^{2N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma)|^2 \right)^{1/2} - MN^{-1/2}.$$

Denote $u = (N+1 + \frac{1}{2}(\alpha + \beta + 1))\theta + \gamma$. Then

$$\begin{aligned} \sum_{n=N+1}^{2N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma)|^2 &= \frac{k(\theta)^2}{2} \sum_{n=0}^{N-1} |\cos(u + n\theta)|^2 \\ &= \frac{Nk(\theta)^2}{4} + \frac{k(\theta)^2}{2} \sum_{n=0}^{N-1} \cos(2u + 2n\theta). \end{aligned}$$

But we have that $|\sum_{n=0}^{N-1} \cos(x + ny)| = |\operatorname{Re} \sum e^{i(x+ny)}| \leq \frac{1}{|\sin y/2|}$, thus

$$\sum_{n=N+1}^{2N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma)|^2 \geq \frac{Nk(\theta)^2}{4} - \frac{k(\theta)^2}{2|\sin \theta|} \geq C'N$$

for some $C' > 0$ and N large enough.

Finally, $S \geq \sqrt{C'N} - MN^{-1/2} \geq C\sqrt{N}$ for $C = \frac{C'}{2}$ and N large enough. \square

Lemma 2.16. *Let $\alpha, \beta \geq 0$ two fixed reals. Let J be a compact interval of $]0, \pi[$. There exists a real $C > 0$ and an integer $m_0 \geq 1$ such that for all $\theta, \phi \in J$, $m \geq m_0$ and $N \geq \frac{m_0}{|\theta - \phi|}$, we have*

$$\left(\sum_{n=m+1}^{m+N} n|P_n^{(\alpha, \beta)}(\cos \theta) - P_n^{(\alpha, \beta)}(\cos \phi)|^2 \right)^{1/2} \geq C\sqrt{N}.$$

Proof. Denote

$$S = \left(\sum_{n=m+1}^{m+N} n|P_n^{(\alpha, \beta)}(\cos \theta) - P_n^{(\alpha, \beta)}(\cos \phi)|^2 \right)^{1/2}.$$

By Darboux's formula (Proposition 2.8), there is a constant M_J such that for all $\theta \in J$ and $n \in \mathbb{N}^*$,

$$|P_n^{(\alpha, \beta)}(\cos \theta) - n^{-\frac{1}{2}}k(\theta) \cos(N\theta + \gamma)| \leq M_J n^{-\frac{3}{2}}.$$

By triangle inequality, we thus have

$$|k(\theta) \cos(N\theta + \gamma) - k(\phi) \cos(N\phi + \gamma)| \leq \sqrt{n}|P_n^{(\alpha, \beta)}(\cos \theta) - P_n^{(\alpha, \beta)}(\cos \phi)| + 2M_J n^{-1}$$

so

$$\frac{1}{2}|k(\theta) \cos(N\theta + \gamma) - k(\phi) \cos(N\phi + \gamma)|^2 \leq n|P_n^{(\alpha, \beta)}(\cos \theta) - P_n^{(\alpha, \beta)}(\cos \phi)|^2 + 4M_J^2 n^{-2}.$$

Thus, we get

$$\begin{aligned} \sum_{n=m+1}^{m+N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma) - k(\phi) \cos(N\phi + \gamma)|^2 &\leq S^2 + 4M_J^2 \sum_{n=m+1}^{m+N} n^{-2} \\ &\leq S^2 + 4M_J^2 \sum_{n=m+1}^{+\infty} n^{-2} \\ &\leq S^2 + 4M_J^2 m^{-1}. \end{aligned}$$

Taking square root, since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we get

$$S \geq \left(\sum_{n=m+1}^{m+N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma) - k(\phi) \cos(N\phi + \gamma)|^2 \right)^{1/2} - 2M_J m^{-1/2}.$$

Denote $u = (m + \frac{1}{2}(\alpha + \beta + 1) + 1)\theta + \gamma$, $v = (m + \frac{1}{2}(\alpha + \beta + 1) + 1)\phi + \gamma$, $a = \frac{k(\theta)}{\sqrt{2}}$ and $b = \frac{k(\phi)}{\sqrt{2}}$. Then

$$\sum_{n=m+1}^{m+N} \frac{1}{2}|k(\theta) \cos(N\theta + \gamma) - k(\phi) \cos(N\phi + \gamma)|^2 = \sum_{n=0}^{N-1} |a \cos(u + n\theta) - b \cos(v + n\phi)|^2.$$

Making use of the formula $2 \cos(x) \cos(y) = \cos(x+y) + \cos(x-y)$, we have

$$\begin{aligned} |a \cos(u + n\theta) - b \cos(v + n\phi)|^2 &= a^2 \cos^2(u + n\theta) + b^2 \cos^2(v + n\phi) - 2ab \cos(u + n\theta) \cos(v + n\phi) \\ &= \frac{1}{2}[a^2 + b^2 - 2ab \cos(u - v + n(\theta - \phi)) + a^2 \cos(2u + 2n\theta) \\ &\quad + b^2 \cos(2v + 2n\phi) - 2ab \cos(u + v + n(\theta + \phi))]. \end{aligned}$$

Again using $|\sum_{n=0}^{N-1} \cos(x + ny)| = |\operatorname{Re} \sum e^{i(x+ny)}| \leq \frac{1}{|\sin y/2|}$, we get

$$\begin{aligned} \frac{2}{N} \sum_{n=0}^{N-1} |a \cos(u + n\theta) - b \cos(v + n\phi)|^2 &\geq \\ &a^2 + b^2 - \frac{2ab}{N|\sin \frac{\theta-\phi}{2}|} - \frac{a^2}{N|\sin \theta|} - \frac{b^2}{N|\sin \phi|} - \frac{2ab}{N|\sin \frac{\theta+\phi}{2}|}. \end{aligned}$$

Now if $N \geq \frac{m_0}{|\theta-\phi|}$ for some m_0 depending on J , we get that

$$\frac{2}{N} \sum_{n=0}^{N-1} |a \cos(u + n\theta) - b \cos(v + n\phi)|^2 \geq \frac{a^2 + b^2}{2} \geq C'$$

where $C' > 0$ is a constant depending on α, β, J .

Finally, $S \geq \sqrt{\frac{C'N}{2}} - 2M_J m^{-1/2} \geq C\sqrt{N}$ for m large enough. \square

Proof of Theorem 2.14. Let $(\varphi_n)_{n \in \mathbb{N}}$ be the family of spherical functions of the pair (G, K) and m_n the dimension of the associated representation of G . Then for $r \in \mathbb{N}$, $x \in]-1, 1[$, we have

$$\|\partial^r T_x\|_{S_p}^p = \sum_{n \geq 0} m_n |\varphi_n^{(r)}(x)|^p.$$

As in (2.13), there is $C_0 > 0$ depending only on the pair (G, K) such that

$$m_n \geq C_0 n^{\dim G/K - 1}$$

for all $n \in \mathbb{N}$. Furthermore, recall that there is $\theta \in]0, \pi[$ such that $x = \cos \theta$ and

$$|\varphi_n^{(r)}(x)| = \frac{\Gamma(\alpha + 1)}{2^r} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)} \frac{\Gamma(\alpha + \beta + n + 1 + r)}{\Gamma(\alpha + \beta + n + 1)} |P_{n-r}^{(\alpha+r, \beta+r)}(\cos \theta)|.$$

But

$$\frac{\Gamma(\alpha + \beta + n + 1 + r)}{\Gamma(\alpha + \beta + n + 1)} = \prod_{j=1}^r (\alpha + \beta + n + j) \geq n^r$$

and as in (2.5), if we define $f(x) = \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}$, there are constants $C_1, C_2 > 0$ such that for $x \geq 0$,

$$C_1 f(x) \leq \Gamma(x + 1) \leq C_2 f(x).$$

Thus for all $n \in \mathbb{N}$,

$$\begin{aligned} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)} &\geq \frac{C_1}{C_2} \frac{n^{n+\frac{1}{2}}}{(n + \alpha)^{n+\alpha+\frac{1}{2}}} \frac{e^{-n}}{e^{-n-\alpha}} \\ &\geq \frac{C_1 e^\alpha}{C_2} n^{-\alpha} \sqrt{\frac{n}{n + \alpha}} \left(1 - \frac{\alpha}{n + \alpha}\right)^{n+\alpha} \\ &\geq C_3 n^{-\alpha} \end{aligned}$$

where $C_3 > 0$ is a constant depending only on α thus on the pair (G, K) .

Combining these estimates, reindexing the sum and using $n - k \geq C_4 n$ for some $C_4 > 0$ and for all $n > k$, we get that there is a constant $C > 0$ which depends only on (G, K) such that

$$\|\partial^r T_x\|_{S_p}^p \geq C \sum_{n \geq 1} n^{p(r-\alpha) + \dim G/K - 1} |P_n^{(\alpha+r, \beta+r)}(\cos \theta)|^p. \quad (2.14)$$

Recall (Section 2.1.1) that $\alpha = \alpha(G, K) = \frac{\dim G/K}{2} - 1$ so that

$$\begin{aligned} p(r - \alpha) + \dim G/K - 1 &= p \left(r - \left(\frac{\dim G/K - 1}{2} - \frac{\dim G/K}{p} \right) + \frac{1}{2} \right) - 1 \\ &= p \left(r - \alpha_p + \frac{1}{2} \right) - 1. \end{aligned}$$

To simplify notations, denote $\kappa = p \left(r - \alpha_p + \frac{1}{2} \right) - 1$. Note that for now, we did not make assumptions on r so we do not know if the right-hand side converges.

For any $N \in \mathbb{N}$, we have by Hölder's inequality that

$$\sum_{n=N+1}^{2N} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta)|^p \geq \frac{\left(\sum_{n=N}^{2N} n |P_n^{(\alpha+r, \beta+r)}(\cos \theta)|^2 \right)^{p/2}}{\left(\sum_{n=N+1}^{2N} n^{\frac{p-2\kappa}{p-2}} \right)^{p/2-1}}. \quad (2.15)$$

By Lemma 2.15, for N large enough, the numerator is greater than $C^p N^{p/2}$. For the denominator, we have

$$\left(\sum_{n=N+1}^{2N} n^{\frac{p-2\kappa}{p-2}} \right)^{p/2-1} \leq \left(N(2N)^{\frac{p-2\kappa}{p-2}} \right)^{\frac{p-2}{2}} \leq C' N^{p-\kappa-1}$$

thus there is $\tilde{C} > 0$ such that

$$\sum_{n=N+1}^{2N} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta)|^p \geq \tilde{C} N^{\kappa+1-\frac{p}{2}}.$$

Then if $r \geq \alpha_p$, we get $\kappa + 1 - \frac{p}{2} \geq 0$, thus by (2.15),

$$\sum_{n=N+1}^{2N} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta)|^p \not\rightarrow_{N \rightarrow \infty} 0$$

so by (2.14), the p -norm of $\partial^r T_x$ is not finite so $\partial^r T_x \notin S_p(L^2(G))$.

Now set $r < \alpha_p$ as in Theorem 2.12. Using what is above, for $x, y \in I$, we have

$$\|\partial^r T_x - \partial^r T_y\|_{S_p}^p = \sum_{n \geq 0} m_n |\varphi_n^{(r)}(x) - \varphi_n^{(r)}(y)|^p.$$

Let $J = \arccos(I)$ which is a compact interval of $]0, \pi[$. There are $\theta, \phi \in J$ such that $x = \cos \theta$ and $y = \cos \phi$. Using the same inequalities on m_n and φ_n as before, we get that there is a constant $C > 0$ which depends only on (G, K) such that

$$\|\partial^r T_x - \partial^r T_y\|_{S_p}^p \geq C \sum_{n \geq 1} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta) - P_n^{(\alpha+r, \beta+r)}(\cos \phi)|^p. \quad (2.16)$$

Set m_0 as in Lemma 2.16 and $m_k = m_0 + kN$. Then we get

$$\|\partial^r T_x - \partial^r T_y\|_{S_p}^p \geq C \sum_{k \geq 0} \sum_{n=m_k+1}^{m_{k+1}} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta) - P_n^{(\alpha+r, \beta+r)}(\cos \phi)|^p. \quad (2.17)$$

By Hölder's inequality, we have that

$$\begin{aligned} \sum_{n=m_k+1}^{m_{k+1}} n^\kappa |P_n^{(\alpha+r, \beta+r)}(\cos \theta) - P_n^{(\alpha+r, \beta+r)}(\cos \phi)|^p &\geq \\ &\frac{\left(\sum_{n=m_k+1}^{m_{k+1}} n |P_n^{(\alpha+r, \beta+r)}(\cos \theta) - P_n^{(\alpha+r, \beta+r)}(\cos \phi)|^2 \right)^{p/2}}{\left(\sum_{n=m_k+1}^{m_{k+1}} n^{\frac{p-2\kappa}{p-2}} \right)^{p/2-1}}. \end{aligned} \quad (2.18)$$

By Lemma 2.16, for $N \geq \frac{m_0}{|\theta-\phi|}$, the numerator is greater than $C'^p N^{p/2}$. For the denominator, first notice that

$$m_{k+1} = m_0 + (k+1)N = N \left(\frac{m_0}{N} + k+1 \right) \leq (\pi + k+1)N$$

so that

$$\begin{aligned} \left(\sum_{n=m_k+1}^{m_{k+1}} n^{\frac{p-2\kappa}{p-2}} \right)^{p/2-1} &\leq \left(N m_{k+1}^{\frac{p-2\kappa}{p-2}} \right)^{\frac{p-2}{2}} \\ &\leq \left((\pi + k+1)^{\frac{p-2\kappa}{p-2}} N^{\frac{p-2\kappa}{p-2}} + 1 \right)^{\frac{p-2}{2}} \\ &\leq (\pi + k+1)^{\frac{p-2\kappa}{2}} N^{p-\kappa-1}. \end{aligned}$$

Using these two inequalities in (2.18) and then in (2.17) we finally get

$$\|\partial^r T_x - \partial^r T_y\|_{S_p}^p \geq CC'^p N^{\kappa+1-\frac{p}{2}} \sum_{k \geq 0} (1 + \pi + k)^{\kappa-\frac{p}{2}}. \quad (2.19)$$

Notice that this sum converges because $\kappa - \frac{p}{2} < -1$. We get

$$\sum_{k \geq 0} (1 + \pi + k)^{\kappa-\frac{p}{2}} \geq \int_0^\infty (1 + \pi + x)^{\kappa-\frac{p}{2}} dx = \frac{(1 + \pi)^{1+\kappa-\frac{p}{2}}}{\frac{p}{2} - \kappa - 1}.$$

Now since we want $N \geq \frac{m_0}{|\theta - \phi|}$, we can choose N so that

$$N \leq \frac{m_0}{|\theta - \phi|} + 1 \leq \frac{m_0 + \pi}{|\theta - \phi|}.$$

Thus taking the p -th root in (2.19) and using that $\frac{\kappa+1}{p} - \frac{1}{2} = r - \alpha_p$, we have

$$\begin{aligned} \|\partial^r T_x - \partial^r T_y\|_{S_p} &\geq C^{1/p} C' \left(\frac{m_0 + \pi}{|\theta - \phi|} \right)^{\frac{\kappa+1}{p} - \frac{1}{2}} \frac{(1 + \pi)^{\frac{1+\kappa}{p} - \frac{1}{2}}}{\left(\frac{p}{2} - \kappa - 1\right)^{1/p}} \\ &\geq C^{1/p} C' \frac{((m_0 + \pi)(1 + \pi))^{r-\alpha_p}}{\left(\frac{p}{2} - \kappa - 1\right)^{1/p}} |\theta - \phi|^{\alpha_p - r}. \end{aligned} \quad (2.20)$$

and finally, we conclude using that $|\theta - \phi| \geq \tilde{C} |\cos \theta - \cos \phi|$ for some $\tilde{C} > 0$. \square

Corollary 2.17. *We have the following optimality results:*

1. If $p \leq 2 + \frac{2}{\dim G/K-1}$ and $\delta \in Q$, $T_\delta \notin S_p(L^2 G)$.
2. If $2 + \frac{2}{\dim G/K-1} < p < +\infty$, the regularity obtained in Theorem 2.12 is optimal.
3. If $p = +\infty$, the regularity obtained in Proposition 2.10 is optimal.
4. Theorem 2.5 is optimal: for any $(r, \delta) > (\lfloor \alpha_\infty \rfloor, \alpha_\infty - \lfloor \alpha_\infty \rfloor)$ in lexicographic order, the family of spherical functions is not bounded in $C^{(r, \delta)}(Q)$.
5. Corollary 2.9 is optimal: with the same notations as above, there are K -bi-invariant matrix coefficients of unitary representations that are not in $C^{(r, \delta)}(Q)$.

Remark 2.18. Even though the regularity of T is optimal, we do not know whether Corollary 2.13 is optimal or not. Indeed, we only say that S_p -multipliers are at least as regular as T , but we cannot construct specific multipliers with the exact regularity of T .

Proof. 1. If $p \leq 2 + \frac{2}{\dim G/K-1}$, then $\alpha_p \leq 0$ so this is the first part of Theorem 2.14.

2. There are two cases to consider. In the first case, if $\alpha_p \in \mathbb{Z}$, we know that

$$T \in C^{(\alpha_p-1, 1-\varepsilon)}(]-1, 1[, S_p(L^2(G)))$$

for any $\varepsilon \in]0, 1[$. By the first part of Theorem 2.14, $T \notin C^{(\alpha_p, 0)}(]-1, 1[, S_p(L^2(G)))$ (because $\partial^{\alpha_p} T_\delta \notin S_p$). Then, assume $\partial^{\alpha_p-1} T$ is Lipschitz on a compact interval J of $]-1, 1[$. Since we assume that $1 < p < +\infty$, $S_p(L^2(G))$ is reflexive, thus by [BL98, Corollary 5.12], the map $\partial^{\alpha_p-1} T : \text{Int}(J) \rightarrow S_p(L^2(G))$ is differentiable

almost everywhere (for the Lebesgue measure on J) which contradicts the fact that $\partial^{\alpha_p} T_\delta \notin S_p(L^2(G))$ for any $\delta \in]-1, 1[$. Thus, $T \notin C^{(\alpha_p-1,1)}(]-1, 1[, S_p(L^2(G)))$.

In the second case, we consider $\alpha_p \notin \mathbb{Z}$. Let $r = \lfloor \alpha_p \rfloor$. Then we know that

$$T \in C^{(r, \alpha_p - r)}(]-1, 1[, S_p(L^2(G))).$$

Let $1 - (\alpha_p - r) > \varepsilon > 0$, and I compact interval of $]-1, 1[$. Assume that there is $C_I > 0$ such that for all $x, y \in I$,

$$\|\partial^r T_x - \partial^r T_y\| \leq C_I |x - y|^{\alpha_p - r + \varepsilon}.$$

Then for $x \neq y \in I$, by the second part of Theorem 2.14, we get

$$C_{I,p} \leq C_I |x - y|^\varepsilon,$$

which is impossible when $x \rightarrow y$.

Thus, for any $\varepsilon > 0$, $T \notin C^{(r, \alpha_p - r + \varepsilon)}(]-1, 1[, S_p(L^2(G)))$.

3. Once again we distinguish two cases. First, if $\alpha_\infty \notin \mathbb{Z}$, let $1 < p < +\infty$ be large enough so that $r = \lfloor \alpha_p \rfloor = \lfloor \alpha_\infty \rfloor$. Then, by the second part of Theorem 2.14, we get for $x, y \in I$ compact subset of $]-1, 1[$ that

$$\|\partial^r T_x - \partial^r T_y\|_{S_p} \geq C_{I,p} |x - y|^{\alpha_p - r}.$$

From the expression of $C_{I,p}$, we see that $\lim_{p \rightarrow +\infty} C_{I,p} = C_I$ exists and is finite. Then we get with $p \rightarrow +\infty$ that

$$\|\partial^r T_x - \partial^r T_y\|_{S_\infty} \geq C_I |x - y|^{\alpha_\infty - r}.$$

Thus, as above, $T \notin C^{(r, \alpha_\infty - r + \varepsilon)}(]-1, 1[, S_\infty(L^2(G)))$ for any $\varepsilon > 0$.

Next, if $\alpha_\infty \in \mathbb{Z}$, we have $T \in C^{(\alpha_\infty - 1, 1)}(]-1, 1[, S_\infty(L^2(G)))$. Let $x = \cos \theta$, $y = \cos \phi$, $\theta, \phi \in [\varepsilon, \pi - \varepsilon]$. We have

$$\begin{aligned} \|\partial^{\alpha_\infty} T_x - \partial^{\alpha_\infty} T_y\|_{S_\infty} &= \sup_n |\varphi_n^{(\alpha_\infty)}(\cos \theta) - \varphi_n^{(\alpha_\infty)}(\cos \phi)| \\ &\geq \left(\frac{1}{N} \sum_{n=m}^{m+N} |\varphi_n^{(\alpha_\infty)}(\cos \theta) - \varphi_n^{(\alpha_\infty)}(\cos \phi)|^2 \right)^{1/2} \\ &\geq C(G, K, \varepsilon) \end{aligned}$$

by Lemma 2.16. Thus, we get that $\partial^{\alpha_\infty} T$ is not continuous, so we showed that $T \notin C^{(\alpha_\infty, 0)}(]-1, 1[, S_\infty(L^2(G)))$.

4. It follows from (3) and Lemma 1.5.

5. It is a consequence of (4) and Lemma 1.10. □

2.2 Higher rank symmetric spaces

2.2.1 The case of a Lie group seen as a symmetric space

We consider G a compact semisimple Lie group and $\Delta(G)$ the diagonal subgroup of $G \times G$. We study the symmetric Gelfand pair $(G \times G, \Delta(G))$. Note that unlike what we did in

Section 2.1, we do not need to assume that the symmetric space is simply connected and we do not even need it to be irreducible (which means G does not need to be simply connected nor simple).

In this case, spherical functions can be described in a more efficient way than what the Cartan-Helgason theorem (Theorem 1.18) tells us. An irreducible representation of $G \times G$ is of the form $(\pi \otimes \sigma, V \otimes W)$ where π, σ are irreducible representations of G . We have that $V \otimes W \simeq \text{Hom}(V^*, W)$, and the representation is given by

$$((\pi \otimes \sigma)(g, g')f)(v) = \sigma(g')(f(\pi(g)^*v)).$$

Assume that there is a non-zero $\Delta(G)$ -invariant vector $\xi \in \text{Hom}(V^*, W)$. Then for all $g \in G$, we have

$$\sigma(g)(\xi(\pi(g)^*v)) = \xi(v)$$

so ξ is G -equivariant from $V^* \rightarrow W$, so by Schur's lemma, $V^* \simeq W$. Thus, the classes of irreducible representations of $G \times G$ with a $\Delta(G)$ -invariant vector are in bijection with the classes of irreducible representations of G . On $\text{Hom}(V, V)$, the scalar product is $\langle f, g \rangle = \text{Tr}(fg^*)$. For (π, V) a (unitary) irreducible representation of G , the associated spherical function of $(G \times G, \Delta(G))$ is

$$\varphi_\pi : (g, g') \mapsto \frac{\langle (\pi \otimes \pi^*)(g, g')(\text{Id}, \text{Id}) \rangle}{\langle \text{Id}, \text{Id} \rangle} = \frac{\langle \pi(g), \pi(g') \rangle}{\dim V} = \frac{\chi_\pi(gg'^{-1})}{\dim V}.$$

So the spherical functions of the pair $(G \times G, \Delta(G))$ are just the normalized characters of G .

Let \mathfrak{g} be the Lie algebra of G . The Lie algebra of $G \times G$ is $\mathfrak{g} \oplus \mathfrak{g}$, and the Lie algebra of $\Delta(G)$ is the subspace

$$\mathfrak{k} = \{(H, H) \mid H \in \mathfrak{g}\}.$$

Then, we have $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{p} = \{(H, -H) \mid H \in \mathfrak{g}\}.$$

Let \mathfrak{b} be a maximal abelian subalgebra of \mathfrak{g} , Φ the root system associated to $(\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$, Φ^+ a choice of positive roots and $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be the associated basis, where $\ell = \text{rank } G$. We have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{b}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_\mathbb{C} \mid \forall H \in \mathfrak{b}_\mathbb{C}, [H, X] = \alpha(H)X\}$ and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.

Let Λ_G be the set of dominant analytically integral element. We know that there is a bijection between Λ_G and the set of equivalence classes of finite-dimensional irreducible representations of G ([Kna02, Thm. 5.110]). For $\lambda \in \Lambda_G$, let π_λ be an associated representation, χ_λ its character and d_λ its dimension. Then, the previous result amount to the following :

Proposition 2.19. *For $\lambda \in \Lambda_G$, define $\varphi_\lambda : (g, g') \mapsto \frac{\chi_\lambda(gg'^{-1})}{d_\lambda}$. Then $\lambda \mapsto \varphi_\lambda$ is a bijection from Λ_G to the set of spherical functions of $((G \times G), \Delta(G))$.*

Now consider $\mathfrak{a} = \{(H, -H) \mid H \in \mathfrak{b}\}$. We have that \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Thus, we can consider $\Sigma_{\mathfrak{a}}$ root system of $\mathfrak{a}_\mathbb{C}$ in $(\mathfrak{g} \times \mathfrak{g})_\mathbb{C}$, whose roots are $\tilde{\alpha} : (H, -H) \mapsto \alpha(H)$ for $\alpha \in \Phi$. We choose as a positive root system $\Sigma_{\mathfrak{a}}^+ = \{\tilde{\alpha} \mid \alpha \in \Phi^+\}$.

Let $\tilde{\mathfrak{g}}_\alpha = \{X \in (\mathfrak{g} \times \mathfrak{g})_\mathbb{C} \mid \forall H \in \mathfrak{a}_\mathbb{C}, [H, X] = \tilde{\alpha}(H)X\}$. Then we have

$$(\mathfrak{g} \times \mathfrak{g})_\mathbb{C} = \mathfrak{a}_\mathbb{C} \oplus (\mathfrak{k}^\mathfrak{a})_\mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} \tilde{\mathfrak{g}}_\alpha$$

and we see that $\mathfrak{k}^{\mathfrak{a}} = \{(H, H) \mid H \in \mathfrak{b}\}$ and for all $\alpha \in \Phi$, $\dim \tilde{\mathfrak{g}}_{\alpha} = 2$ - in fact, $\tilde{\mathfrak{g}}_{\alpha} = (\mathfrak{g}_{\alpha} \times 0) \oplus (0 \times \mathfrak{g}_{-\alpha})$.

Now, recall that we defined Q (Proposition 1.20) as the connected component of the set $\{H \in \mathfrak{a} \mid \forall \alpha \in \Sigma_{\mathfrak{a}}, \alpha(H) \in i\pi\mathbb{Z}\}$ which contained in the Weyl chamber associated to $\Sigma_{\mathfrak{a}}^+$ and containing 0 in its closure. We want to study the regularity of the functions $\psi_{\lambda} = \varphi_{\lambda} \circ \exp|_Q$.

By definition, any positive root $\alpha \in \Phi^+$ is a linear combination of the roots in Δ , with coefficients in \mathbb{N} . We write $\alpha = \sum_{i=1}^{\ell} n_i(\alpha)\alpha_i$. We define

$$\gamma = \min_{1 \leq i \leq \ell} |\{\alpha \in \Phi^+ \mid n_i(\alpha) \geq 1\}|.$$

Remark 2.20. The number γ was used by Cowling and Nevo in [CN01] in estimates of spherical functions of the non-compact dual of $(G \times G)/\Delta(G)$, namely $G_{\mathbb{C}}/G$. A table of the values of γ can be found in [CM89, Appendix]. These values also corresponds to Table A.1 in Appendix A.

As in Section 2.1, we begin by showing that $\Delta(G)$ -bi-invariant matrix coefficients of unitary representations of $G \times G$ are at least of regularity γ .

Theorem 2.21. *The family $(\psi_{\lambda})_{\lambda \in \Lambda_G}$ is bounded in $C^{(\gamma, 0)}(Q)$.*

Proof. Let $\tilde{H} = (H, -H) \in Q$, $H \in \mathfrak{b}$. Then

$$\psi_{\lambda}(\tilde{H}) = \varphi_{\lambda}(\exp(H), \exp(-H)) = \frac{\chi_{\lambda}(\exp(2H))}{d_{\lambda}}.$$

Let q be the Weyl denominator, defined on \mathfrak{b} by $q(H) = \prod_{\alpha \in \Phi^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$. Note that the roots α are in $i\mathfrak{b}^*$, so $\alpha(H) \in i\mathbb{R}$. Thus, if $\alpha(H)/2 \notin i\pi\mathbb{Z}$, we get that $e^{\alpha(H)/2} - e^{-\alpha(H)/2} = 2 \sinh(\alpha(H)/2)$ is non-zero. So for any $\tilde{H} = (H, -H) \in Q$, by definition of Q , $2H$ is such that $q(2H) \neq 0$.

Now, by the Weyl character formula ([Hal03, Thm. 10.14]), if $q(2H) \neq 0$, we have

$$\chi_{\lambda}(e^{2H}) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda + \rho))(2H)}}{q(2H)}$$

where W denote the Weyl group and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Since the map $(H, -H) \mapsto q(2H)$ is smooth and non-zero on Q , by Lemma 1.4, we can study the functions $\tilde{\psi}_{\lambda} : (H, -H) \mapsto q(2H)\psi_{\lambda}(H, -H)$, which will have the same regularity.

Let $k \in \mathbb{N}$, $(\tilde{X}_1, \dots, \tilde{X}_k) \in \mathfrak{a}^k$. Then

$$D^k \tilde{\psi}_{\lambda}(\tilde{H})(\tilde{X}_1, \dots, \tilde{X}_k) = \sum_{w \in W} \frac{\det(w)}{d_{\lambda}} \left(\prod_{j=1}^k (w(\lambda + \rho))(2X_j) \right) e^{(w(\lambda + \rho))(2H)}.$$

Note that $\|w(\lambda + \rho)\| = \|\lambda + \rho\|$ for all $w \in W$. Furthermore, $(w(\lambda + \rho))(2H)$ is pure imaginary, so that $|e^{(w(\lambda + \rho))(2H)}| = 1$. So we have

$$\|D^k \tilde{\psi}_{\lambda}(\tilde{H})\| \leq \frac{2^k |W| \|\lambda + \rho\|^k}{d_{\lambda}}. \quad (2.21)$$

So now, we have to study d_{λ} . By the Weyl dimension formula ([Hal03, Thm. 10.18]), we have

$$d_{\lambda} = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle}.$$

Given that Δ is a basis of the finite-dimensional vector space $\mathfrak{a}_\mathbb{C}^*$, there exists $C > 0$ such that for any $\lambda \in \mathfrak{a}_\mathbb{C}^*$, we have $C\|\lambda\| \leq \max_{1 \leq j \leq \ell} |\langle \alpha_j, \lambda \rangle|$. For $\lambda \in \Lambda_G$ fixed, there exists $j(\lambda)$ such that $\langle \alpha_{j(\lambda)}, \lambda + \rho \rangle = \max_{1 \leq j \leq \ell} \langle \alpha_j, \lambda + \rho \rangle$. Note that since $\lambda + \rho$ is a dominant element, all these scalar products are non-negative integers. Now, for any $\alpha \in \Phi^+$, we have

$$\langle \alpha, \lambda + \rho \rangle = \sum_{j=1}^{\ell} n_j(\alpha) \langle \alpha_j, \lambda + \rho \rangle \geq n_{j(\lambda)}(\alpha) \langle \alpha_{j(\lambda)}, \lambda + \rho \rangle \geq C n_{j(\lambda)} \|\lambda + \rho\|.$$

By definition of γ , there are at least γ positive roots α such that $n_{j(\lambda)}(\alpha) \geq 1$. Choose exactly γ out of them. Let $D = \min_{\alpha} \langle \alpha, \rho \rangle > 0$. For the $|\Phi^+| - \gamma$ remaining roots, we have

$$\langle \alpha, \lambda + \rho \rangle \geq \langle \alpha, \rho \rangle \geq D.$$

Thus, we get that

$$d_\lambda \geq \frac{C^\gamma D^{|\Phi^+| - \gamma} \|\lambda + \rho\|^\gamma}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}. \quad (2.22)$$

Let

$$C_k = \frac{2^k |W| \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}{C^\gamma D^{|\Phi^+| - \gamma}}.$$

Combining (2.21) and (2.22), we have that for any $\lambda \in \Lambda_G$, $\tilde{H} \in Q$,

$$\|D^k \tilde{\psi}_\lambda(\tilde{H})\| \leq C_k \|\lambda + \rho\|^{k - \gamma}.$$

Therefore, as soon as $k \leq \gamma$, the family of differentials of order k are bounded in λ . \square

Corollary 2.22. *Any $\Delta(G)$ -bi-invariant matrix coefficient φ of a unitary representation of $G \times G$ is such that $\varphi \circ \exp \in C^{(\gamma, 0)}(Q)$.*

Proof. It follows from Lemma 1.10. \square

We now prove that Theorem 2.21 is optimal, in the sense that there are $\Delta(G)$ -bi-invariant coefficients of unitary representations that are exactly of regularity γ .

Theorem 2.23. *For any $0 < \delta \leq 1$, the family $(\psi_\lambda)_{\lambda \in \Lambda_G}$ is not bounded in $C^{(\gamma, \delta)}(Q)$. Hence, there exists a K -bi-invariant matrix coefficient of a unitary representation of G that is not in $C^{(\gamma, \delta)}(Q)$.*

Proof. As before, we will prove this for the family $(\tilde{\psi}_\lambda)_{\lambda \in \Lambda_G}$. Assume that there is $\delta > 0$ such that for any compact $L \subset Q$, there is $C_L > 0$ such that for all $\tilde{H}, \tilde{H}' \in L$, $\lambda \in \Lambda_G$,

$$\|D^\gamma \tilde{\psi}_\lambda(\tilde{H}) - D^\gamma \tilde{\psi}_\lambda(\tilde{H}')\| \leq C_L \|\tilde{H} - \tilde{H}'\|^\delta. \quad (2.23)$$

To produce a contradiction, we want to restrict to a subfamily of λ such that the previous estimates were sharp. To give a bound on d_λ , we used that there were at least γ roots non-orthogonal to λ , but in general there might be more. So we will take λ such that there are exactly γ such roots.

Let $\lambda_1, \dots, \lambda_\ell$ be the fundamental weights defined by

$$\frac{2\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}.$$

Let i_0 be such that $\gamma = |\{\alpha \in \Phi^+ \mid n_{i_0}(\alpha) \geq 1\}|$. Up to relabeling, we assume that $i_0 = 1$. Denote also $\beta_1, \dots, \beta_\gamma$ the roots α such that $n_1(\alpha) \geq 1$, $B^+ = \{\beta_1, \dots, \beta_\gamma\}$ and

$B = B^+ \cup (-B^+)$. Then $\langle \lambda_1, \alpha \rangle \neq 0$ if and only if $\alpha \in B$. We will study the subfamily of functions associated to $\{n\lambda_1\}_{n \in \mathbb{N}} \subset \Lambda_G$. For any $\tilde{X}_1, \dots, \tilde{X}_\gamma \in \mathfrak{a}$ unit vectors, we have

$$\begin{aligned} & \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\| \\ & \geq |D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H})(\tilde{X}_1, \dots, \tilde{X}_\gamma) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')(\tilde{X}_1, \dots, \tilde{X}_\gamma)| \\ & \geq \left| \sum_{w \in W} \frac{\det(w)}{d_{n\lambda_1}} \left(\prod_{j=1}^{\gamma} (w(n\lambda_1 + \rho))(2X_j) \right) \right. \\ & \quad \left. [e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')}] \right|. \end{aligned}$$

A root α is an element of $i\mathfrak{b}^*$. There is $Y_\alpha \in i\mathfrak{b}$ such that for any $\lambda \in i\mathfrak{b}^*$, $\langle \alpha, \lambda \rangle = \lambda(Y_\alpha)$. Choose $X_j = iY_{\beta_j}$, up to a normalization constant. Thus,

$$\begin{aligned} & \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\| \geq \\ & C \left| \sum_{w \in W} \frac{\det(w)}{d_{n\lambda_1}} \left(\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle \right) [e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')}] \right| \end{aligned}$$

for some constant $C > 0$ depending only on the root system.

Let $W' = \{w \in W \mid w(B^+) \subset B\}$. Then W' is a subgroup of W , and for $w \in W'$, $w(\beta_i) = \varepsilon_i(w)\beta_{i(w)}$, with $i \mapsto i(w)$ bijection of $\{1, \dots, \gamma\}$ and $\varepsilon_i(w) \in \{\pm 1\}$. Denote also $\varepsilon(w) = \prod_i \varepsilon_i(w)$.

For $w \in W$, we have

$$\begin{aligned} \frac{\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle}{d_{n\lambda_1}} &= \prod_{j=1}^{\gamma} \langle n\lambda_1 + \rho, w^{-1}(\beta_j) \rangle \frac{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle n\lambda_1 + \rho, \alpha \rangle} \\ &= \left(\prod_{j=1}^{\gamma} \langle \rho, \beta_j \rangle \right) \prod_{j=1}^{\gamma} \frac{\langle n\lambda_1 + \rho, w^{-1}(\beta_j) \rangle}{\langle n\lambda_1 + \rho, \beta_j \rangle}. \end{aligned}$$

Now, if $w \in W'$, w^{-1} preserves B^+ up to signs, so we have

$$\frac{\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle}{d_{n\lambda_1}} = \varepsilon(w) \prod_{j=1}^{\gamma} \langle \rho, \beta_j \rangle.$$

Otherwise, if $w \in W \setminus W'$, there is j such that $w^{-1}(\beta_j) \notin B$ and so

$$\langle n\lambda_1 + \rho, w^{-1}(\beta_j) \rangle = \langle \rho, w^{-1}(\beta_j) \rangle$$

is independent of n , and so

$$\frac{\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle}{d_{n\lambda_1}} = O\left(\frac{1}{n}\right).$$

We also have $w(n\lambda_1 + \rho)(2H') \in i\mathbb{R}$ so $|e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')}| \leq 2$. Thus,

setting $C' = C \prod_{j=1}^{\gamma} \langle \rho, \beta_j \rangle$, there is $M > 0$ such that we have for all $n \in \mathbb{N}^*$, $\tilde{H}, \tilde{H}' \in Q$,

$$\begin{aligned} C' & \left| \sum_{w \in W'} \det(w) \epsilon(w) \left[e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')} \right] \right| \\ & \leq C \left| \sum_{w \in W} \frac{\det(w)}{d_{n\lambda_1}} \left(\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle \right) \right. \\ & \quad \left. \left[e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')} \right] \right| \\ & + C \left| \sum_{w \in W \setminus W'} \frac{\det(w)}{d_{n\lambda_1}} \left(\prod_{j=1}^{\gamma} \langle w(n\lambda_1 + \rho), \beta_j \rangle \right) \right. \\ & \quad \left. \left[e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')} \right] \right| \\ & \leq \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\| + \frac{M}{n} \end{aligned}$$

We can further simplify the expression. Indeed, if $j \neq 1$, $w \in W'$, we have $w^{-1}(\alpha_j) \notin B$ since $w(B) = B$. Thus,

$$\langle w(\lambda_1), \alpha_j \rangle = \langle \lambda_1, w^{-1}(\alpha_j) \rangle = 0,$$

so $w(\lambda_1) \in \{\alpha_2, \dots, \alpha_\ell\}^\perp = \mathbb{R}\lambda_1$. Furthermore, w is unitary, so $w(\lambda_1) = s(w)\lambda_1$, $s(w) = \pm 1$. Now denote

$$f(H) = \sum_{w \in W'} \det(w) \epsilon(w) e^{w(p)(2H)}$$

and

$$g(H) = i \sum_{w \in W'} \det(w) \epsilon(w) s(w) e^{w(p)(2H)}.$$

Let $ix = \lambda_1(2H)$ and $iy = \lambda_1(2H')$. We have

$$\begin{aligned} C' & \left| \sum_{w \in W'} \det(w) \epsilon(w) \left[e^{(w(n\lambda_1 + \rho))(2H)} - e^{(w(n\lambda_1 + \rho))(2H')} \right] \right| \\ & = C' |f(H) \cos(nx) + g(H) \sin(nx) - f(H') \cos(ny) - g(H') \sin(ny)|. \end{aligned}$$

Thus, we get that

$$\begin{aligned} C' |f(H) \cos(nx) + g(H) \sin(nx) - f(H') \cos(ny) - g(H') \sin(ny)| \\ \leq \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\| + \frac{M}{n} \end{aligned} \quad (2.24)$$

so

$$\begin{aligned} \frac{C'^2}{2} |f(H) \cos(nx) + g(H) \sin(nx) - f(H') \cos(ny) - g(H') \sin(ny)|^2 \\ \leq \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\|^2 + \frac{M^2}{n^2}. \end{aligned} \quad (2.25)$$

Let m_0, N that will be chosen later. For any $m \geq m_0$, set

$$S = S(m, N, H, H') = \sum_{n=m+1}^{m+N} \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\|^2.$$

Since $\sum_{n=m+1}^{m+N} \frac{1}{n^2} \leq \sum_{n=m+1}^{+\infty} \frac{1}{n^2} = \frac{1}{m}$, from (2.25) we get

$$\begin{aligned} \frac{C'^2}{2} \sum_{n=m+1}^{m+N} |f(H) \cos(nx) + g(H) \sin(nx) - f(H') \cos(ny) - g(H') \sin(ny)|^2 \\ \leq S + \frac{M^2}{m}. \end{aligned} \quad (2.26)$$

Since $|z|^2 \geq (\operatorname{Re}(z))^2$, if we denote

$$a_1 = \frac{C'}{\sqrt{2}} \operatorname{Re}(f(H)), \quad a_2 = \frac{C'}{\sqrt{2}} \operatorname{Re}(g(H)), \quad b_1 = \frac{C'}{\sqrt{2}} \operatorname{Re}(f(H')) \quad \text{and} \quad b_2 = \frac{C'}{\sqrt{2}} \operatorname{Re}(g(H')),$$

as well as $u = (m+1)x$ and $v = (m+1)y$, we have

$$\begin{aligned} \sum_{n=0}^{N-1} \left| a_1 \cos(nx + u) + a_2 \cos(nx + u + \frac{\pi}{2}) - b_1 \cos(ny + v) - b_2 \cos(ny + v + \frac{\pi}{2}) \right|^2 \\ \leq S + \frac{M^2}{m}. \end{aligned} \quad (2.27)$$

Since ρ is strictly dominant, its images under the Weyl group are disjoint. Thus, the functions $H \mapsto e^{w(\rho)(H)}$ are linearly independent in $C(\mathfrak{b})$, so f, g are non-zero. Furthermore, their extensions to $\mathfrak{b}_{\mathbb{C}}$ are holomorphic, so the zeros are isolated. Thus, we can find a compact ball L in Q such that $a_1^2 + a_2^2 + b_1^2 + b_2^2 \geq D > 0$ for any $\tilde{H}, \tilde{H}' \in L$, and a_1, a_2, b_1, b_2 bounded by D' . We can find a smaller ball L' in L such that $x, y, x-y, x+y$ are all in the same compact subset J of $]0, \pi[$, up to $k\pi$, for any $\tilde{H} \neq \tilde{H}' \in L'$.

In that case, the same computations as in Lemma 2.16 but with more terms give that for $N \geq \frac{m_0}{|x-y|}$, where m_0 depends on J, L , we have

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} \left| a_1 \cos(nx + u) + a_2 \cos(nx + u + \frac{\pi}{2}) - b_1 \cos(ny + v) - b_2 \cos(ny + v + \frac{\pi}{2}) \right|^2 \\ \geq \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2}{4} \geq \frac{D}{4}. \end{aligned} \quad (2.28)$$

Thus, combining (2.27) and (2.28), we have $S \geq \frac{DN}{4} - \frac{M^2}{m}$, so for m large enough, we have that for any $\tilde{H} \neq \tilde{H}' \in L'$,

$$S = \sum_{n=m+1}^{m+N} \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\|^2 \geq \frac{DN}{8}. \quad (2.29)$$

But by our first assumption in (2.23), we have

$$S = \sum_{n=m+1}^{m+N} \|D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}) - D^\gamma \tilde{\psi}_{n\lambda_1}(\tilde{H}')\|^2 \leq NC_{L'}^2 \|\tilde{H} - \tilde{H}'\|^{2\delta}. \quad (2.30)$$

Thus, we get for any $\tilde{H} \neq \tilde{H}' \in L'$,

$$\frac{D}{8} \leq C_{L'}^2 \|\tilde{H} - \tilde{H}'\|^{2\delta}$$

which gives a contradiction when $H \rightarrow H'$.

For the matrix coefficient, it then follows directly from Lemma 1.10. \square

As in rank 1, we end this subsection with a regularity result on Schur multipliers. Let λ be the regular representation on $L^2(G \times G)$. We define a function on \overline{Q} by

$$T_H = \int \int_{\Delta(G) \times \Delta(G)} \lambda(k \exp(H)k') dk dk' \in B(L^2(G \times G)).$$

We want to study the regularity of T as a map from Q to $S_p(L^2(G \times G))$.

Proposition 2.24. *The map T belongs to $C^{(\gamma-1,1)}(Q, S_\infty(L^2(G \times G)))$.*

Proof. There is an orthonormal basis of $L^2(G \times G)$ such that for any $H \in Q$, T_H is diagonal and its eigenvalues are $\psi_\lambda(H)$ for $\lambda \in \Lambda_G$. Then, by Theorem 2.21 and Lemma 1.5, we get the result. \square

Proposition 2.25. *Let $p > 2 + \frac{\ell}{\gamma}$, $\gamma_p = \gamma - \frac{\ell+2\gamma}{p}$ and*

$$d_p = \begin{cases} \gamma_p & \text{if } \gamma_p \notin \mathbb{Z} \\ \gamma_p - \varepsilon & \text{if } \gamma_p \in \mathbb{Z} \end{cases}$$

with $\varepsilon > 0$ arbitrarily small. Then the map T belongs to $C^{(r_p, \delta_p)}(Q, S_p(L^2(G \times G)))$ where $r_p = \lfloor d_p \rfloor$ and $\delta_p = d_p - r_p$.

Proof. There is an orthonormal basis such that for any $H \in Q$, T_H is diagonal, with eigenvalues $\psi_\lambda(H)$ of multiplicity the dimension of the representation associated to $\tilde{\lambda} \in \Lambda$ of $G \times G$. Since this representation is $V \otimes V^*$ where V is the representation of G associated to $\lambda \in \Lambda_G$, this multiplicity is d_λ^2 . Thus, we have

$$\|T_H\|_{S_p}^p = \sum_{\lambda \in \Lambda} d_\lambda^2 |\psi_\lambda(H)|^p \in [0, +\infty].$$

Let $k < \gamma$. Then by Theorem 2.21 and Lemma 1.5, $D^k T$ exists and is the map from Q such that for any $H \in Q$, $X_1, \dots, X_k \in \mathfrak{a}$, the map

$$D^k T(H)(X_1, \dots, X_k) \in B(L^2(G \times G))$$

is diagonal in the previous basis with eigenvalues $D^k \psi_\lambda(H)(X_1, \dots, X_k)$ and multiplicities d_λ^2 . Thus, we have

$$\|D^k T(H)\|_{L(\mathfrak{a}^{\otimes k}, S_p)}^p \leq \sum_{\lambda \in \Lambda_G} d_\lambda^2 \|D^k \psi_\lambda(H)\|^p \in [0, +\infty].$$

Let L be a compact subset of Q . Up to replacing L by its convex hull, which is still compact by Carathéodory's theorem and a subset of Q , since Q itself is convex, we can assume that L is convex. Let $H \in L$. By (2.21), there is $C_L > 0$ such that

$$d_\lambda^2 \|D^k \psi_\lambda(H)\|^p \leq C_L \|\lambda + \rho\|^{pk} d_\lambda^{2-p}.$$

Since by assumption, $2 - p < 0$, by (2.22), there exists a constant $C'_L > 0$ such that

$$d_\lambda^2 \|D^k \psi_\lambda(H)\|^p \leq C'_L \|\lambda + \rho\|^{pk+(2-p)\gamma}.$$

Given $\lambda_1, \dots, \lambda_\ell$ the fundamental weights, $\lambda = \sum_{i=1}^\ell n_i(\lambda) \lambda_i$ with $n_i(\lambda) \in \mathbb{N}$ and $\rho = \sum_{i=1}^\ell \lambda_i$. Then, by equivalence of norms, there are $c, C > 0$ such that

$$c \sqrt{\sum_{i=1}^\ell (n_i(\lambda) + 1)^2} \leq \|\lambda + \rho\| \leq C \sqrt{\sum_{i=1}^\ell (n_i(\lambda) + 1)^2}$$

Thus, there is $M_L > 0$ such that

$$\|D^k T(H)\|^p \leq M_L \sum_{n_1, \dots, n_\ell \geq 1} \frac{1}{(\sum n_i^2)^{\frac{(p-2)\gamma - pk}{2}}}.$$

We know that the sum on the right converges whenever

$$(p-2)\gamma - pk > \ell,$$

which is equivalent to

$$\gamma_p = \gamma - \frac{\ell + 2\gamma}{p} > k.$$

In particular, this is the case for $k = r_p = \lfloor d_p \rfloor$, and we get that $\|D^{r_p} T(H)\|$ is bounded by \tilde{M}_L on L .

Now consider $H, H' \in L$, doing the same thing we get

$$\|D^{r_p} T(H) - D^{r_p} T(H')\|^p \leq \sum_{\lambda \in \Lambda_G} d_\lambda^2 \|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\|^p.$$

We use two different bounds. For the first one, by triangle inequality, we have

$$\|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\| \leq \|D^{r_p} \psi_\lambda(H)\| + \|D^{r_p} \psi_\lambda(H')\|$$

so by (2.21) and (2.22) again, there are $C'_1, C_1 > 0$ such that

$$d_\lambda^2 \|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\|^p \leq C'_1 \|\lambda + \rho\|^{pr_p - (p-2)\gamma} \leq \frac{C_1}{(\sum n_i^2)^{\frac{(p-2)\gamma - pr_p}{2}}}.$$

We use this bound when $(\sum n_i^2)^{1/2} > \frac{1}{\|H - H'\|}$.

For the second one, since L is convex, we get by the mean value theorem,

$$\|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\| \leq \sup_L \|D^{r_p+1} \psi_\lambda(x)\| \|H - H'\|$$

so there are $C'_2, C_2 > 0$ such that

$$\begin{aligned} d_\lambda^2 \|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\|^p &\leq C'_2 \|\lambda + \rho\|^{p(r_p+1) - (p-2)\gamma} \|H - H'\|^p \\ &\leq \frac{C_2}{(\sum n_i^2)^{\frac{(p-2)\gamma - p(r_p+1)}{2}}} \|H - H'\|^p. \end{aligned}$$

We use this bound when $(\sum n_i^2)^{1/2} \leq \frac{1}{\|H - H'\|}$.

Assume here that H, H' are close enough, so that $\frac{1}{\|H - H'\|} > \sqrt{\ell}$.

Let $\kappa = (p-2)\gamma - pr_p > \ell$. Let $\mathbf{n} = (n_1, \dots, n_\ell) \in \mathbb{N}^\ell$, $\mathbf{n} \neq 0$. Then for any x in the cube $C(\mathbf{n}) = \prod_{i=1}^\ell [n_i, n_{i+1}]$, we have

$$\frac{1}{\|\mathbf{n} + \mathbf{1}\|^\kappa} \leq \frac{1}{\|x\|^\kappa} \leq \frac{1}{\|\mathbf{n}\|^\kappa}$$

so

$$\frac{1}{\|\mathbf{n} + \mathbf{1}\|^\kappa} \leq \int_{C(\mathbf{n})} \frac{1}{\|x\|^\kappa} dx \leq \frac{1}{\|\mathbf{n}\|^\kappa}. \quad (2.31)$$

Since $\bigcup_{\|\mathbf{n}+\mathbf{1}\| > \frac{1}{\|H-H'\|}} C(\mathbf{n}) \subset \mathbb{R}^\ell \setminus B(0, \frac{1}{\|H-H'\|} - \sqrt{\ell}) = \mathbb{R}^\ell \setminus B$, we get that

$$\begin{aligned} \sum_{\|\mathbf{n}\| > \frac{1}{\|H-H'\|}} \frac{1}{\|\mathbf{n}\|^\kappa} &\leq \int_{\mathbb{R}^\ell \setminus B} \frac{1}{\|x\|^\kappa} dx \\ &\leq D_1 \int_{\frac{1}{\|H-H'\|} - \sqrt{\ell}}^\infty \frac{r^{\ell-1}}{r^\kappa} dr \\ &\leq D_2 \left(\frac{1}{\|H-H'\|} - \sqrt{\ell} \right)^{\ell-\kappa} \\ &\leq D_3 \|H-H'\|^{\kappa-\ell} \end{aligned}$$

where $D_1, D_2, D_3 > 0$ are constants independent of H, H' . Now since $\kappa - \ell = p(\gamma_p - r_p)$, we get that in the first case, there is \tilde{C}_1 such that

$$\sum_{\lambda, \mathbf{n} > \frac{1}{\|H-H'\|}} \|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\|^p \leq \tilde{C}_1 \|H-H'\|^{p(\gamma_p - r_p)} \quad (2.32)$$

Now we turn to the second case. Let $\kappa = (p-2)\gamma - p(r_p + 1) = p(\gamma_p - r_p) - p + \ell$. By the choice of r , we have $\kappa \leq \ell$.

We divide again into three subcases.

First, if $0 < \kappa < \ell$, then we can use again (2.31). This time, $\frac{1}{\|x\|^\kappa}$ is integrable at 0. We have that $\bigcup_{\|\mathbf{n}+\mathbf{1}\| \leq \frac{1}{\|H-H'\|}} C(\mathbf{n}) \subset B(0, \frac{1}{\|H-H'\|})$, so we get that there are $M_1, M_2 > 0$ such that

$$\begin{aligned} \sum_{n_i \geq 1, \|\mathbf{n}\| \leq \frac{1}{\|H-H'\|}} \frac{1}{\|\mathbf{n}\|^\kappa} &\leq \int_{B(0, \frac{1}{\|H-H'\|})} \frac{1}{\|x\|^\kappa} dx \\ &\leq M_1 \int_0^{\frac{1}{\|H-H'\|}} \frac{r^{\ell-1}}{r^\kappa} dr \\ &\leq M_2 \left(\frac{1}{\|H-H'\|} \right)^{\ell-\kappa} \\ &\leq M_2 \|H-H'\|^{p(\gamma_p - r_p) - p}. \end{aligned}$$

If $\kappa \leq 0$, we do the same thing but the inequalities in (2.31) are reversed, so we must integrate on $B(0, \frac{1}{\|H-H'\|} + \sqrt{\ell})$. We still end up with

$$\sum_{n_i \geq 1, \|\mathbf{n}\| \leq \frac{1}{\|H-H'\|}} \frac{1}{\|\mathbf{n}\|^\kappa} \leq M_3 \|H-H'\|^{p(\gamma_p - r_p) - p}$$

for some constant $M_3 > 0$.

In these two subcases, corresponding to $\gamma_p \notin \mathbb{Z}$, we get that there is $\tilde{C}_2 > 0$ such that

$$\sum_{\lambda, \mathbf{n} \leq \frac{1}{\|H-H'\|}} \|D^{r_p} \psi_\lambda(H) - D^{r_p} \psi_\lambda(H')\|^p \leq \tilde{C}_2 \|H-H'\|^{p(\gamma_p - r_p) - p} \|H-H'\|^p \quad (2.33)$$

So combining (2.32) and (2.33), we have for $H, H' \in L$ such that $\|H-H'\| < \frac{1}{\sqrt{\ell}}$ that there is $D_L > 0$ such that

$$\|D^{r_p} T(H) - D^{r_p} T(H')\| \leq D_L \|H-H'\|^{\gamma_p - r_p}.$$

Taking $D'_L = \max(D_L, 2\tilde{M}_L(\sqrt{\ell})^{\gamma_p - r_p})$, we have for any $H, H' \in L$,

$$\|D^{r_p}T(H) - D^{r_p}T(H')\| \leq D'_L \|H - H'\|^{\gamma_p - r_p},$$

which completes the proof in the case $\gamma_p \notin \mathbb{Z}$.

Finally, the last of the three subcases is $\kappa = \ell$, which is equivalent to $\gamma_p \in \mathbb{Z}$. Here, we use again (2.31), but we cannot integrate in 0. So there are constants $E_1, E_2 > 0$ such that

$$\begin{aligned} \sum_{n_i \geq 1, \|\mathbf{n}\| \leq \frac{1}{\|H-H'\|}} \frac{1}{\|\mathbf{n}\|^\ell} &\leq \frac{1}{\sqrt{\ell}} + \int_{B(0, \frac{1}{\|H-H'\|}) \setminus B(0,1)} \frac{1}{\|x\|^\ell} dx \\ &\leq \frac{1}{\sqrt{\ell}} + E_1 \int_1^{\frac{1}{\|H-H'\|}} \frac{r^{\ell-1}}{r^\ell} dr \\ &\leq \frac{1}{\sqrt{\ell}} + E_1 |\ln \|H - H'\||. \end{aligned}$$

Thus there are constants $\tilde{C}_3, \tilde{C}_4 > 0$ such that

$$\sum_{\lambda, \mathbf{n} \leq \frac{1}{\|H-H'\|}} \|D^{r_p}\psi_\lambda(H) - D^{r_p}\psi_\lambda(H')\|^p \leq \tilde{C}_3 \|H - H'\|^p + \tilde{C}_4 \|H - H'\|^p |\ln \|H - H'\|| \quad (2.34)$$

So combining (2.32) (with the exponent which reduces to p) and (2.34), we have for $H, H' \in L$ such that $\|H - H'\| < \frac{1}{\sqrt{\ell}}$ that there are $D_L, D_{L,\varepsilon} > 0$ such that

$$\|D^{r_p}T(H) - D^{r_p}T(H')\| \leq D_L \|H - H'\| |\ln \|H - H'\|| \leq D_{L,\varepsilon} \|H - H'\|^{1-\varepsilon}$$

for any $1 > \varepsilon > 0$. Again, we get the result for any $H, H' \in L$ up to changing the constant $D_{L,\varepsilon}$, and the proof is complete in the case $\gamma_p \in \mathbb{Z}$. \square

Remark 2.26. In Theorem 2.23, we used a specific subfamily to prove optimality. Here, we consider a sum and not a supremum, so we cannot restrict to such a subfamily. The bound we use $d_\lambda^2 |\psi_\lambda|^p$ is only tight for $\lambda = n\lambda_1$, where λ_1 is as in the proof of Theorem 2.23. In general, we can find a better bound for $d_\lambda^2 |\psi_\lambda|^p$ if we know the number of roots orthogonal to λ . Thus, getting an optimal result would require a deeper study of the root system.

Corollary 2.27. *Let $1 < p \leq +\infty$. Let φ be a $\Delta(G)$ -bi-invariant S_p -multiplier of $G \times G$ and $\psi = \varphi \circ \exp$. Then $\psi \in C^{(r_p, \delta_p)}(Q)$.*

Proof. The proof is the same as in Corollary 2.13 using Theorem 2.25. \square

2.2.2 Results for some higher rank symmetric spaces

2.2.2.1 Complex Grassmannians

Let $q \geq p \geq 2$, $G = SU(p+q)$ and a subgroup $K = S(U(p) \times U(q))$ - the case $p = 1$ was treated in the previous sections. Let $M = G/K$. Then M is isomorphic to the Grassmann manifold of p -dimensional subspaces of \mathbb{C}^{p+q} . We have $\dim M = 2pq$ and $\text{rank } M = p$. Let $k = q - p$.

As before, let $P_n^{(\alpha, \beta)}$ denote the Jacobi polynomials and let

$$\tilde{P}_n^{(\alpha, \beta)} = \frac{\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} P_n^{(\alpha, \beta)}$$

be the Jacobi polynomials normalized at 1.

The spherical functions of (G, K) can be found in [Cam06, Section 3]. For a suitable choice of basis and of positive roots, the Weyl chamber is

$$\mathcal{C} = \{(x_1, \dots, x_p) \mid x_1 > \dots > x_p > 0\}.$$

Consider $Q \subset \mathcal{C}$ as before, open in \mathbb{R}^p . Define

$$w(X) = \prod_{i < j} (\cos(x_i) - \cos(x_j)),$$

$$c = 2^{p(p-1)/2} \prod_{j=1}^{p-1} j!(j+k)^{p-j}$$

and

$$c(n) = \left(n + \frac{k+1}{2}\right)^2 - \left(\frac{k+1}{2}\right)^2.$$

Then if $\mu = \sum r_i \mu_i \in \Lambda$, let $m_i = r_i + \dots + r_p$, so that $m_1 \geq m_2 \geq \dots \geq m_p$, and $n_i = m_i + p - i$, so $n_1 > \dots > n_p$. For $X \in Q$, we have

$$\varphi_\mu(\exp(iX)) = c \frac{\det\left(\tilde{P}_{n_i}^{(k,0)}(\cos(x_j))\right)}{w(X) \prod_{i < j} (c(n_i) - c(n_j))}.$$

Since $w(X) \neq 0$ for all $x \in Q$, both $\frac{c}{w}$ and $\frac{w}{c}$ are smooth functions, thus the optimal regularity of the family (φ_μ) is the same as the optimal regularity of the family $(\frac{w}{c}\varphi_\mu)$. Using Lemma 1.3 and since $c(n_i) - c(n_j) = (n_i + n_j + k + 1)(n_i - n_j)$, we want to study the optimal regularity of the functions

$$\begin{aligned} \tilde{Q} &\rightarrow \mathbb{C} \\ \psi_\mu : (t_1, \dots, t_p) &\mapsto \frac{\det\left(\tilde{P}_{n_i}^{(k,0)}(t_j)\right)}{\prod_{i < j} (n_i + n_j + k + 1)(n_i - n_j)}. \end{aligned}$$

We have $\psi_\mu(t) = \frac{\sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \prod_{i=1}^p \tilde{P}_{n_i}^{(k,0)}(t_{\sigma(i)})}{\prod_{i < j} (n_i + n_j + k + 1)(n_i - n_j)}$. Let $D(l_1, \dots, l_p)f = \frac{\partial^{\sum l_i f}}{\partial x_1^{l_1} \dots \partial x_p^{l_p}}$. Then

$$D(l_1, \dots, l_p)\psi_\mu(t) = \frac{\sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \prod_{i=1}^p \partial^{l_{\sigma(i)}} \tilde{P}_{n_i}^{(k,0)}(t_{\sigma(i)})}{\prod_{i < j} (n_i + n_j + k + 1)(n_i - n_j)}.$$

Theorem 2.28. *Let $M = SU(p+q)/S(U(p) \times U(q))$, with $q \geq p \geq 2$. Set*

$$\alpha_\infty = \begin{cases} 2 & \text{if } p = q = 2 \\ q + p - \frac{3}{2} & \text{else} \end{cases}.$$

Then we have

$$r_{opt}(M) \geq (\lfloor \alpha_\infty \rfloor, \alpha_\infty - \lfloor \alpha_\infty \rfloor).$$

Proof. Let L be a compact subset of \tilde{Q} , which we again assume to be convex. Then each t_i is in a compact L' of $] -1, 1[$. From inequality (2.7) in Section 2.1.2, we know that there is a constant $C = C(L', k, l) > 0$ such that $\forall n \in \mathbb{N}, \forall t \in L'$, we have

$$|\partial^l \tilde{P}_n^{(k,0)}(t)| \leq C n^{l-k-\frac{1}{2}}.$$

Thus there is some constant $C = C(L, k, l_1, \dots, l_p)$ such that for all $\mu \in \Lambda$ and $t \in L$

$$|D(l_1, \dots, l_p)\psi_\mu(t)| \leq C \frac{\sum_{\sigma \in \mathfrak{S}_p} \prod_{i=1}^p n_i^{l_{\sigma(i)} - k - \frac{1}{2}}}{\prod_{i < j} (n_i + n_j + k + 1)(n_i - n_j)}. \quad (2.35)$$

Let $I = \{i \mid n_{i+1} < \frac{n_i}{2}\}$ and $J = \llbracket 1, p-1 \rrbracket \setminus I$. Let $i_0 = \min I$. For $j > i_0 \geq i$, we have

$$n_j \leq n_{i_0+1} < \frac{n_{i_0}}{2} \leq \frac{n_i}{2}$$

so $(n_i + n_j + k + 1)(n_i - n_j) \geq \frac{n_i^2}{2}$. On the other hand, for $i < i_0$, there is $C_a > 0$ such that $n_{i+1}^a \leq C_a n_1^a$ for any a . Thus, if $l_1 + \dots + l_p = s$,

$$\begin{aligned} |D(l_1, \dots, l_p)\psi_\mu(t)| &\leq C \sum_{\sigma \in \mathfrak{S}_p} n_1^s n_1^{\sum_{i \leq i_0} (-k - \frac{1}{2} - (p-i) - (p-i_0))} \prod_{i > i_0} n_i^{-k - \frac{1}{2} - (p-i)} \\ &\leq \tilde{C}_{i_0} n_1^{s - \left(i_0 k + \frac{i_0}{2} + i_0 p - \frac{i_0(i_0+1)}{2} + i_0(p-i_0)\right)} \\ &\leq \tilde{C}_{i_0} n_1^{s - (i_0(q+p) - \frac{3}{2}i_0^2)} \end{aligned}$$

for some constant $\tilde{C}_{i_0} > 0$. Let $\kappa = \min_{i \in \llbracket 1, p \rrbracket} i(p+q) - \frac{3}{2}i^2$ and $C' = \max_{i \in \llbracket 1, p \rrbracket} \tilde{C}_i$. Then $\forall \mu \in \Lambda$ and $\forall t \in L$, we have

$$|D(l_1, \dots, l_p)\psi_\mu(t)| \leq C' n_1^{s-\kappa}.$$

Thus, we only need to compute κ . Let $f : x \mapsto (p+q)x - \frac{3}{2}x^2$. If $q \geq 2p$, f is increasing on $\llbracket 1, p \rrbracket$ so its minimum is $f(1)$ and $\kappa = f(1) = q + p - \frac{3}{2}$. If $q \leq 2p$, f is increasing on $\llbracket 1, \frac{p+q}{3} \rrbracket$ and decreasing on $[\frac{p+q}{3}, p]$. Thus, κ is either $f(1)$ or $f(p)$. But

$$\begin{aligned} f(p) &< f(1) \\ \iff pq - \frac{p^2}{2} &< p + q - \frac{3}{2} \\ \iff (p-1)q &< (p-1)\frac{p+3}{2} \\ \iff q &< \frac{p+3}{2}. \end{aligned}$$

But we also have $q \geq p$, thus this last inequality implies $\frac{p+3}{2} > p$ so $p = 2$ and then $q = 2$. Thus, if $p = q = 2$, $\kappa = f(p) = 2$. Otherwise, $\kappa = f(1) = q + p - \frac{3}{2}$.

Now, if we take $\|\cdot\|_1$ on \mathbb{R}^p , this tells us that

$$\|D^s \psi_\mu(t)\| \leq \tilde{C}_s n_1^{s-\kappa}$$

so that if $s \leq \kappa$, $D^s \psi_\mu$ is bounded on L independently of $\mu \in \Lambda$.

If $p = q = 2$, $\kappa = 2$ is an integer so the proof is complete.

Else, $\kappa = r + \frac{1}{2}$, $r = q + p - 2$. If $x, y \in L$, on the one hand,

$$\|D^r \psi_\mu(x) - D^r \psi_\mu(y)\| \leq 2\tilde{C}_r n^{-1/2}.$$

On the other hand, we get

$$\|D^r \psi_\mu(x) - D^r \psi_\mu(y)\| \leq \tilde{C}_{r+1} n^{1/2} \|x - y\|.$$

Thus,

$$\|D^r \psi_\mu(x) - D^r \psi_\mu(y)\| \leq \left(2\tilde{C}_r n^{-1/2}\right)^{1/2} \left(\tilde{C}_{r+1} n^{1/2} \|x - y\|\right)^{1/2} \leq \sqrt{2\tilde{C}_r \tilde{C}_{r+1}} \|x - y\|^{1/2}$$

so that $D^r \psi_\mu$ is $\frac{1}{2}$ -Hölder on L with a constant independent on $\mu \in \Lambda$, so we get the result. \square

Remark 2.29. If in the definition of ψ_λ we replace $(k, 0)$ with $(\alpha, \beta) \in (\mathbb{R}_+)^2$, we get the same result with

$$\kappa = \begin{cases} 2\alpha + 2 & \text{if } p = 2, \alpha < \frac{1}{2} \\ \alpha + 2p - \frac{3}{2} & \text{else} \end{cases}.$$

2.2.2.2 Some related spaces

Consider \mathbb{Z}^ℓ with lexicographic order. A polynomial P in ℓ variables has degree n if $P = \sum_{m \leq n} c_m x^m$, $c_n \neq 0$. If P is symmetric of degree n , then $n_1 \geq n_2 \geq \dots \geq n_\ell$. Let $\Omega = [-1, 1]^\ell$, $\alpha, \beta > -1$ and $\gamma \geq -1/2$. Define on Ω the function

$$w_{\alpha, \beta, \gamma}(x) = \prod_{i=1}^{\ell} (1 - x_i)^\alpha (1 + x_i)^\beta \prod_{i < j} (x_i - x_j)^{2\gamma+1}.$$

If $\ell = 1$, this is the Jacobi weight of parameter (α, β) .

Definition 2.30. The polynomials $(P_n^{(\alpha, \beta, \gamma)})_{n \in \mathbb{N}}$ in ℓ variables are the unique polynomials defined by

- $P_0^{(\alpha, \beta, \gamma)} = 1$,
- $P_n^{(\alpha, \beta, \gamma)}$ is symmetric of degree n and dominant coefficient 1,
- $\int_{\Omega} P_n^{(\alpha, \beta, \gamma)}(x) Q(x) w_{\alpha, \beta, \gamma}(x) dx = 0$ for all Q symmetric of degree $q < n$.

These polynomials are studied in [Vre84]. For certain values, these polynomials can be related to Jacobi polynomials ([Vre84, Thm. 4.5 and 4.6]).

Proposition 2.31. For $\gamma = -\frac{1}{2}$,

$$P_n^{(\alpha, \beta, \gamma)}(x) = \sum_{\sigma \in \mathfrak{S}_\ell} P_{n_1}^{(\alpha, \beta)}(x_{\sigma(1)}) \cdots P_{n_\ell}^{(\alpha, \beta)}(x_{\sigma(\ell)}).$$

Proposition 2.32. Let $A_x^{\alpha, \beta}(m) = \det(P_{m_i}^{(\alpha, \beta)}(x_j))$. Then

$$P_n^{(\alpha, \beta, 1/2)}(x) = \frac{A_x^{(\alpha, \beta)}(n_1 + \ell - 1, n_2 + \ell - 2, \dots, n_\ell)}{A_x^{(\alpha, \beta)}(\ell - 1, \ell - 2, \dots, 0)}.$$

Thus, we can see that, once normalized by 1 at 1, the family of $P_n^{(k, 0, 1/2)}$ are the spherical functions of $(SU(k + 2\ell), S(U(k + \ell) \times U(\ell)))$. It turns out that more families of spherical functions appear as polynomials of this kind ([Vre84, Thm. 4.2]).

Theorem 2.33. Let $\psi_n^{(\alpha, \beta, \gamma)} = \frac{P_n^{(\alpha, \beta, \gamma)}}{P_n^{(\alpha, \beta, \gamma)}(1)}$. Then the functions $\psi_n^{(\alpha, \beta, \gamma)}$ are the spherical functions of (G, K) for the following values of α, β, γ :

| (G, K) | ℓ | α | β | γ |
|--|--------|-----------------|---------|----------|
| $(SO(p + q), S(O(p) \times O(q)))$ | p | $(q - p - 1)/2$ | $-1/2$ | 0 |
| $(SU(p + q), S(U(p) \times U(q)))$ | p | $q - p$ | 0 | $1/2$ |
| $(Sp(p + q), Sp(p) \times Sp(q))$ | p | $2(q - p) + 1$ | 1 | $3/2$ |
| $(Sp(k), U(k))$ | k | 0 | 0 | 0 |
| $(SO(4k), U(2k))$ | k | 0 | 0 | $3/2$ |
| $(SO(4k + 2), U(2k + 1))$ | k | 2 | 0 | $3/2$ |
| $(Sp(k) \times Sp(k), \Delta(Sp(k)))$ | k | $1/2$ | $1/2$ | $1/2$ |
| $(SO(2k + 1) \times SO(2k + 1), \Delta(SO(2k + 1)))$ | k | $1/2$ | $-1/2$ | $1/2$ |

Remark 2.34. Adapting the previous work for general values of $\alpha \geq 0, \beta > -1$, we know that the optimal uniform regularity of the family $(\psi_n^{(\alpha, \beta, 1/2)})$ is at least $\alpha + \frac{1}{2} + 2(\ell - 1)$. From the previous table, this recovers the regularity found in Section 2.2.1 for $(Sp(k) \times Sp(k), \Delta(Sp(k)))$ and $(SO(2k+1) \times SO(2k+1), \Delta(SO(2k+1)))$.

In [Vre84, Thm. 5.1], Vretare shows different formulas expressing $\psi_n^{(\alpha, \beta, \gamma)}$ as a linear combination of different $\psi_n^{(\alpha', \beta', \gamma')}$. If we were able to control the coefficient, we could investigate more families. It turns out that this is the case in 2 variables. Set $\ell = 2$. Vretare showed the following ([Vre84, Thm. 6.2]):

Theorem 2.35. *Let $\alpha, \beta > -1$ and $\gamma \geq -1/2$. Let $n \geq m$. Then*

$$(x - y)^2 \psi_{n,m}^{(\alpha, \beta, \gamma+1)} = b_{20} \psi_{n+2,m}^{(\alpha, \beta, \gamma)} + b_{10} \psi_{n+1,m}^{(\alpha, \beta, \gamma)} + b_{00} \psi_{n,m}^{(\alpha, \beta, \gamma)} + b_{11} \psi_{n+1,m+1}^{(\alpha, \beta, \gamma)} + b_{1-1} \psi_{n+1,m-1}^{(\alpha, \beta, \gamma)}$$

and there is a constant $C = C(\alpha, \beta, \gamma) > 0$ such that $b_{ij} \leq Cn^{-1}(n-m)^{-1}$ for any i, j .

Remark 2.36. From this, in the case $\ell = 2$, we can recover the regularity obtained in Theorem 2.28 from the regularity at least $\alpha + \frac{1}{2}$ for the family $(\psi_n^{(\alpha, \beta, -1/2)})$ that is easy to compute.

With Lemma 1.4, the regularity of the family $(\psi_{n,m}^{(\alpha, \beta, 3/2)})_{n \geq m}$ is the same as the regularity of the family $((x - y)^2 \psi_{n,m}^{(\alpha, \beta, 3/2)})_{n \geq m} = (\phi_{n,m}^{(\alpha, \beta)})$.

Set L a compact subset of $] -1, 1[^2$. Let $l_1, l_2 \in \mathbb{N}$, $s = l_1 + l_2$. Assume first that $m < \frac{n}{2}$. As in Theorem 2.28, there is a constant $C > 0$ such that

$$|D(l_1, l_2) \psi_{n,m}^{(\alpha, \beta, 1/2)}(x)| \leq Cn^{s-\alpha-\frac{5}{2}}$$

thus there is $C' > 0$ such that

$$|D(l_1, l_2) \phi_{n,m}^{(\alpha, \beta)}| \leq C'n^{s-\alpha-\frac{9}{2}}.$$

On the contrary, assume that $n > m \geq \frac{n}{2}$, then there is a constant $D > 0$,

$$|D(l_1, l_2) \psi_{n,m}^{(\alpha, \beta, 1/2)}(x)| \leq Dn^{s-2\alpha-2}$$

thus there is $D' > 0$ such that

$$|D(l_1, l_2) \phi_{n,m}^{(\alpha, \beta)}| \leq D'n^{s-2\alpha-3}.$$

Now depending on (n, m) , we have two possible upper bounds. We want to know which one is the worst, and thus works independently of (n, m) .

We have $2\alpha + 3 \geq \alpha + \frac{3}{2}$ if and only if $\alpha \geq \frac{3}{2}$. Now, adapting again the argument from the proof of Theorem 2.28, we get that the regularity of the family $(\psi_{n,m}^{(\alpha, \beta, 3/2)})_{n \geq m}$ is at least

$$r = \begin{cases} 2\alpha + 3 & \text{if } \alpha < \frac{3}{2} \\ \alpha + \frac{9}{2} & \text{if } \alpha \geq \frac{3}{2} \end{cases}$$

From Theorem 2.33, this gives lower bound for new pairs.

Theorem 2.37. *Let $M = SO(8)/U(4)$, $\text{rank } M = 2$ and $\dim M = 12$. We have*

$$r_{\text{opt}}(M) \geq (3, 0).$$

Let $M = SO(10)/U(5)$, $\text{rank } M = 2$ and $\dim M = 20$. We have

$$r_{\text{opt}}(M) \geq (6, \frac{1}{2}).$$

Let $n \geq 4$, $M = Sp(n)/(Sp(2) \times Sp(n-2))$, $\text{rank } M = 2$ and $\dim M = 8(n-2)$. We have

$$r_{\text{opt}}(M) \geq \begin{cases} (2n-3, \frac{1}{2}) & \text{if } n > 4 \\ (5, 0) & \text{if } n = 4 \end{cases}$$

2.2.3 A conjecture on the optimal regularity

In this section, we will see on an example that there is a difference in the estimates when μ is close to the walls of the Weyl chamber and away from the walls. We will use this to give a conjecture on the optimal regularity in the general case.

Let $G = SU(q+2)$ and $K = S(U(2) \times U(q))$. Let $M = G/K$ of rank 2 and dimension $4q$. In Theorem 2.28, we showed that $r_{\text{opt}}(M) \geq (q, \frac{1}{2})$ if $q > 2$, and $(2, 0)$ otherwise. Furthermore, spherical functions are determined by the highest weight $\mu = r_1\mu_1 + r_2\mu_2$ of the associated representation of G . There is a basis $\{\alpha, \beta\}$ of the root system $\Sigma_{\mathfrak{a}}$ such that any positive root is a linear combination of α, β with positive integral coefficients. The walls of the Weyl chamber are defined by the hyperplanes $H_{\alpha} = \alpha^{\perp}$ and $H_{\beta} = \beta^{\perp}$. We have $\mu_1 \in H_{\alpha}$ and $\mu_2 \in H_{\beta}$. Finally, the positive roots are $\alpha, \beta, 2\alpha, \alpha + \beta, \alpha + 2\beta, 2(\alpha + \beta)$ of multiplicities $2q-4, 2, 1, 2q-4, 2, 1$ respectively (see [Cam06]).

Let $\mu_0 = \mu_1 + \mu_2$. Then, for $n \in \mathbb{N}$, $n\mu_0$ is in a cone with compact base in the open Weyl chamber, so away from the walls. Then,

$$\psi_{n\mu_0}(x, y) = \frac{\tilde{P}_{2n+1}^{(q-2,0)}(x)\tilde{P}_n^{(q-2,0)}(y) - \tilde{P}_{2n+1}^{(q-2,0)}(y)\tilde{P}_n^{(q-2,0)}(x)}{(3n+q)(n+1)}.$$

The family $(\psi_{n\mu_0})_{n \in \mathbb{N}}$ is a subfamily of the family of all spherical functions, and we can show with the same estimates as in Theorem 2.28 that this family is bounded in the Hölder space of regularity $2q-1$.

So, for this subfamily corresponding to highest weight away from the walls, the regularity is strictly better than the regularity of the whole family. Note that being in a cone with compact base means that $\langle n\mu_0, \alpha \rangle, \langle n\mu_0, \beta \rangle$ have the same growth rate as $\|n\mu_0\|$.

In fact, for any of the pairs we considered above, with $\mu_0 = \sum_{i=1}^l \mu_i$, the subfamily corresponding to $\{n\mu_0\}$ is of regularity at least $\frac{\dim M - \text{rank } M}{2}$.

This shows that it should be highest weights close to the walls that gives a bad regularity. Let us see what happens when the highest weight is exactly on a wall, which corresponds to μ being orthogonal to some roots. For this, set either $r_1 = 0$ or $r_2 = 0$.

If $r_1 = 0$, $r_2 = n$, then

$$\psi_n(x, y) = \frac{\tilde{P}_{n+1}^{(q-2,0)}(x)\tilde{P}_{n\mu_2}^{(q-2,0)}(y) - \tilde{P}_{n+1}^{(q-2,0)}(y)\tilde{P}_n^{(q-2,0)}(x)}{(2n+q)}$$

and the uniform regularity of this subfamily is at least $2q-2$. Note that in that case, μ is orthogonal to β , which is the root of the basis with small multiplicity.

If $r_1 = n - 1, r_2 = 0$, since $\tilde{P}_0^{(q-2,0)} = 1$, the expression is easier. We have

$$\psi_{n\mu_1}(x, y) = \frac{\tilde{P}_n^{(q-2,0)}(x) - \tilde{P}_n^{(q-2,0)}(y)}{(n+q+1)n}.$$

Here, we get uniform regularity at least $q + \frac{1}{2}$, so this is where the worst happens (except for $q = 2$). It is not clear whether this is the optimal regularity or not, but we will check that $(q+1)$ -th differential is unbounded in n - so the uniform regularity is at most $q+1$. Let $x = \cos \theta$. From Proposition 2.7 and Proposition 2.8, we have

$$\begin{aligned} \frac{d^r}{dx^r} \tilde{P}_n^{(q-2,0)}(x) &= \frac{\Gamma(q-1)\Gamma(n+1)}{\Gamma(q+n-1)} \frac{d^r}{dx^r} P_n^{(q-2,0)}(x) \\ &= \frac{\Gamma(q-1)\Gamma(n+1)}{\Gamma(q+n-1)} \frac{\Gamma(q+n+r-1)}{2^r \Gamma(q+n-1)} P_{n-r}^{(q-2+r,r)}(x) \\ &= n^{r-q+3/2} u(\theta) \cos(n\theta + \gamma) + O(n^{r-q+1/2}) \end{aligned}$$

where u is a smooth function of $]\ -1, 1[$, γ depends on q, r, θ but not on n . We used the same asymptotics on Γ as in (2.5). Also, $\tilde{P}_n^{(q-2,0)}(y) = O(1)$. Now let $r = q+1$ and consider the partial derivative

$$D(q+1, 0)\psi_n(x, y) = n^{1/2} u(\theta) \cos(n\theta + \gamma) + O(n^{-1/2})$$

which is unbounded in n . So we get that when μ is orthogonal to α , the subfamily is bounded in $C^{(q, \frac{1}{2})}$ but not bounded in C^{q+1} . Thus, we can conclude that it has lower uniform regularity than the previous subfamilies. Note that α is the root with high multiplicity.

So this example shows that when μ is not orthogonal to any roots, spherical functions are well-behaved and we get good estimates as the spectral parameter grows. But when μ becomes orthogonal to some roots, we see that the spherical functions tend to be unbounded in more Hölder spaces. We also see that the optimal uniform regularity decreases with the sum of multiplicities of non-orthogonal roots.

We now try to give a more quantitative interpretation of this. For a root $\alpha \in \Sigma_{\mathfrak{a}}$, let $m(\alpha)$ be its multiplicity. For $\mu \in \Lambda$, define

$$n(\mu) = \sum_{\substack{\alpha \in \Sigma_{\mathfrak{a}}^+ \\ \langle \alpha, \mu \rangle \neq 0}} m(\alpha)$$

and set

$$\kappa(G, K) = \frac{1}{2} \inf_{\mu \in \Lambda \setminus \{0\}} S_{\mu}.$$

We can now formulate Conjecture C:

Conjecture 2.38. Let $r = \lfloor r \rfloor$ and $\delta = \kappa(G, K) - r$. Then $r_{opt}(M) = (r, \delta)$. In terms of spherical functions, the family $(\varphi_{\lambda})_{\lambda \in \Lambda}$ is bounded in $C^{(r, \delta)}(G_r)$ and for any $\delta' > \delta$, $(\varphi_{\lambda})_{\lambda \in \Lambda}$ is not bounded in $C^{(r, \delta')}(G_r)$ where G_r is the set of regular points of (G, K) .

More generally, the optimal uniform regularity of a subfamily of spherical functions indexed by $\Lambda' \subset \Lambda$ should be given by a similar formula involving only weights of Λ' . However, removing a finite number of elements of Λ' will not change the regularity of the family. Furthermore, we can notice with the example above that the behavior of the family $\{(0, n)\}_n$ will be the same as the behavior of $\{(k, n)\}_n$ for k fixed, so what is important is

not the orthogonality of the family with roots, but the boundedness of the scalar products, thus making the formula more complicated.

By [Vre76], there is a basis $\{\alpha_1, \dots, \alpha_\ell\}$ associated to the positive root system $\Sigma_{\mathfrak{a}}^+$ such that the fundamental weights μ_i satisfy

$$\frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, 2\alpha_j \notin \Sigma_{\mathfrak{a}}^+ \\ 2 & \text{if } i = j, 2\alpha_j \in \Sigma_{\mathfrak{a}}^+ \end{cases}$$

Then we have $r = \min_{1 \leq i \leq \ell} \sum_{\langle \alpha, \mu_i \rangle \neq 0} \frac{m(\alpha)}{2}$. Notice that if (G, K) is the compact symmetric pair dual to (H, K) where H is noncompact semisimple and K maximal compact subgroup (as in Section 1.8), then $\kappa(G, K) = \kappa(H)$. Thus, the values of κ are computed in Appendix A. We can verify that these values agree with the results found for rank 1 (Theorem 2.5) and with the lower bound found for some higher rank spaces in Theorem 2.28 and Theorem 2.37.

Furthermore, if $M = M_1 \times M_2$, the root system of M is the direct sum of the root systems of M_i , and so $r(M) = \min(r(M_1), r(M_2))$. Thus, Table A.1 and A.2 from Appendix A are sufficient to compute the value of r for any symmetric space of compact type. Also, this show that the conjecture agrees with the fact that $r_{opt}(M) = \min(r_{opt}(M_i))$.

Finally, assume that $M = (G \times G)/\Delta(G)$. We saw in Subsection 2.2.1 that if Φ^+ is a choice of positive roots with basis $\alpha_1, \dots, \alpha_\ell$, we have

$$r_{opt}(M) = \gamma = \min_{1 \leq i \leq \ell} |\{\alpha \in \Phi^+ \mid n_i(\alpha) \geq 1\}|.$$

We also saw that a choice of positive roots for M was given by $\tilde{\alpha} : (H, -H) \mapsto \alpha(H)$ with $\alpha \in \Phi^+$, and that $m(\tilde{\alpha}) = 2$ for any $\alpha \in \Phi^+$. The bijection $\alpha \mapsto \tilde{\alpha}$ extends to a map $\mathfrak{b}_{\mathbb{C}}^* \mapsto \mathfrak{a}_{\mathbb{C}}^*$, which sends Λ_G to Λ and such that $\langle \tilde{\lambda}, \tilde{\mu} \rangle = \langle \lambda, \mu \rangle$. Thus, from this we get that for any $\lambda \in \Lambda_G$,

$$\frac{1}{2}S_{\tilde{\lambda}} = |\{\alpha \in \Phi^+ \mid \langle \alpha, \lambda \rangle \neq 0\}|.$$

Let $\pi_i, 1 \leq i \leq \ell$ be the fundamental weights of the root system of G , defined by

$$\frac{2\langle \pi_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}.$$

We have that $\lambda \in \Lambda_G$ if and only if $\lambda = \sum_{i=1}^{\ell} m_i(\lambda)\pi_i$, with $m_i(\lambda) \in \mathbb{N}$. Then, for any $\lambda \in \Lambda_G, \alpha \in \Phi^+$,

$$\langle \lambda, \alpha \rangle = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} m_i(\lambda)n_i(\alpha)\langle \pi_i, \alpha_j \rangle = \frac{1}{2} \sum_{i=1}^{\ell} m_i(\lambda)n_i(\alpha)\langle \alpha_i, \alpha_i \rangle.$$

If $\lambda \neq 0$, there is i such that $m_i(\lambda) \neq 0$. Then for any α such that $n_i(\alpha) \neq 0$, we get $\langle \lambda, \alpha \rangle > 0$, so

$$\frac{1}{2}S_{\tilde{\lambda}} \geq |\{\alpha \in \Phi^+ \mid n_i(\alpha) \geq 1\}| \geq \gamma.$$

But on the other hand, $\frac{1}{2}S_{n\pi_{i_0}} = \gamma$ for all $n \in \mathbb{N}^*$ where i_0 is such that

$$\gamma = |\{\alpha \in \Phi^+ \mid n_{i_0}(\alpha) \geq 1\}|.$$

Thus, we get

$$\gamma = \inf_{\tilde{\lambda} \in \Lambda \setminus \{0\}} \frac{1}{2}S_{\tilde{\lambda}},$$

so the regularity found in this case fits again the conjecture.

We will come back to this conjecture in Chapter 4 and obtain some results towards a positive answer.

2.3 Regularity of K -finite matrix coefficients

In this section, we keep the notations introduced in the previous sections. Thus (G, K) is a symmetric compact Gelfand pair with G connected such that G/K is simply connected. Recall that Q is an open subset of \mathfrak{a} such that, by Proposition 1.20, $G = K \exp(\overline{Q})K$, so that G admits a KAK decomposition. Set $G_r = K \exp(Q)K$ the dense open subset of regular points. We want to show that (G_r, K) admits a well-behaved KAK decomposition in the sense of Definition 1.13.

Lemma 2.39. *The map*

$$q: \begin{array}{ccc} K \times K \times Q & \rightarrow & G_r \\ (k_1, k_2, H) & \mapsto & k_1 \exp(H)k_2^{-1} \end{array}$$

is a submersion.

Proof. If $g \in G$, denote L_g and R_g the translations by g on the left and right respectively. Let $m: G \times G \rightarrow G$ be the multiplication map, its tangent map at (a, b) is

$$T_{(a,b)}m: \begin{array}{ccc} T_a G \times T_b G & \rightarrow & T_{ab} G \\ (X_a, X_b) & \mapsto & T_a R_b(X_a) + T_b L_a(X_b) \end{array} .$$

We can identify $T_g G$ with \mathfrak{g} by the isomorphism $T_e L_g$. Under this identification, we have $\forall g, h \in G$, $T_h L_g = \text{Id}$ and $T_h R_g = \text{Ad}(g^{-1})$, so that the tangent map of the multiplication becomes $T_{(a,b)}m(X_a, X_b) = \text{Ad}(b^{-1})(X_a) + X_b$. Furthermore, if $k \in K$, since $L_k(K) = K$, $T_k K \subset T_k G$ is identified with $\mathfrak{k} \subset \mathfrak{g}$. Thus by the chain rule we have

$$T_{(k_1, k_2, H)}q: \begin{array}{ccc} \mathfrak{k} \times \mathfrak{k} \times \mathfrak{a} & \rightarrow & \mathfrak{g} \\ (X_1, X_2, Y) & \mapsto & \text{Ad}(k_2)(\text{Ad}(\exp(-H))(X_1) + T_H \exp(Y)) - X_2 \end{array} .$$

We know that $\text{Ad}(k)$ is an isomorphism of \mathfrak{g} and an isomorphism of \mathfrak{k} in restriction. Furthermore, $T_H \exp: \mathfrak{a} \rightarrow \mathfrak{a}$ is also an isomorphism. Thus, the map $T_{(k_1, k_2, H)}q$ is surjective if and only if $u = \text{Ad}(k_2^{-1}) \circ T_{(k_1, k_2, H)}q \circ (\text{Id}, \text{Ad}(k_2), (T_H \exp)^{-1})$ is surjective. We have

$$u(X_1, X_2, Y) = \text{Ad}(\exp(-H))(X_1) - X_2 + Y.$$

Consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ in eigenspaces of σ . In this decomposition, \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Let $\mathfrak{m} = \mathfrak{k}^{\mathfrak{a}}$ and $\Sigma_{\mathfrak{a}}$ the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$.

For $\lambda \in \Sigma_{\mathfrak{a}}$, let $\mathfrak{g}_{\lambda} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \forall H \in \mathfrak{a}_{\mathbb{C}}, [H, X] = \lambda(H)X\}$. We have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Sigma_{\mathfrak{a}}} \mathfrak{g}_{\lambda}.$$

Let also $\mathfrak{k}_{\lambda} = \mathfrak{k} \cap (\mathfrak{g}_{\lambda} \oplus \mathfrak{g}_{-\lambda})$ and $\mathfrak{p}_{\lambda} = \mathfrak{p} \cap (\mathfrak{g}_{\lambda} \oplus \mathfrak{g}_{-\lambda})$. From [Loo69b, Ch. VI, Prop. 1.4], we get

$$\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\lambda \in \Sigma_{\mathfrak{a}}^+} \mathfrak{k}_{\lambda} = \mathfrak{m} \oplus \mathfrak{l},$$

$$\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma_{\mathfrak{a}}^+} \mathfrak{p}_{\lambda} = \mathfrak{a} \oplus \mathfrak{b}.$$

We also get that for $\lambda \in \Sigma_{\mathfrak{a}}^+$, there is $Z_{\lambda,1}, \dots, Z_{\lambda,r_\lambda}$ a \mathbb{C} -basis of \mathfrak{g}_λ , such that setting $Z_{\lambda,i}^+ = Z_{\lambda,i} + \sigma(Z_{\lambda,i})$ and $Z_{\lambda,i}^- = i(Z_{\lambda,i} - \sigma(Z_{\lambda,i}))$, $\{Z_{\lambda,i}^+\}$ gives an \mathbb{R} -basis of \mathfrak{k}_λ and $\{Z_{\lambda,i}^-\}$ gives an \mathbb{R} -basis of \mathfrak{p}_λ .

Let also H_1, \dots, H_ℓ be a basis of \mathfrak{a} and Y_1, \dots, Y_r a basis of \mathfrak{m} . Then for $H \in \mathfrak{a}$, we have $[H, Y_i] = 0$, $[H, Z_{\lambda,i}^+] = -i\lambda(H)Z_{\lambda,i}^-$ and $[H, Z_{\lambda,i}^-] = i\lambda(H)Z_{\lambda,i}^+$. Thus, we see that

- $u(0, 0, H_i) = H_i$,
- $u(Y_i, 0, 0) = e^{-\text{ad}(H)}(Y_i) = Y_i$,
- $u(0, Y_i, 0) = -Y_i$,
- $u(Z_{\lambda,i}^+, 0, 0) = e^{-\text{ad}(H)}(Z_{\lambda,i}^+) = \sin(i\lambda(H))Z_{\lambda,i}^- + \cos(i\lambda(H))Z_{\lambda,i}^+$,
- $u(0, Z_{\lambda,i}^+, 0) = -Z_{\lambda,i}^+$.

Since $H \in Q$, $\lambda(H) \notin i\pi\mathbb{Z}$ so $\sin(i\lambda(H)) \neq 0$ and u is indeed surjective. \square

The following proposition is found in [Bor98, Ch.V, Thm 3.3].

Proposition 2.40. *Let G be a compact, connected, simply connected Lie group and f an automorphism of G . Then the set of fixed points of f is connected.*

Remark 2.41. This result implies that the subgroup K is automatically connected if G is simply connected.

Lemma 2.42. *Let $M = Z_K(\exp \overline{Q}) = \{k \in K \mid \forall a \in \exp \overline{Q}, ka = ak\}$. Consider the action of $K \times K$ on G by $(k_1, k_2).g = k_1 g k_2^{-1}$. Then for any $H \in Q$, we have*

$$\text{Stab}(\exp H) = \{(k, k) \mid k \in M\} = \Delta(M).$$

Proof. First, we assume that G is simply connected. Let $H \in 2Q$. We have that

$$Z_G(\exp H) = \{g \in G \mid g \exp(H) = \exp(H)g\} = \{g \in G \mid c_{\exp(-H)}(g) = g\}$$

is the set of fixed points of $c_{\exp(-H)}$. Thus, $Z_G(\exp H)$ is connected by Proposition 2.40.

Furthermore,

$$\begin{aligned} \text{Lie}(Z_G(\exp H)) &= \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R}, \exp(tX) \in Z_G(\exp H)\} \\ &= \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R}, c_{\exp(-H)}(\exp(tX)) = \exp(tX)\} \\ &= \{X \in \mathfrak{g} \mid \text{Ad}(\exp(-H))(X) = X\} \\ &= \ker(f_H) \end{aligned}$$

where $f_H(X) = \text{Ad}(\exp(-H))(X) - X$.

Consider the basis $\{Y_i\} \cup \{H_i\} \cup \{Z_{\lambda,i}^+\} \cup \{Z_{\lambda,i}^-\}$ of \mathfrak{g} introduced in Lemma 2.39.

Then, we have

- $f_H(Y_i) = 0$,
- $f_H(H_i) = 0$,
- $f_H(Z_{\lambda,i}^+) = \sin(i\lambda(H))Z_{\lambda,i}^- + (\cos(i\lambda(H)) - 1)Z_{\lambda,i}^+$,
- $f_H(Z_{\lambda,i}^-) = -\sin(i\lambda(H))Z_{\lambda,i}^+ + (\cos(i\lambda(H)) - 1)Z_{\lambda,i}^-$.

Since $H \in 2Q$, for any root λ , $\lambda(H) \notin 2i\pi\mathbb{Z}$, and so $\ker(f_H) = \mathfrak{m} \oplus \mathfrak{a}$.

Thus, $\text{Lie}(Z_G(\exp H))$ does not depend on $H \in 2Q$, and since $Z_G(\exp H)$ is connected, $Z_G(\exp H)$ and $Z_K(\exp H) = Z_G(\exp H) \cap K$ do not depend on $H \in 2Q$, and so $Z_K(\exp H) = M$.

Now take $a = \exp(H) \in Q$, and $(k, k') \in \text{Stab}(a)$, then $ka = ak'$. The automorphism σ of G is such that $K = G^\sigma$, and $\sigma(a) = a^{-1}$, so we get $ka^{-1} = a^{-1}k'$, thus $ka^2 = ak'a = a^2k$. So $k \in Z_K(a^2)$, but $a^2 = \exp(2H)$, $2H \in 2Q$ so $Z_K(a^2) = Z_K(a) = M$. Thus $ka = ak = ak'$, so $k = k'$ and $(k, k') \in \Delta(M)$.

The other inclusion is clear, thus $\text{Stab}(a) = \Delta(M)$.

For the general case, since G/K is assumed to be simply connected, by the Remark 1.19, we have $p : \tilde{G} \rightarrow G$ the universal cover such that $\ker p \subset Z(\tilde{G})^\sigma$ and $\tilde{K} = \tilde{G}^\sigma$. Then the previous case gives that $\text{Stab}_{\tilde{K} \times \tilde{K}}(\exp_{\tilde{G}}(H)) = \Delta(\tilde{M})$ for any $H \in Q$. Clearly, if $(k_1, k_2) \in \text{Stab}_{\tilde{K} \times \tilde{K}}(\exp_{\tilde{G}}(H))$, then the projection $(p(k_1), p(k_2)) \in \text{Stab}_{K \times K}(\exp_G(H))$.

Conversely, let $(k_1, k_2) \in \text{Stab}_{K \times K}(\exp_G(H))$. There exists $\tilde{k}_i \in \tilde{K}$ such that $p(\tilde{k}_i) = k_i$. Then $k_1 \exp_G(H) k_2^{-1} = \exp_G(H)$ implies that there exists $x \in \ker p$ such that

$$\tilde{k}_1 \exp_{\tilde{G}}(H) \tilde{k}_2^{-1} x = \exp_{\tilde{G}}(H).$$

Thus, $(\tilde{k}_2, x^{-1}\tilde{k}_2) \in \Delta(\tilde{M})$ so $\tilde{k}_2 = x^{-1}\tilde{k}_2$, thus $k_1 = k_2 \in Z_K(\exp_G(H))$.

So this tells us that $\text{Stab}_{K \times K}(\exp_G(H)) = p(\text{Stab}_{\tilde{K} \times \tilde{K}}(\exp_{\tilde{G}}(H)))$ does not depend on $H \in Q$ and is equal to $\Delta(Z_K(\exp_G(H)))$ so $Z_K(\exp_G(H)) = M$ and $\text{Stab}_{K \times K}(\exp_G(H)) = \Delta(M)$. \square

The following proposition is a refinement of Proposition 1.20.

Proposition 2.43. *For any $g \in G$, there exists a decomposition*

$$g = k_1(g) \exp(P(g)) k_2(g)^{-1}$$

where $k_1(g), k_2(g) \in K$ and $P(g) \in \mathfrak{a}$. The map $g \mapsto P(g)$ is smooth on G_r . Furthermore, for each $g \in G_r$, there exists a neighborhood U_g of g in G_r and a choice of $g \mapsto k_i(g)$ such that k_i is smooth on U_g , $i = 1, 2$.

Proof. By Lemma 2.42, the map

$$\tilde{q} : \begin{array}{ccc} (K \times K)/\Delta(M) \times Q & \rightarrow & G_r \\ ((k_1, k_2) \bmod M, H) & \mapsto & k_1 \exp(H) k_2^{-1} \end{array}$$

is a well-defined smooth bijection between manifolds of the same dimension.

Let $p : K \times K \rightarrow (K \times K)/\Delta(M)$ be the projection. It is a surjective submersion. Let q be the submersion defined in Lemma 2.39. We have $q = \tilde{q} \circ (p, \text{Id})$. Thus, for any $(x, P) \in (K \times K)/\Delta(M) \times Q$, we have $T_{(x, P)}\tilde{q}$ surjective. But it is a linear map between vector spaces of the same dimension, so it is invertible. Thus, by the local inversion theorem and since \tilde{q} is bijective, \tilde{q} is a smooth diffeomorphism.

Let $(x, P) : G_r \rightarrow (K \times K)/\Delta(M) \times Q$ be a smooth inverse. We get that P is a smooth map. From [Lee03, Proposition 4.26], since p is a submersion, any $(k_1, k_2) \in K \times K$ is in the image of a smooth local section of p . Let $g \in G_r$. Since p is surjective, $x(g) = p(k_1, k_2)$. There exists a neighborhood V of $x(g)$ and a smooth section $s = (s_1, s_2) : V \rightarrow K \times K$ such that $s(x(g)) = (k_1, k_2)$. Let $U = x^{-1}(V)$ neighborhood of g . Then $k_i = s_i \circ x$ is smooth on U and $g = k_1(g) \exp(P(g)) k_2(g)^{-1}$. \square

Corollary 2.44. *Let φ be a K -bi-invariant function on G . Then $\varphi \in C^{(r, \delta)}(G_r)$ if and only if $\varphi \circ \exp \in C^{(r, \delta)}(Q)$.*

Proof. Since \exp is smooth, the first implication is clear. For the converse, assume $\psi = \varphi \circ \exp|_Q \in C^{(r,\delta)}(Q)$. By the previous proposition, the map P is smooth on G_r and $\varphi = \psi \circ P$ by K -bi-invariance, thus $\varphi \in C^{(r,\delta)}(G_r)$ by Lemma 1.3. \square

We may now combine the results in this chapter to prove Theorems A and B.

Proof of Theorem A. By Theorem 2.5 and Corollary 2.44, the spherical functions of (G, K) are bounded in $C^{(\lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor)}(G_r)$. By Lemma 1.10, any K -bi-invariant matrix coefficient of a unitary representation of G is in $C^{(\lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor)}(G_r)$. By Proposition 2.43, (G_r, K) has a well-behaved KAK decomposition, thus by Theorem 1.15, any K -finite matrix coefficient of a unitary representation of G is also in $C^{(\lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor)}(G_r)$. Finally by Corollary 2.17, for any $(r, \delta) > (\lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor)$, there is a K -bi-invariant matrix coefficient of G which is not in $C^{(r,\delta)}(G_r)$. \square

Proof of Theorem B. The proof is identical using now Theorems 2.21 and 2.23. \square

Chapter 3

Regularity of K -finite matrix coefficients of semisimple Lie groups

This chapter is devoted to the study of both flat symmetric pairs and symmetric pairs of noncompact type. The results presented here were published in [Dum25, Section 3 and 4] and [Dum24b, Section 3 and 4]. As explained in Section 1.6 and Section 1.7, the pairs we are interested in are defined in terms of a semisimple Lie group G . We begin with Cartan motion groups: due to the flatness of the symmetric space, the situation is somehow easier. In Section 3.1.1, we prove the existence of a well-behaved KAK decomposition for Cartan motion groups. This allows us to reduce the question to K -bi-invariant coefficients (and thus spherical functions) thanks to Section 1.3. Section 3.1.2 is devoted to the proof of Theorem D. Since the spherical functions are oscillatory integrals (see Section 1.7), we use the stationary phase approximation to obtain estimates. To do so, we make use of the results of [DKV83] where the phase function appearing in these integrals was studied in depth.

We then turn to the study of regularity of K -finite matrix coefficients of a semisimple Lie group G , where K is a maximal compact subgroup of G . The section follows the same construction as for Cartan motion groups. We begin with the KAK decomposition in Section 3.2.1, where we also explain how to work at the level of the Lie algebra (Corollary 3.11). Section 3.2.2 focuses on the proof of Theorem E. As for Cartan motion groups and unlike Chapter 2, we obtain a general results because spherical functions in the noncompact case have a nice representation as an oscillatory integral (see Section 1.6). The key input is again the use of the stationary phase approximation to obtain estimates on these integrals, using the results from [DKV83].

3.1 Cartan motion groups

3.1.1 KAK decomposition

In this section, we prove the existence of a well-behaved KAK decomposition for Cartan motion groups, in order to apply results from Section 1.3.

The following proposition is a consequence of well-known results in the theory of semisimple Lie groups, which are essentially the ingredients used for the KAK decomposition at the level of the semisimple group G itself.

Proposition 3.1. *Let $g \in H = \mathfrak{p} \rtimes K$. Then there exists a unique $a \in \overline{\mathfrak{a}^+}$ and such that $g \in K(a, \text{Id})K$. Furthermore, if k_1, k_2 are such that*

$$g = (0, k_1)(a, \text{Id})(0, k_2^{-1})$$

and if $a \in \mathfrak{a}^+$ then k_1 is unique up to multiplication on the right by an element of $M = Z_K(\mathfrak{a})$.

Proof. Let $g = (x, k)$. Since $(0, k_1)(a, \text{Id})(0, k_2^{-1}) = (\text{Ad}(k_1)(a), k_1 k_2^{-1})$, it suffices to show that for any $x \in \mathfrak{p}$, there exists a unique $a \in \overline{\mathfrak{a}^+}$ such that $x \in \text{Ad}(K)a$.

Existence. Since all maximal abelian subspaces of \mathfrak{p} are conjugated under K , we have $\mathfrak{p} = \bigcup \text{Ad}(k)\mathfrak{a}$ ([Kna02, Thm. 6.51]). Thus, there exists $a_0 \in \mathfrak{a}$ and $k_0 \in K$ such that $x = \text{Ad}(k_0)a_0$. Then by the theory of root systems, there exists $a \in \overline{\mathfrak{a}^+}$ the closure of the positive Weyl chamber and $w \in W$ such that $w.a = a_0$ ([Hal03, Prop. 8.29]). Since $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, w is represented by $k_w \in K$ and $x = \text{Ad}(k_0 k_w)a$.

Uniqueness. Assume that $x = \text{Ad}(k)a = \text{Ad}(k')a'$ for $k, k' \in K$ and $a, a' \in \overline{\mathfrak{a}^+}$. Then a, a' are conjugated under K thus by [Kna02, Lemma 7.38], a, a' are conjugated under $N_K(\mathfrak{a})$. So there is $w \in W$ such that $w.a = a'$ hence by [Hal03, Prop. 8.25], $a = a'$.

Finally, the element k such that $x = \text{Ad}(k)a$ is unique up to multiplication on the right by an element of $Z_K(a)$. When $a \in \mathfrak{a}^+$, $Z_K(a) = M = Z_K(\mathfrak{a})$. \square

By Proposition 3.1, we may define $P : H \rightarrow \overline{\mathfrak{a}^+}$ such that $P(g)$ is the only element of \mathfrak{a}^+ such that $g \in K(P(g), \text{Id})K$. Set

$$H_r = \{g \in H \mid P(g) \in \mathfrak{a}^+\},$$

this is a dense open subset of H that we call the set of regular points of H . We now want to study the regularity of the decomposition.

Lemma 3.2. *The map*

$$q : \begin{array}{ccc} K \times K \times \mathfrak{a}^+ & \rightarrow & H_r \\ (k_1, k_2, a) & \mapsto & (\text{Ad}(k_1)a, k_1 k_2^{-1}) \end{array}$$

is a submersion.

Proof. If $k \in K$, denote L_k and R_k the translations by k on the left and right respectively. Let $m : G \times K \rightarrow K$ be the multiplication map, its differential at (a, b) is

$$T_{(a,b)}m : \begin{array}{ccc} T_a K \times T_b K & \rightarrow & T_{ab} K \\ (X_a, X_b) & \mapsto & T_a R_b(X_a) + T_b L_a(X_b) \end{array} .$$

We can identify $T_k K$ with \mathfrak{k} by the isomorphism $T_e L_k$. Under this identification, we have that for all $g, h \in K$, $T_h L_g = \text{Id}$ and $T_h R_g = \text{Ad}(g^{-1})$, so that the tangent map becomes $T_{(a,b)}m(X_a, X_b) = \text{Ad}(b^{-1})(X_a) + X_b$. Thus by the chain rule we have

$$T_{(k_1, k_2, a)}q : \begin{array}{ccc} \mathfrak{k} \times \mathfrak{k} \times \mathfrak{a} & \rightarrow & \mathfrak{p} \times \mathfrak{k} \\ (X_1, X_2, Y) & \mapsto & (\text{Ad}(k_1)(Y + \text{ad}(X_1)a), \text{Ad}(k_2)(X_1) - X_2) \end{array} .$$

We know that $\text{Ad}(k)$ is an isomorphism of \mathfrak{p} . Thus, the map $T_{(k_1, k_2, H)}q$ is surjective if and only if

$$u : \begin{array}{ccc} \mathfrak{k} \times \mathfrak{a} & \rightarrow & \mathfrak{p} \\ (X, Y) & \mapsto & Y + [X, a] \end{array}$$

is surjective. For $\alpha \in \Sigma^+$, let $\mathfrak{k}^\alpha = \mathfrak{k} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$ and $\mathfrak{p}^\alpha = \mathfrak{p} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$. From [Loo69b, Ch. VI, Prop. 1.4], we get

$$\begin{aligned}\mathfrak{k} &= \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{k}^\alpha = \mathfrak{m} \oplus \mathfrak{l}, \\ \mathfrak{p} &= \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}^\alpha = \mathfrak{a} \oplus \mathfrak{b}.\end{aligned}$$

We also get that for $\alpha \in \Sigma^+$, there exists $Z_{\alpha,1}, \dots, Z_{\alpha,m(\alpha)}$ basis of \mathfrak{g}^α , such that setting $Z_{\alpha,i}^+ = Z_{\alpha,i} + \theta(Z_{\alpha,i})$ and $Z_{\alpha,i}^- = Z_{\alpha,i} - \theta(Z_{\alpha,i})$, $\{Z_{\alpha,i}^+\}$ is a basis of \mathfrak{k}_α and $\{Z_{\alpha,i}^-\}$ is a basis of \mathfrak{p}_α .

Clearly, $\mathfrak{a} \subset \text{Im } u$. Furthermore, for $\alpha \in \Sigma^+$, $1 \leq i \leq m(\alpha)$,

$$\begin{aligned}u(Z_{\alpha,i}^+, 0) &= [Z_{\alpha,i}^+, a] \\ &= [Z_{\alpha,i}, a] + [\theta Z_{\alpha,i}, a] \\ &= [Z_{\alpha,i}, a] + \theta[Z_{\alpha,i}, -a] \\ &= -\alpha(a)Z_{\alpha,i} + \alpha(a)\theta Z_{\alpha,i} \\ &= -\alpha(a)Z_{\alpha,i}^-\end{aligned}$$

Since $a \in \mathfrak{a}^+$, $\alpha(a) \neq 0$ for all $\alpha \in \Sigma^+$ thus $Z_{\alpha,i} \in \text{Im } u$ and u is surjective. \square

Proposition 3.3. *The map $P : H_r \rightarrow \mathfrak{a}^+$ is smooth on H_r . Furthermore, for each $g \in H_r$, there exists a neighborhood U_g of g in H_r and a choice of $g \mapsto k_i(g)$ such that k_i is smooth on U_g , $i = 1, 2$ and for any $g \in U_g$,*

$$g = (0, k_1(g))(P(g), \text{Id})(0, k_2(g)^{-1}).$$

In particular, Propositions 3.1 and 3.3 implies that (H_r, K) has a well-behaved KAK decomposition in the sense of Definition 1.13.

Proof. Let $\Delta(M) = \{(m, m) \mid m \in M\}$ denote the diagonal subgroup of $K \times K$. By Lemma 3.1, the map

$$\tilde{q} : \begin{array}{ccc} (K \times K)/\Delta(M) \times \mathfrak{a}^+ & \rightarrow & H_r \\ ((k_1, k_2) \bmod M, a) & \mapsto & (\text{Ad}(k_1)a, k_1 k_2^{-1}) \end{array}$$

is a well-defined smooth bijection between manifolds of the same dimension.

Let $p : K \times K \rightarrow (K \times K)/\Delta(M)$ be the projection. It is a surjective submersion. Let q be the submersion defined in Lemma 3.2, we have $q = \tilde{q} \circ (p, \text{Id})$. Thus, for any $(x, a) \in (K \times K)/\Delta(M) \times \mathfrak{a}^+$, we have $T_{(x,a)}\tilde{q}$ surjective. But it is a linear map between vector spaces of the same dimension, so it is invertible. Thus, by the local inversion theorem and since \tilde{q} is bijective, \tilde{q} is a smooth diffeomorphism.

Let $(x, P) : G_r \rightarrow (K \times K)/\Delta(M) \times \mathfrak{a}^+$ be a smooth inverse. We get that P is a smooth map. From [Lee03, Proposition 4.26], since p is a submersion, any $(k_1, k_2) \in K \times K$ is in the image of a smooth local section of p . Let $g \in H_r$, since p is surjective, $x(g) = p(k_1, k_2)$. There exists a neighborhood V of $x(g)$ and a smooth section $s = (s_1, s_2) : V \rightarrow K \times K$ such that $s(x(g)) = (k_1, k_2)$.

Let $U = x^{-1}(V)$ be a neighborhood of g , then $k_i = s_i \circ x$ is smooth on U and we get that $g = (0, k_1(g))(P(g), \text{Id})(0, k_2(g)^{-1})$. \square

3.1.2 Boundedness of spherical functions

Theorem 3.4. *Let G be a connected semisimple Lie group with finite center and $H = \mathfrak{p} \rtimes K$ its Cartan motion group. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then the family of positive-definite spherical functions of (H, K) is bounded in $C^{(r, \delta)}(H_r)$.*

Proof. Up to composition with the smooth map P , it suffices to consider the group variable in \mathfrak{a}^+ . Let $\lambda \in \mathfrak{a}^*$, $a \in \mathfrak{a}$ and $t \in \mathbb{R}$. Let D be the differential operator with respect to the variable a . Denote

$$f(\lambda, a, k, t) = e^{it\langle a, \text{Ad}(k)H_\lambda \rangle} \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^+ \times K \times \mathbb{R}).$$

Then by induction on s , there is a polynomial $P \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^+ \times K \times \mathfrak{a}^s)[t]$ of degree s such that for any $X \in \mathfrak{a}^s$,

$$D^s f(\lambda, a, k, t)(X) = P(t)(\lambda, a, k, X) e^{it\langle a, \text{Ad}(k)H_\lambda \rangle}.$$

For $0 \leq j \leq s$, let $g_j(\lambda, a, X) \in C^\infty(K)$ be defined by

$$g_j = \frac{1}{j!} \frac{d^j}{dt^j} \left((D^s f) e^{-it\langle a, \text{Ad}(k)H_\lambda \rangle} \right) \Big|_{t=0}. \quad (3.1)$$

Then we have

$$D^s \varphi_{t\lambda}(a)(X) = \sum_{j=0}^s t^j \int_K e^{it\langle a, \text{Ad}(k)H_\lambda \rangle} g_j(\lambda, a, X)(k) dk. \quad (3.2)$$

We want to apply the method of stationary phase to understand the asymptotics of these integrals as $t \rightarrow \infty$. Let $f_{a, \lambda} : k \mapsto \langle a, \text{Ad}(k)H_\lambda \rangle$ be the phase function. By [DKV83, Prop. 1.2], the critical set of $f_{a, \lambda}$ is $\mathcal{C}_\lambda = \bigcup_{w \in W} k_w K_\lambda$ where $K_\lambda = Z_K(\lambda)$ and k_w is a representative of $w \in W$ (since $M = Z_K(\mathfrak{a}) \subset K_\lambda$ this does not depend on the choice of k_w). By [DKV83, Prop. 1.4], the Hessian of $f_{a, \lambda}$ at $k_w m$, $m \in K_\lambda$, is diagonal in the basis adapted to the orthogonal decomposition of $\mathfrak{p} = \mathfrak{m} \oplus \bigoplus \mathfrak{p}_\alpha$, where $\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, with eigenvalues $-\alpha(H_\lambda)(w\alpha)(a)$ on each \mathfrak{p}_α .

We now proceed as in [DKV83, Prop. 9.2]. Let S be the unit sphere in \mathfrak{a}^* and L a compact subset of \mathfrak{a}^+ . Fix $\lambda_0 \in S$, $a_0 \in L$. Let $k_0 \in \mathcal{C}_{\lambda_0}$, we can find a chart

$$\psi : \begin{array}{ccc} U_{k_0} & \rightarrow & U \times V \subset \mathbb{R}^{\dim \mathcal{C}_{\lambda_0}} \times \mathbb{R}^{n(\lambda_0)} \\ k & \mapsto & (x, y) \end{array}$$

around k_0 such that $\psi(k_0) = (0, 0)$ and \mathcal{C}_λ is given in these local coordinate by $\{y = 0\}$. Treat $f_{a, \lambda}(k)$ as a function of y with parameters a, λ, x . Then $f_{a_0, \lambda_0}(0, \cdot)$ has a unique nondegenerate critical point $y = 0$. Thus, by [Hör83, Thm. 7.7.6], there exists a neighborhood $U_{a_0} \times U_{\lambda_0} \times U_0$ of $(a_0, \lambda_0, 0)$ and a continuous seminorm ν on $C^\infty(K)$, such that for any $a \in U_{a_0}$, $\lambda \in U_{\lambda_0}$, $x \in U_0$, $g \in C^\infty(K)$ and $t \geq 1$,

$$\left| \int_V e^{itf_{a, \lambda}(\psi^{-1}(x, y))} g(\psi^{-1}(x, y)) dy \right| \leq \nu(g) t^{-n(\lambda_0)/2} \leq \nu(g) t^{-\kappa(G)}. \quad (3.3)$$

By compactness we may cover $L \times S$ by a finite number of open subsets $U_{a_0} \times U_{L_0}$ so up to changing the seminorm and U_0 , we may assume (3.3) holds for any $a \in L$, $\lambda \in S$. Similarly by compactness, we can cover K by a finite number of open subsets of the form $U_0 \times V$.

Using a smooth partition of unity and summing estimates in (3.3), we obtain a seminorm ν such that for any $a \in L$, $\lambda \in S$, $g \in C^\infty(K)$ and $t \geq 1$,

$$\left| \int_K e^{itf_{a,\lambda}(k)} g(k) dk \right| \leq \nu(g)t^{-\kappa(G)}. \quad (3.4)$$

Now if $a \in L$, $\lambda \in S$ and $t \leq 1$, then by (3.2) we get that

$$\|D^s \varphi_{t\lambda}(a)\| \leq \sup_{a \in L, \lambda \in S, k \in K \|X_i\|=1} \sum |g_j(\lambda, a, X)(k)| \leq C_{L,s} \quad (3.5)$$

where $C_{L,s} > 0$ depends only on the compact L and the order of differentiation.

On the other hand for $t \geq 1$, combining (3.2) and (3.4), we obtain

$$\begin{aligned} \|D^s \varphi_{t\lambda}(a)\| &= \sup_{\|X_i\|=1} |D^s \varphi_{t\lambda}(a)(X)| \\ &\leq \sup_{\|X_i\|=1} \sum_{j=0}^s t^j \left| \int_K e^{itf_{a,\lambda}(k)} g_j(\lambda, a, X)(k) dk \right| \\ &\leq \sum_{j=0}^s t^j \sup_{\substack{\|X_i\|=1 \\ a \in L, \lambda \in S}} \nu(g_j(\xi, \eta, Y, X)) t^{-\kappa(G)} \\ &\leq D_{L,s} t^{s-\kappa(G)} \end{aligned} \quad (3.6)$$

where $D_{L,s} > 0$ depends again only on the compact L and the order of differentiation.

Thus combining (3.5) and (3.6), for any $\lambda \in \mathfrak{a}^*$, $a \in L$, $s \leq r$,

$$\|D^s \varphi_\lambda(a)\| \leq \max(C_{L,s}, D_{L,s}) = M_{L,s}. \quad (3.7)$$

Thus the differentials of the family of spherical functions are bounded independently on λ up to order r . If $r = \kappa(G)$, the proof is complete.

Otherwise, $\kappa(G) - r = \frac{1}{2}$. Then using (3.6) for $s = r$ and $s = r + 1$, we show that for any $x, y \in L$, $\xi \in S$, $\eta \in C$, $t \geq 1$, we have on the one hand

$$\|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\| \leq \|D^r \varphi_{t\lambda}(x)\| + \|D^r \varphi_{t\lambda}(y)\| \leq 2D_{L,r} t^{-1/2} \quad (3.8)$$

and on the other hand, we get by the mean value theorem that

$$\|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\| \leq \left(\sup_{a \in L} \|D^{r+1} \varphi_{t\lambda}(a)\| \right) \|x - y\| \leq D_{L,r+1} t^{1/2} \|x - y\|. \quad (3.9)$$

Thus, combining (3.8) and (3.9) yields

$$\|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\| \leq (2D_{L,r} D_{L,r+1})^{1/2} \|x - y\|^{1/2}. \quad (3.10)$$

Hence, setting $M_L = \max\left((2D_{L,r} D_{L,r+1})^{1/2}, C_{L,r+1}(\text{diam } L)^{1/2}\right)$, we have that for any $\lambda \in \mathfrak{a}^*$, $x, y \in L$,

$$\|D^r \varphi_\lambda(x) - D^r \varphi_\lambda(y)\| \leq M_L \|x - y\|^{\kappa(G)-r}. \quad \square$$

We will now show that this result is optimal. Before this, we first prove a lemma which essentially states that families of exponential maps are not bounded in Hölder spaces. We will reduce the problem for spherical functions of (H, K) to such functions.

Lemma 3.5. *Let E be a finite-dimensional real vector space, U an open subset of E such that $0 \in \bar{U}$. Let $u_1, \dots, u_n \in E^*$ distinct and non-zero, and $f_1, \dots, f_n : E \rightarrow \mathbb{C}$ continuous functions such that for any U' open subset of U , there is $x \in U'$ such that $\sum |f_j(x)| \neq 0$. Then there exists $C > 0$, $d > 0$, $x \in U$, and an open set V with $0 \in \bar{V}$ such that for all $y = x + h, h \in V$, $m \in \mathbb{N}$ and $N \geq \frac{d}{\|h\|}$,*

$$\frac{1}{N} \sum_{t=m}^{m+N-1} \left| \sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right|^2 \geq C.$$

Proof. Up to multiplying f_j by $x \mapsto e^{imu_j(x)}$, we can assume that $m = 0$ as long as the constants we find depends only on $|f_j|$ and not f_j . We have that

$$\begin{aligned} & \left| \sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right|^2 \\ &= \left(\sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right) \left(\sum_{k=1}^n \overline{f_k(x)} e^{-itu_k(x)} - \overline{f_k(y)} e^{-itu_k(y)} \right) \\ &= \sum_{j,k=1}^n f_j(x) \overline{f_k(x)} e^{it(u_j(x)-u_k(x))} - f_j(x) \overline{f_k(y)} e^{it(u_j(x)-u_k(y))} \\ & \quad - f_j(y) \overline{f_k(x)} e^{it(u_j(y)-u_k(x))} + f_j(y) \overline{f_k(y)} e^{it(u_j(y)-u_k(y))}. \end{aligned}$$

Note that if $z \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\left| \sum_{t=0}^{N-1} e^{itz} \right| \leq \frac{1}{|\sin(z/2)|}$. The set $H = \bigcup_{j \neq k} \ker(u_j - u_k)$ is a finite union of hyperplane, thus $U' = U \cap (E \setminus H)$ is open and non-empty with 0 in its closure.

Let x in U' with $\|x\| \leq \min_{j,k} \frac{\pi}{\|u_j\| + \|u_k\|}$ and such that there is j_0 with $f_{j_0}(x) \neq 0$. Then there is a neighborhood V_x of x in U' and $\varepsilon > 0$ such that for $y \in V_x$ and $j \neq k$, $\varepsilon \leq \left| \frac{u_j(y) - u_k(y)}{2} \right| \leq \pi - \varepsilon$ and $\varepsilon \leq \left| \frac{u_j(x) - u_k(y)}{2} \right| \leq \pi - \varepsilon$.

Then, let $V_0 = V_x - x$, let $h_0 \in V_0 \setminus \bigcup_i \ker u_i$. Let $\eta = \frac{1}{2\|h_0\|} \min_j |u_j(h_0)| > 0$. Let

$$V = V_0 \setminus \{h \in E \mid \forall 1 \leq j \leq n, |u_j(h)| \leq \eta \|h\|\}.$$

Then V is an open subset of U , containing h_0 thus such that $0 \in \bar{V}$. For any $h \in V$, by definition we have $\|h\| < \frac{1}{\eta} |u_j(h)|$ for any $1 \leq j \leq n$.

Hence we get for any $y = x + h, h \in V$, $N \in \mathbb{N}$,

$$\begin{aligned} & \frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right|^2 \geq \\ & \sum_{j=1}^n \left(|f_j(x)|^2 + |f_j(y)|^2 - \frac{|f_j(x)f_j(y)|}{N \left| \sin \frac{u_j(x)-u_j(y)}{2} \right|} \right) \\ & - \sum_{j \neq k} \left(\frac{|f_j(x)f_k(x)|}{N \left| \sin \frac{u_j(x)-u_k(x)}{2} \right|} + \frac{|f_j(y)f_k(y)|}{N \left| \sin \frac{u_j(y)-u_k(y)}{2} \right|} + 2 \frac{|f_j(x)f_k(y)|}{N \left| \sin \frac{u_j(x)-u_k(y)}{2} \right|} \right). \end{aligned}$$

Now, for each of the terms with $j \neq k$, the assumptions on x, y ensures that the arguments in sin are bounded away from $0, \pi$. Furthermore, up to restricting V to a bounded set if

necessary, the functions f_i are bounded. Thus, there is N_0 such that for $N > N_0$, we get

$$\frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right|^2 \geq \frac{|f_{j_0}(x)|^2}{2} - \sum_{j=1}^n \frac{|f_j(x) f_j(x+h)|}{N \left| \sin \frac{u_j(h)}{2} \right|}.$$

Finally, for each j , there is d_j such that for any $h \in V$, $N \geq \frac{d_j}{|u_j(h)|}$,

$$\frac{|f_j(x) f_j(x+h)|}{N \left| \sin \frac{u_j(h)}{2} \right|} \leq \frac{|f_{j_0}(x)|^2}{4}.$$

Thus, for $d > (\max d_j)/\eta$, then for any $y = x + h$, $h \in V$ and $N > \max \left(N_0, \frac{d}{\|h\|} \right)$,

$$\frac{1}{N} \sum_{t=0}^{N-1} \left| \sum_{j=1}^n f_j(x) e^{itu_j(x)} - f_j(y) e^{itu_j(y)} \right|^2 \geq \frac{|f_{j_0}(x)|^2}{4}. \quad \square$$

In order to prove optimality, we will show that a particular subfamily of the whole family of positive-definite spherical functions is already unbounded in Hölder spaces of higher order.

Theorem 3.6. *Let G be a connected semisimple Lie group with finite center and $H = \mathfrak{p} \rtimes K$ its Cartan motion group. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. For any $\delta' > \delta$, the family of positive definite spherical functions of (H, K) is not bounded in $C^{(r, \delta')}(H_r)$.*

Proof. Fix $\lambda \in \mathfrak{a}^*$ such that $n(\lambda) = 2\kappa(G)$, and such that $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Sigma^+$. It suffices to show that the family $(\varphi_{t\lambda})_{t \in \mathbb{R}}$ is not bounded in $C^{(r, \delta')}(H_r)$. As in the proof of Theorem 3.4, let $f_{a, \lambda} : k \mapsto \langle a, \text{Ad}(k)H_\lambda \rangle$ be the phase function. Since $f_{a, \lambda}$ is right K -invariant, we may see the integral on the quotient manifold K/K_λ . More precisely, let $d(k_\lambda)$ be the image of the Haar measure on K under the quotient map $K \rightarrow K/K_\lambda$, then for all $t \geq 1$,

$$\varphi_{t\lambda}(a) = \int_{K/K_\lambda} e^{itf_{a, \lambda}(k)} d(kK_\lambda).$$

This will simplify the proof, as the critical set of $f_{a, \lambda}$ is thus $\mathcal{C} = \bigcup_{w \in W} kwK_\lambda$ a finite set in K/K_λ . As in (3.2), for any $a \in \mathfrak{a}$, $t \geq 1$, $X \in \mathfrak{a}^r$,

$$D^r \varphi_{t\lambda}(a)(X) = \sum_{j=0}^r t^j \int_{K/K_\lambda} e^{itf_{a, \lambda}(k)} g_j(a, X)(k) d(kK_\lambda). \quad (3.11)$$

Let $I_j(a, X, t) = \int_K e^{itf_{a, \lambda}(k)} g_j(a, X)(k) d(kK_\lambda)$. Let W_λ denote the stabiliser of λ under the action of the Weyl group W . Let also

$$\Sigma^+(\lambda) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \lambda \rangle \neq 0\}$$

and

$$\sigma_w = - \sum_{\alpha \in \Sigma^+(\lambda) \neq 0} m(\alpha) \text{sgn}(\langle \alpha, \lambda \rangle) (w\alpha)(Y).$$

Let $d_0 k$ denote the Riemannian measure on K induced by the bi-invariant metric defined by the Killing form on \mathfrak{k} . Let $d_0(kK_\lambda)$ be the volume measure on K/K_λ associated to the (invariant) Riemannian metric induced by the restriction of the inner product $-\langle \cdot, \cdot \rangle$ on \mathfrak{k} .

Denote $\text{Vol}(K/K_\lambda) = \int_{K/K_\lambda} d_0(K/K_\lambda)$. By uniqueness of the invariant measure on K/K_λ , we have $d(kK_\lambda) = \frac{1}{\text{Vol}(K/K_\lambda)} d_0(kK_\lambda)$. For $w \in W$, $g \in C^\infty(K/K_\lambda)$, $a \in \mathfrak{a}^+$, set

$$c_{w,a}(g) = e^{i\frac{\pi}{4}\sigma_w} \prod_{\alpha \in \Sigma^+(\lambda)} \left| \frac{\langle \alpha, \lambda \rangle}{4\pi} (w\alpha)(a) \right|^{-\frac{m(\alpha)}{2}} \frac{1}{\text{Vol}(K/K_\lambda)} g(k_w K_\lambda). \quad (3.12)$$

Then, by the stationary phase approximation ([Hör83, Thm. 7.7.6]), for any $a \in \mathfrak{a}^+$, there is a neighborhood U_a of a in \mathfrak{a}^+ and $D(a) > 0$ such that for any $0 \leq j \leq r$, $t \geq 1$, $a' \in U_a$ and X with $\|X_i\| = 1$ for all i ,

$$\left| I_j(a', X, t) - \sum_{W/W_\lambda} e^{it(w\lambda)(a')} t^{-\kappa(G)} c_{w,a'}(g_j(a', X)) \right| \leq D(a) t^{-\kappa(G)-1}. \quad (3.13)$$

We use that g_j is smooth in all variables hence bounded on compact sets and that the bound is uniform in the parameter a' of the phase function.

The end of the proof relies on Lemma 3.5.

Combining (3.11) with (3.13) for $0 \leq j < r$, for any a there is a neighborhood V_a of a and a constant $C(a)$ such that for any $t \geq 1$, $a' \in V_a$ and X with $\|X_i\| = 1$,

$$|D^r \varphi_{t\lambda}(a')(X) - t^r I_r(a', X, t)| \leq C(a) t^{-1}. \quad (3.14)$$

For X fixed with $\|X_i\| = 1$, let $S_t(x) = \sum_{W/W_\lambda} e^{it(w\lambda)(x)} c_{w,x}(g_r(x, X))$. Combining (3.13) and (3.14), if $t \geq 1$, and $x, y \in U_a \cap V_a$,

$$\begin{aligned} t^{-\delta} |S_t(x) - S_t(y)| &\leq t^r |t^{-\kappa(G)} S_t(x) - I_r(x, X, t)| + t^r |I_r(x, X, t) - I_r(y, X, t)| \\ &\quad + t^r |I_r(y, X, t) - t^{-\kappa(G)} S_t(y)| \\ &\leq 2D(a) t^{-\delta-1} + |t^r I_r(x, H, t) - D^r \varphi_{t\lambda}(x)(X)| \\ &\quad + |D^r \varphi_{t\lambda}(x)(X) - D^r \varphi_{t\lambda}(y)(X)| \\ &\quad + |D^r \varphi_{t\lambda}(y)(X) - t^r I_r(y, X, t)| \\ &\leq 2D(a) t^{-\delta-1} + 2C(a) t^{-1} + |D^r \varphi_{t\lambda}(x)(X) - D^r \varphi_{t\lambda}(y)(X)| \\ &\leq \|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\| + 2(C(a) + D(a)) t^{-1}. \end{aligned} \quad (3.15)$$

Now the functions $c_{w,x}(g_r(x, X))$ are all zero at x if and only if $g_r(x, X)(k_w K_\lambda) = 0$ for all $w \in W$. But

$$g_r(x, X)(kK_\lambda) = \prod_{i=1}^r \langle X_i, \text{Ad}(k)H_\lambda \rangle.$$

Thus given any open subset of \mathfrak{a}^+ , we can choose x, X such that $g_r(x, X)(e) \neq 0$. Thus the hypotheses of Lemma 3.5 hold for the family of functions $f_w : x \mapsto c_{w,x}(g_r(x, X))$, for $U = \mathfrak{a}^+$. Let C, d, x, V be given by Lemma 3.5, $W_x = x + V$ such that for any $y \in W_x$, $m \in \mathbb{N}$, $N \geq \frac{d}{\|x-y\|}$,

$$\sum_{t=m}^{m+N-1} |S_t(x) - S_t(y)|^2 \geq CN. \quad (3.16)$$

From now on, we choose $a = x$ given above. Let $M = 4(C(x) + D(x))^2$, we get from (3.15) that for any $t \geq 1$, $y \in U_x \cap V_x \cap W_x$,

$$\frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \leq \|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\|^2 + Mt^{-2} \quad (3.17)$$

Assume now that the family of positive definite spherical functions of (H, K) is bounded in $C^{(r, \delta')}(a^+)$ for $\delta' > \delta$. In particular, up to reducing $U_x \cap V_x \cap W_x$ to a bounded subset of diameter L if necessary, there is $D > 0$ such that for any $y \in U_x \cap V_x \cap W_x$ and $t \geq 1$,

$$\|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\| \leq D \|x - y\|^{\delta'} \quad (3.18)$$

For y fixed, set m, N such that

$$\frac{1}{\|x - y\|^{\delta'}} \leq m \leq \frac{1}{\|x - y\|^{\delta'}} + 1 \quad (3.19)$$

and

$$\frac{d}{\|x - y\|} \leq N \leq \frac{d}{\|x - y\|} + 1. \quad (3.20)$$

Combining (3.16), (3.17) and (3.18) gives

$$\begin{aligned} \frac{CN}{2(m+N)^{2\delta}} &\leq \sum_{t=m}^{m+N-1} \frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \\ &\leq \sum_{t=m}^{m+N-1} \left(\|D^r \varphi_{t\lambda}(x) - D^r \varphi_{t\lambda}(y)\|^2 + Mt^{-2} \right) \\ &\leq ND^2 \|x - y\|^{2\delta'} + \frac{MN}{m^2} \end{aligned} \quad (3.21)$$

thus

$$\frac{C}{2(m+N)^{2\delta}} \leq D^2 \|x - y\|^{2\delta'} + \frac{M}{m^2} \leq (D^2 + M) \|x - y\|^{2\delta'} \quad (3.22)$$

by (3.19). But by (3.19) and (3.20), we have

$$\begin{aligned} m + N &\leq \frac{d}{\|x - y\|} + 1 + \frac{1}{\|x - y\|^{\delta'}} + 1 \leq \frac{1}{\|x - y\|} \left(d + 2\|x - y\| + \|x - y\|^{1-\delta'} \right) \\ &\leq \frac{1}{\|x - y\|} \left(d + 2L + L^{1-\delta'} \right) \end{aligned} \quad (3.23)$$

hence (3.22) becomes

$$\frac{C}{2(d + 2L + L^{1-\delta'})^{2\delta}} \|x - y\|^{2\delta} \leq (D^2 + M) \|x - y\|^{2\delta'}. \quad (3.24)$$

Since (3.24) holds for any $y \in U_x \cap V_x \cap W_x$ with the constant involved independent from y and $\delta' > \delta$, we get a contradiction as y goes to x (which is possible because $0 \in \bar{V}$ hence $x \in \overline{U_x \cap V_x \cap W_x}$). \square

Finally, we can prove the main theorem as a consequence of these results.

Theorem 3.7. *Let G be a connected semisimple Lie group with finite center and $H = \mathfrak{p} \rtimes K$ its Cartan motion group. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of H is in $C^{(r, \delta)}(H_r)$. Furthermore, this result is optimal in the sense that for any $\delta' > \delta$, there exists a K -finite (even K -bi-invariant) coefficient which is not in $C^{(r, \delta')}(H_r)$.*

Proof. By Propositions 3.1 and 3.3, (H_r, K) has a well-behaved KAK decomposition. Thus by Theorem 1.15, it suffices to consider K -bi-invariant matrix coefficients. The result is then an immediate consequence of Lemma 1.10 and Theorems 3.4 and 3.6. \square

3.2 Semisimple Lie groups

3.2.1 K -bi-invariant functions on Lie groups vs. Lie algebras

Let G be a connected semisimple Lie group with finite center. Given the notations from Section 1.6, G has a KAK decomposition similar to the compact case. This means that the K -bi-invariant functions can be studied as functions on A . The following proposition is [Kna02, Thm. 7.39].

Proposition 3.8 (KAK decomposition). *For any $g \in G$, there are $k_1, k_2 \in K$ and a unique $a \in \overline{A^+}$ such that $g = k_1 a k_2^{-1}$. Furthermore, if $a \in A^+$, k_1 is unique up to multiplication on the right by an element of $M = Z_K(A)$.*

Denote $G_r = KA^+K$. Then G_r is a dense open subset of G which we call the set of regular elements of G . Let also $P : G \rightarrow \overline{\mathfrak{a}^+}$ be the map such that for any $g \in G$, $g \in K \exp P(g)K$, which is well-defined by the above proposition. Note that by definition, $P(G_r) = \mathfrak{a}^+$.

Let φ be a K -bi-invariant function on G . Let also $\psi = \varphi \circ \exp|_{\overline{\mathfrak{a}^+}}$. Then $\varphi = \psi \circ P$ by K -bi-invariance. Hence, we can study ψ and recover information on φ using P . Since we want to study regularity of functions, we will use Lemma 1.3. However, P is not smooth on G , but we will show that it is smooth on G_r (as in Proposition 2.43 in the compact case). The proof is nearly identical to its compact counterpart.

Lemma 3.9. *The map*

$$q : \begin{array}{ccc} K \times K \times \mathfrak{a}^+ & \rightarrow & G_r \\ (k_1, k_2, H) & \mapsto & k_1 \exp(H) k_2^{-1} \end{array}$$

is a submersion.

Proof. If $g \in G$, denote L_g and R_g the translations by g on the left and right respectively. Let $m : G \times G \rightarrow G$ be the multiplication map, its differential at (a, b) is

$$T_{(a,b)}m : \begin{array}{ccc} T_a G \times T_b G & \rightarrow & T_{ab} G \\ (X_a, X_b) & \mapsto & T_a R_b(X_a) + T_b L_a(X_b) \end{array} .$$

We can identify $T_g G$ with \mathfrak{g} by the isomorphism $T_e L_g$. Under this identification, we have that for all $g, h \in G$, $T_h L_g = \text{Id}$ and $T_h R_g = \text{Ad}(g^{-1})$, so that the tangent map becomes $T_{(a,b)}m(X_a, X_b) = \text{Ad}(b^{-1})(X_a) + X_b$. Furthermore, if $k \in K$, since $L_k(K) = K$, $T_k K \subset T_k G$ is identified with $\mathfrak{k} \subset \mathfrak{g}$. Thus by the chain rule we have

$$T_{(k_1, k_2, H)}q : \begin{array}{ccc} \mathfrak{k} \times \mathfrak{k} \times \mathfrak{a} & \rightarrow & \mathfrak{g} \\ (X_1, X_2, Y) & \mapsto & \text{Ad}(k_2)(\text{Ad}(\exp(-H))(X_1) + T_H \exp(Y)) - X_2 \end{array} .$$

We know that $\text{Ad}(k)$ is an isomorphism of \mathfrak{g} and an isomorphism of \mathfrak{k} in restriction. Furthermore, $T_H \exp : \mathfrak{a} \rightarrow \mathfrak{a}$ is also an isomorphism. Thus, the map $T_{(k_1, k_2, H)}q$ is surjective if and only if $u = \text{Ad}(k_2^{-1}) \circ T_{(k_1, k_2, H)}q \circ (\text{Id}, \text{Ad}(k_2), (T_H \exp)^{-1})$ is surjective. We have

$$u(X_1, X_2, Y) = \text{Ad}(\exp(-H))(X_1) - X_2 + Y.$$

For $\alpha \in \Sigma^+$, let $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ and $\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. From [Loo69b, Ch. VI, Prop. 1.4], we get

$$\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha = \mathfrak{m} \oplus \mathfrak{l},$$

$$\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha = \mathfrak{a} \oplus \mathfrak{b}.$$

We also get that for $\alpha \in \Sigma^+$, there exists $Z_{\alpha,1}, \dots, Z_{\alpha,m(\alpha)}$ basis of \mathfrak{g}_α , such that setting $Z_{\alpha,i}^+ = Z_{\alpha,i} + \theta(Z_{\alpha,i})$ and $Z_{\alpha,i}^- = Z_{\alpha,i} - \theta(Z_{\alpha,i})$, $\{Z_{\alpha,i}^+\}$ is a basis of \mathfrak{k}_α and $\{Z_{\alpha,i}^-\}$ is a basis of \mathfrak{p}_α .

Let also H_1, \dots, H_ℓ be a basis of \mathfrak{a} and Y_1, \dots, Y_r a basis of \mathfrak{m} . Then for $H \in \mathfrak{a}$, we have $[H, Y_i] = 0$, $[H, Z_{\alpha,i}^+] = \alpha(H)Z_{\alpha,i}^-$ and $[H, Z_{\alpha,i}^-] = \alpha(H)Z_{\alpha,i}^+$. Thus, we see that

- $u(0, 0, H_i) = H_i$,
- $u(Y_i, 0, 0) = e^{-\text{ad}(H)}(Y_i) = Y_i$,
- $u(0, Y_i, 0) = -Y_i$,
- $u(Z_{\alpha,i}^+, 0, 0) = e^{-\text{ad}(H)}(Z_{\alpha,i}^+) = \cosh(\alpha(H))Z_{\alpha,i}^+ - \sinh(\alpha(H))Z_{\alpha,i}^-$,
- $u(0, Z_{\alpha,i}^+, 0) = -Z_{\alpha,i}^+$.

Since $H \in \mathfrak{a}^+$, $\alpha(H) \neq 0$ for any $\alpha \in \Sigma^+$ and u is indeed surjective. \square

Proposition 3.10. *The map $P : G \rightarrow \overline{\mathfrak{a}^+}$ is smooth on G_r . Furthermore, for each $g \in G_r$, there exists a neighborhood U_g of g in G_r and a choice of $g \mapsto k_i(g)$ such that k_i is smooth on U_g , $i = 1, 2$ and for any $g \in U_g$, $g = k_1(g) \exp(P(g))k_2(g)^{-1}$.*

Proof. Let $\Delta(M) = \{(m, m) \mid m \in M\}$ denote the diagonal subgroup of $K \times K$. By Lemma 3.8, the map

$$\tilde{q} : \begin{array}{ccc} (K \times K)/\Delta(M) \times \mathfrak{a}^+ & \rightarrow & G_r \\ ((k_1, k_2) \bmod M, H) & \mapsto & k_1 \exp(H)k_2^{-1} \end{array}$$

is a well-defined smooth bijection between manifolds of the same dimension.

Let $p : K \times K \rightarrow (K \times K)/\Delta(M)$ be the projection. It is a surjective submersion. Let q be the submersion defined in Lemma 3.9, we have $q = \tilde{q} \circ (p, \text{Id})$. So for any $(x, H) \in (K \times K)/\Delta(M) \times \mathfrak{a}^+$, we have $T_{(x,H)}\tilde{q}$ surjective. But it is a linear map between vector spaces of the same dimension, so it is invertible. Thus, by the local inversion theorem and since \tilde{q} is bijective, \tilde{q} is a smooth diffeomorphism.

Let $(x, P) : G_r \rightarrow (K \times K)/\Delta(M) \times \mathfrak{a}^+$ be a smooth inverse. We get that P is a smooth map. From [Lee03, Proposition 4.26], since p is a submersion, any $(k_1, k_2) \in K \times K$ is in the image of a smooth local section of p . Let $g \in G_r$, since p is surjective, $x(g) = p(k_1, k_2)$. There exists a neighborhood V of $x(g)$ and a smooth section $s = (s_1, s_2) : V \rightarrow K \times K$ such that $s(x(g)) = (k_1, k_2)$. Let $U = x^{-1}(V)$ neighborhood of g , then $k_i = s_i \circ x$ is a smooth map on U and $g = k_1(g) \exp(P(g))k_2(g)^{-1}$. \square

Corollary 3.11. *Let φ be a K -bi-invariant function on G , then $\varphi \in C^{(r,\delta)}(G_r)$ if and only if $\varphi \circ \exp \in C^{(r,\delta)}(\mathfrak{a}^+)$.*

Furthermore, if $(\varphi_i)_{i \in I}$ is a family of K -bi-invariant function on G , then (φ_i) is bounded in $C^{(r,\delta)}(G_r)$ if and only if $(\varphi_i \circ \exp)$ is bounded in $C^{(r,\delta)}(\mathfrak{a}^+)$.

Proof. Since \exp is smooth, the first implication is a consequence of Lemma 1.3. For the converse, assume $\psi = \varphi \circ \exp|_{\mathfrak{a}^+} \in C^{(r,\delta)}(\mathfrak{a}^+)$. By the previous proposition, the map P is smooth on G_r and $\varphi = \psi \circ P$ by K -bi-invariance, thus $\varphi \in C^{(r,\delta)}(G_r)$ by Lemma 1.3. \square

3.2.2 Boundedness of positive definite spherical functions

In this section, we study the spherical functions of the pair (G, K) and their Hölder norms. We will use the estimates obtained by Duistermaat, Kolk and Varadarajan in [DKV83] using the method of stationary phase. The proofs are very similar to the case of Cartan motion groups in Section 3.1.2, but the computations are more involved due to the phase function being more complicated. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we will denote $\psi_\lambda = \varphi_\lambda \circ \exp|_{\mathfrak{a}}$ the spherical function on the Lie algebra.

Theorem 3.12. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Let C be a bounded subset of \mathfrak{a}^* . Then the family of spherical functions ψ_λ of (G, K) with $\text{Im } \lambda \in C$ is bounded in $C^{(r, \delta)}(\mathfrak{a}^+)$.*

Remark 3.13. By Proposition 1.25, this implies that the family of bounded spherical functions, thus the subfamily of positive-definite spherical functions, is bounded in $C^{(r, \delta)}(\mathfrak{a}^+)$.

Proof. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we write $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$ and for $t \in \mathbb{R}$, $\lambda_t = t\xi + i\eta$. Denote $f(\xi, \eta, t, Y, k) = e^{(i\lambda_t - \rho)(H(\exp(Y)k))} \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^* \times \mathbb{R} \times \mathfrak{a} \times K)$. We denote D the operator which differentiates a function with respect to the variable $Y \in \mathfrak{a}$. Let $s \in \mathbb{N}$. Then for any $Y \in \mathfrak{a}$, $X = (X_1, \dots, X_s) \in \mathfrak{a}^s$,

$$D^s \psi_{\lambda_t}(Y)(X) = \int_K D^s f(\xi, \eta, t, Y, k)(X) dk. \quad (3.25)$$

By induction on s , there is a polynomial $P \in C^\infty(\mathfrak{a}^* \times \mathfrak{a}^* \times \mathfrak{a} \times K \times \mathfrak{a}^s)[t]$ of degree s such that

$$D^s f(\xi, \eta, t, Y, k)(X) = P(t) e^{it\xi(H(\exp(Y)k))}. \quad (3.26)$$

For $0 \leq j \leq s$, let $g_j(\xi, \eta, Y, X) \in C^\infty(K)$ be defined by

$$g_j = \frac{1}{j!} \frac{d^j}{dt^j} \left((D^s f) e^{-it\xi(H(\exp(Y)k))} \right) \Big|_{t=0}. \quad (3.27)$$

Then we have

$$D^s \psi_{\lambda_t}(Y)(X) = \sum_{j=0}^s t^j \int_K e^{it\xi(H(\exp(Y)k))} g_j(\xi, \eta, Y, X)(k) dk. \quad (3.28)$$

Let S be the unit sphere in \mathfrak{a}^* and L a compact subset of \mathfrak{a}^+ . Up to replacing L by its convex hull, which is still a compact subset of \mathfrak{a}^+ by Carathéodory's theorem and the convexity of \mathfrak{a}^+ , we can assume that L is convex. We consider $C^\infty(K)$ endowed with the topology given by the family of seminorms $p_i : g \mapsto \sup_{k \in K} \|D^i g(k)\|$.

For $(Y, \xi) \in L \times S$, by [DKV83, Prop. 9.2], there exists a neighborhood $V_{Y, \xi}$ of Y, ξ in $\mathfrak{a} \times \mathfrak{a}^*$ and a continuous seminorm $v_{Y, \xi}$ on $C^\infty(K)$ such that for any $(Y', \xi') \in V_{Y, \xi}$, $g \in C^\infty(K)$, $t \geq 1$,

$$\left| \int_K e^{it\xi'(H(\exp(Y')k))} g(k) dk \right| \leq v_{Y, \xi}(g) t^{-n(\xi)/2} \leq v_{Y, \xi}(g) t^{-\kappa(G)}. \quad (3.29)$$

Now write $\mathfrak{a}^* \times C = ([0, 1]S \times C) \cup (\mathbb{R}_{\geq 1}S \times C)$.

For any $(\xi, \eta) \in S \times C$, $0 \leq t \leq 1$, $Y \in L$, then (3.28) implies

$$\begin{aligned} \|D^s \psi_{\lambda_t}(Y)\| &= \sup_{\|X_i\|=1} |D^s \psi_{\lambda_t}(Y)(X)| \\ &\leq \sup_{\|X_i\|=1} \sum_{j=0}^s t^j \int_K \|g_j(\xi, \eta, Y, X)\|_{\infty} dk \\ &\leq \sup_{\substack{\|X_i\|=1 \\ Y \in L, \xi \in S, \eta \in C, k \in K}} \sum_{j=0}^s |g_j(\xi, \eta, Y, X)(k)| \\ &\leq C_{L,s,C} \end{aligned} \tag{3.30}$$

where $C_{L,s,C} > 0$ is independent from $Y \in L$, $\xi \in S$, $\eta \in C$, using that g_j is smooth hence bounded on compact subsets.

On the other hand, write $L \times S = \bigcup_{(Y,\xi) \in L \times S} V_{Y,\xi}$. Since $L \times S$ is compact, there exists a finite subcover $U_{Y_1, \xi_1}, \dots, U_{Y_n, \xi_n}$. Then for any $(\xi, \eta) \in S \times C$, $t \geq 1$, $Y \in L$,

$$\begin{aligned} \|D^s \psi_{\lambda_t}(Y)\| &= \sup_{\|X_i\|=1} |D^s \psi_{\lambda_t}(Y)(X)| \\ &\leq \sup_{\|X_i\|=1} \sum_{j=0}^s t^j \left| \int_K e^{it\xi(H(\exp(Y)k))} g_j(\xi, \eta, Y, X)(k) dk \right| \\ &\leq \sum_{j=0}^s t^j \max_{1 \leq p \leq n} \sup_{\substack{\|X_i\|=1 \\ Y \in L, \xi \in S, \eta \in L}} v_{Y_p, \xi_p}(g_j(\xi, \eta, Y, X)) t^{-\kappa(G)} \\ &\leq D_{L,s,C} t^{s-\kappa(G)} \end{aligned} \tag{3.31}$$

where $D_{L,s,C} > 0$ is independent from $Y \in L$, $\xi \in S$, $\eta \in C$. Again, the last line comes from the fact that g_j is smooth hence any of its differentials is bounded on compact subsets, and the topology on $C^\infty(K)$ is given by the seminorms p_i , thus if v is a continuous seminorm, there is i such that $v(g) \leq Cp_i(g) = C \sup_{k \in K} \|D^i g(k)\|$.

Thus combining (3.30) and (3.31), for any λ such that $\text{Im } \lambda \in C$, $H \in L$, $s \leq r$,

$$\|D^s \psi_{\lambda}(Y)\| \leq \max(C_{L,s,C}, D_{L,s,C}) = M_{L,s}. \tag{3.32}$$

Thus the differentials of the family of spherical functions are bounded independently on λ such that $\text{Im } \lambda \in C$, up to order r . If $r = \kappa(G)$, the proof is complete.

Otherwise, $\kappa(G) - r = \frac{1}{2}$. Then using (3.31) for $s = r$ and $s = r + 1$, we show that for any $x, y \in L$, $\xi \in S, \eta \in C$, $t \geq 1$, we have on the one hand

$$\|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq \|D^r \psi_{\lambda_t}(x)\| + \|D^r \psi_{\lambda_t}(y)\| \leq 2D_{L,r,C} t^{-1/2} \tag{3.33}$$

and on the other hand, since L is convex, we get by the mean value theorem that

$$\|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq \left(\sup_{Y \in L} \|D^{r+1} \psi_{\lambda_t}(Y)\| \right) \|x - y\| \leq D_{L,r+1,C} t^{1/2} \|x - y\|. \tag{3.34}$$

Thus, combining (3.33) and (3.34) yields

$$\|D^r \psi_{\lambda_t}(x) - D^r \psi_{\lambda_t}(y)\| \leq (2D_{L,r,C} D_{L,r+1,C})^{1/2} \|x - y\|^{1/2}. \tag{3.35}$$

Hence, setting $M_L = \max\left((2D_{L,r,C} D_{L,r+1,C})^{1/2}, C_{L,r+1,C}(\text{diam } L)^{1/2}\right)$, we have that for any λ such that $\text{Im } \lambda \in C$, $x, y \in L$,

$$\|D^r \psi_{\lambda}(x) - D^r \psi_{\lambda}(y)\| \leq M_L \|x - y\|^{\kappa(G)-r}. \quad \square$$

Remark 3.14. If $\text{Im } \lambda$ is allowed to be unbounded, then the functions are not even bounded in $C(\mathfrak{a}^+)$.

We will now show that this result is optimal. The bounds used in the previous theorem are not sharp in general, but for a subfamily where $n(\lambda) = 2\kappa(G)$, they are. Thus, we will show that this particular subfamily is already unbounded in Hölder spaces of higher regularity. For this, we will again make use of Lemma 3.5.

Theorem 3.15. *We keep the notations of Theorem 3.12. For any $\delta' > \delta$, the family of positive definite spherical functions of (G, K) is not bounded in $C^{(r, \delta')}(\mathfrak{a}^+)$.*

Proof of Theorem 3.15. Consider $\lambda \in \mathfrak{a}^*$ such that $n(\lambda) = 2\kappa(G)$, and such that $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Sigma^+$. By Proposition 1.25, $\varphi_{t\lambda}$ is positive definite for any $t \in \mathbb{R}$. As in (3.28), for any $Y \in \mathfrak{a}$, $t \geq 1$, $X \in \mathfrak{a}^r$,

$$D^r \psi_{t\lambda}(Y)(X) = \sum_{j=0}^r t^j \int_K e^{it\lambda(H(\exp(Y)k))} g_j(Y, X)(k) dk. \quad (3.36)$$

Let $I_j(Y, X, t) = \int_K e^{it\lambda(H(\exp(Y)k))} g_j(Y, X)(k) dk$. Let W_λ denote the stabiliser of λ under the action of the Weyl group W and K_Y, K_λ be the centralisers of Y, λ in K . Let also

$$\Sigma^+(\lambda) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \lambda \rangle \neq 0\}$$

and

$$\sigma_w = - \sum_{\alpha \in \Sigma^+(\lambda)} m(\alpha) \text{sgn}(\langle \alpha, \lambda \rangle) (w\alpha)(Y).$$

Let d_0k denote the Riemannian measure on K induced by the bi-invariant metric defined by the Killing form on \mathfrak{k} . Let $\text{Vol}(K) = \int_K d_0k$. We also denote by d_0k the induced Riemannian measure on the submanifold $K_a w K_\lambda$ - the measure coming from the restriction of the Riemannian metric of K to a Riemannian metric on the submanifold. For $w \in W$, $g \in C^\infty(K)$, $Y \in \mathfrak{a}^+$, set

$$c_{w, \alpha}(g) = e^{i\frac{\pi}{4}\sigma_w} \prod_{\alpha \in \Sigma^+(\lambda)} \left| \frac{\langle \alpha, \lambda \rangle}{4\pi} \left(1 - e^{-2(w\alpha)(Y)}\right)^{-\frac{m(\alpha)}{2}} \right| \frac{1}{\text{Vol}(K)} \int_{K_a w K_\lambda} g(k) d_0k. \quad (3.37)$$

Then, by [DKV83, Thm. 9.1], for any $Y \in \mathfrak{a}^+$, there is a neighborhood U_Y of Y in \mathfrak{a}^+ and $D(Y) > 0$ such that for any $0 \leq j \leq r$, $t \geq 1$, $Y' \in U_Y$ and X with $\|X_i\| = 1$ for all i ,

$$\left| I_j(Y', X, t) - \sum_{W/W_\lambda} e^{it(w\lambda)(Y')} t^{-\kappa(G)} c_{w, Y'}(g_j(Y', X)) \right| \leq D(Y) t^{-\kappa(G)-1}. \quad (3.38)$$

We use that g_j is smooth in all variables hence bounded on compacts and that the bound is uniform in the parameter Y' of the phase function.

In particular, combining (3.36) with (3.38) for $0 \leq j < r$, for any Y there is a neighborhood V_Y of Y and a constant $C(Y)$ such that for any $t \geq 1$, $Y' \in V_Y$ and X with $\|X_i\| = 1$,

$$|D^r \psi_{t\lambda}(Y')(X) - t^r I_r(Y', X, t)| \leq C(Y) t^{-1}. \quad (3.39)$$

The end of the proof relies on Lemma 3.5 and is nearly identical to the second part of the proof of Theorem 3.6.

For X fixed with $\|X_i\| = 1$, let $S_t(x) = \sum_{W/W_\lambda} e^{it(w\lambda)(x)} c_{w,x}(g_r(x, X))$. Combining (3.38) and (3.39), if $t \geq 1$, and $x, y \in U_Y \cap V_Y$,

$$\begin{aligned}
t^{-\delta} |S_t(x) - S_t(y)| &\leq t^r |t^{-\kappa(G)} S_t(x) - I_r(x, X, t)| + t^r |I_r(x, X, t) - I_r(y, X, t)| \\
&\quad + t^r |I_r(y, X, t) - t^{-\kappa(G)} S_t(y)| \\
&\leq 2D(Y)t^{-\delta-1} + |t^r I_r(x, H, t) - D^r \psi_{t\lambda}(x)(X)| \\
&\quad + |D^r \psi_{t\lambda}(x)(X) - D^r \psi_{t\lambda}(y)(X)| \\
&\quad + |D^r \psi_{t\lambda}(y)(X) - t^r I_r(y, X, t)| \\
&\leq 2D(Y)t^{-\delta-1} + 2C(Y)t^{-1} + |D^r \psi_{t\lambda}(x)(X) - D^r \psi_{t\lambda}(y)(X)| \\
&\leq \|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\| + 2(C(Y) + D(Y))t^{-1}.
\end{aligned} \tag{3.40}$$

Now the functions $c_{w,x}(g_r(x, X))$ are all zero at x if and only if $g_r(x, X) = 0$ almost everywhere on $\cup K_a w K_\lambda$. Let $f_k(x) = H(\exp(x)k)$, then

$$g_r(x, X)(k) = e^{-\rho(H(\exp(x)k))} \prod_{i=1}^r \lambda(Df_k(x)(X_i)).$$

If $P_{\mathfrak{a}}$ denote the orthogonal projection onto \mathfrak{a} , by [DKV83, Section 5] we have

$$Df_k(x)(X_i) = P_{\mathfrak{a}}(\text{Ad}(t(\exp(x)k)^{-1})(X_i))$$

where $t(g) = a(g)n(g)$ in the Iwasawa decomposition (see (1.9)). The function $k \mapsto f_k(x)$ is left K_x -invariant, and by [DKV83, Prop. 5.6], $k \mapsto \lambda(f_k(x))$ is right K_λ -invariant as $H_\lambda \in \overline{\mathfrak{a}^+}$. Thus, $g_r(x, X)$ is constant on $K_x w K_\lambda$ for any w . Given any open subset of \mathfrak{a}^+ , we can choose x, X such that $g_r(x, X)(e) \neq 0$. Thus the hypotheses of Lemma 3.5 hold for the family of functions $f_w : x \mapsto c_{w,x}(g_r(x, X))$, for $U = \mathfrak{a}^+$. Let C, d, x, V be given by Lemma 3.5, $W_x = x + V$ such that for any $y \in W_x$, $m \in \mathbb{N}$, $N \geq \frac{d}{\|x-y\|}$,

$$\sum_{t=m}^{m+N-1} |S_t(x) - S_t(y)|^2 \geq CN. \tag{3.41}$$

From now on, we choose $Y = x$ given above. Let $M = 4(C(x) + D(x))^2$, we get from (3.40) that for any $t \geq 1$, $y \in U_x \cap V_x \cap W_x$,

$$\frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \leq \|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\|^2 + Mt^{-2} \tag{3.42}$$

Assume now that the family of positive definite spherical functions of (G, K) is bounded in $C^{(r, \delta')}(\mathfrak{a}^+)$ for $\delta' > \delta$. In particular, up to reducing $U_x \cap V_x \cap W_x$ to a bounded subset of diameter L if necessary, there is $D > 0$ such that for any $y \in U_x \cap V_x \cap W_x$ and $t \geq 1$,

$$\|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\| \leq D\|x - y\|^{\delta'} \tag{3.43}$$

For y fixed, set m, N such that

$$\frac{1}{\|x - y\|^{\delta'}} \leq m \leq \frac{1}{\|x - y\|^{\delta'}} + 1 \tag{3.44}$$

and

$$\frac{d}{\|x - y\|} \leq N \leq \frac{d}{\|x - y\|} + 1. \tag{3.45}$$

Combining (3.41), (3.42) and (3.43) gives

$$\begin{aligned} \frac{CN}{2(m+N)^{2\delta}} &\leq \sum_{t=m}^{m+N-1} \frac{t^{-2\delta}}{2} |S_t(x) - S_t(y)|^2 \\ &\leq \sum_{t=m}^{m+N-1} \left(\|D^r \psi_{t\lambda}(x) - D^r \psi_{t\lambda}(y)\|^2 + Mt^{-2} \right) \\ &\leq ND^2 \|x - y\|^{2\delta'} + \frac{MN}{m^2} \end{aligned} \quad (3.46)$$

thus

$$\frac{C}{2(m+N)^{2\delta}} \leq D^2 \|x - y\|^{2\delta'} + \frac{M}{m^2} \leq (D^2 + M) \|x - y\|^{2\delta'} \quad (3.47)$$

by (3.44). But by (3.44) and (3.45), we have

$$\begin{aligned} m + N &\leq \frac{d}{\|x - y\|} + 1 + \frac{1}{\|x - y\|^{\delta'}} + 1 \\ &\leq \frac{1}{\|x - y\|} \left(d + 2\|x - y\| + \|x - y\|^{1-\delta'} \right) \\ &\leq \frac{1}{\|x - y\|} \left(d + 2L + L^{1-\delta'} \right) \end{aligned} \quad (3.48)$$

hence (3.47) becomes

$$\frac{C}{2(d + 2L + L^{1-\delta'})^{2\delta}} \|x - y\|^{2\delta} \leq (D^2 + M) \|x - y\|^{2\delta'}. \quad (3.49)$$

Since (3.49) holds for any $y \in U_x \cap V_x \cap W_x$ with the constant involved independent from y and $\delta' > \delta$, we get a contradiction as y goes to x (which is possible because $0 \in \bar{V}$ hence $x \in \overline{U_x \cap V_x \cap W_x}$). \square

We may now combine our results to prove Theorem E.

Theorem 3.16. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Then any K -finite matrix coefficient of a unitary representation of G is in $C^{(r,\delta)}(G_r)$. Furthermore, for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of G that is not in $C^{(r,\delta')}(G_r)$.*

Proof. By Proposition 3.10, (G_r, K) has a well-behaved KAK decomposition thus the problem is reduced to K -bi-invariant coefficients by Theorem 1.15. The result then follows from Lemma 1.10, Corollary 3.11 and Theorems 3.12 and 3.15. \square

Remark 3.17. If $a = \exp(Y) \in G \setminus G_r$, the behavior around a is more complicated. Indeed by [DKV83], the critical set of the phase functions becomes $\bigcup_W K_a w K_\lambda$. If $\Sigma_w(\lambda, a) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \lambda \rangle (w\alpha)(Y) \neq 0\}$, the the codimension of the critical set is $n_w = \sum_{\alpha \in \Sigma_w(\lambda, a)} m(\alpha)$. In particular, n_w can be smaller than $2\kappa(G)$ so that the regularity of matrix coefficients around a is strictly worse than on G_r .

3.3 Perspectives

The regularity results obtained in this chapter only apply to unitary representations on Hilbert spaces. A question is, can such regularity results be obtained for other classes of representations, say uniformly bounded representations on a Hilbert space? A result

in this direction is [PRdlS22, Theorem 4.6], where it is shown that $SO(n)$ -bi-invariant coefficients of uniformly bounded representations (and even $SO(n)$ -bi-invariant Fourier multipliers) of $SO(n, 1)$ are α_n -Hölder continuous for some real number $\alpha_n > 0$. It is interesting to notice that $\alpha_n < \kappa(SO(n, 1))$, although this result might not be optimal. Any unitary coefficient is of course a Fourier multiplier, but the converse is not true (except when G is amenable). Thus, having an optimal regularity for Fourier multipliers or uniformly bounded representations which is not that of unitary coefficients would imply the construction of Fourier multipliers which behaves very differently from unitary coefficients.

We may ask whether this theorem can be extended to other noncompact symmetric pairs. The proof of [PRdlS22] uses regularity results on $SO(n)$, and embedding of $SO(n)$ into $SO(n, 1)$. Since we studied regularity for compact groups other than $SO(n)$ in Chapter 2 (and will improve this in Chapter 4), an idea could be to look at their embeddings in various groups.

Another direction is to study isometric representations, but on Banach spaces. In any case, a first and fundamental difficulty in all these directions is the lack of a decomposition theory for such representations.

We can also develop further the work done in the flat case. Given any compact Lie group K and any orthogonal representation $\pi : K \rightarrow O(V)$ on a finite-dimensional real vector space V , we can form the semi-direct product $H = V \rtimes K$. In general, there is no reason for H to be the Cartan motion group of a semisimple Lie group. However, (H, K) remains a Gelfand pair, so we may ask if we can find the optimal regularity of K -finite matrix coefficients of unitary representations of H . By Mackey's theory ([Mac49]), we can still find the irreducible representations of H with a nonzero K -invariant vector and thus compute the positive definite spherical functions. On V , they can be expressed by

$$\varphi_\lambda(v) = \int_K e^{i\langle \lambda, \pi(k)v \rangle} dk$$

with $\lambda \in V$, and $\varphi_\lambda = \varphi_{\lambda'}$ if λ, λ' are in the same K -orbit. So the spherical functions are still oscillatory integrals. However, the lack of semisimple structure makes it difficult to understand the orbits, and the critical points of the phase functions.

Restrict to $K = SO(3)$ and π irreducible. In the case of Cartan motion groups, the representation on \mathfrak{p} is irreducible if and only if the ambient semisimple Lie group G is simple. By the classification of simple Lie groups, we see that the only irreducible representations of $SO(3)$ which appear in Cartan motion groups are the 3-dimensional and the 5-dimensional representations. Thus, the first case outside the world of Cartan motion groups to consider seems to be the 7-dimensional representation of $SO(3)$ on the space V_3 of real harmonic homogeneous polynomials of degree 3 in 3 variables. In this specific case, we may be able to make the necessary computations without the additional semisimple structure.

Chapter 4

A local answer to the compact conjecture

The goal of this chapter is to improve the results obtained in Chapter 2 and make a significant step towards Conjecture C. To do so, we will use ideas from Chapter 3 and the stationary phase approximation. Since in the compact case, spherical functions do not have such a well-behaved integral representation, the idea is to use analytic continuation (Lemma 4.4). We then prove Theorem F in two parts: first we obtain that the conjectured regularity is an upper bound for the optimal regularity (Corollary 4.8), and then that the conjecture is valid at least in some open subset (Corollary 4.15). Thanks to previous results in Section 2.2.2, we verify the full conjecture for several new families of groups (Corollary 4.10). Finally, we explain in Section 4.3 how to extend our results to pairs (U, K) where U/K is not simply connected. The results presented in this chapter corresponds to [Dum24b, Section 5].

4.1 An upper bound on regularity

In this chapter, we will use duality to improve our previous results on compact groups with the methods used in Chapter 3. We first recall what we explained in Section 1.8 and introduce some new notations (more details in [Cle88, Hel79]). If \mathfrak{g} is a semisimple real Lie algebra, we introduced a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into eigenspaces of a Cartan involution θ . Let $G_{\mathbb{C}}$ be the simply connected Lie group whose Lie algebra is the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} . Let G, K be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to the subalgebras $\mathfrak{g}, \mathfrak{k}$. Consider $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and U the corresponding analytic subgroup of $G_{\mathbb{C}}$. Then U is a maximal compact subgroup of $G_{\mathbb{C}}$ and is simply connected. Consider also $K_{\mathbb{C}}$ the analytic subgroup corresponding to $\mathfrak{k}_{\mathbb{C}}$. Finally, recall that \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} and $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$. Let $A, A_{\mathbb{C}}, N, N_{\mathbb{C}}$ be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{a}, \mathfrak{a}_{\mathbb{C}}, \mathfrak{n}, \mathfrak{n}_{\mathbb{C}}$.

The involution θ extends to $\mathfrak{g}_{\mathbb{C}}$ and thus induces an involution of $G_{\mathbb{C}}$, also denoted θ . The subgroup K is the subgroup of fixed points of θ in U , so (U, K) is a symmetric Gelfand pair and the symmetric space $M = U/K$ is the compact dual of G/K . Since the Killing form of $\mathfrak{g}_{\mathbb{C}}$ restricted to $\mathfrak{g} \times \mathfrak{g}$ coincides with the Killing form of \mathfrak{g} , we will denote it by $\langle \cdot, \cdot \rangle$ both on \mathfrak{g} and its complexification. It must be noted that $\langle \cdot, \cdot \rangle$ is \mathbb{C} -bilinear on $\mathfrak{g}_{\mathbb{C}}$ and not sesquilinear.

Such pairs (U, K) were studied in Chapter 2. The optimal regularity of K -finite coefficients of U was found in some specific cases and a conjecture was given in the general case. In what follows, we will extend the results using methods similar to what we did above in the non-compact setting.

Let Q be the connected component of $\mathfrak{a}_r = \{H \in \mathfrak{a} \mid \forall \alpha \in \Sigma, \alpha(H) \notin \pi\mathbb{Z}\}$ contained in \mathfrak{a}^+ and whose closure contains 0. Then, there is a KAK decomposition in the group U by Proposition 2.43. Let us restate this decomposition in the notations of duality.

Proposition 4.1. *For any $u \in U$, there exists a decomposition*

$$u = k_1(u) \exp(iP(u)) k_2(u)^{-1}$$

where $k_1(u), k_2(u) \in K$ and $P(u) \in \overline{Q}$. The map $u \mapsto P(u)$ is smooth on the set $U_r = K \exp(iQ)K$. Furthermore, for each $u \in U_r$, there exists a neighborhood V_u of u in U_r and a choice of $u \mapsto k_i(u)$ such that k_i is smooth on V_u , $i = 1, 2$.

The set U_r is a dense open subset of U and we call it the set of regular points. This set will play the same role as G_r in the non-compact case.

As in the previous chapters, we want to study the spherical functions of the pair (U, K) . Let \widehat{U}_K denote the set of classes of irreducible finite-dimensional representations of U with a non-zero K -invariant vector. Let us restate the Cartan-Helgason theorem (Section 1.5) which gives a parametrization of \widehat{U}_K (and thus spherical functions of (U, K)) by a subset of $\mathfrak{a}_\mathbb{C}^*$. Note that since U is compact, any spherical function is positive-definite ([vD09, Thm. 6.5.1]).

Theorem 4.2 (Cartan-Helgason). *Let $\Lambda = \{\mu \in \mathfrak{a}^* \mid \forall \alpha \in \Sigma, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}\}$. Then the map which sends a representation to its highest weight is a bijection from \widehat{U}_K onto Λ .*

There exists $\mu_1, \dots, \mu_\ell \in \mathfrak{a}^*$ such that

$$\frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, 2\alpha_j \notin \Sigma \\ 2 & \text{if } i = j, 2\alpha_j \in \Sigma \end{cases}$$

The elements $(\mu_i)_{1 \leq i \leq \ell}$ are called fundamental weights and $\Lambda = \{\sum m_i \mu_i, m_i \in \mathbb{N}\}$ (see [Vre76]).

Let $\mu \in \Lambda$ and π_μ an irreducible finite-dimensional representation of U with highest weight μ . Let e_K be a unit K -invariant vector. Then $\psi_\mu : u \mapsto \langle \pi_\mu(u)e_K, e_K \rangle$ is a spherical function of (U, K) . Since π_μ is a finite-dimensional representation, it is smooth and its differential induces a representation of \mathfrak{u} , which extends to $\mathfrak{u}_\mathbb{C} = \mathfrak{g}_\mathbb{C}$ and is itself the differential of a representation of the simply connected group $G_\mathbb{C}$. Thus, π_μ extends to a representation of $G_\mathbb{C}$, so ψ_μ is defined on all of $G_\mathbb{C}$. Given the notations of Section 1.6, $\psi_\mu|_G = \varphi_{-i(\mu+\rho)}$.

Thus on G , we know that ψ_μ has an integral representation. But since the Iwasawa decomposition does not extend to $G_\mathbb{C}$, the integral does not have a meaning outside of G . However, even if $K_\mathbb{C} \times A_\mathbb{C} \times N_\mathbb{C} \rightarrow G_\mathbb{C}$ is not a diffeomorphism, it is a diffeomorphism in a neighborhood of the identity $e \in G_\mathbb{C}$ so we can still work there. The following lemma is [Cle76, Lemme 1].

Lemma 4.3. *There exists a neighborhood V of e in $G_\mathbb{C}$ which is invariant by conjugation by K and analytic maps $\kappa : V \rightarrow K_\mathbb{C}$, $n : V \rightarrow N_\mathbb{C}$ and $H : V \rightarrow \mathfrak{a}_\mathbb{C}$ such that*

1. $H(e) = 0$,
2. $\forall g \in V, g = \kappa(g) \exp H(g)n(g)$.

The map H coincides with the Iwasawa projection on $G \cap V$. Since $H(k^{-1}gk) = H(gk)$ for any $g \in G, k \in K$, we can extend the expression of ψ_μ on $G \cap V$ to all of V by analytic continuation ([Cle76, Lemme 3]).

Lemma 4.4. *Let $\mu \in \Lambda$. For any $g \in V, \psi_\mu(g) = \int_K e^{\mu(H(k^{-1}gk))} dk$.*

Unlike the non-compact case, the phase function is now complex-valued. However, for any $u \in U \cap V, \operatorname{Re} \mu(H(k^{-1}uk)) \leq 0$ ([Cle88, Coro. 2.4]) which is the condition to apply the method of stationary phase (see [MS75]).

We will now state the version of the stationary phase approximation we will use, from [Cha74] and [MS75].

Theorem 4.5. *Let (Z, g) be a compact Riemannian manifold of dimension d , dz its volume measure and U an open subset of \mathbb{R}^n . Let $f \in C^\infty(Z)$ and $\phi \in C^\infty(Z \times U)$ be complex-valued functions. Let W_a be the set of critical points of $\phi_a : z \mapsto \phi(z, a)$ for $a \in U$ and assume that $W_a = W$ for any a . Assume also that W is finite, and for any $w \in W$, the Hessian of ϕ_a at w is non-degenerate. Furthermore, suppose that $\operatorname{Re} \phi \leq 0$, with equality at critical points $w \in W$. Set*

$$I(f, a, t) = \int_Z e^{t\phi(z, a)} f(z) dz$$

and fix $a_0 \in U$. Then there exists a semi-norm ν on $C^\infty(Z)$ and an open neighborhood $U' \subset U$ of a_0 such that for any $t \geq 1, a \in U', f \in C^\infty(Z)$,

$$\left| I(f, a, t) - \sum_{w \in W} e^{t\phi(w, a)} t^{-d/2} f(w) \left(\frac{(2\pi)^d}{\det(-\operatorname{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq \nu(f) t^{-\frac{d}{2}-1}$$

where the square root is taken as the branch of the square root which is deformed to 1 under the homotopy $(1-s)(-\operatorname{Hess}_{\phi_a}(w)) + s \operatorname{Id}$.

Proof. We can cover Z by a finite number of chart open subsets $Z_j, j \in J$ which contains at most one element of W . We may assume that $\|D\phi_a(z)\|$ is bounded below on Z_j which does not contain a critical point. Using a partition of unity subordinated to this open cover, we can write

$$I(f, a, t) = \sum_{j \in J} I_j(f, a, t)$$

where

$$I_j(f, a, t) = \int_{Z_j} e^{t\phi(z, a)} f_j(z) dz$$

and $\sum f_j(z) = f(z)$ for any $z \in Z$. In the chart Z_j , consider the local coordinates given by $H_j : Z_j \rightarrow \mathbb{R}^d$, chosen such that if $w \in W \cap Z_j, H_j(w) = 0$. Let $G = \det(g_{i,j})$ where $g_{ij}(z) = g_z(\partial_i, \partial_j)$. Then by definition of the volume measure on Z , we have

$$I_j(f, a, t) = \int_{\mathbb{R}^d} e^{\phi(H_j^{-1}(x), a)} f_j(H_j^{-1}(x)) \sqrt{G(H_j^{-1}(x))} dx.$$

First, if Z_j has no critical points, by [Hör83, Thm. 7.7.1], for any $n > 0$, there exists $C, C' > 0$ such that

$$|I_j(f, a, t)| \leq \frac{C}{t^n} \sum_{|k| \leq n} \sup \|D^k(f_j \sqrt{G})(x)\| \leq \frac{C'}{t^n} \sum_{|k| \leq n} \sup \|D^k(f)(x)\|.$$

We used Leibniz formula to replace $f_j \sqrt{G}$ by f , up to changing the constant C to some C' taking into account norms of differentials of G and of the partition of unity.

If $w \in Z_j$ is critical, by [MS75, Thm. 2.3], [Hör83, Thm. 7.7.5], there exists an open neighborhood $U_j \subset U$ of a_0 such that for any $a_0 \in U_j$,

$$\left| I_j(f, a, t) - e^{t\phi_a(w)} t^{-d/2} f_j(w) \sqrt{G(w)} \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a \circ H_j^{-1}}(0))} \right)^{\frac{1}{2}} \right| \leq v_j(f) t^{-\frac{d}{2}-1}$$

where the square root is taken as in the statement of the theorem. Again, the semi-norm v_j should be applied to $f_j \sqrt{G}$ instead of f , but since it is defined as a differential operator, by Leibniz formula the inequality remains true with f up to changing the semi-norm.

For $u, v \in T_0 \mathbb{R}^d$, at the critical point we have that

$$\text{Hess}_{\phi_a \circ H_j^{-1}}(0)(u, v) = \text{Hess}_{\phi_a}(w)(DH_j^{-1}(0)u, DH_j^{-1}(0)v).$$

Fix an orthonormal basis of $T_w Z$ with respect to the inner product g_w and consider the canonical basis $\frac{\partial}{\partial x_i}|_0$ of $T_0 \mathbb{R}^d$, then

$$\det(-\text{Hess}_{\phi_a \circ H_j^{-1}}(0)) = \det(-\text{Hess}_{\phi_a}(w)) \left(\det DH_j^{-1}(0) \right)^2.$$

But since $\partial_i|_w = DH_j^{-1}(0) \left(\frac{\partial}{\partial x_i}|_0 \right)$, we also have that $(g_{ij}(w))_{1 \leq i, j \leq n} = {}^t A A$ where A is the matrix of $DH_j^{-1}(0)$ in the previous bases, thus $G(w) = \det(DH_j^{-1}(0))^2$. Thus, the previous inequality becomes

$$\left| I_j(f, a, t) - e^{t\phi_a(w)} t^{-d/2} f_j(w) \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq v_j(f) t^{-\frac{d}{2}-1}.$$

Thus by triangular inequality, setting $U' = \cap U_j$, there is a semi-norm ν on $C^\infty(Z)$ such that for any $f \in C^\infty(Z)$, $a \in U'$, and $t \geq 1$,

$$\left| I(f, a, t) - \sum_{w \in W} e^{t\phi(w, a)} t^{-d/2} f(w) \left(\frac{(2\pi)^d}{\det(-\text{Hess}_{\phi_a}(w))} \right)^{\frac{1}{2}} \right| \leq \nu(f) t^{-\frac{d}{2}-1}$$

which concludes the proof. \square

We now compute the critical points and the Hessian of the phase function to make use of the previous theorem. This is mostly an application of the results of [DKV83] on G that we used in Section 3.2.2 and analytic continuation arguments.

For $a \in V$, consider the phase function

$$F_{a, \mu} : \begin{array}{ccc} K & \rightarrow & \mathbb{C} \\ k & \mapsto & \mu(H(k^{-1}ak)) \end{array} .$$

For any k , the map $a \mapsto F_{a,\mu}(k)$ is an analytic continuation of the phase function studied in [DKV83, Section 4]. Denote $H_\mu \in \mathfrak{a}$ the unique vector such that for any $H \in \mathfrak{a}_\mathbb{C}$, $\mu(H) = \langle H, H_\mu \rangle$. Recall that K_μ is the centralizer in K of H_μ . For $\mu \in \Lambda$, $H_\mu \in \overline{\mathfrak{a}^+}$. Then, by [DKV83, Proposition 5.6] and analytic continuation, the map $F_{a,\mu}$ is right K_μ -invariant.

For $x \in G$, let

$$\theta_x : \begin{array}{ccc} K & \rightarrow & K \\ k & \mapsto & \kappa(xk) \end{array} .$$

By uniqueness of the Iwasawa decomposition on G , it is clear that for any $x \in G$ and $k \in K$, $\kappa(xk) = k\kappa(k^{-1}xk)$. Now for $x \in V$, since we can extend κ analytically on V by Lemma 4.3 and V is K -invariant, the element $k\kappa(k^{-1}xk) \in K_\mathbb{C}$ is well-defined and the formula

$$\theta_x : \begin{array}{ccc} K & \rightarrow & K_\mathbb{C} \\ k & \mapsto & k\kappa(k^{-1}xk) \end{array}$$

extends θ on $V \times K$, and $x \mapsto \theta_x(k)$ is analytic on V for each k . Let $a \in V \cap U_r$ such that $a = \exp(iY)$ with $Y \in Q$.

Lemma 4.6. *The set of critical points of $F_{a,\mu}$ is*

$$\mathcal{C}_\mu = \bigcup_{w \in W} k_w K_\mu$$

where $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and k_w is a representative of $w \in W$.

Proof. We identify $T_k K$ with \mathfrak{k} under the isomorphism $T_e L_k$. Then by [DKV83, Lemma 5.1 and Corollary 5.2], for any $x \in G$, the tangent map of $F_{x,\mu}$ at k is

$$T_k F_{x,\mu} : Z \mapsto \langle Z, \text{Ad}(n(k^{-1}xk)^{-1})(H_\mu) \rangle. \quad (4.1)$$

Again by Lemma 4.3, for $k \in K$ and $Z \in \mathfrak{k}$ fixed, the right-hand side of (4.1) extends to a well-defined analytic function of x on V . The map $x \mapsto F_{x,\mu}(k)$ is also analytic on V for $k \in K$ fixed. For $Z \in \mathfrak{k}$,

$$T_k F_{x,\mu}(Z) = \lim_{t \rightarrow 0} \frac{F_{x,\mu}(k \exp(tZ)) - F_{x,\mu}(k)}{t}$$

and the convergence is uniform for x in a compact set. Thus, as a uniform limit of analytic functions, $T_k F_{x,\mu}(Z)$ is analytic in $x \in V$. Thus, both sides of (4.1) extends analytically to functions of $x \in V$ which coincides on $G \cap V$, thus by uniqueness of analytic continuation, (4.1) holds for any $x \in V$.

Similar arguments of analytic continuation will be used several times in what follows. We will not give full details as the proofs are identical.

By [DKV83, Prop. 5.4], any $k \in \mathcal{C}_\mu$ is a critical point of $F_{a,\mu}$ for $a \in A$. For $\mu \in \Lambda$, $k \in \mathcal{C}_\mu$, $Z \in \mathfrak{k}$ fixed, the map $a \mapsto T_k F_{a,\mu}(Z)$ is analytic in $a \in V \cap A_\mathbb{C}$ and zero on A . Thus, by analytic continuation it is identically 0 and k is a critical point of $F_{a,\mu}$.

Conversely, let k be a critical point of $F_{a,\mu}$. Then $\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) \in \mathfrak{g}_\mathbb{C}$ is orthogonal to \mathfrak{k} so $\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) \in \mathfrak{p}_\mathbb{C}$. Since $H_\mu \in \mathfrak{a} \subset \mathfrak{p}_\mathbb{C}$, we get that

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{p}_\mathbb{C}.$$

Furthermore, for any $X \in \mathfrak{n}_\mathbb{C}$, $\text{Ad}(\exp(X))(H_\mu) = e^{\text{ad}(X)}(H_\mu) = H_\mu \pmod{\mathfrak{n}_\mathbb{C}}$. Since $\exp(\mathfrak{n}_\mathbb{C})$ generates $N_\mathbb{C}$, we get $\text{Ad}(n)(H_\mu) = H_\mu \pmod{\mathfrak{n}_\mathbb{C}}$ for any $n \in N_\mathbb{C}$. Thus,

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{n}_\mathbb{C}$$

so

$$\text{Ad}(n(k^{-1}ak)^{-1})(H_\mu) - H_\mu \in \mathfrak{p}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C} = \{0\}.$$

So we get

$$\text{Ad}(n(k^{-1}ak))(H_\mu) = H_\mu. \quad (4.2)$$

From [DKV83, Lemma 1.1 and Lemma 5.9], we also have for $a' = \exp(Y') \in A$, $k \in K$ and $Z \in \mathfrak{k}$ that

$$T_k F_{a',\mu}(Z) = -\langle [Y', \text{Ad}(\theta_{a'}(k))(H_\mu)], \tilde{Z} \rangle$$

where $\tilde{Z} = (\sinh \text{ad}(Y')/\text{ad}(Y')) \circ \text{Ad}(k)(Z)$. By analytic continuation, we extend this expression to V . In particular for $a = \exp(iY)$, we get

$$T_k F_{a,\mu}(Z) = -i\langle [Y, \text{Ad}(\theta_a(k))(H_\mu)], \tilde{Z} \rangle$$

where $\tilde{Z} = (\sin \text{ad}(Y)/\text{ad}(Y)) \circ \text{Ad}(k)(Z)$.

We claim that the map $Z \mapsto \tilde{Z}$ is an isomorphism of \mathfrak{k} . Since $\text{Ad}(k)$ is an isomorphism of \mathfrak{k} , it suffices to show that

$$T = (\sin \text{ad}(Y)/\text{ad}(Y)) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} \text{ad}(Y)^{2n}$$

is an isomorphism of \mathfrak{k} . Consider the basis of \mathfrak{k} used in the proof of Lemma 3.9. Let Y_1, \dots, Y_r be a basis of $\mathfrak{m} = \mathfrak{k}^\mathfrak{a}$. For $\alpha \in \Sigma^+$, let $Z_{\alpha,1}, \dots, Z_{\alpha,m(\alpha)}$ be a basis of \mathfrak{g}_α . Let $Z_{\alpha,i}^+ = Z_{\alpha,i} + \theta(Z_{\alpha,i}) \in \mathfrak{k}$. Then $(Y_i)_{1 \leq i \leq r} \cup (Z_{\alpha,i}^+)_{\alpha \in \Sigma^+, 1 \leq i \leq m(\alpha)}$ is a basis of \mathfrak{k} . In this basis, T is a diagonal operator and its eigenvalues are nonzero. Indeed, for any $1 \leq i \leq r$, we have $T(Y_i) = Y_i$ - since Y_i commutes with \mathfrak{a} . For any $\alpha \in \Sigma^+$ and $1 \leq i \leq m(\alpha)$, we have $\text{ad}(Y)(Z_{\alpha,i}) = \alpha(Y)Z_{\alpha,i}$ and $\text{ad}(Y)\theta(Z_{\alpha,i}) = -\alpha(Y)\theta(Z_{\alpha,i})$. Thus, $\text{ad}(Y)^2(Z_{\alpha,i}^+) = \alpha(Y)^2 Z_{\alpha,i}^+$. Hence,

$$T(Z_{\alpha,i}^+) = \sum_{n \in \mathbb{N}} \frac{(-1)^n \alpha(Y)^{2n}}{(2n+1)!} Z_{\alpha,i}^+ = \frac{\sin(\alpha(Y))}{\alpha(Y)} Z_{\alpha,i}^+.$$

Since we assumed $Y \in Q$, $\alpha(Y) \notin \pi\mathbb{Z}$ for any $\alpha \in \Sigma^+$ and thus the claim is proved and T is an isomorphism.

Hence if k a critical point of $F_{a,\mu}$, we get that $[Y, \text{Ad}(\theta_a(k))(H_\mu)] \in \mathfrak{g}_\mathbb{C}$ is orthogonal to \mathfrak{k} thus $[Y, \text{Ad}(\theta_a(k))(H_\mu)] \in \mathfrak{p}_\mathbb{C}$. But since $\theta_a(k) \in K_\mathbb{C}$, we also have $\text{Ad}(\theta_a(k))(H_\mu) \in \mathfrak{p}_\mathbb{C}$ thus

$$[Y, \text{Ad}(\theta_a(k))(H_\mu)] \in [\mathfrak{p}_\mathbb{C}, \mathfrak{p}_\mathbb{C}] \cap \mathfrak{p}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \cap \mathfrak{p}_\mathbb{C} = \{0\}. \quad (4.3)$$

Since $k^{-1}ak \in V$, we have by definitions of κ, H, n that

$$ak = k\kappa(k^{-1}ak) \exp(H(k^{-1}ak))n(k^{-1}ak) = \theta_a(k) \exp(H(k^{-1}ak))n(k^{-1}ak).$$

Thus combining (4.2) and (4.3), we get that

$$\begin{aligned} [Y, \text{Ad}(ak)(H_\mu)] &= [Y, \text{Ad}(\theta_a(k)) \circ \text{Ad}(\exp(H(k^{-1}ak))) \circ \text{Ad}(n(k^{-1}ak))(H_\mu)] \\ &= [Y, \text{Ad}(\theta_a(k))(H_\mu)] \\ &= 0 \end{aligned}$$

Since $\text{Ad}(a)$ is an automorphism of Lie algebra and $A_\mathbb{C}$ is abelian, we get

$$[\text{Ad}(a^{-1})(Y), \text{Ad}(k)(H_\mu)] = [Y, \text{Ad}(k)(H_\mu)] = 0.$$

Now since $k \in K$ and Y is regular, we conclude with [DKV83, Prop. 1.2] that $k \in \mathcal{C}_\mu$. \square

We now see $F_{a,\mu}$ as a function defined on K/K_μ , whose distinct critical points are $k_w K_\mu$ for $w \in W/W_\mu$ - thus, there are only finitely many such points. For $\alpha \in \Sigma^+$, let $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Denote $\Sigma^+(\mu) = \{\alpha \in \Sigma^+ \mid \langle \alpha, \mu \rangle \neq 0\}$ and

$$\mathfrak{l}_\mu = \bigoplus_{\alpha \in \Sigma^+(\mu)} \mathfrak{k}_\alpha.$$

Then $\dim K/K_\mu = \dim(\mathfrak{l}_\mu) = \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) = n(\mu)$. Let $F_\alpha : \mathfrak{l}_\mu \rightarrow \mathfrak{k}_\alpha$ be the orthogonal projection.

Let $w \in W/W_\mu$. The value at a critical point of the phase is $F_{a,\mu}(k_w K_\mu) = i\mu(w^{-1}Y) = i(w\mu)(Y)$. By analytic continuation and [DKV83, Prop. 6.5], the Hessian at $k_w K_\mu$ is given by

$$\text{Hess}_{F_{a,\mu}}(k_w K_\mu)(Y, Z) = -\langle Y, L_{a,\mu,w}(Z) \rangle$$

where

$$L_{a,\mu,w} = -\frac{1}{2} \sum_{\alpha \in \Sigma^+(\mu)} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) F_\alpha \quad (4.4)$$

and $-\langle \cdot, \cdot \rangle$ is positive-definite on \mathfrak{k} . Finally, to apply the method of stationary phase ([MS75, Cha74]), we need to compute the square root of $\det(-L_{a,\mu,w})^{-1}$ which is continuously deformed to 1 by the homotopy $s \text{Id} + (1-s)(-L_{a,\mu,w})$.

If $w\alpha \in \Sigma^+$, we have

$$\frac{1}{2} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) = i \langle \alpha, \mu \rangle e^{-i(w\alpha)(Y)} \sin((w\alpha)(Y))$$

and $0 < (w\alpha)(Y) < \pi$ so we take $e^{i\frac{\pi}{4}} \langle \alpha, \mu \rangle^{\frac{1}{2}} e^{-i\frac{(w\alpha)(Y)}{2}} \sin((w\alpha)(Y))^{\frac{1}{2}}$ as a square root.

If $w\alpha \in -\Sigma^+$, we have

$$\frac{1}{2} \langle \alpha, \mu \rangle \left(1 - e^{-2(w\alpha)(iY)}\right) = -i \langle \alpha, \mu \rangle e^{-i(w\alpha)(Y)} |\sin((w\alpha)(Y))|$$

since $-\pi < (w\alpha)(Y) < 0$, so we take $e^{-i\frac{\pi}{4}} \langle \alpha, \mu \rangle^{\frac{1}{2}} e^{-i\frac{(w\alpha)(Y)}{2}} |\sin((w\alpha)(Y))|^{\frac{1}{2}}$ as a square root.

Let $\sigma_w = \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) \text{sgn}((w\alpha)(Y))$. We get

$$\det(-L_{a,\mu,w})^{-\frac{1}{2}} = e^{-i\sigma_w \frac{\pi}{4}} e^{i(w\rho_\mu)(Y)} \prod_{\alpha \in \Sigma^+(\mu)} \langle \alpha, \mu \rangle^{-\frac{m(\alpha)}{2}} |\sin((w\alpha)(Y))|^{-\frac{m(\alpha)}{2}} \quad (4.5)$$

where $\rho_\mu = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mu)} m(\alpha) \alpha$.

Theorem 4.7. *For $\mu \in \Lambda$, let $\Psi_\mu = \psi_\mu \circ \exp|_{iQ}$. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then for any $\delta' > \delta$, the family $(\Psi_\mu)_{\mu \in \Lambda}$ is not bounded in $C^{(r,\delta')}(iQ)$.*

The proof of this result is a variation of the proof of Theorem 3.15 using the expression of spherical functions of (U, K) from Lemma 4.4 and the method of stationary phase for complex-valued phase functions.

Proof. By (1.8), we see that $\kappa(G)$ is attained on fundamental weights, which are elements of Λ . Thus, we can choose $\mu \in \Lambda$ be such that $n(\mu) = 2\kappa(G)$. Let V be as above given by Lemma 4.3 and V' an open subset of Q such that $a = \exp(iY) \in V$ for any $Y \in V'$. Let $d(kK_\mu)$ be the image of the Haar measure on K on K/K_μ . Then for any $n \in \mathbb{N}$ and $Y \in V'$, $\Psi_{n\mu}(iY) = \int_{K/K_\mu} e^{n\mu(H(k^{-1} \exp(iY)k))} d(kK_\mu)$. As in the proof of Theorem 3.15,

for any $X \in \mathfrak{a}^r$ there are functions $g_j(Y, X) \in C^\infty(K/K_\mu)$ which do not depend on $n \in \mathbb{N}$ such that

$$D^r \Psi_{n\mu}(iY)(X) = \sum_{j=0}^r n^j \int_{K/K_\mu} e^{nF_{a,\mu}(k)} g_j(Y, X)(k) d(kK_\mu).$$

Let $I_j(Y, X, n) = \int_{K/K_\mu} e^{nF_{a,\mu}(k)} g_j(Y, X)(k) d(kK_\mu)$ for $1 \leq j \leq r$. We want to evaluate this integral as n goes to infinity. Let $d_0(kK_\mu)$ be the volume measure on K/K_μ associated to the (invariant) Riemannian metric induced by the restriction of the inner product $-\langle \cdot, \cdot \rangle$ on \mathfrak{k} . Denote $\text{Vol}(K/K_\mu) = \int_{K/K_\mu} d_0(K/K_\mu)$. By uniqueness of the invariant measure on K/K_μ , we have $d(kK_\mu) = \frac{1}{\text{Vol}(K/K_\mu)} d_0(kK_\mu)$. If $w \in W/W_\mu$, $g \in C^\infty(K/K_\mu)$, let

$$c_{w,Y}(g) = \frac{(2\pi)^{n(\mu)/2}}{\text{Vol}(K/K_\mu)} \det(-L_{\exp(iY),\mu,w})^{-\frac{1}{2}} g(k_w K_\mu).$$

By the method of stationary phase (Theorem 4.5) and the computations on the phase functions $F_{a,\mu}$, there is a neighborhood U_Y of Y and a constant $D(Y) \geq 0$ such that for any $1 \leq j \leq r$, for any $n \in \mathbb{N}^*$, $Y' \in U_Y$ in V' and $X \in \mathfrak{a}^r$ with $\|X_i\| = 1$ for all i ,

$$\left| I_j(Y', X, n) - \sum_{w \in W/W_\mu} e^{in(w\mu)(Y')} n^{-\frac{n(\mu)}{2}} c_{w,Y'}(g_j(Y', X)) \right| \leq D(Y) n^{-\frac{n(\mu)}{2}-1}. \quad (4.6)$$

We used that $g_j : V' \times \mathfrak{a}^r \times K \rightarrow \mathbb{C}$ is smooth, so that $\nu(g_j(Y', X))$ is bounded when Y', X stay in compact sets. Using the previous inequality in the cases $1 \leq j < r$, for any $Y \in V'$, there is a neighborhood V_Y of Y and a constant $C(Y) > 0$ such that for any $n \geq 1$, $Y' \in V_Y$ and X with $\|X_i\| = 1$ for all $1 \leq i \leq r$,

$$|D^r \Psi_{n\mu}(iY')(X) - n^r I_r(Y', X, n)| \leq C(Y) n^{-1}. \quad (4.7)$$

Let $S_n(x) = \sum_{w \in W/W_\mu} e^{in(w\mu)(x)} c_{w,x}(g_r(x, X))$. Since $n(\mu) = 2\kappa(G)$ by choice of μ , combining (4.6) with $j = r$ and (4.7) yields that for any $n \in \mathbb{N}^*$ and $x, y \in U_Y \cap V_Y$,

$$\begin{aligned} n^{-\delta} |S_n(x) - S_n(y)| &\leq n^r |n^{-\kappa(G)} S_n(x) - I_r(x, X, n)| + n^r |I_r(x, X, n) - I_r(y, X, n)| \\ &\quad + n^r |I_r(y, X, n) - n^{-\kappa(G)} S_n(y)| \\ &\leq 2D(Y) n^{r-\kappa(G)-1} + |n^r I_r(x, H, n) - D^r \Psi_{n\lambda}(x)(X)| \\ &\quad + |D^r \Psi_{n\lambda}(x)(X) - D^r \Psi_{n\lambda}(y)(X)| \\ &\quad + |D^r \Psi_{n\lambda}(y)(X) - n^r I_r(y, X, n)| \\ &\leq 2D(Y) n^{-\delta-1} + 2C(Y) n^{-1} + |D^r \Psi_{n\lambda}(x)(X) - D^r \Psi_{n\lambda}(y)(X)| \\ &\leq \|D^r \Psi_{n\lambda}(x) - D^r \Psi_{n\lambda}(y)\| + 2(C(Y) + D(Y)) n^{-1}. \end{aligned} \quad (4.8)$$

The hypotheses of Lemma 3.5 hold for the family of functions $x \mapsto c_{w,x}(g_r(x, X))$, for $U = V'$. Let C, d, x, V be given by Lemma 3.5, $W_x = x + V$ such that for any $y \in W_x$, $m \in \mathbb{N}$ and $N \geq \frac{d}{\|x-y\|}$,

$$\sum_{t=m}^{m+N-1} |S_n(x) - S_n(y)|^2 \geq CN. \quad (4.9)$$

The end of the proof follows as in Theorem 3.15. \square

Corollary 4.8. *Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Then for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of a unitary representation of U which is not in $C^{(r,\delta')}(U_r)$.*

Proof. By Theorem 4.7, the family of spherical functions viewed on the Lie algebra is not bounded in $C^{(r,\delta)}(Q)$. By Proposition 4.1 and Lemma 1.3, it follows that the family of spherical functions of (U, K) is not bounded in $C^{(r,\delta)}(U_r)$. Thus, the result follows from Lemma 1.10. \square

In Section 2.2.3, we made a conjecture on the expected optimal regularity of coefficients of (U, K) , which should be $\kappa(G)$.

Conjecture 4.9. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Any K -finite matrix coefficient of a unitary representation of U is in $C^{(r,\delta)}(U_r)$ and this regularity is optimal.

By Theorem 1.15, it suffices to look at K -bi-invariant matrix coefficients since (U_r, K) has a well-behaved KAK decomposition (Proposition 2.43). This conjecture has been proven in rank 1 and when U/K is isomorphic to a Lie group (which is equivalent to G being a complex Lie group) in Chapter 2 (Theorems A and B). Corollary 4.8 gives a partial result towards this conjecture : the optimal regularity cannot be greater than $\kappa(G)$. Furthermore, the conjecture is also shown in some new specific cases.

Corollary 4.10. *The conjecture is true for*

- $(U, K) = (SU(p+q), S(U(p) \times U(q)))$ with $q \geq p \geq 2$, which corresponds to $G = SU(p, q)$;
- $(U, K) = (SO(8), U(4))$, which corresponds to $G = SO^*(8)$;
- $(U, K) = (SO(10), U(5))$, which corresponds to $G = SO^*(10)$;
- $(U, K) = (Sp(n), Sp(2) \times Sp(n-2))$ with $n \geq 4$, which corresponds to $G = Sp(2, n-2)$.

Proof. This is a consequence of Corollary 4.8 and Theorems 2.28 and 2.37. \square

4.2 Towards a lower bound

We now know that the optimal regularity of K -finite matrix coefficient of U on U_r is at most $\kappa(G)$, with equality in many cases. We will conclude this chapter by showing that equality holds in all cases, but only in some open subset of U_r .

In Theorem 4.7, we used the method of stationary phase to obtain a sharp estimate of the spherical functions, but only on a subfamily. In order to obtain a lower bound on regularity, we need estimates for all $\mu \in \Lambda$. Clearly, we know that we cannot obtain sharp estimate of this kind. Indeed, as μ varies, the critical submanifold of the phase function $F_{a,\mu}$ varies in dimension.

Thus, we cannot treat μ as a parameter and use stationary phase approximation to obtain uniform estimate when μ is close to some μ_0 . However, in a local chart, we can see K in coordinates (x, y) such that K_μ is given by $\{y = 0\}$, and treat x, a, μ as parameters, as done in [DKV83, Prop. 9.2] and used in Theorem 3.12. This leads to brutal estimates, as we forget what happens in the x coordinate, but sufficient for our purposes.

A second issue arises in the statement of stationary phase used before (Theorem 4.5). When choosing a parameter $\mu_0 \in \Lambda$, we need to consider a neighborhood of μ_0 in \mathfrak{a}^* . However, a necessary hypothesis is that the real part of the phase is non-positive, which is only true on Λ , thus there may not exist a neighborhood of μ_0 such that this is true.

To deal with this last issue, we begin our proof with another statement of the stationary phase approximation which is well-suited to holomorphic phase functions - instead of simply C^∞ as in Theorem 4.5.

Theorem 4.11 ([Sjö82, Thm 2.8 and Rem. 2.10],[BJ20, Prop. 1.3]). *Let $U \times V$ be an open subset of $\mathbb{C}^n \times \mathbb{C}^k$ and Γ a k -dimensional real submanifold of V with boundary. Let $\Phi, u : U \times V \rightarrow \mathbb{C}$ be holomorphic functions. For $x \in U$, denote $\Phi_x : y \mapsto \Phi(x, y)$. Let $x_0 \in U$. Assume that there exists a unique critical point y_0 of Φ_{x_0} in Γ , which is non-degenerate and in the interior of Γ and that $\Phi(x_0, y_0) = 0$. Furthermore, assume that $\text{Im } \Phi(x_0, y) \geq 0$ for any $y \in \Gamma$, and $\text{Im } \Phi(x_0, y) > 0$ for any $y \in \partial\Gamma$.*

Then, there exists an open neighborhood $U' \subset U$ of x_0 such that for any $x \in U'$, the map Φ_x has a unique critical point $y_c(x)$ close to y_0 and $x \mapsto y_c(x)$ is holomorphic. Furthermore, there exists $C > 0$ such that for any $x \in U'$, $t \geq 1$,

$$\left| e^{-it\Phi(x, y_c(x))} \int_{\Gamma} e^{it\Phi(x, y)} u(x, y) dy \right| \leq C \|u\|_{\infty, U' \times \Gamma} t^{-k/2}.$$

If we want to estimate the integral, we need to understand the term $e^{-it\Phi(x, y_c(x))}$. It is clear that this term is bounded below as $t \rightarrow +\infty$ except when $\text{Im } \Phi(x, y_c(x)) < 0$. But since $y_c(x)$ is not necessarily real, even for x real, it is *a priori* not enough to know the sign of $\text{Im } \Phi(x, y)$ on real points y (which is our only hypothesis). The following lemma takes care of this problem.

Lemma 4.12 ([BJ20, Lem. 1.16]). *Let Φ be as in Theorem 4.11 and $y_c : U' \rightarrow V$ given by the theorem. Assume that $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\text{Im Hess}_{\Phi_{x_0}}(y_0)$ is positive. Then there exists an open neighborhood $U'' \subset U'$ of x_0 such that for any $x \in U'' \cap \mathbb{R}^n$ with $\text{Im } \Phi(x, y) \geq 0$ for all $y \in V \cap \mathbb{R}^k$, then*

$$\text{Im } \Phi(x, y_c(x)) \geq 0.$$

Remark 4.13. In [BJ20], this lemma is stated with the hypothesis that $\text{Im } \Phi(x, y) \geq 0$ for **all** x, y real - and thus of course, $\text{Im } \Phi(x, y_c(x)) \geq 0$ for all x real. However it is clear in the proof that we can remove this assumption by adding positivity of the Hessian matrix, and we get the result only for x real where the assumption holds. This is exactly what will allow us to take care of parameters μ_0 on the boundary of $\tilde{\Lambda}$.

We now come back to the Lie group setting, and we will use all the notations introduced in Section 4.1. Notice that since the subgroup K is compact, there exists a relatively compact open subset \tilde{K} of $K_{\mathbb{C}}$ such that $K \subset \tilde{K}$. By compactness, up to shrinking the open subset V given by Lemma 4.3, we may assume that V is invariant by conjugation by elements of \tilde{K} . Thus, the element $F_{g, \mu}(k) = \mu(H(k^{-1}gk)) \in \mathbb{C}$ is well-defined for any $g \in V$, $k \in \tilde{K}$, $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ and is holomorphic in all variables.

Recall also that there are fundamental weights $\mu_1, \dots, \mu_{\ell} \in \mathfrak{a}^*$ such that

$$\Lambda = \left\{ \sum_{i=1}^{\ell} m_i \mu_i \mid m_i \in \mathbb{N} \right\}$$

and that for any $u \in U \cap V$, $\mu \in \Lambda$,

$$\text{Re } \mu(H(u)) \leq 0.$$

Thus, defining the cone

$$\tilde{\Lambda} = \left\{ \sum_{i=1}^{\ell} t_i \mu_i \mid t_i \in \mathbb{R}_+ \right\},$$

it is clear that for any $u \in U \cap V$, $\mu \in \tilde{\Lambda}$,

$$\text{Re } \mu(H(u)) \leq 0.$$

Furthermore, the function $\psi_{\mu} : g \mapsto \int_K e^{\mu(H(k^{-1}gk))} dk$ is also well-defined for any $\mu \in \mathfrak{a}_{\mathbb{C}}^*$.

Theorem 4.14. For $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\Psi_{\mu} = \psi_{\mu} \circ \exp|_{iQ}$. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Let $Q_0 = \{Y \in Q \mid \exp(iY) \in V\}$. Then the family $(\Psi_{\mu})_{\mu \in \Lambda}$ is bounded in $C^{(r,\delta)}(iQ_0)$.

Proof. First consider $\tilde{Q} = \{Y \in \mathfrak{a}_{\mathbb{C}} \mid \exp(Y) \in V\}$. Let $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ and $Y \in \tilde{Q}$. Then the phase function $F_{\exp Y, \mu} : k \mapsto \mu(H(k^{-1} \exp(Y)k))$ is well-defined on \tilde{K} and holomorphic in all variables. Let $t \in \mathbb{R}$, consider $\Psi_{t\mu}(Y) = \int_K e^{tF_{a,\mu}(k)} dk$. Let $s \in \mathbb{N}$ and $X \in \mathfrak{a}_{\mathbb{C}}^r$. For $0 \leq j \leq s$, we can consider the functions

$$g_j(\mu, Y, X) : k \in \tilde{K} \mapsto \frac{1}{j!} \frac{d^j}{dt^j} \left(\left(D^s(e^{tF_{a,\mu}(k)})(Y)(X) \right) e^{-tF_{a,\mu}(k)} \right) \Big|_{t=0}$$

where D is the differential operator with respect to the variable Y . The functions g_j are holomorphic in all variables. As in Theorem 3.12, we get that

$$D^s \Psi_{t\mu}(Y)(X) = \sum_{j=0}^s t^j \int_K e^{tF_{a,\mu}(k)} g_j(\mu, Y, X)(k) dk. \quad (4.10)$$

Let $S = \{\mu \in \tilde{\Lambda} \mid \|\mu\| = \min \|\mu_i\|\}$ which is a compact set. For now, fix $\mu_0 \in S$, $Y_0 \in Q_0$, $a_0 = \exp(iY_0)$. Let $\mathcal{C} = \mathcal{C}_{\mu_0}$ be the critical set of F_{a_0, μ_0} computed in Lemma 4.6. The critical set \mathcal{C}_{μ} varies with μ , but we can choose a neighborhood V_{μ_0} of μ_0 in $\mathfrak{a}_{\mathbb{C}}^*$ such that for any $\mu \in V_{\mu_0} \cap \mathfrak{a}$, $\mathcal{C}_{\mu} \subset \mathcal{C}$.

Consider a tubular neighborhood of \mathcal{C} , that is to say a vector bundle $\pi : E \rightarrow \mathcal{C}$ together with a map $J : E \rightarrow K$ such that if 0_E denote the zero section of the vector bundle, $J(0_E(x)) = x$ and J is a diffeomorphism from an open neighborhood Ω of $0_E(\mathcal{C})$ to an open neighborhood of \mathcal{C} (we refer to [Lee03] for more details on the construction of such a bundle). For any $k_0 \in \mathcal{C}$, consider an open neighborhood U_{1,k_0} which is a local trivialization of the bundle. Then $U_{2,k_0} = J(\Omega \cap \pi^{-1}(U_{1,k_0}))$ is a neighborhood of k_0 in K . Notice that if $x \in U_{2,k} \cap U_{2,k'}$, then $\pi(J^{-1}(x)) \in U_{1,k} \cap U_{1,k'}$. Let V_{k_0} be a neighborhood of k_0 in \tilde{K} containing U_{2,k_0} , which we may assume up to reducing U_{2,k_0} to be the domain of an analytic chart

$$H_{k_0} : \begin{array}{ccc} V_{k_0} & \rightarrow & \mathbb{C}^{\dim(\mathcal{C})} \times \mathbb{C}^{n(\mu_0)} \\ k & \mapsto & (z_1, z_2) \end{array}$$

defined in such a way that $H_{k_0}(k_0) = (0, 0)$, $H_{k_0}(U_{2,k_0}) = H_{k_0}(V_{k_0} \cap K) = H(V_{k_0}) \cap (\mathbb{R}^{\dim(\mathcal{C})} \times \mathbb{R}^{n(\mu_0)})$ and \mathcal{C} is given in those local coordinates by $H_{k_0}(U_{2,k_0}) \cap \{z_2 = 0\}$. Indeed, by the computations in Lemma 4.6, there exists $w \in W$ such that $k_0 \subset wK_{\mu_0}$, and we may assume that $U_{1,k_0} \cap \mathcal{C} \subset wK_{\mu_0}$. Then set $O = \exp_K^{-1}(w^{-1}U_{2,k_0})$, $O_{\mathbb{C}} = O + iO$ open subset of $\mathfrak{k}_{\mathbb{C}}$ and $V_{k_0} = w \exp_{K_{\mathbb{C}}}(O_{\mathbb{C}})$ - up to shrinking at each step so that the exponential map is a diffeomorphism in the neighborhoods considered. Then if we consider a decomposition $\mathfrak{k} = \mathfrak{k}_{\mu} \oplus F$, the chart $H_{k_0}(k) = \exp_{K_{\mathbb{C}}}^{-1}(w^{-1}k)$ satisfies the requirements.

Let $x = (Y, \mu, z_1) \in V_1 = \tilde{Q} \times V_{\mu_0} \times \text{pr}_1(H_{k_0}(V_{k_0}))$, $y = z_2 \in V_2 = \text{pr}_2(H_{k_0}(V_{k_0}))$. We will treat x as a parameter and apply the stationary phase only in the y coordinate. Let $x_0 = (Y_0, \mu_0, 0)$ and $\Phi(x, y) = -i(F_{\exp(iY), \mu}(H_{k_0}^{-1}(z_1, z_2)) - F_{a_0, \mu_0}(k_0))$. Then $\Phi_{x_0} : y \mapsto \Phi(x_0, y)$ has a unique critical point $y_0 = 0$ in V_2 , which is non-degenerate. Since $\mu_0 \in \tilde{\Lambda}$ and $\text{Re } F_{a_0, \mu_0}(k_0) = 0$, we also have $\text{Im } \Phi(x_0, y) \geq 0$ for any $y \in V_2$. Furthermore, by (4.4), the imaginary part of the Hessian matrix of Φ_{x_0} at y_0 is positive-definite. By Taylor's formula, this implies that $\text{Im } \Phi(x_0, y) > 0$ for y real and close to y_0 . Thus, we can choose a small ball $\Gamma_{k_0} \subset V_2 \cap \mathbb{R}^{n(\mu_0)}$ centered around y_0 such that $\text{Im } \Phi(x_0, y) > 0$ for $y \in \partial\Gamma_{k_0}$. We can now apply Theorem 4.11 and Lemma 4.12. Thus, there is an open neighborhood $V_1' \subset V_1$ of x_0 , which we may assume to be of the form $V_{Y_0}'(k_0) \times V_{\mu_0}'(k_0) \times V_0'(k_0)$, such that for any $x \in V_1'$, the map Φ_x has a unique critical point $y_c(x)$. Up to shrinking, we may assume

that $V'_0(k_0)$ is a ball centered at 0, whose closure is contained in V_{k_0} . For $x = (Y, \mu, z_1)$ real and such that $\mu \in \tilde{\Lambda}$, Lemma 4.12 ensures that $\text{Im } \phi(x, y_c(x)) \geq 0$. Finally, there is a constant $C_{Y_0, \mu_0, k_0} > 0$ such that for any holomorphic map g on $V_1 \times V_2$, for any $t \geq 1$ and $x = (Y, \mu, z_1) \in V'_1$ real with $\mu \in \tilde{\Lambda}$,

$$\left| \int_{\Gamma_{k_0}} e^{it\phi(x,y)} g(x, y) dx \right| \leq C_{Y_0, \mu_0, k_0} \|g\|_\infty t^{-n(\mu_0)/2} \leq C_{Y_0, \mu_0, k_0} \|g\|_\infty t^{-\kappa(G)}. \quad (4.11)$$

For the last inequality, we used that $\kappa(G) \leq \frac{n(\mu)}{2}$ for any $\mu \in \mathfrak{a}^*$ by definition.

Let $\Omega_{k_0} = H_{k_0}^{-1}((V'_0(k_0) \cap \mathbb{R}^{\dim \mathcal{C}}) \times \{0\}) = H_{k_0}^{-1}(V'_0(k_0) \times \dot{\Gamma}_{k_0}) \cap \mathcal{C}$. Then Ω_{k_0} is an open neighborhood of k_0 in \mathcal{C} , contained in U_{1, k_0} . We will construct an "almost disjoint" cover of \mathcal{C} . Since $\mathcal{C} = \bigcup_{k \in \mathcal{C}} \Omega_k$ is compact, we can extract a finite subcover $\Omega_{k_1}, \dots, \Omega_{k_p}$. Set $V_1 = \Omega_{k_1}$ and inductively define $V_j = \Omega_{k_j} \cap (\mathcal{C} \setminus \bigcup_{i < j} \bar{V}_i)$. This construction yields p disjoint open subsets V_j of \mathcal{C} such that $\mathcal{C} = \bigcup_{j=1}^p \bar{V}_j = \bigcup_{j=1}^p V_j \cup \bigcup_{j=1}^p \partial V_j$. We have that

$$\begin{aligned} \partial V_j &\subset \partial \Omega_{k_j} \cup \partial \left(\mathcal{C} \setminus \bigcup_{i < j} \bar{V}_i \right) \\ &= \partial \Omega_{k_j} \cup \partial \bigcup_{i < j} \bar{V}_i \\ &\subset \partial \Omega_{k_j} \cup \bigcup_{i < j} \partial \bar{V}_i \\ &\subset \partial \Omega_{k_j} \cup \bigcup_{i < j} \partial V_i. \end{aligned}$$

Since $\partial V_1 = \partial \Omega_1$, recursively we obtain

$$\partial V_j \subset \bigcup_{i \leq j} \partial \Omega_{k_i}.$$

But $\partial \Omega_{k_j}$ is contained in the embedded image of a $\dim \mathcal{C} - 1$ dimensional sphere, thus it has Riemannian volume 0. This implies that each ∂V_j has volume 0.

Let $N_j = H_{k_j}^{-1}(H_{k_j}(\bar{V}_j) \times \Gamma_{k_j}) \subset U_{2, k_j}$ and $D = \bigcup_{j=1}^p N_j$. Since the \bar{V}_j 's are disjoint up to negligible sets, and contained in U_{1, k_j} local trivialization of the tubular neighborhood, then the N_j 's are also disjoint up to negligible sets (for the volume measure on K). Indeed, $N_i \cap N_j \subset J(\Omega \cap \pi^{-1}(\partial V_j))$. Note that $\mathcal{C} \subset \dot{D}$ and set $K' = K \setminus \dot{D}$. Then $K' \cap D = \partial D \subset \bigcup_j H_{k_j}^{-1}(H_{k_j}(\partial \bar{V}_j) \times \Gamma_{k_j}) \cup H_{k_j}^{-1}(H_{k_j}(\bar{V}_j) \times \partial \Gamma_{k_j})$ which has volume 0 again.

Denote $A_{Y_0} = Q \cap \bigcap_{j=1}^p V'_{Y_0}(k_j)$ and $S_{\mu_0} = \bigcap_{j=1}^p V'_{\mu_0}(k_j)$, by finiteness of the intersection, these are still open neighborhoods of Y_0, μ_0 respectively, and we may assume, up to shrinking if necessary, that they are both bounded. Up to reducing once again A_{Y_0} to a ball, it will be convenient to assume it is convex.

By assumption, for $a = \exp(iY)$ with $Y \in Q$ and μ real in S_{μ_0} , $\mathcal{C}_{\mu_0} \subset \mathcal{C}$ so $F_{a, \mu}$ has no critical point in K' since $\mathcal{C} \cap K' = \emptyset$. By compactness of K' and since with these parameters, $\text{Re } F_{a, \mu}(k) \leq 0$ for any $k \in K'$, this implies that $\|T_k F_{\exp(iY), \mu}\|^2 - \text{Re } F_{a, \mu}(k)$ is bounded below by $\delta > 0$ for $k \in K'$, $Y \in A_{Y_0}$ and $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$. Thus, by [Hör83, Thm. 7.7.1], for any $n \in \mathbb{N}$, there exists a semi-norm ν_{n, Y_0, μ_0} on $C^\infty(K')$ such that for any $g \in C^\infty(K')$, $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$ and $t \geq 1$,

$$\left| \int_{K'} e^{tF_{\exp(iY), \mu}(k)} g(k) dk \right| \leq \nu_{n, Y_0, \mu_0}(g) t^{-n}. \quad (4.12)$$

Now, as in Theorem 4.7, let d_0k denote the volume measure on K associated to the invariant Riemannian metric h induced by the inner product $-\langle \cdot, \cdot \rangle$ and $\text{Vol}(K) = \int_K d_0K$. Then the Haar measure dk is $\frac{1}{\text{Vol}(K)}d_0k$. Let also $G(k) = \det(h_k(\partial_i, \partial_j))$. Denote also

$$\tilde{g}_{j,m}(\mu, Y, X)(z_1, z_2) = g_j(\mu, Y, X)(k)\sqrt{G(k)}$$

where $(z_1, z_2) = H_{k_m}(k)$. Since by construction, the set of points counted several times is negligible, we can write that for any $1 \leq j \leq s$,

$$\begin{aligned} & \left| \int_K e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \leq \left| \int_{K'} e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \quad + \sum_{m=1}^p \left| \int_{N_m} e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \leq \left| \int_{K'} e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \\ & \quad + \frac{1}{\text{Vol}(K)} \sum_{m=1}^p \left| \int_{H_{k_m}(\overline{V}_m)} \int_{\Gamma_{k_m}} e^{it\phi((Y, \mu, z_1), z_2)} \tilde{g}_{j,m}(\mu, Y, X)(z_1, z_2) dz_2 dz_1 \right| \end{aligned}$$

We can now apply (4.11), and (4.12) with $n = \lceil \kappa(G) \rceil$ to get that for any $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$, $X \in \mathfrak{a}^s$ and $t \geq 1$,

$$\begin{aligned} \left| \int_K e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| & \leq \nu_{\lceil \kappa(G) \rceil, Y_0, \mu_0}(g_j(\mu, Y, X)) t^{-\lceil \kappa(G) \rceil} \\ & \quad + \sum_{m=1}^p \frac{C_{Y_0, \mu_0, k_i} \text{Vol} H_{k_m}(\overline{V}_m)}{\text{Vol}(K)} \|\tilde{g}_{j,m}(\mu, Y, X)\|_{\infty} t^{-\kappa(G)} \end{aligned}$$

Now since the functions g_j are smooth in all variables, they are bounded in $C^\infty(K)$ when μ, Y, X remain bounded, thus there exists a constant $D_{j, Y_0, \mu_0} > 0$ such that for any $Y \in A_{Y_0}$, $\mu \in S_{\mu_0} \cap \tilde{\Lambda}$, $X \in \mathfrak{a}^s$ with $\|X_i\| = 1$ and any $t \geq 1$,

$$\left| \int_K e^{tF_{\exp(iY), \mu}(k)} g_j(\mu, Y, X)(k) dk \right| \leq D_{j, Y_0, \mu_0} t^{-\kappa(G)}. \quad (4.13)$$

Since S is compact and $S \subset \bigcup_{\mu_0 \in S} S_{\mu_0}$, we can extract a finite cover $S_{\mu_1}, \dots, S_{\mu_q}$. Set

$$M_{Y_0, s} = \max_{1 \leq k \leq q} \sum_{j=1}^s D_{j, Y_0, \mu_k}.$$

Combining (4.13) with (4.10), for any $Y \in A_{Y_0}$, $\mu \in S$ and any $t \geq 1$,

$$\|D^s \Psi_{t\mu}(Y)\| = \sup_{\|X_i\|=1} |D^s \Psi_{t\mu}(Y)(X)| \leq M_{Y_0, s} t^{s-\kappa(G)}. \quad (4.14)$$

Note that $\Lambda \setminus \{0\} \subset \mathbb{R}_{\geq 1} S$, so the previous equation immediately implies that for any $\mu \in \Lambda \setminus \{0\}$ and $s \leq r$,

$$\|D^s \Psi_{t\mu}(Y)\| \leq M_{Y_0, s}. \quad (4.15)$$

Thus if $\kappa(G)$ is an integer, the proof is complete.

Otherwise, $\delta = \kappa(G) - r = \frac{1}{2}$. Then applying (4.14) with $s = r$ and triangular inequality gives that for any $Y, Y' \in A_{Y_0}$, $\mu \in S$, $t \geq 1$,

$$\|D^r \Psi_{t\mu}(Y) - D^r \Psi_{t\mu}(Y')\| \leq 2M_{Y_0, r} t^{-1/2}.$$

On the other hand, using (4.14) with $s = r + 1$ and the mean value theorem gives that for any $Y, Y' \in A_{Y_0}$, $\mu \in S$, $t \geq 1$,

$$\|D^r \Psi_{t\mu}(Y) - D^r \Psi_{t\mu}(Y')\| \leq \sup_{Z \in A_{Y_0}} \|D^{r+1} \Psi_{t\mu}(Z)\| \|Y - Y'\| \leq M_{Y_0, r+1} t^{1/2} \|Y - Y'\|.$$

Combining both estimates, we get that for any $Y, Y' \in A_{Y_0}$, $\mu \in \Lambda \setminus \{0\}$,

$$\|D^r \Psi_{\mu}(Y) - D^r \Psi_{\mu}(Y')\| \leq (2M_{Y_0, r} M_{Y_0, r+1})^{1/2} \|Y - Y'\| \quad (4.16)$$

which completes the proof when $\kappa(G)$ is not an integer. \square

Corollary 4.15. *Consider (U, K) as before. Let $r = \lfloor \kappa(G) \rfloor$ and $\delta = \kappa(G) - r$. Let $Q_0 = \{Y \in Q \mid \exp(iY) \in V\}$ and $U_0 = K \exp(iQ_0)K \subset U_r$. Then any K -finite matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(U_0)$.*

Proof. By Theorem 4.14, Proposition 4.1 and Lemma 1.3, the family of spherical functions of (U, K) is bounded in $C^{(r, \delta)}(U_0)$. By Lemma 1.10, it follows that any K -bi-invariant matrix coefficient of a unitary representation of U is in $C^{(r, \delta)}(U_0)$. Finally, Proposition 2.43 and Theorem 1.15 allows to extend this regularity to K -finite coefficients. \square

Remark 4.16. Combining Corollaries 4.8 and 4.15, we get that Conjecture 4.9 is true for any compact symmetric pair (U, K) , but only in some open subset U_0 and not all of U_r .

4.3 Non simply connected groups

Until that point, we assumed U to be simply connected. We will now prove that we can drop this assumption and extend our results to any symmetric pair (U, K) of compact type.

First, we recall that by Lemma 1.17, the spherical functions of (U, K) depend only on the symmetric space $M = U/K$. Thus, as soon as M is simply connected, our results extends to the pair (U, K) . However, when M is no longer simply connected, the situation becomes more complicated: the pair (U, K) may have fewer spherical functions. For example, consider the pair $(SO(3), SO(2))$, whose associated symmetric space is the sphere and is simply connected. Then its spherical functions are the Legendre polynomials. Consider then the pair $(SO(3), S(O(1) \times O(2)))$, the associated symmetric space is the real projective plane. The spherical functions of this pair are the Legendre polynomials of *even* degree ([CW75]). We see on this example that the pair associated to $\mathbb{R}P^2$ has less spherical functions than its universal cover S^2 , but still enough to obtain the same behavior. We will now formalize this fact in the general case.

Let U be a connected semisimple compact Lie group, let σ be an involutive automorphism of U and K a subgroup such that $(U^\sigma)_0 \subset K \subset U^\sigma$. Let $M = U/K$ be the associated symmetric space. Let \tilde{U} be the universal cover of U , σ induces an automorphism of \tilde{U} and the subgroup $\tilde{K} = \tilde{U}^\sigma$ is connected (by [Bor98, Ch.V, Thm 3.3]). Then $\tilde{M} = \tilde{U}/\tilde{K}$ is simply connected and is the universal cover of M . If $p: \tilde{U} \rightarrow U$ is the covering map, then $F = \ker p$ is a subgroup of the finite group $Z(\tilde{U})$ and $p(\tilde{K}) = K_0$.

Recall from Section 1.5 that σ induces a decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$. We chose \mathfrak{a} a maximal abelian subspace of \mathfrak{p} and considered the root system $\Sigma_{\mathfrak{a}}$ of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{u}_{\mathbb{C}}$. Let $\Sigma_{\mathfrak{a}}^+$ a choice of

positive roots. We also chose \mathfrak{h} a Cartan subalgebra of \mathfrak{u} containing \mathfrak{a} , $\Sigma_{\mathfrak{h}}$ the root system of $\mathfrak{h}_{\mathbb{C}}$ and $\Sigma_{\mathfrak{h}}^+$ a choice of positive roots, such that $\Sigma_{\mathfrak{a}}^+ = \{\phi|_{\mathfrak{a}} \mid \phi \in \Sigma_{\mathfrak{h}}^+, \phi|_{\mathfrak{a}} \neq 0\}$.

We recall some facts on the representation theory of compact Lie groups (see [Kna02, Ch. V]). We say that $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ is a dominant element if for any $\alpha \in \Sigma_{\mathfrak{h}}^+$, $\langle \mu, \alpha \rangle \geq 0$. We say that μ is (algebraically) integral if

$$\forall \alpha \in \Sigma_{\mathfrak{h}}, 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

We say that μ is analytically integral if for any $H \in \mathfrak{h}$ such that $\exp(H) = 1$, $\mu(H) \in 2\pi i\mathbb{Z}$. Let $P(\mathfrak{u})$ be the set of dominant integral element and $P(U)$ the set of dominant analytically integral elements. Note that $P(U) \subset P(\mathfrak{u})$ and that $P(\tilde{U}) = P(\mathfrak{u})$ ([Kna02, Thm. 5.107]). Finally, define a partial order on dominant integral element: λ is higher than μ if $\lambda - \mu$ is a combination of roots in $\Sigma_{\mathfrak{h}}^+$ with coefficients in \mathbb{N} .

By the theorem of the highest weight ([Kna02, Thm. 5.110]), there is a one-one correspondence between the set \hat{U} of finite-dimensional irreducible representations of U (up to isomorphism) and $P(U)$. More precisely, if π is a finite-dimensional representation of \mathfrak{u} on V , then V is the direct sum of its weight space, i.e. $V = \bigoplus V^{(\lambda)}$ where

$$V^{(\lambda)} = \{v \in V \mid \forall H \in \mathfrak{h}, \pi(H)v = \lambda(H)v\}.$$

If π is a finite-dimensional representation of U , we still denote the representation of \mathfrak{u} induced by its differential π . A highest weight of π is a weight which is higher than every other weight appearing in the decomposition of V . Then the theorem of the highest weight states that

1. If (π, V) is an irreducible representation of U , there exists a unique highest weight λ , and $\lambda \in P(U)$;
2. If two representations have the same highest weight, they are isomorphic;
3. For any $\lambda \in P(U)$, there exists an irreducible representation $(\pi_{\lambda}, V_{\lambda})$ with highest weight λ , and the weight space of λ is one-dimensional.

Of course, this theorem applies to \tilde{U} with the set of dominant integral elements $P(\mathfrak{u})$ (equivalently, to the Lie algebra \mathfrak{u}).

If $\lambda \in P(U)$, denote v_{λ} a generator of the weight space of λ . If $\lambda, \mu \in P(U)$, consider the tensor representation $V = V_{\lambda} \otimes V_{\mu}$. Then the decomposition into irreducible representations of V is

$$V = \bigoplus V_{\nu}^{\oplus m_{\nu}}$$

with $m_{\lambda+\mu} = 1$, and the weight space of $\lambda + \mu$ is generated by $v_{\lambda} \otimes v_{\mu}$.

Finally, recall that by the Cartan-Helgason theorem (Theorem 1.18), if we set

$$\Lambda = \left\{ \mu \in i\mathfrak{a}^* \mid \forall \alpha \in \Sigma_{\mathfrak{a}}^+, \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N} \right\},$$

then the map which sends a representation to its highest weight is a bijection from $\hat{U}_{\tilde{K}}$ to Λ , where $\hat{U}_{\tilde{K}}$ is the subset of classes of irreducible representations of \tilde{U} with a nonzero \tilde{K} -invariant vector.

We now want to understand the set Λ_M of highest weight of representations of U with a nonzero K -invariant vector.

Proposition 4.17. *Let $\Lambda_M = \{\lambda \in P(U) \mid V_\lambda^K \neq \{0\}\}$. Then there exists $N \in \mathbb{N}$ such that*

$$N\Lambda \subset \Lambda_M \subset \Lambda.$$

Proof. One inclusion is easy: if $\lambda \in \Lambda_M$, there exists an irreducible representation $\pi_\lambda \in \widehat{U}_K$ of highest weight λ . Let ξ_λ be a generator of the space of K -invariant vector. Consider the composition $\tilde{\pi}_\lambda = \pi_\lambda \circ p$, then $\tilde{\pi}_\lambda \in \widehat{U}$ and ξ_λ is $p^{-1}(K)$ -invariant. Since $\tilde{K} \subset p^{-1}(K_0) \subset p^{-1}(K)$, $\lambda \in \Lambda$.

For the other inclusion, we first claim that there is $N_0 \in \mathbb{N}$ such that $N_0P(\mathfrak{u}) \subset P(U)$. Indeed, if $\lambda \in P(\mathfrak{u})$, there is an irreducible representation $\pi_\lambda \in \widehat{U}$ on V_λ . Let v_λ be a nonzero vector of weight λ . Let $F = \ker p \subset Z(\tilde{U})$. Then $\lambda \in P(U)$ if and only if π_λ descends to U , i.e. $\pi_\lambda|_F = \text{Id}$. Now since F is central in \tilde{U} and π_λ is irreducible, by Schur's lemma F acts by scalar multiplication. Thus, there exists a morphism $\tau_\lambda : F \rightarrow S^1$ such that $\pi_\lambda(x) = \tau_\lambda(x) \text{Id}$ for any $x \in F$. Let $n \in \mathbb{N}$, consider $v_{n\lambda} = v_\lambda^{\otimes n} \in V_{n\lambda}$ seen as a subrepresentation of $V_\lambda^{\otimes n}$. Then on the one hand, for any $x \in F$, $\pi_{n\lambda}(x)v_{n\lambda} = \tau_{n\lambda}(x)v_{n\lambda}$. On the other hand, $\pi_{n\lambda}$ acts as the tensor representation so

$$\pi_{n\lambda}(x)v_{n\lambda} = (\pi_\lambda(x)v_\lambda)^{\otimes n} = (\tau_\lambda(x))^n v_\lambda^{\otimes n} = \tau_\lambda(x^n)v_{n\lambda}.$$

Thus, we obtain that for any $\lambda \in P(\mathfrak{u})$, $x \in F$, $n \in \mathbb{N}$, $\tau_{n\lambda}(x) = \tau_\lambda(x^n)$. Let $N_0 = |F|$ the order of the finite group F , then $\tau_{N_0\lambda} = 1$ for any $\lambda \in P(\mathfrak{u})$, so $\pi_{N_0\lambda}$ induces an irreducible representation of U . Thus,

$$N_0P(\mathfrak{u}) \subset P(U). \quad (4.17)$$

Let $\lambda \in \Lambda \cap P(U)$. Then there exists an irreducible representation (π_λ, V_λ) of U such that $\tilde{\pi}_\lambda = \pi_\lambda \circ p$ is an irreducible representation of \tilde{U} , and a nonzero \tilde{K} -invariant vector ξ_λ . By the proof of the Cartan-Helgason theorem ([Wol07, Thm. 11.4.10]), the generator ξ_λ can be chosen as $\Pi(v_\lambda)$ where Π is the projection on the space of K_0 -invariant vectors. In particular, this implies that $\langle \xi_\lambda, v_\lambda \rangle \neq 0$. Clearly since $p(\tilde{K}) = K_0$, ξ_λ is K_0 -invariant. We want to understand when ξ_λ is actually invariant under the whole subgroup K and not only its connected component.

Since (U, K_0) is also a Gelfand pair, the subspace $V_\lambda^{K_0}$ is one-dimensional, generated by ξ_λ (Proposition 1.7). The connected component K_0 is a normal subgroup of K , so we get that for any $k \in K$, $\pi_\lambda(k)\xi_\lambda$ is also a K_0 -invariant vector, thus a multiple of ξ_λ . Thus K acts on $V_\lambda^{K_0}$ by scalar multiplication. Since K_0 acts trivially, this induces a morphism

$$\chi_\lambda : K/K_0 \rightarrow S^1$$

where K/K_0 is a finite group. Then ξ_λ is K -invariant if and only if this character χ_λ is trivial. We will use a similar strategy as in the case of τ to prove that χ is multiplicative. Let $\lambda, \mu \in \Lambda \cap P(U)$. We claim that

$$\chi_{\lambda+\mu} = \chi_\lambda \chi_\mu.$$

As explained before, consider $V_{\lambda+\mu}$ as a subrepresentation of $W = V_\lambda \otimes V_\mu$.

Let $\xi = \xi_\lambda \otimes \xi_\mu$. Then for any $k \in K$, we have that

$$(\pi_\lambda \otimes \pi_\mu)(k)\xi = \chi_\lambda(kK_0)\xi_\lambda \otimes \chi_\mu(kK_0)\xi_\mu = \chi_\lambda(kK_0)\chi_\mu(kK_0)\xi. \quad (4.18)$$

However ξ need not be in $V_{\lambda+\mu}$, so we cannot argue directly as above. As a U -representation,

$$W = \bigoplus V_\nu^{\oplus m_\nu}$$

and thus, the space of K_0 -invariant vector is

$$W^{K_0} = \bigoplus_{\nu} \bigoplus_{i=1}^{m_{\nu}} V_{\nu}^{K_0}.$$

On each summand $V_{\nu}^{K_0}$, either $V_{\nu}^{K_0}$ is trivial, or it is one-dimensional generated by ξ_{ν} and K/K_0 acts by multiplication by χ_{ν} . Since $\xi \in W^{K_0}$, there exists a unique decomposition

$$\xi = \sum_{\nu} \sum_{i=1}^{m_{\nu}} a_{\nu,i} \xi_{\nu}.$$

Then for any $k \in K$,

$$(\pi_{\lambda} \otimes \pi_{\mu})(k)\xi = \sum_{\nu} \sum_{i=1}^{m_{\nu}} a_{\nu,i} \pi_{\nu}(k)\xi_{\nu} = \sum_{\nu} \sum_{i=1}^{m_{\nu}} a_{\nu,i} \chi_{\nu}(k) \xi_{\nu}. \quad (4.19)$$

By uniqueness of the decomposition of ξ , we obtain combining (4.18) and (4.19) that for any ν such that there is i with $a_{\nu,i} \neq 0$, then

$$\chi_{\nu} = \chi_{\lambda} \chi_{\mu}.$$

It now suffices to show that $a_{\lambda+\mu,1} \neq 0$ - recall that $m_{\lambda+\mu} = 1$. But

$$\langle \xi, v_{\lambda} \otimes v_{\mu} \rangle = \langle \xi_{\lambda}, v_{\lambda} \rangle \langle \xi_{\mu}, v_{\mu} \rangle \neq 0$$

so ξ has a nonzero $V_{\lambda+\mu}$ -component. Thus, we obtain $\chi_{\lambda+\mu} = \chi_{\lambda} \chi_{\mu}$ so for any $n \in \mathbb{N}$, $\lambda \in \Lambda \cap P(U)$, $\chi_{n\lambda} = \chi_{\lambda}^n$.

Let $N_1 = |K/K_0|$ be the order of the finite group K/K_0 , then $\chi_{N_1\lambda} = 1$ for all $\lambda \in \Lambda \cap P(U)$ thus $\pi_{N_1\lambda}$ has a nonzero K -invariant vector. Thus,

$$N_1(\Lambda \cap P(U)) \subset \Lambda_M. \quad (4.20)$$

Set $N = N_0 N_1$, combining (4.17) and (4.20), we obtain

$$N\Lambda = N(\Lambda \cap P(\mathfrak{u})) \subset N_1(\Lambda \cap N_0 P(\mathfrak{u})) \subset N_1(\Lambda \cap P(U)) \subset \Lambda_M$$

which completes the proof. \square

Theorem 4.18. *Let (U, K) be a symmetric pair of compact type, let $\tilde{M} = \tilde{U}/\tilde{K}$ be the universal cover of U/K with \tilde{U} the universal cover of U . Then (\tilde{U}, \tilde{K}) is the compact dual of the pair (G, \tilde{K}) for some semisimple Lie group with finite center G . Let $r = \lfloor \kappa(G) \rfloor$, $\delta = \kappa(G) - r$. Then*

- (i) *there exists an open set V in U such that any K -finite matrix coefficient of U is in $C^{(r,\delta)}(V)$;*
- (ii) *for any $\delta' > \delta$, there exists a K -bi-invariant matrix coefficient of a unitary representation of U which is not in $C^{(r,\delta')}(p(\tilde{U}_r))$.*

Furthermore, the regularity in (i) can be improved to all of $p(\tilde{U}_r)$ whenever (\tilde{U}, \tilde{K}) is a pair of rank 1, of the form $(\tilde{K} \times \tilde{K}, \tilde{K})$ or one of the pairs of Corollary 4.10.

Proof. (i). This is the easy part: let $p : \tilde{U} \rightarrow U$ be the covering map. If φ is a K -finite coefficient of a representation π , then $\varphi \circ p$ is a \tilde{K} -finite coefficient of \tilde{U} . Thus by Corollary 4.15, there exists an open set U_0 such that $\varphi \circ p \in C^{(r,\delta)}(U_0)$. Since p is a local diffeomorphism, for any $x \in V = p(U_0)$, we can find a neighborhood $U_x \subset V$, and an open $\tilde{U}_x \subset U_0$ such that $p : \tilde{U}_x \rightarrow U_x$ is a diffeomorphism. Thus by Lemma 1.3, $\varphi \in C^{(r,\delta)}(U_x)$. Thus $\varphi \in C^{(r,\delta)}(V)$.

To obtain the improvement in specific cases, we use Theorems A, B, 2.28 and 2.37.

(ii). If $\delta' > \delta$, we know from Theorem 4.7 that the family $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$ of spherical functions of (\tilde{U}, \tilde{K}) is not bounded in $C^{(r,\delta)}(U_r)$. Let N be given by Proposition 4.17 so that $N\Lambda \subset \Lambda_{U/K}$. Then it is easy to see from the proof of Theorem 4.7 that the subfamily $(\tilde{\varphi}_\lambda)_{\lambda \in N\Lambda}$ is already not bounded in $C^{(r,\delta)}(U_r)$. But since $N\Lambda \subset \Lambda_{U/K}$, $\tilde{\varphi}_\lambda = \varphi_\lambda \circ p$ with φ_λ a spherical function of (U, K) . Since p is smooth, by Lemma 1.3, $(\varphi_\lambda)_{\lambda \in N\Lambda}$ is not bounded in $C^{(r,\delta)}(p(U_r))$. We then conclude with Lemma 1.10 as usual. \square

4.4 Perspectives

In Chapter 2, in addition to unitary coefficients, we proved regularity results for K -bi-invariant S_p -multipliers when U/K is rank 1 or a Lie group (Corollaries 2.13 and 2.27). Since in this chapter we extended the results on unitary coefficients to any symmetric compact pair (U, K) , a natural direction would be to obtain regularity for S_p -multipliers for any pair (at least in some open subset of U_r).

The strategy would be again to consider the map

$$T : g \mapsto \int_K \int_K \lambda(kgk') dk dk',$$

prove some result on T as a map into $S_p(L^2(G))$ using spherical functions and relate the regularity of T to that of an S_p -multiplier (look back to Section 2.1.3 for more details on this strategy). However, as pointed out in Remark 2.26, in order to understand spherical functions uniformly, our estimates for "regular" spherical functions are too brutal - meaning that although $\varphi_{t\lambda}$ decays in $t^{-n(\lambda)/2}$, we bound this by $t^{-\kappa(G)}$ for every λ . In order to get an optimal result on T , we need to be much more precise. A result in the spirit of [DKV83, Thm. 11.1], but for compact groups, would be a first step. Then, one would have to find a way to sum these estimates without loosing precision - which we were not able to do in Proposition 2.25 in the easier case. Finally, even if we obtain the optimal regularity α_p of T , we only deduce that S_p -multipliers are at least α_p -regular. To prove optimality in this case, we must construct multipliers with specific regularity, which is difficult.

A second direction is of course to extend the results of this chapter to all of U_r for the remaining pairs, but the same proof cannot work. It is clear that for any $g \in K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ - in particular for any $a \in U_r \cap A_{\mathbb{C}}$ - we can consider an analytic extension of H in a neighborhood of g . But since g is not a fixed point of the action by conjugation, this neighborhood cannot be chosen K -invariant as in Lemma 4.3, thus we cannot get the integral expression of spherical functions around g to work with.

However Clerc gave in [Cle88] a multivalued analytic extension of H to all of $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$, as well as an integral formula of spherical functions of (U, K) very similar to Lemma 4.4. But the domain of integration in the expression of $\varphi_\mu(g)$ is now an open subset K_g of K . Thus, the lack of compactness does not allow for a similar proof.

When μ is regular - meaning that $\langle \mu, \alpha \rangle \neq 0$ for any root α - Clerc managed to reduce this integral to a compact subset and get precise estimate of spherical functions (compare

with [DKV83, Thm. 11.1]). However, this is not sufficient for our purposes, and this does not work when μ is not regular.

Part II

Property (TTT) , wq-cocycles and quasi-homomorphisms

Chapter 5

Property (TTT) and quasi-homomorphism rigidity

In this chapter, we study a strengthening of property (T) introduced by Ozawa ([Oza11]). This new property, called (TTT) , deals with objects that do not fully retain the group structure, unlike representations, but instead exhibit a looser connection. The main goal of this chapter is to prove that higher rank simple algebraic group over local fields and their lattices have property (TTT) . After giving some motivations for the definition of property (TTT) in Section 5.1, we first prove that lattices inherits property (TTT) in Section 5.2. Section 5.3 is devoted to the proof of property (TTT) in the case of the symplectic group $Sp_4(\mathbb{K})$. We then proceed to the general case in Section 5.4 and prove Theorem H. Finally, we end this chapter with some comments on the case of simple Lie groups with infinite center. This chapter is self-contained and is primarily based on the work presented in [Dum24a].

5.1 Introduction

Let G be a locally compact second countable group. It is well-known that by the Delorme-Guichardet theorem, G has Kazhdan property (T) if and only if every continuous affine isometric action on a real Hilbert space has a fixed point (this property is called FH , see [BdlHV08, Ch. 2] for more details). Let α be such an action on H , then for any $g \in G$, $\xi \in H$, there exists $\pi(g) \in \mathcal{U}(H)$ and $b(g) \in H$ such that

$$\alpha(g)\xi = \pi(g)\xi + b(g).$$

The map π is a (strongly continuous) unitary representation of G and $b : G \rightarrow H$ is a continuous map verifying

$$\forall g, h \in G, b(gh) = b(g) + \pi(g)b(h).$$

Such a map is called a cocycle. Then G has property (T) if and only if every cocycle is bounded.

Let $\varphi : G \rightarrow G'$ be a continuous group morphism. Assume that G has property (T) and that b is a proper cocycle on G' (i.e. for any $M \geq 0$, $\{x \mid \|b(x)\| \leq M\}$ is compact in G'). Then $\varphi(G)$ is a relatively compact subset of G' .

Definition 5.1. If G, G' are locally compact group, a quasi-homomorphism is a map $\varphi : G \rightarrow G'$ such that $\{\varphi(gh)^{-1}\varphi(g)\varphi(h) \mid g, h \in G\}$ is a relatively compact subset of G' .

Obviously, property (T) does not imply anything on (continuous) quasi-homomorphisms since such a map does not respect the group law. Thus, if we want to study such maps, we need to introduce a stronger property.

Burger and Monod linked bounded cohomology of a group with so-called "rough actions". A (continuous) map $b : G \rightarrow H$ is a quasi-cocycle if there exists a unitary representation π such that

$$\sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty.$$

There is a 1 – 1 correspondence between the translation part of rough actions and quasi-cocycles. In [BM99], Burger and Monod introduced a strengthening of property (T): G has property (TT) if every quasi-cocycle is bounded.

If G has property (TT), then all quasi-homomorphisms $G \rightarrow \mathbb{R}$ (usually called "quasi-morphisms") are bounded. Indeed such a map is a quasi-cocycle associated to the trivial representation. However, this strengthening is not enough to understand quasi-homomorphisms with non-commutative targets. Since the linear part π of a quasi-cocycle must still be a representation, quasi-cocycles are not preserved by composition with quasi-homomorphisms.

In this chapter, we study yet another stronger property introduced by Ozawa ([Oza11]).

Definition 5.2. Let G be a locally compact group and H a Hilbert space. We say that a Borel locally bounded (i.e. bounded on compact subsets) map $b : G \mapsto H$ along with a Borel map $\pi : G \mapsto \mathcal{U}(H)$ is a wq-cocycle if

$$\sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty.$$

Definition 5.3. Let G be a locally compact group and A a subgroup of G .

The pair (G, A) has relative property (TTT) if any wq-cocycle on G is bounded on A .

The group G has property (TTT) if (G, G) has relative property (TTT).

Let $\varphi : G \rightarrow G'$ be a (continuous) quasi-homomorphism and b a wq-cocycle on G' . Then $b \circ \varphi$ is a wq-cocycle on G . Thus, property (TTT) allows to study quasi-homomorphisms where the target group is no longer \mathbb{R} (see [Oza11, Thm. A]).

Ozawa showed that for any local field \mathbb{K} , the group $SL_3(\mathbb{K})$ has property (TTT) as well as all its lattices ([Oza11, Thm. B]). But to obtain his result for lattices, he made use of a stronger property (see Section 5.2.2 for its definition). Thus, it was not clear to him whether in general, property (TTT) passes to lattices that are not cocompact. We show in Section 5.2 that this is true and prove Theorem G, which we recall here.

Theorem G. *Let G be a locally compact second countable group and Γ a lattice in G . Then G has property (TTT) if and only if Γ has property (TTT).*

Our main result is Theorem H which is an extension of Ozawa's result on SL_n to higher rank simple algebraic groups.

Theorem H. *Let G be a connected simple algebraic group over a local field \mathbb{K} with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property (TTT).*

We follow the same idea as the classical proof of property (T) for these group: in Section 5.4 we reduce the proof to the cases of the classical groups SL_3 and Sp_4 . As said before, it is already known that SL_3 has property (TTT). We prove that for any local field \mathbb{K} , $Sp_4(\mathbb{K})$ has property (TTT) in Theorem 5.13.

Finally, Theorem H applies to higher rank simple Lie groups with finite center. On the other hand, when G has infinite center, it is well-known that G has an unbounded quasi-morphism $\phi : G \rightarrow \mathbb{R}$ (see [BG92, Prop. 6]). In particular, G does not have property (TT) nor (TTT) . However, we show in Proposition 5.25 that the unbounded wq-cocycles of G are completely controlled by the unbounded wq-cocycles of its center.

5.2 Related analytical properties

5.2.1 Positive definite kernels and completely bounded norm

Let G be a locally compact second countable group endowed with a left Haar measure. A function $\theta \in L^\infty(G \times G)$ is a positive definite kernel if for any $\xi \in L^1(G)$, $\int \theta(x, y) \xi(x) \overline{\xi(y)} dx dy \geq 0$. Equivalently, θ is a positive definite kernel if and only if there exists a separable Hilbert space H and a measurable map $F : G \rightarrow H$ such that $\theta(x, y) = \langle F(x), F(y) \rangle$ almost everywhere (see [BO08, Appendix D]). If θ is continuous, F can be taken continuous and equality holds everywhere. We say that θ is normalized if $\theta(x, x) = 1$ for any $x \in G$.

The following inequality will be useful throughout the chapter. Let θ be a normalized positive definite kernel on G . Let $x, y, z \in G$. We have

$$\begin{aligned} |\theta(x, z) - \theta(y, z)| &= |\langle F(x) - F(y), F(z) \rangle| \\ &\leq \|F(x) - F(y)\| \\ &\leq \left(\|F(x)\|^2 + \|F(y)\|^2 - \langle F(x), F(y) \rangle - \langle F(y), F(x) \rangle \right)^{1/2} \\ &\leq \left(2 - \theta(x, y) - \overline{\theta(x, y)} \right)^{1/2} \\ &\leq \sqrt{2} |\theta(x, y) - 1|^{1/2}. \end{aligned}$$

Let $\theta \in L^\infty(G \times G)$. We define the cb-norm of θ by

$$\|\theta\|_{cb} = \inf \{ \|P\| \|Q\| : P, Q \in L^\infty(G; H), \theta(x, y) = \langle P(x), Q(y) \rangle \}.$$

5.2.2 Property (T_P) and (T_Q)

Let G be a locally compact second countable group and A a subgroup of G . As explained in the previous section, relative property (TTT) is a natural strengthening of the fixed-point property FH . Although (T) and FH are formally equivalent by the Delorme-Guichardet theorem, it is often easier to work with the analytic version: recall that if $\pi : G \rightarrow \mathcal{U}(H)$ is a unitary representation, π almost has invariant vectors if for any compact subset and any $\varepsilon > 0$, there exists a unit vector $\xi \in H$ which is (Q, ε) -invariant, that is to say such that

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon.$$

The pair (G, A) has relative property (T) if whenever π almost has invariant vectors, π has a nonzero A -invariant vector. It is a classical result that (G, A) has relative property (T) if and only if there exists a compact set S and a constant $\kappa = \kappa(G, S)$ such that whenever π is a unitary representation which has an (S, κ) -invariant vector, π has a nonzero A -invariant vector - in fact, this is taken as a definition in [BdlHV08]. We say that (S, κ) is a Kazhdan pair. The following lemma is a straightforward consequence of the definition (see [BdlHV08, Proposition 1.1.9]).

Lemma 5.4. *Let (S, κ) be a Kazhdan pair for (G, A) and π a unitary representation. For any unit vector ξ , if*

$$\sup_{x \in S} \|\pi(x)\xi - \xi\| < \delta$$

then

$$\sup_{a \in A} \|\pi(a)\xi - \xi\| < 2\kappa^{-1}\delta.$$

This means a if (G, A) has relative property (T), for any $\varepsilon > 0$, there exists δ such that any vector which is (S, δ) -invariant is (A, ε) -invariant. The converse is actually true. Indeed, for any π with almost invariant vectors, pick $\varepsilon < 1$, δ associated and ξ which is (S, δ) -invariant. Then the unique vector of minimal norm in $\text{Conv}\{\pi(a)\xi\}$ is nonzero and A -invariant.

To obtain an analytic version of property (TTT), we want to assume that π is no longer a representation - but some similar object - in the definition of property (T). Of course, invariant vectors are no longer a good notion to consider, but ε -invariant vectors are. This led Ozawa to the following definition.

Definition 5.5 ([Oza11]). The pair (G, A) has relative property (T_Q) if for all $\varepsilon > 0$, there exist $\delta > 0$ and a compact subset $K \subset G$ such that for every Borel map $\pi : G \mapsto \mathcal{U}(H)$ and every unit vector $\xi \in H$ verifying

$$\sup_{g, h \in G} \|\pi(gh)\xi - \pi(g)\pi(h)\xi\| < \delta \quad (5.1)$$

and

$$\sup_{g \in K} \|\pi(g)\xi - \xi\| < \delta \quad (5.2)$$

then

$$\sup_{x \in A} \|\pi(x)\xi - \xi\| < \varepsilon. \quad (5.3)$$

If π is a unitary representation and ξ a unit vector, the map $\theta : (x, y) \mapsto \langle \pi(x)\xi, \pi(y)\xi \rangle$ is a positive definite normalized kernel which is left G -invariant. Invariance of the vector translate into a property of the kernel: given a subset Q of G , ξ is (Q, δ) -invariant if and only if $\theta(x, 1) \simeq_\delta 1$ for any $x \in Q$. By the Gelfand-Naimark-Segal construction ([BdlHV08, Thm. C.4.10]), any left G -invariant positive definite kernel is actually of this form. Thus, property (T) can be interpreted in terms of kernels. Since left invariance is due to π being a representation, another way to generalize property (T) to non-representation is to loosen left-invariance of kernels. This led Ozawa to another definition.

Definition 5.6 ([Oza11]). The pair (G, A) has relative property (T_P) if for all $\varepsilon > 0$, there are $\delta > 0$ and a compact subset $K \subset G$ such that for any $\theta : G \times G \mapsto \mathbb{C}$ Borel normalized positive definite kernel verifying

$$\sup_{g \in G} \|\theta(g \cdot, g \cdot) - \theta\|_{cb} < \delta \quad (5.4)$$

and

$$\sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta \quad (5.5)$$

then

$$\sup_{x, y \in A} |\theta(x, y) - 1| < \varepsilon. \quad (5.6)$$

Remark 5.7. As pointed out by Ozawa ([Oza11, Section 3]), it is enough to consider only continuous kernels instead of Borel kernels. Furthermore, the hypothesis (5.5) can be weakened to

$$\sup_{x \in K} |\theta(x, 1) - 1| < \delta. \quad (5.7)$$

Indeed, if θ verifies (5.4) and (5.7), then for any $g, h \in G$ with $g^{-1}h \in K$, there is $x \in K$ such that $h = gx$ so

$$\begin{aligned} |\theta(g, h) - 1| &= |\theta(g, gx) - 1| \\ &\leq |\theta(g, gx) - \theta(1, x)| + |\theta(x, 1) - 1| \\ &< 2\delta \end{aligned}$$

so θ verifies (5.5) up to replacing δ by 2δ .

To sum up, we defined property (TTT) as a natural strengthening of property FH , as well as two generalizations of property (T) . Recall that property FH is equivalent to property (T) - for second countable locally compact groups. We may ask ourselves if all three strengthenings defined by Ozawa are equivalent. Ozawa proved the following implications between these strengthenings of property (T) ([Oza11, Thm. 1]).

Theorem 5.8.

$$\text{rel. property } (T_P) \implies \text{rel. property } (TTT) \implies \text{rel. property } (T_Q).$$

A direct proof that $(T_P) \implies (T_Q)$ was almost explained above: given the data of (π, ξ) , one can construct a positive definite kernel $\theta = \langle \pi(\cdot)\xi, \pi(\cdot)\xi \rangle$. It is then routine to check that we can apply (T_P) to θ and obtain the result. However, due to the lack of left invariance in our kernels, we cannot directly link them to maps π by the GNS construction, which is an obstacle to prove that $(T_Q) \implies (T_P)$. Nevertheless, we will prove that these properties are all equivalent. To do so, we first establish some preliminary results from functional analysis related to kernels.

5.2.3 Measurable factorization

Let X be a σ -finite measure space such that $L^2(X)$ is a separable Hilbert space, for example X a locally compact second countable group. Then $L^1(X)$ is also separable. If E is a separable Banach space, a function $\phi : X \rightarrow E^*$ is w^* -measurable if $x \mapsto [\phi(x)](v)$ is measurable for any $v \in E$. Since E is separable, let (x_n) be a dense sequence in the unit sphere of E . Then

$$\|\phi(\cdot)\| = \sup_{n \in \mathbb{N}} |\phi(\cdot)(x_n)|$$

is measurable, as the supremum of measurable functions. Thus, we can define $L^p_\sigma(X; E^*)$ as the space of w^* -measurable functions $\phi : X \rightarrow E^*$ such that

$$\|\phi\|_p = \|\|\phi(\cdot)\|\|_p < +\infty$$

(see [DU77] for more details). By Pettis measurability theorem ([DU77, Ch. II, Thm. 2]), if E^* is separable and $\phi : X \rightarrow E^*$ is such that $x \mapsto u(\phi(x))$ is measurable for any $u \in E^{**}$, then ϕ is Bochner measurable. This implies that when E is a separable reflexive Banach space, the space $L^p_\sigma(X; E^*)$ coincides with the space $L^p(X; E^*)$ of (Bochner) measurable functions. This holds more generally when E^* has the Radon-Nikodym property (see [DU77, Ch. IV]).

Let E, F be two Banach spaces. We denote $E \hat{\otimes} F$ the completion of $E \otimes F$ for the projective tensor norm (see [DU77, Ch. VIII]). When E, F are separable, this is a separable Banach space. By [DU77, Ch. VIII.2, Coro. 2], there is an isometric isomorphisms

$$(E \hat{\otimes} F)^* \simeq B(E, F^*) \quad (5.8)$$

and $\phi : E \hat{\otimes} F \rightarrow \mathbb{C}$ corresponds to the unique bounded operator $u : E \rightarrow F^*$ such that $\forall x, y \in E \times F$, $\phi(x \otimes y) = u(x)(y)$. Thus, we can define the spaces $L_\sigma^\infty(X; B(E, F^*))$ when E, F are separable Banach spaces.

Let E be a Banach space. By [DU77, Ch. VIII.1, Ex.10], the natural embedding $L^1(X) \otimes E \rightarrow L^1(X; E)$ extends to an isometric isomorphism

$$L^1(X) \hat{\otimes} E \simeq L^1(X; E). \quad (5.9)$$

Furthermore, if E is separable, the map

$$\begin{aligned} L_\sigma^\infty(X; E^*) &\rightarrow L^1(X; E)^* \\ \xi &\mapsto u \mapsto \int_X [\xi(x)](u(x)) dx \end{aligned} \quad (5.10)$$

is an isometric isomorphism (see [Coi17, Thm. 1.16] or [Pis16, Prop. 2.20, 2.26 and Thm. 2.29]).

Let H be a separable Hilbert space. Combining (5.9) and (5.10), a function ξ in $L^\infty(X; H) = L_\sigma^\infty(X; H^*)$ corresponds to a functional ϕ on $L^1(X) \hat{\otimes} H$, which is defined on simple tensors $u \otimes y \in L^1(X) \otimes H$ by

$$\phi(u \otimes y) = \int_X u(x) \langle \xi(x), y \rangle dx.$$

Thus by (5.8), the map

$$T : \begin{aligned} L^\infty(X; H) &\rightarrow B(L^1(X), H) \\ \xi &\mapsto u \mapsto \int_X u(x) \xi(x) dx \end{aligned} \quad (5.11)$$

is an isometric isomorphism.

Let E, F be two separable Banach spaces. The above properties give isometric isomorphisms

$$\begin{aligned} L_\sigma^\infty(X; B(E, F^*)) &\simeq L_\sigma^\infty(X; (E \hat{\otimes} F)^*) && \text{by (5.8)} \\ &\simeq L^1(X; E \hat{\otimes} F)^* && \text{by (5.10)} \\ &\simeq (L^1(X) \hat{\otimes} (E \hat{\otimes} F))^* && \text{by (5.9)} \\ &\simeq (E \hat{\otimes} L^1(X) \hat{\otimes} F)^* \\ &\simeq (E \hat{\otimes} L^1(X; F))^* && \text{by (5.9)} \\ &\simeq B(E, L^1(X, F)^*) && \text{by (5.8)} \\ &\simeq B(E, L_\sigma^\infty(X; F^*)) && \text{by (5.10)} \end{aligned}$$

and following the path of isomorphisms shows that

$$\begin{aligned} L_\sigma^\infty(X; B(E, F^*)) &\rightarrow B(E, L_\sigma^\infty(X; F^*)) \\ \xi &\mapsto u \mapsto \xi(\cdot)(u) \end{aligned} \quad (5.12)$$

is an isometric isomorphism.

Let

$$\Gamma_2 \left(L^1(X), L^\infty(X) \right) = \left\{ T \in B(L^1(X), L^\infty(X)) \left| \begin{array}{l} T = SR \text{ where } R \in B(L^1(X), H), \\ S \in B(H, L^\infty(X)) \text{ for some} \\ \text{separable Hilbert space } H \end{array} \right. \right\}$$

with norm $\gamma(T) = \inf \|S\| \|R\|$. Let $z \in L^1(X) \otimes L^1(X)$. We define

$$\|z\|_* = \inf \left(\sum \|u_i\|^2 \right)^{1/2} \left(\sum \|v_i\|^2 \right)^{1/2}$$

where the infimum runs over all finite families $(u_i), (v_i)$ such that for $\xi, \eta \in (L^1(X))^*$,

$$|(\xi \otimes \eta)(z)| \leq \left(\sum |\xi(u_i)|^2 \right)^{1/2} \left(\sum |\eta(v_i)|^2 \right)^{1/2}.$$

Then, $\|\cdot\|_*$ is a norm on $L^1(X) \otimes L^1(X)$. By [Pis86, Thm. 2.8 and Coro. 2.9], there is an isometric isomorphism

$$\Gamma_2 \left(L^1(X), L^\infty(X) \right) \simeq \left(L^1(X) \otimes_* L^1(X) \right)^* \quad (5.13)$$

where $L^1(X) \otimes_* L^1(X)$ is the completion of the tensor product $L^1(X) \otimes L^1(X)$ for the norm $\|\cdot\|_*$. Thus, this space has a separable predual and we can consider the spaces of w^* -measurable functions $L^\infty(Y; \Gamma_2(L^1(X), L^\infty(X)))$ with respect to this predual.

If $\varphi \in L^\infty(X \times X)$, we can define $r_\varphi \in B(L^1(X), L^\infty(X))$ by

$$r_\varphi(f)(s) = \int_X f(t) \varphi(t, s) dt.$$

By [Spr04, Thm. 3.3], φ is a Schur multiplier if and only if $r_\varphi \in \Gamma_2(L^1(X), L^\infty(X))$, and in that case, $\|\varphi\|_{cb} = \gamma(r_\varphi)$.

Let $\phi \in L^\infty(X \times X \times X)$ and denote $\phi_x = \phi(\cdot, x, \cdot)$. Such a map defines an operator

$$\tilde{\phi} \in L^\infty \left(X; B \left(L^1(X), L^\infty(X) \right) \right)$$

by

$$\tilde{\phi}(x)(u) = \int_X \phi(t, x, \cdot) u(t) dt = r_{\phi_x}(u).$$

Proposition 5.9. *Let G be a locally compact second countable group. Let $\theta \in L^\infty(G \times G)$ be a positive definite kernel on G such that for any $g \in G$, $\|g\theta - \theta\|_{cb} \leq \delta$. Consider the map $\phi \in L^\infty(G \times G \times G)$ defined by $\phi(x, g, y) = \theta(gx, gy) - \theta(x, y)$. Then there exists a separable Hilbert space H and two functions $a, b \in L^\infty(G; B(L^1(G), H))$ such that for almost every $g \in G$ and for every $u, v \in L^1(G)$,*

$$\left[\tilde{\phi}(g)(u) \right] (v) = \langle a(g)(u), b(g)(v) \rangle$$

with $\|a\|_\infty \|b\|_\infty \leq \delta$.

Proof. Since ϕ_g is a Schur multiplier for any $g \in G$, we have

$$\tilde{\phi} \in L^\infty \left(G; \Gamma_2 \left(L^1(G), L^\infty(G) \right) \right)$$

with $\|\tilde{\phi}\|_{\infty, \Gamma_2} = \sup_{g \in G} \gamma(r_{\phi_g}) \leq \delta$.

The result is then a direct consequence of [CLMS21, Thm 5.1]. □

Lemma 5.10. *Let H be a separable Hilbert space, X, Y measured spaces such that the spaces $L^2(X)$ and $L^2(Y)$ are separable and Y is complete. Let $\alpha, \beta \in L^\infty_\sigma(Y; B(L^1(X), H))$ be two maps such that for almost every $y \in Y$ and every $u, v \in L^1(X)$,*

$$\langle \alpha(y)(u), \alpha(y)(v) \rangle = \langle \beta(y)(u), \beta(y)(v) \rangle. \quad (5.14)$$

Then there exists a map $\pi : Y \rightarrow \mathcal{U}(H \oplus \ell^2(\mathbb{N}))$ which is measurable when the group $\mathcal{U}(H \oplus \ell^2(\mathbb{N}))$ is endowed with the Borel σ -algebra coming from the strong operator topology, such that for almost all $y \in Y$, for all $u \in L^1(X)$, $U_y(\alpha(y)(u)) = \beta(y)(u)$.

Proof. First, by (5.12), the map

$$\begin{array}{ccc} L^\infty_\sigma(Y; B(L^1(X), H)) & \rightarrow & B(L^1(X), L^\infty_\sigma(Y; H)) \\ \alpha & \mapsto & u \mapsto \alpha(\cdot)(u) \end{array}$$

is an isometric isomorphism. Furthermore, $L^\infty_\sigma(Y; H) = L^\infty(Y; H)$ since H is a separable Hilbert space. Thus, for $u \in L^1(X)$, the maps $y \mapsto \alpha(y)(u)$ and $y \mapsto \beta(y)(u)$ are measurable.

Set $H' = H \oplus \ell^2(\mathbb{N})$. Since $L^1(X)$ is separable, we can consider $(u_n)_{n \in \mathbb{N}}$ a dense sequence in $L^1(X)$. Denote Y' a conull set in Y such that (5.14) holds for all $y \in Y'$.

If $y \in Y$, define $H_y = \overline{\alpha(y)(L^1(X))}$, then the sequence $(\alpha(y)(u_n))_{n \in \mathbb{N}}$ is dense in H_y . We apply the Gram-Schmidt process to this family: set $a_0(y) = \alpha(y)(u_0)$ which is measurable. If we have constructed $a_0(y), \dots, a_{n-1}(y)$ such that

$$\text{span}(a_0(y), \dots, a_{n-1}(y)) = \text{span}(\alpha(y)(u_0), \dots, \alpha(y)(u_{n-1}))$$

and each a_k is measurable, we set

$$a_n(y) = \alpha(y)(u_n) - \sum_{k < n, a_k(y) \neq 0} \frac{\langle a_k(y), \alpha(y)(u_n) \rangle}{\|a_k(y)\|^2} a_k(y).$$

Recursively, this give a family of vectors $(a_n(y))_{n \in \mathbb{N}}$ which for each y contains an orthogonal basis of H_y and some zero vectors. Since $\{y \mid a_n(y) \neq 0\}$ is measurable, replacing $a_n(y)$ by $a_n(y)/\|a_n(y)\|$ on this set still gives a measurable function, and now $(a_n(y))$ contains an orthonormal basis and some zero vectors for each $y \in Y$.

With the same process, we construct for each $y \in Y$ a family $(b_n(y))_{n \in \mathbb{N}}$ containing an orthonormal basis of $K_y = \overline{\beta(y)(L^1(X))}$ and some zero vectors such that for each $n \in \mathbb{N}$, $y \mapsto b_n(y)$ is measurable.

The crucial point is that using the hypothesis (5.14), for any $y \in Y'$ we have

$$a_n(y) = 0 \iff b_n(y) = 0 \quad (5.15)$$

and

$$a_n(y) = \sum_{k=0}^n \lambda_k(y) \alpha(y)(u_k) \iff b_n(y) = \sum_{k=0}^n \lambda_k(y) \beta(y)(u_k). \quad (5.16)$$

Now, consider an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H and an orthonormal basis $(f_n)_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$. Since $\ell^2(\mathbb{N})$ has uncountable dimension, there exists $(f'_n)_{n \in \mathbb{N}}$ such that $(f_n) \cup (f'_n)$ is linearly independent. Let $(g_n) = (e_n + f'_n) \cup (f_n)$. This is a total family in $H' = H \oplus \ell^2(\mathbb{N})$. Let

$$c_n(y) = P_{H_y^\perp}(g_n) = g_n - \sum \langle a_n(y), g_n \rangle a_n(y)$$

and

$$d_n(y) = P_{K_y^\perp}(g_n) = g_n - \sum \langle b_n(y), g_n \rangle b_n(y).$$

As limits of measurable functions, c_n, d_n are measurable since Y is complete. The family $(c_n(y))_{n \in \mathbb{N}}$ is total in H_y^\perp , and linearly independent. Indeed, if there is a relation $\sum_{i=1}^n \lambda_i c_i(y) = 0$, then $\sum_{i=1}^n \lambda_i g_i \in H_y$, but $(\text{span}(g_n)_{n \in \mathbb{N}}) \cap H = \{0\}$ by construction, so $\lambda_i = 0$ for any $1 \leq i \leq n$.

Similarly, the family $(d_n(y))$ is total in K_y^\perp and linearly independent. Thus, applying the Gram-Schmidt process produces $(a'_n(y))_{n \in \mathbb{N}}$ and $(b'_n(y))_{n \in \mathbb{N}}$, which are also measurable functions and an orthonormal basis of H_y^\perp, K_y^\perp respectively.

For $y \in Y'$, we have two orthonormal bases of $H' = H \oplus \ell^2(\mathbb{N})$. Thus, there is a unique unitary map U_y sending $a_n(y)$ to $b_n(y)$ and $a'_n(y)$ to $b'_n(y)$, using (5.15) to ensure that U_y is well-defined on the zero vectors in $(a_n(y))_{n \in \mathbb{N}}$. On $Y \setminus Y'$, we set $U_y = \text{Id}$.

Using (5.16) we show recursively that for any $y \in Y', n \in \mathbb{N}$,

$$U_y(\alpha(y)(u_n)) = \beta(y)(u_n).$$

Thus, by density of (u_n) and continuity of U_y , we get that for any $u \in L^1(X)$,

$$U_y(\alpha(y)(u)) = \beta(y)(u).$$

Let $\xi \in H'$. Then for $y \in Y'$,

$$\xi = \sum_{n \geq 0} (\langle a_n(y), \xi \rangle a_n(y) + \langle a'_n(y), \xi \rangle a'_n(y))$$

so

$$U_y \xi = \sum_{n \geq 0} (\langle a_n(y), \xi \rangle b_n(y) + \langle a'_n(y), \xi \rangle b'_n(y)).$$

Again since Y is complete, $y \mapsto U_y \xi$ is measurable as a pointwise limit almost everywhere of measurable functions.

Since this is true for any $\xi \in H'$ and since H' is separable, this implies that $y \mapsto U_y$ is measurable for the strong operator topology on $\mathcal{U}(H')$. \square

5.2.4 Equivalence between properties

Theorem 5.11. *If G is a second countable locally compact group and A a subgroup of G , then if (G, A) has relative property (T_Q) , (G, A) has relative property (T_P) .*

Proof. Let $\varepsilon > 0$ and θ be a continuous positive definite normalized kernel verifying (5.4) and (5.5) for some δ, K to be determined later. By definition, there exists a separable Hilbert space H and a continuous map $\xi : G \rightarrow H$ such that $\forall g, h \in G, \theta(g, h) = \langle \xi(g), \xi(h) \rangle$ and $\forall g \in G, \|\xi(g)\| = 1$.

By Proposition 5.9, there exists a separable Hilbert space H' and two functions $a, b \in L^\infty_\sigma(G; B(L^1(G), H'))$ such that for almost every $g \in G$ and for all $u, v \in L^1(G)$,

$$\int_{G \times G} (\theta(gx, gy) - \theta(x, y)) u(x)v(y) dx dy = \langle a(g)(u), b(g)(v) \rangle$$

with $\|a\|_\infty \|b\|_\infty < \delta$. Up to multiplying a, b by some constant, we can actually assume that $\|a\|_\infty < \sqrt{\delta}$ and $\|b\|_\infty < \sqrt{\delta}$.

With the notation of (5.11), we also get

$$\int_{G \times G} (\theta(gx, gy) - \theta(x, y)) u(x)v(y) dx dy = \langle T(g^{-1}\xi)(u), T(g^{-1}\xi)(v) \rangle - \langle T(\xi)(u), T(\xi)(v) \rangle.$$

But then, setting $\tilde{a}(g)(u) = \frac{a(g)(u)+b(g)(u)}{2}$ and $\tilde{b}(g)(u) = \frac{a(g)(u)-b(g)(u)}{2}$, we also have $\|\tilde{a}\|_\infty < \sqrt{\delta}$ and $\|\tilde{b}\|_\infty < \sqrt{\delta}$. In the space $H \oplus H'$, we have for almost every $g \in G$ and any $u, v \in L^1(G)$ that

$$\langle (T(\xi)(u), \tilde{a}(g)(u)), (T(\xi)(v), \tilde{a}(g)(v)) \rangle = \langle (T(g^{-1}\xi)(u), \tilde{b}(g)(u)), (T(g^{-1}\xi)(v), \tilde{b}(g)(v)) \rangle.$$

We apply Lemma 5.10 to $X = Y = G$ and

$$\alpha(g)(u) = (T(\xi)(u), \tilde{a}(g)(u)), \beta(g)(u) = (T(g^{-1}\xi)(u), \tilde{b}(g)(u)),$$

to get a map $\pi : G \rightarrow \mathcal{U}(H \oplus H' \oplus \ell^2(\mathbb{N}))$ which is measurable for the completion of the Borel σ -algebra on G , and such that for almost every $g \in G$ and every $u \in L^1(G)$,

$$\pi(g)(T(\xi)(u), \tilde{a}(g)(u)) = (T(g^{-1}\xi)(u), \tilde{b}(g)(u)). \quad (5.17)$$

Then,

$$\begin{aligned} \|T(g^{-1}\xi)(u) - \pi(g)T(\xi)(u)\| &\leq \|\tilde{b}(g)(u)\| + \|(T(g^{-1}\xi)(u), \tilde{b}(g)(u)) - \pi(g)T(\xi)(u)\| \\ &\leq \sqrt{\delta}\|u\| + \|\pi(g)(T(\xi)(u), \tilde{a}(g)(u)) - \pi(g)T(\xi)(u)\| \\ &\leq \sqrt{\delta}\|u\| + \|\tilde{a}(g)(u)\| \\ &\leq 2\sqrt{\delta}\|u\|. \end{aligned}$$

But since

$$\pi(g)T(\xi)(u) = \pi(g) \int_G u(x)\xi(x) dx = \int_G u(g)\pi(g)(\xi(x)) dx = T(\pi(g) \circ \xi)(u),$$

we get that for almost every g ,

$$\|T(g^{-1}\xi - \pi(g) \circ \xi)(u)\| \leq 2\sqrt{\delta}\|u\|$$

thus

$$\|T(g^{-1}\xi - \pi(g) \circ \xi)\|_{B(L^1(X), H)} \leq 2\sqrt{\delta}.$$

Since T is an isometry and ξ is continuous, for almost every $g \in G$ and for all $x \in G$,

$$\|\xi(gx) - \pi(g)\xi(x)\| \leq 2\sqrt{\delta}. \quad (5.18)$$

We want to change π so that (5.17) holds everywhere and π is a Borel map. We proceed as in [Oza11]. Let M be a Borel subset of G of measure zero such that (5.17) holds for all $g \in G \setminus M$. There exists also a Borel subset N of measure zero such that π is Borel $G \setminus N$. By regularity of the Haar measure, there exists a G_δ set of measure zero $N' = \bigcap_n U_n$ such that $M \cup N \subset N'$. Let K be any compact neighborhood of G and consider the map multiplication map $m : (G \setminus N') \times (K \setminus N') \rightarrow G$. Since N' has zero measure and K positive measure, m is surjective. Furthermore, for any $g \in G$,

$$\begin{aligned} m^{-1}(\{g\}) &= \{(gk^{-1}, k) \mid k \in K \setminus N', xk^{-1} \in G \setminus N'\} \\ &= \bigcup_{p, q \in \mathbb{N}} \{(gk^{-1}, k) \mid k \in K \cap U_p^c\} \cap ((G \setminus U_n) \times K) \end{aligned}$$

so $m^{-1}(\{g\})$ is σ -compact. Thus, applying the Lusin-Novikov uniformization theorem ([Kec12, Thm.35.46]), there exists a Borel section $s : G \rightarrow (G \setminus N') \times (K \setminus N')$ of m . Then $t = p_K \circ s : G \rightarrow K$ is a Borel map such that $\forall g \in G, gt_g^{-1}, t_g \in G \setminus N'$.

Set $\tilde{\pi}(g) = \pi(gt_g^{-1})\pi(t_g)$, this is a Borel map and $\forall g \in G$,

$$\begin{aligned} \|\xi(gx) - \tilde{\pi}(g)\xi(x)\| &\leq \|\xi(gx) - \pi(gt_g^{-1})\xi(t_gx)\| \\ &\quad + \|\pi(gt_g^{-1})\xi(t_gx) - \pi(gt_g^{-1})\pi(t_g)\xi(x)\| \\ &\leq 2\sqrt{\delta} + \|\xi(t_gx) - \pi(t_g)\xi(x)\| \\ &\leq 4\sqrt{\delta} \end{aligned}$$

since (5.18) holds for t_g and gt_g^{-1} .

Let $\xi = \xi(e)$. Let us show that (5.1) and (5.2) hold for the pair $(\tilde{\pi}, \xi)$, in order to apply relative property (T_Q) .

By hypothesis (5.5), we have for any $g \in G, x \in K$,

$$|\theta(g, gx) - 1| = |\langle \xi(g), \xi(gx) \rangle - 1| < \delta.$$

Thus,

$$\begin{aligned} \|\xi(gx) - \xi(g)\|^2 &= \left| \|\xi(gx)\|^2 + \|\xi(g)\|^2 - \langle \xi(gx), \xi(g) \rangle - \langle \xi(g), \xi(gx) \rangle \right| \\ &\leq |1 - \langle \xi(gx), \xi(g) \rangle| + |1 - \langle \xi(g), \xi(gx) \rangle| \\ &\leq 2|\langle \xi(g), \xi(gx) \rangle - 1| \\ &\leq 2\delta. \end{aligned}$$

Hence, for any $g \in G, x \in K$, we have

$$\|\xi(gx) - \xi(g)\| < \sqrt{2\delta}. \quad (5.19)$$

Then, if $x \in K$, we have

$$\|\tilde{\pi}(x)\xi - \xi\| \leq \|\tilde{\pi}(x)\xi(e) - \xi(x)\| + \|\xi(x) - \xi(e)\| \leq (4 + \sqrt{2})\sqrt{\delta} = \delta'$$

by (5.18) and (5.19).

Let $g, h \in G$. We have that

$$\begin{aligned} \|\tilde{\pi}(gh)\xi - \tilde{\pi}(g)\tilde{\pi}(h)\xi\| &\leq \|\tilde{\pi}(gh)\xi(e) - \xi(gh)\| + \|\xi(gh) - \tilde{\pi}(g)\xi(h)\| \\ &\quad + \|\tilde{\pi}(g)\xi(h) - \tilde{\pi}(g)\tilde{\pi}(h)\xi(e)\| \\ &< 4\sqrt{\delta} + 4\sqrt{\delta} + \|\xi(h) - \tilde{\pi}(h)\xi(e)\| \\ &< 12\sqrt{\delta} = \delta''. \end{aligned}$$

Now since (G, A) has relative property (T_Q) , choosing K associated to ε in (T_Q) and δ small enough so that δ', δ'' are associated to ε , we get that (5.1) and (5.2) hold for $(\tilde{\pi}, \xi)$. Then we have by relative (T_Q) (5.3) that for any $x \in A$,

$$\|\tilde{\pi}(x)\xi - \xi\| = \|\tilde{\pi}(x)\xi(e) - \xi(e)\| < \varepsilon.$$

Let $x, y \in A$. We have that

$$\begin{aligned} |\theta(x, y) - 1|^2 &= |\langle \xi(x), \xi(y) \rangle - 1|^2 \\ &= (1 - \langle \xi(x), \xi(y) \rangle)(1 - \overline{\langle \xi(x), \xi(y) \rangle}) \\ &= 1 - \langle \xi(x), \xi(y) \rangle - \overline{\langle \xi(x), \xi(y) \rangle} + |\langle \xi(x), \xi(y) \rangle|^2 \\ &\leq 2 - \langle \xi(x), \xi(y) \rangle - \overline{\langle \xi(x), \xi(y) \rangle} \\ &= \|\xi(x) - \xi(y)\|^2 \end{aligned}$$

thus

$$\begin{aligned}
|\theta(x, y) - 1| &\leq \|\xi(x) - \xi(y)\| \\
&\leq \|\xi(x) - \tilde{\pi}(x)\xi(e)\| + \|\tilde{\pi}(x)\xi(e) - \xi(e)\| \\
&\quad + \|\xi(e) - \tilde{\pi}(y)\xi(e)\| + \|\tilde{\pi}(y)\xi(e) - \xi(y)\| \\
&\leq 4\sqrt{\delta} + \varepsilon + \varepsilon + 4\sqrt{\delta} = \varepsilon'
\end{aligned}$$

by relative (T_Q) and by (5.18).

Hence, we showed (5.6) for ε' , so (G, A) has relative property (T_P) . \square

It was shown in [Oza11] that both (T_P) and (T_Q) passes to lattices, but as noticed in the introduction, it was not clear whether (TTT) passes to non cocompact lattices. The equivalence of these three properties immediately implies Theorem G.

Corollary 5.12. *Let G be a locally compact group and Γ a lattice in G . Then G has (TTT) if and only if Γ has (TTT).*

5.3 The symplectic group $Sp_4(\mathbb{K})$

Let \mathbb{K} be a local field. We consider the symplectic group

$$Sp_4(\mathbb{K}) = \{g \in GL_4(\mathbb{K}) \mid {}^t g J g = J\}$$

where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Let also $S^{2*}(\mathbb{K}^2)$ be the vector space of symmetric bilinear form on \mathbb{K}^2 which can be identified with the space of symmetric matrices in $M_2(\mathbb{K})$. Then the group $SL_2(\mathbb{K})$ acts on $S^{2*}(\mathbb{K}^2)$ by $g.B = gB^t g$.

Consider the subgroup

$$G_2 = \left\{ g_A = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \mid A \in SL_2(\mathbb{K}) \right\} \simeq SL_2(\mathbb{K})$$

as well as the two subgroups

$$N_2^+ = \left\{ X_B^+ = \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \mid B \in M_2(\mathbb{K}), {}^t B = B \right\}$$

and

$$N_2^- = \left\{ X_B^- = \begin{pmatrix} I_2 & 0 \\ B & I_2 \end{pmatrix} \mid B \in M_2(\mathbb{K}), {}^t B = B \right\}.$$

Then the maps

$$\iota_1 : \begin{array}{ccc} SL_2(\mathbb{K}) \times S^{2*}(\mathbb{K}^2) & \rightarrow & Sp_4(\mathbb{K}) \\ (A, B) & \mapsto & X_B^+ g_A \end{array}$$

and

$$\iota_2 : \begin{array}{ccc} SL_2(\mathbb{K}) \times S^{2*}(\mathbb{K}^2) & \rightarrow & Sp_4(\mathbb{K}) \\ (A, B) & \mapsto & X_B^- g_A^{-1} \end{array}$$

define two group embeddings of $SL_2(\mathbb{K}) \times S^{2*}(\mathbb{K}^2)$ with N_2^+, N_2^- as images of $S^{2*}(\mathbb{K}^2)$.

It is known that the pair $(SL_2(\mathbb{K}) \times S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$ has relative property (T) (see [BdlHV08, Coro. 1.5.2]) thus by [Oza11, Prop. 3], it has relative property (T_P) .

Theorem 5.13. *Let \mathbb{K} be a local field. The group $Sp_4(\mathbb{K})$ has property (T_P) .*

We first need a Mautner type lemma adapted to the context of "almost invariance" instead of the usual invariance.

Lemma 5.14. *Let G be a locally compact group, $\theta : G \times G \rightarrow \mathbb{C}$ a normalized positive definite kernel such that $\sup_{g \in G} \|g \cdot \theta - \theta\|_{cb} < \varepsilon$. Let $x, y \in G$ be such that*

$$|\theta(y^{-1}xy, 1) - 1| < \varepsilon \quad \text{and} \quad |\theta(y, 1) - 1| < \varepsilon.$$

Then

$$|\theta(x, 1) - 1| < 2\varepsilon + 4\varepsilon^{1/2}.$$

Proof. First, note that for any $g \in G$,

$$|\theta(gy, g) - 1| \leq |\theta(gy, g) - \theta(y, 1)| + |\theta(y, 1) - 1| < 2\varepsilon.$$

We have

$$\begin{aligned} |\theta(x, 1) - 1| &\leq |\theta(x, 1) - \theta(y^{-1}xy, 1)| + |\theta(y^{-1}xy, 1) - 1| \\ &< |\theta(x, 1) - \theta(y^{-1}x, y^{-1})| + |\theta(y^{-1}x, y^{-1}) - \theta(y^{-1}xy, 1)| + \varepsilon \\ &< 2\varepsilon + |\theta(y^{-1}x, y^{-1}) - \theta(y^{-1}x, 1)| + |\theta(y^{-1}x, 1) - \theta(y^{-1}xy, 1)| \\ &< 2\varepsilon + \sqrt{2}|\theta(1, y^{-1}) - 1|^{1/2} + \sqrt{2}|\theta(y^{-1}xy, y^{-1}x) - 1|^{1/2} \\ &< 2\varepsilon + 2\sqrt{2}(2\varepsilon)^{1/2}. \end{aligned} \quad \square$$

We are now ready to prove Theorem 5.13

Proof of Theorem 5.13. Let $\varepsilon > 0$ and (K_0, δ) associated to ε in property (T_P) for the pair

$$\left(SL_2(\mathbb{K}) \rtimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2) \right).$$

We may assume $\delta < \varepsilon$. Consider ι_1 and ι_2 the embeddings of $SL_2(\mathbb{K}) \rtimes S^{2*}(\mathbb{K}^2)$ into $G = Sp_4(\mathbb{K})$, and set $K = \iota_1(K_0) \cup \iota_2(K_0)$.

Let θ be a normalized positive definite kernel on $Sp_4(\mathbb{K})$, that we may assume continuous (by a Remark in [Oza11, Section 3]), such that

$$\sup_{g \in G} \|g \cdot \theta - \theta\|_{cb} < \delta$$

and

$$\sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta.$$

Then by relative property (T_P) for $(SL_2(\mathbb{K}) \rtimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$, we get that

$$\sup_{s \in N_2^+ \cup N_2^-} |\theta(s, 1) - 1|.$$

Consider the subgroup

$$H = \left\{ \left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid ad - bc = 1 \right) \right\} \simeq SL_2(\mathbb{K})$$

and its two subgroups $N^+ = \{g \in H \mid a = d = 1, c = 0\}$, $N^- = \{g \in H \mid a = d = 1, b = 0\}$. Since $N^+ \cup N^- \subset N_2^+ \cup N_2^-$, for any $s \in N^+ \cup N^-$, $|\theta(s, 1) - 1| < \varepsilon$.

If $g \in H, s \in N^+ \cup N^-$, we have

$$\begin{aligned} |\theta(gs, g) - 1|^2 &\leq 2|\theta(gs, g) - 1| \\ &\leq 2(|\theta(gs, g) - \theta(s, 1)| + |\theta(s, 1) - 1|) \\ &< 2(\delta + \varepsilon) < 4\varepsilon. \end{aligned}$$

But every element g in H can be written as a product of at most 3 elements of $N^+ \cup N^-$ (these corresponds to the transvections in $SL_2(\mathbb{K})$). Thus, we get that for any $g \in H$,

$$|\theta(g, 1) - 1| \leq 4\varepsilon^{1/2} + \varepsilon = \varepsilon'.$$

For any $\lambda \in \mathbb{K}$, the matrix $d_\lambda = \text{Diag}(\lambda, 1, \lambda^{-1}, 1)$ is an element of H . For $x \in \mathbb{K}$, consider the matrices

$$a(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \quad \text{and} \quad a'(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $x \in \mathbb{K}$ fixed. If $\lambda^{-1} \rightarrow 0$, we have $d_\lambda^{-1}a(x)d_\lambda \rightarrow 1$. In particular, by continuity of θ , there is λ such that $|\theta(d_\lambda^{-1}a(x)d_\lambda, 1) - 1| < \varepsilon'$. Thus, by Lemma 5.14, we have

$$|\theta(a(x), 1) - 1| < 2\varepsilon' + 4\sqrt{\varepsilon'} = \varepsilon''.$$

Similarly, if $\lambda \rightarrow 0$, we have $d_\lambda^{-1}a'(x)d_\lambda \rightarrow 1$ and thus $|\theta(a'(x), 1) - 1| < \varepsilon''$.

Finally, there is some integer ℓ such that any element $g \in G$ is a product of at most ℓ elements of $a(\mathbb{K}) \cup a'(\mathbb{K}) \cup N_2^+ \cup N_2^-$ (see [Neu03]).

Thus for any $g \in G$,

$$|\theta(g, 1) - 1| \leq 2\ell\sqrt{\varepsilon''}$$

which shows that G has (T_P) . □

5.4 Algebraic groups over local fields

We now know that $SL_3(\mathbb{K})$ and $Sp_4(\mathbb{K})$ have property (TTT). As in the classical proof of property (T), we want to use this fact to show that any almost \mathbb{K} -simple algebraic group of rank at least 2 has (TTT), where \mathbb{K} is a local field. Before that, we need to show that (TTT) is stable under some operations.

If G_1, G_2 are two locally compact group, a quasi-homomorphism is a Borel map $\varphi : G_1 \rightarrow G_2$ which is regular (i.e. the image of a compact subset of G_1 is relatively compact) and such that $\{\varphi(gh)^{-1}\varphi(g)\varphi(h) \mid g, h \in G_1\}$ is a relatively compact of G_2 .

Proposition 5.15. *Let G_1, G_2 be two locally compact groups. Let $\varphi : G_1 \mapsto G_2$ be a surjective quasi-homomorphism. If G_1 has (TTT), then G_2 has (TTT).*

Proof. Let b be Borel wq-cocycle on G_2 . Then since φ is a quasi-homomorphism, $b \circ \varphi$ is a wq-cocycle on G_1 , hence bounded by (TTT). Since φ is surjective, b is bounded. □

Proposition 5.16. *Let G be a second countable locally compact group, $N \triangleleft G$ a closed normal subgroup. If N and G/N have (TTT), then G has (TTT).*

Proof. Let b be a wq-cocycle on G , and let

$$D = \sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$$

be its defect. Then $b|_N$ is a wq-cocycle on N , hence bounded by C by property *(TTT)*. By [Mac52, Lemma 1.1], there exists a Borel section $\sigma : G/N \rightarrow G$ which is regular, meaning that the image of any compact subset of G/N is relatively compact in G . Denote $n_g = g^{-1}\sigma(gN)$. Set $\tilde{b} = b \circ \sigma$ and $\tilde{\pi} = \pi \circ \sigma$. Then \tilde{b} is a wq-cocycle on G/N associated to $\tilde{\pi}$. Indeed, if $g, h \in G$,

$$\begin{aligned} & \|\tilde{b}(ghN) - \tilde{b}(gN) - \tilde{\pi}(gN)\tilde{b}(hN)\| \\ &= \|b(ghn_{gh}) - b(gn_g) - \pi(gn_g)b(hn_h)\| \\ &\leq \|b(ghn_{gh}) - b(gn_g h n_h)\| + D \\ &\leq \|b(ghn_{gn}) - b(ghn')\| + D \\ &\leq \|b(ghn_{gh}) - b(gh) - \pi(gh)b(n_{gh})\| + \|b(gh) - b(ghn')\| + \|b(n_{gh})\| + D \\ &\leq \|b(ghn') - b(gh) - \pi(gh)b(n')\| + \|b(n')\| + 2D + C \\ &\leq 3D + 2C, \end{aligned}$$

using that N is a normal subgroup, thus $n' = h^{-1}n_g h n_h \in N$. Since G/N has property *(TTT)*, \tilde{b} is bounded by C' . Thus, for any $g \in G$,

$$\|b(g)\| \leq \|b(gn_g) - b(g) - \pi(g)b(n_g)\| + \|b(gn_g)\| + \|b(n_g)\| \leq D + C + C'$$

so b is bounded on G , and thus G has property *(TTT)*. \square

In [Oza11, Thm. 6], Ozawa showed that a lattice in a group with property *(TTT)* inherits property *(TTT)*. In fact, his proof also shows the following results.

Theorem 5.17. [Oza11, Thm. 6] *Let H be a closed subgroup of G locally compact second countable such that there exists a finite Borel measure on G/H invariant under the action of G . If G has property *(TTT)*, then H has property *(TTT)*.*

We will now turn to algebraic groups. By algebraic group, we will always mean an affine algebraic group realized as an algebraic subgroup of GL_n . We will use the notations of [Mar91, Ch. I], where more details can be found.

Lemma 5.18. *Let \mathbb{K} be a local field, G a connected semisimple \mathbb{K} -group and \tilde{G} its simply connected cover (in the algebraic sense). Then $G(\mathbb{K})$ has *(TTT)* if and only if $\tilde{G}(\mathbb{K})$ has *(TTT)*.*

Proof. Let $\pi : \tilde{G} \rightarrow G$ be a central \mathbb{K} -isogeny. Then by [Mar91, Ch. I, Thm. 2.3.4], $\pi(\tilde{G}(\mathbb{K}))$ is a closed normal subgroup of $G(\mathbb{K})$ such that $G(\mathbb{K})/\pi(\tilde{G}(\mathbb{K}))$ is compact (thus has *(TTT)* as well as a finite Borel measure invariant by $G(\mathbb{K})$).

Furthermore, $\tilde{G}(\mathbb{K})/(\ker \pi)(\mathbb{K}) \rightarrow \pi(\tilde{G}(\mathbb{K}))$ is a homeomorphism. Thus, since the subgroup $(\ker \pi)(\mathbb{K})$ is finite hence has *(TTT)*, by Propositions 5.15 and 5.16, $\tilde{G}(\mathbb{K})$ has *(TTT)* if and only if $\pi(\tilde{G}(\mathbb{K}))$ has *(TTT)*.

By Proposition 5.16, $\pi(\tilde{G}(\mathbb{K}))$ has *(TTT)* implies $G(\mathbb{K})$ has *(TTT)*. Conversely, by Theorem 5.17, if $G(\mathbb{K})$ has *(TTT)*, so does $\pi(\tilde{G}(\mathbb{K}))$. \square

In [dC09], Yves de Cornulier studied lengths on algebraic groups and showed the following result. A semigroup length on G is a map $\ell : G \rightarrow \mathbb{R}_+$ which is locally bounded and such that $\forall x, y \in G, \ell(xy) \leq \ell(x) + \ell(y)$.

Theorem 5.19. [dC09, Thm. 1.4] *Let G be an almost \mathbb{K} -simple algebraic group over a local field \mathbb{K} , then every semigroup length on $G(\mathbb{K})$ is bounded or proper.*

To prove Theorem H, we will show using (TTT) on SL_3 and Sp_4 that a certain length is not proper, thus is bounded.

Theorem 5.20. *Let \mathbb{K} be a local field, G a connected almost \mathbb{K} -simple \mathbb{K} -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property (TTT) .*

Proof. By [Mar91, Ch.I, Prop. 1.6.2], G contains an almost \mathbb{K} -simple \mathbb{K} -subgroup H whose (algebraic) simply connected cover is SL_3 or Sp_4 . Thus, by [Oza11, Thm.5], Theorem 5.13 and Lemma 5.18, $H(\mathbb{K})$ has property (TTT) .

Let b be a wq-cocycle on $G(\mathbb{K})$. Let $C = \sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\| < +\infty$. Then $b|_{H(\mathbb{K})}$ is a wq-cocycle on $H(\mathbb{K})$ hence is bounded by (TTT) .

Consider the function $\ell : g \mapsto \|b(g)\| + C$. We have that $\ell(gh) \leq \ell(g) + \ell(h)$. Furthermore, ℓ is locally bounded since by definition b is. Then, by [dC09, Thm. 1.4], ℓ is either proper or bounded. But b is bounded on $H(\mathbb{K})$ which is not relatively compact, thus b is bounded. \square

Remark 5.21. Let G be a connected simple Lie group with finite center of rank at least 2. Then G is locally isomorphic to the group of \mathbb{R} -point of an almost- \mathbb{R} -simple algebraic group, thus has (TTT) .

Corollary 5.22. *Let \mathbb{K} be a local field, G a connected almost \mathbb{K} -simple \mathbb{K} -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Let Γ be a lattice in $G(\mathbb{K})$. Then Γ has (TTT) .*

Proof. The result is a direct consequence of Theorem 5.20 and the fact that (TTT) passes to lattices. \square

Let $\varphi : G \rightarrow G'$ be a quasi-homomorphism. As noticed by Ozawa in [Oza11], if b is a wq-cocycle on G' , then $b \circ \varphi$ is a wq-cocycle on G . Hence, if G has property (TTT) and there exists b a proper wq-cocycle on G' (i.e. such that $\{g \mid \|b(g)\| \leq n\}$ is relatively compact in G' for any $n \in \mathbb{N}^*$), then any quasi-homomorphism $G \rightarrow G'$ has a relatively compact image.

Corollary 5.23. *Let Γ be a lattice in a higher rank almost \mathbb{K} -simple algebraic group. Then any quasi-homomorphism $\Gamma \rightarrow G'$ where G' admits a proper wq-cocycle has relatively compact image.*

This applies in particular when G' has the Haagerup property, or when G' is hyperbolic. Thus, it gives another proof of [FK16, Coro. 4.3].

5.5 Simple Lie groups with infinite center

In the previous section, we showed that any connected simple Lie group with finite center of rank at least 2 has (TTT) . We say that a quasi-homomorphism $\Phi : G \rightarrow \mathbb{R}$ is homogeneous if for any $g \in G, n \in \mathbb{N}$, $\Phi(g^n) = n\Phi(g)$. In that case, if g, h commute, then $\Phi(gh) = \Phi(g) + \Phi(h)$. Let G be a connected simple Lie group with infinite center $Z(G)$ and rank at least 2. Then by [BG92, Prop. 6], the space of homogeneous quasimorphisms is one dimensional. In particular, a nonzero element of this space, called a Barge-Ghys quasimorphism, is a wq-cocycle (and even a quasi-cocycle) which is unbounded, thus G does not have property (TTT) (and (TT) as well).

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G and \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Let A, K be the analytic subgroups of G with Lie algebras $\mathfrak{a}, \mathfrak{k}$ respectively. Then $G = KAK$ as in the finite center case. However, note that K is not compact. Indeed, $Z(G) \subset K$ is an infinite discrete subgroup, but $K/Z(G)$ is compact.

The following lemma is due to Yves de Cornulier and Mikael de la Salle in an unpublished note. We here reproduce their proof.

Lemma 5.24. *Let G be a connected simple Lie group with infinite center. There exists a Borel regular section $s : G/Z(G) \rightarrow G$ such that*

$$S = \{s(ghZ(G))(s(gZ(G))s(hZ(G)))^{-1} \mid g, h \in G\}$$

is finite and $s(\exp_{\text{Ad}(G)}(X)) = \exp_G(X)$ for any $X \in \mathfrak{a}$.

Proof. Let Φ be a Barge-Ghys quasimorphism, normalized by $\Phi(Z(G)) \subset \mathbb{Z}$. Since Φ is homogeneous, we can define $s(gZ(G))$ by g if $\Phi(g) \in [-\frac{1}{2}, \frac{1}{2}]$. Since Φ is a quasimorphism, there is $C > 0$ such that $|\Phi(gh) - \Phi(g) - \Phi(h)| \leq C$. But we have

$$\begin{aligned} |\Phi(s(xy)s(y)^{-1}s(x)^{-1})| &\leq 2C + |\Phi(s(xy))| + |\Phi(s(y)^{-1})| + |\Phi(s(x)^{-1})| \\ &\leq 2C + \frac{3}{2} \end{aligned}$$

bounded independently of $x, y \in G/Z(G)$. Since $\{s(xy)s(y)^{-1}s(x)^{-1}\} \subset Z(G)$, it is finite. \square

Note that $s(1) = 1$. We want to study wq-cocycles on G , up to bounded functions. Let H be a Hilbert space and $\pi : G \rightarrow \mathcal{U}(H)$ be fixed. Let

$$Z_w(G, \pi) = \{b : G \rightarrow H \mid b \text{ wq-cocycle for } \pi\}$$

and $B_w(G, \pi)$ the subspace of bounded Borel functions. We want to understand the space $H_w(G, \pi) = Z_w(G, \pi)/B_w(G, \pi)$.

Let $i : Z(G) \rightarrow G$ denote the inclusion. Right-composition by i induces a map

$$i_* : H_w(G, \pi) \rightarrow H_w(Z(G), \pi).$$

Denote $z_g = gs(gZ(G))^{-1} \in Z(G)$.

Proposition 5.25. *The map i_* is injective and*

$$i_*(H_w(G, \pi)) = \left\{ [b] \mid \sup_{g \in G, z \in Z(G)} \|\pi(g)b(z) - \pi(z_g)b(z)\| < +\infty \right\}.$$

Proof. Let b be a wq-cocycle with defect D such that $i_*[b] = 0$. Then $b \circ i$ is bounded. The map $b \circ s$ is also a wq-cocycle on $G/Z(G)$. Indeed, since s is Borel regular, $b \circ s$ is Borel and locally bounded. Furthermore, if $g, h \in G/Z(G)$, then

$$\begin{aligned} \|b(s(gh)) - b(s(g)) - \pi(s(g))b(s(h))\| &\leq \|b(s(gh)) - b(s(g)s(h))\| + D \\ &\leq \|b((s(g)s(h))^{-1}s(gh))\| + 2D \end{aligned}$$

which is bounded in g, h since S is finite. But then since $G/Z(G)$ has finite center, by Theorem 5.20, $b \circ s$ is bounded.

For any $g \in G$, denote $z_g = gs(gZ(G))^{-1} \in Z(G)$. Thus

$$\|b(g)\| \leq D + \|b(s(gZ(G)))\| + \|b(z_g)\|$$

so b is bounded and $[b] = 0$.

Let \tilde{b} be a wq-cocycle on $Z(G)$ with defect D . Suppose that there exists a wq-cocycle b on G with defect D' such that $i_*[b] = [\tilde{b}]$. Then, for any $g \in G, z \in Z(G)$, using that z, z_g commute with G ,

$$\begin{aligned} \|\pi(g)b(z) - \pi(z_g)b(z)\| &\leq \|\pi(g)b(z) + b(z_g s(gZ(G))) - b(gz) \\ &\quad - b(z_g s(gZ(G))) + b(z_g) + \pi(z_g)b(s(gZ(G))) \\ &\quad + b(gz) - b(z_g) - \pi(z_g)b(zs(gZ(G))) \\ &\quad + \pi(z_g)b(zs(gZ(G))) - \pi(z_g)b(z) - \pi(z_g)\pi(z)b(s(gZ(G))) \\ &\quad - \pi(z_g)b(s(gZ(G))) + \pi(z_g)\pi(z)b(s(gZ(G)))\| \\ &\leq 4D' + 2\|b(s(gZ(G)))\|. \end{aligned}$$

But since $b \circ s$ is bounded, we get that

$$\sup_{g \in G, z \in Z(G)} \|\pi(g)b(z) - \pi(z_g)b(z)\| < +\infty.$$

Now since $b|_{Z(G)} - \tilde{b}$ is bounded by assumption, we get the necessary condition

$$\sup_{g \in G, z \in Z(G)} \|\pi(g)\tilde{b}(z) - \pi(z_g)\tilde{b}(z)\| < +\infty \quad (5.20)$$

Finally, we show that condition (5.20) is sufficient. So let \tilde{b} be a wq-cocycle such that (5.20) holds and let C be the supremum. Define $b(g) = \tilde{b}(gs(gZ(G))^{-1}) = \tilde{b}(z_g)$ which is Borel and locally bounded. Then b is a wq-cocycle. Indeed, if $g, h \in G$, then

$$\begin{aligned} \|b(gh) - b(g) - \pi(g)b(h)\| &= \|\tilde{b}(z_{gh}) - \tilde{b}(z_g) - \pi(g)\tilde{b}(z_h)\| \\ &\leq \|\tilde{b}(z_{gh}) - \tilde{b}(z_g) - \pi(z_g)\tilde{b}(z_h)\| + \|\pi(g)\tilde{b}(z) - \pi(z_g)\tilde{b}(z)\| \\ &\leq \|\tilde{b}(z_{gh}) - \tilde{b}(z_g z_h)\| + D + C \\ &\leq \|\tilde{b}((z_g z_h)^{-1} z_{gh})\| + 2D + C. \end{aligned}$$

But $(s(gZ(G))s(hZ(G)))^{-1}s(ghz(G)) = (z_g z_h)^{-1}z_{gh}$ so since S is finite,

$$\sup_{g, h \in G} \|\tilde{b}((z_g z_h)^{-1} z_{gh})\| < +\infty.$$

Finally, for any $z \in Z(G)$, $b(z) = \tilde{b}(zs(1)^{-1}) = \tilde{b}(z)$ so that $i_*[b] = [\tilde{b}]$. \square

Remark 5.26. In particular, any wq-cocycle on $Z(G)$ associated with $\pi : Z(G) \rightarrow \mathcal{U}(H)$ induces a wq-cocycle on G , associated to $\pi' : G \rightarrow \mathcal{U}(H)$ defined by

$$\pi'(g) = \pi(gs(gZ(G))^{-1}) = \pi(z_g).$$

Furthermore, any wq-cocycle on G is bounded on A , since $A \subset s(G/Z(G))$.

Chapter 6

Banach versions of property (TTT)

In Chapter 5, we focused on the study of wq-cocycles taking values into Hilbert spaces. In this chapter, we introduce Banach valued wq-cocycles and study the associated version of property (TTT) : if E is a Banach space, a group G has property (FFF_E) if all E -valued wq-cocycles are bounded. This property is also a generalization of the Banach fixed-point property (F_E) introduced in [BFGM07]. We begin by introducing this new property and recalling some known results in the case of property (F_E) . We then define in Section 6.2 the class of Banach spaces that we will be interested in, namely super-reflexive Banach spaces. We explain some of their geometric properties that will be useful in the study of property (FFF_E) . In Section 6.3, we define analogues of property (T_Q) and (T_P) (see Section 5.2) in the context of Banach spaces. We study the relationship between the different formulations. In Section 6.3.3, we prove that under some conditions, lattices inherits property (FFF_E) and obtain Theorem I. In Section 6.4, we restrict ourselves to L^p -spaces. Finally, Section 6.5 is devoted to the case of Banach spaces with a compatible Hilbert norm. We prove that any lattice in a higher rank simple algebraic group has property $(T_{Q,\mathcal{H}})$ and obtain Theorem K. The results in this chapter are part of a work in progress and have not yet been published.

6.1 Motivation

Let G be a locally compact second countable group and E a Banach space. Let $O(E)$ be the group of linear isometries of E and $\text{Isom}(E)$ the group of affine isometries of E . A version of property (T) for Banach spaces is introduced in [BFGM07, CDH10].

Definition 6.1. The group G has property (F_E) if for any action $\alpha : G \curvearrowright E$ by affine isometries, α has a fixed point.

The group G has property (T_E) if for any continuous representation $\pi : G \rightarrow O(E)$, the induced representation $\pi' : G \rightarrow O(E/E^{\pi(G)})$ does not almost have invariant vectors.

If \mathcal{E} is a class of Banach spaces, G has $(T_{\mathcal{E}})$ (respectively $(F_{\mathcal{E}})$) if G has (T_E) (respectively (F_E)) for any $E \in \mathcal{E}$.

If \mathcal{H} is the class of all Hilbert spaces, $(T_{\mathcal{H}})$ is the usual Kazhdan property (T) , and $(F_{\mathcal{H}})$ is property FH . Recall that by the Delorme-Guichardet theorem, FH and (T) are equivalent (for locally compact second countable groups). However, it is no longer true that if G has property (T_E) , then G has property (F_E) ([BFGM07, Ex. 2.22]).

In the other direction, it is still true that property (F_E) implies property (T_E) ([BFGM07, Thm. 1.3]). When G is super-reflexive (see Section 6.2), the existence of a fixed point is equivalent to the existence of a bounded orbit. Let $\alpha : G \rightarrow \text{Isom}(E)$ be a isometric affine action. Then for any $g \in G, \xi \in E$,

$$\alpha(g)\xi = \rho(g)\xi + b(g)$$

where ρ is a linear isometric representation and b is a cocycle with values in E . If E is super-reflexive, G has property (F_E) if and only if every cocycle $b : G \rightarrow E$ is bounded.

In the spirit of Burger and Monod for property (TT) , Mimura introduced a quasification of (F_E) .

Definition 6.2. A quasi-action of G on E is a map $\beta : G \rightarrow \text{Isom}(E)$ such that

$$\sup_{g,h \in G, \xi \in E} \|\beta(gh)\xi - \beta(g)\beta(h)\xi\|.$$

If $\pi : G \rightarrow O(E)$ is a continuous representation, a quasi- π -cocycle is a map $b : G \rightarrow E$ such that

$$\sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\|.$$

The group G has property (FF_E) if all quasi-action have bounded orbits, equivalently if all quasi-cocycles are bounded.

We point out that it is clear by the previous discussion on cocycles that when E is super-reflexive, property (FF_E) implies property (F_E) , but it might no longer be true for general Banach spaces.

We now naturally introduce a Banach version of property (TTT) that will be the main topic of this chapter.

Definition 6.3. Let G be a locally compact group and A a closed subgroup of G . Let E be a Banach space. We say that a Borel locally bounded (i.e. bounded on compact subsets) map $b : G \rightarrow E$ along with a Borel map $\rho : G \rightarrow O(E)$ is a wq-cocycle if

$$\sup_{g,h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| < +\infty.$$

The pair (G, A) has relative property (FFF_E) if every wq-cocycle $b : G \rightarrow E$ is bounded on A .

We wish to investigate property (FFF_E) for higher rank simple algebraic groups.

Let G be a locally compact non-compact group. Let

$$L_0^1(G) = \left\{ f \in L^1(G) \mid \int_G f(g) dg = 0 \right\}.$$

Then G acts isometrically on $L_0^1(G)$ without fixed point ([BFGM07, Ex. 2.23]). Furthermore, it is shown in [HP06] that every locally compact second countable group admits a proper isometric action on the reflexive strictly convex space $\bigoplus_{n=1}^{\infty} L^{2n}(G)$.

On the other hand, it was conjectured in [BFGM07] that any higher rank simple algebraic group (and their lattices) have (F_E) for any super-reflexive Banach space E . This was answered by the affirmative by Lafforgue and Liao in the non-Archimedean case ([Laf08, Lia14]) and very recently by de Laat and de la Salle (improving on [Opp23]) in the real case ([dLdS23]).

It is also known that $SL_n(\mathbb{R})$, $n \geq 5$, has property (FF_E) for any super-reflexive Banach space ([dLMdlS16, Section 5],[Opp23, Coro. 9.12]).

Thus, our comments on cocycles and these results encourage us to restrict ourselves to the class of super-reflexive Banach spaces. We will define these spaces in the next section and give some of their geometric properties that will be of interest to us in the study of property (FFF_E) .

We should point out that there exists results for fixed-point property of higher rank simple algebraic groups outside the class of super-reflexive spaces, related to Lafforgue's strong property (T) ([Laf09, dlS16, dLdlS15]).

6.2 Super-reflexive Banach spaces

If E is a Banach space, let $S(E)$ be its unit sphere, $B_1(E)$ its unit ball and $O(E)$ the group of all linear invertible isometries.

Definition 6.4. Let $(E, \|\cdot\|)$ be a Banach space and $B_1(E)$ its unit ball. Let

$$d_{\|\cdot\|}(\varepsilon) = \inf \left\{ 1 - \frac{\|\xi - \eta\|}{2} \mid \xi, \eta \in B_1(E), \|\xi - \eta\| \geq \varepsilon \right\} \quad (6.1)$$

and

$$r_{\|\cdot\|}(\tau) = \sup \left\{ \frac{\|\xi + \eta\| + \|\xi - \eta\|}{2} - 1 \mid \xi \in B_1(E), \|\eta\| \leq \tau \right\}. \quad (6.2)$$

If there is no ambiguity, we may denote $d(\varepsilon)$ and $r(\tau)$. The functions d and r are called the *modulus of convexity* and *modulus of smoothness* respectively.

1. The space E is said to be uniformly convex (or uc) if for any $0 < \varepsilon < 2$, $d(\varepsilon) > 0$.
2. The space E is said to be uniformly smooth (or us) if $\lim_{\tau \rightarrow 0} r(\tau)/\tau = 0$.
3. The space E is said to be ucus if it is both uc and us.
4. The space E is said to be super-reflexive if it admits a compatible ucus norm.

Example 6.5. Any Hilbert space is ucus. We denote \mathcal{H} the class of all Hilbert space.

Let $1 \leq p \leq +\infty$. By L^p space, we mean a space $L^p(X, \mu)$ where X is a standard Borel space and μ is a σ -finite measure on X . Any L^p space with $1 < p < +\infty$ is ucus. The spaces L^1 and L^∞ are not super-reflexive (indeed, they are not reflexive). We denote by L^p the class of all L^p spaces.

We denote $[\mathcal{H}]$ the class of all super-reflexive spaces with a compatible Hilbert norm. We also denote $[\mathcal{H}]_M$ the subclass of all spaces with a norm ratio $\leq M$, that is to say with

$$\max \left(\frac{\|x\|_H}{\|x\|_B}, \frac{\|x\|_B}{\|x\|_H} \right) \leq M$$

where $\|\cdot\|_B, \|\cdot\|_H$ are the original and the Hilbert norm respectively.

Lemma 6.6 ([BL98]). *Let $(E, \|\cdot\|)$ be a Banach space and $(E^*, \|\cdot\|_*)$ its dual. For any $\tau > 0$,*

$$r_{\|\cdot\|}(\tau) = \sup_{0 < \varepsilon < 2} \frac{\varepsilon\tau}{2} - d_{\|\cdot\|_*}(\varepsilon).$$

In particular, E is uniformly smooth if and only if E^ is uniformly convex.*

Let $\langle \cdot, \cdot \rangle : E \times E^* \rightarrow \mathbb{C}$ denote the duality bracket of E .

Definition 6.7. Let E be a real Banach space and $S(E)$ its unit sphere. Then for $\xi \in S(E)$, there exists a unique element $\xi^* \in S(E^*)$ such that $\langle \xi, \xi^* \rangle = 1$. The map $\xi \mapsto \xi^*$ is called the duality mapping.

Remark 6.8. The duality mapping can be extended to all of E by multiplication by a real number, but it has no linearity property in general.

The duality mapping can be defined under the weaker hypothesis that E^* is strictly convex.

Lemma 6.9 ([BL98]). *Let E be a uniformly smooth Banach space. Let $0 < \varepsilon < 2$, then for any $\xi, \eta \in S(E)$ with $\|\xi - \eta\| \leq \varepsilon$, we have*

$$\|\xi^* - \eta^*\| \leq 2 \frac{r(2\varepsilon)}{\varepsilon}.$$

In particular, the duality mapping is uniformly continuous.

When dealing with representations, the following lemma plays a key role to go from bounded orbits to fixed points.

Lemma 6.10. *Let E be a uniformly convex Banach space. For any bounded subset $B \subset E$, there exists a unique point $C(B) \in E$ which is the unique x minimizing*

$$\inf\{r > 0 : B \subset \overline{B(x, r)}\}.$$

The point $C(B)$ is called the circumcentre or Chebyshev center of B .

Definition 6.11. A representation ρ of a group G on a Banach space E is uniformly bounded if

$$\|\rho\| = \sup_{g \in G} \|\rho(g)\| < +\infty.$$

We will say that a map $\rho : G \rightarrow GL(E)$ is uniformly bounded if

$$\|\rho\| = \sup_{T \in \langle \rho(G) \rangle} \|T\| < +\infty$$

where $\langle \rho(G) \rangle$ is the subgroup of $GL(E)$ generated by $\rho(G)$.

Remark 6.12. Note that the definition we use of a uniformly bounded map is far stronger than $\rho : G \rightarrow B(E)$ being bounded. If ρ is a representation, this is equivalent because $\rho(G) = \langle \rho(G) \rangle$ is a group. But in general, we want to check that any product of operators remains in a bounded set.

To motivate this definition, think of the following example. Let T be an invertible operator on a Hilbert space and ρ the map sending a single point to T . Then ρ is uniformly bounded if and only if the representation of \mathbb{Z} sending n to T^n is uniformly bounded, if and only if there exists an operator S such that STS^{-1} is unitary. The last part Dixmier's unitarization of uniformly bounded representations of amenable groups.

The following proposition shows that with this definition, uniformly bounded maps on super-reflexive spaces can be made isometric up to changing the norm.

Proposition 6.13 ([BFGM07]). *Let E be a super-reflexive Banach space and G a uniformly bounded group of invertible linear maps. Then there exists a compatible vector norm $\|\cdot\|$ on E such that $G \subset O(E, \|\cdot\|)$.*

In particular, if $\rho : G \rightarrow GL(E)$ is a uniformly bounded map, there exists a compatible vector norm $\|\cdot\|$ on E such that $\rho : G \rightarrow O(E, \|\cdot\|)$.

Since most our maps are not continuous but only measurable, we may ask ourselves whether our constructions will preserve measurability of $O(E)$ -valued map.

Let E be a Banach space. As in the definition of continuous representations, asking for measurability with respect to the operator norm is not a good idea. On the space $B(E)$ of bounded operators, there are two topologies we want to consider. The strong operator topology (SOT) is the locally convex topology generated by the seminorms $T \mapsto \|Tx\|$ for $x \in E$. The weak operator topology (WOT) is the locally convex topology generated by the seminorms $T \mapsto |\langle Tx, y \rangle|$ for $x \in E, y \in E^*$. Consider a map $\rho : G \rightarrow O(E)$. There are four "natural" notions of measurability we may want to consider.

Definition 6.14. We say that $\rho : G \rightarrow O(E)$ is

- (i) SOT-measurable if ρ is measurable for the Borel σ -algebra of $(O(E), SOT)$;
- (ii) WOT-measurable if ρ is measurable for the Borel σ -algebra of $(O(E), WOT)$;
- (iii) strongly measurable if $\rho_x : g \mapsto \rho(g)x$ is measurable for $x \in E$;
- (iv) weakly measurable if $\rho_{x,y} : g \mapsto \langle \rho(g)x, y \rangle$ is measurable for any $x \in E, y \in E^*$.

The notion of measurability we want to use is SOT-measurability. In Chapter 5, in Lemma 5.10 for example, we used the other definitions but it turned out that since we were working with a separable Hilbert space, these were all equivalent. In general, this need not be the case.

Since SOT is stronger than WOT, it is clear that (i) \Rightarrow (ii). Similarly, (iii) \Rightarrow (iv). Furthermore, by definition of SOT, $T \mapsto Tx$ is continuous thus (i) \Rightarrow (iii). Similarly, (ii) \Rightarrow (iv).

If E is separable, then SOT is metrizable on $O(E)$ and one can prove that (iii) \Rightarrow (i). If E is reflexive and separable, then (iv) \Rightarrow (ii). However even though our spaces are all reflexive, we work with non-separable spaces in general.

The following is due to Megrelishvili in [Meg01].

Proposition 6.15. *Let E be a Banach space with the Radon-Nikodym Property. The strong operator topology and the weak operator topology coincide on every bounded subgroup G of $GL(E)$.*

In particular, reflexive spaces and separable dual spaces have the RNP, thus (i) \Leftrightarrow (ii). For any of the super-reflexive spaces we will consider, SOT and WOT measurability will thus be equivalent.

6.3 Analytic properties

Let $1 \leq p < +\infty$. Let $(E_i)_{i \in \mathbb{N}}$ be a sequence of Banach spaces, we define the ℓ_p direct sum by

$$\bigoplus_p E_i = \left\{ x = (x_i)_{i \in \mathbb{N}} \mid \|x\|_p^p = \sum \|x_i\|_{E_i}^p < +\infty \right\}$$

and

$$\bigoplus_\infty E_i = \left\{ x = (x_i)_{i \in \mathbb{N}} \mid \|x\|_\infty = \sup \|x_i\|_{E_i} < +\infty \right\}.$$

Then $\bigoplus_p E_i$ endowed with the corresponding norm is a Banach space. We say that a class \mathcal{E} of Banach spaces is stable by ℓ^p direct sum if for any $(E_i)_{i \in \mathbb{N}} \subset \mathcal{E}$, $\bigoplus_p E_i \in \mathcal{E}$.

Note that there are many other ways to make direct sums of Banach spaces, using different norms, but we will restrict to ℓ^p direct sums. If all spaces are Hilbert spaces, then the ℓ^2 direct sum is the Hilbert direct sum.

6.3.1 Property $(T_{Q,\mathcal{E}})$

Definition 6.16. Let G be a locally compact second countable group and A a closed subgroup. Let E be a Banach space.

The pair (G, A) has relative property $(T_{Q,E})$ if for all $\varepsilon > 0$ and for any Borel map $\rho : G \mapsto O(E)$, there exist $\delta > 0$ and a compact subset $K \subset G$ such that for every unit vector $\xi \in E$ verifying

$$\sup_{g,h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{g \in K} \|\rho(g)\xi - \xi\| < \delta$$

then

$$\sup_{x \in A} \|\rho(x)\xi - \xi\| < \varepsilon.$$

If \mathcal{E} is a class of Banach spaces, (G, A) has $(T_{Q,\mathcal{E}})$ if (G, A) has $(T_{Q,E})$ for any $E \in \mathcal{E}$.

A group G has $(T_{Q,E})$ (resp. $(T_{Q,\mathcal{E}})$) if (G, G) has relative $(T_{Q,E})$ (resp. relative $(T_{Q,\mathcal{E}})$).

Remark 6.17. It suffices to consider maps such that $\rho(1) = \text{Id}$. Indeed, we claim that the property holds as soon as it holds for maps with $\rho(1) = \text{Id}$. Let $\varepsilon > 0$, $\rho : G \rightarrow O(E)$ and $\xi \in S(E)$. Consider $\tilde{\rho} : g \mapsto \rho(g)\rho(1)^{-1}$ and $\tilde{\xi} = \rho(1)\xi$. Let (δ, K) be given by the property for $\tilde{\rho}$ and $\varepsilon/2$. We may assume $1 \in K$. Let $\delta_0 = \frac{1}{2} \min(\varepsilon, \delta)$. Then if

$$\sup_{g,h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta_0$$

and

$$\sup_{g \in K} \|\rho(g)\xi - \xi\| < \delta_0,$$

we get

$$\begin{aligned} \|\tilde{\rho}(gh)\tilde{\xi} - \tilde{\rho}(g)\tilde{\rho}(h)\tilde{\xi}\| &= \|\rho(gh)\xi - \rho(g)\rho(1)^{-1}\rho(h)\xi\| \\ &< \delta_0 + \|\rho(g)\rho(h)\xi - \rho(g)\rho(1)^{-1}\rho(h)\xi\| \\ &\leq \delta_0 + \|\rho(1)\rho(h)\xi - \rho(h)\xi\| \\ &< 2\delta_0 \leq \delta \end{aligned}$$

for any $g, h \in G$ and

$$\|\tilde{\rho}(g)\tilde{\xi} - \tilde{\xi}\| = \|\rho(g)\xi - \rho(1)^{-1}\xi\| < 2\delta_0$$

for any $g \in K$. Thus

$$\|\tilde{\rho}(a)\tilde{\xi} - \tilde{\xi}\| < \varepsilon/2$$

for any $a \in A$ and so

$$\|\rho(a)\xi - \xi\| < \varepsilon.$$

This fact will be useful to lighten some heavy proofs in Section 6.5.

Remark 6.18. Notice that unlike the original definition for Hilbert spaces, δ in the definition of $(T_{Q,\mathcal{E}})$ depends not only on ε , but also on the map ρ and the space $E \subset \mathcal{E}$. If \mathcal{E} is stable under ℓ^p direct sum, δ depends only on ε (this is mostly the same as the proof of the existence of a Kazhdan constant for property (T) groups). Thus for the class \mathcal{H} of all Hilbert spaces, we recover Definition 5.5. However, the class of all super-reflexive spaces and the class $[\mathcal{H}]$ are not stable under direct sum: for example in $[\mathcal{H}]$, the norm ratio may tend to infinity. But let $E \in [\mathcal{H}]$, then $\bigoplus_2 E \in [\mathcal{H}]$ - this is because $[\mathcal{H}]_M$ is stable by ℓ^2 direct sum for any M - thus in $(T_{Q,\mathcal{E}})$, δ can still be taken independent of ρ if $\bigoplus_p E \in \mathcal{E}$, but not necessarily of E .

Proposition 6.19. *Let E be a ucus space. Let A be a closed normal subgroup of a locally compact group G . If (G, A) has relative property $(T_{Q,E})$, then (G, A) has relative property (T_E) .*

Proof. Let $\rho : G \rightarrow O(E)$ be a (strongly continuous) isometric representation. We want to show that the quotient representation $\rho' : G \rightarrow E/E^{\rho(A)}$ does not almost have invariant vectors. By [BFGM07, Prop. 2.6 and 2.10], there exists a closed G -invariant subspace E' such that $E = E^{\rho(G)} \oplus E'$ and such that E' is isomorphic to $E/E^{\rho(A)}$ as a G -representation (see also [BFGM07, Remark 5.3] to see that the canonical complement E' which is defined as an A -representation is actually $\rho(G)$ -invariant).

Assume that $\rho : G \rightarrow O(E')$ almost has invariant vectors. Let $\varepsilon = 1$, let (δ, K) associated to ε in relative $(T_{Q,E})$ for ρ, ε . There exists $\xi \in S(E') \subset S(E)$ such that $\|\rho(x)\xi - \xi\| < \delta$ for any $x \in K$. Since ρ is a representation, the second condition is trivial. Thus by $(T_{Q,E})$, for any $a \in A$, $\|\rho(a)\xi - \xi\| < 1$. Consider $B = \overline{\text{Conv}}(\{\rho(a)\xi \mid a \in A\}) \subset E'$. By the lemma of the center, there exists a unique $\eta \in B$ of minimal norm. Then since $\rho(a)$ is an isometry, by uniqueness, for any $a \in A$, $\rho(a)\eta = \eta$ so $\eta \in E^{\rho(A)}$. Thus $\eta = 0$. But since $\|\rho(a)\xi - \xi\| < 1$ for any $a \in A$, we get that $\|\eta - \xi\| < 1$, contradiction. \square

The following proposition mirrors the proof that FH implies (T) (see [BdlHV08, Prop. 2.4.5]).

Proposition 6.20. *Let E be a Banach space, then property $(FFF_{\bigoplus_p E})$ implies property $(T_{Q,E})$.*

In particular if \mathcal{E} is a class of Banach spaces stable under direct sum of an infinite number of copies of the same space, then property $(FFF_{\mathcal{E}})$ implies $(T_{Q,\mathcal{E}})$.

Proof. Suppose that (G, A) does not have relative property $(T_{Q,E})$. Since G is locally compact second countable, it is σ -compact, so we can take a sequence of increasing compact subsets Q_n such that $G = \bigcup Q_n$. There exists $\varepsilon > 0$ and a Borel map $\rho : G \rightarrow O(E)$ such that for any $n \in \mathbb{N}$, there exists a vector $\xi_n \in E$ such that

$$\sup_{g,h \in G} \|\rho(gh)\xi_n - \rho(g)\rho(h)\xi_n\| < 2^{-n}$$

and

$$\sup_{x \in Q_n} \|\rho(x)\xi_n - \xi_n\| < 2^{-n}$$

but

$$\sup_{a \in A} \|\rho(a)\xi_n - \xi_n\| \geq \varepsilon.$$

Let $E' = \bigoplus_p E$ and define $b : G \rightarrow E'$ by

$$b(g) = \sum n(\rho_n(g)\xi_n - \xi_n).$$

Then

$$\|b(g)\|^p = \sum_{n \geq 0} n^p \|\rho(g)\xi_n - \xi_n\|^p.$$

For any $g \in G$, there exists $N \in \mathbb{N}$ such that $g \in Q_N$, thus for any $n \geq N$,

$$\|\rho(g)\xi_n - \xi_n\| < 2^{-n}$$

so

$$\|b(g)\|^p \leq \sum_{n=0}^{N-1} n^p \|\rho(g)\xi_n - \xi_n\|^p + \sum_{n \geq N} \frac{n^p}{2^{np}} < +\infty$$

and so $b(g) \in E'$. In fact, this also shows that b is locally bounded (using Baire's theorem and local compactness of G to get that the Q_n 's have non-empty interior after some n_0). As a pointwise limit of Borel maps, b is also Borel. Define $\tilde{\rho} : G \rightarrow O(E')$ by $\tilde{\rho}(g)(\xi_n)_{n \in \mathbb{N}} = (\rho(g)\xi_n)_{n \in \mathbb{N}}$. Then

$$\begin{aligned} \|b(gh) - b(g) - \tilde{\rho}(g)b(h)\|^p &= \sum_{n \geq 0} n^p \|\rho(gh)\xi_n - \xi_n - (\rho(g)\xi_n - \xi_n) - \rho(g)(\rho(h)\xi_n - \xi_n)\|^p \\ &= \sum_{n \geq 0} n^p \|\rho(gh)\xi_n - \rho(g)\rho(h)\xi_n\|^p \\ &\leq \sum_{n \geq 0} \frac{n^p}{2^{np}} < +\infty \end{aligned}$$

so b is a wq-cocycle. But for any $M > 0$, there exists n such that $n\varepsilon > M$ and $a \in A$ such that

$$\|\rho(a)\xi_n - \xi_n\| \geq \varepsilon$$

so

$$\|b(a)\| \geq M$$

and b is unbounded. Thus (G, A) does not have relative property $(FFF_{E'})$. \square

Remark 6.21. We can check that we can allow ρ to vary with n in the proof, thus this actually proves that $(FFF_{\bigoplus_p E})$ implies a stronger version of $(T_{Q,E})$ where δ depends only on $\varepsilon > 0$ and not on ρ . Compare with Remark 6.18 to see that it makes sense if we consider classes of Banach spaces.

Proposition 6.22. *Let E be a ucus Banach space. Then (G, A) has relative property $(T_{Q,E})$ if and only if (G, A) has relative property (T_{Q,E^*})*

Proof. Since E is reflexive, it suffices to prove a single direction. Let r, r_* be the moduli of smoothness of E, E^* . Assume that (G, A) has relative property (T_{Q,E^*}) . Let $\rho : G \rightarrow O(E)$ be a Borel map, $\varepsilon > 0$ and $\xi \in S(E)$, such that

$$\sup_{g,h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{x \in K} \|\rho(x)\xi - \xi\| < \delta$$

for δ, K to be determined. If $T \in B(E)$, define the adjoint T^* as the unique operator on E^* such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle.$$

If $T \in O(E)$, then $T^* \in O(E^*)$. By reflexivity, $T \mapsto T^*$ is WOT-continuous. By Proposition 6.15, The restriction

$$\begin{array}{ccc} O(E) & \rightarrow & O(E^*) \\ T & \mapsto & T^* \end{array}$$

is SOT-continuous. Define a map $\rho^* : G \rightarrow O(E^*)$ by $\rho^*(g) = \rho(g^{-1})^*$. Then by composition of measurable maps, ρ^* is SOT-measurable. Since the duality mapping $*$: $S(E) \rightarrow S(E^*)$ is bijective, we can check that for any $x \in E$, $\rho^*(g)x^* = (\rho(g^{-1})^{-1}x)^*$. We may assume that $1 \subset K$. Since $\rho(g^{-1})$ is an isometry, we get that

$$\begin{aligned} \|\rho(g^{-1})^{-1}\xi - \rho(g)\xi\| &= \|\xi - \rho(g^{-1})\rho(g)\xi\| \\ &\leq \|\xi - \rho(1)\xi\| + \|\rho(g^{-1}g)\xi - \rho(g^{-1})\rho(g)\xi\| \\ &\leq 2\delta. \end{aligned} \tag{6.3}$$

Thus if $k \in K$, we obtain

$$\begin{aligned} \|\rho^*(k)\xi^* - \xi^*\| &\leq \|\rho^*(k)\xi^* - (\rho(k)\xi)^*\| + \|(\rho(k)\xi)^* - \xi^*\| \\ &\leq \|(\rho(k^{-1})\xi)^* - (\rho(k)\xi)^*\| + \|(\rho(k)\xi)^* - \xi^*\| \\ &\leq \frac{r(4\delta)}{\delta} + 2\frac{r(2\delta)}{\delta} \end{aligned} \tag{6.4}$$

by Lemma 6.9 and (6.3). Furthermore, for any $g, h \in G$,

$$\begin{aligned} \|\rho(g)\rho(h)\xi - \rho(g^{-1})^{-1}\rho(h^{-1})^{-1}\xi\| &\leq \|\rho(g)\rho(h)\xi - \rho(g^{-1})^{-1}\rho(h)\xi\| \\ &\quad + \|\rho(g^{-1})^{-1}\rho(h)\xi - \rho(g^{-1})^{-1}\rho(h^{-1})^{-1}\xi\| \\ &\leq \|\rho(g^{-1})\rho(g)\rho(h)\xi - \rho(h)\xi\| + \|\rho(h)\xi - \rho(h^{-1})^{-1}\xi\| \\ &\leq \|\rho(g^{-1})\rho(g)\rho(h)\xi - \rho(g^{-1})\rho(gh)\xi\| \\ &\quad + \|\rho(g^{-1})\rho(gh)\xi - \rho(h)\xi\| + 2\delta \\ &\leq 4\delta \end{aligned} \tag{6.5}$$

an so

$$\begin{aligned} \|\rho^*(gh)\xi^* - \rho^*(g)\rho^*(h)\xi^*\| &= \|(\rho((gh)^{-1})^{-1}\xi)^* - (\rho(g^{-1})^{-1}\rho(h^{-1})^{-1}\xi)^*\| \\ &\leq \|(\rho((gh)^{-1})^{-1}\xi)^* - (\rho(gh)\xi)^*\| \\ &\quad + \|(\rho(gh)\xi)^* - (\rho(g)\rho(h)\xi)^*\| \\ &\quad + \|(\rho(g)\rho(h)\xi)^* - (\rho(g^{-1})^{-1}\rho(h^{-1})^{-1}\xi)^*\| \\ &\leq \frac{r(4\delta)}{\delta} + 2\frac{2\delta}{\delta} + \frac{r(8\delta)}{2\delta} \end{aligned} \tag{6.6}$$

by Lemma 6.9 and (6.3),(6.5).

Let $\varepsilon' > 0$ and (δ', K) associated to property (T_{Q,E^*}) for the pair (G, A) . By uniform smoothness of E , we can choose δ small enough so that by (T_{Q,E^*}) and (6.4),(6.6),

$$\sup_{a \in A} \|\rho^*(a)\xi^* - \xi^*\| = \|(\rho(a^{-1})^{-1}\xi)^* - \xi^*\| < \varepsilon'.$$

Then

$$\|\rho(a)\xi - \xi\| \leq \|\rho(a)\xi - \rho(a^{-1})^{-1}\xi\| + \|\rho(a^{-1})^{-1}\xi - \xi\| \leq 2\delta + 2\frac{r_*(2\varepsilon')}{\varepsilon'}.$$

Again by uniform smoothness of E^* , we may choose δ, ε' small enough so that

$$\sup_{a \in A} \|\rho(a)\xi - \xi\| < \varepsilon$$

and thus (G, A) has relative property $(T_{Q,E})$. \square

6.3.2 Property $(T_{P,\mathcal{E}})$

Definition 6.23. Let (X, μ) be a σ -finite measured space. Let E be a ucus Banach space and $\theta \in L^\infty(X \times X)$. Then θ is an E -kernel if there exists a measurable map $F : X \rightarrow S(E)$ such that $\theta(x, y) = \langle F(x), F(y)^* \rangle$ almost everywhere.

Let \mathcal{E} be a class of ucus spaces such that for any $E_1, E_2 \in \mathcal{E}$, $E_1 \oplus_p E_2 \in \mathcal{E}$ for some p . If $\theta \in L^\infty(X \times X)$, we define the \mathcal{E} -cb-norm by

$$\|\theta\|_{\mathcal{E}\text{-cb}} = \inf \{ \|P\| \|Q\| \mid P \in L^\infty(X; E), Q \in L^\infty(X; E^*), E \in \mathcal{E}, \theta(x, y) = \langle P(x), Q(y) \rangle \}.$$

Denote $V^\infty(X, \mu; \mathcal{E}) = \{ \theta \in L^\infty(X \times X) \mid \|\theta\|_{\mathcal{E}\text{-cb}} < +\infty \}$ endowed with the \mathcal{E} -cb norm.

Remark 6.24. We need to consider a class of Banach spaces stable by direct sum of two Banach spaces for $\|\cdot\|_{\mathcal{E}\text{-cb}}$ to verify triangular inequality and be a norm.

As with the cb-norm (see Section 5.2.3), the space of map that are \mathcal{E} -cb is isometric to a space of operators that factor through \mathcal{E} .

Definition 6.25. Let (X, μ) be a σ -finite measured space. Let

$$\Gamma_{\mathcal{E}}(L^1(X), L^\infty(X)) = \left\{ T \in B(L^1(X), L^\infty(X)) \left| \begin{array}{l} T = SR \text{ where } R \in B(L^1(X), E), \\ S \in B(E, L^\infty(X)) \text{ for } E \in \mathcal{E} \end{array} \right. \right\}$$

with norm $\gamma_{\mathcal{E}}(T) = \inf \|S\| \|R\|$.

Proposition 6.26. Let $u \in L^\infty(X \times X)$. We can define an operator $T_u \in B(L^1(X), L^\infty(X))$ by

$$T_u(f)(x) = \int_X f(t)u(t, x) d\mu(t).$$

Then $u \mapsto T_u$ maps $V^\infty(X, \mu; \mathcal{E})$ isometrically onto $\Gamma_{\mathcal{E}}(L^1(X), L^\infty(X))$.

Proof. Assume that $u \in V^\infty(X, \mu; \mathcal{E})$, $u(x, y) = \langle P(x), Q(y) \rangle$ with $P \in L^\infty(X; E)$ and $Q \in L^\infty(X; E^*)$. Then

$$T_u(f) = \left\langle \int_X f(t)P(t) d\mu(t), Q(\cdot) \right\rangle.$$

Let $R : f \mapsto \int_X f(t)P(t) d\mu(t)$ and $S : v \mapsto \langle v, Q(\cdot) \rangle$. Then $R \in B(L^1(X), E)$ with $\|R\| \leq \|P\|_\infty$ and $S \in B(E, L^\infty(X))$ with $\|S\| \leq \|Q\|_\infty$. It is clear that $T_u = SR$, thus $T_u \in \Gamma_{\mathcal{E}}(L^1(X), L^\infty(X))$ with $\gamma_{\mathcal{E}}(T) \leq \|u\|_{\mathcal{E}\text{-cb}}$.

Conversely, let $T \in \Gamma_{\mathcal{E}}(L^1(X), L^\infty(X))$, $T = SR$ with $R \in B(L^1(X), E)$, $S \in B(E, L^\infty(X))$. By (5.12) with $F = \mathbb{C}$, we get that there is a unique $Q \in L^\infty(X, E^*)$ such that $S(v) = \langle v, Q(\cdot) \rangle$ (note that since E is uniformly convex, it has the Radon-Nikodym Property, so $L^\infty(X; E^*)$ coincides with $L^\infty_\sigma(X; E)$). The map Q is defined for $x \in X$, $v \in E$ by $Q(x)(v) = S(v)(x)$ and $\|Q\|_\infty = \|S\|$.

Furthermore, since $E = E^{**}$, combining (5.8), (5.9), (5.10), there is an isometric isomorphism

$$\begin{array}{ccc} L^\infty(X, E) & \rightarrow & B(L^1(X), E) \\ P & \mapsto & f \mapsto \int_X f(t)P(t) d\mu(t) \end{array}.$$

Fix P such that $R(f) = \int_X f(t)P(t) d\mu(t)$ for any $f \in L^1(X)$. Let $u = \langle P(\cdot), Q(\cdot) \rangle$. Then $u \in V^\infty(X, \mu; \mathcal{E})$ and $\|u\|_{\mathcal{E}\text{-cb}} \leq \|R\| \|S\|$. By construction, $T_u = T$. \square

Definition 6.27. Let G be a locally compact group and A be a closed subgroup. Let \mathcal{E} be a class of ucus spaces such that for any $E_1, E_2 \in \mathcal{E}$, $E_1 \oplus_p E_2 \in \mathcal{E}$ for some p .

The pair (G, A) has relative property $(T_{P,\mathcal{E}})$ if for all $\varepsilon > 0$, there are $\delta > 0$ and a compact subset $K \subset G$ such that for any $\theta : G \times G \mapsto \mathbb{C}$ E -kernel verifying

$$\sup_{g \in G} \|\theta(g \cdot, g \cdot) - \theta\|_{\mathcal{E}-cb} < \delta$$

and

$$\sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta$$

then

$$\sup_{x, y \in A} |\theta(x, y) - 1| < \varepsilon.$$

Remark 6.28. As for property (T_P) , it suffices for the second condition that

$$\sup_{x \in K} |\theta(x, 1) - 1| < \delta$$

(see Remark 5.7).

Proposition 6.29. Let \mathcal{E} be a class of ucus spaces such that for any $E_1, E_2 \in \mathcal{E}$, $E_1 \oplus_p E_2 \in \mathcal{E}$ for some p . If (G, A) has property $(T_{P,\mathcal{E}})$, then (G, A) has property $(T_{Q,\mathcal{E}})$.

Proof. Let $\varepsilon > 0$ and $E \in \mathcal{E}$. Let $\pi : G \rightarrow O(E)$ be an SOT-measurable map and $\xi \in S(E)$ such that

$$\sup_{g, h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{g \in K} \|\rho(g)\xi - \xi\| < \delta$$

for some δ, K to be determined later. Since E is uniformly convex, we can define the E -kernel

$$\theta : (x, y) \mapsto \langle \rho(x)\xi, (\rho(y)\xi)^* \rangle$$

using the duality mapping. We may assume that $1 \in K$, so that for any $x \in K$,

$$\begin{aligned} |\theta(x, 1) - 1| &= |\langle \rho(x)\xi, (\rho(1)\xi)^* \rangle - 1| \\ &= |\rho(x)\xi - \rho(1)\xi, (\rho(1)\xi)^*| \\ &\leq \|\rho(x)\xi - \rho(1)\xi\| \\ &\leq \|\rho(x)\xi - \xi\| + \|\rho(1)\xi - \xi\| \\ &\leq 2\delta. \end{aligned} \tag{6.7}$$

If $T \in B(E)$, define the adjoint T^* as the unique operator on E^* such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle.$$

If $T \in O(E)$, then $T^* \in O(E^*)$. For any $g, x, y \in G$, we get

$$\begin{aligned} \theta(gx, gy) - \theta(x, y) &= \langle \rho(gx)\xi, (\rho(gy)\xi)^* \rangle - \langle \rho(x)\xi, (\pi(y)\xi) \rangle \\ &= \langle \rho(gx)\xi, (\rho(gy)\xi)^* \rangle - \langle \rho(g)\rho(x)\xi, (\rho(g)^{-1})^*(\rho(y)\xi)^* \rangle \\ &= \langle \rho(gx)\xi - \rho(g)\rho(x)\xi, (\rho(gy)\xi)^* \rangle \\ &\quad + \langle \rho(g)\rho(x)\xi, (\rho(gy)\xi)^* - (\rho(g)^{-1})^*(\rho(y)\xi)^* \rangle \end{aligned} \tag{6.8}$$

Since $\langle \rho(g)\rho(y)\xi, (\rho(g)^{-1})^*(\rho(y)\xi)^* \rangle = 1$ and $(\rho(g)^{-1})^*(\rho(y)\xi)^* \in S(E^*)$, by uniqueness we get $(\rho(g)\rho(y)\xi)^* = (\rho(g)^{-1})^*(\rho(y)\xi)^*$. Thus since $\|\rho(gy)\xi - \rho(g)\rho(y)\xi\| \leq \delta$, by Lemma 6.9, we get

$$\|(\rho(gy)\xi)^* - (\rho(g)^{-1})^*(\rho(y)\xi)^*\| = \|(\rho(gy)\xi)^* - (\rho(g)\rho(y)\xi)^*\| \leq 2\frac{r(2\delta)}{\delta}.$$

So we obtain that

$$\|g^{-1}\theta - \theta\|_{\mathcal{E}-cb} \leq \delta + 2\frac{r(2\delta)}{\delta}.$$

Let $\varepsilon' > 0$ to be determined later, let (δ', K) be associated with ε' in $(T_{P,\mathcal{E}}$ for (G, A) . Since E is uniformly smooth, we can choose δ small enough so that $\max\left(2\delta, 2\frac{r(2\delta)}{\delta}\right) < \delta'$. Then by $(T_{P,\mathcal{E}}$ for the pair (G, A) , we get that

$$\sup_{a \in A} |\theta(a, 1) - 1| < \varepsilon'$$

so

$$\langle \rho(a)\xi, (\rho(1)\xi)^* \rangle > 1 - \varepsilon'.$$

By [BL98, Prop. A.5], we get by convexity of the norm that

$$\langle x, y^* \rangle \leq \|x + y\| - \|y\|$$

for any $x, y \in E$. Thus,

$$\|\rho(a)\xi + \rho(1)\xi\| > 1 - \varepsilon' + \|\rho(1)\xi\| = 2 - \varepsilon'$$

so

$$1 - \frac{\|\rho(a)\xi + \rho(1)\xi\|}{2} < \frac{\varepsilon'}{2}.$$

If we choose $\varepsilon' < 2d(\varepsilon/2)$, which is possible since by uniform convexity of E , $d(\varepsilon/2) > 0$, then by definition of d we get that

$$\|\pi(g)\xi - \pi(1)\xi\| < \varepsilon$$

so

$$\|\rho(a)\xi - \xi\| \leq \|\rho(a)\xi - \rho(1)\xi\| + \|\rho(1)\xi - \xi\| < \frac{\varepsilon}{2} + \delta < \varepsilon$$

as soon as δ is chosen smaller than $\frac{\varepsilon}{2}$. \square

Remark 6.30. The value of δ depends on the space $E \in \mathcal{E}$, since it depend both on the modulus of smoothness of E and the modulus of convexity, as it should be. However, once E is fixed, it does not depend on the choice of the map ρ . As pointed out in Remark 6.18, this is irrelevant as soon as \mathcal{E} is stable under direct sum of an infinite number of copies of the same space. If that is not the case, we actually proved that $(T_{P,\mathcal{E}})$ implies a statement stronger than $(T_{Q,\mathcal{E}})$. For the classes of spaces we want to consider, there is no distinction.

Notice however that to make sense of $(T_{P,\mathcal{E}})$, we had to restrict our spaces to uniformly convex spaces. Indeed, the definitions really depend on the norm we chose, so we cannot assume spaces to be super-reflexive, which is a problem.

Finally, we point out that by Theorem 5.11, property (T_P) is equivalent to property (T_Q) . This is *a priori* no longer true for Banach spaces. The comment above show that for the equivalence to hold, we should at least restrict to nice classes of Banach spaces, or slightly alter the definitions. But even then, the proof of Theorem 5.11 uses a lot of the Hilbertian structure and cannot be transposed to the Banach case.

6.3.3 Inheritance to lattices

We know from Theorem G that if Γ is a lattice in G , then G has property (TTT) if and only if Γ has property (TTT) . However, the proof relies on the equivalence of property (TTT) with the analytic properties and the fact that lattices inherits the analytic properties (T_P) and (T_Q) ([Oza11, Thm. 6]).

However in the context on Banach spaces, these properties are no longer equivalent. By adapting the proofs given by Ozawa in the Hilbertian context, we can still obtain some partial results.

Let G be locally compact second countable group and Γ lattice in G . By [Mac52, Lemma 1.1], there exists a Borel section $\sigma : G/\Gamma \rightarrow G$ which is regular, meaning that the image of any compact subset of G/Γ is relatively compact in G . Such a section defines a map $\beta : G \times G/\Gamma \rightarrow \Gamma$ by

$$\beta(g, x) = \sigma(x)^{-1}g\sigma(g^{-1}x).$$

This map satisfies as cocycle relation, in the sense that for any $g, h \in G$ and $x \in G/\Gamma$,

$$\beta(gh, x) = \beta(g, x)\beta(h, g^{-1}x).$$

Proposition 6.31. *Let H be a closed subgroup of G locally compact second countable such that there exists a finite Borel measure on G/H invariant under the action of G . Let B be a Banach space and $E = L^p(G/\Gamma; B)$. Then*

1. *If H has property $(T_{Q,B})$, G has property $(T_{Q,B})$.*
2. *If G has property $(T_{Q,E})$, H has property $(T_{Q,B})$.*

Proof. The proof is identical to [Oza11, Thm. 6] up to inducing on L^p instead of L^2 . \square

To obtain similar results for property (FFF_E) , we need to be able to induce wq-cocycles to L^p . When the lattice Γ is cocompact, a well-chosen fundamental domain makes it easy to prove that the induced map does indeed belong to L^p . To deal with other lattices, we need integrability conditions (see for example [BFGM07, Section 8] for the case of property (F_E)).

Definition 6.32. [BFGM07, Def. 8.2] Let $p > 0$. The lattice Γ is p -integrable if Γ is finitely generated and for some (equivalently, for any) finite generating set S , there is a Borel section $\sigma : G/\Gamma \rightarrow G$ such that the boundary of the fundamental domain $\mathcal{D} = \sigma(G/\Gamma)$ has zero measure and for any $g \in G$,

$$\int_{G/\Gamma} \ell_S(\beta(g, x))^p dx < +\infty$$

where ℓ_S is the word length associated to S .

Irreducible lattices in higher rank semisimple algebraic groups are known to be p -integrable for any $1 \leq p < +\infty$.

In our context, we want more than p -integrability: we wish for the integral to be locally bounded in G . This was defined in [CR09, Def. 30] as uniform p -integrability. We actually show that this comes automatically with p -integrability.

Lemma 6.33. *Let Γ be a p -integrable lattice in G locally compact second countable. Then for any compact subset C of G , the map*

$$F : g \mapsto \left(\int_{G/\Gamma} \ell_S(\beta(g, x))^p dx \right)^{\frac{1}{p}}$$

is bounded on C .

Proof. Consider the fundamental domain $\mathcal{D} = \sigma(G/\Gamma)$, we have that $\mathcal{D} \setminus \mathring{\mathcal{D}}$ has zero measure by assumption. Let $g \in G$ and $(g_n)_{n \in \mathbb{N}}$ such that $g_n \xrightarrow{n \rightarrow +\infty} g$. Let us show that $\beta(g_n, x)$ converges to $\beta(g, x)$ for μ -almost every x . By the cocycle property of β , it suffices to check at the identity $g = 1$. For any $x \in \mathring{\mathcal{D}}$, there exists n_0 such that for any $n \geq n_0$, $g_n^{-1}x \in \mathring{\mathcal{D}}$. Then

$$\beta(g_n, x\Gamma) = \sigma(x\Gamma)^{-1}g_n\sigma(g_n^{-1}x\Gamma) = x^{-1}g_n g_n^{-1}x = 1$$

for $n \geq n_0$.

Then by Fatou's lemma,

$$F(g) \leq \liminf F(g_n).$$

Thus F is lower semicontinuous, so for any $n \in \mathbb{N}$, $Q_n = \{g \mid F(g) \leq n\}$ is closed. By p -integrability, $G = \bigcup_n Q_n$ so by Baire's theorem, there exists n such that Q_n has nonempty interior. Let $g_0 \in U_0 = \overset{\circ}{Q}_n$, then F is bounded by n on U_0 .

For any $g \in G$, $gg_0^{-1}U_0$ is a neighborhood of g . By the cocycle property of β and G -invariance of the measure, F is subadditive, i.e. for any $g, h \in G$,

$$F(gh) \leq F(g) + F(h).$$

Thus F is bounded on $gg_0^{-1}U_0$ by $F(gg_0^{-1}) + n$.

So F is bounded in a neighborhood of any point, hence bounded on any compact subset of G . \square

Theorem 6.34. *Let G be a locally compact second countable group. Let Γ be a lattice in G . Let B be a Banach space and $E = L^p(G/\Gamma; B)$. Then*

1. *If Γ has property (FFF_B) , G has property (FFF_B) .*
2. *If G has property (FFF_E) and Γ is p -integrable, then Γ has property (FFF_B) .*

Remark 6.35. In particular, if \mathcal{E} is a class of super-reflexive Banach spaces such that for any $B \in \mathcal{E}$, $L^p(G/\Gamma; B) \in \mathcal{E}$, then property $(FFF_{\mathcal{E}})$ for G is equivalent to property $(FFF_{\mathcal{E}})$ for Γ . This is the case for the classes of Banach spaces we considered. For L^p spaces, this is clear. For $[\mathcal{H}]$, if E has a compatible norm, so does $L^2(X; E)$. Finally, if B is ucus, so is $E = L^p(X; B)$ ([BFGM07, Lem. 8.6]).

Proof. 1. Let $b : G \rightarrow B$ be a wq-cocycle, with $\rho : G \rightarrow O(B)$ such that

$$\sup_{g, h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| < D.$$

Since $b|_{\Gamma}$ is a wq-cocycle for Γ , by property (FFF_B) , $b|_{\Gamma}$ is bounded by C_1 . As above, choose a regular section $\sigma : G/\Gamma \rightarrow G$ and let β be the associated cocycle. Let μ be the G -invariant probability measure on G/Γ . Let X_0 be a compact subset of G/Γ such that $\mu(X_0) \geq \frac{2}{3}$. Since the section σ is regular and X_0 is compact, then $\sigma(X_0)$ is relatively compact in G . Since b is locally bounded, b is bounded on $\sigma(X_0) \cup \sigma(X_0)^{-1}$ by C_2 . Let $g \in G$, we have that

$$\mu(X_0 \cap gX_0) \geq \mu(X_0) + \mu(gX_0) - 1 \geq \frac{1}{3} > 0.$$

In particular, $X_0 \cap gX_0 \neq \emptyset$. Then for any $g \in G$, there exists $x \in X_0$ such that $g^{-1}x \in X_0$ and thus

$$\begin{aligned} \|b(g)\| &= \|b(\sigma(x)\beta(g,x)\sigma(g^{-1}x)^{-1})\| \\ &\leq \|b(g) - b(\sigma(x)) - \rho(\sigma(x))b(\beta(g,x)\sigma(g^{-1}x)^{-1})\| \\ &\quad + \|b(\sigma(x))\| + \|b(\beta(g,x)\sigma(g^{-1}x)^{-1})\| \\ &\leq D + C_2 + D + \|b(\beta(g,x))\| + \|b(\sigma(g^{-1}x)^{-1})\| \\ &\leq 2D + C_1 + 2C_2. \end{aligned}$$

so b is bounded.

2. Let $b : \Gamma \rightarrow B$ be a wq-cocycle, with

$$\sup_{g,h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| < D.$$

We want to induce to a wq-cocycle $\tilde{b} : G \rightarrow E = L^p(G/\Gamma; B)$. Set

$$\tilde{b}(g)(x) = b(\beta(g,x))$$

and define $\tilde{\rho} : G \rightarrow O(E)$ by

$$(\tilde{\rho}(g)f)(x) = \rho(\beta(g,x))f(g^{-1}x).$$

We need to show that \tilde{b} is a wq-cocycle.

We first check that \tilde{b} is well-defined. Let S be a finite generating set of Γ and $M = \max_{s \in S} \|b(s)\|$. Then since $\|b(gh)\| \leq D + \|b(g)\| + \|b(h)\|$ and by the cocycle property of β ,

$$\|b(\beta(g,x))\| \leq (D + M)\ell_S(\beta(g,x)).$$

Thus by p -integrability of Γ , $\tilde{b}(g) \in E$ for any $g \in G$. By p -integrability and Lemma 6.33, if C is a compact subset of G , we have

$$\|\tilde{b}(g)\| \leq (D + M) \left(\int_{G/\Gamma} \ell_S(\beta(g,x)) dx \right)^{\frac{1}{p}}$$

which is bounded on C . Thus, \tilde{b} is locally bounded.

Since Γ is countable, $B_0 = \overline{\text{span } b(\Gamma)}$ is a separable subspace. Thus \tilde{b} has range in the separable subspace $E_0 = L^p(X; B_0)$. It suffices to show that \tilde{b} is Borel measurable as a map into E_0 . By [Pis16, Prop. 2.20, 2.26 and Thm. 2.29], since E_0 is separable, $E_0^* = L_w^q(G/\Gamma; B_0^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ (recall that L_w^q denote the space of w^* -measurable functions, see Section 5.2.3). For any $f \in E_0^*$,

$$\langle \tilde{b}(g), f \rangle = \int_{G/\Gamma} \langle b(\beta(g,x)), f(x) \rangle dx$$

which is a measurable function of G by Fubini, since β is measurable. Thus \tilde{b} is weakly measurable and separably valued so by Pettis measurability theorem ([DU77, Ch. II, Thm. 2]), \tilde{b} is Bochner measurable hence Borel.

The map $\tilde{\rho}$ is the composition of the left regular representation on E , which is continuous, and the map τ defined for any $f \in E$ by

$$(\tau(g)f)(x) = \rho(\beta(g,x))f(x).$$

We want to show that $\tau : G \rightarrow O(E)$ is continuous in the strong operator topology. By definition of SOT, it suffices to show that for any $f \in E$, the map $g \mapsto \tau(g)f$ is continuous. Since G is second-countable, it suffices to show sequential continuity. Let $g \in G$ and $(g_n)_{n \in \mathbb{N}}$ such that $g_n \xrightarrow{n \rightarrow +\infty} g$. Let $f_n = \tau(g_n)f$. With the same argument as in the proof of Lemma 6.33, $\beta(g_n, \cdot) \rightarrow \beta(g, \cdot)$ almost everywhere, so f_n converges almost everywhere to $\tau(g)f$. Furthermore, $\|f_n(x)\| = \|f(x)\|$ so (f_n) is dominated by a function in $L^p(G/\Gamma)$, independently of $n \in \mathbb{N}$. Then by dominated convergence, $\int_{G/\Gamma} \|f_n(x) - (\tau(g)f)(x)\|^p dx \rightarrow 0$ so f_n converges to $\tau(g)f$. So we showed that τ hence $\tilde{\rho}$ is continuous (in particular, Borel).

Finally, we check that \tilde{b} verifies the wq-cocycle identity. For any $g, h \in G$,

$$\begin{aligned} \|\tilde{b}(gh) - \tilde{b}(g) - \tilde{\rho}(g)\tilde{b}(h)\|^p &= \int_{G/\Gamma} \|b(\beta(gh, x)) - b(\beta(g, x)) - \rho(\beta(g, x))b(\beta(h, g^{-1}x))\|^p dx \\ &\leq D^p \end{aligned}$$

using the cocycle property of β .

Since G has property (FFF_E), \tilde{b} is bounded on G . Set $\ell : \Gamma \rightarrow \mathbb{R}_+$ by $\ell(\gamma) = \|b(\gamma)\| + D$. Then $\ell(gh) \leq \ell(g) + \ell(h)$. Then

$$\begin{aligned} L(g) &= \int_{G/\Gamma} \ell(\beta(g, x)) dx \\ &\leq \int_{G/\Gamma} \|b(\beta(g, x))\| dx + D \\ &\leq \left(\int_{G/\Gamma} \|b(\beta(g, x))\|^p dx \right)^{1/p} + D \\ &\leq \|\tilde{b}(g)\| + D \end{aligned}$$

using Hölder's inequality and the fact that G/Γ is a probability space. Thus L is bounded on G and by [Oza11, Coro. 9], ℓ hence b is bounded on Γ . \square

Remark 6.36. In this proof, we showed that $\tilde{\rho}$ is not just Borel but actually continuous. Yet, we only proved that \tilde{b} is measurable. However, we can do better if we put a slightly stronger assumption on Γ .

Assume that Γ is not only p -integrable, but also q -integrable for some $q > p$. Then $\tilde{b} : G \rightarrow E = L^p(G/\Gamma; B)$ is continuous. Indeed, with the same argument as before for $\tilde{\rho}$, we know that if $g_n \rightarrow g$, then $\tilde{b}(g_n) \rightarrow \tilde{b}(g)$ almost everywhere. However, we could not find a domination to upgrade this to a convergence in E . But by q -integrability and Lemma 6.33, we get a bound

$$\|\tilde{b}(g_n)\|_q = \left(\int_{G/\Gamma} \|b(\beta(g_n, x))\|^q dx \right)^{\frac{1}{q}} \leq (D + M) \left(\int_{G/\Gamma} \ell(\beta(g_n, x))^q dx \right)^{\frac{1}{q}} \leq C$$

when n greater than some n_0 . By Egorov's theorem ([Fol13, Thm. 2.33]), for any $\varepsilon > 0$, there exists a measurable set $X_\varepsilon \subset G/\Gamma$ such that $\mu(X_\varepsilon) \leq \varepsilon$ and the convergence

$\tilde{b}(g_n) \rightarrow \tilde{b}(g)$ is uniform on $(G/\Gamma) \setminus X_\varepsilon$. Then for $n \geq n_0$, using Hölder's inequality,

$$\begin{aligned} \|\tilde{b}(g_n) - \tilde{b}(g)\| &= \left(\int_{G/\Gamma} \|b(\beta(g_n, x)) - b(\beta(g, x))\|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{X_\varepsilon^c} \|b(\beta(g_n, x)) - b(\beta(g, x))\|^p dx + \int_{X_\varepsilon} \|b(\beta(g_n, x)) - b(\beta(g, x))\|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\varepsilon^p + \mu(X_\varepsilon)^{1-\frac{p}{q}} \|\tilde{b}(g_n) - \tilde{b}(g)\|_q^p \right)^{\frac{1}{p}} \\ &\leq \left(\varepsilon^p + \varepsilon^{1-\frac{p}{q}} (2C)^p \right)^{\frac{1}{p}}. \end{aligned}$$

Since $q > p$, as ε can be taken arbitrarily small, we get that $\tilde{b}(g_n)$ converges to $\tilde{b}(g)$ in E , so \tilde{b} is continuous.

As a by-product of this remark, we get that if G is a group which has a uniformly q -integrable lattice Γ and \mathcal{E} is a class of Banach space stable under $L^p(G/\Gamma; -)$, with $p < q$, then property $(FFF_{\mathcal{E}})$ as we defined is equivalent to the same property restricted to continuous wq-cocycles.

6.4 The case of L^p -spaces

The Banach spaces $L^p(X, \mu)$ possesses some special properties that are useful in the study of (FFF_{L^p}) . The following theorem, due to Banach and Lamperti, classifies isometries of such spaces ([FJ02, Thm. 3.25]).

Theorem 6.37 (Banach-Lamperti). *Let $1 < p < +\infty$, $p \neq 2$. For any $U \in O(L^p(X, \mu))$, there exists a measurable and measure-class preserving map $T : X \rightarrow X$ and a measurable function h such that $|h(x)| = 1$ almost everywhere and such that for any $f \in L^p(X, \mu)$,*

$$(Uf)(x) = f(T(x))h(x) \left(\frac{dT_*\mu}{d\mu}(x) \right)^{\frac{1}{p}}.$$

The Mazur map can be used to relate L^p spaces for different values of p ([BL98, Thm. 9.1]).

Proposition 6.38. *Let (X, μ) be a σ -finite measure space. Let $1 \leq p, q < +\infty$. Consider the Mazur map*

$$M_{p,q} : \begin{array}{ccc} L^p(X, \mu) & \rightarrow & L^q(X, \mu) \\ f & \mapsto & \operatorname{sgn}(f)|f|^{\frac{p}{q}} \end{array}.$$

Then $M_{p,q}$ induces a uniformly continuous homeomorphism $M_{p,q} : S(L^p) \rightarrow S(L^q)$.

We can prove as in [BFGM07, Thm. A] implications between properties (T_{Q, L^p}) , for $1 \leq p < +\infty$. The first proposition makes use of the Mazur map.

Proposition 6.39. *Let $1 \leq p, q < +\infty$ with $p \neq 2$. If (G, A) has relative property (T_{Q, L^q}) then (G, A) has relative property (T_{Q, L^p}) .*

Proof. Assume that (G, A) has relative property (T_{Q, L^q}) . Let $\varepsilon > 0$, $\rho : G \rightarrow O(L^p)$ Borel, $\xi \in S(L^p)$ such that

$$\sup_{g, h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{x \in K} \|\rho(x)\xi - \xi\| < \delta$$

for δ, K to be determined.

Since $p \neq 2$, by Banach-Lamperti's description of $O(L^p)$ (Theorem 6.37), the map $U \mapsto M_{pq}UM_{qp}$ sends $O(L^p)$ to $O(L^q)$. Set $\tilde{\rho} : g \mapsto M_{pq}\rho(g)M_{qp}$ and $\tilde{\xi} = M_{qp}\xi \in S(L^q)$. Then $\tilde{\rho} : G \rightarrow O(L^q)$ is a Borel map. Furthermore, for any $x \in K, g, h \in G$,

$$\|\tilde{\rho}(x)\tilde{\xi} - \tilde{\xi}\| = \|M_{pq}\rho(x)\xi - M_{pq}\xi\|$$

and

$$\|\tilde{\rho}(gh)\tilde{\xi} - \tilde{\rho}(g)\tilde{\rho}(h)\tilde{\xi}\| = \|M_{pq}\rho(gh)\xi - M_{pq}\rho(g)\rho(h)\xi\|.$$

Let $\varepsilon' > 0$ to be determined later. By uniform continuity of M_{pq} and property (T_{Q,L^q}) , we can chose δ small enough so that

$$\sup_{a \in A} \|\tilde{\rho}(a)\tilde{\xi} - \tilde{\xi}\| < \varepsilon'.$$

By uniform continuity of M_{qp} , we can choose ε' small enough so that

$$\sup_{a \in A} \|\rho(a)\xi - \xi\| < \varepsilon$$

and so (G, A) has relative property (T_{Q,L^p}) . \square

Proposition 6.40. *Let $1 < p < +\infty$. If (G, A) has relative property (T_{Q,L^p}) , then (G, A) has relative property (T_Q) .*

Proof. Assume that (G, A) has relative property (T_{Q,L^p}) . Let $\varepsilon > 0$, \mathcal{H} a Hilbert space, $\pi : G \rightarrow O(\mathcal{H})$ Borel, $\xi \in S(\mathcal{H})$ such that

$$\sup_{g, h \in G} \|\pi(gh)\xi - \pi(g)\pi(h)\xi\| < \delta$$

and

$$\sup_{x \in K} \|\pi(x)\xi - \xi\| < \delta$$

for δ, K to be determined.

We will use the Gaussian functor construction (see [BdlHV08, Appendix A.7] for the construction and its properties). There exists a probability space (X, μ) and a linear map $v \in H \mapsto \hat{v} \in L^0(X, \mu)$ such that \hat{v} is a Gaussian random variable of variance $\|v\|^2$, and for any $T \in O(\mathcal{H})$, there exists an automorphism $\hat{T} \in \text{Aut}(X, \mu)$ which verifies $\hat{\xi} \circ \hat{T}^{-1} = \widehat{T\xi}$. Furthermore, the map $O(\mathcal{H}) \rightarrow \text{Aut}(X, \mu)$ is continuous. Thus, we may define a Borel map

$$\hat{\pi} : G \rightarrow O(L^p(X, \mu))$$

such that for any $f \in L^p(X, \mu)$, $\hat{\pi}(g)f = f \circ \widehat{\pi(g)}^{-1}$. Note that for any $v \in \mathcal{H}$, since \hat{v} is a Gaussian random variable, $\hat{v} \in L^p(X, \mu)$ and $\|\hat{v}\|_p = C_p\|v\|$ for some constant C_p which depends only on p . Thus, choose K and δ such that $(K, C_p\delta)$ is associated to $(C_p\varepsilon, \hat{\pi})$ in (T_{Q,L^p}) . Then for any $g, h \in G$,

$$\|\hat{\pi}(gh)\hat{\xi} - \hat{\pi}(g)\hat{\pi}(h)\hat{\xi}\|_p = \|\widehat{\pi(gh)\xi - \pi(g)\pi(h)\xi}\|_p < C_p\delta$$

and for any $x \in K$,

$$\|\hat{\pi}(x)\hat{\xi} - \hat{\xi}\|_p = \|\widehat{\pi(x)\xi - \xi}\|_p < C_p\delta.$$

Thus, relative property (T_{Q,L^p}) for the pair (G, A) gives that for any $a \in A$,

$$\|\hat{\pi}(a)\hat{\xi} - \hat{\xi}\|_p < C_p \varepsilon$$

thus

$$\|\pi(a)\xi - \xi\| < \varepsilon$$

which completes the proof. \square

Remark 6.41. We point out that the last two propositions are very different in nature. In Proposition 6.39, the proof shows that for a fixed standard Borel space (X, μ) , property $(T_{Q,L^q(X,\mu)})$ implies property $(T_{Q,L^p(X,\mu)})$ when $p \neq 2$. On the other hand, in Proposition 6.40, we use property (T_{Q,L^p}) for some abstract measured space given by the Gaussian construction, to obtain property (T_Q) . In particular, there is no reason for property $(T_{Q,\ell^p(\mathbb{N})})$ to imply property (T_Q) - even though any separable Hilbert space is isometric to $\ell^2(\mathbb{N})$. In fact if we consider true representations, it is known that property (T_{ℓ^p}) for $p \neq 2$ is strictly weaker than property (T) ([BO14]).

We can also show that for $p \leq 2$, the Hilbertian property (TTT) implies property (FFF_{L^p}) . As in [BFGM07, Thm. 1.3], the proof uses the crucial fact that for such values of p , the map $(f, g) \mapsto \|f - g\|^p$ is conditionally of negative type. Thus L^p embeds into the unit sphere of a Hilbert space and the proof is then similar to the proof that (T_P) implies (TTT) ([Oza11, Thm. 1]).

Proposition 6.42. *Let $1 \leq p \leq 2$. If (G, A) has property (TTT) then (G, A) has property (FFF_{L^p}) .*

Proof. Let $b : G \rightarrow L^p(X)$ be a wq-cocycle for the Borel map $\rho : G \rightarrow O(L^p(X))$. By Theorem 5.11, (G, A) has property (T_P) . Let $\varepsilon = \frac{1}{2}$, let (δ, K) be associated to ε in (T_P) for the pair (G, A) . Up to rescaling, we may assume that

$$\sup_{g \in K} \|b(g)\| \leq \delta_0$$

and

$$\sup_{g, h \in G} \|b(gh) - b(g) - \rho(g)b(h)\| \leq \delta_0$$

for $\delta_0 > 0$ to be determined later.

The map $\Psi : (f, g) \mapsto \|f - g\|_p^p$ is conditionally of negative type on $L^p(X) \times L^p(X)$. By Schoenberg's theorem ([BdlHV08, Thm. C.3.2], for any $s \geq 0$, the kernel $(f, g) \mapsto e^{-s\|f-g\|^p}$ is positive definite. By the GNS construction for $s = 1$, there exists a Hilbert space H and a map $U : L^p(X) \rightarrow S(H)$ such that for any $f, g \in L^p(X)$, $\langle U(f), U(g) \rangle = e^{-\|f-g\|^p}$.

Set $\theta(x, y) = \langle U(b(x)), U(b(y)) \rangle = e^{-\|b(x)-b(y)\|^p}$, this is a positive definite normalized kernel. Let $x, y \in G$ such that $x^{-1}y \in K$. Then

$$\|b(y) - b(x)\| \leq \|b(y) - b(x) - \rho(x)b(x^{-1}y)\| + \|b(x^{-1}y)\| \leq 2\delta_0$$

thus

$$\sup_{x^{-1}y \in K} |\theta(x, y) - 1| \leq 1 - e^{-2^p \delta_0^p}.$$

Furthermore for any $g, x, y \in G$,

$$\theta(gx, gy) = \langle U(b(gx)), U(b(gy)) \rangle$$

and

$$\begin{aligned}\theta(x, y) &= e^{-\|b(x)-b(y)\|^p} = e^{-\|b(g)+\rho(g)b(x)-(b(g)+\rho(g)b(y))\|^p} \\ &= \langle U(b(g) + \rho(g)b(x)), U(b(g) + \rho(g)b(y)) \rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\theta(gx, gy) - \theta(x, y) &= \langle U(b(gx)) - U(b(g) + \rho(g)b(x)), U(b(gy)) \rangle \\ &\quad + \langle U(b(g) + \rho(g)b(x)), U(b(gy) - U(b(g) + \rho(g)b(y))) \rangle\end{aligned}$$

so that

$$\|g^{-1}\theta - \theta\|_{cb} \leq 2 \sup_{x \in G} \|U(b(gx)) - U(b(g) + \rho(g)b(x))\|.$$

But

$$\|U(b(gx)) - U(b(g) + \rho(g)b(x))\|^2 = 2 - 2e^{-\|b(gx)-b(g)-\rho(g)b(x)\|^p} \leq 2 - 2e^{-\delta_0^p}.$$

Thus

$$\sup_{g \in G} \|g^{-1}\theta - \theta\|_{cb} \leq 2\sqrt{2}(1 - e^{-\delta_0^p})^{1/2}.$$

Choose δ_0 small enough, then we can apply property (T_P) to get that

$$\sup_{x, y \in A} |\theta(x, y) - 1| < \frac{1}{2}.$$

In particular, for any $a \in A$,

$$1 - \theta(a, 1) = 1 - e^{-\|b(a)-b(1)\|^p} \leq \frac{1}{2}$$

so

$$\|b(a) - b(1)\|^p \leq \ln(2)$$

so b is bounded on A . □

In [Mim11, Thm. 1.3], Mimura shows that "up to increasing the rank", property (F_E) can be obtained from property (T_E) . More precisely, he proves that relative property (T_E) for $(SL_n(\mathbb{R}) \times \mathbb{R}^n, \mathbb{R}^n)$ implies relative property (F_E) for $(SL_{n+1}(\mathbb{R}) \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. An even stronger result is given in [Mim11, Thm. 6.4], [dLMdlS16, Prop. 5.2]: relative property (T_E) for $(SL_n(\mathbb{R}) \times \mathbb{R}^n, \mathbb{R}^n)$ implies relative property (FFF_E) for $(SL_{n+1}(\mathbb{R}) \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Thus, a statement on almost invariant vectors of a representation implies a statement on quasi-cocycles. We believe that with similar ideas, it may be possible to show that $(T_{Q,E})$ implies (FFF_E) "up to increasing the rank". We were not able to prove such a statement, the main difficulty being that the value $\delta(\varepsilon)$ given by $(T_{Q,E})$ can decrease much faster than ε .

6.5 Banach spaces isomorphic to Hilbert spaces

Let A be a locally compact abelian group and consider a continuous action of a locally compact group H on A . Let $G = H \rtimes A$ be the associated semi-direct product. It is well-known that relative property (T) for the pair (G, A) can be linked with invariant means on the Pontryagin dual \hat{A} of A . It was proven by Shalom ([Sha99, Thm. 5.5]) that if the only H -invariant mean on \hat{A} is the Dirac δ_0 , then (G, A) has relative property (T) .

In [Ioa10, CT11], a complete equivalent characterization of relative property (T) for (G, A) is given in terms of H -invariant means on \hat{A} . As shown by Ozawa ([Oza11, Prop. 3]), this equivalence extends to property (T_P) (and thus (T_Q) and (TTT)).

In this section, we want to show property $(T_{Q, [\mathcal{H}]})$ for these semi-direct products under some assumptions on means. Note that if $\pi : G \rightarrow O(E)$ is a measurable map, where E has a compatible Hilbert norm $\|\cdot\|$, the group $\langle \pi(G) \rangle$ is contained in $O(E)$. Then we can see π as a map $G \rightarrow B(E, \|\cdot\|)$ and so $\langle \pi(G) \rangle$ is uniformly bounded. Thus property $(T_{Q, [\mathcal{H}]})$ is equivalent to a statement of property (T_Q) for uniformly bounded maps on Hilbert spaces. If π is a true isometric representation on E , then it is a uniformly bounded representation on $(E, \|\cdot\|)$.

Before stating the result, let us recall some details on complex measures and functional analysis (see [Rud74, Ch. 6]). Let (X, Σ) be a measured space. A complex measure is a map $\mu : \Sigma \rightarrow \mathbb{C}$ such that for any disjoint sequence $(A_n)_{n \in \mathbb{N}}$, $\mu(\bigcup A_n) = \sum_{n \geq 0} \mu(A_n)$. Notice that this implies the absolute convergence of $\sum \mu(A_n)$. We define the total variation measure $|\mu|$ by

$$|\mu|(A) = \sup \left\{ \sum |\mu(A_k)| \mid (A_k) \text{ partition of } A \right\}.$$

Then $|\mu|$ is a finite positive measure and the smallest positive measure ν such that for all $A \in \Sigma$, $|\mu(A)| \leq \nu(A)$. It is easy to show that $|\mu_1 + \mu_2| \leq |\mu_1| + |\mu_2|$ so

$$\left| |\mu_1| - |\mu_2| \right| \leq |\mu_1 - \mu_2|.$$

Furthermore, μ has a polar decomposition: there exists a measurable map $\theta : X \rightarrow \mathbb{R}$ such that

$$d\mu = e^{i\theta} d|\mu|.$$

We define a norm on the space of all complex measures by

$$\|\mu\|_{TV} = |\mu|(X).$$

Assume now that X is locally compact and Σ is its Borel σ -algebra. Let $M(X)$ be the space of complex measures μ such that $|\mu|$ is regular. Then $(M(X), \|\cdot\|_{TV})$ is naturally isometric to $C_0(X)^*$ through integration against μ .

Let also $\mathcal{L}^\infty(X)$ be the space of all bounded measurable functions on X with the supremum norm. Then $\mathcal{L}^\infty(X)^*$ is identified with the space of *finitely additive* finite complex measures. A mean is a map $m \in \mathcal{L}^\infty(X)^*$ which is positive and such that $m(1) = 1$.

Lemma 6.43. *Let $G = H \rtimes A$ be the semi-direct product of a locally compact second countable abelian group A by a locally compact second countable group H . Assume that the only H -invariant mean on \hat{A} is δ_0 where 0 is the neutral element in the Pontryagin dual \hat{A} of A . Then for any $\varepsilon > 0$, there exists $\delta > 0$ and a compact subset $Q \subset H$ such that for any $\mu \in M(\hat{A})$ verifying*

$$\sup_{h \in Q} \|h_*\mu - \mu\|_{TV} < \delta \|\mu\|_{TV}$$

then

$$\sup_{a \in A} \left| \int_{\hat{A}} (\chi(a) - 1) d\mu(\chi) \right| < \varepsilon \|\mu\|_{TV}.$$

Proof. By σ -compactness, we can consider an increasing sequence $(Q_n)_{n \in \mathbb{N}}$ of compact subset such that $H = \bigcup Q_n$. By contradiction, assume there is $\varepsilon > 0$ such that for any compact subset $Q \subset H$ and any $\delta > 0$, there exists $\mu \in M(\hat{A})$ with

$$\sup_{h \in Q} \|h_*\mu - \mu\|_{TV} < \delta \|\mu\|_{TV}$$

and

$$\sup_{a \in A} \left| \int_{\hat{A}} (\chi(a) - 1) d\mu(\chi) \right| \geq \varepsilon \|\mu\|_{TV}.$$

In particular for any $n \in \mathbb{N}^*$, we may consider μ_n the measure given by the pair $(Q_n, \frac{1}{n})$. Then $\tilde{\mu}_n = \frac{|\mu_n|}{\|\mu_n\|_{TV}}$ is a positive probability measure. Since $|h_*\mu_n| = h_*|\mu_n|$, we get that for any $h \in Q_n$,

$$\|h_*\tilde{\mu}_n - \tilde{\mu}_n\|_{TV} = \frac{\| |h_*\mu_n| - |\mu_n| \|_{TV}}{\|\mu_n\|_{TV}} \leq \frac{\|h_*\mu_n - \mu_n\|_{TV}}{\|\mu_n\|_{TV}} < \frac{1}{n}. \quad (6.9)$$

Then by Banach-Alaoglu theorem, there is a subnet $(\tilde{\mu}_i)_{i \in I}$ converging to a mean $\nu \in \mathcal{L}^\infty(\hat{A})^*$. Since $H = \bigcup Q_n$ and (Q_n) is increasing, by (6.9), we obtain that ν is H -invariant hence by hypothesis, $\nu = \delta_0$. Thus, the sequence $(\tilde{\mu}_i(\hat{A} - \{0\}))_{i \in I}$ converges to 0.

But by assumption, we have for any $n \in \mathbb{N}$ that

$$\begin{aligned} \varepsilon &\leq \frac{1}{\|\mu_n\|_{TV}} \left| \int_{\hat{A}} (\chi(a) - 1) d\mu_n(\chi) \right| \\ &\leq \frac{1}{\|\mu_n\|_{TV}} \int_{\hat{A}} |\chi(a) - 1| d|\mu_n|(\chi) \\ &\leq \frac{2}{\|\mu_n\|_{TV}} \int_{\hat{A}} \mathbf{1}_{\chi(a) \neq 1} d|\mu_n|(\chi) \\ &\leq 2\tilde{\mu}_n(\hat{A} - \{0\}) \end{aligned}$$

which is a contradiction. \square

We also need to recall some properties of abelian groups. Since A is abelian, it is amenable so we can consider a Følner sequence $(F_n)_{n \in \mathbb{N}}$ and denote $|F_n|$ its measure. Let ω be a free ultrafilter on \mathbb{N} . If $f \in L^\infty(A)$, define

$$f_n = \frac{1}{|F_n|} \int_{F_n} f(a) d\mu(a).$$

Since $(f_n)_{n \in \mathbb{N}}$ is bounded, we can consider the ultralimit

$$m(f) = \lim_{\omega} f_n.$$

Then because $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence, m is an invariant mean on $L^\infty(A)$.

If $\theta \in L^\infty(A \times A)$ is continuous and left-invariant (i.e. there exists $\varphi : A \rightarrow \mathbb{C}$ such that $\theta(x, y) = \varphi(x^{-1}y)$) and has finite cb-norm, then the map

$$m_\theta : \begin{array}{ccc} L^1(A) & \rightarrow & L^1(A) \\ f & \mapsto & \varphi f \end{array}$$

extends to the reduced C^* -algebra $C_r^*(A)$ with $\|m_\theta\| \leq \|\theta\|_{cb}$ ([Oza11, Section 2]). Since A is abelian, the Fourier transform gives an isomorphism $C_r^*(A) \rightarrow C_0(\hat{A})$ where \hat{A} is the Pontryagin dual of A . If we see the trivial character 0_A as a linear functional on $C_0(\hat{A})$, then $0_A \circ m_\theta \in C_0(\hat{A})^* = M(\hat{A})$ is represented by a complex measure μ_θ such that

$$\|\mu_\theta\| = \|\theta\|_{cb}$$

and

$$\varphi(a) = \int_{\hat{A}} \chi(a) d\mu_\theta(\chi).$$

If a locally compact group G acts continuously on A by group automorphisms, the map $(g \cdot \theta) : (x, y) \mapsto \theta(g^{-1} \cdot x, g^{-1} \cdot y)$ is also an A -invariant map represented by $g \cdot \varphi : a \mapsto \varphi(g^{-1} \cdot a)$ with the same cb-norm as θ and one can check that

$$\mu_{g \cdot \theta} = g_* \mu.$$

We can now prove Theorem J. In what follows, when $G = H \ltimes A$ is a semi-direct product, we will denote the action of $g \in G$ on itself by left translations by $(g, x) \mapsto gx$. On the other hand, to avoid confusion, we will denote the action by conjugation on the normal subgroup A by $(g, x) \mapsto g \cdot x = gxg^{-1}$. We will do the same for the actions on function spaces on G and A .

Theorem 6.44. *Let $G = H \ltimes A$ be the semi-direct product of a locally compact second countable abelian group A by a locally compact second countable group H . Assume that the only H -invariant mean on \hat{A} is δ_0 where 0 is the neutral element in the Pontryagin dual \hat{A} of A . Then the pair (G, A) has relative property $(T_{Q, [\mathcal{H}]})$.*

Proof. Let $(H, \|\cdot\|_B)$ be a Banach space with a compatible Hilbert norm $\|\cdot\|_H$ and $\varepsilon > 0$. Let $\rho : G \rightarrow O(H, \|\cdot\|_B)$ be a Borel map, $\xi \in H$ with $\|\xi\|_B = 1$ such that

$$\sup_{g, h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\|_B < \delta,$$

$$\sup_{x \in Q} \|\rho(x)\xi - \xi\|_B < \delta$$

for some compact subset $Q \subset G$ and $\delta > 0$ to be determined later. We may assume Q to be symmetric and containing a neighborhood of the unit 1 , and that $\rho(1) = \text{Id}$ (Remark 6.17).

By Proposition 6.13, we may assume that the norm $\|\cdot\|_B$ is ucus while keeping $\rho(g)$ isometric for any g . Denote $\langle \cdot, \cdot \rangle$ the duality bracket in $(H, \|\cdot\|_B)$ and $(\cdot | \cdot)$ the inner product associated to $\|\cdot\|_H$. Let also $\|\cdot\|_{B^*}$ be the dual norm on H^* of $\|\cdot\|_B$. Let $M > 0$ be the norm ratio, i.e. for any $x, y \in H$,

$$M^{-1} \|\cdot\|_B \leq \|\cdot\|_H \leq M \|\cdot\|_B.$$

Then we also have

$$M^{-1} \|\cdot\|_{B^*} \leq \|\cdot\|_{H^*} \leq M \|\cdot\|_{B^*}.$$

If $\phi \in H^*$, by Riesz representation theorem, there exists a unique $\eta_\phi \in H$ such that for any $x \in H$,

$$\langle x, \phi \rangle = \phi(x) = (x | \eta_\phi)$$

and

$$\|\eta_\phi\|_H = \|\phi\|_{H^*} \leq M \|\phi\|_{B^*}.$$

Set $\theta(x, y) = \langle \rho(x)\xi, (\rho(y)\xi)^* \rangle$. Then θ is a $[\mathcal{H}]$ -kernel, but we will see that we can understand its completely bounded norm in the sense given in Section 5.2, i.e. with Hilbert spaces. However, we must keep in mind that θ need not be a positive definite kernel, which will force us to use complex measures. First,

$$\theta(x, y) = \langle \rho(x)\xi, (\rho(y)\xi)^* \rangle = \left(\rho(x)\xi | \eta_{(\rho(y)\xi)^*} \right)$$

so that

$$\|\theta\|_{cb} \leq \|\rho(x)\xi\|_H \|\eta_{(\rho(y)\xi)^*}\|_H \leq M^2.$$

On the other hand,

$$\|\theta\|_\infty = 1 \leq \|\theta\|_{cb}.$$

We can also use the same computations as in the proof of Proposition 6.29. By (6.7), we obtain that

$$\sup_{x \in Q} |\theta(x, 1) - 1| < 2\delta.$$

By (6.8), for any $g, x, y \in G$,

$$\begin{aligned} \theta(gx, gy) - \theta(x, y) &= \langle \rho(gx)\xi - \rho(g)\rho(x)\xi, (\rho(gy)\xi)^* \rangle \\ &\quad + \langle \rho(g)\rho(x)\xi, (\rho(gy)\xi)^* - (\rho(g)\rho(y)\xi)^* \rangle \\ &= \left(\rho(gx)\xi - \rho(g)\rho(x)\xi \mid \eta_{(\rho(gy)\xi)^*} \right) \\ &\quad + \left(\rho(g)\rho(x)\xi \mid \eta_{(\rho(gy)\xi)^* - (\rho(g)\rho(y)\xi)^*} \right). \end{aligned}$$

Thus

$$\begin{aligned} \|g^{-1}\theta - \theta\|_{cb} &\leq \sup_{x \in G} \|\rho(gx)\xi - \rho(g)\rho(x)\xi\|_H \|\eta_{(\rho(gy)\xi)^*}\|_H \\ &\quad + \sup_{y \in G} \|\rho(g)\rho(x)\xi\|_H \|\eta_{(\rho(gy)\xi)^* - (\rho(g)\rho(y)\xi)^*}\|_H \\ &\leq M^2\delta + M^2 \sup_{y \in G} \|(\rho(gy)\xi)^* - (\rho(g)\rho(y)\xi)^*\|_{B^*} \\ &\leq M^2\delta + 2M^2 \frac{r(2\delta)}{\delta} = \delta' \end{aligned}$$

where r is the modulus of smoothness of $(H, \|\cdot\|_B)$.

We can also check that if $g, h \in Q$, then

$$\theta(xg, yh) - \theta(x, y) = \langle \rho(xg)\xi - \rho(x)\xi, (\rho(yh)\xi)^* \rangle + \langle \rho(x)\xi, (\rho(yh)\xi)^* - (\rho(y)\xi)^* \rangle \quad (6.10)$$

Since $\|\rho(xg)\xi - \rho(x)\xi\|_B \leq \|\rho(xg)\xi - \rho(x)\rho(g)\xi\|_B + \|\rho(x)\rho(g)\xi - \rho(x)\xi\|_B < 2\delta$, we get by the same estimates as above that

$$\sup_{g, h \in Q} \|\theta(\cdot g, \cdot h) - \theta\|_{cb} \leq 2M^2\delta + M^2 \frac{r(4\delta)}{\delta} = \delta''. \quad (6.11)$$

We then use arguments similar to [Oza11, Prop. 3]. If $a, x, y \in A, g \in Q$, define

$$\theta_a^g(x, y) = \frac{1}{\text{vol}(Q)^2} \int_{Q \times Q} \theta(agxg^{-1}q, agyg^{-1}q') dq dq'.$$

Then (θ_a^g) is a family of equicontinuous functions on $A \times A$ and such that by (6.10) and (6.11),

$$\|\theta_a^g - \theta\|_{cb} \leq \delta' + \delta''.$$

Since for any $x, y \in A, g \in Q$, the map $a \mapsto \theta_a^g(x, y)$ is in $L^\infty(A)$, we can apply the invariant mean to get

$$\tilde{\theta}^g(x, y) = \lim_{\omega} \frac{1}{|F_n|} \int_{F_n} \theta_a^g(x, y) da.$$

Since $g^{-1} \cdot \theta_a^1 = \theta_a^g$ for any a , we get that $g^{-1} \cdot \tilde{\theta}^1 = \tilde{\theta}^g$. The function $\tilde{\theta}^g$ is an A -invariant kernel on $A \times A$ and

$$\|\tilde{\theta}^g - \theta\| \leq \delta' + \delta'' = \tilde{\delta}.$$

Let μ be the measure on \hat{A} associated to the A -invariant completely bounded map $\tilde{\theta}^1$, then $(g^{-1})_*\mu$ is the measure associated to $\tilde{\theta}^g$. We have that

$$\|\mu\|_{TV} = \|\tilde{\theta}^1\|_{cb} \in [\|\theta\|_{cb} - \tilde{\delta}, \|\theta\|_{cb} + \tilde{\delta}] \subset [1 - \tilde{\delta}, M^2 + \tilde{\delta}].$$

Then for any $g \in Q$,

$$\|g_*\mu - \mu\|_{TV} \leq \|\tilde{\theta}^{g^{-1}} - \tilde{\theta}^1\| \leq 2\tilde{\delta}.$$

We used here the assumption that Q is symmetric. Let $\varepsilon_0 = \frac{d(\varepsilon)}{M^2+1}$ where d is the modulus on convexity of $(H, \|\cdot\|_B)$. Then $\varepsilon_0 > 0$ by uniform convexity. Let (Q_0, δ_0) be associated to ε_0 from Lemma 6.43. Assume that $Q_0 \subset Q$. By uniform smoothness, we may choose δ small enough so that $\tilde{\delta} \leq \min(\frac{1}{2}, \delta_0, d(\varepsilon))$. Then for any $g \in Q_0$,

$$\|g_*\mu - \mu\|_{TV} \leq 2\tilde{\delta} \leq \delta_0 \|\mu\|_{TV}.$$

Thus by Lemma 6.43, we obtain that

$$\sup_{a \in A} \left| \int_{\hat{A}} (\chi(a) - 1) d\mu(\chi) \right| < \varepsilon \|\mu\|_{TV} \leq (M^2 + 1)\varepsilon_0.$$

Thus for any $a \in A$,

$$|\theta(a, 1) - 1| \leq |\theta(a, 1) - \tilde{\theta}^1(a, 1)| + |\tilde{\theta}^1(a, 1) - 1| \leq \tilde{\delta} + \left| \int_{\hat{A}} (\chi(a) - 1) d\mu(\chi) \right| \leq \tilde{\delta} + (M^2 + 1)\varepsilon_0.$$

Again with computations as in Proposition 6.29, we obtain that

$$1 - \frac{\|\rho(a)\xi + \xi\|_B}{2} \leq \frac{\tilde{\delta} + (M^2 + 1)\varepsilon_0}{2} \leq d(\varepsilon)$$

for any $a \in A$, thus

$$\sup_{a \in A} \|\rho(a)\xi - \xi\|_B \leq \varepsilon$$

which completes the proof. \square

By Proposition 6.19, we immediately obtain the following corollary on property (T) for uniformly bounded representations. We do not know if this was already known.

Corollary 6.45. *Let $G = H \rtimes A$ be the semi-direct product of a locally compact second countable abelian group A by a locally compact second countable group H . Assume that the only H -invariant mean on \hat{A} is δ_0 where 0 is the neutral element in the Pontryagin dual \hat{A} of A . Then the pair (G, A) has relative property $(T_{[\mathcal{H}]})$.*

With Theorem 6.44 and the classical proof scheme for property (T) , we can obtain property $(T_{Q, [\mathcal{H}]})$ for simple algebraic groups. We first apply our result to classical semi-direct products (see Section 5.3 for the definition of $SL_2(\mathbb{K}) \rtimes S^{2*}(\mathbb{K}^2)$).

Proposition 6.46. *Let \mathbb{K} be a local field. Then*

- $(SL_2(\mathbb{K}) \rtimes \mathbb{K}^2, \mathbb{K}^2)$ has relative property $(T_{Q, [\mathcal{H}]})$;
- $(SL_2(\mathbb{K}) \rtimes S^{2*}(\mathbb{K}^2), S^{2*}(\mathbb{K}^2))$ has relative property $(T_{Q, [\mathcal{H}]})$.

Proof. By [BdlHV08, Prop. 1.4.12 and 1.5.1], these pairs verify the hypothesis on invariant means. The result is then a direct consequence of Theorem 6.44. \square

Proposition 6.47. *Let \mathbb{K} be a local field. Then*

- $SL_3(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$;
- $Sp_4(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$.

Proof. The proof is nearly identical to [Oza11, Thm. 5] and Theorem 5.13, using Proposition 6.46 to get the appropriate result on the embedded semi-direct products. \square

Finally, to extend the result to simple algebraic groups, we need a few lemmas as in Section 5.4. We will only sketch the proofs as they are very similar, although we worked with wq-cocycles in Section 5.4, while here we work with maps $\rho : G \rightarrow O(E)$.

Recall G_1, G_2 are two locally compact group, a quasi-homomorphism is a Borel map $\varphi : G_1 \rightarrow G_2$ which is regular (i.e. the image of a compact subset of G_1 is relatively compact) and such that $\{\varphi(gh)^{-1}\varphi(g)\varphi(h) \mid g, h \in G_1\}$ is a relatively compact of G_2 .

Proposition 6.48. *Let G_1, G_2 be two locally compact groups. Let E be a Banach space. Let $\varphi : G_1 \rightarrow G_2$ be a surjective quasi-homomorphism. If G_1 has $(T_{Q,E})$, then G_2 has $(T_{Q,E})$.*

Proof. Let $\varepsilon > 0$, $\rho : G_2 \rightarrow O(E)$ Borel. Let $\xi \in S(E)$. Consider $\tilde{\rho} = \rho \circ \varphi$. Let Q be a compact subset of G_1 and $\delta > 0$ associated to $(\varepsilon, \tilde{\rho})$ in $(T_{Q,E})$ for G_1 . Let $Q' = \overline{\varphi(Q) \cup \{\varphi(gh)^{-1}\varphi(g)\varphi(h) \mid g, h \in G_1\}}$ compact subset of G_2 , $\delta' = \delta/5$. Then one can check that if ρ has the hypotheses of $(T_{Q,E})$ for (Q', δ') , we can apply the property to $\tilde{\rho}$ so that

$$\sup_{x \in G_1} \|\tilde{\rho}(x)\xi - \xi\| = \sup_{g \in G_2} \|\rho(g)\xi - \xi\| < \varepsilon$$

by surjectivity of φ . \square

Proposition 6.49. *Let G be a second countable locally compact group, $N \triangleleft G$ a closed normal subgroup. Let E be a Banach space. If N and G/N have $(T_{Q,E})$, then G has $(T_{Q,E})$.*

Proof. Let $\varepsilon > 0$, $\rho : G \rightarrow O(E)$ Borel. By [Mac52, Lemma 1.1], there exists a Borel section $\sigma : G/N \rightarrow G$ which is regular, meaning that the image of any compact subset of G/N is relatively compact in G . Let $(Q_{G/N}, \delta_{G/N})$ be given by property $(T_{Q,E})$ for the group G/N with $\varepsilon_{G/N} = \frac{\varepsilon}{3}$ and $\tilde{\rho} = \rho \circ \sigma$. Let (Q_N, δ_N) be given by property $(T_{Q,E})$ for the group N with $\varepsilon_N = \min\left(\frac{\varepsilon}{3}, \frac{\delta_{G/N}}{5}\right)$ and $\rho|_N$. Finally, let $Q = Q_N \cup \overline{\sigma(Q_{G/N})}$ compact subset of G and $\delta = \min\left(\frac{\varepsilon}{3}, \frac{\delta_{G/N}}{5}, \delta_N\right)$.

Then one can check that for any $\xi \in S(E)$, if

$$\sup_{g, h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{x \in Q} \|\rho(x)\xi - \xi\| < \delta,$$

we obtain

$$\sup_{n \in N} \|\rho(n)\xi - \xi\| < \varepsilon_N$$

and

$$\sup_{x \in G/N} \|\tilde{\rho}(x)\xi - \xi\| < \varepsilon_{G/N}.$$

Finally, since $g^{-1}\sigma(gN) \in N$, we obtain

$$\sup_{g \in G} \|\rho(g)\xi - \xi\| < \varepsilon$$

and G has $(T_{Q,E})$. \square

Lemma 6.50. *Let \mathbb{K} be a local field, G a connected semisimple \mathbb{K} -group and \tilde{G} its simply connected cover (in the algebraic sense). Let \mathcal{E} be a class of Banach spaces with the property that there exists $1 < p < +\infty$ such that for any $E \in \mathcal{E}$ and any (X, μ) a σ -finite standard Borel space, $L^p(X; E) \in \mathcal{E}$. Then $G(\mathbb{K})$ has $(T_{Q, \mathcal{E}})$ if and only if $\tilde{G}(\mathbb{K})$ has $(T_{Q, \mathcal{E}})$.*

Proof. The proof is identical to that of Lemma 5.18 but using Propositions 6.31, 6.48 and 6.49. \square

Lemma 6.51. (*Mautner lemma, revisited*) *Let G be a locally compact group, E a Banach space, $\rho : G \rightarrow O(E)$ and $\xi \in S(E)$. Let $\varepsilon > 0$, assume that there exists a neighborhood U of 1 and $a \in G$ such that*

$$\sup_{g, h \in G} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \varepsilon,$$

$$\sup_{x \in U} \|\rho(x)\xi - \xi\| < \varepsilon$$

and

$$\sup_{n \in \mathbb{Z}} \|\rho(a^n)\xi - \xi\| < \varepsilon.$$

Then for any $g \in G$ such that $a^n g a^{-n} \xrightarrow{n \rightarrow +\infty} 1$,

$$\|\rho(g)\xi - \xi\| < 6\varepsilon.$$

Proof. Let $g \in G$, $n \in \mathbb{Z}$, then

$$\begin{aligned} \|\rho(g)\xi - \xi\| &\leq \|\rho(g)\xi - \rho(g)\rho(a^{-n})\xi\| + \|\rho(g)\rho(a^{-n})\xi - \rho(a^{-n})\xi\| + \|\rho(a^{-n})\xi - \xi\| \\ &\leq 2\varepsilon + \|\rho(g)\rho(a^{-n})\xi - \rho(a^{-n})\xi\| \\ &\leq 2\varepsilon + \|\rho(g)\rho(a^{-n})\xi - \rho(ga^{-n})\xi\| + \|\rho(ga^{-n})\xi - \rho(a^{-n})\xi\| \\ &\leq 3\varepsilon + \|\rho(a^n)\rho(ga^{-n})\xi - \rho(a^n)\rho(a^{-n})\xi\| \\ &\leq 3\varepsilon + \|\rho(a^n)\rho(ga^{-n})\xi - \rho(a^n g a^{-n})\xi\| + \|\rho(a^n g a^{-n})\xi - \rho(a^n)\rho(a^{-n})\xi\| \\ &\leq 4\varepsilon + \|\rho(a^n g a^{-n})\xi - \xi\| + \|\xi - \rho(a^n)\rho(a^{-n})\xi\| \\ &\leq 5\varepsilon + \|\rho(a^n g a^{-n})\xi - \xi\|. \end{aligned}$$

Now by assumption, $a^n g a^{-n} \xrightarrow{n \rightarrow +\infty} 1$ so there is n such that $a^n g a^{-n} \in U$ thus

$$\|\rho(a^n g a^{-n})\xi - \xi\| < \varepsilon$$

which completes the proof. \square

We now have all the ingredients to prove Theorem K.

Theorem 6.52. *Let \mathbb{K} be a local field, G a connected almost \mathbb{K} -simple \mathbb{K} -group with $\text{rank}_{\mathbb{K}} G \geq 2$. Then $G(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$.*

Let Γ be a lattice in $G(\mathbb{K})$, then Γ has property $(T_{Q, [\mathcal{H}]})$.

Proof. By Lemma 6.50, we may assume G to be simply connected. By [Mar91, Ch.I, Prop. 1.6.2], G contains an almost \mathbb{K} -simple \mathbb{K} -subgroup H whose (algebraic) simply connected cover is SL_3 or Sp_4 . Thus, by Proposition 6.47 and Lemma 6.50, $H(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$.

Let $\varepsilon > 0$, let $\rho : G \rightarrow O(E)$ with $E \in [\mathcal{H}]$. Let (Q_0, δ_0) be given by property $(T_{Q, [\mathcal{H}]})$ for $H(\mathbb{K})$ with ε . Let Q be a compact subset of $G(\mathbb{K})$ containing a neighborhood of 1 and Q_0 . Let $\delta = \min(\delta_0, \varepsilon)$. Let $\xi \in S(E)$ such that

$$\sup_{g, h \in G(\mathbb{K})} \|\rho(gh)\xi - \rho(g)\rho(h)\xi\| < \delta$$

and

$$\sup_{x \in Q} \|\rho(x)\xi - \xi\| < \delta.$$

Then by property $(T_{Q, [\mathcal{H}]})$,

$$\sup_{x \in H(\mathbb{K})} \|\rho(x)\xi - \xi\| < \varepsilon.$$

As in [Mar91, Ch.III, Thm. 5.3], let T be a maximal \mathbb{K} -split torus in H , there exists a semisimple element $a \in T(\mathbb{K})$ such that $\text{Ad}(a)$ has an eigenvalue different from 1.

By [Mar91, Ch.II, Prop. 3.3.(a)], a defines two subgroups V_a, V_a^- such that ([Mar91, Ch.II, Lem. 3.1.(a)]) for any $g \in V_a$, $a^n g a^{-n} \rightarrow 1$ and for any $g \in V_a^-$, $a^{-n} g a^n$. Since Q contains a neighborhood of 1 and

$$\sup_{n \in \mathbb{Z}} \|\rho(a^n)\xi - \xi\| < \varepsilon,$$

we obtain by Lemma 6.51 that for any $g \in S = V_a \cup V_a^-$,

$$\|\rho(g)\xi - \xi\| < 6\varepsilon.$$

Since G is simply connected almost \mathbb{K} -simple with $\text{rank}_{\mathbb{K}} G > 0$, by [Mar91, Ch.I, Prop. 1.5.4.(iii)] and [Mar91, Ch.I, Thm. 2.3.1.(a)], S generates $G(\mathbb{K})$. Now consider the generating set $\tilde{S} = S \cup \{a^n, n \in \mathbb{Z}\}$, then \tilde{S} is non-compact and

$$\sup_{g \in \tilde{S}} \|\rho(g)\xi - \xi\| < 6\varepsilon.$$

By [dC09, Prop. 1.3 and Thm. 1.4], \tilde{S} boundedly generates $G(\mathbb{K})$, meaning that there exists n such that $G = \tilde{S}^n$. Thus for any $g \in G(\mathbb{K})$,

$$\|\rho(g)\xi - \xi\| < n(6\varepsilon + \delta) \leq 7n\varepsilon$$

thus $G(\mathbb{K})$ has property $(T_{Q, [\mathcal{H}]})$.

If Γ is a lattice in $G(\mathbb{K})$, the result then follows immediately from Proposition 6.31. \square

Appendix A

Computation of $\kappa(G)$

This appendix is devoted to the computation of κ , defined in Section 1.6 for noncompact symmetric pairs and in Section 2.2.3 for compact symmetric pairs. If G is semisimple with finite center, then $G/Z(G) = \prod_{i=1}^n G_i$ with G_i simple. Then,

$$\kappa(G) = \min_{i \text{ s.t. } G_i \text{ not compact}} \kappa(G_i).$$

Indeed, the restricted root system of G is the direct sum of the restricted root system of the G_i 's, and the restricted root system of a compact group is trivial (because it is its own maximal compact subgroup). Furthermore, if (U, K) is the compact dual pair of (G, K) , then $\kappa(G) = \kappa(U, K)$. Thus, it suffices to compute κ for noncompact simple Lie groups.

Table A.1 deals with complex simple Lie groups (viewed as real Lie groups) and their compact real form K - which is also a maximal compact subgroup. The compact dual pair is thus $(K \times K, K)$. In this case, all multiplicities of roots are 2. In Table A.2, M is the compact symmetric space dual to G/K , that is to say U/K where U is a compact real form of G . The computation uses the classification and multiplicities for such symmetric spaces given in [Loo69b, Ch. VII].

Table A.1: Values of $\kappa(G)$ for complex simple Lie groups.

| G | K | $\dim G$ | $\text{rank } G$ | $\kappa(G)$ |
|--------------------------|------------------------|--------------|------------------|---|
| $SL(n, \mathbb{C})$ | $SU(n), n \geq 2$ | $2(n^2 - 1)$ | $n - 1$ | $n - 1$ |
| $SO(2n + 1, \mathbb{C})$ | $SO(2n + 1), n \geq 1$ | $2n(2n + 1)$ | n | $2n - 1$ |
| $Sp(2n, \mathbb{C})$ | $Sp(n), n \geq 1$ | $2n(2n + 1)$ | n | $2n - 1$ |
| $SO(2n, \mathbb{C})$ | $SO(2n), n \geq 2$ | $2n(2n - 1)$ | n | 1 if $n = 2$ 3 if $n = 3$ $2n - 2$ else |
| $(G_2)_{\mathbb{C}}$ | G_2 | 28 | 2 | 5 |
| $(F_4)_{\mathbb{C}}$ | F_4 | 104 | 4 | 15 |
| $(E_6)_{\mathbb{C}}$ | E_6 | 156 | 6 | 16 |
| $(E_7)_{\mathbb{C}}$ | E_7 | 266 | 7 | 27 |
| $(E_8)_{\mathbb{C}}$ | E_8 | 496 | 8 | 57 |

Table A.2: Values of $\kappa(G)$ for real simple Lie groups.

| | M | G | rank G | $\kappa(G)$ |
|-------------|---|----------------------|--------------|--|
| <i>AI</i> | $SU(n)/SO(n), n \geq 2$ | $SL(n, \mathbb{R})$ | $n - 1$ | $\frac{n-1}{2}$ |
| <i>AII</i> | $SU(2n)/Sp(n), n \geq 2$ | $SU^*(2n)$ | $n - 1$ | $2(n - 1)$ |
| <i>AIII</i> | $SU(p+q)/S(U(p) \times U(q)), p+q \geq 3$ | $SU(p, q)$ | $\min(p, q)$ | 2 if $p = q = 2$ $p + q - \frac{3}{2}$ else |
| <i>BDI</i> | $SO(p+q)/SO(p) \times SO(q), p+q \geq 3$ | $SO_0(p, q)$ | $\min(p, q)$ | $\frac{1}{2}$ if $p = q = 2$ $\frac{3}{2}$ if $p = q = 3$ $\frac{p+q}{2} - 1$ else |
| <i>CI</i> | $Sp(n)/U(n), n \geq 1$ | $Sp(2n, \mathbb{R})$ | n | $n - \frac{1}{2}$ |
| <i>CII</i> | $Sp(p+q)/Sp(p) \times Sp(q), p+q \geq 2$ | $Sp(p, q)$ | $\min(p, q)$ | 5 if $p = q = 2$ $2(p+q) - \frac{5}{2}$ else |
| <i>DIII</i> | $SO(4n)/U(2n), n \geq 1$ | $SO^*(4n)$ | n | $n \left(n - \frac{1}{2} \right)$ if $n \leq 3$ $4n - \frac{7}{2}$ if $n > 3$ |
| | $SO(4n+2)/U(2n+1), n \geq 1$ | $SO^*(4n+2)$ | n | $4n - \frac{3}{2}$ |
| | <i>EI</i> | $E_{6(6)}$ | 6 | 8 |
| | <i>EII</i> | $E_{6(2)}$ | 4 | $\frac{21}{2}$ |
| | <i>EIII</i> | $E_{6(-14)}$ | 2 | $\frac{21}{2}$ |
| | <i>EIV</i> | $E_{6(-26)}$ | 2 | 8 |
| | <i>EV</i> | $E_{7(7)}$ | 7 | $\frac{27}{2}$ |
| | <i>EVI</i> | $E_{7(-5)}$ | 4 | $\frac{33}{2}$ |
| | <i>EVII</i> | $E_{7(-24)}$ | 3 | $\frac{27}{2}$ |
| | <i>EVIII</i> | $E_{8(8)}$ | 8 | $\frac{57}{2}$ |
| | <i>EIX</i> | $E_{8(-24)}$ | 4 | $\frac{57}{2}$ |
| | <i>FI</i> | $F_{4(4)}$ | 4 | $\frac{15}{2}$ |
| | <i>FII</i> | $F_{4(-20)}$ | 1 | $\frac{15}{2}$ |
| | <i>G</i> | $G_{2(2)}$ | 2 | $\frac{5}{2}$ |

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Notations

Lie groups and Lie algebras

| | |
|---|---|
| G, \mathfrak{g} | Lie group and its Lie algebra |
| $G_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}$ | Complexification of G, \mathfrak{g} |
| U | Compact real form of G when G is non-compact |
| G_r, U_r | Set of regular points of G, U |
| θ | Cartan involution of \mathfrak{g} |
| G^{σ} | Fixed points of an automorphism σ of G |
| $\mathfrak{k}, \mathfrak{p}$ | Eigenspaces of θ associated to $+1, -1$ respectively |
| K | Maximal compact subgroup of G |
| \mathfrak{a} | Maximal abelian subspace of \mathfrak{p} |
| $\Sigma_{\mathfrak{a}}, \Sigma, \Phi$ | Root systems (depending on the context) |
| \mathfrak{g}_{α} | Root space associated to the root α |
| $m(\alpha)$ | Multiplicity of α (i.e. $\dim \mathfrak{g}_{\alpha}$) |
| W | Weyl group |
| K, A, N | Iwasawa decomposition |
| \mathfrak{a}^+ | Positive Weyl chamber |
| $H : G \rightarrow \mathfrak{a}$ | Iwasawa projection |
| Ad, ad | Adjoint representation of G, \mathfrak{g} |
| $Z_G(M)$ | Centralizer in G of a subgroup M |

| | |
|-------------|--|
| $\Delta(G)$ | Diagonal subgroup $\{(g, g) \in G \times G \mid g \in G\}$ |
|-------------|--|

Vector spaces, Banach spaces

| | |
|----------------------|---|
| $\mathcal{U}(H)$ | Unitary group of the Hilbert space H |
| \mathcal{H}^K | Subspace of K -invariant vectors in \mathcal{H} |
| $C^{(r, \alpha)}$ | Hölder space of maps f such that $D^r f$ is α -Hölder continuous |
| $E \hat{\otimes} F$ | Projective tensor product of E, F |
| $L^p_\sigma(X; E^*)$ | L^p -space of w^* -measurable maps $X \rightarrow E^*$ |
| $L^p(X; E)$ | L^p -space of Bochner measurable maps $X \rightarrow E$ |
| $\ \cdot\ _{cb}$ | Completely bounded norm or cb-norm |
| $\Gamma_2(E, F^*)$ | Subspace of $B(E, F^*)$ of Hilbert-factorable operators |
| $\gamma(T)$ | Factorisation norm of $T \in \Gamma_2(E, F^*)$ |
| $O(E)$ | Group of linear isometries of E |
| $\text{Isom}(E)$ | Group of affine isometries of E |
| $S(E)$ | Unit sphere of E |
| $d_{\ \cdot\ }$ | Modulus of convexity of E |
| $r_{\ \cdot\ }$ | Modulus of smoothness of E |
| $[\mathcal{H}]$ | Class of Banach spaces with compatible Hilbert norm |
| $\bigoplus_p E_i$ | ℓ^p direct sum of the spaces $(E_i)_{i \in \mathbb{N}}$ |

Miscellaneous

| | |
|-----------------|---|
| $r_{opt}(M)$ | Optimal regularity for a pair associated to the symmetric space M |
| $\kappa(G)$ | Real number defined in terms of the root system |
| \mathbb{K} | Local field |
| $G(\mathbb{K})$ | Group of \mathbb{K} -points of the linear algebraic group G |

$SL_n(\mathbb{K})$ | Special linear group ($\subset M_n(\mathbb{K})$)

$Sp_{2n}(\mathbb{K})$ | Symplectic group ($\subset M_{2n}(\mathbb{K})$)

Some rigidity properties of semisimple groups : regularity of matrix coefficients and rigidity of quasi-homomorphisms

Abstract: This manuscript presents the work carried out during my thesis on some properties of semisimple Lie groups. The manuscript is divided into two independent parts.

In the first part, we study the regularity of K -finite coefficients of unitary representations of a Lie group G for various pairs (G, K) . The main question we aim to answer is the following: what is the largest $\alpha \geq 0$ for which any such coefficient is locally α -Hölder continuous? Although there seems to be no reason for α to be anything other than 0, Lafforgue observed in his work on strong property (T) that for the pair $(SO(3), SO(2))$, the answer is at least $\frac{1}{2}$. In the first chapter, we explain how to reduce the question to one of harmonic analysis on the spherical functions of Gelfand pairs. The second chapter deals with pairs (G, K) where G is compact and G/K is a symmetric space of compact type. We give a complete answer when $G = K \times K$ or G/K is of rank 1. We also conjecture what happens in higher rank. In the third chapter, we consider non-compact groups: we study the pairs (G, K) where G is a semisimple Lie group with finite center and K a maximal compact subgroup, as well as (H, K) where H is the Cartan motion group associated to G . The last chapter of this part comes back to the compact case: we give a partial answer to our conjecture using techniques of stationary phase developed in the previous chapter.

The second part is devoted to the study of property (TTT) , which is a strengthening of property (T) introduced by Ozawa. A measurable locally bounded map b from G to a Hilbert space is a wq-cocycle if there exists a measurable map $\rho : G \rightarrow \mathcal{U}(H)$ such that the defect $b(gh) - b(g) - \rho(g)b(h)$ is bounded on $G \times G$. A group G has property (TTT) if every wq-cocycle is bounded. This is a vast generalization of the fixed-point property for affine isometric actions on Hilbert spaces. Property (TTT) for a group G implies strong rigidity statements for quasi-homomorphisms $G \rightarrow G'$. For example, when G' is hyperbolic, the image of such a (continuous) quasi-homomorphism is relatively compact. In the fifth chapter, we prove that lattices inherit this property and that lattices in higher rank simple algebraic groups over local fields have property (TTT) , which was known only in the case of SL_n . In the final chapter, we introduce property (FFF_E) , which is defined as boundedness of wq-cocycles with a Banach space E replacing the Hilbert space as the target space. We also define a slightly weaker property, property $(T_{Q,E})$, which deals with almost invariant vectors. Under some integrability conditions, we also prove that lattices inherit these properties. Finally, we study these properties in the case of super-reflexive Banach space. We consider in particular the class $[\mathcal{H}]$ of Banach spaces with a compatible Hilbert norm, in the case of higher rank simple algebraic groups over a local field.

Keywords: Semisimple groups ; Unitary representations ; Quasi-homomorphisms ; Property (T) ; Fixed point properties.

Quelques propriétés de rigidité des groupes semi-simples : régularité de coefficients matriciels et rigidité de quasi-homomorphismes

Résumé: Ce manuscrit présente les travaux réalisés au cours de ma thèse sur certaines propriétés des groupes de Lie semi-simples. Il est divisé en deux parties indépendantes.

Dans la première partie, nous étudions la régularité des coefficients K -finis des représentations unitaires d'un groupe de Lie G pour diverses paires (G, K) . La question principale que nous cherchons à résoudre est la suivante : quel est le plus grand $\alpha \geq 0$ pour lequel tout coefficient K -fini est localement α -Hölderien ? Bien qu'il n'y ait a priori aucune raison pour que α soit différent de 0, Lafforgue a observé dans ses travaux sur la propriété (T) renforcée, que pour la paire $(SO(3), SO(2))$, la réponse est au moins $\frac{1}{2}$. Dans le premier chapitre, nous expliquons comment ramener la question originale à de l'analyse harmonique sur les fonctions sphériques des paires de Gelfand. Le deuxième chapitre traite des paires (G, K) où G est compact et G/K est un espace symétrique de type compact. Nous donnons une réponse complète lorsque $G = K \times K$ ou lorsque G/K est de rang 1. Nous conjecturons également ce qui se passe en rang supérieur. Dans le troisième chapitre, nous considérons le cas des groupes non compacts : nous étudions les paires (G, K) où G est un groupe de Lie semi-simple de centre fini et K un sous-groupe compact maximal, ainsi que les paires (H, K) où H est le groupe de Cartan associé à G . Dans le dernier chapitre de cette partie, nous donnons une réponse partielle à notre conjecture sur les paires compactes en utilisant des techniques de phase stationnaire développées dans le cas non compact.

La seconde partie est consacrée à l'étude de la propriété (TTT) , un renforcement de la propriété (T) introduit par Ozawa. Une application mesurable localement bornée b de G dans un espace de Hilbert est un wq-cocycle s'il existe une application mesurable $\rho : G \rightarrow \mathcal{U}(H)$ tel que le défaut $b(gh) - b(g) - \rho(g)b(h)$ est borné sur $G \times G$. Un groupe G a la propriété (TTT) si tout wq-cocycle est borné. Il s'agit d'une généralisation forte de la propriété de point fixe pour les actions affines isométriques sur des espaces de Hilbert. La propriété (TTT) pour un groupe G entraîne des résultats de rigidité forts pour les quasi-homomorphismes $G \rightarrow G'$. Par exemple, lorsque G' est hyperbolique, l'image d'un tel quasi-homomorphisme (continu) est relativement compacte. Dans le cinquième chapitre, nous prouvons que les réseaux héritent de cette propriété et que les réseaux dans les groupes algébriques simples de rang supérieur sur un corps local ont la propriété (TTT) , ce qui n'était connu auparavant que pour SL_n . Dans le dernier chapitre, nous introduisons la propriété (FFF_E) , qui est définie comme la bornitude des wq-cocycles lorsque l'espace de Hilbert est remplacé par un espace de Banach E . Nous définissons également une propriété légèrement plus faible, la propriété $(T_{Q,E})$, qui concerne les vecteurs presque invariants. Sous certaines conditions d'intégrabilité, nous montrons que les réseaux héritent aussi de ces propriétés. Enfin, nous étudions ces propriétés dans le cas des espaces de Banach super-réflexifs. Nous nous intéressons en particulier à la classe $[\mathcal{H}]$ des espaces de Banach admettant une norme hilbertienne compatible, dans le cas des groupes algébriques simples de rang supérieur sur un corps local.

Mots clés: Groupes semi-simples ; Représentations unitaires ; Quasi-homomorphismes ; Propriété (T) ; Propriétés de point fixe.

Image en couverture : Projection of E_8 root system onto Coxeter plane. Crédit image : J. Gregory Moxness.

