
Mod p étale cohomology in p -adic geometry

Author : Guillaume PIGNON--YWANNE

Ecole normale supérieure de Lyon

Advisors : Frédéric DEGLISE, Gabriel DOSPINESCU and Arthur-César
LE BRAS

Examiners : Benoît STROH and Jared WEINSTEIN

Jury : Wiesława NIZIOŁ, Sandra ROZENSZTAJN and
Benjamin SCHRAEN

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Abstract

In this thesis, we generalize the recent result of Poincaré Duality for mod p étale cohomology of smooth and proper rigid analytic varieties defined over a complete algebraically closed extension C of $\mathbb{Q}_p$, proven independently by Mann and Gabber-Zavvalov. Specifically, we establish said Poincaré duality for smooth, partially proper rigid analytic varieties over C whose compactly supported étale cohomology groups satisfy a primitive comparison result, in the sense that the natural map $H_{\text{ét},c}^n(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\text{ét},c}^n(X, \mathcal{O}_X^+/p)$ is an almost isomorphism for all integers n .

As an application, we recover Poincaré duality for Zariski-open subsets of smooth proper varieties, established by Li-Reinecke-Zavvalov, using different methods. We moreover prove Poincaré duality for p -adic period domains, leading to the computation of their étale $\mathbb{F}_p$ -cohomology groups, generalizing a known result of Colmez-Dospinescu-Nizioł for Drinfeld's symmetric spaces. Finally, we obtain a classification result for rigid analytic curves satisfying primitive comparison with compact support, and we precisely explain the failure of this primitive comparison for the open unit disk, for which Poincaré duality is known not to hold.

The proof of our main result makes extensive use of Mann's six-functor formalism for quasicohherent, solid, almost $\mathcal{O}^+/p$ -coefficients, together with novel comparison results between Mann's formalism and the classical theory of étale cohomology, in the sense of Huber and Scholze.

In order to compute the compactly supported étale $\mathcal{O}^+/p$ -cohomology of p -adic period domains, we expand on an argument due to Orlik, which relies on an explicit geometric stratification of their complement inside a flag variety. In the specific case of Drinfeld's symmetric spaces, we also present an alternative method, adapting a computation of Schneider-Stuhler.

Keywords : p -adic geometry, mod p étale cohomology, Poincaré duality, primitive comparison, period domains, six-functor formalism.

Résumé

Dans cette thèse, nous généralisons la dualité de Poincaré pour la cohomologie étale modulo p des variétés analytiques rigides propres et lisses, définies sur une extension complète et algébriquement close C de $\mathbb{Q}_p$, obtenue indépendamment par Mann et Gabber-Zavvalov. Plus spécifiquement, nous établissons une dualité de Poincaré pour les variétés analytiques rigides lisses et partiellement propres sur C pour lesquelles les groupes de cohomologie étale à support compact vérifient un résultat de comparaison primitive, au sens où l'application naturelle $H_{\text{ét},c}^n(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\text{ét},c}^n(X, \mathcal{O}_X^+/p)$ est un presque isomorphisme pour tout entier n .

En guise d'application, nous retrouvons la dualité de Poincaré pour les ouverts de Zariski de variétés propres et lisses, établie via des méthodes différentes par Li-Reinecke-Zavvalov. Nous établissons également cette dualité pour les espaces de périodes p -adiques, permettant ainsi de calculer leur cohomologie étale à coefficients dans $\mathbb{F}_p$, généralisant un résultat de Colmez-Dospinescu-Nizioł pour les espaces symétriques de Drinfeld. Enfin, nous obtenons un résultat de classification des courbes rigides analytiques satisfaisant la comparaison primitive à support compact, et nous expliquons précisément pourquoi cette comparaison échoue dans le cas du disque unité ouvert, qui ne vérifie pas la dualité de Poincaré.

La preuve de notre résultat principal repose sur la puissance du formalisme des six-foncteurs à coefficients dans des presque- $\mathcal{O}^+/p$ -modules quasi-cohérents et solides, construit par Mann, que nous enrichissons de résultats de comparaison avec la théorie classique de la cohomologie étale, au sens de Huber et Scholze.

Afin de calculer la cohomologie $\mathcal{O}^+/p$ à support compact des espaces de périodes p -adiques, nous généralisons une approche due à Orlik, qui repose sur une stratification géométrique explicite de leur complémentaire dans une variété de drapeaux. Dans le cas particulier des espaces symétriques de Drinfeld, nous présentons également une autre approche du calcul, adaptée d'une méthode de Schneider-Stuhler.

Mots-clés : géométrie p -adique, cohomologie étale modulo p , dualité de Poincaré, comparaison primitive, espaces de périodes, formalisme des six-foncteurs.

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Introduction

Notations

In this introduction :

- p denotes a fixed prime number.
- $\mathbb{Q}$ denotes the field of rational numbers.
- $\mathbb{C}$ denotes the field of complex numbers.
- K denotes a complete extension of the field of p -adic numbers $\mathbb{Q}_p$, endowed with the extension of the p -adic norm $|\cdot|_p$.
- C denotes a complete algebraically closed field containing K , together with the extended norm, still denoted $|\cdot|_p$. Let $\mathcal{O}_C = \{x \in C, |x|_p \leq 1\}$ be its ring of integers, and $\mathfrak{m}_C = \{x \in C, |x|_p < 1\}$ be the unique maximal ideal of $\mathcal{O}_C$.

1.1 Motivation and the main result

Tate discovered a new category of analytic-algebraic objects with a structure rich enough to make possible the impossible : analytic continuation over totally disconnected ground fields - [BGR].

The first modern attempt at formulating a theory of p -adic geometry was conceived by John Tate in the late 1950's, as part of his work on the uniformization of elliptic curves over p -adic fields. Explicit computations suggested that some class of elliptic curves over C could be realized as a *quotient* of the punctured affine line C^* by a discrete group action. However, such a statement cannot properly be expressed in the language of algebraic geometry, as the quotient map is *transcendental*, i.e., not locally expressible as quotients of polynomial functions.

This statement would be a natural p -adic analogue¹ of the uniformization of complex elliptic curves, which can be written as a quotient of $\mathbb{C}$ by a lattice Λ , with a holomorphic quotient map, given by the Weierstrass function $\wp_\Lambda$ and its derivative. This allows to view complex elliptic curves as *analytic* objects – namely, Riemann surfaces – a theory whose rigidity leads to precise classification results of complex elliptic curves, which is far from being the case over incomplete topological fields such as $\mathbb{Q}$.

This observation led Tate to develop an analogue of complex analytic geometry over p -adic fields, with functions being locally defined as converging power series. One main challenge compared to the complex setting is that, as a topological field, C is totally disconnected, so that no nontrivial open subsets are connected.

Hence, naively adapting the definitions from complex geometry does not yield a reasonable notion of connectedness.² Tate's approach to p -adic analytic geometry is rather inspired by its algebraic counterpart,

¹By passing to the exponential map, one can see complex elliptic curves as quotients of $\mathbb{C}^*$. One needs to work with $\mathbb{C}_p^*$ at the p -adic level, as the p -adic exponential function is not globally defined.

²There is a notion of K -analytic *manifold*, defined by local charts given by analytic functions (defined for example in [Ser92]), but they admit no reasonable (not totally disconnected) topology.

namely, the theory of schemes that was being developed concurrently by Grothendieck.³ He defined a notion of "rigid-analytic spaces" as glued together from the spectra (in the sense of maximal ideals) of *affinoid algebras*, defined as quotients of the Tate algebras :

$$T_n(C) = C \langle X_1, \dots, X_n \rangle = \left\{ \sum_{k_1, \dots, k_n \in \mathbb{N}} a_{k_1, \dots, k_n} X_1^{k_1} X_2^{k_2} \dots X_n^{k_n} \mid a_{k_1, \dots, k_n} \in C, \lim_{k_1, \dots, k_n \rightarrow \infty} |a_{k_1, \dots, k_n}|_p = 0 \right\}.$$

For example, $\text{Spm}(T_n(C))$ defines the closed unit (poly)disk of radius 1 of dimension n over C (together with an adequate topology and structure sheaf). $T_n(C)$, in turn, should be understood as the ring of power series that converge on this disk.⁴

The equivalent of the Zariski topology makes this disk disconnected, contradicting the geometric intuition. Hence, Tate introduced a more *rigid* topology, called the admissible topology, obtained by restricting the allowed open coverings, leading to the notion of *rigid analytic varieties*. Tate's theory of rigid analytic theory resembles in many aspects the theory of schemes but encapsulates fundamentally *different*⁵ spaces, of *analytic* nature, such as the closed unit disk described above.⁶

One of the key features of this theory is étale cohomology. Introduced in [SGA4] in the setup of schemes, étale cohomology of constant sheaves yields a purely algebraic way of encoding precise information about the *shape* of a scheme, by understanding its étale covers - an algebraic analogue to covering maps. In particular, the étale cohomology of a nice⁷ scheme over $\mathbb{C}$ recovers the *Betti* (singular, de Rham, ...) cohomology of its complex analytification, viewed as a complex manifold. The idea that one could associate topological information to a scheme defined over an arbitrary base field turned out to be very successful, as étale cohomology over *finite fields* became a key ingredient in the proof of the Weil conjectures.

From there, one may try and construct a version of étale cohomology for rigid-analytic varieties, which, as in the complex case, should carry some information about the geometry of the variety – starting from its number of connected components – which is already a surprising feat over a totally disconnected base field. This program was laid out in [Ber93], [JJP96] and [Hub96].

Consider a fixed ring of coefficients $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$, for a prime ℓ . Etale cohomology with coefficients in Λ is well-behaved whenever $\ell \neq p$, in the sense that it admits a *six-functor formalism*, and, hence, behaves similarly to the scheme-theoretic setup. In particular, it satisfies Poincaré duality, in the following sense.

Theorem 1.1.1. [Hub96, Cor 7.5.6]

Let X be a (separated, taut) rigid analytic variety over C that is smooth of pure dimension d . Let $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$, for a prime $\ell \neq p$. Then, for all $k \in \mathbb{N}$, there exists a natural isomorphism⁸ :

$$H_{\text{ét}}^k(X, \Lambda(d)) \cong \text{Hom}_{\Lambda} \left(H_{\text{ét}, c}^{2d-k}(X, \Lambda), \Lambda \right).$$

³After some initial reluctance, made famous in a letter to Serre, Grothendieck contributed directly to the foundations of rigid-analytic geometry.

⁴Indeed, elements $x_1, \dots, x_n \in C$ define a natural evaluation mapping $ev_{x_1, \dots, x_n} : T_n(C) \rightarrow C$ if and only if $|x_i|_p \leq 1$ for all i . This follows from the non-archimedean property of the norm $|\cdot|_p$.

⁵One may embed schemes into rigid analytic varieties, since polynomials are power series. The technical statement is the existence of an analytification functor from schemes of locally finite type over C to rigid-analytic varieties, respecting many geometric properties (smoothness, being an immersion for finite type morphisms ...), and satisfying the analogue of Serre's GAGA theorem in the complex analytic setup - see [Con99].

⁶In particular, Tate was able to make sense of the quotient $C^*/<q^{\mathbb{Z}}>$ as an analytic torus, constructed by gluing together annuli of different radii, and classify elliptic curves with j -invariant $|j| > 1$ as such analytic tori, completing his initial motivation.

⁷i.e. smooth and locally of finite type.

⁸The more classical formulation is the existence of a perfect pairing between cohomology and compactly supported cohomology group. This is equivalent to the presentation above when cohomology groups are finitely dimensional, which is not the case in the above setup (This already fails when X is an infinite disjoint union of points).

In the above, $H_{\acute{e}t}^\bullet$ denotes étale cohomology groups, $H_{\acute{e}t,c}^\bullet$ denote compactly supported étale cohomology groups, and (d) denotes a Tate twist.

The statement of Poincaré duality has a long history, starting from the foundations of algebraic topology in the *Analysis Situs*, in what we would call today simplicial homology, and has since become a key feature of many cohomology theories. In particular, a similar statement holds for singular cohomology and for étale cohomology of schemes in characteristic zero, with no restriction on the characteristic of the torsion coefficients.⁹

One may wonder whether the above statement still holds when $\ell = p$. As is shown by the next example, Poincaré duality does not hold in general.

Example 1.1.2. Let $\mathbb{D}_C = \text{Spm}(T_1(C))$ be the closed unit disk of radius 1, which is smooth of pure dimension $d = 1$. An explicit computation yields :

$$H_{\acute{e}t,c}^i(\mathbb{D}_C, \mathbb{F}_p) \cong \begin{cases} \{0\} & \text{if } i = 0 \\ \{0\} & \text{if } i = 1 \\ M & \text{if } i = 2 \end{cases} \quad \text{and} \quad H_{\acute{e}t}^i(\mathbb{D}_C, \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } i = 0 \\ N & \text{if } i = 1 \\ \{0\} & \text{if } i = 2, \end{cases}$$

where M and N are infinite dimensional $\mathbb{F}_p$ -vector spaces.¹⁰

However, recent results by L.Mann [Man22] and B.Zavvalov [Zav24b] establish Poincaré duality when $\ell = p$ for smooth and *proper*¹¹ rigid-analytic varieties. Moreover, [LRZ24] extends this result to so-called Zariski-compactifiable¹² varieties, which are Zariski open subsets of proper varieties. Furthermore, explicit computations of [CDHN21] and [CDN21] suggest that Poincaré duality should also hold for Drinfeld's symmetric spaces, which are *not* Zariski-compactifiable.

Hence, there should be some *intrinsic* reason to why Zariski-compactifiable spaces and Drinfeld's symmetric spaces satisfy Poincaré duality, while other very natural spaces, such as the closed (or open¹³) unit disk, do not.

In this thesis, we partially answer this question by introducing a *criterion*, that can be computed on compactly supported étale cohomology groups¹⁴, and that implies Poincaré duality.

The main result of this thesis is (a more general version of)¹⁵ the following :

Theorem 1.1.3. (Simplified version of [theorem 1.4.1](#))

Let X be a partially proper¹⁶ rigid analytic variety over C that is smooth of pure dimension d . Assume furthermore that X satisfies **primitive comparison with compact support**, i.e., for all $i \in \mathbb{N}$, the natural morphism :

$$H_{\acute{e}t,c}^i(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\acute{e}t,c}^i(X, \mathcal{O}_X^+/p) \quad (1.1)$$

⁹A similar " $\ell \neq p$ " restriction holds for schemes over a field of characteristic p .

¹⁰Moreover, the dimension of M and N depends on the cardinal of C . This also shows that other expected properties of cohomology, such as smooth base change, fail when $\ell = p$.

¹¹This is the analogue of *compactness* in algebraic or analytic geometry. For proper spaces, compactly supported cohomology coincides with cohomology.

¹²These varieties are sometimes known as "almost proper". This terminology is somewhat unfortunate, as it is unrelated to the language of almost mathematics, that will appear shortly.

¹³The open unit disk is defined as the union of *closed* unit disks of radii < 1 , cf. [example 2.1.18](#).

¹⁴For partially proper spaces, compactly supported étale cohomology groups are defined as derived functors of defined, for partially proper spaces, as the derived functors of compactly supported sections, i.e. sections that are zero outside of a proper subspace.

¹⁵The general version establishes a Galois-equivariant Poincaré duality for varieties which admit a model over K , and allows coefficients in overconvergent étale sheaves (provided that primitive comparison with compact support holds for said coefficients).

¹⁶A rigid analytic variety is partially proper if it is universally specializing, separated, and locally of weakly finite type. This condition includes many non proper examples such as the open unit disk, the analytic affine line, and Drinfeld's symmetric spaces. Partially proper should be thought of as spaces *without boundary*, and quasicompact partially proper spaces are proper.

is an almost¹⁷ isomorphism of $\mathcal{O}_C/p$ -modules.

Then X satisfies Poincaré duality, in the sense that, for all $0 \leq k \leq 2d$, there exists an isomorphism :

$$H_{\text{ét}}^k(X, \mathbb{F}_p(d)) \cong \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(X, \mathbb{F}_p), \mathbb{F}_p \right), \quad (1.2)$$

where (d) denotes a Tate twist.

Here, the étale sheaf $\mathcal{O}_X^+/p$ computes the mod p reduction of the sheaf of analytic functions on X of norm bounded by 1. Its definition and relevance will be explained in the next paragraph.

Furthermore, this thesis aims to show that the criterion above :

- Can be checked in practice ;
- Recovers Poincaré duality for proper and Zariski-compactifiable varieties ;
- Confirms Poincaré duality for Drinfeld's symmetric spaces ;
- Extends Poincaré duality to p -adic period domains (in which setting it was unknown), allowing to compute new $\mathbb{F}_p$ -cohomology groups.¹⁸

In the following two paragraphs, we motivate and introduce the necessarily tools to sketch the proof of our main theorem. Then, in [section 1.4](#), we present the other results of this thesis, which mainly consist in computation of the étale cohomology of the sheaf $\mathcal{O}^+/p$, and consequences of the above theorem.

In order to understand the relevance of [equation \(1.1\)](#), and especially the role played by the sheaf $\mathcal{O}^+/p$, we first need to talk about the *primitive comparison* theorem [[Sch12a](#), Thm 5.1].

1.2 Primitive comparison and perfectoid spaces

At this point, it is long overdue to mention that rigid analytic varieties are not the only incarnation of p -adic analytic spaces. Let us mention a few of the important concepts.

- *Formal* geometry, whose relationship to rigid geometry was studied by M.Raynaud, studies spaces at the integral level, i.e. defined over $\mathcal{O}_K$. They admit a *generic* fiber, which is a rigid-analytic space over K , as well as a *special* fiber, which is a scheme over $\mathcal{O}_K/p$.
- *Berkovich* geometry, introduced by V.Berkovich, solves the problem of connectedness by adding *more points* to rigid-analytic varieties, yielding a more Euclidean topology, which is Hausdorff, locally compact, and locally path connected.
- *Adic* geometry, introduced by R.Huber, adds even more points, leading to a topology resembling the one of schemes.¹⁹ It allows to work on more general bases, and allows to consider formal schemes, their special and their generic fibers in a unified setup.

These theories are far from being independent. There exists a fully faithful functor from (nice) Berkovich spaces to rigid analytic varieties, and from (nice) rigid analytic varieties to adic spaces. Moreover, there exist fully developed theories of étale cohomology in the Berkovich and adic setup, developed respectively in [[Ber93](#)] and [[Hub96](#)], compatibly to these functors.

¹⁷This is to be taken in the sense of Faltings. A morphism of $\mathcal{O}_C/p$ -modules is an almost isomorphism if its kernel and cokernel are killed by all p^ε . In this introduction, we mostly treat almost isomorphisms as if they were isomorphisms, for the sake of simplicity.

¹⁸In that setup, étale cohomology groups are not finite dimensional. Our more general version of [theorem 1.1.3](#) allows to establish an equivariant version of Poincaré duality, with respect to the action of a Galois group as well as a p -adic Lie group ; cf. [theorem 1.4.1](#).

¹⁹All adic spaces are locally *spectral*, in the sense that they are locally homeomorphic to an affine scheme (in a non-constructive manner).

As adic spaces will be the default language²⁰ of the rest of this thesis, let us give some technical baseline of the formalism. In some sense, the passage from rigid to adic geometry is similar to the passage from the classical theory of algebraic varieties to the theory of schemes.

Adic spaces are locally ringed spaces that are locally *affinoid*, i.e. locally of the form $\mathrm{Spa}(A, A^+)$, for some complete topological²¹ ring A , and an open subring A^+ of *integral elements*. Moreover, Spa denotes the *valuative spectrum*²², defined by equivalence classes of valuations $|\cdot|_v$ on A , satisfying $|\cdot|_v \leq 1$ on A^+ , endowed with an adequate topology.

Example 1.2.1. *The closed adic unit disk over C is $\mathrm{Spa}(C\langle T \rangle, \mathcal{O}_C\langle T \rangle)$.*

More generally, there exists a fully faithful functor $\mathrm{Spm}(C) \mapsto \mathrm{Spa}(C, C^\circ)$ from rigid analytic varieties to adic spaces, where $C^\circ = \{x \in C, \{x^n\}_{n \in \mathbb{N}} \text{ is bounded}\}$.

Adic spaces admit two structure sheaves, namely $\mathcal{O} : \mathrm{Spa}(R, R^+) \mapsto R$, and the integral structure sheaf $\mathcal{O}^+ : \mathrm{Spa}(R, R^+) \mapsto R^+$. In the following, we consider the sheaf $\mathcal{O}^+/p$, defined as the mod p reduction of the étale sheaf $\mathcal{O}^+$; or, equivalently, the sheafification²³ of the étale presheaf $\mathrm{Spa}(R, R^+) \mapsto R^+/p$.

Hence, the étale cohomology of $\mathcal{O}^+/p$, encodes some mod p *coherent* information, regarding bounded analytic functions on étale coverings of X . This compares with étale $\mathbb{F}_p$ -cohomology, which encodes information about the *shape* of the variety, by the primitive comparison theorem [Sch12a, Thm 5.1] :

Theorem 1.2.2. *Let X be a smooth²⁴ proper rigid analytic variety over C . Then, for any $k \geq 0$, the natural morphism :*

$$H_{\text{ét}}^k(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét}}^k(X, \mathcal{O}^+/p)$$

is an almost isomorphism. Moreover, all $H_{\text{ét}}^k(X, \mathbb{F}_p)$ are finite dimensional.

The primitive comparison theorem lies at the heart of the proofs of Poincaré duality for proper spaces as given by L.Mann and B.Zavyalov, who establish some form of Poincaré duality for $\mathcal{O}^+/p$ -coefficients, before descending to $\mathbb{F}_p$ coefficients by the primitive comparison theorem.²⁵ Note that the properness assumption is necessary, as primitive comparison (for classical and compactly supported étale cohomology) fails even for the closed unit disk, as well as the open unit disk.

Hence, our main result **theorem 1.1.3** shows that Poincaré duality holds for (smooth, partially proper) adic spaces that satisfy the equivalent of primitive comparison for compactly supported cohomology. In order to apply it, one needs a practical way to compute the cohomology of the sheaf $\mathcal{O}^+/p$.

The sheaf $\mathcal{O}^+/p$ resembles the structure sheaf of schemes over fields of characteristic p , for which the equivalent of the primitive comparison theorem is the less-celebrated [Sta23, Lemma 0A3P], stemming from Artin-Schreier theory. In this setup, étale cohomology of the structure sheaf is easy to compute, since :

²⁰This is, in great part, to encapsulate perfectoid spaces, which will be introduced below.

²¹There are some additional conditions, cf. **definition 2.1.7**.

²²There is a natural comparison with the usual spectrum of prime ideals $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spec}(A)$, taking the kernel of norms

²³This sheafification is needed, even on the closed disk introduced in **example 1.1.2**, as $H_{\text{ét}}^0(\mathbb{D}_C, \mathcal{O}^+/p) \not\cong \mathcal{O}_C\langle T \rangle/p$, cf. **section 2.6**.
Passing to almost coefficients does not solve this issue.

²⁴The smoothness assumption has been removed in [Sch13, Sec 3] using resolution of singularities. In this thesis, we work with smoothness assumptions throughout.

²⁵From the Artin-Schreier exact sequence of étale sheaves on $X : 0 \rightarrow (\mathbb{F}_p)_X \rightarrow \mathcal{O}_X^+/p \xrightarrow{x \mapsto x^p - x} \mathcal{O}_X^+/p \rightarrow 0$, so that $\mathbb{F}_p$ -cohomology should be able to be recovered from $\mathcal{O}^+/p$ -cohomology by taking Frobenius invariants. One must be very careful with this kind of statements, as the almost isomorphism in the primitive comparison theorem need not be compatible with the Frobenius morphism. One usually solves this issue by passing to the tilted structure sheaf $\mathcal{O}_X^b$, but this becomes trickier outside of the proper case.

- Étale cohomology of coherent sheaves coincides with its sheaf cohomology on the underlying topological space, with the usual Zariski topology.
- For any affine scheme $Z = \text{Spec}(R)$:

$$H_{\text{Zar}}^i(Z, \mathcal{O}_Z) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \quad (1.3)$$

In other words, the sheaf $\mathcal{O}$ is *étale-locally* acyclic and can hence be computed by an explicit Čech complex; taking Frobenius invariants yields $\mathbb{F}_p$ -cohomology groups.

In p -adic geometry, the situation is a bit more complicated. While, by Tate's acyclicity theorem, the analogue of [equation \(1.3\)](#) holds for the étale (or analytic) cohomology of $\mathcal{O}$ on affinoid spaces, it does not hold for $\mathcal{O}^+$, nor $\mathcal{O}^+/p$.

The key input was introduced by Scholze, following ideas of Faltings : The sheaf $\mathcal{O}^+/p$ is *pro-étale-locally* (almost) acyclic.

Let us start with an example of a pro-étale cover. Consider the p -adic torus²⁶, given by the rigid analytic variety $\mathbb{T}_C = \text{Spm}(C \langle T, T^{-1} \rangle)$. It admits a natural tower of étale coverings, each of degree p :

$$\cdots \rightarrow \text{Spm}(C \langle T^{\pm 1/p^3} \rangle) \rightarrow \text{Spm}(C \langle T^{\pm 1/p^2} \rangle) \rightarrow \text{Spm}(C \langle T^{\pm 1/p} \rangle) \rightarrow \mathbb{T}_C.$$

While the projective limit of this tower does not exist as a rigid analytic variety, it makes sense as an adic space, as the so-called perfectoid torus :

$$\tilde{\mathbb{T}}_C = \text{Spa}(C \langle T^{\pm 1/p^\infty} \rangle, \mathcal{O}_C \langle T^{\pm 1/p^\infty} \rangle).$$

This is an example of a *perfectoid space*, which are highly ramified and non-noetherian adic spaces (see [definition 2.2.1](#) for a precise definition), and the morphism $\tilde{\mathbb{T}}_C \rightarrow \mathbb{T}_C$ is pro-étale, as a projective limit of étale morphisms. The $\mathcal{O}^+/p$ -cohomology of perfectoid spaces is easy to compute, at least up to almost isomorphism, by the following almost acyclicity result [[Sch17](#), Prop 8.5.iii] :

Proposition 1.2.3. *Let $Z = \text{Spa}(R, R^+)$ be an affinoid perfectoid space. Then :*

$$H_{\text{et}}^i(Z, \mathcal{O}^+/p) \stackrel{a}{\cong} \begin{cases} R^+/p & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$$

where $\stackrel{a}{\cong}$ denotes an almost isomorphism.

More generally, any so-called adic space X over a field²⁷ admits a surjective *pro-étale* morphism from a perfectoid space $f : \tilde{X} \rightarrow X$, which can be written as a projective limit of étale morphisms, and can be thought of as a pro- p covering of X ²⁸. While f is not étale, it can be shown that the cohomology of $\mathcal{O}^+/p$

²⁶The same image does not work for the perfectoid unit ball, as the natural morphism $\text{Spm}(C \langle T^{1/p} \rangle) \rightarrow \text{Spm}(C \langle T \rangle)$ is not étale at 0. One can view the torus as a *punctured* unit disk, removing the problematic point. A perfectoid cover of the unit disk will be discussed in [section 2.6](#).

²⁷The correct notion is that the adic space needs to be analytic.

²⁸Recall that any nice topological space Y admits a universal covering $\tilde{Y} \rightarrow Y$ from a simply connected (hence, cohomologically simpler) space $\tilde{Y}$. The analogy only goes so far, as, here, there is no canonical choice of a perfectoid covering of X . Note that, in [[Heu21](#), Def 4.6], Heuer defines a canonical *pro-finite étale* covering, but the covering space is not necessarily perfectoid.

might as well be computed with pro-étale covers [Sch17, Prop 14.7]. In particular, from the covering $\{\tilde{X} \rightarrow X\}$, one gets the Čech-to-cohomology spectral sequence :

$$E_2^{s,t} = H_{\acute{e}t}^s \left(\underbrace{\tilde{X} \times_X \cdots \times_X \tilde{X}}_{t \text{ times}}, \mathcal{O}^+/p \right) \implies H_{\acute{e}t}^{s+t}(X, \mathcal{O}^+/p), \quad (1.4)$$

where all of the spaces $\tilde{X} \times_X \cdots \times_X \tilde{X}$ are perfectoid, so that their cohomology can be computed from an affinoid cover and [proposition 1.2.3](#).

The purpose of this is twofold. Firstly, it gives us an explicit blueprint for computing $\mathcal{O}^+/p$ -cohomology groups. Secondly, from a more conceptual point of view, it gives us a new way of thinking about $\mathcal{O}^+/p$ -cohomology and associated concepts : start from perfectoid spaces, and *descend* to rigid analytic varieties. This leads to the concept of *diamonds*, which are p -adic spaces defined as *quotients* of perfectoid spaces, via pro-étale equivalence relations, introduced in [Sch17].²⁹

A striking application of this philosophy is due to Lucas Mann in his proof of Poincaré duality in [Man22]. He established a fully well-behaved theory of solid, almost, quasicoherent $\mathcal{O}^+/p$ -cohomology, satisfying a six-functor formalism, at the cost of additional layers of abstraction. As the use of this theory is crucial to the proof of our main result [theorem 1.1.3](#), let us present it in more detail.

1.3 Mann's six-functor formalism and a sketch of proof

A six-functor formalism is the data of a closed symmetric monoidal derived category of *sheaves* $\mathcal{D}(X)$ attached to any geometric object X , in a *highly functorial manner*, the compatibility between those functors encoding many properties of cohomology, such as proper base change, a Künneth formula, Poincaré duality³⁰ ...

In particular, any morphism $f : X \rightarrow Y$ induces two covariant morphisms, a pushforward f_* and a proper pushforward $f_!$:

$$f_*, f_! : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$$

which encode *relative* versions of cohomology. In particular, they should compute cohomology groups (resp. compactly supported cohomology groups) when Y is a geometric point.

Following ideas of [Sch19], one could not expect a geometrically sound³¹ coherent six-functor formalism in p -adic geometry without restricting to *complete* objects in some topological sense.³² Hence, Mann uses the formalism of condensed mathematics due to Clausen-Scholze, where *solid* objects correspond to the *complete* objects.

Combining this idea with the ones of the previous paragraph, Mann defines a six-functor formalism with *solid, almost* $\mathcal{O}^+/p$ -coefficients in a way that is *tailored* to make functorial and descent properties work. Rather than explicitly defining a category $\mathcal{D}(X)$ for all X , Mann defines it for (some) perfectoid spaces as follows, and extends the definition for arbitrary spaces using descent.

²⁹They are, in some sense, the analytic equivalent of the notion of algebraic spaces.

³⁰The ℓ -adic version Poincaré duality from [theorem 1.1.1](#) (as well as most known versions of Poincaré duality) stems from a six-functor formalism.

³¹In algebraic geometry, it is possible to make a six-functor formalism encoding coherent cohomology in some degenerate way that does not match the geometry ; for example, by declaring every quasicompact morphism to be cohomologically proper.

³²The main issue is that the proper pushforward along open immersions should commute with products ; which fails for the étale sheaf $\mathcal{O}^+/p$, as the localisation of rings does not commute with products.

To a (strictly totally disconnected) affinoid perfectoid space $\mathrm{Spa}(A, A^+)$ over K , Mann defines :

$$\mathcal{D}_{\square}^a(\mathrm{Spa}(A, A^+), \mathcal{O}^+/p) := \mathcal{D}_{\square}^a(A^+/p)$$

as the localization with respect to almost isomorphisms of the (infinity) derived category of *solid* A^+/p -modules. This first glues to perfectoid spaces, and extends to more general spaces by *descent*. In particular, for any pro-étale surjection $\tilde{X} \rightarrow X$ onto an adic space, where $\tilde{X}$ is perfectoid, we *define* (compare with [equation \(1.4\)](#)) :

$$\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p) = \varprojlim_{[n] \in \Delta} \mathcal{D}_{\square}^a\left(\underbrace{\tilde{X} \times_X \cdots \times_X \tilde{X}}_{n \text{ times}}, \mathcal{O}^+/p\right),$$

where the limit is taken along the simplex category. Here, the right hand side consists of (the infinity categorical version of) a *descent datum* : Letting $\tilde{X}_n$ be the n -th fiber product of $\tilde{X}$ over X (which is perfectoid), an element of the above limit is the data of $M_1 \in \mathcal{D}_{\square}^a(\tilde{X}_0)$, together with an isomorphism between the two pullbacks to $\mathcal{D}_{\square}^a(\tilde{X}_2)$, a homotopy in $\mathcal{D}_{\square}^a(\tilde{X}_3)$ realizing the cocycle condition, etc.

The main properties are recapitulated in the following [\[Man22, Thm 1.2.1\]](#).

Theorem 1.3.1. *There exists a unique sheaf of ∞ -categories :*

$$X \in \{\text{Diamonds over } C\} \mapsto \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p),$$

extending the above definition for strictly totally disconnected perfectoid spaces.

Each $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$ is a closed symmetric monoidal category, with unit $\mathcal{O}_X^{+,a}/p$. This construction admits a six-functor formalism ; the pushforward and proper pushforward are denoted respectively f_*^M and $f_!^M$.

Hence, we have at our disposal an abstract, difficult to compute, yet well-behaved notion of almost solid $\mathcal{O}^+/p$ -cohomology, which is fully explicit for affinoid perfectoid spaces – although it is not clear a priori whether f_*^M and $f_!^M$ recovers classical $\mathcal{O}^+/p$ -cohomology (resp. compactly generated cohomology) groups. In order to extract results in classical $\mathcal{O}^+/p$ cohomology (and, ultimately, in $\mathbb{F}_p$ -cohomology), one may try and establish *compatibility results* between the classical étale formalism, and Mann’s formalism.

Let us state two important such results appearing in Mann’s thesis.

Proposition 1.3.2. (cf. [\[Man22, Sec. 3.3 and 3.9\]](#)).

For any diamond X over K , the category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$ relates to usual sheaves in two important ways :

- There exists a "Riemann-Hilbert" functor :

$$(-) \otimes \mathcal{O}_X^{+,a}/p : \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p) \quad (1.5)$$

from the derived category of overconvergent³³ étale $\mathbb{F}_p$ -sheaves on X .

- The full subcategory $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_{\omega}$ of discrete objects (i.e. the ones admitting a trivial solid structure) identifies naturally with $(\mathcal{D}_{\text{ét}}^{\wedge}(X, \mathcal{O}_X^+/p)^{\text{oc}})^a$, the localization with respect to almost isomorphisms of the (left completed) derived category of overconvergent étale sheaves of $\mathcal{O}_X^+/p$ -modules on X . In the following, we freely identify the two.

In this thesis, we obtain the two following additional compatibility results, which broadly claim that, under extra hypotheses, the cohomology groups (resp. compactly supported cohomology groups), as computed

³³An étale sheaf is said to be overconvergent if its stalks on affinoid fields $\mathrm{Spa}(k, k^+)$ do not depend on the choice of k^+ .

in Mann's category, recover the classical cohomological groups (resp. compactly supported cohomology groups), up to an almost isomorphism.

Proposition 1.3.3. (cf. [corollary 3.5.6](#) and [proposition 4.3.1](#)).

Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be an adic space.³⁴

We let $R\Gamma_{\acute{e}t}(X, \mathcal{O}^+/p)$ (resp. $R\Gamma_{\acute{e}t,c}(X, \mathcal{O}^+/p)$) denote the complex computing the étale cohomology groups (resp. compactly supported étale cohomology groups) of the sheaf $\mathcal{O}^+/p$, as constructed by Huber in [\[Hub96\]](#).

- There is an almost isomorphism :

$$\left(f_*^M \mathcal{O}_X^+/p\right)_\omega \xrightarrow{\sim} R\Gamma_{\acute{e}t}(X, \mathcal{O}^+/p),$$

where $(-)_\omega$ is the functor forgetting the solid topology.

In particular, if X is quasi-compact, the $(-)_\omega$ functor is unnecessary, and the above recovers [\[Man22, Example 3.3.17\]](#).

- If X is furthermore partially proper, there is an almost isomorphism :

$$f_!^M \mathcal{O}_X^{+,a}/p \xrightarrow{\sim} R\Gamma_{\acute{e}t,c}(X, \mathcal{O}_X^+/p).$$

While the first point follows essentially directly from Mann's formalism, the second one is more subtle, and lies at the technical heart of this thesis. It relies on establishing localization triangles of the form :

$$R\Gamma_c(X, \mathcal{O}_X^+/p) \rightarrow R\Gamma(X, \mathcal{O}_X^+/p) \rightarrow \varinjlim_{U \in \Phi_X} R\Gamma(X \setminus U, \mathcal{O}_{X \setminus U}^+/p),$$

in both Mann's six-functor formalism and an appropriate almost quasi-coherent version of standard étale cohomology. Here, the colimit is taken along quasicompact open subsets $U \subset X$, and $X \setminus U$ admits the structure of a locally spatial diamond by [lemma A.1.3](#). We may then compare the middle and right-most terms using the first point of [proposition 1.3.3](#), which concludes.

From there, the hypothesis of primitive comparison with compact support of [equation \(1.1\)](#) amounts to comparing the $\mathbb{F}_p$ -étale cohomology with objects computed in Mann's category.

Proposition 1.3.4. (cf. [proposition 5.2.2](#))

Under the hypothesis of the above proposition, X satisfies primitive comparison with compact support with coefficients in $\mathcal{L}$ if and only if the natural map :

$$R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C^{+,a}/p \rightarrow f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \quad (1.6)$$

is a isomorphism in $\mathcal{D}_{\square}^a(\mathcal{O}_C/p)_\omega$, where $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ is the structure map. Here, both tensor products are given by the Riemann-Hilbert functor.

The following computation follows directly from properties of the six-functor formalism and properties of the Riemann-Hilbert functor³⁵ :

Proposition 1.3.5. (cf. [theorem 3.3.1](#)).

Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a smooth (locally noetherian, analytic) adic space that is pure of dimension d .

³⁴This works more generally for locally spatial diamonds. In this case, we replace the classical notion of étale cohomology with the one of [\[Sch17\]](#).

³⁵We use the fact that the Riemann-Hilbert functor is fully faithful onto the category $\mathcal{D}_{\square}^a(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}^+/p)^\varphi$ whose objects are almost, solid, derived $\mathcal{O}_C/p$ -modules equipped with a Frobenius, and whose morphisms are Frobenius-equivariant maps.

Let $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\text{Spa}(C, \mathcal{O}_C), \mathbb{F}_p)$ such that there exists a Frobenius-equivariant almost isomorphism :

$$f_!^M (\mathcal{O}_X^{+,a}/p) \xrightarrow{\sim} \mathcal{F} \otimes \mathcal{O}_C^{+,a}/p.$$

There is an isomorphism in $\mathcal{D}_{\text{ét}}(\text{Spa}(C, \mathcal{O}_C), \mathbb{F}_p)$:

$$R\Gamma_{\text{ét}}(X, \mathbb{F}_p)(d)[2d] \simeq \underline{\text{RHom}}_{\mathbb{F}_p}(\mathcal{F}, \mathbb{F}_p),$$

where (d) denotes a Tate twist, $[2d]$ denotes the shift of complexes, and $\underline{\text{RHom}}_{\mathbb{F}_p}$ denotes the internal Hom in the derived category of étale sheaves of $\mathbb{F}_p$ -vector spaces on X .

Combining the results above leads to a proof of our main result, as stated in [theorem 1.1.3](#). Indeed, assuming that a partially proper X satisfies primitive comparison with compact support, by [proposition 1.3.4](#), there is an isomorphism :

$$R\Gamma_{\text{ét},c}(X, \mathbb{F}_p) \otimes \mathcal{O}_C^{+,a}/p \xrightarrow{\sim} f_!^M \mathcal{O}_X^{+,a}/p$$

which is compatible with the Frobenius. We may hence apply [proposition 1.3.5](#) with $\mathcal{F} = R\Gamma_{\text{ét},c}(X, \mathbb{F}_p)$, yielding :

$$\underline{\text{Hom}}_{\mathbb{F}_p}(R\Gamma_{\text{ét},c}(X, \mathbb{F}_p), \mathbb{F}_p) \cong R\Gamma_{\text{ét},c}(X, \mathbb{F}_p)(d)[2d].$$

Passing to cohomology groups yields Poincaré duality as in [equation \(1.2\)](#).

1.4 Examples and non-examples of Poincaré duality

Before mentioning the other important results on this thesis, let us state our main result in full generality, and compare it with the version of [theorem 1.1.3](#).

Theorem 1.4.1. (cf. [theorem 5.2.4](#))

Let $f : X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a partially proper rigid-analytic variety over K that is smooth of pure dimension d . Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -modules on X . Let X_C denote the base change of X to C , and $\mathcal{L}_C$ denote the base change of $\mathcal{L}$ to X_C .

Assume that X_C satisfies primitive comparison with compact support with respect to $\mathcal{L}_C$, in the sense that, for all $i \in \mathbb{N}$, the natural morphism :

$$H_{\text{ét},c}^i(X_C, \mathcal{L}_C) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét},c}^i(X_C, \mathcal{L}_C \otimes \mathcal{O}_X^+/p)$$

is an almost isomorphism of $\mathcal{O}_C/p$ -modules.

Then, for any $0 \leq k \leq 2d$, there exists a $\text{Gal}(\overline{K}/K)$ -equivariant natural isomorphism :

$$H_{\text{ét}}^k(X_C, \mathcal{L}_C^\vee(d)) \cong \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(X_C, \mathcal{L}_C), \mathbb{F}_p \right).$$

Where $\mathcal{L}_C^\vee = \underline{\text{RHom}}_{\mathbb{F}_p}(\mathcal{L}_C, \mathbb{F}_p)$ is the dual sheaf of $\mathcal{L}_C$, and (d) denotes a Tate twist.

Moreover, if X_C admits a group action by any group G for which $\mathcal{L}_C$ is equivariant, then the Poincaré-Duality isomorphism is G -equivariant.

Compared to the version of [theorem 1.1.3](#) :

- This version of Poincaré duality is equivariant with respect to various group actions on X_C , starting from the Galois action.
- This version of Poincaré duality works for arbitrary overconvergent étale sheaves, provided that primitive comparison is satisfied for said sheaf. Note that, even for proper X , one should not expect primitive comparison to hold for such general coefficients, as the most general known version of primitive comparison holds for Zariski-constructible coefficients, in the sense of [BH21].³⁶

Compared to the simplified version presented above, the proof of this version adds little difficulty, and follows from carefully extending [proposition 1.3.3](#), [proposition 1.3.4](#) and [proposition 1.3.5](#) to this setup.

The first point is particularly relevant, as, for many applications in this thesis, cohomology groups will be infinitely dimensional, so that establishing an abstract isomorphism is not very insightful. This version allows us to compute étale cohomology groups with the additional structure of a representation.

Let us now present the other main results obtained in this thesis.

Firstly, we recover the Poincaré duality for Zariski-compactifiable spaces, as established in [LRZ24, Cor 1.1.2] :

Corollary 1.4.2. (cf. [corollary 5.3.2](#)).

Let X be a smooth proper rigid analytic variety over K of pure dimension d , and $U \subset X$ be a Zariski-open subset. Let $\mathcal{L}$ be an étale $\mathbb{F}_p$ -local system on U .

Let U_C denote the base change of U to C , and $\mathcal{L}_C$ be the base change of $\mathcal{L}$ to U_C .

Then U_C satisfies primitive comparison with compact support with respect to $\mathcal{L}_C$, so that, for any $0 \leq k \leq 2d$, there is a natural $\text{Gal}(\overline{K}/K)$ -isomorphism :

$$H_{\text{ét}}^k(U_C, \mathcal{L}_C^\vee(d)) \cong \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(U_C, \mathcal{L}_C), \mathbb{F}_p \right).$$

The proof of the above follows from the primitive comparison theorem for Zariski-constructible coefficients on X , together with a version of excision for étale sheaves, obtained in [proposition A.2.1](#).

Furthermore, we explicitly compute $\mathcal{O}^+/p$ -cohomology groups of the closed unit disk of various radius, by taking an explicit pro-étale covering, and applying the method described in [section 1.2](#). We obtain the following :

Proposition 1.4.3. (cf. [proposition 2.6.1](#))

For $r, n \in \log(\Gamma)$, consider the inclusion of $\mathbb{D}_C(|p^{n+r}|) \subset \mathbb{D}_C(|p^n|)$, which induces a map in $\mathcal{O}^+/p$ -cohomology. We have the following :

- If $r \geq p/(p-1)$, the image of $H_{\text{ét}}^0(\mathbb{D}_C(|p^n|), \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^0(\mathbb{D}_C(|p|^{n+r}), \mathcal{O}^+/p)$ is almost isomorphic to $\mathcal{O}_C/p$.
- If $r \geq 1$, the image of $H_{\text{ét}}^1(\mathbb{D}_C(|p^n|), \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^1(\mathbb{D}_C(|p|^{n+r}), \mathcal{O}^+/p)$ is almost zero.
- For all r , if $t \geq 2$, $H_{\text{ét}}^t(\mathbb{D}_C(|p|^r), \mathcal{O}^+/p)$ is almost zero.

By excision in $\mathbb{P}_C^1$, we may deduce :

Corollary 1.4.4. (cf. [proposition 2.7.4](#))

Let $\mathring{\mathbb{D}}_C := \bigcup_{r \in \mathbb{N}^*} \mathbb{D}_K(|\pi^{1/r}|)$ denote the open unit disk over C .

³⁶The original version due to Scholze in [Sch12a, Thm 5.1] was formulated with local systems, and the extension to the Zariski-constructible setup is due to [Zav24a, Lemma 7.3.7].

Then $H_{\text{ét},c}^1(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p)$ and $H_{\text{ét},c}^2(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p)$ are not almost finitely generated, and not almost free as $\mathcal{O}_C/p$ -modules. In particular, $\mathring{\mathbb{D}}_C$ does not satisfy primitive comparison with compact support.

The open unit disk is known not to satisfy Poincaré duality, by [LRZ24, Ex. 6.3.3], so that the above result was expected.

Actually, for rigid analytic curves, we obtain the following characterisation of primitive comparison with compact support, and one can easily check that the open unit disk does not satisfy (iii).

Theorem 1.4.5. (cf. *theorem 5.5.1*).

Let X be a smooth, connected, partially proper rigid-analytic curve over C . Assume furthermore that X is of finite genus, in the sense of [Liu87, Sec. 1].³⁷

The following are equivalent :

- (i) X satisfies primitive comparison with compact support with respect to $\mathbb{F}_p$.
- (ii) The natural map $H_{\text{ét}}^0(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\text{ét}}^0(X, \mathcal{O}_X^+/p)$ is an almost isomorphism.
- (iii) $\mathcal{O}^+(X) \cong \mathcal{O}_C$
- (iv) There exists a smooth projective algebraic curve Y over C , such that X is open in Y^{an} , and the complement is compact for the canonical topology of Y^{an} .

A key ingredient of the proof of the above is a result of Qing Liu [Liu87, Theorem 3.3], which proves the implication (iii) $\implies$ (iv).

In the second part of this thesis, we establish primitive comparison with compact support for a class of partially proper varieties of higher dimension admitting a nontrivial action of p -adic Lie groups : the p -adic period domains, in the sense of Rapoport-Zink [RZ96, Def 1.36].

Let us first state our result for Drinfeld symmetric spaces, which are important examples of such period domains :

Proposition 1.4.6. (Special case of *theorem 1.4.7*)

Let $\Omega_{\mathbb{Q}_p}^d = \mathbb{P}_{\mathbb{Q}_p}^n \setminus \bigcup_{H \in \mathcal{H}} H$ be the Drinfeld symmetric space of dimension d over $\mathbb{Q}_p$, where $\mathcal{H}$ denotes the set of $\mathbb{Q}_p$ -rational hyperplanes. Let Ω_C^d be its base change to C , admitting a natural $\text{GL}_{d+1}(C)$ action.

Then, Ω_C^d satisfies primitive comparison with compact support with coefficients in $\mathbb{F}_p$.

In particular, Ω_C^d satisfies Poincaré duality, so that, for any $0 \leq k \leq 2d$, there is a natural $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \times \text{GL}_{d+1}(\mathbb{Q}_p)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(\Omega_C^d, \mathbb{F}_p(d)) \cong \text{Hom}_{\mathbb{F}_p}(H_{\text{ét},c}^{2d-k}(\Omega_C^d, \mathbb{F}_p), \mathbb{F}_p).$$

It follows from the computation of compactly supported cohomology of Drinfeld symmetric spaces in [CDHN21, Cor 6.8] that the étale cohomology of Drinfeld symmetric spaces with $\mathbb{F}_p$ -coefficients is given by the Tate-twisted dual of a generalised Steinberg representation (denoted Sp for "special").

Indeed, for all $0 \leq k \leq d$:

$$H_{\text{ét}}^k(\Omega_C^d, \mathbb{F}_p) \cong \text{Sp}_k^*(\mathbb{F}_p)(k).$$

This recovers the computation of [CDN21, Thm 1.1]. Note that the approach of loc.cit. does not stem from a Poincaré duality result, but rather an explicit computation using formal models and A_{inf} -cohomology, in the sense of Bhatt-Morrow-Scholze [BMS16].

³⁷The genus of a curve is defined as the supremum of the genus of its affinoid open subsets, for which it computes as the genus of their canonical reduction, viewed as an algebraic curve. If X is of the form (iv) below, its genus is simply the genus of Y .

We give two different proofs of the above statement :

1. The first one is based on adapting the proof strategy of Schneider-Stühler [SS91] to $\mathcal{O}^+/p$ -coefficients. This relies on the "stalk-invariance" property of $\mathcal{O}^+/p$ -cohomology, in the form :

$$\varprojlim_{r \in \mathbb{N}} H_{\text{ét}}^k(X \times \mathbb{D}_C(|p^r|), \mathcal{O}^+/p) \stackrel{a}{\cong} H_{\text{ét}}^k(X, \mathcal{O}^+/p)$$

for smooth proper X . This follows from [proposition 1.4.3](#), together with a weak Künneth formula extracted from Mann's six-functor formalism, see [section 6.2](#).

2. The second one is by adapting the method of S.Orlik in [Orl05a].

The second method is more general, as it works for more general period domains, whose definition is detailed in [chapter 7](#). Let us state the result.

Let $\check{\mathbb{Q}}_p$ denote the completion of the maximal unramified extension of $\mathbb{Q}_p$. Consider the datum $(G, [b], \{\mu\})$ of :

- A connected and quasi-split reductive algebraic group G over $\mathbb{Q}_p$.
- An element $[b]$ of the Kottwitz set $B(G)$ of σ -conjugacy classes in $G(\check{\mathbb{Q}}_p)$, associated to some $b \in G(\check{\mathbb{Q}}_p)$ that is assumed to be basic and of s -decent³⁸ for some integer $s \geq 1$.
- A conjugacy class $\{\mu\}$ of geometric cocharacters of G .

Let E be the field of definition of $\{\mu\}$, also known as its reflex field. The datum of $(G, \{\mu\})$ induces a flag variety $\mathcal{F}(G, \{\mu\})$ defined over E , classifying par-equivalence classes of cocharacters in $\{\mu\}$.

Let $\check{E} = E \cdot \check{\mathbb{Q}}_p$, and J_b be the inner form of G defined as the σ -centralizer of b . The period domain associated to $(G, [b], \{\mu\})$, denoted $\mathcal{F}^{wa}(G, [b], \{\mu\})$, is the partially proper open subset of $\mathcal{F}(G, \{\mu\}) \otimes_E \check{E}$ parameterizing weakly admissible filtrations of type $\{\mu\}$ on N_b . It admits a canonical model over $E_s := E \cdot \mathbb{Q}_{p^s}$, still denoted $\mathcal{F}^{wa}(G, [b], \{\mu\})$, and is endowed with a natural action of $J_b(\mathbb{Q}_p)$.

We may now state our result :

Theorem 1.4.7. (cf. [theorem 8.0.1](#))

Let $(G, b, [\mu])$ be as above, and $\mathcal{F}^{wa}(G, [b], \{\mu\})$ be the associated period domain over E_s . Let C be the completion of the algebraic closure of E_s , and $\mathcal{F}_C^{wa}$ be the base change of $\mathcal{F}^{wa}(G, [b], \{\mu\})$ to C .

Let $j : \mathcal{F}^{wa}(G, [b], \{\mu\}) \rightarrow \mathcal{F}(G, \{\mu\})$ be the open immersion, with base change to $C : j_C : \mathcal{F}_C^{wa}(G, [b], \{\mu\}) \rightarrow \mathcal{F}_C(G, \{\mu\})$.

Let $\mathcal{L}$ be a local system on $\mathcal{F}(G, \{\mu\})$, and $j_C^* \mathcal{L}_C$ be its pullback to $\mathcal{F}_C^{wa}$.

Then, $\mathcal{F}_C^{wa}$ satisfies primitive comparison with compact support with $j_C^* \mathcal{L}_C$ -coefficients.

It hence satisfies Poincaré duality, i.e., letting d be its dimension, for all $0 \leq k \leq 2d$, there is a natural $\text{Gal}(\overline{E_s}/E_s) \times J_b(\mathbb{Q}_p)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(\mathcal{F}_C^{wa}, (j_C^* \mathcal{L}_C)^\vee(d)) \cong \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(\mathcal{F}_C^{wa}, j_C^* \mathcal{L}_C), \mathbb{F}_p \right).$$

The proof of the above theorem relies on adapting the computation of étale cohomology with compact support of $\mathcal{F}_C^{wa}$, done in [CDHN21] with $\mathbb{F}_p$ -coefficients, to $\mathcal{O}^+/p$ coefficients. This relies on a stratification of the complement of the inclusion of $\mathcal{F}_C^{wa}$ inside the flag variety $\mathcal{F}_C$ due to [Orl05a]. Using geometric

³⁸The decency hypothesis is not a constraint, as any conjugacy class contains a decent element.

arguments, we reduce to establishing a comparison result for Schubert cells, which are proper, hence the primitive comparison theorem applies.

In particular, using the computation of the compactly supported cohomology of period domains in [CDHN21, Thm 1.2], the above theorem allows us to compute the étale cohomology of $\mathcal{F}_C^{wa}$ with $\mathbb{F}_p$ -coefficients, in a $\text{Gal}(\overline{E_s}/E_s) \times J_b(\mathbb{Q}_p)$ -manner, which were unknown (outside of Drinfeld's symmetric spaces).

Since the statement of this result requires the introduction of lots of additional notations, we postpone it to [corollary 8.4.1](#).

1.5 Structure of the manuscript

In [chapter 2](#), we start by giving precise definition of some concepts that were mentioned in this introduction : adic and perfectoid spaces, diamonds, (pro)-étale sites, and the formalism of almost mathematics. We then turn our attention to the explicit computation of the $\mathcal{O}^+/p$ of the closed unit disks and associated spaces, as in [proposition 1.4.3](#) and [corollary 1.4.4](#).

In [chapter 3](#), we introduce the necessary tools from Mann's six-functor formalism. We also establish the version of Poincaré duality on the essential image of the Riemann-Hilbert functor, as exposed in [proposition 1.3.5](#), and we establish the first compatibility result of [proposition 1.3.3](#).

In [chapter 4](#), we expend on new comparison results between the objects appearing in Mann's six-functor formalism and classical objects, centered around the proof of [proposition 1.3.3](#).

In [chapter 5](#), we deduce our main result [theorem 1.4.1](#) from the above, and give our first applications. We recover Poincaré duality for Zariski-compactifiable spaces as in [corollary 1.4.2](#), and we establish the criterion for rigid analytic curves exposed in [theorem 1.4.5](#).

In [chapter 6](#), we compute the compactly supported $\mathcal{O}^+/p$ -cohomology of Drinfeld's symmetric spaces, adapting a method from Schneider-Stuhler.

In [chapter 7](#), we define p -adic period domains, and, in [chapter 8](#), we compute their compactly supported $\mathcal{O}^+/p$ -cohomology.

This document contains two appendices. The first one, [Appendix A](#), contains technical results regarding the topology and étale cohomology of locally special diamonds, and is mostly used in [chapter 4](#). We also obtain a general excision result, which is used throughout this thesis.

The second one, [Appendix B](#), contains technical results regarding Tits buildings and the geometric realisation of simplicial profinite spaces, which are useful in section [chapter 8](#), and, to a lesser extent, [chapter 6](#). Its primary goal is to make more precise an argument of Orlik, and laying the first stones of a bridge between the 2 proofs of [proposition 1.4.6](#).

1.6 Conventions and notations

Throughout this thesis, p denotes a fixed prime number. Unless mentioned otherwise, K denotes a complete extension of $\mathbb{Q}_p$, $\overline{K}$ denotes an algebraic closure of K , and C denotes a complete algebraically closed field containing K (that will sometimes be assumed to be the completion of $\overline{K}$).

The symbol $:=$ is used to define objects, and $\mathbb{N}$ denotes the set of nonnegative integers. The symbol $\cong$ denotes a canonical isomorphism, while $\simeq$ denotes a specified isomorphism.

Using the natural embeddings, we canonically identify rigid analytic varieties with their associated analytic adic space (cf. [Hub96, 1.1.11.(d)]), analytic adic spaces with their associated locally spatial diamond (cf. [Sch17, Def 15.5]), and locally spatial diamonds with their associated small v-stack (cf. [Sch17, Remark 12.2]). We try as much as possible to state results for the most general type of space in the above list.

For an abelian category $\mathcal{A}$, we use the upright $D(\mathcal{A})$ to denote its 1-derived category, and the calligraphic $\mathcal{D}(\mathcal{A})$ to denote its associated ∞ -derived category. Following [Man22], we work with homotopical notation, i.e., for $\mathcal{C} \in \mathcal{D}(\mathcal{A})$, we let $\pi_k(\mathcal{C}) \in \mathcal{A}$ denotes its k -th homology group.

Recall that, in a stable ∞ -category, a square :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

is cartesian if and only if it is cocartesian. When this is the case, we say that $X \rightarrow Y \rightarrow Z$ forms an exact sequence (this corresponds to a distinguished triangle on the homotopy category). Moreover, a reduced functor between stable infinity categories preserves fiber sequences if and only if it preserves cofiber sequences, and, when this is the case, we say that the functor is exact.

We use the subscript or superscript "M" to denote the functors $f_M^*, f_*^M, f_!^M, \dots$, as constructed by Mann in [Man22]. We use the letter "H" to denote functors as defined by Huber in [Hub96], and "S" to denote their improvements to the setup of diamonds as constructed in [Sch17].

Mann's functors are always implicitly derived, whereas, the functors $f_H^*, f_!^H, f_S^*, f_!^S, \dots$ are not. We denote $Rf_H^*, Rf_!^H, Rf_S^*, Rf_!^S, \dots$ their derived version.

Acknowledgements

A part of this thesis (namely, section 2.4, chapters 3, 4, 5, 7, 8 and appendix A) already appeared in the preprint "Mod p Poincaré duality for p -adic period domains", available at <https://arxiv.org/pdf/2512.25029>.

No generative AI was used as part of the redaction of the present document, nor of the research leading to it.

The $\mathcal{O}^+/p$ -cohomology of disks

Content of this chapter

In this chapter, we start by introducing many concepts that were motivated in the introduction, namely adic spaces, perfectoid spaces, diamonds, (pro)-étale sites.

In [section 2.4](#), we expand on foundational results of [\[MW20a\]](#), by defining a structure sheaf $\mathcal{O}^+/p$ on general locally spatial diamonds.

After recalling the necessary notions from almost mathematics, we then compute the $\mathcal{O}^+/p$ -cohomology of some classical one-dimensional rigid analytic spaces, following the method laid out in the introduction.

The results are as follows :

1. We show that, for r large enough¹, there is an almost isomorphism :

$$\mathrm{Im} \left(H_{\text{ét}}^n(\mathbb{D}_C, \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^n(\mathbb{D}_C(|p^r|), \mathcal{O}^+/p) \right) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases},$$

where $\mathbb{D}_C(|p^r|)$ denotes the closed unit disk of radius $|p^r|$ (defined in [example 2.1.16](#)). This result is used in [chapter 6](#), where we deduce a *stalk-invariance* property.

2. As a consequence, we show that the adic affine line $\mathbb{A}_C$ satisfies the primitive comparison theorem.
3. We show that the *open* unit disk $\mathring{\mathbb{D}}_C$, defined as the increasing union of closed adic disks of radii $R < 1$ (cf. [example 2.1.18](#)), does *not* satisfy primitive comparison with compact support.²

While these results do not appear in the literature, the methods used are very classical.

¹We give a precise bound on r depending on n , cf. [proposition 2.6.11](#).

²This is good news, since the open unit disk is known not to satisfy Poincaré duality.

2.1 Recollection on adic geometry

Let us start by defining basic concepts of adic geometry, which are used throughout this thesis. As many of these concepts are classical, or were already motivated in the introduction, we give a rather streamlined presentation.

In this section, we fix a complete and nonarchimedean field K , that is either of mixed characteristic $(0, p)$ or of characteristic p , endowed with the norm $|\cdot|_K$, and pseudo-uniformizer π_K (simply denoted $|\cdot|$ and π when there is no ambiguity).

We let $\mathcal{O}_K = \{x \in K, |x|_K \leq 1\}$ denote the ring of integers of K , which admits a unique maximal ideal, $\mathfrak{m}_K = \{x \in K, |x|_K < 1\}$.

2.1.1 Definition of adic spaces

Let us start by recalling the definition of adic spaces.

The classical reference for the material presented here is [Hub96] ; we rather follow the presentation of [SW20]. Good introductions also include [Wed15], [Mor19], and [Hü24].

Definition 2.1.1. A topological ring A is said to be **Huber** if it admits an open subring $A_0 \subset A$ whose topology is I -adic, for some finitely generated ideal I of A_0 . Such a ring A_0 is called a **ring of definition** of A , and I is called an **ideal of definition** of A .

A subset $M \subset A$ is called **bounded** if $M \subset a \cdot A_0$ for some $a \in A$ and some ring of definition A_0 . We say that an element $x \in A$ is **power-bounded** if the set $\{x^n\}_{n \in \mathbb{N}}$ is bounded. We let A° denote the subset of power-bounded elements.

Definition 2.1.2. An **Huber pair** is a pair (A, A^+) consisting of a Huber ring A and an open and integrally closed subring $A^+ \subset A^\circ$.

A **morphism** of Huber pairs $(A, A^+) \rightarrow (B, B^+)$ is a continuous ring morphism $\varphi : A \rightarrow B$ such that the $\varphi(A^+) \subset B^+$.

Definition 2.1.3. A Huber ring A is said to be **Tate** if it contains a topologically nilpotent unit. We say that a Huber pair (A, A^+) is Tate if A is.

As example, the pairs $(K\langle T \rangle, \mathcal{O}_K\langle T \rangle)$ and $(K\langle T \rangle, \mathcal{O}_K + \mathfrak{m}_K\langle T \rangle)$ are Huber pairs, which are Tate, as π is a topologically nilpotent unit.³

Definition 2.1.4. Let (A, A^+) be a Huber pair. A **continuous valuation**⁴ on A is a morphism $|\cdot| : A \rightarrow \Gamma \sqcup \{0\}$ onto some totally ordered abelian group Γ (denoted multiplicatively), such that, for all $a, b \in A$:

1. $|0| = 0, |1| = 1$.
2. $|a \cdot b| = |a| \cdot |b|$ and $|a + b| \leq \max(|a|, |b|)$.
3. For all $\gamma \in \Gamma$, the set $\{a \in A, |a| \leq \gamma\}$ is open.

We then let :

$$\mathrm{Spa}(A, A^+) = \{ \text{Continuous valuations } |\cdot| \text{ on } A, \forall x \in A^+, |x| \leq 1 \} / \sim,$$

³The Tate algebras are considered to be endowed with the π -adic topology (in fact, $K\langle T \rangle$ is the completion of $K[T]$ with respect to the π -adic topology).

⁴While the convention is to speak of them as valuations, it would be perhaps more intuitive to call them norms, as they are written multiplicatively.

where two valuations are equivalent if they define the same order relation on A . We denote the valuations symbolically as $f \mapsto |f(x)|$ for $x \in \text{Spa}(A, A^+)$.

We endow $\text{Spa}(A, A^+)$ with the topology generated by **rational subsets**, defined as :

$$U \left(\frac{f_1, \dots, f_n}{g} \right) = \{x \in \text{Spa}(A, A^+), \forall i, |f_i(x)| \leq |g(x)| \text{ and } |g(x)| \neq 0\},$$

for $(f_1, \dots, f_n)$ generating A as an ideal, and $g \in A$.

For any Huber pair (A, A^+) , the topology of $\text{Spa}(A, A^+)$ is **spectral** ; i.e. there exists a (non-explicit) ring B with a homeomorphism $\text{Spa}(A, A^+) \simeq \text{Spec}(B)$, where $\text{Spec}(B)$ is endowed with the Zariski topology.

Definition 2.1.5. Let (A, A^+) be a Huber pair, and $X = \text{Spa}(A, A^+)$.

Let A_0 be a ring of definition of A , and I an ideal of definition.

We define two structure presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on X , given on $U \left(\frac{f_1, \dots, f_n}{g} \right)$ by :

- $\mathcal{O}_X(U) = A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$ is the completion of $A_0 \left[\frac{f_1}{g}, \dots, \frac{f_n}{g} \right]$ with the topology generated by powers of the ideal $I \cdot A_0 \left[\frac{f_1}{g}, \dots, \frac{f_n}{g} \right]$;
- $\mathcal{O}_X^+(U)$ is the completion of the integral closure of $A^+ \left[\frac{f_1}{g}, \dots, \frac{f_n}{g} \right]$, viewed as a subring of $A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$.

By construction, any rational open U is affinoid and $U \cong \text{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$.

For any $W \subset X$, we let :

$$\mathcal{O}_X^+(W) = \varprojlim_{U \subset W, U \text{ rational}} \mathcal{O}_X^+(U),$$

and likewise for $\mathcal{O}_X$. The data of $x \in \text{Spa}(A, A^+)$ induces a valuation on each $\mathcal{O}_X^+(U)$, inducing a valuation on the local ring $\mathcal{O}_{X,x} = \varprojlim_{x \in U} \mathcal{O}_X(U)$.

An **affinoid adic space** is the data of the topological space $\text{Spa}(A, A^+)$ for some complete Huber pair (A, A^+) , endowed with the topology, structure sheaves, and equivalence classes on local rings defined above.

Definition 2.1.6. We say that (A, A^+) is **sheafy** if the presheaf $\mathcal{O}_X$ is a sheaf. When this is the case, $\mathcal{O}_X^+$ is also a sheaf.

All of the Huber pairs that appear in this thesis are sheafy. This does not come for free however, and requires extra assumption ; usually either of noetherianity, or of being perfectoid.⁵

We finally arrive at our definition of adic spaces :

Definition 2.1.7. An **adic space** is the data of a triple $(X, \mathcal{O}_X, \{|\cdot|_x\}_{x \in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X , and, for all $x \in X$, $|\cdot|_x$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$; that is locally isomorphic to an affinoid adic space $\text{Spa}(A, A^+)$ for some sheafy, complete⁶ Huber pair (A, A^+) .

⁵All perfectoid spaces are sheafy [Sch12b, Thm 6.3], as well as all stably uniform adic spaces [BV15, Thm 7]. See also [SW20, Sec 3.4] for a discussion on non sheafy adic spaces.

⁶For any Huber pair (A, A^+) , $\text{Spa}(A, A^+)$ is equal to the spectrum of the completions $\text{Spa}(\hat{A}, \hat{A}^+)$. We may hence assume that all Huber pairs are complete.

A **morphism** of adic spaces is a morphism of locally ringed spaces such that the morphism of sheaves is continuous and compatible with the valuations. The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ is fully faithful.

We will also need the following notion :

Definition 2.1.8. We say that a point x of an adic space X is **analytic** if there exists an open neighborhood U of x such that $\mathcal{O}_X(U)$ is Tate.

We say that an adic space is **analytic** if all of its points are analytic.

It follows from definitions (cf. [Hü24, Lemma 6.6]) that any adic space over a field is analytic.

As in the setup of schemes, we say that an adic space is quasi-compact or quasi-separated if its underlying topological space is. We say that a morphism of adic spaces is an open immersion, if the underlying map of locally ringed space is.

An important issue with the theory of adic spaces, compared to the setup of schemes, is that the closed complement of an open adic subspace need not admit an analytic structure⁷, outside of the specific case of so-called Zariski-open subset, whose complement is Zariski-closed.

Definition 2.1.9. Let $F \subset X$ be a subset of a locally noetherian adic space.

We say that F is **Zariski-closed** in X if F is locally of the form $\mathrm{Spa}(A/J, A^+/(J \cap A^+))$, where J is an ideal of A . We say that some $U \subset X$ is Zariski-open if its complement admits a structure of a Zariski-closed subset.

Zariski-open subsets hence locally correspond to the locus on non-vanishing of a function, i.e. $U = \{f \neq 0\}$. They are not rational opens, but can be seen as the increasing union of the rational opens $\{|f| \leq \varepsilon\}$.

2.1.2 Classical properties of adic spaces

Let us define a notion of noetherianity for adic spaces.

Definition 2.1.10. An adic space is said to be **locally noetherian** if it is locally of the form $\mathrm{Spa}(A, A^+)$ for some A such that $A \langle T_1, \dots, T_n \rangle$ is noetherian for all $n \in \mathbb{N}$.

In the book [Hub96], Huber works under a running assumption to guarantee that every space appearing in the book is sheafy, which often reduces to being locally noetherian. When applying results of the standard theory, we thus often make a locally noetherian assumption, although it is not clear precisely when this hypothesis is needed.

Let us introduce two subtle finiteness properties of adic spaces.

Definition 2.1.11. A morphism of adic spaces $f : X \rightarrow Y$, with analytic Y , is :

- **Locally of $^+$ -weakly finite type** if it is locally given by $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ with complete B , corresponding to some continuous ring map $\varphi : B \rightarrow A$ such that φ factors through⁸ a continuous, surjective and open map $B \langle T_1, \dots, T_n \rangle \rightarrow A$, and such that there exists a finite subset $E \subset B$ such that B^+ is the smallest ring of integral elements containing $\varphi(A^+) \cup E$.
- **Locally of finite type** if it is locally of $^+$ -weakly finite type and it is locally of the form $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ such that there exists an open subring C of B containing B^+ , such that B^+ is integral over C and $f : A^+ \rightarrow C$ factors through a continuous, surjective and open map $A^+ \langle T_1, \dots, T_n \rangle \rightarrow C$.

⁷We will see in lemma A.1.3 that passing to the language of locally spatial diamonds somewhat solves this issue, by constructing diamantine structure on the complement of any generalizing open subset

⁸This is not the standard definition, but follows from the criterion [Hub94, Lemma 3.3.(iii)] (as X is assumed to be analytic.)

In this thesis, we **define** a quasi-separated rigid-analytic variety over K as a quasi-separated, locally of finite type, adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$.

Remark 2.1.12. The classical definition of rigid-analytic varieties due to Tate, is in [Tat71]. In [Hub96], Huber defines a fully faithful functor :

$$\begin{aligned} r : \{\text{Rigid analytic spaces over } K\} &\rightarrow \{\text{Adic spaces over } \mathrm{Spa}(K, \mathcal{O}_K)\} \\ \mathrm{Spm}(A) &\mapsto \mathrm{Spa}(A, A^\circ) \end{aligned}$$

This induces an equivalence of category between quasi-separated rigid analytic varieties over K and locally of finite type quasi-separated adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, reviewed in [Hub96, p. 1.11].

Hence, we may view quasi-separated rigid analytic varieties as a full subcategory of adic spaces ; this is the point of view taken here.

We may now define a notion of smoothness, that is very similar to the one for schemes.

Definition 2.1.13. We say that a morphism $f : X \rightarrow Y$ of adic spaces over K is **smooth** if it is locally of finite type and, for any Huber pair (A, A^+) and ideal $I \subset A$ such that $I^2 = \{0\}$, the natural map :

$$\mathrm{Hom}_Y(\mathrm{Spa}(A, A^+), X) \rightarrow \mathrm{Hom}_Y(\mathrm{Spa}(A/I, A^+/(I \cap A^+)^c), X)$$

is surjective (where $A^+/(I \cap A^+)^c$ denotes the integral closure of $A^+/(I \cap A^+)$ in A/I).

We may now define a notion of (partial) properness.

Definition 2.1.14. We say that a morphism of analytic adic spaces $f : X \rightarrow Y$ is **partially proper** if it is :

1. Locally of $^+$ -weakly finite type, in the sense of definition 2.1.11).
2. **Separated**, i.e. the diagonal $\Delta(X) \subset X \times_Y X$ is closed.
3. **Universally specializing**, i.e. for every morphism of adic spaces $g : Y' \rightarrow Y$, the projection $\pi : X \times_Y Y' \rightarrow Y'$ is **specializing**, in the sense that its image is closed under specializations (i.e. for all $x \in \pi(X)$, $\overline{\{x\}} \subset \pi(X)$).

We say that f is **proper** if it is locally of $^+$ -weakly finite type, separated and universally closed (i.e., for every $g : Y' \rightarrow Y$, the projection $X \times_Y Y' \rightarrow Y'$ is closed).

Note that, by definition, smooth and partially proper adic spaces over a field are automatically rigid analytic varieties.

Clearly, proper spaces are partially proper. Partially proper spaces are the analytic equivalent of the rather obscure notion of *essentially proper* morphism of schemes, defined in [Rk 18.10.20, EGA4].

It is however a crucial notion in adic geometry. This discrepancy can be explained by the following.

Proposition 2.1.15. [Hub96, Lemma 1.3.4] Let f be a partially proper (resp. essentially proper) morphism of analytic adic spaces (resp. schemes).

Then f is quasi-compact if and only if f is proper.

In algebraic geometry, most spaces of interest are qcqs (quasi-compact and quasi-separated)⁹, which is far from being the case in adic geometry.

We conclude this section with a few notable examples of some one-dimensional partially proper adic spaces, that will appear throughout this thesis.

⁹When introducing essentially proper spaces, Grothendieck names the notable counterexamples of Picard schemes and Néron-Severi schemes.

2.1.3 Examples of adic spaces

We refer the reader to [Hü24] for a detailed introduction to these spaces.

Example 2.1.16. Let Γ denote the value group of K . For any $r \in \log(\Gamma)$, we define the closed disk of radius $|\pi|^r$ over K as :

$$\begin{aligned} \mathbb{D}_K(|\pi|^r) &= \mathrm{Spa}(K\langle\pi^{-r}T\rangle, \mathcal{O}_K\langle\pi^{-r}T\rangle) \\ &= \left\{ \sum_{n \in \mathbb{N}} a_n \pi^{-rn} T^n, |a_n| \rightarrow_{n \rightarrow \infty} 0 \right\} \cong \left\{ \sum_{n \in \mathbb{N}} b_n T^n, |b_n \pi^{rn}| \xrightarrow{n \rightarrow \infty} 0 \right\} \end{aligned}$$

There are natural open immersions $\mathbb{D}_K(|\pi|^r) \subset \mathbb{D}_K(|\pi|^s)$ whenever $r \geq s$, given by the inclusion of the rational subset $|T| \leq |\pi|^r$.

Remark 2.1.17. For any r, s , the disks $\mathbb{D}_K(|\pi|^r)$ and $\mathbb{D}_K(|\pi|^s)$ are naturally isomorphic, by scaling the formal parameter.

By abuse of notation¹⁰, we'll also let $\mathbb{D}_K(|\pi|^r) := \mathbb{D}_K(|\pi|^r)$.

Example 2.1.18. We let $\mathring{\mathbb{D}}_K := \bigcup_{r \in \mathbb{N}^*} \mathbb{D}_K(|\pi^{1/r}|)$ denote the **open unit disk** over K .

We let $\mathbb{A}_K := \bigcup_{r \in \mathbb{N}} \mathbb{D}_K(|\pi^{-r}|)$ denote the **adic affine line** over K .

Both the open unit disk and the affine line are examples of *partially proper* spaces, yet not proper. Indeed, they are not quasicompact, as we cannot extract a finite subcover of the affinoid cover defining them.

Example 2.1.19. The projective line $\mathbb{P}_K^1$ is defined as the gluing of two closed disks, with coordinate noted T and S respectively, by identifying the boundaries $|T| = 1$ and $|S| = 1$ via the map $T \mapsto S^{-1}$.

One of the key features of adic geometry is the presence of canonical compactifications. In particular, we will need the compactified closed unit disk (see [LRZ24, Sec. 4.2]).

Example 2.1.20. The canonical compactification of $\mathbb{D}_K$, denoted $\mathbb{D}_K^c$, is defined as :

$$\mathbb{D}_K^c = \mathrm{Spa}(K\langle T \rangle, \mathcal{O}_K + \mathfrak{m}_K\langle T \rangle).$$

It is a proper space, with an open immersion $\mathbb{D}_K \rightarrow \mathbb{D}_K^c$; and is furthermore universal for those properties. The closed complement inside $\mathbb{P}_K^1$ of the disk $\mathbb{D}_K$ centered around zero is the compactified disk $\mathbb{D}_K^c$, centered at infinity.

2.2 Perfectoid spaces and tilting

Let us now define perfectoid spaces, as introduced by Scholze in [Sch12b]. Recall the typical example of the perfectoid torus

$$\tilde{\mathbb{T}}_C = \mathrm{Spa}\left(C\langle T^{\pm 1/p^\infty} \rangle, \mathcal{O}_C\langle T^{1/p^\infty} \rangle\right)$$

from the introduction.

¹⁰This is allowed by the fact that, for two pseudo-uniformizers π, π' with $|\pi| = |\pi'|$, the spaces $\mathbb{D}_K(|\pi|^r)$ and $\mathbb{D}_K(|\pi'|^r)$ are naturally isomorphic.

We follow the approach of [SW20, Sec. 6].¹¹

Definition 2.2.1. A ring R is said to be **perfectoid** if :

1. The set R° of power-bounded elements of R is bounded in R .
2. There exists a topologically nilpotent unit $\varpi \in R$ such that ϖ^p divides p in R° . Such an element ϖ is called a pseudo-uniformizer.
3. The Frobenius morphism induces an isomorphism $R^\circ/\pi \rightarrow R^\circ/\pi^p$.

Definition 2.2.2. A **perfectoid space** is an adic space that can be covered by $\mathrm{Spa}(R, R^+)$ for some perfectoid rings R . Morphisms of perfectoid spaces are simply morphism of adic spaces.

A weird, yet useful example of perfectoid spaces, is given by profinite topological spaces, as follows.

Example 2.2.3. Let S be a profinite space. Letting $S_C := \mathrm{Spa}(\mathrm{Cont}(S, C), \mathrm{Cont}(S, \mathcal{O}_S))$ defines a perfectoid space over C .

Let us now define our notion of (pro-)étale morphisms, and the associated (pro-)étale sites. We will do so directly at the level of *diamonds*, so that we start from the perfectoid level.

Definition 2.2.4. A morphism of perfectoid spaces $X \rightarrow Y$ is **finite étale** if, for any affinoid open $\mathrm{Spa}(B, B^+) \subset Y$, the pullback $X \times_Y \mathrm{Spa}(B, B^+)$ is of the form $\mathrm{Spa}(A, A^+)$ where A is a finite étale B -algebra, and A^+ is the integral closure of the image of B^+ in A .

A morphism of perfectoid spaces $X \rightarrow Y$ is **étale** if it is locally the composition of an open immersion and a finite étale map.

Definition 2.2.5. A morphism $f : X \rightarrow Y$ of perfectoid spaces is **affinoid pro-étale** if X and Y are affinoid, and we can write $X = \varprojlim X_i \rightarrow Y$ as a (small) cofiltered inverse limit of affinoid spaces, such that the induced map $f_i : X_i \rightarrow Y$ is étale.

A morphism $f : X \rightarrow Y$ of perfectoid spaces is **pro-étale** if it is locally affinoid pro-étale.

We may hence define an étale and pro-étale site on every perfectoid space X , whose objects are étale (resp. pro-étale) maps $Y \rightarrow X$, and covers are given by jointly surjective families of étale (resp. pro-étale) maps.

Any perfectoid space admits a pro-étale map from a so-called *strictly totally disconnected* perfectoid space, as follows.

Definition 2.2.6. Let X be a perfectoid space. We say that X is **strictly totally disconnected** if it is qcqs and every étale cover splits. Equivalently (cf. [Sch17, Prop 7.16]), it is affinoid and its connected components are of the form $\mathrm{Spa}(C, C^+)$ for some algebraically closed field C .

In particular, the étale site of a strictly totally disconnected is equivalent to its topological site. Pro-étale locally, every perfectoid space is strictly totally disconnected, as follows.

Lemma 2.2.7. [Sch17, Lemma 7.18]. Let X be an affinoid perfectoid space. Then it admits a surjective, universally open map $\tilde{X} \rightarrow X$ from a strictly totally disconnected affinoid perfectoid space.

The historic motivation of perfectoid theory is that they allow for a bridge between mixed characteristic $(0, p)$ and characteristic p , through the *tilting* operation.

¹¹Note that this is not the original definition of [Sch12b]. Many similar notions co-exist in the litterature, see for example [BMS16, Sec 3] for a short survey. The definition chosen here is tailored for geometric applications.

Definition 2.2.8. Any perfectoid ring R admits a **tilt**, which is a ring of characteristic p defined by :

$$R^\flat := \varprojlim_{x \mapsto x^p} R.$$

We view an element of $R^\flat$ as a compatible sequence $(x_n)_{n \in \mathbb{N}}$, such that $x_{n+1}^p = x_n$. The ring $R^\flat$ is endowed with the termwise multiplication, and addition defined by :

$$(x_n) + (y_n) = \left(\varprojlim_{k \rightarrow \infty} (x_{k+n} + y_{k+n})^{p^k} \right)_{n \in \mathbb{N}}.$$

There is a natural multiplicative map $\sharp : R^\flat \rightarrow R$ defined by $(x_0, x_1, \dots)^\sharp = x_0$.

Our basic example is the following.

Example 2.2.9. Fix F is a complete extension of $\mathbb{Q}_p$, and let C be the completion of an algebraic closure of F . Then C is perfectoid, and its tilt $C^\flat$ admits the pseudo-uniformizer $p^\flat$, and norm $|x|_{C^\flat} := |x^\flat|_C$.

This notion of tilting geometrizes to perfectoid spaces, and the tilt functor respects étale sites. (cf. [Sch12b, Prop. 6.17], [Sch12b, Thm 7.12])

Proposition 2.2.10. The functor $\mathrm{Spa}(R, R^+) \mapsto \mathrm{Spa}(R^\flat, R^{+, \flat})$ glues to a functor $X \mapsto X^\flat$ from perfectoid spaces over C to perfectoid spaces over $C^\flat$.

It moreover induces an equivalence of étale sites.

Let us now introduce diamonds, viewed as quotients of perfectoid spaces by pro-étale equivalence relations, as defined in [Sch17] (see also [SW20]).

2.3 Diamonds and (pro)-étale sites

We let Perf be the category of all¹² perfectoid spaces of characteristic p . We consider its attached big pro-étale $\mathrm{Perf}_{\mathrm{pro\acute{e}t}}$, whose objects are perfectoid spaces, and covers are jointly surjective families of pro-étale maps.

We may now introduce our notion of locally spatial diamonds.

Definition 2.3.1. A **diamond** is a sheaf Y on the pro-étale site $\mathrm{Perf}_{\mathrm{pro\acute{e}t}}$ that can be written as $Y = X/R$, where X is a perfectoid space (identified with the sheaf it represents), and $R \subset X \times X$ is a representable equivalence relation such that the two projections $s, t : R \rightarrow X$ are pro-étale.

Its underlying space is the topological space $|Y| := |X|/|R|$. A diamond Y is said to be **locally spatial** if $|Y|$ admits a basis of quasi-compact open subsets.

We may define a notion of étale and (quasi)¹³-pro-étale morphisms between diamonds, by descent from the perfectoid setup. This will lead to our notion of (pro)étale site of analytic adic spaces, as we will see in [definition 2.3.6](#) that analytic adic spaces embed naturally into diamond.

¹²One should work with a fixed cutoff cardinal κ , and restrict to spaces of size bounded by κ .

¹³Such maps are called quasi pro-étale since a quasi-pro-étale morphism between perfectoid space is not necessarily pro-étale. Nevertheless, we think about them as pro-étale morphisms.

Definition 2.3.2. Let $f : X \rightarrow Y$ be a morphism of diamonds. We say that f is *étale* (resp. *quasi-pro-étale*) if f is locally separated and, for every strictly totally disconnected perfectoid space $Z \rightarrow Y$, the pullback $X \times_Y Z$ is representable, and the based change map $X \times_Y Z \rightarrow Z$ is étale (resp. pro-étale).

This allows us to define a notion of étale and pro-étale sites. The notion of surjectivity is to be understood as stacks.

Definition 2.3.3. Let X be a locally spatial diamond. Its *étale site*, denoted $X_{\text{ét}}$, is the site whose objects are locally separated étale maps from diamonds $Y' \rightarrow Y$, and coverings are jointly surjective étale maps.

For the étale site, we restrict to locally spatial diamonds as, otherwise, it is unclear whether the site has enough objects.

Definition 2.3.4. Let X be a diamond. Its *quasi-pro-étale site*, denoted $X_{\text{qproét}}$, is the site whose objects are locally separated quasi-pro-étale maps from diamonds $Y' \rightarrow Y$, and coverings are jointly surjective étale maps.

There is a natural comparison map between the two sites. An important technical result is as follows [Sch17, Prop. 14.8] :

Proposition 2.3.5. Let Y be a locally spatial diamond, and $(\mu_*^Y, \mu_Y^*) : Y_{\text{ét}}^\sim \rightarrow Y_{\text{qproét}}^\sim$ denote the associated geometric morphism of topos.

Then, for any sheaf of abelian groups $\mathcal{F}$ on $Y_{\text{ét}}$, $\mathcal{F} \cong \nu_*^Y \nu_Y^* \mathcal{F}$ and, for all $i \geq 1$, $R^i \nu_*^Y \nu_Y^* \mathcal{F} = 0$

In particular, the cohomology of étale sheaves is the same when computed on the étale or pro-étale site.

We finish this paragraph by explaining how to embed analytic adic spaces into locally spatial diamonds, as follows (cf. [Sch17, Def 15.5]).

Definition 2.3.6. Let X be an analytic adic space over $\mathbb{Z}_p$. We let $X^\diamond$ be the locally spatial diamond, whose represented sheaf on Perf is defined as :

$$X^\diamond : Y \in \text{Perf} \mapsto \{(Y^\sharp, \iota), f : Y^\sharp \rightarrow Y\} / \sim,$$

where, for $Y \in \text{Perf}$, the pair $(Y^\sharp, \iota)$ is an *untilt* of Y , i.e. $Y^\sharp$ is a perfectoid ring and ι is an isomorphism : $\iota : (Y^\sharp)^\flat \simeq Y$, and f is a morphism of adic spaces. The equivalence relation $\sim$ denotes isomorphisms between such untilts.

This defines a locally spatial diamond by [Sch17, Lemma 15.6].

Definition 2.3.7. Let X be an analytic adic space. We **define** its *étale* (resp. *pro-étale*) site of an analytic adic space as the étale (resp. quasi-pro-étale) site of its associated diamond $X^\diamond$.

It follows from [Sch17, Lemma 15.6] that this coincides with the more classical notion of étale sites, due to Huber¹⁴. For the pro-étale site, this definition does not coincide with the one of [Sch12a, Defi. 3.9], but it follows from the arguments of the proof of [Bra17, Prop. 2.3.7] that the cohomology groups coincide.

Definition 2.3.8. For a Huber pair (A, A^+) over $\mathbb{Z}_p$, we let $\text{Spd}(A, A^+)$ be the spatial diamond $\text{Spa}(A, A^+)^\diamond$.

By abuse of notation, we also $\text{Spd}(\mathbb{Z}_p)$ be the pro-étale sheaf, mapping a perfectoid space X to the equivalence class of its untilts. It is not a diamond.

Let us now define an étale sheaf $\mathcal{O}^+/p$ on a general locally spatial diamond.

¹⁴It also agrees with the generalization to the non-noetherian setting of [KL15, Defi. 8.2.19].

2.4 The structure sheaf $\mathcal{O}^+/p$ for locally spatial diamonds

For an analytic adic space X , the equivalence of étale sites shows that an object of site $X_{\text{ét}}$ can be viewed as an analytic adic space Y étale over X . Defining $Y \mapsto \mathcal{O}_Y^+(Y)$ defines an étale sheaf on $X_{\text{ét}}$. We let $\mathcal{O}_X^+/p$ be the sheafification of $Y \mapsto \mathcal{O}^+(Y)/p$, that can be pulled back to a sheaf $\nu_X^*(\mathcal{O}_X^+/p)$ on $X_{\text{proét}}$.

In [MW20b, Sec 2], Mann and Werner endow every diamond X over $\text{Spd}(\mathbb{Z}_p)$ with a quasi-pro-étale sheaves, $\hat{\mathcal{O}}^+$, defined as the of the presheaf mapping a perfectoid space Y over X to $\mathcal{O}^+(Y^\sharp)$, where $Y^\sharp$ is the untillt of Y associated to the composition $Y \rightarrow X \rightarrow \text{Spd}(\mathbb{Z}_p)$. We likewise define a diamond over $\text{Spd}(\mathbb{Q}_p)$ with a mod p sheaf $\hat{\mathcal{O}}^+/p$, viewed as the sheafification of $Y \mapsto \mathcal{O}^+(Y^\sharp)/p$ (this is equivalent to the mod p reduction of $\tilde{\mathcal{O}}$).

As is shown in the proof of [MW20b, Lemma 2.8], if X is a locally noetherian analytic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, the restriction $\nu_X^* \hat{\mathcal{O}}^+/p$ to the étale site coincides with the classical étale structure sheaf $\mathcal{O}_X^+/p$ (see also [Zav24a, Sec 6] for a detailed overview of these notions).

In the following, we show that, more generally, if $X \rightarrow \text{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$ is a locally spatial diamond, the pro-étale sheaf $\hat{\mathcal{O}}^+_X/p$ comes from the pullback of an étale sheaf, that we may then reasonably call $\mathcal{O}_X^+/p$.¹⁵

Recall the following lemma :

Lemma 2.4.1. *Let $f : X \rightarrow Y$ be a quasi-pro-étale morphism of locally spatial diamonds, and $\mathcal{F} \in Y_{\text{qproét}}$. Then, the pullback $f^*\mathcal{F}$ is given by the restriction of $\mathcal{F}$ to $X_{\text{qproét}}$, i.e, for any quasi-pro-étale $U \rightarrow Y$, $f^*\mathcal{F}(U) = \mathcal{F}(U \rightarrow X)$, where we consider the quasi-pro-étale composition $U \rightarrow Y \rightarrow X$.*

The same result holds by replacing "quasi-pro-étale" by "étale" everywhere.

Proof. By the general formalism (cf. [Sta23, Section 00VC], or [Sta23, Section 03PZ] in the étale case), the inverse image $f^*\mathcal{F}$ is given by the sheafification of the presheaf :

$$U/X \mapsto \varinjlim_{(V, \varphi) \in (I_U)^{op}} \mathcal{F}(V/Y),$$

where I_U denotes the category of pairs (V, φ) defined by :

- $\text{Ob}(I_U) = \{(V, \varphi) : V \in Y_{\text{qproét}}, \varphi : U \rightarrow V \times_X Y\}$
- $\text{Hom}((V, \varphi), (V', \varphi')) = \{g : V \rightarrow V' \text{ quasi-pro-étale}, g_Y \circ \varphi = \varphi'\}$ where g_Y denotes the base change $V \times_X Y \xrightarrow{g \times_X \text{id}_Y} V' \times_X Y$ of g .

Since f is quasi-pro-étale, we may view any U/X as quasi-pro-étale over Y , so that we may view $(U, \varphi_U : U \rightarrow U \times_X Y)$ as an object of I_U . Moreover, for any $(V, \varphi_V) \in I_U$, there is a natural morphism $g : U \rightarrow V$ given by the composition $U \rightarrow V \times_X Y \rightarrow V$, that is quasi-pro-étale by the two-out-of-three property [Sch17, Prop 10.4] applied to the composition $U \rightarrow V \rightarrow Y$.

It is straightforward to check that this induces a morphism $(U, \varphi_U) \rightarrow (V, \varphi_V)$ in I_U , so that (U, φ_U) is an initial object of I_U , i.e. a final object of $(I_U)^{op}$. Hence, the colimit alongside $(I_U)^{op}$ simply computes $\mathcal{F}(U \rightarrow Y)$, which clearly defines a sheaf, so that there is no need to sheafify. This concludes. $\square$

Proposition 2.4.2. *Let X a locally spatial diamond over $\text{Spd}(\mathbb{Q}_p)$.*

Then, there exists an étale sheaf on $X_{\text{ét}}$, denoted $\mathcal{O}_X^+/p$, such that $\hat{\mathcal{O}}^+_X/p \cong \nu_X^* \mathcal{O}_X^+/p$.

¹⁵This is claimed in the proof of [Sch17, Thm 25.1], but, as far as I know, the proof of this statement does not appear explicitly in the litterature.

Proof. Since this property is Zariski-local over X , we may assume that X is spatial, so that there exists a surjective quasi-pro-étale morphism $f : Y \rightarrow X$ from a strictly totally disconnected perfectoid space, by [Sch17, Pro 11.24].

The construction of $\hat{\mathcal{O}}^+$ is compatible with restrictions, so that, by lemma 2.4.1, $f^*\hat{\mathcal{O}}_X^+$ is simply $\hat{\mathcal{O}}_Y^+$. Since pullbacks commute with small colimits, the same holds for $\hat{\mathcal{O}}^+/p$.

Since Y is perfectoid, it follows from the proof of [MW20b, Lemma 2.7] that the pro-étale sheaf $\hat{\mathcal{O}}_Y^+$ actually comes from the pullback of an étale sheaf, and so does $\hat{\mathcal{O}}_Y^+/p$. Hence, by [Sch17, Thm 14.12], the sheaf $\hat{\mathcal{O}}_X^+/p$ on X comes from the pullback of an étale sheaf, that we denote $\mathcal{O}_X^+/p$. $\square$

We will also use the following compatibility lemma.

Lemma 2.4.3. *Let $f : X \rightarrow Y$ be a quasi-pro-étale morphism of locally spatial diamonds over $\mathrm{Spd}(\mathbb{Q}_p)$.*

Then $\mathcal{O}_Y^+/p \cong f_{\acute{e}t}^* \mathcal{O}_X^+/p$, where $f_{\acute{e}t}^*$ denotes the pullback of étale sheaves along f .

Proof. We start by establishing the commutativity of the following diagram :

$$\begin{array}{ccc} X_{\acute{e}t}^{\sim} & \xleftarrow{f_{\acute{e}t}^*} & Y_{\acute{e}t}^{\sim} \\ \nu_X^* \downarrow & & \downarrow \nu_Y^* \\ X_{qpro\acute{e}t}^{\sim} & \xleftarrow{f_{qpro\acute{e}t}^*} & Y_{qpro\acute{e}t}^{\sim} \end{array}$$

All of the four arrows in this diagram are induced from morphisms of sites, which are themselves induced by continuous functors satisfying the conditions of [Sta23, Proposition 00X6]. Hence, it suffices to consider the following diagram :¹⁶

$$\begin{array}{ccc} X_{\acute{e}t} & \xleftarrow{Z \mapsto Z \times_X Y} & Y_{\acute{e}t} \\ id \downarrow & \swarrow \text{dashed} & \downarrow id \\ X_{qpro\acute{e}t} & \xleftarrow{Z \mapsto Z \times_X Y} & Y_{qpro\acute{e}t} \end{array}$$

Here, the vertical arrows are obtained by viewing any étale cover as a pro-étale one, so that the diagram obviously commutes. It follows from functoriality of morphisms of sites [Sta23, Lemma 03CB] that both compositions $\nu_Y^* \circ f_{\acute{e}t}^*$ and $f_{qpro\acute{e}t}^* \circ \nu_X^*$ are equal to the pullback along the morphism of site induced by the diagonal $X_{\acute{e}t} \rightarrow Y_{qpro\acute{e}t}$, so that the first diagram commutes.

Let us now prove that $f_{\acute{e}t}^* \mathcal{O}_Y^+/p \cong \mathcal{O}_X^+/p$. From the lemma 2.4.1 and the proof of proposition 2.4.2, it follows that $f_{qpro\acute{e}t}^* \hat{\mathcal{O}}_Y^+/p \cong \hat{\mathcal{O}}_X^+/p$. By commutation of the above diagram, we have :

$$\nu_X^* \mathcal{O}_X^+/p \cong \hat{\mathcal{O}}_X^+/p \cong f_{qpro\acute{e}t}^* \nu_Y^* \hat{\mathcal{O}}_Y^+/p \cong \nu_X^* f_{\acute{e}t}^* \mathcal{O}_Y^+/p.$$

Finally, by [Sch17, Prop 14.8], $\nu_*^X \nu_X^* \cong id$, so that $\mathcal{O}_X^+/p \cong f_{\acute{e}t}^* \mathcal{O}_Y^+/p$. $\square$

In what follows, we then identify the étale and pro-étale sheaves $\mathcal{O}_X^+/p$ and $\hat{\mathcal{O}}_X^+/p$ for locally spatial diamond over $\mathrm{Spf}(\mathbb{Q}_p)$. By proposition 2.3.5, so that their cohomology in the étale or pro-étale sites coincide.

¹⁶Here, the arrows are continuous functors, and the associated morphisms of site are contravariant.

2.5 A tiny bit of almost mathematics

The notion of almost mathematics, due to Faltings, has been extensively studied in [GR03]. See also [Zav24a] for an account of the derived version of the theory.

In this thesis, we only encounter almost setups with respect to $\mathfrak{m}_C$. We make the definition only in that setup.

Definition 2.5.1. Let A be an $\mathcal{O}_C$ -module.

We say that A is **almost zero** if, for all pseudo-uniformizer π of C and any $a \in A$, $\pi \cdot a = 0$.

Let $f : A \rightarrow B$ be a morphism of $\mathcal{O}_C$ -modules. We say that f is *almost injective* (resp. *almost surjective*) if $\ker(f)$ (resp. $\operatorname{Coker}(f)$) is almost zero.

We say that f is an *almost isomorphism* if it is both almost injective and almost surjective. When two modules are quasi-isomorphic, we denote $A \stackrel{a}{\simeq} B$.

The following criterion is useful in practice.

Lemma 2.5.2. A morphism $f : A \rightarrow B$ between two $\mathcal{O}_C$ -algebras is an almost isomorphism if and only if $f \otimes_{\mathcal{O}_C} \mathfrak{m}_C : A \otimes_{\mathcal{O}_C} \mathfrak{m}_C \rightarrow B \otimes_{\mathcal{O}_C} \mathfrak{m}_C$ is an isomorphism.

Proof. This is clear. □

For A an $\mathcal{O}_C$ -algebra, we may define a category of *almost* A -modules as the localization of the category of A -modules with respect to quasi-isomorphisms. This defines a functor $(-)^a : A\text{-Mod} \rightarrow A^a\text{-Mod}$, that admits both a left and a right adjoint, denoted respectively $(-)_!$ and $(-)_*$.

We may now state the following crucial result for the computation of $\mathcal{O}^+/p$ -cohomology.

Proposition 2.5.3. Let $X = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space. Then :

$$H_{\text{ét}}^i(X, \mathcal{O}^+/p) \stackrel{a}{\cong} \begin{cases} R^+/p & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$$

where $\stackrel{a}{\cong}$ denotes an almost isomorphism.

Proof. This follows from the analogue statement for $\mathcal{O}^+$ -coefficients [Sch17, Lemma 8.8], together with the long exact sequence associated to the short exact sequence $0 \rightarrow \mathcal{O}_X^+ \xrightarrow{\times p} \mathcal{O}_X^+ \rightarrow \mathcal{O}_X^+/p \rightarrow 0$ of étale sheaves over X . □

The following lemmas will be useful throughout this thesis.

Lemma 2.5.4. Let (R, R^+) be a Huber pair over C , and R° denote the subring of power-bounded elements of R . Then the natural injection $R^+ \hookrightarrow R^\circ$ is an almost isomorphism, with respect to the almost setup associated to the maximal ideal of C .

Proof. The ring R° is the union of all possible rings of definition, so that the injection is well defined. Let $x \in R$, and ϖ be a pseudo-uniformizer of C . We need to show that $\operatorname{Coker}(R^+ \rightarrow R^\circ)$ is killed by ϖ , i.e. $x \cdot \varpi \in R^+$. Clearly, $\varpi \cdot x \in R^{\circ\circ} \subset R^+$.

Since R^+ is an open neighborhood of zero, and $(\varpi \cdot x)^n \rightarrow 0$, there exists n large enough such that $(\varpi \cdot x)^n \in R^+$. Now, R^+ is integrally closed, so that $\varpi \cdot x \in R^+$. $\square$

Lemma 2.5.5. *Let π be a pseudo-uniformizer of $\mathbb{C}$. Let M be an $\mathcal{O}_C/\pi$ -module, and $u_n \in \mathbb{R}_+^*$ be a decreasing sequence converging to zero such that, for all $n \in \mathbb{N}$, $M/u_n M$ is of u_{n+1} -torsion. Then M is almost zero.*

Proof. Let $r \in \log(\Gamma)$. The sequence $(u_n + u_{n+1})$ is decreasing and converging to zero, such that there exists $n \in \mathbb{N}$ such that $(u_n + u_{n+1}) \leq r$.

By hypothesis, $\pi^{u_n + u_{n+1}} \cdot M = 0$, so that $\pi^r M = 0$, which concludes. $\square$

We will need the following Mittag-Leffler condition in the almost category (see also [GR03, Lemma 2.4.2.iii] for a similar result under a weak "almost surjectivity" hypothesis), in order to guarantee the exactness of some projective limits.

Lemma 2.5.6. *Let M_n be a sequence of almost finitely presented $\mathcal{O}/p^+$ -modules, with $f_n : M_{n+1} \rightarrow M_n$.*

Assume that the natural inclusion $\text{Im}(f_n) \subset \bigcap_{p \in \mathbb{N}} \text{Im}(f_{n+p} \circ \cdots \circ f_n)$ is an almost isomorphism.

Then $R^1 \varprojlim_{n \in \mathbb{N}} M_n$ is almost zero.

Proof. By definition, for all r , $\pi^r \cdot R^1 \varprojlim_{n \in \mathbb{N}} M_n = R^1 \varprojlim_{n \in \mathbb{N}} (\pi^r \cdot M_n)$. The family $(\pi^r \cdot M_n)$ satisfies the usual Mittag-Leffler condition (i.e. the family $\pi^r \cdot \text{Im}(f_{n+p} \circ \cdots \circ f_n)_{p \geq 0}$ stabilizes for all n), so that, by [Sta23, Lemma 0598], the $R^1 \varprojlim$ vanishes. $\square$

We are now finally equipped to compute the cohomology of the unit disk. We follow the method laid out in the introduction, by constructing an explicit perfectoid cover.

2.6 The cohomology of the unit disk

Recall that C is a complete and algebraically closed extension of $\mathbb{Q}_p$. The goal of this paragraph is to establish the following :

Proposition 2.6.1. *For $r, n \in \log(\Gamma)$, consider the inclusion of $\mathbb{D}_C(|p|^{n+r}) \subset \mathbb{D}_C(|p|^n)$, which induces a map in $\mathcal{O}^+/p$ -cohomology. We have the following :*

- *If $r \geq p/(p-1)$, the image of $H_{\text{ét}}^0(\mathbb{D}_C(|p|^n), \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^0(\mathbb{D}_C(|p|^{n+r}), \mathcal{O}^+/p)$ is almost isomorphic to $\mathcal{O}_C/p$.*
- *If $r \geq 1$, the image of $H_{\text{ét}}^1(\mathbb{D}_C(|p|^n), \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^1(\mathbb{D}_C(|p|^{n+r}), \mathcal{O}^+/p)$ is almost zero.*
- *For all r , if $t \geq 2$, $H_{\text{ét}}^t(\mathbb{D}_C(|p|^r), \mathcal{O}^+/p)$ is almost zero.*

By [remark 2.1.17](#), up to scaling everything, we may assume that $n = 0$.

2.6.1 The geometric picture

Let us fix a *framing*, i.e. an embedding of the closed unit disk into the torus¹⁷ $\mathbb{T}_C$:

$$\mathbb{D}_C = \text{Spa}(C\langle T \rangle, \mathcal{O}_C\langle T \rangle) \hookrightarrow \mathbb{T}_C = \text{Spa}(C\langle X^{\pm 1} \rangle, \mathcal{O}_C\langle X^{\pm 1} \rangle)$$

¹⁷This choice of embedding already appears in [Heu21, Lemma 6.10].

induced by $X \mapsto 1 + pT$ (and $X^{-1} \mapsto \frac{1}{1 + pT} = \sum_{n \in \mathbb{N}} (-1)^n p^n T^n \in \mathcal{O}_C \langle T \rangle$).

Recall that the torus admits a natural pro-étale surjection from the so-called *perfectoid torus*, $\tilde{\mathbb{T}}_C := \mathrm{Spa}(C \langle T^{\pm 1/p^\infty} \rangle, \mathcal{O}_C \langle T^{\pm 1/p^\infty} \rangle)$. Consider the cartesian square :

$$\begin{array}{ccc} \tilde{\mathbb{D}}_C & \longrightarrow & \tilde{\mathbb{T}}_C \\ \downarrow & & \downarrow \\ \mathbb{D}_C & \longrightarrow & \mathbb{T}_C. \end{array}$$

The morphism $\tilde{\mathbb{D}}_C \rightarrow \mathbb{D}_C$ is a pro-étale covering (as it is the base change of a pro-étale covering). The associated Čech-to-cohomology spectral sequence yields :

$$E_2^{s,t} = H_{\mathrm{pro\acute{e}t}}^s \left(\underbrace{\tilde{\mathbb{D}}_C \times_{\mathbb{D}_C} \cdots \times_{\mathbb{D}_C} \tilde{\mathbb{D}}_C}_{t \text{ times}}, \mathcal{O}^+/p \right) \implies H_{\mathrm{pro\acute{e}t}}^{s+t}(X, \mathcal{O}^+/p).$$

By [proposition 2.5.3](#), pro-étale cohomology of $\mathcal{O}^+/p$ coincides with étale $\mathcal{O}^+/p$ -cohomology. Let us make explicit the spaces appearing on the left hand side.

Fix a compatible system of p^n -th roots of unity : $(\zeta_p, \zeta_{p^2}, \dots)$ in C .

We consider the continuous action of the topological group $\mathbb{Z}_p$ acting on $\mathcal{O}_C \langle T^{\pm 1/p^\infty} \rangle$, such that its topological generator $\gamma = 1$ acts via $\gamma \cdot T^{1/p^n} = \zeta_{p^n} \cdot T^{1/p^n}$. Its fixed points are exactly $\mathcal{O}_C \langle T^{\pm 1} \rangle$, so that $\tilde{\mathbb{T}}_C/\mathbb{Z}_p \cong \mathbb{T}_C$.

It then follows from [\[Sch17, Prop 11.3.\(ii\)\]](#) that the morphism $\tilde{\mathbb{T}}_C \rightarrow \mathbb{T}_C$ is a *Galois cover* of group $\mathbb{Z}_p$, i.e. $\tilde{\mathbb{T}}_C \times_{\mathbb{T}_C} \tilde{\mathbb{T}}_C \cong \tilde{\mathbb{T}}_C \times \mathbb{Z}_p$. Hence :

$$\begin{aligned} \tilde{\mathbb{D}}_C \times_{\mathbb{D}_C} \tilde{\mathbb{D}}_C &\cong (\tilde{\mathbb{T}}_C \times_{\mathbb{T}_C} \mathbb{D}_C) \times_{\mathbb{D}_C} (\tilde{\mathbb{T}}_C \times_{\mathbb{T}_C} \mathbb{D}_C) \\ &\cong \mathbb{D}_C \times_{\mathbb{T}_C} (\tilde{\mathbb{T}}_C \times_{\mathbb{T}_C} \tilde{\mathbb{T}}_C) \\ &\cong \tilde{\mathbb{D}}_C \times \mathbb{Z}_p. \end{aligned}$$

This leads to the following :

Lemma 2.6.2. *There is a spectral sequence :*

$$E_2^{s,t} = H_{\mathrm{cont}}^s \left(\mathbb{Z}_p, H_{\acute{e}t}^t \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \right) \implies H_{\acute{e}t}^{s+t}(\mathbb{D}_C, \mathcal{O}^+/p),$$

where H^* denotes continuous group cohomology (defined as the cohomology of the complex of continuous cochains, cf. [\[Ser97, Sec 2.2\]](#)). Here $H_{\acute{e}t}^t(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p)$ is endowed with the discrete topology.

Proof. For all $s \in \mathbb{N}^*$, iterating the above computation yields :

$$\underbrace{\tilde{\mathbb{D}}_C \times_{\mathbb{D}_C} \cdots \times_{\mathbb{D}_C} \tilde{\mathbb{D}}_C}_{s \text{ times}} \cong \tilde{\mathbb{D}}_C \times \mathbb{Z}_p^{s-1}$$

We write $\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n \mathbb{Z}$, so that $\tilde{\mathbb{D}}_C \times \mathbb{Z}_p^{s-1} \cong \varprojlim_{n \in \mathbb{N}} \tilde{\mathbb{T}}_C \times (\mathbb{Z}/p^n \mathbb{Z})^{p-1}$ (since fiber products commute with limits). It follows that :

$$\begin{aligned} H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C \times \mathbb{Z}_p^{s-1}, \mathcal{O}^+/p \right) &\cong \varinjlim_{n \in \mathbb{N}} H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C \times (\mathbb{Z}/p^n \mathbb{Z})^{s-1}, \mathcal{O}^+/p \right) \\ &\cong \varinjlim_{n \in \mathbb{N}} \bigoplus_{(\mathbb{Z}/p^n \mathbb{Z})^{s-1}} H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \\ &\cong \varinjlim_{n \in \mathbb{N}} \text{Cont} \left((\mathbb{Z}/p^n \mathbb{Z})^{s-1}, H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \right) \\ &\cong \text{Cont} \left((\mathbb{Z}_p)^{s-1}, H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \right). \end{aligned}$$

Where we used [Sch17, Prop 14.9], the commutation of étale cohomology with colimits, and lemma B.1.10, respectively.

Hence, the Čech complex of equation (2.6.1) is the complex computing the continuous group cohomology $H_{\text{cont}}^* \left(\mathbb{Z}_p, H_{\text{ét}}^t \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \right)$. This concludes the proof. $\square$

It will then suffice to understand the cohomology groups of $\tilde{\mathbb{D}}_C$, endowed with the $\mathbb{Z}_p$ -action. The following lemma shows that it suffices to do up to an equivariant almost isomorphism.

Lemma 2.6.3. *Let G be a profinite topological group $f : M \rightarrow N$ be a G -equivariant almost isomorphism of discrete $\mathcal{O}_C$ -modules.*

For all $n \in \mathbb{N}$, f induces an almost isomorphism $H_{\text{cont}}^n(G, M) \xrightarrow{a} H_{\text{cont}}^n(G, N)$.

Proof. Composition with f induces a map $\text{Cont}(G^\bullet, M) \rightarrow \text{Cont}(G^\bullet, N)$ at the level of complexes, and we need to show that its kernel and cokernel are almost zero. Since the functor $\text{Cont}(G^n, -)$ is exact on discrete modules (this is clear if G is finite, and exactness commutes with direct limits), it suffices to check that, if M is an almost zero $\mathcal{O}_C$ -module, $\text{Cont}(G^n, M)$ is almost zero as an $\mathcal{O}_C$ -module.

This is clear, as, for all $r \in \log(\Gamma)$ and $f : G^n \rightarrow M$, $(\pi^r f)(-) = \pi^r \cdot f(-) = 0$. $\square$

2.6.2 Reduction to an explicit complex

We may now deduce the following.

Corollary 2.6.4. *$H_{\text{ét}}^*(\mathbb{D}_C, \mathcal{O}^+/p)$ is computed, up to an almost isomorphism, as the cohomology of the complex*

$$H_{\text{cont}}^\bullet \left(\mathbb{Z}_p, H_{\text{ét}}^0 \left(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p \right) \right).$$

Proof. It follows from lemma 2.6.3 that we only need to identify $H_{\text{ét}}^t(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p)$ up to an (equivariant) almost isomorphism. Since $\tilde{\mathbb{D}}_C$ is affinoid perfectoid, it follows from proposition 2.5.3 that its étale cohomology is almost trivial in degree $t \geq 1$.

The spectral sequence from lemma 2.6.2 passes to the almost category, in which it degenerates at page 2, admitting only one nonzero line. This concludes. $\square$

Let us write $\tilde{\mathbb{D}}_C = \text{Spa}(R, R^+)$, where (cf. [Hub96, Prop 1.2.2.(a)]) :

$$R = C \left\langle X^{1/p^\infty} \right\rangle \otimes_{C\langle X^{\pm 1} \rangle} C \left\langle \frac{X-1}{p} \right\rangle \cong C \left\langle X^{1/p^\infty} \right\rangle \left\langle \frac{X-1}{p} \right\rangle.$$

It follows from [Heu24, Rk 1.2.9] that $R^+ = R^\circ$.

Remark 2.6.5. Note that one can adapt what we have done so far if $\mathbb{D}_C$ is replaced by $X = \mathrm{Spa}(C \langle T \rangle, A^+)$ for any ring of definition A^+ .

Indeed, we may consider the same framing, and it follows from the description of fiber products of adic spaces (cf. [Hub96, Prop 1.2.2.a]) that the fiber product $X \times_{\mathbb{T}_C} \tilde{\mathbb{T}}_C$ is of the form $\mathrm{Spa}(R, R')$ for R as above, and some ring of definition R' of R .

It moreover then follows from lemma 2.5.4 and lemma 2.6.3 that

$$H_{\acute{e}t}^*(\mathrm{Spa}(C \langle T \rangle, A^+), \mathcal{O}^+/p) \stackrel{a}{\cong} H_{\acute{e}t}^*(\mathbb{D}_C, \mathcal{O}^+/p).$$

It follows from proposition 2.5.3 that $H_{\acute{e}t}^0(\tilde{\mathbb{D}}_C, \mathcal{O}^+/p) \stackrel{a}{\cong} R^+/p$, so that it suffices to determine R^+ , and understand the associated $\mathbb{Z}_p$ -action. These computations can be done after tilting, since, by definition, $R^+/p \cong (R^+)^{\flat}/p^{\flat}$.

The following is an integral variant of [Heu21, Lemma 6.10] :

Lemma 2.6.6. There is an isomorphism :¹⁸

$$\tilde{\mathbb{D}}_C^{\flat} \simeq \mathrm{Spa} \left(C^{\flat} \left\langle Z^{1/p^{\infty}} \right\rangle, \mathcal{O}_C^{\flat} \left\langle Z^{1/p^{\infty}} \right\rangle \right),$$

where Z is a formal variable corresponding to $\frac{X^{\flat} - 1}{p^{\flat}}$.

Here, $X^{\flat} = (X, X^{1/p}, X^{1/p^2}, \dots) \in R^{\flat}$, and $p^{\flat} = (p, p^{1/p}, p^{1/p^2}, \dots)$ is a pseudo-uniformizer of $C^{\flat}$. Moreover, $\tilde{\mathbb{D}}_C$ admits a natural $\mathbb{Z}_p$ -action, induced by

$$\gamma \cdot Z^{1/p^n} = \frac{\varepsilon^{1/p^n} (X^{\flat})^{1/p^n} - 1}{(p^{\flat})^{1/p^n}} = \varepsilon Z^{1/p^n} + \frac{\varepsilon^{1/p^n} - 1}{(p^{\flat})^{1/p^n}},$$

where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in C^{\flat}$.

Proof. $\tilde{\mathbb{D}}_C$ corresponds to the rational open subset $U \left(\frac{X-1}{p} \right)$ of the perfectoid torus. Hence, by [Sch12b, Lemma 6.4], $\tilde{\mathbb{D}}_C^{\flat}$ is the rational open subset $U \left(\frac{Y-1}{p^{\flat}} \right)$ of the tilted perfectoid torus :

$$\tilde{\mathbb{T}}_C^{\flat} = \mathrm{Spa} \left(C^{\flat} \left\langle Y^{\pm 1/p^{\infty}} \right\rangle, \mathcal{O}_C^{\flat} \left\langle Y^{\pm 1/p^{\infty}} \right\rangle \right),$$

where the formal variable Y corresponds to the tilt of X .

In particular, the $\mathbb{Z}_p$ -action is given by $\gamma \cdot Y^{1/p^n} = \varepsilon^{1/p^n} Y^{1/p^n}$.

Write $\tilde{\mathbb{D}}_C^{\flat} \cong \mathrm{Spa}(S, S^+)$. Since $\tilde{\mathbb{D}}_C$ is of the form $\mathrm{Spa}(R, R^\circ)$, it follows from [SW20, Lemma 6.2.2] that $S^+ = (R^\circ)^{\flat} = (R^{\flat})^\circ = S^\circ$. By [Sch12b, Lemma 6.4] :¹⁹

$$S \cong C^{\flat} \left\langle Y^{\pm 1/p^{\infty}} \right\rangle \left\langle \frac{Y-1}{p^{\flat}} \right\rangle.$$

¹⁸This gives an untilted version $\mathbb{D}_C \cong \mathrm{Spa} \left(C \left\langle U^{1/p^{\infty}} \right\rangle, \mathcal{O}_C \left\langle U^{1/p^{\infty}} \right\rangle \right)$; although it is unclear how to express U as a function of the coordinate T of the disk. In particular, the natural morphism $\tilde{\mathbb{D}}_C \rightarrow \mathbb{D}_C$ is not induced by $T \mapsto U$, as such a morphism is not pro-étale at the origin.

¹⁹This lemma also gives $S^\circ \stackrel{a}{\cong} \mathcal{O}_C^{\flat} \left\langle Y^{\pm 1/p^{\infty}} \right\rangle \left\langle \left(\frac{Y-1}{p^{\flat}} \right)^{1/p^{\infty}} \right\rangle$. The method shows here upgrades this almost isomorphism to an isomorphism.

There is a natural almost isomorphism $S \xrightarrow{a} \mathcal{O}^b \langle Z^{1/p^\infty} \rangle$, sending Z to $\frac{Y-1}{p^b}$. This naturally maps S° to $\mathcal{O}_C^b \langle Z^{1/p^\infty} \rangle$, hence the result, and the description of the action. $\square$

Lemma 2.6.7. *The Koszul²⁰ complex, viewed in cohomological degrees 0 and 1*

$$\left[\mathcal{O}_C^b / p^b \left[Z^{1/p^\infty} \right] \xrightarrow{\gamma-1} \mathcal{O}_C^b / p^b \left[Z^{1/p^\infty} \right] \right]$$

computes $H_{\text{ét}}^\bullet(\mathbb{D}_C, \mathcal{O}^+ / p)$, up to almost isomorphism.

$$\text{Explicitly, } H_{\text{ét}}^n(\mathbb{D}_C, \mathcal{O}^+ / p) \xrightarrow{a} \begin{cases} \text{Ker}(\gamma - 1) & \text{if } n = 0 \\ \text{Coker}(\gamma - 1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Proof. Recall the complex of **corollary 2.6.4**. The space $\widetilde{\mathbb{D}}_C = \text{Spa}(R, R^\circ)$ is affinoid perfectoid, with tilt $\widetilde{\mathbb{D}}_C^b = \text{Spa}(R^b, R^{\circ,b})$, which satisfies $R^\circ / p = R^{\circ,b} / p^b$, so that $H_{\text{ét}}^0(\widetilde{\mathbb{D}}_C, \mathcal{O}^+ / p) \xrightarrow{a} H_{\text{ét}}^0(\widetilde{\mathbb{D}}_C^b, \mathcal{O}^+ / p^b)$ by **proposition 2.5.3**.

By **lemma 2.6.6** and **lemma 2.6.3**, we have :

$$\begin{aligned} H_{\text{ét}}^0(\widetilde{\mathbb{D}}_C, \mathcal{O}^+ / p) &\xrightarrow{a} R^\circ / p \left(\mathcal{O}^+ \left(\text{Spa} \left(C^b \langle Z^{1/p^\infty} \rangle, \mathcal{O}_C^b \langle Z^{1/p^\infty} \rangle \right) \right) \right) / p^b \\ &\xrightarrow{a} \mathcal{O}_C^b \langle Z^{1/p^\infty} \rangle / p^b \\ &\xrightarrow{a} \mathcal{O}_C^b / p^b \left[Z^{1/p^\infty} \right]. \end{aligned}$$

Since $M = \mathcal{O}_C^b / p^b \left[Z^{1/p^\infty} \right]$ is discrete, it follows from [BMS16, Lemma 7.3] that the continuous group cohomology coincides²¹ with the cohomology of the discrete group $\mathbb{Z}$. For clarity, we let $G = \mathbb{Z}$, so that there is an isomorphism $\mathbb{Z}[G] \cong \mathbb{Z}[t^{\pm 1}]$ with the Laurent ring of polynomials.

Consider the acyclic resolution $\mathbb{Z}[t^{\pm 1}] \xrightarrow{-\times(t-1)} \mathbb{Z}[t^{\pm 1}]$ of $\mathbb{Z}$ as a $\mathbb{Z}[G]$ -module. Recall that group cohomology can be interpreted as the derived functor of the functor

$$N \in \{G\text{-Mod}\} \mapsto M^G \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Applying $\text{Hom}_{\mathbb{Z}[G]}(-, M)$ to the above resolution yields the desired complex. $\square$

Let us make the action of $(\gamma - 1)$ fully explicit.

Lemma 2.6.8. *The action of $\gamma - 1$ on $\mathcal{O}_C^b / p^b [Z^{1/p^\infty}]$ is given, for $k \in \mathbb{N}^*$ prime to p and $n \in \mathbb{Z}$, by :*

$$(\gamma - 1) \cdot Z^{k/p^n} = \begin{cases} 0 & \text{if } n < 0 \\ \left((\varepsilon Z)^{1/p^n} + \frac{\varepsilon^{1/p^n} - 1}{(p^b)^{1/p^n}} \right)^k - Z^{k/p^n} & \text{if } n \geq 0 \end{cases}$$

Proof. Recall that Z corresponds to $\frac{X^b - 1}{p^b}$, so that Z^{1/p^n} corresponds to $\frac{(X^b)^{1/p^n} - 1}{(p^b)^{1/p^n}}$.

²⁰In this setup, one does not need the theory of Koszul complexes, as this is simply the complex of [Bro82, eq. III.I.4.5]. For computation in higher dimension, when X embeds in some $\mathbb{T}_C^n$, the natural construction is to use a Koszul complex (cf. [BMS16, Lemma 7.3]), hence the terminology used here.

²¹This also holds for modules that can be written as $\varprojlim_{n \in \mathbb{N}} M_n$ for discrete p^n -torsion modules M_n .

Moreover, $\gamma \cdot (X^b)^{1/p^n} = \varepsilon^{1/p^n} (X^b)^{1/p^n}$.

By definition, γ is a morphism of algebras, so that :

$$\begin{aligned} \gamma \cdot Z^{k/p^n} &= \left(\gamma \cdot Z^{1/p^n} \right)^k \\ &= \left(\gamma \cdot \frac{(X^b)^{1/p^n} - 1}{(p^b)^{1/p^n}} \right)^k \\ &= \left(\frac{\varepsilon^{1/p^n} (X^b)^{1/p^n} - 1}{(p^b)^{1/p^n}} \right)^k \\ &= \left(\frac{\varepsilon^{1/p^n} [(p^b)^{1/p^n} Z^{1/p^n} + 1] - 1}{(p^b)^{1/p^n}} \right)^k \\ &= \left(\varepsilon^{1/p^n} Z^{1/p^n} + \frac{\varepsilon^{1/p^n} - 1}{(p^b)^{1/p^n}} \right)^k. \end{aligned}$$

For $n < 0$, the following lemma shows that both $\varepsilon^{1/p^n} - 1$ and $\frac{\varepsilon^{1/p^n} - 1}{(p^b)^{1/p^n}}$ are divisible by p^b , hence zero in $\mathcal{O}_C^b/p^b[Z^{1/p^\infty}]$. $\square$

Lemma 2.6.9. For all $n \geq 1$ and k prime to p , $v_p(\zeta_{p^n}^k - 1) = \frac{1}{p^{n-1} \cdot (p-1)}$.

Moreover, for all $n \in \mathbb{Z}$ and k prime to p ,

$$v_{p^b}(\varepsilon^{k/p^n} - 1) = \frac{1}{p^{n-1}(p-1)} \text{ and } v_{p^b}\left(\frac{\varepsilon^{k/p^n} - 1}{(p^b)^{k/p^n}}\right) = \frac{1}{p^n(p-1)}.$$

Proof. For any $n \geq 1$, let :

$$P_n(X) = \prod_{1 \leq k \leq p^n-1} (X - \zeta_{p^n}^k) = \frac{X^{p^n} - 1}{X - 1} = \sum_{r=0}^{p^n-1} X^r.$$

Hence, $n = P_n(1) = \prod_{k=1}^{p^n-1} (1 - \zeta_{p^n}^k)$. We prove the first point by induction on n .

For $n = 1$, this yields $1 = \sum_{k=1}^{p-1} v_p(1 - \zeta_p^k)$, where all the terms of the sum admit the same p -adic valuation, as they are Galois conjugated, which yields the result.

For $n \geq 2$, we write : $P_n(X) = \prod_{d=1}^n \Phi_{p^d}(X)$, where $\Phi_{p^d}(X) = \prod_{1 \leq k \leq p^d, p \nmid k} (X - \zeta_{p^d}^k)$ is the cyclotomic polynomial of order p^d . Note that $\Phi_{p^d}(X)$ is irreducible, so that all its roots are conjugated, hence admit the same valuation. We compute :

$$\begin{aligned} n = v_p(P_n(1)) &= \sum_{d=1}^n \varphi(p^d) \cdot v_p(\zeta_{p^d} - 1) \\ &= \varphi(p^n) \cdot v_p(\zeta_{p^n} - 1) + \sum_{d=1}^{n-1} \frac{1}{p^{d-1}(p-1)} \varphi(p^d). \end{aligned}$$

Where $\varphi(n) = \#\{m \in \mathbb{N}, \gcd(m, n) = 1\}$ denotes Euler's totient function. It is well known that, for all $k \in \mathbb{N}^*$, $\varphi(p^k) = p^{k-1}(p-1)$, since p is prime. This gives the first point.

Moreover, by the definition of tilting : $(\varepsilon^{k/p^n} - 1) = (x_0, x_1, x_2, \dots)$ where :

$$x_l = \lim_{r \rightarrow \infty} \left(\zeta_{p^{r+l}}^{k/p^n} - 1 \right)^{p^r} = \lim_{r \rightarrow \infty} \left(\zeta_{p^{r+l+n}}^k - 1 \right)^{p^r}.$$

All terms appearing in the limit are of valuation $\frac{1}{p^{k+n-1}(p-1)}$, so that the same holds for the limit. Recall that $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots)$, which yields the result. $\square$

From this description, it becomes clear that these cohomology groups are hard to compute ; in particular, both $\text{Ker}(\gamma - 1)$ and $\text{Coker}(\gamma - 1)$ are infinitely dimensional over $\mathcal{O}_C/p$.

Remark 2.6.10. Both $\text{Ker}(\gamma - 1)$ and $\text{Coker}(\gamma - 1)$ are **not** free as $\mathcal{O}_C/p$ -modules.

Indeed, $(p^\flat)^{1-(p-1)} \cdot Z \in \text{Ker}(\gamma - 1)$ is of $(p^\flat)^{1/(p-1)}$ -torsion, but not divisible by any positive power of $p^\flat$. Likewise, Z is of $(p^\flat)^{1/(p-1)}$ -torsion in $\text{Coker}(\gamma - 1)$, but not divisible.

2.6.3 Restriction to smaller disks and H^0

In what follows, we compute the image of morphisms induced via restriction to smaller disks, for $r \in \log(\Gamma) \cap \mathbb{R}_+^*$:

$$H_{\text{ét}}^n(\mathbb{D}_C, \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^n(\mathbb{D}_C(|p|^r), \mathcal{O}^+/p)$$

According to [Sch12a, Lemma 5.6], this image is *almost* finitely presented over $\mathcal{O}_C/p$.

Recall that the inclusion of the rational subset $\mathbb{D}_C(|p|^r) \subset \mathbb{D}_C$ defined in [example 2.1.16](#), corresponds to the inclusion of the rational subset $|T| \leq |p|^r$.

Consider the diagram with cartesian squares :

$$\begin{array}{ccccc} \tilde{\mathbb{D}}_C(|p|^r) & \longrightarrow & \tilde{\mathbb{D}}_C & \longrightarrow & \tilde{\mathbb{T}}_C \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}_C(|p|^r) & \longrightarrow & \mathbb{D}_C & \longrightarrow & \mathbb{T}_C. \end{array}$$

The inclusion $\tilde{\mathbb{D}}_C(|p|^r) \subset \tilde{\mathbb{D}}_C$ corresponds to the morphism

$$C \left\langle Y'^{\pm 1/p^\infty} \right\rangle \left\langle \left(\frac{Y-1}{p} \right)^{1/p^\infty} \right\rangle \rightarrow C \left\langle Y'^{\pm 1/p^\infty} \right\rangle \left\langle \left(\frac{Y'-1}{p^r} \right)^{1/p^\infty} \right\rangle$$

sending $Y^{\pm 1/p^n}$ to $Y'^{\pm 1/p^n}$ and $\left(\frac{Y-1}{p} \right)^{1/p^n}$ to $p^{r/p^n} \cdot \left(\frac{Y'-1}{p^r} \right)^{1/p^n}$.

There is an isomorphism $\tilde{\mathbb{D}}_C(|p|^r) \simeq \tilde{\mathbb{D}}_C$ sending $\frac{Y'-1}{p^r}$ to $\frac{Y-1}{p}$, as in [remark 2.1.17](#), through this identification, the inclusion $\tilde{\mathbb{D}}_C^b(|p|^r) \rightarrow \tilde{\mathbb{D}}_C^b$ corresponds to the endomorphism of $\tilde{\mathbb{D}}_C^b$ given by :

$$\Phi_r : \begin{cases} \mathcal{O}_C^b \left\langle Z^{1/p^\infty} \right\rangle \rightarrow \mathcal{O}_C^b \left\langle Z^{1/p^\infty} \right\rangle \\ Z^{1/p^n} \mapsto (p^\flat)^{r/p^n} Z^{1/p^n} \end{cases}$$

Note that the formation of the Koszul complex can clearly made to be functorial, so that, using [lemma 2.6.7](#), the result [proposition 2.6.1](#) reduces to the following.

Proposition 2.6.11. Consider the diagram :

$$\begin{array}{ccc}
\mathcal{O}_C^b/p^b [Z^{1/p^\infty}] & \xrightarrow{\Psi_r} & \mathcal{O}_C^b/p^b [Z^{1/p^\infty}] \\
\downarrow \gamma-1 & & \downarrow \gamma-1 \\
\mathcal{O}_C^b/p^b [Z^{1/p^\infty}] & \xrightarrow{\Psi_r} & \mathcal{O}_C^b/p^b [Z^{1/p^\infty}],
\end{array}$$

Where Ψ_r is the mod p^b reduction of Φ_r . Then :

1. Assume that $r \geq p/(p-1)$. The image of the induced $\Psi_r : \text{Ker}(\gamma-1) \rightarrow \text{Ker}(\gamma-1)$ is isomorphic to $\mathcal{O}_C^b/p^b$.
2. Assume that $r \geq 1$. The image of the induced $\Psi_r : \text{Coker}(\gamma-1) \rightarrow \text{Coker}(\gamma-1)$ is almost zero.

Let us start by the first point.

Lemma 2.6.12. Assume that $r \geq p/(p-1)$. The image of $\Psi_r : \text{Ker}(\gamma-1) \rightarrow \text{Ker}(\gamma-1)$ is $\mathcal{O}_C^b/p^b$.

Proof. Let $F = \sum_{k,n} a_{k,n} Z^{k/p^n} \in \text{Ker}(\gamma-1)$, and $\Psi_r(F) = \sum_{k,n} b_{k,n} Z^{k/p^n}$, where $b_{k,n} = (p^b)^{kr/p^n} a_{k,n}$. We will show that all non-constant terms of $\Psi_r(F)$ are zero.

Assume by contradiction that $\Psi_r(F)$ admits non-constant terms. Let $n_0 \geq 1$ be maximal such that there exists some k prime to p such that $b_{k,n}$ is nonzero mod p^b . Let $k_0 \geq 1$ be maximal among such k .

By lemma 2.6.8, for all (k, n) , there exists $c_{k,i,n}$ such that :

$$(\gamma-1)(a_{k,n} Z^{k/p^n}) = a_{k,n} \cdot (\varepsilon^{1/p^n} - 1) Z^{k/p^n} + \sum_{i=0}^{n-1} c_{k,i,n} Z^{i/p^n}$$

By maximality of (n_0, k_0) , the coefficient of $Z^{k_0/p^{n_0}}$ in $(\gamma-1)F$ is exactly $a_{k_0,n_0} \cdot (\varepsilon^{1/p_0^{n_0}} - 1)$, which is assumed to be zero mod p^b .

By lemma 2.6.9, it follows that $v_{p^b}(a_{k_0,n_0}) \geq 1 - \frac{1}{p^{\varphi(n_0)}}$.

Hence, $v_{p^b}(b_{k_0,n_0}) = v_{p^b}(a_{k_0,n_0}) + \frac{r \cdot k_0}{p^{n_0}} \geq 1 - \frac{1}{p^{\varphi(n_0)}} + \frac{r}{p^{n_0}} \geq 1 - \frac{1}{p^{n_0-1}(p-1)} + \frac{r}{p^{n_0}}$.

Since $r \geq p/(p-1)$, it follows that $\frac{r}{p^{n_0}} \geq \frac{1}{p^{n_0-1}(p-1)}$, so that b_{k_0,n_0} is zero modulo p^b . This contradicts our hypothesis. Hence, $\Psi_r(F)$ is constant.

Moreover, Ψ_r restrict to the identity on constant functions, so that it is surjective on $\mathcal{O}_C^b/p^b$. $\square$

2.6.4 Vanishing of the H^1

We now consider the second point of proposition 2.6.11. We start with the following lemma.

Corollary 2.6.13. For all $m \leq n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, there exists $G_{k,n,m} \in \mathcal{O}_C^b/p^b [Z^{1/p^\infty}]$ and $c_{k,n,m} \in \mathcal{O}_C^b/p^b$ such that :

$$(p^b)^{1/\varphi(p^m)} \cdot (p^b Z)^{k/p^n} = (\gamma-1)(G_{k,n,m}) + c_{k,n,m}$$

And $v_{p^b}(c_{k,n,m}) \geq \frac{1}{\varphi(p^m)}$.

Proof. Fixing $m \leq n \in \mathbb{N}^*$, we proceed by induction on k . In order to simplify the notations, we let $\mu = \varepsilon - 1$, so that $\mu^{1/p^n} = \varepsilon^{1/p^n} - 1$ for all $n \geq 1$.

The computation of [lemma 2.6.8](#) with $k = 1$ shows that :

$$(\gamma - 1) \cdot Z^{1/p^n} = \mu^{1/p^n} \cdot Z^{1/p^n} + \frac{\mu^{1/p^n}}{(p^b)^{1/p^n}}.$$

Multiplying everything by $\frac{(p^b)^{1/\varphi(p^m)}}{\mu^{1/p^n}}$ (of valuation $\frac{1}{\varphi(p^m)} - \frac{1}{\varphi(p^n)} \geq 0$) yields :

$$(p^b)^{1/\varphi(p^m)} Z^{1/p^n} = (\gamma - 1) \cdot \left(\frac{(p^b)^{1/\varphi(p^m)}}{\mu^{1/p^n}} Z^{1/p^n} \right) - (p^b)^{1/\varphi(p^m)-1/p^n}. \quad (2.1)$$

Multiplying everything further by $(p^b)^{1/p^n}$ yields the result for $k = 1$.

Let $k \geq 1$, and assume that the result holds for all $1 \leq k' \leq k - 1$. If k is divisible by p , this reduces to a result for lower m, n , so that we may assume $p \nmid k$.

For simplicity, we let $C = \frac{\varepsilon^{k/p^n} - 1}{\varepsilon^{1/p^n} - 1}$ and $D = \frac{\varepsilon^{1/p^m} - 1}{(p^b)^{1/\varphi(p^m)}}$. Both are invertible.

The computation of [lemma 2.6.8](#) shows that :

$$\begin{aligned} (\gamma - 1) \cdot Z^{k/p^n} &= \left((\varepsilon Z)^{1/p^n} + \frac{\mu^{1/p^n}}{(p^b)^{1/p^n}} \right)^k - Z^{k/p^n} \\ &= (\varepsilon^{k/p^n} - 1) Z^{k/p^n} + \sum_{i=0}^{k-1} \binom{k}{i} (\varepsilon Z)^{i/p^n} \left(\frac{\mu^{1/p^n}}{(p^b)^{1/p^n}} \right)^{k-i}. \end{aligned}$$

Multiplying everything by $(p^b)^{k/p^n} \cdot \frac{\mu^{1/p^m}}{\mu^{1/p^n}}$ yields :

$$\begin{aligned} C \cdot D \cdot (p^b)^{1/\varphi(p^m)} \cdot (p^b)^{k/p^n} \\ = (\gamma - 1) \cdot \left((p^b)^{k/p^n} \cdot \frac{\mu^{1/p^m}}{\mu^{1/p^n}} Z^{k/p^n} \right) - \frac{\mu^{1/p^m}}{\mu^{1/p^n}} \sum_{i=0}^{k-1} \binom{k}{i} (\varepsilon p^b Z)^{i/p^n} (\mu^{1/p^n})^{k-i}, \end{aligned}$$

where the last term rewrites as :

$$\begin{aligned} \frac{\mu^{1/p^m}}{\mu^{1/p^n}} \sum_{i=0}^{k-1} \binom{k}{i} (\varepsilon p^b Z)^{i/p^n} (\mu^{1/p^n})^{k-i} \\ = (\mu^{1/p^m}) (\mu^{1/p^n})^{k-1} + D \cdot \sum_{i=1}^{k-1} \binom{k}{i} \varepsilon^{i/p^n} \cdot (\mu^{1/p^n})^{k-1-i} \cdot \left[(p^b)^{1/\varphi(p^m)} \cdot (p^b)^{i/p^n} \right] \end{aligned}$$

The induction hypothesis applies to the bracket, and $(\gamma - 1)$ is $\mathcal{O}_C^b/p^b$ -linear, so that we get the desired form. Since $(\mu^{1/p^m}) (\mu^{1/p^n})^{k-1}$ is of valuation larger than $1/\varphi(p^m)$ by [lemma 2.6.9](#), this concludes. $\square$

We may now prove the second point of [proposition 2.6.11](#).

Proposition 2.6.14. *Assume that $r \geq 1$. Then, Ψ_r is almost zero on $\text{Coker}(\gamma - 1)$.*

Proof. We may assume that $r = 1$ (as Ψ_r factors through Ψ_1 for $r \geq 1$).

Let $M = \mathcal{O}_C^b/p^b [Z^{1/p^\infty}]$, and let $N = \text{Coker}(\gamma - 1)$.

Consider the induced morphism $\Psi_r : N \rightarrow N$. We will show that its image is almost zero. We apply the criterion of [lemma 2.5.5](#), so that it suffices to establish that, for all $m \in \mathbb{N}^*$, $N/(p^\flat)^{1/\varphi(p^m)}N$ is of $(p^\flat)^{1/\varphi(p^{m+1})}$ -torsion.

Let $F = \sum_{k,n} a_{k,n} Z^{k/p^n} \in M$, and $G = \Psi_1(F) = \sum_{k,n} a_{k,n} (p^\flat)^{k/p^n} Z^{k/p^n}$.

It suffices to show that there exists $c \in \mathcal{O}_C/p^\flat$ such that

$$(p^\flat)^{1/p^{\varphi(m+1)}} G + c(p^\flat)^{1/p^{\varphi(m)}} \cdot Z^{1/p} \in \text{Im}(\gamma - 1).$$

Modulo $(p^\flat)^{1/\varphi(m)}$, all coefficients $a_{k,n} (p^\flat)^{k/p^n}$ where $k \neq 0$ and $n \leq m$ of G are zero, so that, so that we may assume that $F = a_0 + \sum_{(k,n), n > m} a_{k,n} Z^{k/p^n}$.

We then apply [corollary 2.6.13](#) to all nonconstant and nonzero coefficients of G , so that there exists a constant c_0 such that

$$(p^\flat)^{1/\varphi(p^{m+1})} \cdot (G - a_0) + c_0 (p^\flat)^{1/\varphi(m+1)} \in \text{Im}(\gamma - 1).$$

The [equation \(2.1\)](#) for $m = n$ shows that :

$$(p^\flat)^{1/\varphi(p^m) - 1/p^m} = (\gamma - 1) \cdot \left(\frac{(p^\flat)^{1/\varphi(p^m)}}{\mu^{1/p^m}} Z^{1/p^m} \right) - (p^\flat)^{1/\varphi(p^m)} Z^{1/p^m}.$$

A quick computation shows that $1/\varphi(p^m) - 1/p^m = 1/\varphi(p^{m+1})$. Hence, [equation \(2.6.4\)](#) follows with $c = a_0 - c_0$. $\square$

This concludes the proof of [proposition 2.6.1](#).

2.7 Cohomology of the affine space and open unit disk

We deduce from the above the computation of the cohomology of the affine space.

Proposition 2.7.1. *Let $\mathbb{A}_C$ denote the adic affine line over C . We have :*

$$H_{\text{ét}}^n(\mathbb{A}_C, \mathcal{O}^+/p) \cong \begin{cases} \mathcal{O}_C/p & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

Proof. By definition, $\mathbb{A}_C = \bigcup_{r \in \mathbb{N}} \mathbb{D}_C(|p^{-r}|)$. We let $f_r : \mathbb{D}_C(|p^{-r}|) \rightarrow \mathbb{A}_C$ be the associated open immersions, so that :

$$R\Gamma_{\text{ét}}(\mathbb{D}_C(|p^{-r}|), \mathcal{O}^+/p) = R\Gamma_{\text{ét}}(\mathbb{A}_C, f_{r,*} \mathcal{O}^+/p)$$

Since the $\mathbb{D}_C(|p^{-r}|)$ cover $\mathbb{A}_C$, for any étale sheaf $\mathcal{F}$ on $\mathbb{A}_C$, $\mathcal{F} \cong \varprojlim_{r \in \mathbb{N}} f_{r,*} f_r^* \mathcal{F}$ (this can be checked on geometric points, where it follows from [lemma A.2.2](#)).

Then, [[Sta23](#), [Lemma 0D6K](#)] yields a short exact sequence, for all $n \in \mathbb{N}$:

$$0 \rightarrow R^1 \varprojlim H_{\text{ét}}^{n-1}(\mathbb{A}_C, f_{r,*} \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^n(\mathbb{A}_C, R \varprojlim f_{r,*} \mathcal{O}^+/p) \rightarrow \varprojlim H_{\text{ét}}^n(\mathbb{A}_C, f_{r,*} \mathcal{O}^+/p) \rightarrow 0.$$

For $n = 1, 2$, it follows from [proposition 2.6.1](#) that the images of the transition morphisms of the system $H_{\text{ét}}^{n-1}(\mathbb{A}_C, f_{r,*}\mathcal{O}^+/p)$ almost stabilize. This family hence satisfies the Mittag-Leffler condition in the almost category, so that, by [lemma 2.5.6](#), $R^1\varprojlim$ is almost zero.

$$\text{Hence, } H_{\text{ét}}^n(R\varprojlim_r f_{r,*}\mathcal{O}^+/p) \stackrel{a}{\cong} \varprojlim H_{\text{ét}}^n(\mathbb{A}_C, f_{r,*}\mathcal{O}^+/p) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

It then suffices to show that $R\varprojlim_{r \in \mathbb{N}} f_{r,*}\mathcal{O}^+/p \stackrel{a}{\cong} \varprojlim_{r \in \mathbb{N}} f_{r,*}\mathcal{O}^+/p$, in the category of sheaves of almost $\mathcal{O}_C/p$ -modules on $\mathbb{A}_C$. We apply [\[Sta23, Lemma 0BKY\]](#) to the pro-étale site of $\mathbb{A}_C$ (as we can compute $\mathcal{O}^+/p$ -cohomology on the pro-étale site), with basis given by strictly totally disconnected affinoid perfectoid spaces (who have trivial higher cohomology).

Let $U \rightarrow \mathbb{A}_C$ be a map from a strictly totally disconnected affinoid perfectoid space. It suffices to show that $R^1\varprojlim H_{\text{ét}}^0(U, f_{r,*}\mathcal{O}^+/p)$ vanishes.

By definition of pushforward, $H_{\text{ét}}^0(U, f_{r,*}\mathcal{O}^+/p) = H_{\text{ét}}^0(U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r}|), \mathcal{O}^+/p)$. We will show that the morphisms

$$H_{\text{ét}}^0(U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r}|), \mathcal{O}^+/p) \rightarrow H_{\text{ét}}^0(U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r-1}|), \mathcal{O}^+/p)$$

are surjective, so that the system is Mittag-Leffler, hence admit trivial $R^1\varprojlim$ by [\[Sta23, Lemma 0598\]](#).

The morphisms $\mathbb{D}_C(|p^{-r}|) \rightarrow \mathbb{A}_C$ is a quasi-compact open immersion, so that the same holds for the base change $U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r}|) \rightarrow U$. Since U is strictly totally disconnected, its étale site is equivalent to its topological site.

Topologically, U is profinite, and $U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r}|)$ corresponds to a quasi-compact open subset of U

It follows from the second point of [\[Sta23, Proposition 0A3G\]](#) that the morphism $H^0(U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r}|), \mathcal{O}^+/p) \rightarrow H^0(U \times_{\mathbb{A}_C} \mathbb{D}_C(|p^{-r-1}|), \mathcal{O}^+/p)$ is surjective. This concludes. $\square$

Remark 2.7.2. In [\[Ber95, Lemma 2.2\]](#), Berkovich shows that the affine space is universally acyclic for étale $\mathbb{F}_p$ -cohomology of constant sheaves, relying on a statement of Lütkebohmert [\[Lüt93, Thm 2.1\]](#), claiming that finite étale coverings of a disk are trivial near the origin.

The proof of the result of Berkovich cannot directly be applied to $\mathcal{O}^+/p$ -coefficients, as one needs to control non-finite étale covers.

Remark 2.7.3. Another approach for the computation of the étale cohomology of the affine space is to use excision when embedded into the projective space $\mathbb{P}_C^1$, whose complement is simply a point. From the primitive comparison theorem, this yields the compactly supported cohomology groups of $\mathbb{A}_C$, and we may deduce the above result using Poincaré duality for $\mathcal{O}^+/p$ -coefficients.

This is done in [example 5.3.3](#), and explored for general rigid analytic curves in [section 5.5](#).

Proposition 2.7.4. Let $\mathring{\mathbb{D}}_C$ denote the open unit disk over C , as defined in [example 2.1.18](#). The natural map

$$H_{c,\text{ét}}^i(\mathring{\mathbb{D}}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \not\stackrel{a}{\cong} H_{c,\text{ét}}^i(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p)$$

is **not** an almost isomorphism for $i \in \{1, 2\}$.

Proof. Consider the open immersion $\mathring{\mathbb{D}}_C \hookrightarrow \mathbb{P}_C^1$. By construction, the closed complement of this open subset identifies to the universal compactification of the closed unit disk centered at infinity (exposed in [example 2.1.20](#)) : $\mathbb{D}_C^c = \mathrm{Spa}(C \langle S \rangle, \mathcal{O}_C + S \cdot \mathfrak{m}_C \langle S \rangle)$.

The excision triangle, established in [proposition A.2.1](#), yields the following diagram with exact²² rows :

$$\begin{array}{ccccc} R\Gamma_{\acute{e}t,c}(\mathring{\mathbb{D}}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p & \rightarrow & R\Gamma_{\acute{e}t}(\mathbb{P}_C^1, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p & \rightarrow & R\Gamma_{\acute{e}t}(\mathbb{D}_C^{1,c}, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_{\acute{e}t,c}(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p) & \longrightarrow & R\Gamma_{\acute{e}t,c}(\mathbb{P}_C^1, \mathcal{O}^+/p) & \longrightarrow & R\Gamma_{\acute{e}t}(\mathbb{D}_C^c, \mathcal{O}^+/p). \end{array}$$

It follows from the primitive comparison theorem that the middle vertical arrow is an isomorphism.

By [remark 2.6.5](#), $H_{\acute{e}t}^*(\mathbb{D}_C^c, \mathcal{O}^+/p) \xrightarrow{a} H_{\acute{e}t}^*(\mathbb{D}_C, \mathcal{O}^+/p)$. Moreover, by [[Hub96](#), Corro 2.6.7], we have an isomorphism $H_{\acute{e}t}^*(\mathbb{D}_C^c, \mathbb{F}_p) \cong H_{\acute{e}t}^*(\mathbb{D}_C, \mathbb{F}_p)$.

Recall (or cf. [proposition 6.1.3](#)) that $H_{\acute{e}t}^1(\mathbb{P}_C^1, \mathbb{F}_p) = 0$. It follows from definition of compactly supported cohomology [[Hub96](#), Def 5.2.1] that $H_{\acute{e}t,c}^0(\mathring{\mathbb{D}}_C, \mathbb{F}_p) = 0$, as they are no nontrivial compactly supported global sections of the constant étale sheaf $\mathbb{F}_p$.

This yields the following diagram, with exact rows :

$$\begin{array}{ccccccc} 0 \rightarrow H_{\acute{e}t}^0(\mathbb{P}_C^1, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & H_{\acute{e}t,c}^0(\mathring{\mathbb{D}}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & H_{\acute{e}t,c}^1(\mathring{\mathbb{D}}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow h \\ * \longrightarrow & H_{\acute{e}t}^0(\mathbb{P}_C^1, \mathcal{O}^+/p) & \longrightarrow & H_{\acute{e}t}^0(\mathbb{D}_C^c, \mathcal{O}^+/p) & \longrightarrow & H_{\acute{e}t,c}^1(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p) & \longrightarrow 0 \end{array}$$

If h were an almost isomorphism, it would follow from the five lemma (applied in the almost category) that g is an isomorphism, contradicting [remark 2.6.10](#).

Likewise, we also get the following diagram with exact rows :

$$\begin{array}{ccccccc} 0 \rightarrow H_{\acute{e}t}^1(\mathbb{D}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & H_{\acute{e}t}^2(\mathring{\mathbb{D}}_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & H_{\acute{e}t}^2(\mathbb{P}_C^1, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p & \rightarrow & 0 \\ & & \downarrow f' & & \downarrow g' & & \downarrow h' \\ 0 \longrightarrow & H_{\acute{e}t}^1(\mathbb{D}_C, \mathcal{O}^+/p) & \longrightarrow & H_{\acute{e}t,c}^2(\mathring{\mathbb{D}}_C, \mathcal{O}^+/p) & \longrightarrow & H_{\acute{e}t}^2(\mathbb{P}_C^1, \mathcal{O}^+/p) & \longrightarrow 0 \end{array}$$

If h' were an almost isomorphism, so would g' by the five lemma, contradicting [remark 2.6.10](#). This concludes. $\square$

Remark 2.7.5. Note that the open unit disk does not satisfy Poincaré duality, as noted in [[LRZ24](#), Rk 6.4.11]. It was then indeed expected not to satisfy primitive comparison with compact support, as to not contradict our main result [theorem 5.2.3](#).

²²This exactness is taken in the ∞ -categorical sense, i.e. the lines are (co)fiber sequences. This can also be interpreted as exact triangles on the homotopy category.

A crash course on Mann's six functor formalism

Content of this chapter

In this chapter, we review some of the main ideas and results of Mann's six-functor formalism [Man22], focusing on the comparison results with classical objects in étale cohomology.

We claim no originality for the majority of this section, with the notable exception of the following two results, which, while they are essentially contained in [Man22], do not appear explicitly, and will be crucial in the rest of the thesis (namely in chapter 4).

The first one establishes a form of Poincaré duality on the essential image of the Riemann-Hilbert functor (a simplified version was presented in the introduction proposition 1.3.5) :

Theorem 3.0.1. (cf. theorem 3.3.1)

Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a smooth locally noetherian adic space that is pure of dimension d . Let $\mathcal{L} \in \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}}$.

Assume that there exists $\mathcal{F}_{\mathcal{L}} \in \mathcal{D}_{\text{ét}}(\mathrm{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)$ and an isomorphism in $\mathcal{D}_{\square}^a(\mathrm{Spa}(K, \mathcal{O}_K), \mathcal{O}_K^+/p)^{\varphi}$:

$$f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+a}/p) \simeq \mathcal{F}_{\mathcal{L}} \otimes \mathcal{O}_K^{+a}/p.$$

Then, there is an isomorphism in $\mathcal{D}_{\text{ét}}(\mathrm{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)$:

$$Rf_*^H \mathcal{L}^{\vee}(d)[2d] \simeq \mathcal{F}_{\mathcal{L}}^{\vee}.$$

Where $\mathcal{F}^{\vee} = \underline{\mathrm{RHom}}_{\mathcal{D}(X_{\text{ét}}, \mathbb{F}_p)}(\mathcal{F}, \mathbb{F}_p)$.

The proof of the above is purely a formal juggling with the properties of the six-functor formalism, together with the fully faithfulness of the Riemann-Hilbert functor.

We also obtain the following comparison result between the cohomology computed in Mann's category and in the classical setting.

Corollary 3.0.2. (cf. corollary 3.5.6)

Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)^{\diamond}$ be a locally spatial diamond. Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -vector spaces over X .

Then, there is a natural isomorphism :

$$J_C((f_*^M(\mathcal{L} \otimes \mathcal{O}_X^{+a}/\pi))_{\omega}) \cong (Rf_*^S(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^{+}/\pi))^a,$$

where $(-)_{\omega}$ denotes a discretization functor (i.e. forgets about the condensed structure).

In this section, all categories are ∞ -categories, and all functors live implicitly at the derived level.

3.1 The functors $\mathcal{D}_{\square}^a(-, \mathcal{O}^+/\pi)$ and $\mathcal{D}_{\square}^a(-, \mathcal{O}^+/\pi)^\varphi$

In his thesis [Man22], Lucas Mann associates, to any diamond (more generally, to any untilted small v-stack) X , with a fixed pseudo-uniformizer π , an ∞ -category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ consisting of *complexes of almost, quasi-coherent, solid* sheaves of $\mathcal{O}_X^+/\pi$ -modules on X .

For an analytic adic space X , objects of $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ are not explicit - rather, this category is constructed by gluing from an adequate basis of the pro-étale topology, given by (a loosely improved class of) totally disconnected perfectoid spaces, for which the construction is as follows :

Definition 3.1.1. *Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space that is of weakly perfectly finite type¹ over some totally disconnected² perfectoid space (such as C), and let π be a pseudo-uniformizer of A .*

Then, let $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) := \mathcal{D}_{\square}^a(A^+/\pi)$ be the derived ∞ -category of solid almost A^+/π -modules, constructed as follows :

- *Let $\mathcal{D}(A^+/\pi)^{cond}$ denote the ∞ -derived category of the category of condensed A^+/π -modules, viewed as sheaves on extremally disconnected compact Hausdorff sets, as in [Sch19].³*
- *Let $\mathcal{D}_{\square}(A^+/\pi) \subset \mathcal{D}(A^+/\pi)^{cond}$ be the full subcategory formed by solid objects, with respect to A^+/π .*
- *We say that a morphism $f : M \rightarrow N$ in $\mathcal{D}_{\square}(A^+/\pi)$ is an almost isomorphism if, in $\mathrm{fib}(f)$, the multiplication by any pseudo-uniformizer is zero.*

Finally, let $\mathcal{D}_{\square}^a(A^+/\pi)$ be the localization of $\mathcal{D}_{\square}(A^+/\pi)$ with respect to almost isomorphisms.

An object of $\mathcal{D}_{\square}^a(A^+/\pi)$ can be thought of a complex of solid (i.e. "complete" for some topology) A^+/π -modules, up to a (derived) almost isomorphism.

Remark 3.1.2. *More generally, Mann defines and studies analogue over animated rings. In this more general setup, one replaces the derived category by the category of "modules" over the animated ring, in the sense of Lurie's higher algebra [Lur17, Def 7.1.1.2]. In this thesis, we always consider static (i.e. not animated) rings, so that this is simply a derived category by [Lur17, Remark 7.1.1.6].*

Recall that small v-stacks, defined in [Sch17], form a generalization of analytic adic spaces (and of diamonds). In order to properly endow a small v-stack X with a sheaf $\mathcal{O}_X^+/\pi$, we need to chose an untilt⁴ $X^\sharp$ of X , and a pseudo-uniformizer π . Let $\mathrm{vStack}_{\pi}^{\sharp}$ denote the associated category.

Proposition 3.1.3. ([Man22, Thm 1.2.1]) *There exists a unique hypercomplete v-sheaf of ∞ -categories :*

$$(X, X^\sharp, \pi) \in (\mathrm{vStack}_{\pi}^{\sharp})^{op} \mapsto \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) \in \mathrm{Cat}_{\infty}$$

extending the map defined in definition 3.1.1. This construction admits a six-functor formalism.

While we do not precisely define what a six-functor formalism is, let us present the main characteristic properties, as in [Man22, Theorem 1.2.4]. We refer the reader interested in general six-functor formalisms to [Man22, Sec A.5] or [Sch26].

¹This is the perfectoid variant of the notion of finite type introduced in definition 2.1.11, we ask that the morphism of algebra factors through a continuous, open and surjective map $B \left\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty} \right\rangle \rightarrow A$.

²This is a weaker notion than the one of strictly totally disconnected, where we instead ask that every open cover splits, rather than all étale covers

³In [Man22], this is simply denoted $\mathcal{D}(A^+/\pi)$.

⁴Note that, contrarily to what the notation might suggest, the sheaf $\mathcal{O}_X^+/\pi$ depends on the choice of the untilt $X^\sharp$, and should maybe rather be denoted $\mathcal{O}_{X^\sharp}^+/\pi$.

Proposition 3.1.4. The mapping⁵ $X \mapsto \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ has the following properties :

1. For any X , the category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ is closed symmetric monoidal, i.e. admits a tensor product $-\otimes^M -$, which admits a right adjoint, given by an internal Hom functor $\underline{\text{Hom}}^M$.
For any X , the unit of the tensor product over X is $\mathcal{O}_X^{+,a}/\pi$, sometimes denoted $\mathbb{1}_X$.
2. Any morphism $f : X \rightarrow Y$ induces adjoint functors :

$$f_*^M : \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) \rightleftarrows \mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/\pi) : f^*_M,$$

where the left adjoint f^*_M is called a pullback functor, and f_*^M is a pushforward. Both are compatible with composition, and f^*_M preserves the unit object.

3. Any morphism⁶ $f : X \rightarrow Y$ induces an adjoint pair :

$$f_!^M : \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) \rightleftarrows \mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/\pi) : f^!_M,$$

where the left adjoint $f_!^M$ is called a proper pushforward, and $f^!_M$ is an exceptional pullback. Both are compatible with composition. Both are compatible with composition of bdc morphisms.

4. Let $f : X \rightarrow Y$ be a bdc morphism. If f is proper, then $f_!^M = f_*^M$. If f is étale, then $f^!_M = f^*_M$.
5. (Projection formula) Let $f : X \rightarrow Y$ be a bdc morphism, $\mathcal{M} \in \mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/\pi)$ and $\mathcal{N} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$. Then there exists a natural isomorphism :

$$f_!^M(\mathcal{N} \otimes f^*_M(\mathcal{M})) \cong f_!^M \mathcal{N} \otimes \mathcal{M}.$$

6. (Proper base change). Consider the following cartesian square :

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & Y' \end{array}$$

Assume that f is bdc. Then, there is a natural isomorphism of functors :

$$g^*_M \circ f_!^M \cong f'^!_M \circ g'^!_M.$$

7. (Smooth base change)⁷. Consider the same cartesian square as above. Assume that f is bdc. Then there is a natural isomorphism of functors :

$$f^!_M f^{\circ M}_* \simeq f'^!_M \circ g'^!_M$$

8. Let $f : X \rightarrow Y$ be a smooth morphism of locally noetherian analytic spaces over $\mathbb{Q}_p$, of pure dimension d . Then, there is a canonical isomorphism $f^!_M \mathcal{O}_Y^+/\pi \cong f^*_M \mathcal{O}_Y^+/\pi(d)[2d]$, where (d) denotes a Tate twist, and $[2d]$ denotes a shift of complexes.

As is classical whenever we have a six-functor formalism, we may define a notion of cohomology (resp. cohomology with compact support) of some X as the pushforward f_*^M (resp $f_!^M$) of the unit object $\mathcal{O}_X^+/\pi$, where $f : X \rightarrow \text{Spa}(C, \mathcal{O}_C)$ is a structural morphism to a point. In this specific setup, it should be interpreted as a form of "almost étale $\mathcal{O}_X^+/\pi$ -cohomology", that should hopefully, at least in some cases,

⁵For simplicity, we simply denote $(X, X^\sharp, \pi)$ as X .

⁶Bdc morphisms are defined as locally compactifiable (i.e written locally as the composition of an étale and a proper map) and " p -cohomologically bounded". This is a technical assumption that is satisfied by every morphism of rigid analytic varieties, and by any morphism of analytic adic spaces that is locally weakly of finite type, cf. [Man22, Prop 3.5.14].

⁷This statement can be deduced from proper base change, by passing to right adjoints.

coincide, up to an almost isomorphism, with the standard cohomology groups of the étale sheaf $\mathcal{O}_X^+/\pi$. A main goal is to obtain such comparison results.

In order to descend from $\mathcal{O}^+/\pi$ -cohomology groups to some $\mathbb{F}_p$ -cohomology groups, we need to take some form of Frobenius invariants. Assuming that $\pi \mid p$, the classical approach is to start from the Artin-Schreier exact sequence for sheaves on X :

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}_X^+/\pi \xrightarrow{x \mapsto x^p - x} \mathcal{O}_X^+/\pi \rightarrow 0,$$

and consider some form of associated long exact sequence in cohomology (or, rather, a derived refinement).

The key technical difficulty is Mann's category only captures the information of the $\mathcal{O}^+/\pi$ -cohomology up to an *almost* isomorphism, which need not commute with taking Frobenius invariants. This is not an artifact of Mann's formalism, as, in practice, most computations of $\mathcal{O}^+$ or $\mathcal{O}^+/\pi$ -cohomology relies on the fundamental result of *almost* vanishing of cohomology affinoid perfectoid spaces [Sch17, Lemma 8.8], which only holds in up to almost isomorphism.

This idea can be adapted in a less naive way. Whenever $\pi \mid p$, the unit object $\mathcal{O}_X^+/\pi \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ is naturally endowed with a Frobenius endomorphism φ , which endows the whole category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ with a Frobenius endomorphism φ_M^* .

Definition 3.1.5. A φ -module in $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ is the datum of an object $C \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$, together with a specified isomorphism $\varphi_M^* C \simeq C$. We let

$$\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi} = \text{eq} \left(\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) \xrightleftharpoons[id]{\varphi_M^*} \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) \right)$$

be the category of φ -modules in $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi}$.

Remark 3.1.6. Note that $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi}$ is very far from being a full subcategory of $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$. Indeed, morphisms in $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi}$ are Frobenius-equivariant morphisms.

There is a canonical "forgetful" functor $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$, that forgets the φ -module structure.

Moreover, the assignment $X \mapsto \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi}$ is particularly well behaved, since it also satisfies a six-functor formalism, and all properties of [proposition 3.1.4](#) are satisfied with $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)^{\varphi}$ in place of $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$.

We conclude this section by establishing a Künneth isomorphism, as well as a formal Poincaré duality statement, that are formally deduced from the properties of six-functors in [proposition 3.1.4](#). These proofs are highly classical (cf. [HM24, Ex. 1.0.1 and 1.0.2]).

We recall these computations, as they display the power of a six-functor formalism.

Proposition 3.1.7. Let X, Y be two bdc's untilted small v -stack over C , with structure morphisms denoted p_X, p_Y respectively. Let $\Delta : X \times Y \rightarrow \text{Spa}(C, \mathcal{O}_C)$ be the diagonal morphism. Then :

$$\Delta_!^M(\mathbb{1}_{X \times Y}) \cong p_{X,!}^M \mathbb{1}_X \otimes p_{Y,!}^M \mathbb{1}_Y.$$

Proof. Consider the cartesian square :

$$\begin{array}{ccc}
X \times Y & \xrightarrow{f_X} & Y \\
\downarrow f_Y & \searrow \Delta & \downarrow p_Y \\
X & \xrightarrow{p_X} & \mathrm{Spa}(C, \mathcal{O}_C)
\end{array}$$

Then, we compute :

$$\Delta_!^M \mathbb{1}_{X \times Y} = \Delta_!^M (f_X^{*,M} \mathbb{1}_X \otimes f_Y^{*,M} \mathbb{1}_Y) \cong (p_X^M)_! \circ (f_Y)_!^M (f_{X,M}^* \mathbb{1}_X \otimes f_{Y,M}^* \mathbb{1}_Y).$$

We may then apply the projection formula, so that :

$$(p_X^M)_! \circ (f_Y)_!^M (f_{X,M}^* \mathbb{1}_X \otimes f_{Y,M}^* \mathbb{1}_Y) \cong (p_X^M)_! (\mathbb{1}_Y \otimes f_{Y,M}^M f_{X,M}^* \mathbb{1}_X)$$

We then apply proper base change, so that :

$$(p_X^M)_! (\mathbb{1}_Y \otimes f_{Y,M}^M f_{X,M}^* \mathbb{1}_X) \cong (p_X^M)_! (\mathbb{1}_Y \otimes p_{X,M}^* p_{Y,!}^M \mathbb{1}_X)$$

Apply once again the projection formula :

$$(p_X^M)_! (\mathbb{1}_Y \otimes p_{X,M}^* p_{Y,!}^M \mathbb{1}_X) \cong p_{X,!}^M \mathbb{1}_X \otimes p_{Y,!}^M \mathbb{1}_Y.$$

This is the desired result. □

We will study more subtle variants of this Künneth isomorphism in [section 6.2](#).

We now obtain a version of Poincaré duality

Proposition 3.1.8. *Let $X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a smooth morphism of locally noetherian analytic adic spaces of pure dimension d . We have Poincaré duality in Mann's category, in the following sense :*

$$\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M \mathbb{1}_X, \mathbb{1}_C) \cong f_*^M \mathbb{1}_X(d)[2d]$$

Proof. For any $M \in \mathcal{D}_{\square}^a(\mathcal{O}_C/p)$, we have :

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(M, f_*^M f_M^* \mathbb{1}_C(d)[2d]) \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(M, f_*^M f_M^! \mathbb{1}_C) && \text{By smoothness} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M f_M^*, \mathbb{1}_C) && \text{By adjunction} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M (f_M^* \otimes^M \mathbb{1}_X), \mathbb{1}_C) && \text{Since } \mathbb{1}_X \text{ is the tensor unit} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M \mathbb{1}_X \otimes^M \mathbb{1}_C) && \text{By the projection formula} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}^M(M, \underline{\mathrm{Hom}}(f_!^M \mathbb{1}_X, \mathbb{1}_C)) && \text{By adjunction}
\end{aligned}$$

We conclude by Yoneda's lemma. □

We also get the following *relative* variant, sometimes known as Verdier duality.

Proposition 3.1.9. *Let $f : X \rightarrow Y$ be a bdc map between two v -stacks over C .*

Then, for any $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$ and $\mathcal{G} \in \mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/p)$, we have :

$$\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/p)}^M(f_! \mathcal{F}, \mathcal{G}) \cong f_* \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}^M(\mathcal{F}, f^! \mathcal{G}).$$

Proof. This is [Man22, Cor 3.6.17]. The proof follows the same steps as [proposition 3.1.8](#). $\square$

3.2 Classical $\mathbb{F}_p$ -étale sheaves and the Riemann-Hilbert adjunction

Let us start by recalling some facts about the classical theory of étale sheaves in analytic geometry, as considered by Huber in [Hub96], and refined by Scholze in [Sch17].

Fix S a locally noetherian analytic adic space, and some $n \in \mathbb{N}$. To any locally noetherian adic space X over S , Huber associates in [Hub96] a 1-category $D_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$, as the derived category of étale $\mathbb{Z}/n\mathbb{Z}$ -sheaves on X . He shows that, whenever n is invertible in $\mathcal{O}_S^+$ (and when restricting to spaces of finite type over S , the setup $D(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ admits a six-functor formalism.

Remark 3.2.1. While Huber works 1-categorically, and the notion of "six-functor formalism" was not properly formalized yet, his constructions can be translated in an ∞ -categorical setup, and can be shown to be an actual six-functor formalism. This has been checked thoroughly by Zavyalov in [Zav25, Sec 8] (see also [Zav23, Sec 6.1]). In particular, it satisfies all the properties of [proposition 3.1.4](#). In what follows, we always work ∞ -categorically.

When n is not invertible in $\mathcal{O}_S^+$, most of the constructions still apply, but some key properties are lacking. In particular, there is no general proper base change (see [Man22, p. 4] for a crucial counter-example), nor a general projection formula.

Remark 3.2.2. Most of Huber's theory (at least in the $\ell \neq p$ setup) has been generalized by Scholze in [Sch17] in the generality of small v -stacks, and the constructions coincide for locally noetherian analytic adic spaces. We will use both theories, usually preferring the more modern notations of [Sch17].

Let X be a locally spatial diamond, and $X_{\text{ét}}$ the associated étale site. Following [Sch17, Prop 14.15], we let $\mathcal{D}(X_{\text{ét}}, \mathbb{F}_p)$ be the ∞ -derived category of sheaves of $\mathbb{F}_p$ -modules over $X_{\text{ét}}$. This category need not be left-complete in general⁸, and we denote its left completion $\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)$.

Remark 3.2.3. If X is of p -cohomologically finite dimension, the category $\mathcal{D}(X_{\text{ét}}, \mathbb{F}_p)$ is already left-complete by [Sch17, Prop 20.17].

This is functorial, in the sense that for any morphism $f : X \rightarrow Y$ of locally spatial diamonds, there is an adjoint pair of functors, constructed by Scholze in [Sch17] :

$$Rf_*^S : \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p) \rightleftarrows \mathcal{D}_{\text{ét}}(Y, \mathbb{F}_p) : Rf_S^*.$$

If X and Y are locally noetherian analytic adic spaces, these notions coincide with the ones defined by Huber : $Rf_*^S = Rf_*^H$ and $Rf_S^* = Rf_H^*$.

Moreover, for any X , the category $\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)$ admits a well behaved derived tensor product $\otimes_{\mathbb{F}_p}^{\mathbb{L}}$, which admits a right adjoint, a derived internal Hom, denoted⁹ $\underline{\text{RHom}}_{\mathbb{F}_p}$.

Let us now introduce overconvergent sheaves.

⁸Recall that a stable ∞ -category $\mathcal{C}$ with a t -structure is said to be left-complete if, for all objects A of $\mathcal{C}$, the natural morphism $A \rightarrow \varprojlim_{n \in \mathbb{Z}} \tau^{\geq n} A$ is an isomorphism, where $\tau^{\geq n}$ denotes the truncation. The left completion of $\mathcal{C}$ is then defined by $\hat{\mathcal{C}} = \varprojlim_{n \in \mathbb{Z}} \tau^{\geq n} \mathcal{C}$, which remains a stable ∞ -category.

⁹This is neither the notation of [Hub96] nor [Sch17], but mimics the one of Mann.

Definition 3.2.4. An étale complex $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)$ is said to be **overconvergent** if, for any quasi-pro-étale morphism $\bar{x} : \text{Spa}(C, C^+) \rightarrow X$ from an algebraically closed perfectoid field C with open bounded valuation subring C^+ , the natural map :

$$\mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}^\circ}$$

is an isomorphism, where $\bar{x}^\circ$ denotes the natural composition $\text{Spa}(C, C^\circ) \rightarrow \text{Spa}(C, \mathcal{O}_C) \xrightarrow{\bar{x}} X$.

We let $\mathcal{D}(X_{\text{ét}}, \mathbb{F}_p)^{\text{oc}} \subset \mathcal{D}(X_{\text{ét}}, \mathbb{F}_p)$ be the full subcategory of overconvergent étale $\mathbb{F}_p$ -sheaves on X , and denote $\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}}$ its left completion. It is stable under pullbacks and (derived) tensor product.

Let us now introduce the Riemann-Hilbert and φ -invariants functor, as well as a few of their properties. Both of these functors are defined by descent, so that we start from the local situation. Let us recall the construction from [Man22, Prop. 3.9.8].

Let $X = \text{Spa}(A, A^+)$ be a strictly totally disconnected perfectoid space with pseudo-uniformizer $\pi \in A^+$. On such a space, étale covers split, so that is a natural equivalence of categories :

$$\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}} \cong \mathcal{D}(\mathbb{F}_p(X))$$

with the derived category of $\mathbb{F}_p(X)$ -modules, where $\mathbb{F}_p(X)$ denotes the ring of continuous maps $|X| \rightarrow \mathbb{F}_p$.

The natural morphism of étale sheaves $(\mathbb{F}_p)_X \rightarrow \mathcal{O}_X^+/\pi$ induces a map :

$$\mathbb{F}_p(X) \cong H^0(X, (\mathbb{F}_p)_X) \rightarrow H^0(X, \mathcal{O}_X^+/\pi) \cong A^+/\pi$$

where the last isomorphism follows from the vanishing of higher étale cohomology groups on strictly totally disconnected perfectoid spaces. This map induces a morphism of analytic rings $\mathbb{F}_p(X) \rightarrow A^{+,a}/\pi$.

Definition 3.2.5. The Riemann-Hilbert functor $\mathcal{D}(\mathbb{F}_p(X))_\omega \rightarrow \mathcal{D}_{\square}^a(X, A^+/\pi)^\varphi$ is defined as the base change $(-) \otimes_{\mathbb{F}_p(X)} A^{+,a}/\pi$ along the morphism of analytic rings constructed above, in the sense of [Man22, Def 2.12.23.(c)], with the σ -structure on $A^{+,a}/\pi$ given by the Frobenius morphism.

This local construction glues to the following :

Definition 3.2.6. Let K be a complete extension of $\mathbb{Q}_p$, with pseudo-uniformizer π such that $\pi|p$. For X an untilted locally spatial diamond over $\text{Spd}(K)$, Mann defines [Man22, Def 3.9.21] :

1. A "Riemann-Hilbert" functor $- \otimes \mathcal{O}_X^{+,a}/\pi : \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^{+,a}/\pi)^\varphi$.
2. Its left adjoint, a functor of φ -invariants $(-)^\varphi : \mathcal{D}_{\square}^a(X, \mathcal{O}_X^{+,a}/\pi)^\varphi \rightarrow \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}}$.

When $X = \text{Spa}(K, \mathcal{O}_K)$ with pseudo-uniformizer π , we simply denote the Riemann-Hilbert functor as $- \otimes \mathcal{O}_K^{+,a}/\pi$.

Remark 3.2.7. This Riemann-Hilbert functor can be seen as an analogue of the one introduced by Bhatt-Lurie in [BL17], relating $\mathbb{F}_p$ -étale sheaves over a fixed scheme X of characteristic p with quasi-coherent modules over X equipped with a Frobenius.

The key fact for our purposes is as follows :

Proposition 3.2.8. ([Man22, Thm 3.9.23]) The Riemann-Hilbert functor is fully faithful.

Moreover, when restricting both sides of the equivalence to the full subcategories of dualizable objects, L.Mann proves that the Riemann-Hilbert functor induces an equivalence of category. In [Man22], this is the result used by Mann to establish his form of Poincaré duality. Dualizable objects of $\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}}$

can be seen to correspond to *perfect complexes*, i.e. the ones that are locally quasi-isomorphic to bounded complexes of finite dimension.

In this thesis, we are interested in the étale cohomology of non-proper spaces, for which cohomology groups may be infinite dimensional, and, hence, non dualizable. We will nonetheless prove that the Poincaré duality holds on the essential image of the Riemann-Hilbert functor.

3.3 Poincaré duality on the essential image of Riemann-Hilbert

For any $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)$, we define its dual $\mathcal{F}^\vee := \underline{\text{RHom}}_{\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)}(\mathcal{F}, (\mathbb{F}_p)_X)$, where $(\mathbb{F}_p)_X$ denotes the constant sheaf associated to $\mathbb{F}_p$ on $X_{\text{ét}}$.

We establish the following :

Theorem 3.3.1. *Let K be a complete extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ be its ring of integers, and let π be a pseudo-uniformizer, such that $\pi|_p$. Let $f : X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a smooth locally noetherian adic space that is pure of dimension d . Let $\mathcal{L} \in \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{\text{oc}}$.*

Assume that there exists $\mathcal{F}_{\mathcal{L}} \in \mathcal{D}_{\text{ét}}(\text{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)$ and an isomorphism in $\mathcal{D}_{\square}^a(\text{Spa}(K, \mathcal{O}_K), \mathcal{O}_K^+/\pi)$:

$$f_!^M(\mathcal{L} \otimes \mathcal{O}_X^+/\pi) \simeq \mathcal{F}_{\mathcal{L}} \otimes \mathcal{O}_K^+/\pi.$$

Then, there is an isomorphism in $\mathcal{D}_{\text{ét}}(\text{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)$:

$$Rf_*^H \mathcal{L}^\vee(d)[2d] \simeq \mathcal{F}_{\mathcal{L}}^\vee.$$

Note that, if f is additionally proper and $\mathcal{L}$ is a perfect (e.g. a local system), the primitive comparison theorem [Man22, Cor 3.9.24] applies, so that the assumption is automatically verified with $\mathcal{F}_{\mathcal{L}} = Rf_*^M \mathcal{L}$ (this is overconvergent since f is quasi-compact). In that setup, this recovers [Man22, Cor 3.10.22].

Proof. By taking stalks from $\text{Spa}(\overline{K}, \mathcal{O}_{\overline{K}})$, we may freely identify $\mathcal{D}_{\text{ét}}(\text{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)$ with the derived ∞ -category of $\mathbb{F}_p$ -vector spaces equipped with a $\text{Gal}(\overline{K}/K)$ -action, with its natural symmetric monoidal structure. In particular, all étale sheaves on $\text{Spa}(K, \mathcal{O}_K)$ are necessarily overconvergent.

The Riemann-Hilbert functor is symmetric monoidal and commutes with pullbacks. Hence, for any $A \in \mathcal{D}_{\text{ét}}(\text{Spa}(K, \mathcal{O}_K), \mathbb{F}_p) = \mathcal{D}_{\text{ét}}(\text{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)^{\text{oc}}$, we have :

$$\begin{aligned} f_!^M \left((\mathcal{L} \otimes_{\mathbb{F}_p}^{\mathbb{L}} Rf_H^* A) \otimes \mathcal{O}_X^+/\pi \right) &\cong f_!^M \left((\mathcal{L} \otimes \mathcal{O}_X^+/\pi) \otimes^M (Rf_H^* A \otimes \mathcal{O}_X^+/\pi) \right) \\ &\cong f_!^M \left((\mathcal{L} \otimes \mathcal{O}_X^+/\pi) \otimes^M f_M^*(A \otimes \mathcal{O}_K^+/\pi) \right) \\ &\cong (f_!^M(\mathcal{L} \otimes \mathcal{O}_X^+/\pi)) \otimes^M (A \otimes \mathcal{O}_K^+/\pi) \\ &\simeq (\mathcal{F}_{\mathcal{L}} \otimes \mathcal{O}_K^+/\pi) \otimes^M (A \otimes \mathcal{O}_K^+/\pi) \\ &\cong (\mathcal{F}_{\mathcal{L}} \otimes_{\mathbb{F}_p}^{\mathbb{L}} A) \otimes \mathcal{O}_K^+/\pi. \end{aligned}$$

where the third isomorphism follows from the projection formula (cf. fifth point of [proposition 3.1.4](#)), and the fourth one follows from the hypothesis. From there, we compute :

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}_{\acute{e}t}(\mathrm{Spa}(K, \mathcal{O}_K), \mathbb{F}_p)}(A, Rf_*^H \mathcal{L}^\vee(d)[2d]) \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)}(Rf_H^* A, \mathrm{RHom}_{\mathbb{F}_p}(\mathcal{L}, \mathbb{F}_p)(d)[2d]) && \text{By adjunction} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}}(Rf_H^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}, \mathbb{F}_p(d)[2d]) && \text{By adjunction} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_X^{+a}/\pi)^\varphi}((Rf_H^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi, \mathcal{O}_X^{+a}/\pi(d)[2d]) && \text{By full faithfulness} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_X^{+a}/\pi)^\varphi}((Rf_H^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi, f_M^! \mathcal{O}_K^{+a}/\pi) && \text{By the [proposition 3.1.4](#)} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_X^{+a}/\pi)^\varphi}(f_M^!((Rf_H^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi), \mathcal{O}_K^{+a}/\pi) && \text{By adjunction} \\
& \simeq \mathrm{Hom}_{\mathcal{D}_{\square}^a(\mathcal{O}_X^{+a}/\pi)^\varphi}(\mathcal{F}_{\mathcal{L}} \otimes_{\mathbb{F}_p}^{\mathbb{L}} A) \otimes \mathcal{O}_K^{+a}/\pi, \mathcal{O}_K^{+a}/\pi) && \text{By the above} \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}}(\mathcal{F}_{\mathcal{L}} \otimes_{\mathbb{F}_p}^{\mathbb{L}} A, \mathbb{F}_p) && \text{By fully faithfulness } (-) \otimes \mathcal{O}_K^{+a}/p \\
& \cong \mathrm{Hom}_{\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)}(A, \mathrm{RHom}_{\mathbb{F}_p}(\mathcal{F}_{\mathcal{L}}, \mathbb{F}_p)). && \text{By adjunction}
\end{aligned}$$

We conclude using Yoneda's lemma. $\square$

The following corollary is the version used in practice.

Corollary 3.3.2. *Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and $\mathcal{O}_C$ be its ring of integers. Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a smooth locally noetherian adic space that is pure of dimension d .*

Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -vector spaces on X . Assume that there exists some $\mathcal{F}_{\mathcal{L}} \in \mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C, \mathcal{O}_C), \mathbb{F}_p)$ and an isomorphism in $\mathcal{D}_{\square}^a(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^+/p)$:

$$f_M^!(\mathcal{L} \otimes \mathcal{O}_X^+/p) \simeq \mathcal{F}_{\mathcal{L}} \otimes \mathcal{O}_C^+/p.$$

Then, for all $0 \leq k \leq 2d$, there exists an isomorphism :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee)(d) \simeq \mathrm{Hom}_{\mathbb{F}_p}(\pi_{k-2d}(\mathcal{F}_{\mathcal{L}}), \mathbb{F}_p).$$

Proof. We apply [theorem 3.3.1](#), and then take homotopy groups π_{k-2d} on both sides. Since $\mathcal{L}$ is concentrated in degree zero, so is $\mathcal{L}^\vee$, and we compute :

$$\pi_{2d-k}(Rf_*^H(\mathcal{L}^\vee)(d)[2d]) \cong R^k f_*^H \mathcal{L}^\vee(d) \cong H_{\acute{e}t}^k(X, \mathcal{L}^\vee)(d)$$

Moreover, $\mathbb{F}_p$ is an injective object of $\mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C, \mathcal{O}_C), \mathbb{F}_p) \cong \mathcal{D}(\mathbb{F}_p)$ the derived category of $\mathbb{F}_p$ -modules, so that we get :

$$\pi_{2d-k}(\mathrm{RHom}_{\mathcal{D}(\mathbb{F}_p)}(\mathcal{F}_{\mathcal{L}}, \mathbb{F}_p)) = \mathrm{Hom}_{\mathbb{F}_p}(\pi_{k-2d}(\mathcal{F}_{\mathcal{L}}), \mathbb{F}_p)$$

This yields the desired result. $\square$

Remark 3.3.3. *The above computation can be made functorially in $\mathcal{L}$, provided that $\mathcal{F}_{\mathcal{L}}$ can be chosen functorially in $\mathcal{L}$. In particular, if a group G acts analytically on X or Y such that :*

- $\mathcal{L}$ is G -equivariant.
- For all $g \in G$, there is a natural isomorphism morphism $\phi_{g,M}^* \mathcal{F}_{\mathcal{L}} \cong \mathcal{F}_{\phi_{g,H}^* \mathcal{L}}$, compatible with the action of G , where $\phi_g : x \in X \mapsto g \cdot x$.

- The isomorphism in the hypothesis is G -equivariant.

Then the Poincaré duality isomorphism is G -equivariant.

3.4 Quasi-coherent standard almost $\mathcal{O}^+/\pi$ -étale sheaves

This section follows [Man22, Sec 3.3].

Let X be an untilted locally spatial diamond over a complete extension K of $\mathbb{Q}_p$, with pseudo-uniformizer π . We will define a left-complete category¹⁰ $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)$ of almost quasi-coherent étale sheaves of $\mathcal{O}_X^+/\pi$ -modules over $X_{\text{ét}}$, that is compatible with pushforwards and pullbacks.

Let $f : X \rightarrow Y$ be a morphism of such untilted locally spatial diamonds. While the étale pushforward f_*^S of an $\mathcal{O}_X^+/\pi$ -module is naturally an $\mathcal{O}_Y^+/\pi$ -module, the pullback f_S^* does not preserve $\mathcal{O}^+/\pi$ -modules, so that we should consider a different notion of pullback of quasi-coherent sheaves.

Let us first recall the setup in the non-almost case, following [Sta23, Chapter 03A4].

Definition 3.4.1. Let X be an untilted locally spatial diamond over a complete extension K of $\mathbb{Q}_p$, with pseudo-uniformizer π .

We may view the sheaf $\mathcal{O}_X^+/\pi$ as a sheaf of rings on $X_{\text{ét}}$, and consider the ringed topos $(\text{Sh}(X_{\text{ét}}), \mathcal{O}_X^+/\pi)$ associated to the ringed site $(X_{\text{ét}}, \mathcal{O}_X^+/\pi)$.

Every morphism of locally spatial diamonds $X \rightarrow Y$ induces a morphism of ringed sites $(X, \mathcal{O}_X^+/\pi) \rightarrow (Y, \mathcal{O}_Y^+/\pi)$. This defines a pullback functor:¹¹

$$f_S^{*,qcoh} \mathcal{F} = \mathcal{O}_X^+/\pi \otimes_{f_S^* \mathcal{O}_Y^+/\pi} f_S^* \mathcal{G}.$$

We let $\text{Mod}(\mathcal{O}_X^+/\pi)$ denote the abelian (1)-category of sheaves of $\mathcal{O}_X^+/\pi$ -modules on $X_{\text{ét}}$, and $\mathcal{D}(X_{\text{ét}}, \mathcal{O}_X^+/\pi)$ be its derived ∞ -category. Any morphism $f : X \rightarrow Y$ induces an adjoint pair :

$$Rf_*^S : \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/\pi) \rightleftarrows \mathcal{D}_{\text{ét}}(Y, \mathcal{O}_Y^+/\pi) : Rf_S^{*,qcoh}.$$

Defining the almost version $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)$, it is a bit more tricky, as the object " $\mathcal{O}_X^{+a}/\pi$ " does not exist as a sheaf of classical rings in $\text{Sh}(X_{\text{ét}}, \mathbb{F}_p)$. Instead, we may define a classical sheaf $\mathcal{F}$ of $\mathcal{O}_X^{+a}/\pi$ -modules on X as a sheaf on $X_{\text{ét}}$, valued in the category of almost $\mathcal{O}_X^+/\pi(X)$ -modules, together with the compatible structure of an $\mathcal{O}_X^+/\pi(U)$ -module on $\mathcal{F}(U)$ for all $U \in X_{\text{ét}}$.

We then let $\mathcal{D}(X_{\text{ét}}, \mathcal{O}_X^{+a}/\pi)$ be the derived ∞ -category of the above category, and $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)$ be its left completion, as in [Man22, Def 3.3.12]. See also [GR03, Paragraph 5.5.1] for the classical theory of almost étale sheaves over schemes.

Here are the main properties that will be useful to us (cf. [Man22, Lemma 3.3.13]).

Proposition 3.4.2. Let $f : X \rightarrow Y$ be a morphism of locally spatial diamonds over a complete nonarchimedean field K , with pseudo-uniformizer π . Then, there exists an adjoint pairs of functors :

$$Rf_*^S : \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi) \rightleftarrows \mathcal{D}_{\text{ét}}(Y, \mathcal{O}_Y^{+a}/\pi) : Rf_S^{*,qcoh}.$$

¹⁰This is denoted $\text{Shv}^\wedge(X_{\text{ét}}, \mathcal{O}_X^{+a}/\pi)$ in [Man22].

¹¹Note that, what we denote f^* here is usually denoted f^{-1} in a quasi-coherent context, while what we denote $f^{*,qcoh}$ is usually just referred to as f^* .

Moreover, there is a natural t -exact almostification functor :

$$(-)^a : \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/\pi) \rightarrow \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi).$$

It is computed as $\mathcal{F}^a(U) = \mathcal{F}(U)^a$, where the second " a " denotes the almostification for derived $\mathcal{O}_X^+/\pi(X)$ -modules. The functor $(-)^a$ is compatible with the formation of pushforward and pullbacks.

Remark 3.4.3. Note that there is a natural morphism $- \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/\pi : \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p) \rightarrow \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/\pi)$, obtained by left-completing the derived tensor product. Composing this with the almostification functor $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/\pi) \rightarrow \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)$ defines a functor that is closely related to the Riemann-Hilbert functor, cf. [proposition 3.5.4](#).

The same definition as in [definition 3.2.4](#) allows us to consider overconvergent objects in $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)$, and we denote the full subcategory of overconvergent étale sheaves of $\mathcal{O}_X^{+a}/\pi$ -modules as $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)^{oc}$ (cf. [\[Man22, Def 3.3.14\]](#) for more detail).

Note that this is really a notion of overconvergence for *almost* sheaves, in particular, the sheaf $\mathcal{O}^+/\pi$ itself is not overconvergent when viewed as an étale sheaf of $\mathbb{F}_p$ -modules. It is, however, *almost* overconvergent.

Let us now explain how to identify this category with the subcategory of discrete objects in $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$.

3.5 Discrete objects inside $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$

Recall that there is a fully faithful inclusion of rings into condensed rings (viewed as sheaves of rings on the category of profinite spaces), given by constant sheaves, which factors through solid objects. The essential image of this functor is the full subcategory of *discrete* rings.

For any strictly totally disconnected affinoid perfectoid space $\text{Spa}(A, A^+)$, we may consider the subcategory $\mathcal{D}_{\square}^a(A^+/\pi)_{\omega} \subset \mathcal{D}_{\square}^a(A^+/\pi)$ of discrete (derived) A^+/π -modules. This construction descends to a subfunctor of discrete objects $\mathcal{D}_{\square}^a(\mathcal{O}_X^+/\pi)_{\omega} \subset \mathcal{D}_{\square}^a(\mathcal{O}_X^+/\pi)$ for any untilted v -stack X , defined as follows.

Definition 3.5.1. (cf. [\[Man22, Def 3.2.17\]](#))

Let X be a locally spatial diamond, with pseudo-uniformizer π .

We let $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \subset \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ be the full subcategory formed by objects $\mathcal{F}$ such that, for any map $f : Y \rightarrow X$ from a totally disconnected affinoid perfectoid space, the pullback $f_M^* \mathcal{F}$ is discrete.

This inclusion admits a natural right adjoint, as follows.

Proposition 3.5.2. For any untilted small v -stack X , the subcategory $\mathcal{D}_{\square}^a(\mathcal{O}_X^+/\pi)_{\omega}$ is stable under colimits.

Hence, the inclusion $\mathcal{D}_{\square}^a(\mathcal{O}_X^+/\pi)_{\omega} \subset \mathcal{D}_{\square}^a(\mathcal{O}_X^+/\pi)$ admits a right adjoint, called the discretization and denoted $(-)_{\omega}$.

The key result is now that discrete objects identify with overconvergent standard almost étale sheaves, in the following sense.

Proposition 3.5.3. Let X be an untilted locally spatial diamond, with pseudo-uniformizer π . Assume moreover that X admits a map to an affinoid perfectoid space. Then, there is an equivalence of ∞ -categories :

$$J_X : \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \cong \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+a}/\pi)^{oc}$$

The functor J_X^{-1} should be thought of as a global section functor, while J_X maps a module to the associated quasi-coherent sheaf.

It is functorial in the sense that, for any morphism $f : X \rightarrow Y$ of untilted locally spatial diamonds over an affinoid perfectoid space, there is a canonical isomorphism of functors $\mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/\pi)_{\omega} \rightarrow \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+,a}/\pi)^{oc} :$

$$J_X \circ f_M^* \cong Rf_S^{qcoh,*} \circ J_Y.$$

Proof. This is [Man22, Prop 3.3.16]. Note that the hypothesis that X admits a map to an affinoid perfectoid space allows implies that X admits "enough pseudo-uniformizers", in the sense of [Man22, Def 3.2.6]. $\square$

If $X = \text{Spa}(C, \mathcal{O}_C)$, we denote for simplicity $J_C := J_{\text{Spa}(C, \mathcal{O}_C)}$.

At this point, we constructed tools for two natural functors $\mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{oc} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi) :$ the Riemann-Hilbert functor, and the almostification of the tensor product of classical étale sheaves. We show that they actually coincide (this was informally remarked in [Zav23, Paragraph 6.2]).

Proposition 3.5.4. *Let X be an untilted locally spatial diamond, with pseudo-uniformizer π . Suppose that X admits a map to an affinoid perfectoid space.*

Let $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{oc}$. Then, the following square commutes :

$$\begin{array}{ccc} \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)^{oc} & \xrightarrow{- \otimes \mathcal{O}_X^{+,a}/\pi} & \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \\ (- \otimes_{\mathbb{F}_p}^{\mathbb{F}_p} \mathcal{O}_X^+/\pi)^a \downarrow & & \downarrow \\ \mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^{+,a}/\pi)^{oc} & \xrightarrow{J_X^{-1}} & \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \end{array}$$

Here, the top horizontal arrow is the Riemann-Hilbert functor of [definition 3.2.6](#), the rightmost vertical arrow is the restriction to discrete objects of the forgetful functor described in [remark 3.1.6](#), the leftmost vertical arrow is the functor described in [remark 3.4.3](#) (it preserves overconvergent objects, since taking stalks commutes with tensor products), and the lower horizontal arrow is the identification of [proposition 3.5.3](#).

Proof. It follows from [Man22, Lemma 3.9.18.(i)], the proof of [Man22, Prop 3.3.16], [Man22, Remark 3.9.11] and [Man22, Lemma 3.2.19.(i)] that the four categories of the above diagram satisfy pro-étale descent, so that it suffices to check the commutation on a basis of strictly totally disconnected affinoid perfectoid spaces.

For such spaces, using the identifications from [Man22, Lemma 3.3.15.(ii)] [Man22, Lemma 3.9.18.(iii)], this follows from the definition of the Riemann-Hilbert functor as the composition of $(-)^a$ and a base change functor. $\square$

Moreover, since the equivalence J_C is compatible with pullback, it is also stable by the right adjoints of such maps. This give the following compatibility with pushforwards :

Corollary 3.5.5. *Let $f : X \rightarrow \text{Spa}(C, \mathcal{O}_C)^{\diamond}$ be a locally spatial diamond. Let $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$.*

Then, there is a natural isomorphism : $J_C((f_*^M \mathcal{F})_{\omega}) \cong Rf_*^S(J_X(\mathcal{F}))$.

Proof. By [proposition 3.5.3](#), the two functors $f_M^* \circ J_C^{-1}$ and $J_X^{-1} \circ Rf_S^{qcoh,*}$ define two naturally isomorphic functors $\mathcal{D}_{\text{ét}}(\text{Spa}(C, \mathcal{O}_C/p), \mathbb{F}_p) \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$. We will deduce the result above by passing to right adjoints. Since J_X and J_C are equivalences of categories, they are naturally adjoint to their inverse.

Recall that f_M^* preserves discrete objects by definition. By composition of right adjoints, the pullback $f_M^* : \mathcal{D}_{\square}^a(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^+/\pi)_{\omega} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$ restricted to discrete objects admits the right adjoint $(f_M^*)_{\omega}$, where $(-)_\omega$ is the discretization functor described in [proposition 3.5.2](#).

All étale sheaves over C are overconvergent, and $Rf_S^{qcoh,*} : \mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^+/\pi) \rightarrow \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^+/\pi)^{oc}$ admits the right adjoint Rf_*^S .

Since naturally isomorphic functors admit naturally isomorphic adjoints, there is a natural isomorphism $J_C \circ ((f_M^*)_{\omega}) \cong Rf_*^S \circ J_X$, which concludes. $\square$

In particular, we have the following :

Corollary 3.5.6. *Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)^{\diamond}$ be a locally spatial diamond. Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -vector spaces over X .*

Then, there is a natural isomorphism :

$$J_C((f_*^M(\mathcal{L} \otimes \mathcal{O}_X^+/\pi))_{\omega}) \cong (Rf_*^S(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/\pi))^a.$$

Proof. This follows directly from [proposition 3.5.4](#) and [corollary 3.5.5](#) applied to $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_X^+/\pi$, together with the commutation of $(-)^a$ with pushforwards. $\square$

Remark 3.5.7. *In the setup of the corollary above, if f is furthermore quasi-compact, then f_*^M preserves discrete objects by [Man22, Lemma 3.3.10.(ii)].*

Letting $\mathcal{L} = \mathbb{F}_p$, we get $f_*^M \mathcal{O}_X^{+,a}/p \cong J_C^{-1}((Rf_*^S \mathcal{O}_X^+/p)^a)$.

This recovers a variant of [Man22, Ex 3.3.17]. Informally, this means that $f_*^M \mathcal{O}_X^{+,a}/p$ computes, up to an almost isomorphism, the classical object $R\Gamma_{\acute{e}t}(X, \mathcal{O}_X^+/p)$.

We conclude this section with a description of the internal $\underline{\mathrm{Hom}}$ inside the subcategory of discrete objects, that will be used in [section 5.4](#).

Lemma 3.5.8. *For any untitled v -stack X with pseudo-uniformizer π , the category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$ is closed symmetric monoidal, with internal Hom functor defined by :*

$$\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}}(A, B) := \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)}(i(A), i(B))_{\omega}$$

where $i : \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \rightarrow \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ denotes the natural inclusion, and $(-)_{\omega}$ denotes its right adjoint.

Proof. By [Man22, Lemma 3.2.19.(ii)], the category $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$ is symmetric monoidal with tensor product given by restriction of the one of $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$ (so that i commutes with tensor products). It suffices to prove that said tensor product admits a right adjoint.

The functor i is fully faithful, as it is the inclusion of a full subcategory ([Man22, Def 3.2.17]).

For any $N, M_1, M_2 \in \mathcal{D}(X, \mathcal{O}_X^+/p)_\omega$, we compute :

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_\omega} \left(N, \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}(i(M_1), i(M_2))_\omega \right) \\
& \stackrel{a}{\cong} \mathrm{Hom}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)} \left(i(N), \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}(i(M_1), i(M_2)) \right) \\
& \stackrel{a}{\cong} \mathrm{Hom}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)} (i(N) \otimes i(M_1), i(M_2)) \\
& \stackrel{a}{\cong} \mathrm{Hom}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)} (i(N \otimes M_1), i(M_2)) \\
& \stackrel{a}{\cong} \mathrm{Hom}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_\omega} (N \otimes M_1, M_2).
\end{aligned}$$

This yields the desired result. $\square$

On the level of geometric points, this internal Hom functor can be made fully explicit.

Lemma 3.5.9. *Let C be a complete algebraically closed field. The functor of lemma 3.5.8 on $\mathcal{D}_{\square}^a(\mathcal{O}_C/p)_\omega$:*

$$\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p)_\omega^a}(M^a, N^a) \stackrel{a}{\cong} \underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p\text{-Mod})}(M, N)^a.$$

Proof. The category $\mathcal{D}_{\square}^a(\mathcal{O}_C/p)_\omega^a$ is simply the almostification of the derived category of $\mathcal{O}_C/p$ -modules, for which an internal Hom functor was constructed by Zavyalov in [Zav24a, Def 2.4.5]. Since the tensor products coincide in both settings (as they are simply the almostification of the classical derived tensor product, see [Man22, Lemma 2.2.10.(v)], [Man22, Prop 2.1.11] and [Zav24a, Prop. 2.4.13]), the internal Hom coincides.

The first isomorphism now follows from Zavyalov's description of the internal Hom [Zav24a, Def 2.4.4], together with [Zav24a, Prop 2.4.8]. $\square$

Remark 3.5.10. *It is also possible to deduce lemma 3.5.9 directly from Mann's construction. We can reduce to the derived internal Hom by establishing showing the commutation of the almostification $(-)^a$ with the discretization $(-)_{\omega}$.*

It suffices to show that the left adjoints $(-)_{!}$ and i commute, which follows from an explicit computation.

Compatibility with classical étale cohomology

Content of this chapter

This chapter is the technical heart of the present thesis. Its goal is to identify Mann's $f_!^M$ and Huber's $Rf_!^H$ (whose construction will be shortly recalled in paragraph [section 4.2](#)) for partially proper morphisms f . The precise statement is as follows :

Proposition 4.0.1. (cf. [proposition 4.5.1](#))

Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a partially proper locally noetherian analytic adic space over C . Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -modules on X .

There is a natural isomorphism in $\mathcal{D}_{\text{ét}}(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^{+,a}/p)$:

$$J_C (f_!^M (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p)) \cong (Rf_!^H (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a,$$

where J_C denotes the identification functor from [proposition 3.5.3](#).

In order to do so, we prove that $f_!^M$ preserves discrete objects, and establish localization : sequences in both Mann's formalism and a quasicohherent version of $\mathcal{O}^+/p$ -sheaves :

$$R\Gamma_c(X, \mathcal{O}^+/p) \rightarrow R\Gamma(X, \mathcal{O}^+/p) \rightarrow \varinjlim_{i \in I} R\Gamma(X \setminus U_i, \mathcal{O}^+/p), \quad (4.1)$$

where the colimit is taken a well-chosen covering of X , constructed in [Appendix A](#) (namely, [proposition A.3.3](#)). We may now deduce the result for $f_!$ from the comparison for the middle and rightmost term established in [corollary 3.5.6](#).

In [Appendix A](#), we also contains a discussion on how to endow complements of partially proper open subvarieties with a structure of a locally spatial diamond, as well as a localization sequence for open-closed decomposition, both results that will be used throughout this chapter. We encourage the reader to familiarise themselves with [Appendix A](#) before diving into the present chapter.

4.1 Overview

The proof of [proposition 4.0.1](#) follows the following program :

Proof sketch 4.1.1. Fix an open covering of X by partially proper open subsets $(U_i)_{i \in I}$, as constructed in [proposition A.3.3](#). We proceed in four steps, as follows :

1. Prove that, for any standard étale sheaf $\mathcal{F}$ of $\mathcal{O}_X^+/p$ -modules on X , there is an exact sequence in $\mathcal{D}_{\text{ét}}(\text{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^+/p)$:

$$Rf_!^H \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ i_*^{X \setminus U_i, S} \circ i_S^{X \setminus U_i, *} \mathcal{F},$$

where, $i^{X \setminus U_i}$ denotes the closed immersion of locally spatial diamonds $X \setminus U_i \rightarrow X$.

2. Prove that $f_!^M$ preserves discrete objects.
3. Prove that, for $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_\omega$, there is an exact sequence in $\mathcal{D}_{\square}^a(\text{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^+/p)_\omega$:

$$f_!^M \mathcal{F} \rightarrow (f_*^M \mathcal{F})_\omega \rightarrow \varinjlim_{i \in I} \left(f_*^M \circ i_*^{X \setminus U_i, M} \circ i_M^{X \setminus U_i, *} \mathcal{F} \right)_\omega,$$

where the notations are the same as in the first point.

4. Identify the second and third terms of the two exact sequences above, and deduce an isomorphism between the first terms.

We will tackle the different points in order.

The first point is an essentially standard result (cf. [\[AGN25, Cor 2.17\]](#)), but is usually written for rigid analytic varieties, where the colimit is taken along a covering by quasi-compact open subsets, and the complement is taken in the rigid-analytic sense, as in [lemma A.1.1](#).

4.2 Proper pushforward for almost quasi-coherent classical sheaves

First, let us recall Huber's definition of direct image with compact support $Rf_!^H$ [\[Hub96, Def 5.2.1\]](#) for a partially proper morphism f .

For any separated morphism of locally noetherian adic spaces $f : X \rightarrow Y$, we may define a left exact functor $f_!^H : X_{\text{ét}}^\sim \rightarrow Y_{\text{ét}}^\sim$ by :

$$f_!^H \mathcal{F} : U \in Y_{\text{ét}} \mapsto \{s \in \Gamma(U \times_Y X, \mathcal{F}), \text{supp}(s) \text{ is proper over } U\}.$$

Whenever f is étale, the functor $f_!^H$ is exact, is left adjoint to f_H^* by [\[Hub96, Lemma 2.7.6.\(i\) and Prop 5.2.4\]](#). When f is proper, $f_!^H = f_*^H$.

For a general f , we should not expect the derived functors of $f_!^H$ to behave nicely - rather, for morphisms f that can be written as a composition $f = p \circ i$ of a proper morphism p and an open immersion i , we should define the derived pushforward $Rf_!^H$ as $Rp_*^H \circ i_!^H$, where $i_!^H$ implicitly denotes the extension of the exact functor to the derived category.

However, for partially proper morphism f of analytic adic spaces, the derived functors of $f_!^H$ admit reasonable properties. In [Hub96], Huber considers a right derived functor $R^+f_!^H : D^+(X_{\text{ét}}, \mathbb{Z}) \rightarrow D^+(Y_{\text{ét}}, \mathbb{Z})$ at the level of the bounded below derived 1-categories, and additionally constructs a fully derived functor $Rf_!^H : D(X_{\text{ét}}, \mathbb{Z}) \rightarrow D(Y_{\text{ét}}, \mathbb{Z})$ under some finite dimensionality assumption on f . However, using [Sta23, Lemma 070K] and the discussion at the beginning of [Sta23, Section 07A5], we may define a right derived functor $Rf_!^H : D(X_{\text{ét}}, \mathbb{F}_p) \rightarrow D(Y_{\text{ét}}, \mathbb{F}_p)$ for any partially proper morphism f , as the unbounded derived functors of $f_!$. Indeed, the categories of modules on a site form Grothendieck abelian categories (cf. [Sta23, Section 07A5]).

This construction naturally admits an ∞ -categorical enhancement (cf. [Lur17, Sec 1.3.5]), i.e. there exists a canonical morphism of ∞ -categories $Rf_!^H : \mathcal{D}(X_{\text{ét}}, \mathbb{F}_p) \rightarrow \mathcal{D}(Y_{\text{ét}}, \mathbb{F}_p)$, that induces the morphism above at the level of homotopy categories, and it passes to left completions to a morphism $Rf_!^H : \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p) \rightarrow \mathcal{D}_{\text{ét}}(Y, \mathbb{F}_p)$. Note that all of Huber's functors have been extended to the ∞ -categorical setting in [Zav25, Sec 9].

Likewise, we can check that, for any partially proper morphism $f : X \rightarrow Y$ and any sheaf $\mathcal{F}$ of $\mathcal{O}_X/p$ -modules on X , the sheaf $f_!^H \mathcal{F}$ admits a natural structure of an $\mathcal{O}_Y/p$ -module, obtained by viewing every section $f_!^H \mathcal{F}(U)$ as a submodule of $\Gamma(U \times_Y X)$.

Hence, the derived functor $Rf_!^H$ induces a functor $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/p) \rightarrow \mathcal{D}_{\text{ét}}(Y, \mathcal{O}_Y^+/p)$.

Remark 4.2.1. This approach is not the one of Scholze [Sch17, Def 22.13]. In loc. cit, he defines, for j an open immersion, $j_!^S$ as the left adjoint of j_S^* . For a general f , the pushforward $f_!^S$ is defined by Kan extension from the setup where one may write $f = p \circ j$ as the composition of a proper map and an open immersion (with $p_!^S := p_*^S$).

It is however clear from definitions that, if $j : X \rightarrow Y$ is a separated open immersion between locally noetherian adic spaces, $j_H^* \cong j_S^*$ so that, by uniqueness of left adjoints, $j_!^H \cong j_!^S$.

Let us establish the following result.

Lemma 4.2.2. Let $f : X \rightarrow Y$ be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y . Let $(U_i)_{i \in I}$ be an increasing covering of X by partially proper open subsets as constructed in proposition A.3.3.

Then, for any sheaf $\mathcal{F}$ of abelian groups over $X_{\text{ét}}$, there is a natural isomorphism in $\mathcal{D}_{\text{ét}}(X, \mathbb{Z})$:

$$\varinjlim_{i \in I} Rf_*^H \circ j_{U_i!}^H \circ j_{U_i, H}^* \mathcal{F} \cong Rf_!^H \mathcal{F}.$$

Remark 4.2.3. For technical reasons, the result is stated with a colimit alongside partially proper open, but the colimit is the same when taken along qcqs open subsets, as one can take intertwining cofinal coverings as in proposition A.3.3.

Proof. For any $i \in I$, there is a natural morphism $Rj_{U_i!}^H j_{U_i, H}^* \mathcal{F} \rightarrow \mathcal{F}$ induced by the counit of the adjunction. Everything lives in the bounded below subcategory $\mathcal{D}^+(X_{\text{ét}}, \mathbb{F}_p) \subset \mathcal{D}_{\text{ét}}(X, \mathbb{F}_p)$, so that we may forget about the left completions.

It suffices to check that it induces an isomorphism on homotopy groups. By the proof of [Lur17, Prop 1.3.5.21], since we're working in the derived ∞ -category of a Grothendieck abelian category, the functors π_k commute with filtered colimits, so that, for all $n \in \mathbb{N}$:

$$\pi_{-n} \left(\varinjlim_{i \in I} Rf_*^H \circ j_{U_i!}^H \circ j_{U_i,H}^* \mathcal{F} \right) \cong \varinjlim_{i \in I} R^n f_*^H \circ j_{U_i!}^H \circ j_{U_i,H}^* \mathcal{F}.$$

We easily check that the covering $(U_i)_{i \in I}$ satisfies the conditions of [Hub96, Lemma 5.3.3.(ii)], so that the result follows from [Hub96, Lemma 5.3.3.(iii)]. $\square$

We may now establish the announced exact sequence.

Proposition 4.2.4. *Let $f : X \rightarrow Y$ be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y . Let $(U_i)_{i \in I}$ be an increasing covering of X by partially proper open subsets as constructed in proposition A.3.3.*

For any $i \in I$, we let Z_i be the complement of the partially proper U_i , viewed as a closed locally spatial subdiamond of $X^\diamond$ by lemma A.1.3. Let $i_{Z_i} : Z_i \rightarrow X^\diamond$ be the associated closed immersion. Then, for any sheaf $\mathcal{F}$ of abelian groups over $X_{\text{ét}}$, there is a natural exact sequence in $\mathcal{D}_{\text{ét}}(Y, \mathbb{Z})$:

$$Rf_!^H \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ i_{Z_i*}^S \circ i_{Z_i,S}^* \mathcal{F}.$$

Proof. Using the localization sequence established in proposition A.2.1, there is a short exact sequence of sheaves of abelian groups on $Y_{\text{ét}}$:

$$0 \rightarrow j_{U_i!}^S j_{U_i,S}^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_{Z_i*}^S i_{Z_i,S}^* \mathcal{F} \rightarrow 0.$$

This induces an exact sequence in $\mathcal{D}_{\text{ét}}(X, \mathbb{Z})$. Applying the exact functor Rf_*^H yields the exact sequence :

$$Rf_*^H \circ j_{U_i!}^S \circ j_{U_i,S}^* \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow Rf_*^H \circ i_{Z_i*}^S \circ i_{Z_i,S}^* \mathcal{F}.$$

We then form the colimit over I . Since the morphisms from passing from i to some $j > i$ are induced by (co)units of the adjunctions, the colimit diagram is homotopy coherent, and, since colimits of exact sequences remain exact (as they can be viewed as cofiber sequences), we have :

$$\varinjlim_{i \in I} Rf_*^H \circ j_{U_i!}^S \circ j_{U_i,S}^* \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ i_{Z_i*}^S \circ i_{Z_i,S}^* \mathcal{F}.$$

By remark 4.2.1, the functors $j_{U_i!}^*$ and $j_{U_i!}$ coincide in Huber and Scholze's formalism, so that the first term identifies to $Rf_!^H \mathcal{F}$ by lemma 4.2.2, which concludes. $\square$

Now, we prove a version for sheaves of $\mathcal{O}^+/p$ -modules.

Corollary 4.2.5. *Consider the setup of proposition 4.2.4 over K , and let $\mathcal{F}$ be an étale sheaf of $\mathcal{O}_X^+/p$ -modules on X . Then, there is an exact sequence :*

$$Rf_!^H \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ i_{*,H}^{Z_i} \circ i_{Z_i,H}^{*,qcoh} \mathcal{F}.$$

Proof. By [lemma A.2.5](#), the quasi-coherent pullback $i_{Z_i, H}^{*, qcoh}$ coincides with the pullback $i_{Z_i, H}^*$ as étale sheaves, so that this directly follows from [proposition 4.2.4](#), since the natural forgetful functor $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/p) \rightarrow \mathcal{D}_{\text{ét}}(X, \mathbb{Z})$ respects exact sequences. $\square$

We conclude this paragraph by noting that we can deduce from the above a version that remains in the realm of analytic adic spaces, in the spirit of [\[AGN25, Cor 2.17\]](#). Following [lemma A.1.2](#), we replace the Z_i with the interior of the complement of some qcqs open subspaces. This is what is done in the following :

Corollary 4.2.6. *Let $f : X \rightarrow Y$ be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y . Let $(X_i)_{i \in I}$ be an increasing covering of X by quasicompact open subsets.*

For any $i \in I$, let $Y_i = (X \setminus X_i)^\circ = X \setminus \overline{X_i}$, which admits an analytic structure as an open subset of X (and is the rigid-analytic complement of X_i , via [lemma A.1.2](#)). Let $j_{Y_i} : Y_i \rightarrow X$ denote the associated open immersion. Then, for any étale sheaf $\mathcal{F}$ of abelian groups on $X_{\text{ét}}$, there is an exact sequence :

$$Rf_!^H \mathcal{F} \rightarrow Rf_*^H \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ j_{Y_i}^H \circ j_{Y_i, H}^* \mathcal{F}.$$

Proof. Up to reducing I , we may assume that, for any i , there is only finitely many j such that $j < i$. Following proof of [proposition A.3.3](#), we may construct partially proper open subsets $(U_i)_{i \in I}$ such that $X_i \subset U_i \subset X_j$ for all $i < j$, satisfying the properties of [proposition A.3.3](#).

We reuse the notations from [proposition 4.2.4](#). It suffices to check that there is a canonical isomorphism $\varinjlim_{i \in I} Rf_*^H \circ i_{F_i}^H \circ i_{F_i, H}^* \mathcal{F} \rightarrow \varinjlim_{i \in I} Rf_*^H \circ j_{Z_i}^H \circ j_{Z_i, H}^* \mathcal{F}$.

From the properties of the covering, for any $i < j$ in I , we have $Y_j \subset Z_j \subset Y_i$, so that both families of subsets are cofinal in one another. Since the category $\mathcal{D}(X_{\text{ét}}, \mathbb{Z})$ admits all colimits, the colimit alongside alternating Z and Y 's exists, and can be computed alongside one family or the other, so that both two colimits coincide. $\square$

4.3 Mann's proper pushforward preserves discrete objects

Let us proceed with the established program in [proof sketch 4.1.1](#). This paragraph is dedicated to the proof of the following :

Proposition 4.3.1. *Let $f : X \rightarrow Y$ be a partially proper morphism of locally noetherian analytic adic spaces over K .*

Then, for any $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_\omega$, the sheaf $f_!^M \mathcal{F}$ is discrete in $\mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/p)$.

To avoid busy notations, until the end of this paragraph, all functors are implicitly in the sense of Mann, and we drop the superscript "M".

We will reduce the case where f factors as the composition of a partially proper open immersion and a proper morphism. The pushforward by a proper morphism preserves discrete objects by [\[Man22, Lemma 3.3.10.\(ii\)\]](#), so that it suffices to deal with open immersions. We use the following lemma :

Lemma 4.3.2. *Let $(I, \leq)$ be a filtered partial order, and $(U_i)_{i \in I}$ be an increasing open covering of a locally spatial diamond X over K . For any $i \in I$, let $g_i : U_i \rightarrow X$ denote the open immersion.*

Then, there is a natural isomorphism $\mathcal{F} \cong \varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F}$ for any $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$.

Proof. For any $j \in I$, $g_j^* = g_j^!$ admits the left adjoint $g_j^!$. The counit of the adjunctions defines a morphism $\varinjlim_{i \in I} g_i^! g_i^* \mathcal{F} \rightarrow \mathcal{F}$.

Let $Y := \bigsqcup_{i \in I} U_i$. By (hyper-)v-descent [Man22, Thm 1.2.1], we have :

$$\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p) \cong \varprojlim_{n \in \Delta} \mathcal{D}_{\square}^a(Y_n, \mathcal{O}_{Y_n}^+/p) \text{ where } Y_n = Y^{\times n} = \bigsqcup_{i_1, \dots, i_n \in I} U_{i_1} \times_X \cdots \times_X U_{i_n}.$$

Hence, it suffices to check the isomorphism after pulling back to every $U_{i_1} \times_X \cdots \times_X U_{i_n}$, for fixed $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in I$. Since all the U_i are open subsets of X , the fiber product is simply the intersection, and, since I is filtered, every such morphism $U_{i_1} \times_X \cdots \times_X U_{i_n} \rightarrow X$ factors through U_j for some large enough $j \in I$. Hence, it suffices to prove that, for all $j \in J$, the map :

$$g_j^* \left(\varinjlim_{i \in I} g_i^! g_i^* \mathcal{F} \right) \rightarrow g_j^* \mathcal{F}$$

is an isomorphism. Since g_j^* admits a right adjoint, it commutes with colimits, so that it suffices to prove that :

$$\varinjlim_{i \in I} g_j^* g_i^! g_i^* \mathcal{F} \cong g_j^* \mathcal{F}$$

Since the covering is increasing, the fiber product $U_i \times_X U_j$ is $U_{\min(i,j)}$. Since I is filtered, the subset $J = \{i \in I, i \geq j\}$ is cofinal in I , so that we may compute the limit alongside J . For any $i \in J$, consider the following cartesian diagram :

$$\begin{array}{ccc} U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & X \end{array}$$

By proper base change (cf. [Man22, Thm 1.2.4]), we see that, for all $i \in J$, $g_j^* g_i^! g_i^* \mathcal{F} \cong g_j^* \mathcal{F}$. Hence, the colimit along J is trivial, which concludes. $\square$

Remark 4.3.3. Under the setup of lemma 4.3.2, a similar argument shows that : $\mathcal{F} \cong \varprojlim_{i \in I} g_{i*}^M g_{iM}^* \mathcal{F}$ for any $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$.

Moreover, using a technical lemma on $(\infty, 2)$ -categories [HM24, p. D.4.7], we may improve the adjunction triangle $g_{i!}^M \dashv g_{i,M}^* \dashv g_{i*}^M$ to the expected adjunctions between the following functors :

- The map :

$$(\mathcal{F}_i)_{i \in I} \in \varprojlim \mathcal{D}_{\square}^a(U_i, \mathcal{O}_{U_i}^+/p) \mapsto \varinjlim_{i \in I} g_i^! \mathcal{F}_i \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p);$$

- A right adjoint the above map :

$$\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p) \mapsto (g_i^* \mathcal{F})_{i \in I} \in \varprojlim_{i \in I} \mathcal{D}_{\square}^a(U_i, \mathcal{O}_{U_i}^+/p);$$

- A right adjoint of the above map :

$$(\mathcal{F}_i)_{i \in I} \in \varprojlim_{i \in I} \mathcal{D}_{\square}^a(U_i, \mathcal{O}_{U_i}^+/p) \mapsto \varinjlim_{i \in I} g_{i*} \mathcal{F}_i \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p).$$

Here, the limits $\varprojlim_{i \in I} \mathcal{D}_{\square}^a(U_i, \mathcal{O}_{U_i}^+/p)$ is taken along pullbacks. The above proposition shows that the counit of the first adjunction and the unit of the second one are isomorphisms, both these statements being equivalent (cf. [Lur25, Corollary 02FF]), to the full faithfulness of the middle arrow.

Let us now prove that partially proper open immersions preserve discrete objects. We start by doing so for totally disconnected perfectoid spaces. Note that the proof follows roughly [Zav23, Lemma 6.2.1 and 6.2.2], which moreover assumes U to be a Zariski open subset.

Lemma 4.3.4. *Let X be a totally disconnected perfectoid space over K and $j : U \rightarrow X$ be a partially proper open immersion. Then $j_!$ preserves discrete objects.*

Proof. The map j is partially proper, hence specializing, in the sense that the image $j(U)$ is stable under specialization. We let $V = |j(U)|$.

Consider the natural quotient map $\pi : |X| \rightarrow \pi_0(X)$. Since X is a totally disconnected perfectoid space, by definition, every connected component of $|X|$ admits a unique closed point, to which any other point in the connected component specializes. Since V is open and stable under specializations, V contains the connected component of every $x \in U$. Hence, $V = \pi^{-1}(\pi(V))$. Since π is a quotient map, it follows that $\pi(V)$ is open in $\pi(X)$.

By [Sta23, Lemma 0906], $\pi_0(X)$ is profinite, so that it admits a basis of clopen subsets by lemma B.1.12. Thus, $\pi(V)$ can be written as a filtered union of quasi-compact clopen sets ; so that the same holds for $V = \pi^{-1}(\pi(V))$. Hence, there exists a filtered poset I such that $V = \bigcup_{i \in I} V_i$ is a filtered union of clopen subsets of $|X|$. For all $i \in I$, let $U_i = j^{-1}(V_i) \subset U$, which admits an analytic structure as an open subset of U .

Consider the map $j_i : U_i \rightarrow X$, which is the inclusion of a clopen subset of X . Let $\mathcal{F} \in \mathcal{D}_{\square}^a(U, \mathcal{O}_U^+/p)_{\omega}$.

By the lemma 4.3.2, $\mathcal{F} \cong \varinjlim_i j_{i!} j_i^* \mathcal{F}$, so that $j_! \mathcal{F} \cong j_! \varinjlim_i j_{i!} j_i^* \mathcal{F} \cong \varinjlim_i (j \circ j_i)_! j_i^* \mathcal{F}$,¹ since $j_!$ commutes with colimits (as it admits a right adjoint).

By definition, pullback preserves discrete objects, and $j \circ j_i$ is the immersion of clopen subset of X , so that $(j \circ j_i)_! = (j \circ j_i)_*$, which, by [Man22, 3.3.10.(ii)], preserves discrete objects. Finally, by [Man22, lemma 3.2.19.(ii)], (small) colimits of discrete objects remain discrete, so that $j_! \mathcal{F}$ is discrete. $\square$

Let us now establish the result for general partially proper open immersions.

Lemma 4.3.5. *Let $j : U \rightarrow X$ be a partially proper open immersion of locally spatial diamonds over K . Then $j_!$ preserves discrete objects.*

Proof. Fix a discrete object $\mathcal{F} \in \mathcal{D}_{\square}^a(U, \mathcal{O}_U^+/p)$. Let us reduce to the situation where X is a totally disconnected perfectoid space.

As discreteness is defined by descent (cf [Man22, Def 3.2.17]), we need to show that, for every morphism $f : \tilde{X} \rightarrow X$ from a totally disconnected perfectoid space $\tilde{X}$, the sheaf $f^* j_! \mathcal{F}$ is discrete in $\mathcal{D}_{\square}^a(\tilde{X}, \mathcal{O}_{\tilde{X}}^+/p)$. Consider the cartesian square below :

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{j'} & \tilde{X} \\ f' \downarrow & & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

¹By unraveling the proof of [Man22, Lemma 3.6.2], this is actually the definition of $j_!$ for arbitrary open immersions.

By proper base change [Man22, Thm 1.2.4.(iv)], $f^* j_! \mathcal{F} \cong j'_! f'^* \mathcal{F}$.

By definition [Sch17, Def 10.7], j' is a separated open immersion. Note that the valuative criteria from [Sch17, Def 18.4] is automatically stable under base change, so that j' remains partially proper, and, by lemma 4.3.4, $j'_!$ preserves discrete objects.

Finally, pullbacks preserve discrete objects by definition, so that $j'_! f'^* \mathcal{F}$ is discrete, which concludes. $\square$

We are now equipped to prove proposition 4.3.1.

Proof. (of proposition 4.3.1). Using proposition A.3.3, we write $X = \bigcup_{i \in I} X_i$ as an increasing filtered cover by qcqs open subsets, such that, for any $i < j$, the inclusion $X_i \subset X_j$ factors through a partially proper U_i and such that the map $X_i \rightarrow X_j$ factors through Huber's universal compactification $\overline{X}_i^Y$ of X_i over Y . Denote $f_i : X_i \rightarrow X$ and $g_i : U_i \rightarrow X$ the associated open immersions.

By lemma 4.3.2, there is an isomorphism $\mathcal{F} \cong \varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F}$.

Hence, $f_! \mathcal{F} = f_! \left(\varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F} \right) = \varinjlim_{i \in I} (f \circ g_i)_! g_i^* \mathcal{F}$, since left adjoints commute with colimits. By [Man22], Lemma 3.2.19.(ii), a (small) colimit of discrete objects remains discrete, so that, as pullbacks preserve discrete objects, it suffices to show that every $(f \circ g_i)_!$ preserves discrete objects.

Here, $g_i : U_i \rightarrow X$ is the inclusion of a partially proper open subset in X . Moreover, the partially proper map $f \circ g_i$ factors through X_i , then, for any $i < j$, through the adic compactification $\overline{X}_j^Y$ of X_j over Y , by the universal property of universal compactifications [Hub96, Def 5.1.1]. We then have the following commutative diagram, where $U_i \rightarrow Y$ is partially proper :

$$\begin{array}{ccccc}
 & & g_i & & \\
 & & \curvearrowright & & \\
 U_i & \hookrightarrow & \overline{X}_j^Y & \hookrightarrow & X \\
 & \searrow & \searrow & & \downarrow f \\
 & & & & Y
 \end{array}$$

By [Hub96, Cor 5.1.6], the map $\overline{X}_j^Y \rightarrow Y$ is proper, so that, by the two-out-of-three property for partially proper morphisms [Hub96, Lemma 1.10.17.(vi)], the open immersion $U_i \rightarrow \overline{X}_j^Y$ is additionally partially proper.

Hence, the composition $g_i \circ f$ factors as the composition of a partially proper open immersion and a proper morphism, and both g_i and f preserve discrete objects by the lemma 4.3.5 and [Man22, Lemma 3.3.10.(ii)] respectively. This concludes. $\square$

4.4 Localization sequence in $\mathcal{D}_{\square}^a(-, \mathcal{O}^+/p)$

We start with the following localization sequence for open-closed decomposition, which is an analogue, in Mann's formalism, of proposition A.2.1.

Proposition 4.4.1. *Let X be an untilted locally spatial diamond over K , together with a partially proper open immersion $j : U \rightarrow X$. By lemma A.1.3, the complement $F = |X| \setminus j(|U|)$ admits a structure of a locally spatial diamond, and we let $i : F \rightarrow X$ be the corresponding closed immersion.*

Let $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$. Then, there is a natural exact sequence in $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$:

$$j_!^M j_M^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*^M i_M^* \mathcal{F}.$$

As in the previous section, every functor will implicitly be in the sense on Mann. Note that the proof resembles the one of [Zav23, Lemma 6.2.2], which only considers Zariski open subsets.

Proof. The morphisms are naturally (co)-unit morphism, so that it suffices to prove the exactness of the sequence. This can be checked on a basis, i.e. after pullback to any totally disconnected perfectoid space. By proper base change, we may assume that X admits a map to a perfectoid space².

Since $\mathcal{O}_X^+/p$ is the unit of Mann's tensor product $\otimes$ on X , we may write $\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_X^+/p$ and, using the projection formula (cf. [proposition 3.1.4](#)), the sequence reduces to :

$$\mathcal{F} \otimes j_! j^* \mathcal{O}_X^+/p \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes i_* i^* \mathcal{O}_X^+/p.$$

Since tensoring with $\mathcal{F}$ is exact, we reduced to the case $\mathcal{F} = \mathcal{O}_X^+/p$, which is discrete. By the above [proposition 4.3.1](#) and [Man22, Lemma 3.3.10], all objects appearing in the triangle are discrete. Recall that there is a functorial equivalence of categories $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)_{\omega} \cong \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+,a}/p)^{oc}$, cf. [proposition 3.5.3](#). It then suffices to check that the above is exact as a sequence of almost étale sheaves, which are étale sheaves valued in an abelian category.

Since the étale site $X_{\acute{e}t}$ admits enough points by [Sch17, Prop 14.3], it suffices to check the isomorphism after pullback to geometric points of the form $\mathrm{Spa}(C, C^+)$, for some complete algebraically closed C , with open bounded valuation subring C^+ . Here, the pullback is to be taken in the quasi-coherent sense, and it coincides with f_M^* by [proposition 3.5.3](#) (we implicitly use the identification of [proposition 3.5.3](#)). Let $f : \mathrm{Spa}(C, C^+) \rightarrow X$ be such a morphism.

By proper base change applied to the same diagram as in the proof of [proposition A.2.1](#), we reduce to the case where X is a totally disconnected perfectoid space, and U is a partially proper open subset of X , and F is its complement. Following the proof of [proposition A.2.1](#), U is either empty or $U = X$, so that, in both cases, the exactness is trivial. $\square$

Let us now construct the announced exact sequence.

Lemma 4.4.2. *Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a partially proper analytic adic space. Let $(U_i)_{i \in I}$ be an open covering of X by partially proper open subsets, as in [proposition A.3.3](#).*

For any $i \in I$, we let Z_i be the complement of the partially proper U_i , viewed as a closed locally spatial subdiamond of $X^{\diamond}$ by [lemma A.1.3](#). Let $i_{Z_i} : Z_i \rightarrow X^{\diamond}$ be the associated closed immersion. Then, for $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)$, there is a natural exact sequence :

$$f_! \mathcal{F} \rightarrow f_* \mathcal{F} \rightarrow \varinjlim_{i \in I} f_* i_{Z_i,*} i_{Z_i}^{*,M} \mathcal{F}.$$

Note that this resembles [ABM24, Prop 6.2.5]. The proof follows the same ideas.

²This hypothesis is needed to use [proposition 3.5.3](#).

Proof. Let $j_{U_i} : U_i \rightarrow X$ denote the associated open immersions. Starting from the excision triangle from the above [proposition 4.4.1](#), we apply f_* and take colimits, as in the proof of [proposition 4.2.4](#). This yields the following exact sequence :

$$\lim_{i \in I} f_* \circ j_{U_i}! \circ j_{U_i}^* \mathcal{F} \rightarrow f_* \mathcal{F} \rightarrow \lim_{i \in I} f_* \circ i_{Z_i}^* \circ i_{Z_i}^* \mathcal{F}.$$

Let $(X_i)_{i \in I}$ be a covering of X by quasi-compact opens subsets, associated to $(U_i)_{i \in I}$ as in [proposition A.3.3](#).

Let us now establish that $f_* \circ j_{U_i}! \cong f_! \circ j_{U_i}!$.

Consider the following diagram for any $i < j \in I$, where we let $Y := \mathrm{Spa}(K, \mathcal{O}_K)$, and $\overline{X_j}^Y$ denote the universal adic compactification of X_j over Y :

$$\begin{array}{ccccc} U_i & \xrightarrow{j_{U_i}} & X & \xrightarrow{f} & Y \\ & \searrow k_{i,j} & \uparrow \pi_j & \nearrow p_j & \\ & & \overline{X_j}^Y & & \end{array}$$

By [[Hub96](#), Cor 5.1.6], since X_j is quasi-compact, p_j is proper, and so is π_j by [[Hub96](#), Lemma 1.10.17.(vi)] since f is separated. Hence :

$$f_! \circ j_{U_i}! = p_j! \circ k_{i,j}! = p_j^* \circ k_{i,j}! = f_* \circ \pi_j! \circ k_{i,j}! = f_* \circ j_{U_i}!.$$

Hence, using [lemma 4.3.2](#) and the fact that $f_!$ commutes with colimits (as it admits a right adjoint), we have

$$\lim_{i \in I} f_* j_{U_i}! j_{U_i}^* \mathcal{F} \cong \lim_{i \in I} f_! j_{U_i}! j_{U_i}^* \mathcal{F} \cong f_! \left(\lim_{i \in I} j_{U_i}! j_{U_i}^* \mathcal{F} \right) \cong f_! \mathcal{F}$$

So that the first term of the above exact sequence identifies with $f_! \mathcal{F}$, which concludes. $\square$

Let us now apply the discretization functor, to get the sequence announced in [proof sketch 4.1.1](#).

Corollary 4.4.3. *In the situation above, there is a natural exact sequence :*

$$f_! \mathcal{F} \rightarrow (f_* \mathcal{F})_\omega \rightarrow \lim_{i \in I} \left(f_* i_{Z_i,*} i_{Z_i}^{*,M} \mathcal{F} \right)_\omega.$$

Proof. Apply the exact functor $(-)_\omega$ from [proposition 3.5.2](#) to previous result, which commutes with colimits by the [lemma 4.4.6](#) below. The object $f_! \mathcal{F}$ is discrete by [proposition 4.3.1](#), which concludes. $\square$

Remark 4.4.4. As in [corollary 4.2.6](#), we may rewrite the right hand side as a colimit alongside the complement of any reasonable covering, using a cofinality argument.

It then suffices to check the commutation of $(-)_\omega$ with colimits. The following two lemmas are purely technical, and not very insightful.

Lemma 4.4.5. *Let $\mathcal{C}$ be an ∞ -category that admits colimits and such that finite products coincide with finite coproducts (for example, an additive category). Let κ be a strong limit cardinal.*

Let $\mathrm{Cond}(\mathcal{C})$ be the category of condensed objects in $\mathcal{C}$ as constructed in [[Sch19](#), p.15], and $(-)_\kappa : \mathrm{Cond}(\mathcal{C}) \rightarrow \mathrm{Cond}(\mathcal{C})_\kappa$ be the functor obtained by restricting to sheaves on κ -small extremally disconnected sets.

Then, the functor $(-)_\kappa$ commutes with colimits.

Proof. Let $(\mathcal{F}_i)_{i \in I} \in \text{Cond}(\mathcal{C}) = \varinjlim_{\lambda} \text{Cond}(\mathcal{C})_{\lambda}$, where the colimit is computed along strict limit cardinals. Recall from [Sch19, Sec 2] that objects of $\text{Cond}(\mathcal{C})$ can be seen as $\mathcal{C}$ -valued sheaves from the category of extremally disconnected compact Hausdorff sets, and that colimits are given by sheafifying the pointwise colimit.

The sheaf condition for a presheaf on such extremally disconnected rewrites as mapping finite disjoint unions to finite products. Since finite products coincide with finite coproducts in $\mathcal{C}$, the pointwise colimit of sheaves remains a sheaf, and there is no need to sheafify.

Hence, for all κ -small extremally disconnected set S , the evaluation $\mathcal{F} \in \text{Cond}(\mathcal{C}) \mapsto \mathcal{F}(S)$ commutes with colimits, so that $(-)_\kappa$ commutes with colimits. $\square$

Lemma 4.4.6. *Let (A, A^+) be a perfectoid pair with pseudo-uniformizer π , and κ be a strong limit cardinal. Let $\mathcal{D}_{\square}^a(A^+/\pi)$ be Mann's category of solid almost A^+/π -modules, and $(-)_{\kappa,a} : \mathcal{D}_{\square}^a(A^+/\pi) \rightarrow \mathcal{D}_{\square}^a(A^+/\pi)_{\kappa}$ be the restriction to κ -small objects constructed in [Man22, Def 3.2.20].*

Then $(-)_{\kappa,a}$ commutes with colimits.

Proof. Since A^+/π is discrete, it follows from [Man22, Def 2.3.28] that the almostification functor $(-)^a : \mathcal{D}_{\square}(A^+/\pi) \rightarrow \mathcal{D}_{\square}^a(A^+/\pi)$ restricts to a functor $(-)^{a,\kappa} : \mathcal{D}_{\square}(A^+/\pi)_{\kappa} \rightarrow \mathcal{D}_{\square}^a(A^+/\pi)_{\kappa}$, that is essentially surjective by definition [Man22, Def 2.3.3].

Consider the following commutative diagram :

$$\begin{array}{ccc} \mathcal{D}_{\square}(A^+/\pi)_{\kappa} & \xrightarrow{I_{\kappa}} & \mathcal{D}_{\square}(A^+/\pi) \\ (-)^{a,\kappa} \downarrow & & \downarrow (-)^a \\ \mathcal{D}_{\square}^a(A^+/\pi)_{\kappa} & \xrightarrow{I_{\kappa}^a} & \mathcal{D}_{\square}^a(A^+/\pi). \end{array}$$

Here, I_{κ} and I_{κ}^a are the inclusion of full subcategories, and the functor $(-)^{\kappa,a}$ that we are interested in is defined as the right adjoint of I_{κ}^a . Recall that the vertical arrows are localization functors, and that $(-)^a$ admits both a fully faithful left adjoint $(-)_{!}^a$ and a fully faithful right adjoint $(-)_{*}^a$. Likewise, $(-)^{a,\kappa}$ admits fully faithful left and right adjoints denoted respectively $(-)_{!}^{\kappa,a}$ and $(-)_{*}^{\kappa,a}$.

The morphism I_{κ} admits a right adjoint, that we denote $(-)_{\kappa}^{\square}$, defined as the restriction to the full subcategories of solid objects of the functor $(-)_{\kappa} : \text{Cond}(\mathcal{D}(A^+/\pi)) \rightarrow \text{Cond}(\mathcal{D}(A^+/\pi))_{\kappa}$ considered in lemma 4.4.5, that commutes with colimits by said lemma (here, $\mathcal{D}(A^+/\pi)$ denotes the ∞ -derived category of the abelian category of A^+/π -modules). Hence $(-)_{\kappa}^{\square}$ also commutes with colimits.

Since $(-)^{a,\kappa}$ admits the fully faithful left adjoint $(-)_{!}^{\kappa,a}$, the unit morphism $id \rightarrow (-)^{a,\kappa} \circ (-)_{!}^{\kappa,a}$ is a natural isomorphism. It then follows from the commutation of the diagram that $I_{\kappa}^a \cong (-)^a \circ I_{\kappa} \circ (-)_{!}^{\kappa}$, where all three functors in the composition admit a right adjoint, so that we may write $(-)_{\kappa,a} \cong (-)^{a,\kappa} \circ (-)_{\kappa}^{\square} \circ (-)_{*}$.

Since the functor $(-)_{\kappa}^{\square}$ respects almost isomorphisms (as this is a pointwise condition), it follows from the universal property of localizations [Lur08, Prop 5.2.7.12] that there exists a functor $J : \mathcal{D}_{\square}^a(A^+/\pi) \rightarrow \mathcal{D}_{\square}^a(A^+/\pi)_{\kappa}$ such that $J \circ (-)^a \cong (-)^{a,\kappa} \circ (-)_{\kappa}^{\square}$.

As the right adjoint $(-)_{*}$ of $(-)_{!}^a$ is fully faithful, the counit morphism $(-)^a \circ (-)_{*} \rightarrow id$ is a natural isomorphism, so that $J \cong J \circ (-)^a \circ (-)_{*} \cong (-)^{a,\kappa} \circ (-)_{\kappa}^{\square} \circ (-)_{*}$, hence there is a natural isomorphism $J \cong (-)_{\kappa,a}$. Moreover, the co-unit $(-)^a \circ (-)_{!} \rightarrow id$ is a natural isomorphism, so that :

$$(-)_{\kappa,a} \cong J \cong J \circ (-)^a \circ (-)_{!} \cong (-)^{a,\kappa} \circ (-)_{\kappa}^{\square} \circ (-)_{!}.$$

The functors $(-)^{a,\kappa}$ and $(-)_!$ commutes with colimits as they are left adjoints, and $(-)^{\square}_{\kappa}$ commutes with colimits by the argument earlier in the proof. Hence $(-)^{\kappa,a}$ commutes with colimits. $\square$

4.5 Identification of Mann and Huber's proper pushforwards

We may now prove the announced statement [proposition 4.0.1](#) :

Proposition 4.5.1. *Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a partially proper locally noetherian analytic adic space over a complete algebraically closed extension C of $\mathbb{Q}_p$.*

Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -modules on X . Then, there is a natural isomorphism :

$$J_C (f_!^M (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p)) \cong (Rf_!^H (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a$$

in $\mathcal{D}_{\text{ét}}(\mathrm{Spa}(C, \mathcal{O}_C), \mathcal{O}_C^{+,a}/p)$, where J_C denotes the identification from [proposition 3.5.3](#).

Proof. Let $(X_i)_{i \in I}$ be a covering of X by partially proper open subsets, as in [proposition A.3.3](#). For any i , let Z_i be the complement of X_i in X , viewed as a locally spatial diamond, and let $i_{Z_i} : Z_i \rightarrow X$ be the associated closed immersion. Applying the colimit-preserving exact functor J_C to the exact sequence from [corollary 4.4.3](#) yields the following exact sequence :

$$J_C (f_!^M (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p)) \rightarrow J_C (f_*^M (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p)_{\omega}) \rightarrow \varinjlim_{i \in I} J_C \left((f_*^M \circ i_{Z_i,*}^M \circ i_{Z_i,M}^* (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p))_{\omega} \right). \quad (4.2)$$

Applying the colimit-preserving exact functor $(-)^a$ to the exact sequence constructed in [corollary 4.2.5](#), we get the following exact sequence :

$$(Rf_!^H (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a \rightarrow (Rf_*^H (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a \rightarrow \varinjlim_{i \in I} (Rf_*^H \circ i_{Z_i,*}^S \circ i_{Z_i,S}^* (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a. \quad (4.3)$$

By [corollary 3.5.6](#), there is a natural isomorphism between the middle terms of both sequences.

We now prove that, for all $i \in I$, there is a natural isomorphism :

$$J_C \left((f_*^M \circ i_{Z_i,*}^M \circ i_{Z_i,M}^* (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p))_{\omega} \right) \cong (Rf_*^H \circ i_{Z_i,*}^S \circ i_{Z_i,S}^* (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a.$$

We may compute the left hand side as :

$$\begin{aligned} & J_C \left((f_*^M \circ i_{Z_i,*}^M \circ i_{Z_i,M}^* (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p))_{\omega} \right) \\ & \cong Rf_*^S i_{Z_i,*}^S J_{Z_i} (i_{Z_i,M}^* (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^{+,a}/p)) && \text{By corollary 3.5.5} \\ & \cong Rf_*^S i_{Z_i,*}^S i_{Z_i,S}^{*,qcoh} (J_C (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^{+,a}/p)) && \text{By proposition 3.5.3} \\ & \cong Rf_*^S i_{Z_i,*}^S i_{Z_i,S}^{*,qcoh} ((\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p)^a) && \text{By proposition 3.5.4} \\ & \cong (Rf_*^S i_{Z_i,*}^S i_{Z_i,S}^{*,qcoh} (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a && \text{By proposition 3.4.2.} \end{aligned}$$

Moreover, $Rf_*^S \cong Rf_*^H$ since f is a morphism between analytic adic spaces, and, by [lemma A.2.5](#), $i_{Z_i,S}^{*,qcoh} \cong i_{Z_i,S}^*$. Hence, there is a canonical isomorphism between the rightmost terms of the exact sequences [equation \(4.2\)](#) and [equation \(4.3\)](#) :

$$\varinjlim_{i \in I} J_C \left((f_*^M \circ i_{Z_i}^M \circ i_{Z_i,M}^* (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p))_\omega \right) \cong \varinjlim_{i \in I} (Rf_*^H \circ i_{Z_i}^S \circ i_{Z_i,S}^* (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p))^a$$

that is compatible with the isomorphism between the middle terms.

There is then a natural isomorphism between the fibers, which concludes. $\square$

Passing to cohomology groups yields the following.

Corollary 4.5.2. *Let $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a partially proper locally noetherian analytic adic space, and $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -modules on X .*

Then, for all $n \in \mathbb{N}$, there is a natural isomorphism :

$$\pi_{-n} \left(J_C \left(f_!^M (\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \right) \right) \stackrel{a}{\cong} H_{\acute{e}t,c}^n (X, \mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p)^a.$$

Proof. This follows directly from [proposition 4.5.1](#), together with the fact that $(-)^a$ is t -exact. $\square$

Primitive comparison with compact support

Content of this chapter

In this chapter, we reap the benefits of the technical results of the previous chapters.

We start in [section 5.1](#) with a reminder on the notion of Zariski-constructible coefficients, and a proof of primitive comparison in that setup, by piecing together arguments scattered throughout the litterature.

In [section 5.2](#), we finally define our notion of primitive comparison with compact support, and prove our main result : primitive comparison with compact support implies Poincaré duality. Let us recall the statement (cf. [theorem 5.2.3](#)) :

Theorem. *Let X be a partially proper rigid analytic variety over C that is smooth of pure dimension d .*

Let $\mathcal{L}$ be an overconvergent sheaf of $\mathbb{F}_p$ -modules on X , and assume that X satisfies primitive comparison with compact support with respect to $\mathcal{L}$.

Then, for any $0 \leq k \leq 2d$, there exists a natural isomorphism :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee(d)) \simeq \mathrm{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathbb{F}_p),$$

where $\mathcal{L}^\vee = \underline{\mathrm{Hom}}_{\mathbb{F}_p}(\mathcal{L}, \mathbb{F}_p)$ and (d) denotes a Tate twist.

The proof follows directly from the main result of [chapter 4](#), together with the fully faithfulness of the Riemann-Hilbert functor from [corollary 3.3.2](#).

We then deduce some consequences of this result, and more generally of the hypothesis of primitive comparison with compact support.

In [section 5.3](#), we recover Poincaré duality for Zariski-compactifiable rigid analytic varieties, as was shown in [[LRZ24](#), Cor 1.1.2]. This follows directly from our main result, together with the localization triangle for open-closed decomposition established in [proposition A.2.1](#), and with primitive comparison for Zariski-constructible coefficients.

In [section 5.4](#), we show that, assuming primitive comparison with compact support, we can extract from Mann's formalism a Poincaré duality statement for classical $\mathcal{O}^+/p$ -cohomology. From there, under some extra finiteness hypothesis, we can deduce a primitive comparison statement for non compactly supported étale cohomology.

While this result is interesting by itself, the specific application we have in mind is in [section 5.5](#), where it is used to prove "(i) $\implies$ (ii)" of the following classification result :

Theorem 5.0.1. (cf. [theorem 5.5.1](#)) Let X be a smooth, connected, partially proper rigid-analytic curve over C . Assume furthermore that X is of finite genus, in the sense of [[Liu87](#), Sec. 1].¹

Then, the following are equivalent :

- (i) X satisfies primitive comparison with compact support with respect to $\mathbb{F}_p$.
- (ii) The natural map $H_{\acute{e}t}^0(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\acute{e}t}^0(X, \mathcal{O}^+/p)$ is an almost isomorphism.
- (iii) $\mathcal{O}^+(X) \cong \mathcal{O}_C$
- (iv) There exists a smooth projective algebraic curve Y over C , such that X is open in Y^{an} , and the complement is compact for the canonical topology of Y^{an} .

The typical example of a curve satisfying (iv) is Drinfeld's upper half plane $\Omega_C^1 = \mathbb{P}_C^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

The proof of "(iii) $\implies$ (iv)" uses a striking result of Qing Liu [[Liu87](#), Thm 3.3.].

The proof of (ii) $\implies$ (iii) is easy, and the proof of (iv) $\implies$ (i) follows from a direct computation based on the localization sequence for open-closed decomposition, obtained in [proposition A.2.1](#).

¹The genus of a curve is defined as the supremum of the genus of its affinoid open subsets, for which it computes as the genus of their canonical reduction, viewed as an algebraic curve.

5.1 Primitive comparison theorem for Zariski-constructible coefficients

Let us start by recalling the notion of Zariski-constructible complexes from [BH21], and the primitive comparison result for them.

Definition 5.1.1. Let X be a rigid analytic space over a nonarchimedean field K . A sheaf $\mathcal{F}$ of $\mathbb{F}_p$ -vector spaces on $X_{\text{ét}}$ is Zariski-constructible if X admits a locally finite stratification $X = \bigcup_{i \in I} X_i$ into Zariski locally closed subsets X_i such that, for all $i \in I$, $\mathcal{F}|_{X_i}$ is locally constant and admits finite stalks.

We let $D_{zc}^b(X_{\text{ét}}, \mathbb{F}_p) \subset D^b(X_{\text{ét}}, \mathbb{F}_p)$ be the full subcategory of bounded complexes $\mathcal{F}$ of étale sheaves whose homotopy sheaves $\pi_n(\mathcal{F})$ are Zariski-constructible for all $n \in \mathbb{Z}$.

Note that any Zariski-constructible sheaf is automatically overconvergent.

Remark 5.1.2. Recall that, for any finite dimensional² partially proper morphism of locally noetherian analytic adic spaces $f : X \rightarrow Y$ and any $\mathcal{F} \in D^b(X_{\text{ét}}, \mathbb{F}_p)$, there is a natural morphism :

$$Rf_!^H \mathcal{F} \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_Y^+/p \cong Rf_!^H \left(\mathcal{F} \otimes_{\mathbb{F}_p}^{\mathbb{L}} f_H^* \mathcal{O}_Y^+/p \right) \rightarrow Rf_!^H \left(\mathcal{F} \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p \right)$$

Where the isomorphism is the projection formula of [Hub96, Prop 5.5.1.(iv)] and the second map is induced by the natural morphism $f_H^* \mathcal{O}_Y^+/p \rightarrow \mathcal{O}_X^+/p$.

In the following, we will need the following version of the primitive comparison theorem :

Theorem 5.1.3. Let $f : X \rightarrow \text{Spa}(K, \mathcal{O}_K)$ be a proper rigid analytic variety, and let $\mathcal{F} \in D_{zc}^b(X, \mathbb{F}_p)$. Then, for all $n \in \mathbb{N}$, the natural morphism :

$$(R^n f_*^H \mathcal{F}) \otimes_{\mathbb{F}_p} \mathcal{O}_K/p \rightarrow R^n f_*^H (\mathcal{F} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p)$$

is an almost isomorphism. In particular, if C is the completion of $\overline{K}$, the natural morphism :

$$H_{\text{ét}}^n(X_C, \mathcal{F}_C) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét}}^n(X_C, \mathcal{F}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p)$$

is a $\text{Gal}(\overline{K}/K)$ -equivariant almost isomorphism, where the Galois action on the left hand side is diagonal.

Proof. By [BH21, Prop 3.6], the category $\mathcal{F} \in D_{zc}^b(X, \mathbb{F}_p)$ is generated by objects of the form $g_* \mathbb{F}_p$ for finite morphism $g : Y \rightarrow X$, and, by the arguments of the proof of [Zav24a, Lemma 7.3.6], it suffices to check the isomorphism for $\mathcal{F} = g_* \mathbb{F}_p$. Using [Zav24a, Cor 7.2.8], we may replace f by $g \circ f$, so that it suffices to prove the statement for $\mathcal{F} = \mathbb{F}_p$.

By [Sch17, Prop. 14.3], it suffices to check the isomorphism on stalks $\text{Spa}(C', C'^+)$, which reduces to $\text{Spa}(C', \mathcal{O}_{C'})$ following the beginning of the proof of [Sch12a, Thm. 5.1]. The result then follows from [Sch13, Thm 3.17].

The second statement follows from taking stalks with respect to the geometric point $\text{Spa}(C, \mathcal{O}_C) \rightarrow X$, and using proper base change as in [Sch17, Thm 16.10], together with the identification of pullbacks of the structure sheaf $\mathcal{O}^+/p$ by quasi-pro-étale morphisms as in lemma 2.4.1. $\square$

²A morphism is said to be finite-dimensional if it is of finite transcendence degree. This hypothesis is used in [Hub96] to ensure that complexes are bounded in the derived category, and is probably not needed.

Remark 5.1.4. Without the smoothness assumption, one really needs to work with rigid analytic varieties. In particular primitive comparison does not hold for proper locally noetherian analytic adic spaces, see [ABM24, Example 5.2.5.(b)] for a counterexample.

Remark 5.1.5. A version of a relative primitive comparison theorem for the Gross-Hopkins period map appears in [JLH20, Prop. 5.2.2]. This seems to be unrelated to our setup.

5.2 Primitive comparison with compact support and Poincaré duality

Let us define our notion of primitive comparison with compact support.

Definition 5.2.1. Let X be a finite dimensional partially proper locally noetherian analytic adic space over C , and $\mathcal{L}$ be an overconvergent sheaf of $\mathbb{F}_p$ -vector spaces on X .

We say that X satisfies primitive comparison with compact support with respect to $\mathcal{L}$ if the morphism from [remark 5.1.2](#) is an almost isomorphism, or equivalently if, for all $i \in \mathbb{N}$, the induced map :

$$H_{\acute{e}t,c}^i(X, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\acute{e}t,c}^i(X, \mathcal{L} \otimes \mathcal{O}_X^+/p)$$

is an almost isomorphism of $\mathcal{O}_C/p$ -modules.

The following reformulation will be useful. This allows to view primitive comparison with compact support as a compatibility statement between Huber's étale cohomology of $\mathcal{L}$ and Mann's $f_!^M$.

Proposition 5.2.2. Under the hypothesis of the above proposition, X satisfies primitive comparison with compact support with coefficients in $\mathcal{L}$ if and only if the natural map :

$$R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C^{+,a}/p \rightarrow f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \quad (5.1)$$

is a isomorphism in $\mathcal{D}_{\square}^a(\mathcal{O}_C/p)_\omega$, where $f : X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ is the structure map. Here, both tensor products are given by the Riemann-Hilbert functor.

Proof. Let us start by defining the map. Recall that, by [proposition 3.5.4](#), there is a natural isomorphism

$$J_C^{-1} \left(\left(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \right)^a \right) \cong R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C^{+,a}/p.$$

Moreover, since f is partially proper, we proved in [proposition 4.5.1](#) the identification :

$$J_C^{-1} \left(\left(Rf_!^H(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p) \right)^a \right) \cong f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p).$$

Hence, the morphism [equation \(5.1\)](#) is given, through the above identifications, by the almostification of the map from [remark 5.1.2](#). The map of [equation \(5.1\)](#) is an almost isomorphism if and only if the map :

$$\left(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \right)^a \rightarrow \left(R\Gamma_{\acute{e}t,c}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p) \right)^a$$

is an isomorphism, which is clearly equivalent to primitive comparison with compact support by taking homotopy groups.

□

We may now deduce our main result :

Theorem 5.2.3. *Let X be a partially proper rigid analytic variety over C that is smooth of pure dimension d .*

Let $\mathcal{L}$ be an overconvergent sheaf of $\mathbb{F}_p$ -modules on X , and assume that X satisfies primitive comparison with compact support with respect to $\mathcal{L}$.

Then, for any $0 \leq k \leq 2d$, there exists a natural isomorphism :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee(d)) \simeq \mathrm{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathbb{F}_p).$$

Proof. By [proposition 5.2.2](#), primitive comparison with compact support rewrites as :

$$f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \simeq Rf_!^H \mathcal{L} \otimes \mathcal{O}_C^{+,a}/p.$$

We then apply [corollary 3.3.2](#) with $\mathcal{F}_{\mathcal{L}} = R\Gamma_{\acute{e}t,c}(X, \mathcal{L})$, yielding isomorphisms :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee(d)) \stackrel{a}{\simeq} \mathrm{Hom}_{\mathbb{F}_p}(\pi_{k-2d}(R\Gamma_{\acute{e}t,c}(X, \mathcal{L})), \mathbb{F}_p).$$

This concludes. □

We can also make the following Galois-equivariant version.

Theorem 5.2.4. *Let X be a partially proper rigid analytic variety over K that is smooth of pure dimension d .*

Let $\mathcal{L}$ be an overconvergent sheaf of $\mathbb{F}_p$ -modules on X .

Let C be the completion of $\overline{K}$, X_C denote the base change of X to C , and $\mathcal{L}_C$ denote the base change of $\mathcal{L}$ to X_C . Assume that X_C satisfies primitive comparison with compact support with respect to $\mathcal{L}_C$.

Then, for any $0 \leq k \leq 2d$, there exists a $\mathrm{Gal}(\overline{K}/K)$ -equivariant natural isomorphism :

$$H_{\acute{e}t}^k(X_C, \mathcal{L}_C^\vee(d)) \simeq \mathrm{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X_C, \mathcal{L}_C), \mathbb{F}_p).$$

Proof. The isomorphism follows from [theorem 5.2.3](#), so that it suffices to establish Galois equivariance.

By construction, $\mathcal{L}_C$ is $\mathrm{Gal}(\overline{K}/K)$ equivariant, and so is the morphism of [proposition 5.2.2](#).

Let ψ_g denote the action on X_C of an element $g \in \mathrm{Gal}(\overline{K}/K)$. The induced action on étale sheaves on X_C is then given by the pullback $\psi_{g,H}^*$. It follows from base change with respect to zero-dimensional morphisms [[Hub96](#), Thm 5.4.6] that $\psi_{g,H}^* Rf_!^H \mathcal{L} \cong Rf_!^H \psi_{g,H}^* \mathcal{L}$.

The equivariance then follows from [remark 3.3.3](#). □

Remark 5.2.5. *The same argument shows that the Poincaré duality isomorphism can be made G -equivariant, for any group G acting on X_C such that $\mathcal{L}_C$ is G -equivariant.*

As a first consequence of our main result, let us recover Poincaré duality for Zariski-compactifiable varieties.

5.3 Poincaré duality for Zariski-compactifiable varieties

Let $U \subset X$ be an open subvariety. Using the localization triangle with respect to an open-closed decomposition obtained in [proposition A.2.1](#), verifying primitive composition with compact support for U reduces to a computation on X , and on the complement $X \setminus U$.

The precise result is as follows.

Proposition 5.3.1. *Let $f : X \rightarrow \mathrm{Spa}(K, \mathcal{O}_K)$ be a proper rigid-analytic variety, and let C be the completion of $\overline{K}$. Let $j : U \rightarrow X$ be the open immersion from a partially proper open subset, with complement F , endowed with the induced structure of a locally spatial diamond as in [lemma A.1.3](#).*

Let $i : F \rightarrow X$ be the closed immersion, and $h = f \circ i$. Denote by X_C, U_C , and F_C the base change of X, U and F to C , and i_C, j_C the associated immersions. Let $f_C : X_C \rightarrow C$ be the structural morphism of X_C , and $h_C = f_C \circ i_C$.

Let $\mathcal{L}$ be an overconvergent étale sheaf of $\mathbb{F}_p$ -vector spaces on X , and $\mathcal{L}_C$ its pullback to X_C . Assume that, for all $n \in \mathbb{N}$, the natural morphisms :

$$H_{\text{ét}}^n(X_C, \mathcal{L}_C) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét}}^n(X_C, \mathcal{L}_C \otimes \mathcal{O}^+/p) \quad (5.2)$$

$$H_{\text{ét}}^n(F_C, i_C^* \mathcal{L}_C) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét}}^n(F_C, i_C^* \mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}^+/p) \quad (5.3)$$

are almost isomorphisms. Assume furthermore that X is smooth of pure dimension d .

Then, for any $0 \leq k \leq 2d$, there exists a natural $\mathrm{Gal}(\overline{K}/K)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(U_C, j_C^* \mathcal{L}_C^\vee(d)) \cong \mathrm{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(U_C, j_C^* \mathcal{L}_C), \mathbb{F}_p \right).$$

Proof. In this proof, every functor is taken in the sense of [\[Sch17\]](#).

From the localization sequence [proposition A.2.1](#) applied to $\mathcal{L}_C$, we get the following exact sequence in $\mathcal{D}_{\text{ét}}(X_C, \mathbb{F}_p)$:

$$Rj_{C!} j_C^* \mathcal{L}_C \rightarrow \mathcal{L}_C \rightarrow Ri_{C*} i_C^* \mathcal{L}_C$$

Likewise, by applying the localisation sequence to $\mathcal{L}_C \otimes \mathcal{O}_{X_C}^+/p$, we get the following exact sequence in $\mathcal{D}_{\text{ét}}(X_C, \mathcal{O}_{X_C}^+/p)$:

$$Rj_{C!} (j_C^* \mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) \rightarrow \mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p \rightarrow Ri_{C*} i_C^* (\mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p)$$

We may then apply the exact functor $Rf_{C*} = Rf_{C!}$ to both sequences, and extend the scalars of the first one to $\mathcal{O}_C/p$. By functoriality of localization sequences, and using the functorial morphism constructed in [remark 5.1.2](#), we obtain the following diagram with exact rows :

$$\begin{array}{ccccc} (R(f_C \circ j_C)_! j_C^* \mathcal{L}_C) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p & \longrightarrow & Rf_{C*} \mathcal{L}_C \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p & \longrightarrow & (R(f_C \circ i_C)_* i_C^* \mathcal{L}_C) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p \\ \downarrow & & \downarrow & & \downarrow \\ R(f_C \circ j_C)_! j_C^* (\mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) & \rightarrow & Rf_{C*} (\mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) & \rightarrow & R(f_C \circ i_C)_* i_C^* (\mathcal{L}_C \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) \end{array}$$

By [equation \(5.2\)](#), the middle vertical arrow induces an almost isomorphism in the derived category.

Using [equation \(5.3\)](#), the rightmost vertical arrow is an almost isomorphism. Indeed, pullbacks commute with tensor products, and, by [lemma 2.4.3](#) (since generalizing closed immersions are quasi-pro-étale), we have an isomorphism :

$$Rh_{C*} i_C^* (\mathcal{L}_C \otimes \mathcal{O}_{X_C}^+ / p) \cong Rh_{C*} (i_C^* \mathcal{L}_C \otimes \mathcal{O}_{F_C}^+ / p).$$

Hence the leftmost vertical arrow is an almost isomorphism (this is the five lemma in cohomology groups). By the same argument as the one used for the right hand side, this corresponds to the fact that the following morphisms :

$$H_{\text{ét},c}^n(U_C, j^* \mathcal{L}_C \otimes \mathcal{O}_{U_C}^+ / p) \rightarrow H_{\text{ét},c}^i(U_C, j^* \mathcal{L}_C) \otimes \mathcal{O}_C / p$$

are almost isomorphisms for all $i \in \mathbb{N}$. We conclude by [theorem 5.2.4](#). $\square$

We may deduce Poincaré duality for Zariski-compactifiable rigid analytic varieties.

Corollary 5.3.2. *Let X be a smooth proper rigid analytic variety over K , and $U \subset X$ be a Zariski open subset. Let $j : U \rightarrow X$ denote the associated open immersion.*

Let C be the completion of $\overline{K}$. Then, for any étale $\mathbb{F}_p$ -local system $\mathcal{L}$ on U , U_C satisfies Poincaré duality with respect to $\mathcal{L}_C$, i.e there is a natural $\text{Gal}(\overline{K}/K)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(U_C, \mathcal{L}_C^\vee(d)) \cong \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^{2d-k}(U_C, \mathcal{L}_C), \mathbb{F}_p \right).$$

Proof. Let $F = X \setminus U$, which is a Zariski closed subset of X (viewed with its reduced structure), making it a proper rigid analytic variety over K . Since j is a Zariski open immersion, it follows from [\[BH21, Prop 3.11\]](#) that $j_! \mathcal{L}_C$ is Zariski constructible on X_C .

Moreover, it follows from base change [\[Sch17, Thm 19.1\]](#) that $j_C^* j_{C!} \mathcal{L}_C \cong \mathcal{L}_C$.

We will use the criterion from [proposition 5.3.1](#) (and the associated notations) to the overconvergent sheaf $j_!^H \mathcal{L}$.

By base change [\[Sch17, Cor 16.10\]](#), $i_C^* j_{C!} \mathcal{L} = 0$, so that the almost isomorphism of [equation \(5.3\)](#) is trivial.

It then suffices to show that the following morphisms are almost isomorphism :

$$H_{\text{ét}}^n(X_C, j_{C!} \mathcal{L}_C) \otimes_{\mathbb{F}_p} \mathcal{O}_C / p \rightarrow H_{\text{ét}}^n(j_{C!} \mathcal{L}_C \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_{X_C}^+ / p).$$

This follows directly from the primitive comparison theorem for Zariski-constructible coefficients, as stated in [theorem 5.1.3](#). $\square$

This recovers [\[LRZ24, Cor 1.1.2\]](#). Note that the approach of loc.cit. uses different tools, as it barely uses perfectoid methods, and, rather, relies on a study of formal models, and of trace formalisms. In particular, it barely relies on the theory of perfectoid spaces.

In particular, we may compute étale cohomology of the affine line.

Example 5.3.3. The affine line $\mathbb{A}_C$ satisfies Poincaré duality with coefficients in any $\mathbb{F}_p$ -local system. We have the following :

$$H_{\acute{e}t,c}^i(\mathbb{A}_C, \mathbb{F}_p) \cong \begin{cases} 0 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{F}_p & \text{if } i = 2 \end{cases} \quad \text{and} \quad H_{\acute{e}t}^i(\mathbb{A}_C, \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ 0 & \text{if } i = 2, \end{cases}$$

Proof. The affine line is Zariski-open inside $\mathbb{P}_C^1$, with complement given by the point, so that [corollary 5.3.2](#) applies.

The compactly supported étale cohomology can be computed directly by excision [proposition A.2.1](#), using the fact that the cohomology of the point is concentrated in degree zero, and that the cohomology of $\mathbb{P}_C^1$ is classically $R\Gamma_{\acute{e}t}(\mathbb{P}_C^1, \mathbb{F}_p) = \mathbb{F}_p \oplus \mathbb{F}_p[2]$.

The non-compactly supported cohomology groups can be deduced by duality. $\square$

We will see in a general statement for rigid-analytic curves. Before doing that, we need to explore other consequences of primitive comparison with compact support.

5.4 Primitive comparison with and without support

We start by establishing a Poincaré duality result for classical $\mathcal{O}^+/p$ -coefficients, under the hypothesis of primitive comparison with compact support.

Proposition 5.4.1. Let X be a smooth partially proper rigid analytic variety over C of pure dimension d and satisfies primitive comparison with compact support with respect to some $\mathcal{L} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}$.

Assume moreover than $\mathcal{L}$ is dualizable³ inside $\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}$.

Then, for all $0 \leq k \leq 2d$, there is an almost isomorphism :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee \otimes \mathcal{O}_X^+/p(d)) \stackrel{a}{\cong} \text{Hom}_{\mathcal{O}_C/p} \left(H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p \right).$$

This is a statement of Poincaré duality for $\mathcal{O}^+/p$ -coefficients. Note that we really need the primitive comparison with compact support hypothesis in the proof, even if the $\mathbb{F}_p$ -cohomology does not appear explicitly in the result. The proof is purely formal, but surprisingly tricky.

Proof. In order to simplify the notations, we do not write down the functor J_C from [proposition 3.5.3](#), and implicitly identify both sides.

By relative Poincaré duality in Mann's category (cf. [proposition 3.1.9](#)), we have :

$$\underline{\text{Hom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M(\mathcal{L} \otimes \mathbb{1}_X), \mathbb{1}_C) \cong f_*^M \underline{\text{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}(\mathcal{L} \otimes \mathbb{1}_X, \mathbb{1}_X(d)[2d]) \quad (5.4)$$

By primitive comparison with compact support (cf. [proposition 5.2.2](#)), we have :

$$R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C^{+,a}/p \cong f_!^M(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p).$$

³By [\[Man22, Lemma 3.9.20\]](#), this is equivalent to $\mathcal{L}$ being perfect, which is defined by descent from the usual notion of perfect complexes in a derived category. If $\mathcal{L}$ is concentrated in degree 0, this is equivalent to being finite dimensional.

We will apply the discretization functor $(-)_\omega$, then take homotopy groups of the above. We start by focusing on the left hand side of [equation \(5.4\)](#).

Hence, by [lemma 3.5.8](#), [proposition 3.5.4](#) and [lemma 3.5.9](#) :

$$\begin{aligned} \left(\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!(\mathcal{L} \otimes \mathbb{1}_X), \mathbb{1}_C) \right)_\omega &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p)_\omega^a}(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C^a/p, \mathcal{O}_C^a/p) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p)_\omega^a}\left(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C^a/p, \mathcal{O}_C^a/p\right) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p)_\omega}\left(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p, \mathcal{O}_C/p\right)^a \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{O}_C/p)_\omega}\left(R\Gamma_{\acute{e}t,c}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p\right)^a \end{aligned}$$

Here, the category $\mathcal{D}(\mathcal{O}_C/p)_\omega$ is simply the derived category of $\mathcal{O}_C/p$ -modules. As we are only interested in the homotopy groups, the internal $\underline{\mathrm{Hom}}$ functor might as well be computed in the 1-derived category $D(\mathcal{O}_C/p)$, where it satisfies (cf. [\[Sta23, Equation 0A64\]](#))

$$\pi_n \underline{\mathrm{Hom}}_{D(\mathcal{O}_C/p)}(C_\bullet, D_\bullet) = \mathrm{Hom}_{D(\mathcal{O}_C/p)}(C_\bullet, D_\bullet[n]).$$

We hence have :

$$\pi_k \left(\left(\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M(\mathcal{L} \otimes \mathbb{1}_X), \mathbb{1}_C) \right)_\omega \right)^a \cong \mathrm{Hom}_{\mathcal{O}_C/p}(R\Gamma_{\acute{e}t,c}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p[k])^a.$$

By primitive comparison with compact support, the complex $R\Gamma_{\acute{e}t,c}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p) \cong R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p$ is free, hence projective, as a complex of $\mathcal{O}_C/p$ -modules, so that :

$$\mathrm{Hom}_{D(\mathcal{O}_C/p)}(R\Gamma_{\acute{e}t,c}(X, \mathcal{L}) \otimes \mathcal{O}_C/p, \mathcal{O}_C/p[k]) \cong \mathrm{Hom}_{\mathcal{O}_C/p}(H_{\acute{e}t,c}^k(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p).$$

Finally, we have :

$$\pi_k \left(\left(\underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)}(f_!^M(\mathcal{L} \otimes \mathbb{1}_X), \mathbb{1}_C) \right)_\omega \right)^a \cong \mathrm{Hom}_{\mathcal{O}_C/p}(H_{\acute{e}t,c}^k(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p). \quad (5.5)$$

We now turn our attention to the right hand side of [equation \(5.4\)](#). Since $\mathcal{L}$ is assumed to be dualizable, it follows from [\[Man22, Thm 3.9.23\]](#) that :

$$f_*^M \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}(\mathcal{L} \otimes \mathcal{O}_X^+/p, \mathcal{O}_X^+/p(d)[2d]) \cong \underline{\mathrm{Hom}}_{\mathcal{D}(X, \mathbb{F}_p)}(\mathcal{L}, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^{+,a}/p(d)[2d] \cong \mathcal{L}^\vee \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^{+,a}/p(d)[2d]$$

We may now apply [corollary 3.5.6](#), so that :

$$f_*^M \underline{\mathrm{Hom}}_{\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/p)}(\mathcal{L} \otimes \mathcal{O}_X^+/p, \mathcal{O}_X^+/p(d)[2d])_\omega \cong (Rf_*^H(\mathcal{L}^\vee \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p)(d)[2d])^a. \quad (5.6)$$

We apply π_{2d-k} . Since $(-)^a$ is t-exact,

$$\pi_{2d-k} (Rf_*^H(\mathcal{L}^\vee \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p)(d)[2d])^a = H_{\acute{e}t}^k(X, \mathcal{L}^\vee)(d). \quad (5.7)$$

The result now follows from combining [equation \(5.4\)](#), [equation \(5.5\)](#), [equation \(5.6\)](#) and [equation \(5.7\)](#). $\square$

Remark 5.4.2. It's unclear what we can say without the hypothesis of primitive comparison with compact support ; the issue is that there might be non-trivial higher Ext-groups between $\mathcal{O}_C/p$ -modules. In particular, the short exact sequence :

$$0 \rightarrow \mathcal{O}_C/p \xrightarrow{\cdot p^{1/2}} \mathcal{O}_C/p^{1/2} \rightarrow \mathcal{O}_C/p \rightarrow 0$$

does not split as $\mathcal{O}_C/p$ -modules.

Under moreover that cohomology groups are finite dimensional, we show that primitive comparison with compact support implies primitive comparison without support.

Corollary 5.4.3. Let $X \rightarrow \mathrm{Spa}(C, \mathcal{O}_C)$ be a smooth partially proper rigid analytic variety, of pure dimension d , that satisfies primitive comparison with compact support with respect to some $\mathcal{L} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}$.

Then, for all $k \in \mathbb{N}$ such that $H_{\acute{e}t}^{2d-k}(X, \mathcal{L})$ is of finite dimension over $\mathbb{F}_p$, there exists an almost isomorphism :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee) \otimes \mathcal{O}_C/p \xrightarrow{\sim} H_{\acute{e}t}^k(X, \mathcal{L}^\vee \otimes \mathcal{O}_X^+/p).$$

Proof. We may apply Poincaré duality for $\mathcal{L} \otimes \mathcal{O}^+/p$ -coefficients as in [proposition 5.4.1](#), yielding :

$$H_{\acute{e}t}^k(X, \mathcal{L}^\vee \otimes \mathcal{O}_X^+/p(d)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_C/p} \left(H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p), \mathcal{O}_C/p \right).$$

By hypothesis of primitive comparison with compact support, $H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L} \otimes \mathcal{O}_X^+/p) \xrightarrow{\sim} H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}) \otimes \mathcal{O}_C^+/p$, so that :

$$\begin{aligned} H_{\acute{e}t}^k(X, \mathcal{L}^\vee \otimes \mathcal{O}_X^+/p(d)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_C/p} (H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}) \otimes \mathcal{O}_C/p, \mathcal{O}_C/p) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p} (H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathcal{O}_C/p). \end{aligned}$$

Since $H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L})$ is finite dimensional over $\mathbb{F}_p$, it is dualizable, so that

$$\mathrm{Hom}_{\mathbb{F}_p} (H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathcal{O}_C/p) \cong \mathrm{Hom}_{\mathbb{F}_p} (H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathbb{F}_p) \otimes \mathcal{O}_C/p.$$

The result now follows from Poincaré duality with coefficients in $\mathcal{L}$, as in [theorem 5.2.3](#). □

Remark 5.4.4. It is unclear what one can say without the finite dimensionality hypothesis.

If X is an infinite disjoint union of copies of $\mathrm{Spa}(C, \mathcal{O}_C)$ indexed by some I , we have $H_{\acute{e}t}^0(X, \mathbb{F}_p) \cong \prod_I \mathbb{F}_p$ and $H_{\acute{e}t}^0(X, \mathbb{F}_p) \cong \prod_I \mathcal{O}_C/p$.

Getting a comparison statement would require a notion of completed tensor product such that $(\prod_I \mathbb{F}_p) \hat{\otimes}_{\mathcal{O}_C/p} \cong (\prod_I \mathcal{O}_C/p)$. That the solid tensor product is not suited for this purpose.

For a more geometric example, Drinfeld's upper half plane Ω_C^1 . It is known that $H_{\acute{e}t}^1(\Omega_C^1, \mathbb{F}_p(1)) = \mathrm{Sp}(\mathbb{F}_p)^*$ is a dual Steinberg representation. We will see that Ω_C^1 satisfies primitive comparison with compact support with respect to $\mathbb{F}_p$, so that :

$$H_{\acute{e}t}^1(X, \mathcal{O}_X^+/p(1)) \xrightarrow{\sim} (\mathrm{Sp}(\mathbb{F}_p) \otimes \mathcal{O}_C/p)^*$$

is the $\mathcal{O}_C$ -linear dual of the Steinberg representation with $\mathcal{O}_C$ -coefficients, which is not the dual Steinberg representation with coefficients in $\mathcal{O}_C/p$.

5.5 Primitive comparison for rigid analytic curves

One may wonder which rigid analytic curves satisfy primitive comparison with compact support, and, hence, Poincaré duality. We give a full characterisation for curves of finite genus.

Theorem 5.5.1. *Let X be a smooth, connected, partially proper rigid-analytic curve over C . Assume furthermore that X is of finite genus, in the sense of [Liu87, Sec. 1].⁴*

The following are equivalent :

- (i) X satisfies primitive comparison with compact support with respect to $\mathbb{F}_p$.
- (ii) The natural map $H_{\text{ét}}^0(X, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\text{ét}}^0(X, \mathcal{O}_X^+/p)$ is an almost isomorphism.
- (iii) $\mathcal{O}^+(X) \cong \mathcal{O}_C$
- (iv) There exists a smooth projective algebraic curve Y over C , such that X is open in Y^{an} , and the complement is compact for the canonical topology of Y^{an} .

Proof. (i) $\implies$ (ii). It follows from [theorem 5.2.3](#) that X satisfies Poincaré duality with $\mathbb{F}_p$ coefficients, and, in particular (ignoring Tate twists) :

$$H_{\text{ét}}^0(X, \mathbb{F}_p) \simeq \text{Hom}_{\mathbb{F}_p} \left(H_{\text{ét},c}^2(X, \mathbb{F}_p), \mathbb{F}_p \right),$$

where $H_{\text{ét}}^0(X, \mathbb{F}_p) \cong \mathbb{F}_p$ since X is connected. Hence $H_{\text{ét},c}^2(X, \mathbb{F}_p) \cong \mathbb{F}_p$ is finite dimensional, and the result follows from [corollary 5.4.3](#).

(ii) $\implies$ (iii). By definition of the étale sheaf $\mathcal{O}^+$, $H_{\text{ét}}^0(X, \mathcal{O}^+) = \mathcal{O}^+(X)$. There is a natural injection $\mathcal{O}_C \hookrightarrow \mathcal{O}^+(X)$, inducing an almost isomorphism $\mathcal{O}_C/p \xrightarrow{a} \mathcal{O}^+(X)/p$ by hypothesis. Since $\mathcal{O}^+(X)$ is p -torsion free, this is an isomorphism. Since both $\mathcal{O}_C$ and $\mathcal{O}^+(X)$ are p -adically complete, this lifts to an isomorphism $\mathcal{O}_C \cong \mathcal{O}^+(X)$.

(iii) $\implies$ (iv). This is contained in [Liu87, Thm 3.3.].

(iv) $\implies$ (i). We let S denote the complement of X in Y^{an} , viewed as a locally spatial diamond. We will prove the result by using the criterion from [proposition 5.3.1](#), stemming from excision. The topology on the analytification Y^{an} is profinite for the canonical topology, as this is locally induced by the one on C , so that S is profinite.

The set S with a structure of a locally spatial sub-diamond of $(Y^{\text{an}})^\diamond$ over C , by [lemma A.1.3](#). It even admits the structure of an affinoid perfectoid space, as constructed in [example 2.2.3](#) :

$$S_C := \text{Spa} \left(\mathcal{C}^0(S, C), \mathcal{C}^0(S, \mathcal{O}_C) \right). \quad (5.8)$$

The proof reduces to the following lemma, which is a primitive comparison result for profinite spaces.⁵ $\square$

Lemma 5.5.2. *Let S be a profinite set, viewed as a perfectoid space S_C over C as in [equation \(5.8\)](#). Then, for all $i \in \mathbb{N}$, the natural map*

$$H_{\text{ét}}^i(S_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \rightarrow H_{\text{ét}}^i(S_C, \mathcal{O}^+/p)$$

is an almost isomorphism. It is moreover trivial outside of degree 0.

⁴The genus of a curve is defined as the supremum of the genus of its affinoid open subsets, for which it computes as the genus of their canonical reduction, viewed as an algebraic curve. If X is of the form (iv) below, its genus is simply the genus of Y .

⁵Profinite spaces are proper when viewed as diamonds by [Sch17, Rk 18.2], so that primitive comparison coincides with primitive comparison with compact support.

Note that this essentially appeared in the proof of [JLH20, Prop. 5.2.2].

Proof. Write S as a projective limit $S = \varprojlim_{i \in I} S_i$, for finite S_i . By [Sch17, Prop 14.9], for any étale sheaf $\mathcal{F}$ on S_C , there is a natural isomorphism

$$H_{\text{ét}}^{\bullet}(S_C, \mathcal{F}) \cong \varinjlim_{i \in I} H_{\text{ét}}^{\bullet}(S_{i,C}, \mathcal{F}_i),$$

where $\mathcal{F}_i$ denotes the pullback of $\mathcal{F}$ to $S_{i,C}$, the perfectoid space associated to S_i .

For any $i \in I$, $S_{i,C}$ is a finite disjoint union of geometric points $\text{Spa}(C, \mathcal{O}_C)$, so that their étale cohomology is the cohomology of the underlying topological space S_i . There admit no higher cohomology, and $H_{\text{ét}}^0(S_{i,C}, \mathcal{F}) = \text{Fun}(S_i, \mathcal{F}(\text{Spa}(C, \mathcal{O}_C)))$.

Hence, we compute :

- $H_{\text{ét}}^0(S_C, \mathbb{F}_p) \cong \varinjlim_{i \in I} \text{Fun}(S_i, \mathbb{F}_p) \cong \text{LC}(S, \mathbb{F}_p)$ by lemma B.1.10.
- $H_{\text{ét}}^0(S_C, \mathcal{O}_{S_C}^+/p) \cong \varinjlim_{i \in I} \text{Fun}(S_i, \mathcal{O}_C/p) \cong \text{LC}(S_C, \mathcal{O}_C/p)$ by lemma B.1.10.

The result now follows from lemma B.1.11. □

The point (iii) of theorem 5.5.1 can be reinterpreted as a p -adic analogue of the "Liouville property" from complex geometry, claiming that all bounded holomorphic functions are constant. Even finding an analogue of (iv) $\implies$ (iii) in complex geometry is very complicated, as, even for spaces of the form $\mathbb{P}_{\mathbb{C}}^1 \setminus S$, the Liouville property depends on the Hausdorff dimension of S (cf. [You15, Sec. 2]).

Remark 5.5.3. *It is a bit surprising that, in the p -adic case, the result (iii) $\implies$ (iv) holds without assuming that C is spherically closed.*

As a typical example, let X be rigid analytic variety obtained by removing a point of type 4 from $\mathbb{P}_C^1$ (viewed as a Berkovich space). This admits the structure of a rigid analytic variety, with a morphism to $\mathbb{P}_C^1$ that is bijective on rigid points, but it not an isomorphism - X is hence not of the form $\mathbb{P}_C^1 \setminus S$.

One can show, via explicit computations, that X indeed admits non constant bounded holomorphic functions on X , so that $\mathcal{O}^+(X) \neq \mathcal{O}_C$ (this is [GP80, Ch. IV, Prop. 1.3.2]).

As a corollary, we deduce Poincaré duality for Drinfeld's upper half plane. Note that it is somewhat the typical example of a space satisfying (iv).

Corollary 5.5.4. *Let C be the completion of $\overline{K}$, and algebraic closure of $\mathbb{Q}_p$, and $\Omega_C = \mathbb{P}_C^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$ denote Drinfeld's upper half plane over C .*

Then Ω_C satisfies Poincaré duality. For $0 \leq k \leq 2$, there is a natural $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(\Omega_C, \mathbb{F}_p(1)) \cong \text{Hom}(H_{\text{ét},C}^{2-k}(\Omega_C, \mathbb{F}_p), \mathbb{F}_p).$$

Proof. Since $\mathbb{P}^1(\mathbb{Q}_p)$ is compact, Ω_C^1 satisfies primitive comparison with respect to $\mathbb{F}_p$, follows directly from the above proposition and theorem 5.2.4. □

Note that one can easily check that the above morphism is also $\text{GL}_2(\mathbb{Q}_p)$ -equivariant, using the functoriality established in remark 5.2.5.

In the next chapter, we extend primitive comparison with compact support to Drinfeld's symmetric spaces of arbitrary dimension, for which the computation is very geometrically and combinatorially involved.

The $\mathcal{O}^+/p$ -cohomology of Drinfeld's symmetric spaces

Content of this chapter

In this chapter, we compute the $\mathcal{O}^+/p$ -cohomology of Drinfeld's symmetric spaces, adapting the strategy of [SS91]. The main result is as follows :

Proposition 6.0.1. *Let $d \in \mathbb{N}$, K be a complete extension of $\mathbb{Q}_p$, and C be a complete algebraically closed extension of K .*

Let $\mathcal{H}^d$ denote the set of K -rational hyperplanes of $\mathbb{P}_C^d$, and $\Omega_K^d = \mathbb{P}_K^d \setminus \bigcup_{H \in \mathcal{H}^d} H$ be Drinfeld's symmetric space of dimension d over K .

Then, for all $n \geq 0$, there is a $\mathrm{GL}_{n+1}(K)$ -equivariant¹ almost isomorphism :

$$H_{\acute{e}t,c}^n(\Omega_C^d, \mathcal{O}^+/p) \stackrel{a}{\cong} \mathrm{Sp}_{2d-n}(\mathbb{F}_p) \otimes \mathcal{O}_C/p,$$

where $\mathrm{Sp}_{2d-n}(\mathbb{F}_p)$ denotes the $(2d-n)$ -th generalized Steinberg representation of $\mathrm{GL}_{d+1}(K)$, valued in $\mathbb{F}_p$.

By [CDHN21], the compactly supported étale $\mathbb{F}_p$ -cohomology of Ω_C^d is given by $\mathrm{Sp}_{2d-n}(\mathbb{F}_p)$, so that the above proposition is a weak form² of primitive comparison with compact support. The full comparison with compact support, will be obtained in chapter 8, where we instead adapt a method of S.Orlik (which shares many technical similarities).

The main interest of the method presented here is that, besides being more explicit, it relies on a mostly axiomatic point of view on cohomology, which can be adapted *mutatis mutandis* to other cohomology theories.³ In particular, the arguments given here can easily be adapted for ℓ -adic or de Rham cohomology, and are expected to work for solid $\mathcal{O}^p$ -cohomology, as developed in [ABM24].⁴

The two main differences with [SS91] are as follows :

- In loc.cit. the authors work on non-compactly supported étale cohomology, by considering limit of *relative cohomology*, relative to intermediary spaces. Here, we use the same geometric construction, but work on compactly supported étale cohomology instead, by considering the colimit of the cohomology of the complement of such spaces, in a dual manner.⁵ This relies on the excision result established in proposition A.2.1.

¹For simplicity, we do not compute the Tate twist here. This will be handled in chapter 8.

²We do not show that the comparison morphism $H_{\acute{e}t,c}^n(\Omega_C^d, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\acute{e}t,c}^n(\Omega_C^d, \mathcal{O}^+/p)$ is an almost isomorphism. This will be handled in chapter 8.

³The method of chapter 8 is well-suited for computing the cohomology of many étale sheaves, but crucially uses the étale formalism.

⁴See the relationship with problems of chromatic homotopy theory, at least in dimension 1, as in [Man25]. See also [Bar+25], where results of chromatic homotopy theory are obtained by considering the condensed $\hat{\mathcal{O}}^+$ -cohomology of Drinfeld's symmetric spaces.

⁵Actually, it would be more accurate to say that the computation of [SS91] is dual to the one performed here.

- In loc.cit. the authors work under the assumption that the cohomology theory is invariant with respect to the open unit disk, i.e. $H^*(X) \simeq H^*(X \times \mathring{\mathbb{D}}_C)$. Here, we work with a less restrictive assumption of *stalk*-invariance, in the sense that :

$$\varprojlim_{r \rightarrow \infty} H^*(X \times \mathbb{D}_C(|p^r|)) \simeq H^*(X)$$

for smooth proper X (see [proposition 6.3.1](#)). For $\mathcal{O}^+/p$ -cohomology, this will follow from the computation in [proposition 2.6.1](#), together with some weak version of the Künneth formula.

It would be possible to perform most of the computation in the language of Mann's six-functor formalism. For readability, we chose to formulate the results in the classical language of étale cohomology, using Mann's category in the proofs when necessary, in [section 6.2](#).

This chapter relies lightly on the appendix [section B.1](#), which gives some notions regarding the geometric realization of simplicial profinite spaces and its cohomology, as well as topologized Tits buildings.

Many computations can only be made up to almost isomorphism. This can make the notations a bit heavy, but poses essentially no extra difficulty (the only non-trivial part is [lemma 6.5.3](#)).

6.1 The geometry of the complement

Let K be a complete extension of $\mathbb{Q}_p$, with value group Γ_K , and C be a complete algebraically closed field containing K . Let $\mathcal{H}^d$ denote the set of K -rational hyperplanes of $\mathbb{P}_C^d$, and $\Omega_K^d = \mathbb{P}_K^d \setminus \bigcup_{H \in \mathcal{H}^d} H$ be Drinfeld's symmetric space of dimension d over K , and Ω_C^d denote its base change to C .⁶

Convention 6.1.1. By convention (following [SS91]), in this chapter, we always write projective coordinates $[z_0 : \cdots : z_d] \in \mathbb{P}^d(C)$ such that :

1. For all i , $|z_i| \leq 1$
2. There exists an i_0 such that $|z_{i_0}| = 1$.

Let $F = \mathbb{P}_C^n \setminus \Omega_C^d$, viewed as a closed topological subset of $\mathbb{P}_C^n$. The starting point is the following excision exact sequence in the derived category of almost $\mathcal{O}_C/p$ -modules, as (the almost version) of the one established in [proposition A.2.1](#), together with [lemma 4.2.2](#) :

$$R\Gamma_{\acute{e}t,c}(\Omega_C^d, \mathcal{O}^+/p)^a \rightarrow R\Gamma_{\acute{e}t}(\mathbb{P}_C^d, \mathcal{O}^+/p)^a \rightarrow \varinjlim_{F \subset U} R\Gamma_{\acute{e}t}(U, \mathcal{O}^+/p)^a \quad (6.1)$$

where the colimit is taken over qcqs open neighborhoods of F .

Remark 6.1.2. Note that the right-most term can be seen as the étale cohomology of F , viewed as a locally spatial diamond by [lemma A.1.3](#).

We stick with the weaker version written above, as this allows us to stay close to the language of rigid-analytic geometry used in [SS91], and this generalizes better to the setup of solid $\mathcal{O}^b$ -cohomology, as the version of above is the only available version of excision in that setup (cf. [ABM24, Prop 6.2.5]).

The étale cohomology of $\mathbb{P}_C^d$ is well-known.

Proposition 6.1.3. For any $n \in \mathbb{N}$:

$$H_{\acute{e}t}^n(\mathbb{P}_C^d, \mathcal{O}^+/p) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } n \text{ is even and } 0 \leq n \leq 2d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the primitive comparison theorem [Sch12a, Thm 5.1], the natural morphism

$$H_{\acute{e}t}^n(\mathbb{P}_C^d, \mathbb{F}_p) \otimes \mathcal{O}_C/p \rightarrow H_{\acute{e}t}^n(\mathbb{P}_C^d, \mathcal{O}^+/p)$$

is an almost isomorphism.

The $\mathbb{F}_p$ -cohomology of the scheme $\mathbb{P}_C^d$ is classical (this follows from the fact that étale cohomology in characteristic zero is a Weil cohomology theory), and this passes to the analytic setup using [Hub96, Thm 3.7.2]. $\square$

Our goal is ultimately to understand the leftmost term of [equation \(6.1\)](#), so that we will focus on computing the rightmost term.

We start by considering standard neighborhood of K -rational hyperplanes.

⁶Even though most of these notions depend on K , we drop it from the notation.

Definition 6.1.4. Let $\varepsilon \in \Gamma_K \cap]0, 1]$, typically of the form $|p|^n$ for $n \in \mathbb{N}$, and $H \in \mathcal{H}^d$. There exists a unique (up to a unit) unimodular (i.e. of norm one) linear form $\ell_H : \mathcal{O}_C^{d+1} \rightarrow \mathcal{O}_C$ such that $H = \{z \in \mathbb{P}^d(C), \ell_H(z) = 0\}$.

We let $H(\varepsilon) = \{z \in \mathbb{P}^d(C), |\ell_H(z)| \leq \varepsilon\}$ be the ε -neighborhood of H . It is the C -points of a quasi-compact open subset of $\mathbb{P}_C^d$.

For all $\varepsilon \in \Gamma_K \cap]0, 1]$, we may write $\bigcup_{H \in \mathcal{H}^d} H \subset \bigcup_{H \in \mathcal{H}^d} H(\varepsilon)$. Since the left hand side is compact as a closed subset of $\mathbb{P}_C^d$, the covering can be refined by a finite covering.

In fact, one can produce an explicit subcovering, as follows :

Definition 6.1.5. We say that two hyperplanes $H, H' \in \mathcal{H}^d$ are **congruent mod p^n** for $n \in \mathbb{N}$ if ℓ_H and $\ell_{H'}$ are equal modulo p^n .

We let $\mathcal{H}_n$ denote the quotient of $\mathcal{H}^d$ by the above equivalence relation, which is finite. There are natural inclusions $\mathcal{H}_{n+1} \subset \mathcal{H}_n$.

Recall that, if V is a vector space, the space of hyperplanes in $\mathbb{P}(V)$ naturally identifies with $\mathbb{P}(V^*)$. Hence, there are natural isomorphisms :

$$\mathcal{H}_n \cong \mathbb{P} \left((\mathcal{O}_C^{d+1})^* / p^n \right).$$

These are compatible with the inclusions, and $\varprojlim_{n \in \mathbb{N}} \mathcal{H}_n \cong \mathcal{H}^d$.

The following description will be useful.

Proposition 6.1.6. Any $H, H' \in \mathcal{H}^d$ are congruent modulo p^n iff $H(|p|^n) = H'(|p|^n)$.

Proof. This is [SS91, Sec.1, Lemma 2]. □

Note that any $H(|p|^n)$ is naturally isomorphic (up to a linear endomorphism of $\mathbb{P}_C^n$) to $\mathbb{P}_C^{d-1} \times \mathbb{D}_C(|p|^n)$, where $\mathbb{D}_C(|p|^n)$ denotes the closed unit ball of radius $|p|^n$, as defined in [example 2.1.16](#). More generally, we obtain the following statement regarding intersections of such neighborhoods :

Proposition 6.1.7. Let $H_0, \dots, H_k \in \mathcal{H}^d$.

There exists a unique integer $0 \leq s \leq d$ and some unique $\alpha_0 \leq \dots \leq \alpha_s \in \log(\Gamma_K) \cap \mathbb{R}_+^*$, together with a linear transformation φ of $\mathbb{P}_C^d$ such that, for all $n \in \mathbb{N}$:

$$\varphi : H_0(|p|^n) \cap \dots \cap H_k(|p|^n) \simeq \{[z_0 : \dots : z_d] \in \mathbb{P}^d(C), \forall 0 \leq i \leq s, |z_i| \leq |p|^{n-\alpha_i}\}.$$

Proof. This is [SS91, Sec.1, Lemma 5]. We recall the proof, as we will need some of its ideas.

Let $M = \sum_{i=0}^k \mathcal{O}_C \cdot \ell_{H_i}$, viewed as an $\mathcal{O}_C$ -submodule of $(\mathcal{O}_C^{d+1})^*$. By definition,

$$H_0(|p|^n) \cap \dots \cap H_k(|p|^n) = \{z \in \mathbb{P}_C^d, \forall \ell \in M, |\ell(z)| \leq |p|^n\}. \quad (6.2)$$

By the stacked basis theorem (as $\mathcal{O}_C$ is a principal ideal domain), we may pick a basis $\ell_0, \dots, \ell_d$ of $(\mathcal{O}_C^{d+1})^*$ and some unique $\alpha_0 \leq \dots \leq \alpha_s$ in $\log(\Gamma_K)$ such that $(p^{\alpha_0} \ell_0, \dots, p^{\alpha_s} \ell_s)$ is a basis of M .

Applying the linear transformation mapping the basis $(\ell_0, \dots, \ell_d)$ of $(\mathcal{O}_C^{d+1})^*$ to the standard basis $([z_0 : \dots : z_d] \mapsto z_i)_{0 \leq i \leq d}$ yields the result. □

Definition 6.1.8. For $H_0, \dots, H_k \in \mathcal{H}^d$, let $M = \sum_{i=0}^k \mathcal{O}_C \cdot \ell_{H_i}$ be as in the proof above.

We let $rk_n(H_0, \dots, H_k) \in \llbracket 0, d+1 \rrbracket$ be the **rank** of $M/p^n M$, defined as its minimal number of generators as an $(\mathcal{O}_C^{d+1})^*$ -module.

We may now state the following.

Proposition 6.1.9. Let $H_0, \dots, H_k \in \mathcal{H}^d$, and $(\alpha_0, \dots, \alpha_s)$ be given by [proposition 6.1.7](#). Let $r = rk_n(H_0, \dots, H_k)$.

1. If $r = d+1$, $H_0(|p|^n) \cap \dots \cap H_k(|p|^n) = \emptyset$.
2. If $r \leq d$, there is a natural C -linear endomorphism of $\mathbb{P}_C^d$ sending :

$$H_0(|p|^n) \cap \dots \cap H_k(|p|^n) \simeq \mathbb{P}_C^{d-r} \times \prod_{i=0}^{r-1} \mathbb{D}_C^1(|p|^{n-\alpha_i}).$$

Proof. We reuse the notations of the proof of [proposition 6.1.7](#).

Let $r' = \max\{0 \leq i \leq s, \alpha_i \leq n\}$. We start by showing that $r' = r$.

Since $(p^{\alpha_0} \ell_0, \dots, p^{\alpha_s} \ell_s)$ is a basis of M , the quotient $M/p^n M$ is generated by $(p^{\alpha_i} \cdot \bar{\ell}_i)_{0 \leq i \leq r'}$, where $\bar{\ell}_i$ denotes the mod p^n reduction of ℓ_i . Hence $r' \geq r$.

Assume by contradiction that $r' > r$, i.e. there exists a generating family $(\bar{\ell}'_0, \dots, \bar{\ell}'_{r'-1})$ of $M/p^n M$. Take arbitrary lifts ℓ'_i of $\bar{\ell}'_i$ in M .

By construction, the family $(\ell'_i)_{0 \leq i \leq r'-1} \sqcup (p^{\alpha_i-n} \cdot \ell_i)_{r'+1 \leq i \leq s}$ generates M , which is impossible, since M is free of rank $s+1$. Hence $r = r'$.

The first point is now clear from the description of [equation \(6.2\)](#).

The second point follows from [proposition 6.1.7](#), letting $|z_d| = 1$, and noting that the condition $|z_i| \leq p^{n-\alpha_i}$ is vacuous exactly when $n \geq \alpha_i$ (under [convention 6.1.1](#)). $\square$

For $H_0, \dots, H_k \in \mathcal{H}^d$, $rk_n(H_0, \dots, H_k)$ increases as n grows larger, and it converges to the element denoted s in [proposition 6.1.7](#). We then denote it $rk_\infty(H_0, \dots, H_k)$.

Remark 6.1.10. For every n and $H_0, \dots, H_k \in \mathcal{H}^d$, $rk_n(H_0, \dots, H_k)$ depends only on the class of $H_0, \dots, H_k$ in $\mathcal{H}_n$.

For $n \in \mathbb{N}$, let $\Omega_n = \bigcup_{H \in \mathcal{H}^d} H(|p|^n) = \bigcup_{H \in \mathcal{H}_n} H(|p|^n)$. When n grows larger, this forms a decreasing set of qcqs open neighborhoods of F . The family $\{\Omega_n\}_{n \in \mathbb{N}}$ is furthermore cofinal in the set of such open neighborhoods.

Hence the colimit in [equation \(6.1\)](#) rewrites as :

$$\varinjlim_{F \subset U} R\Gamma_{\acute{e}t, c}(U, \mathcal{O}^+/p) \cong \varinjlim_{n \in \mathbb{N}} R\Gamma_{\acute{e}t}(\Omega_n, \mathcal{O}^+/p) = \varinjlim_{n \in \mathbb{N}} R\Gamma_{\acute{e}t} \left(\bigcup_{H \in \mathcal{H}_n} H(|p|^n), \mathcal{O}^+/p \right). \quad (6.3)$$

When $d = 1$, it follows from [proposition 6.1.9](#) that the covering $\bigcup_{H \in \mathcal{H}_n} H(|p|^n)$ is disjoint, so that the cohomology can be computed easily.

For general d , the open subset are not necessarily disjoint. However, the intersections are of a known form by [proposition 6.1.7](#), and the combinatorics of such intersections can be encoded in a spectral sequence.

Lemma 6.1.11. For all $n \in \mathbb{N}$, there exists a spectral sequence of $\mathcal{O}_C/p$ -modules :

$$\bigoplus_{H_0, \dots, H_s \in \mathcal{H}_n} H_{\text{ét}}^t \left(\bigcap_{k=0}^s H_k(|p|^n), \mathcal{O}^+/p \right) \xRightarrow{E_1^{s,t}(n)} E^{s+t}(n) \xRightarrow{H_{\text{ét}}^{s+t} \left(\bigcup_{H \in \mathcal{H}_n} H(|p|^n), \mathcal{O}^+/p \right)}$$

Moreover, the differential morphism $d_1 : E_1^{s,t}(n) \rightarrow E_1^{s+1,t}(n)$ is given by the alternating sums of the maps induced by removing the i -th hyperplane.

Proof. This is the Čech-to-cohomology spectral sequence⁷ [Sta23, Theorem 03OW] associated to the étale covering $\bigsqcup_{H \in \mathcal{H}_n} H(|p|^n) \rightarrow \Omega_n$. $\square$

Remark 6.1.12. It would be possible to construct this spectral sequence by working directly in the Mann’s formalism, although the (∞) -categorical language makes it more cumbersome.

This is done by applying the Bousfield-Kan spectral sequence (cf. [Lur17, Variant 1.2.4.9]) in the opposite category of almost $\mathcal{O}_C/p$ -modules, together with [proposition 3.5.3](#).

Moreover, this process is compatible with natural inclusion $H(|p|^{n+1}) \subset \tilde{H}(|p|^n)$ (where, for $H \in \mathcal{H}_{n+1}$, $\tilde{H}$ denotes the class of H in $\mathcal{H}_n$). This yields morphisms $E_1(n) \rightarrow E_1(n+1)$.

Moreover, the convergence of spectral sequences naturally commutes with colimits and localizations, so that :

$$\left(\varinjlim_{n \in \mathbb{N}} E_1^{\bullet, \bullet}(n) \right)^a \xRightarrow{\quad} \left(\varinjlim_{n \in \mathbb{N}} E^{\bullet}(n) \right)^a. \quad (6.4)$$

We may rewrite $\varinjlim_{n \in \mathbb{N}} E_1^{\bullet, \bullet}(n)$ as $\varinjlim_{n \in \mathbb{N}} \text{Im} (E_1^{\bullet, \bullet}(n) \rightarrow E_1^{\bullet, \bullet}(n+3))$,⁸ so that it suffices to study the image of the morphisms increasing n .

By [proposition 6.1.9](#), this reduces to studying the image of morphisms :

$$H_{\text{ét}}^* \left(\mathbb{P}_C^{d-r} \times \prod \mathbb{D}_C(|p|^{n-\alpha_i}), \mathcal{O}^+/p \right) \rightarrow H_{\text{ét}}^* \left(\mathbb{P}_C^{d-|I_n|} \times \prod \mathbb{D}_C(|p|^{n+3-\alpha_i}), \mathcal{O}^+/p \right) \quad (6.5)$$

in the almost category.

Recall that the image of the restriction morphism in $\mathcal{O}^+/p$ -cohomology between closed disks was studied in [proposition 2.6.1](#). We may then deduce the necessary statement by a variant of the Künneth formula, even though the spaces considered are not proper.

6.2 A weak Künneth formalism

This section relies heavily on Mann’s six-functor formalism. All spaces considered will be qcqs, so that, by [remark 3.5.7](#), for any rigid analytic variety $f : X \rightarrow \text{Spa}(C, \mathcal{O}_C)$, the object $f_*^M 1_X$ is discrete and computes the classical étale cohomology $R\Gamma_{\text{ét}}(X, \mathcal{O}^+/p)$, up to almost isomorphism.

Recall the Künneth’s isomorphism for proper spaces.

Proposition 6.2.1. *Let $(X_i)_{i \in I}$ be a finite family of proper adic spaces over C . Then, there exists a natural morphism in $\mathcal{D}_{\text{ét}}(\mathcal{O}_C/p)^a$:*

⁷Note that this is not the most standard form of the Čech-to-cohomology spectral sequence, as one usually writes down the second page.

⁸The integer 3 is chosen for reasons that will become clear in [proposition 6.3.2](#).

$$\bigotimes_{i \in I} R\Gamma_{\text{ét}}(X_i, \mathcal{O}^+/p)^a \rightarrow R\Gamma_{\text{ét}}\left(\prod X_i, \mathcal{O}^+/p\right)^a.$$

Proof. This is [proposition 3.1.7](#), together with the fact that, for proper objects, the cohomology computed in Mann's category coincides with standard étale cohomology [remark 3.5.7](#). $\square$

We start by defining a *Künneth map* for not necessarily proper spaces (that need not be an isomorphism).

Proposition 6.2.2. *Let $(X_i)_{i \in I}$ be a finite family of qcqs smooth rigid analytic varieties over C . There exists a natural morphism in $\mathcal{D}_{\text{ét}}(\mathcal{O}_C/p)^a$:*

$$\bigotimes_{i \in I} R\Gamma_{\text{ét}}(X_i, \mathcal{O}^+/p)^a \rightarrow R\Gamma_{\text{ét}}\left(\prod X_i, \mathcal{O}^+/p\right)^a.$$

It coincides with the map from Künneth isomorphism from [proposition 6.2.1](#) if the X_i are additionally proper. It is moreover functorial in the X_i .

Proof. By induction on $|I|$, we may assume that $|I| = 2$. Let X_1, X_2 be two smooth and qcqs rigid analytic varieties. We work implicitly in Mann's formalism, dropping the "M" superscripts for clarity. As all spaces are quasi-compact, it follows from [remark 3.5.7](#) that f_*^M computes classical étale cohomology groups.

Consider the cartesian diagram :

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{f'} & X_2 \\ \downarrow g' & \searrow \Delta & \downarrow g \\ X_1 & \xrightarrow{f} & \text{Spa}(C, \mathcal{O}_C) \end{array}$$

For any map $h : X \rightarrow Y$ and $\mathcal{F} \in \mathcal{D}_{\square}^a(X, \mathcal{O}^+/p)$, $\mathcal{G} \in \mathcal{D}_{\square}^a(Y, \mathcal{O}^+/p)$, the unit $h^* \circ h_* \rightarrow \text{id}$ of the adjunction $h^* \dashv h_*$ yields a map :

$$h^*(h_*\mathcal{F} \otimes \mathcal{G}) \cong h^*h_*\mathcal{F} \otimes h^*\mathcal{G} \rightarrow \mathcal{F} \otimes (h^*\mathcal{G}).$$

This yields, by adjunction, a "projection formula map" :

$$h_*\mathcal{F} \otimes \mathcal{G} \rightarrow h_*(\mathcal{F} \otimes h^*\mathcal{G}). \quad (6.6)$$

By smooth base change (cf. [proposition 3.1.4](#)), for all $\mathcal{F} \in \mathcal{D}_{\square}^a(X_2, \mathcal{O}^+/p)$; there is a natural isomorphism $f^*g_*\mathcal{F} \cong g'_*f'^*\mathcal{F}$. We may then construct a Künneth morphism as the composition of the following maps :

$$\begin{aligned} f_*\mathbb{1}_{X_1} \otimes g_*\mathbb{1}_{X_2} &\rightarrow f_*(\mathbb{1}_{X_1} \otimes f^*g_*\mathbb{1}_{X_2}) && \text{By equation (6.6)} \\ &\cong f_*(\mathbb{1}_{X_1} \otimes g'_*f'^*\mathbb{1}_{X_2}) && \text{By smooth base change} \\ &\rightarrow f_*(g'_*(g'^*\mathbb{1}_{X_2} \otimes f'^*\mathbb{1}_{X_1})) && \text{By equation (6.6)} \\ &\cong \Delta_*(g'^*\mathbb{1}_{X_2} \otimes f'^*\mathbb{1}_{X_1}) && \text{By composition} \\ &\cong \Delta_*\mathbb{1}_{X_1 \times X_2} && \text{Since } \mathbb{1} \text{ is the unit object.} \end{aligned}$$

This extends the result for proper space, as the computation is the same than the proof of [proposition 3.1.7](#). This construction is clearly functorial. $\square$

Proposition 6.2.3. *Let X, Y be two smooth rigid analytic varieties over C , of pure dimension d_X and d_Y respectively.*

Assume that X is proper, and Y is qcqs.

Then, the map constructed in [proposition 6.2.2](#) induces an almost isomorphism :

$$R\Gamma_{\acute{e}t}(X, \mathcal{O}^+/p) \otimes R\Gamma_{\acute{e}t}(Y, \mathcal{O}^+/p) \xrightarrow{a} R\Gamma_{\acute{e}t}(X \times Y, \mathcal{O}^+/p).$$

Proof. As in the proof above, we go through Mann's six-functor formalism, and drop the M superscripts.

Consider the cartesian square :

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_Y} & X \\ \downarrow \pi_Y & \searrow \Delta & \downarrow f_X \\ Y & \xrightarrow{f_Y} & \mathrm{Spa}(C, \mathcal{O}_C) \end{array}$$

The product $X \times Y$ is smooth over C of dimension $d_X + d_Y$, so that Poincaré duality in the sense of [proposition 3.1.8](#) yields :

$$\begin{aligned} \Delta_* \mathbb{1}_{X \times Y} &\xrightarrow{a} \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)^\varphi}(\Delta_* \mathbb{1}_{X \times Y}(d_X + d_Y)[2d_X + 2d_Y], \mathbb{1}_C) \\ &\xrightarrow{a} \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)^\varphi}(f_{X,*} \mathbb{1}_X(d_X)[2d_X] \otimes f_{Y,*} \mathbb{1}_Y(d_Y)[2d_Y], \mathbb{1}_C) \\ &\xrightarrow{a} \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)^\varphi}\left(f_{X,*} \mathbb{1}_X(d_X)[2d_X], \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)^\varphi}(f_{Y,*} \mathbb{1}_Y(d_Y)[2d_Y], \mathbb{1}_C)\right), \end{aligned}$$

where the second isomorphism is Künneth's formula for compactly supported cohomology (as in [proposition 3.1.7](#)), and the third one follows from (internal) adjunction.

Since X is proper, its étale cohomology complex is perfect, hence dualizable. Hence, we have, by [[Man22](#), Lemma 3.7.4.iii] :

$$\begin{aligned} &\underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C^+/p)^\varphi}\left(f_{X,*} \mathbb{1}_X(d_X)[2d_X], \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C/p)^\varphi}(f_{Y,*} \mathbb{1}_Y(d_Y)[2d_Y], \mathbb{1}_C)\right) \\ &\xrightarrow{a} \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C^+/p)^\varphi}(f_{X,*} \mathbb{1}_X(d_X)[2d_X], \mathbb{1}_C) \otimes \underline{\mathrm{RHom}}_{\mathcal{D}_{\square}^a(\mathcal{O}_C^+/p)^\varphi}(f_{Y,*} \mathbb{1}_Y(d_Y)[2d_Y], \mathbb{1}_C) \\ &\xrightarrow{a} (f_X)_* \mathbb{1}_X \otimes (f_Y)_* \mathbb{1}_Y \end{aligned}$$

Where the last line followed by re-applying Poincaré duality as in [proposition 3.1.8](#). This yields the desired result by [remark 3.5.7](#). $\square$

6.3 Identification of the transition morphisms

We may now study the image of the morphism of [equation \(6.5\)](#).

Let us start by isolating the following "stalk-invariance" property :

Proposition 6.3.1. *Let X a smooth proper rigid analytic variety over C . Then, for all $n \geq 0$, there is an almost isomorphism :*

$$H_{\acute{e}t}^n(X, \mathcal{O}^+/p) \xrightarrow{a} \varprojlim_{k \in \mathbb{N}} H_{\acute{e}t}^n(X \times \mathbb{D}_C(|p^k|), \mathcal{O}^+/p)$$

Proof. This follows from [proposition 6.2.3](#) and [proposition 2.6.1](#). $\square$

Proposition 6.3.2. Let $c, b, s \in \mathbb{N}$, $j_1, \dots, j_b \in \mathbb{N}$, $i_0, \dots, i_s \in \mathbb{N}$, and let $r \in \log(\Gamma_K)$ such that $r > p/(p-1)$.

For any $n \in \mathbb{N}$, the image of the morphism :

$$\begin{aligned} \Phi : H_{\text{ét}}^n \left(\mathbb{P}_C^c \times \mathbb{P}_C^b \times \prod_{k=0}^s \mathbb{D}_C(|p|^{i_k}), \mathcal{O}^+/p \right)^a \\ \longrightarrow H_{\text{ét}}^n \left(\mathbb{P}_C^c \times \prod_{k=1}^b \mathbb{D}_C(|p|^{j_k}) \times \prod_{k=0}^s \mathbb{D}_C(|p|^{i_k+r+1}), \mathcal{O}^+/p \right)^a \end{aligned}$$

is given by :

$$\text{Im}(\Phi) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } n \text{ is even and } 0 \leq n \leq 2c; \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by [proposition 6.1.3](#), this is the result one would obtain if $\mathcal{O}^+/p$ cohomology were $\mathbb{D}_C$ -invariant.

Proof. The morphisms $\mathbb{D}_C(|p|^{j_k}) \rightarrow \mathbb{P}_C$ factors through $\mathbb{D}_C(|p|^{j_k-r})$, so we may assume that $b = 0$.

Using [proposition 6.2.3](#), together with the cohomology of projective space computed in [proposition 6.1.3](#), we reduce to the case $c = 0$.

It then suffices to prove that :

$$\begin{aligned} \text{Im} \left(H_{\text{ét}}^n \left(\prod_{k=0}^s \mathbb{D}_C(|p|^{i_k}), \mathcal{O}^+/p \right)^a \rightarrow H_{\text{ét}}^n \left(\prod_{k=0}^s \mathbb{D}_C(|p|^{i_k+r}), \mathcal{O}^+/p \right)^a \right) \\ \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For any $k \in \mathbb{N}$, we let $\mathbb{D}_C^c(|p|^k)$ be the canonical compactification of $\mathbb{D}_C(|p|^k)$, viewed as an adic space. There are natural inclusions $\mathbb{D}_C(|p|^{k+1}) \subset \mathbb{D}_C^c(|p|^{k+1}) \subset \mathbb{D}_C(|p|^k)$.

By [proposition 6.2.2](#), we have the following commutative diagram :

$$\begin{array}{ccc} \bigotimes R\Gamma_{\text{ét}}(\mathbb{D}_C(|p|^{i_k}), \mathcal{O}^+/p)^a & \longrightarrow & R\Gamma_{\text{ét}}(\prod \mathbb{D}_C(|p|^{i_k}), \mathcal{O}^+/p)^a \\ \downarrow & \swarrow \text{dashed} & \downarrow \\ \bigotimes R\Gamma_{\text{ét}}(\mathbb{D}_C^c(|p|^{i_k+1}), \mathcal{O}^+/p)^a & \xrightarrow{\cong} & R\Gamma_{\text{ét}}(\prod \mathbb{D}_C^c(|p|^{i_k+1}), \mathcal{O}^+/p)^a \\ \downarrow & & \downarrow \\ \bigotimes R\Gamma_{\text{ét}}(\mathbb{D}_C(|p|^{i_k+1}), \mathcal{O}^+/p)^a & \longrightarrow & R\Gamma_{\text{ét}}(\prod \mathbb{D}_C(|p|^{i_k+1}), \mathcal{O}^+/p)^a \\ \downarrow f & & \downarrow \\ \bigotimes R\Gamma_{\text{ét}}(\mathbb{D}_C(|p|^{i_k+r+1}), \mathcal{O}^+/p)^a & \dashrightarrow & R\Gamma_{\text{ét}}(\prod \mathbb{D}_C(|p|^{i_k+r+1}), \mathcal{O}^+/p)^a \end{array}$$

Where the second horizontal arrow is an isomorphism by [proposition 6.2.1](#).

The desired morphism can be obtained as the composition of the dashed arrows, in particular, it factors through f . It follows from [proposition 2.6.1](#) that f is zero, outside of degree zero.

In degree zero, every morphism of the diagram induces the identity on the submodule corresponding to constant functions, identified to $\mathcal{O}_C/p$. Hence $\mathcal{O}_C/p$ is included in the image of the composition. By [proposition 2.6.1](#), the image of f in degree zero is $\mathcal{O}_C/p$, so that the image of the composition is exactly $\mathcal{O}_C/p$. This concludes. $\square$

We may now go back to the spectral sequence, constructed in [lemma 6.1.11](#).

Lemma 6.3.3. *Fix some $H_{i_0}, \dots, H_{i_s} \in \mathcal{H}_{n+3}$. We let $\tilde{H}_{i_k}$ denote their natural image in $\mathcal{H}_n$. The image of the morphism :*

$$\Psi : H_{\acute{e}t}^t \left(\bigcap_{k=0}^s \tilde{H}_{i_k}(|p|^n), \mathcal{O}^+/p \right)^a \rightarrow H_{\acute{e}t}^t \left(\bigcap_{k=0}^s H_{i_k}(|p|^{n+3}), \mathcal{O}^+/p \right)^a$$

is given by :

$$\text{Im}(\Psi) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } t \text{ is even and } t/2 \leq d - rk_{n+3}(H_{i_0}, \dots, H_{i_k}). \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $r = rk_{n+3}(H_{i_0}, \dots, H_{i_k})$.

If $r = d + 1$, $\bigcap_{k=0}^s H_{i_k}(|p|^{n+3}) = \emptyset$, and the result vacuously holds.

Assume that $r \leq d$. By [proposition 6.1.9](#), up to a linear endomorphism of $\mathbb{P}_C^d$, we have :

$$\bigcap_{k=0}^p H_{i_k}(|p|^n) \cong \mathbb{P}_C^{d-r+b} \times \prod_{i=0}^{r-b-1} \mathbb{D}_C^1(|p|^{n-\alpha_i}),$$

with $b = r - rk_n(\tilde{H}_{i_0}, \dots, \tilde{H}_{i_k})$. This linear endomorphism restricts to some :

$$\bigcap_{k=0}^p \tilde{H}_{i_k}(|p|^{n+3}) \cong \mathbb{P}_C^{d-r} \times \prod_{i=0}^{r-1} \mathbb{D}_C^1(|p|^{n+3-\alpha_i}).$$

The result now follows from [proposition 6.3.2](#). $\square$

We arrive at the following :

Proposition 6.3.4. *For $s, t \geq 0$ and $n \in \mathbb{N}$, we let*

$$\text{Im} \left(E_1^{s,t}(n) \rightarrow E_1^{s,t}(n+3) \right) \stackrel{a}{\cong} \begin{cases} \bigoplus_{(H_0, \dots, H_s) \in Y_s^{(n+3, d-t/2)}} \mathcal{O}_C/p & \text{if } t \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

where $Y_s^{(n,t)} = \{H_0, \dots, H_s \in \mathcal{H}_n, rk_n(H_0, \dots, H_s) \leq t\}$.

For simplicity, we let $F_1^{s,t}(n) := \text{Im} \left(E_1^{s,t}(n) \rightarrow E_1^{s,t}(n+3) \right)^a$. For fixed n , this admits the structure of the first page of spectral sequence as s, t vary.

By construction of $E_{\bullet}^{s,t}$ and [equation \(6.3\)](#), we have :

$$\varinjlim_{n \in \mathbb{N}} F_1^{s,t}(n) \implies \varinjlim_{n \in \mathbb{N}} H_{\acute{e}t}^{s+t}(\Omega_n, \mathcal{O}^+/p)^a$$

Remark 6.3.5. There is a natural $\mathrm{GL}_{d+1}(K)$ -action on both sides. This spectral sequence is equivariant with respect to the action.⁹

The next step is to understand the differential morphisms appearing in $F_1^{\bullet,\bullet}$. This has to do with the simplicial structure of the $Y_s^{(n,t)}$.

Before moving on to the next section, let us isolate a small result.

Lemma 6.3.6. For $0 \leq k \leq d-1$, the natural morphism :

$$\mathcal{O}_C/p \xrightarrow{a} H_{\acute{e}t}^{2k}(\mathbb{P}_C^d, \mathcal{O}^+/p) \rightarrow \varinjlim_{n \in \mathbb{N}} H_{\acute{e}t}^{2k}(\Omega_n, \mathcal{O}^+/p)$$

is almost injective.

Proof. It suffices to prove the injectivity for all fixed n . Consider the map :

$$\mathcal{O}_C/p \xrightarrow{a} H_{\acute{e}t}^{2k}(\mathbb{P}_C^d, \mathcal{O}^+/p) \rightarrow H_{\acute{e}t}^{2k}(\Omega_n, \mathcal{O}^+/p) \rightarrow H_{\acute{e}t}^{2k}(H(|p|^n), \mathcal{O}^+/p)$$

for some arbitrary $H \in \mathcal{H}_n$. We will show that the composition is almost injective.

Since $H(|p|^n) \cong \mathbb{D}_C(|p|^n) \times \mathbb{P}_C^{d-1}$, by **proposition 6.2.3**, this reduces to showing that the map $H_{\acute{e}t}^{2k}(\mathbb{P}_C^n, \mathcal{O}^+/p) \rightarrow H_{\acute{e}t}^{2k}(\mathbb{P}_C^{n-1}, \mathcal{O}^+/p)$ is an isomorphism for $k < n$.

This follows from a relative version of the primitive comparison theorem, together with a known explicit computation for $\mathbb{F}_p$ -cohomology (which reduces to the algebraic setting). $\square$

6.4 The simplicial structure on hyperplanes

Our next step is to understand the morphisms appearing in **proposition 6.3.4**, in term of the simplicial structure on $Y_s^{(n,t)}$.

Recall that we defined $Y_s^{(n,t)} = \{H_0, \dots, H_s \in \mathcal{H}_n, rk_n(H_0, \dots, H_s) \leq t\}$.

Varying s , this admits a natural structure as a simplicial set, with face maps d_i given by removing the i -th hyperplane, and degeneracy maps given by duplicating one.

We define $\mathcal{C}^s(Y^{(n,t)}, \mathcal{O}_C/p) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[Y_s^{(n,t)}], \mathcal{O}_C/p)$. Varying s , this naturally admits a structure of a cosimplicial $\mathcal{O}_C/p$ -module.

Then, **proposition 6.3.4** rewrites as :

Proposition 6.4.1. There is an almost isomorphism of $\mathcal{O}_C/p$ -modules :

$$F_1^{s,t}(n) \xrightarrow{a} \begin{cases} \mathcal{C}^s(Y^{(n+3, d-t/2)}, \mathcal{O}_C/p) & \text{if } t \text{ is even, } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The differential morphisms $d_1 : F_1^{s,t}(n) \rightarrow F_1^{s+1,t}(n)$ induced by the one on E_1 have been made explicit in **lemma 6.1.11**. Hence, the second page $F_2^{s,t}(n)$ can be computed as follows.

⁹Note however that the action does not stabilize the Ω_n , but it passes to the colimit (cf. [SS91, 2nd remark at the end of §3]).

Proposition 6.4.2. We have :

$$F_2^{s,t}(n) \stackrel{a}{\cong} \begin{cases} H^s(\mathcal{C}^\bullet(Y^{(n+3,d-t/2)}, \mathcal{O}_C/p)^\sim) & \text{if } t \text{ is even, } t \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $(\mathcal{C}^\bullet(Y^{(n+3,d-t/2)}, \mathcal{O}_C/p))^\sim$ denotes the cochain complex associated to the cosimplicial object $\mathcal{C}^\bullet(Y^{(n+3,d-t/2)}, \mathcal{O}_C/p)$, with differential obtained by taking alternate sums of codegeneracy maps (cf. [definition B.1.2](#)).

When t is even, this is the complex computing the simplicial cohomology groups of the simplicial set $Y_r^{(n+3,d-t/2)}$, with $\mathcal{O}_C/p$ -coefficients.

We are interested in what happens when $n \rightarrow \infty$. Let $F_r^{\bullet,\bullet}(\infty) := \varinjlim_{n \in \mathbb{N}} F_r^{\bullet,\bullet}(n)$.

For fixed s, t , we let $Y_s^{(t)} = \varprojlim_{n \in \mathbb{N}} Y_s^{(n,t)}$, endowed with the limit topology, making it a profinite space. By construction, this can be explicitated as :

$$Y_r^{(t)} \cong \{H_0, \dots, H_r \in \mathcal{H}^d, rk_\infty(H_0, \dots, H_r) \leq t\},$$

Moreover, the limit topology coincides with the topology as a subset of $(\mathbb{P}^d(C)^*)^{r+1}$. Since the degeneracy and face maps are clearly continuous, varying r , this admits a natural structure of a simplicial profinite space.

Let $|Y^{(t)}|^{pro}$ denote the geometric realization of the simplicial profinite space $Y_\bullet^{(t)}$, as defined in the [section B.1](#).

Lemma 6.4.3. $\varinjlim_{n \in \mathbb{N}} F_1^{s,t}(n) \stackrel{a}{\cong} \begin{cases} \text{LC}(Y_s^{(d-t/2)}, \mathcal{O}_C/p) & \text{if } t \text{ is even, } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

Proof. This follows from [proposition 6.4.1](#) and [lemma B.1.10](#). □

As in [proposition 6.4.2](#), the second page is given by the cohomology of the complex. This can be interpreted geometrically.

Proposition 6.4.4. We have : $\varinjlim_{n \in \mathbb{N}} F_2^{s,t}(n) \stackrel{a}{\cong} \begin{cases} H_{\text{Shf}}^s(|Y^{(d-t/2)}|^{pro}, \underline{\mathbb{F}}_p) \otimes \mathcal{O}_C/p & \text{if } t \text{ is even, } t \geq 0 \\ 0 & \text{otherwise,} \end{cases}$

where $H_{\text{Shf}}^s(|Y^{(d-t/2)}|^{pro}, \underline{\mathbb{F}}_p)$ denotes the sheaf cohomology of the constant sheaf $\mathbb{F}_p$.

Proof. It follows from [proposition B.1.3](#) that, for t even and $t \geq 0$:

$$\varinjlim_{n \in \mathbb{N}} F_2^{s,t}(n) \stackrel{a}{\cong} H_{\text{Shf}}^s(|Y^{(d-t/2)}|^{pro}, \underline{\mathcal{O}_C/p}).$$

By [lemma B.1.11](#), the terms appearing in the complex [lemma 6.4.3](#) are free $\mathcal{O}_C/p$ -modules. By flatness, cohomology commutes with tensor products, hence the result. □

It suffices to understand the sheaf cohomology of $|Y^{(d-t/2)}|$, with $\mathbb{F}_p$ -coefficients. This was done in [\[SS91\]](#); let us recall the computation.

We introduce the following auxiliary complex :

Definition 6.4.5. Let $\mathcal{T}_r^{(k)}$ be the simplicial profinite space defined, for $1 \leq k \leq d$, by :

$$\mathcal{T}_r^{(k)} = \{W_0 \subset W_1 \subset \cdots \subset W_r \subset (K^{d+1})^* \text{ such that } 1 \leq \dim_K(W_i) \leq k\}.$$

This admits a simplicial structure when r varies.

We endow each $\mathcal{T}_r^{(k)}$ with the finest topology such that the natural $GL_{d+1}(K)$ -action induces homeomorphisms $GL_{d+1}(K) \cdot \tau \cong GL_{d+1}(K)/P_\tau(K)$.

Remark 6.4.6. When $k = d$, this is exactly the Tits building of GL_{d+1} , described in [example B.2.2](#). Its topology will be generalized in [definition B.2.3](#).

Let $|\mathcal{T}_r^{(k)}|^{pro}$ be the geometric realization of the above, viewed as a simplicial profinite space. Note that it admits a natural $GL_{d+1}(K)$ -action, as all $\mathcal{T}_r^{(k)}$ do, and that the face and degeneracy maps are clearly $GL_{d+1}(K)$ -equivariant.

Proposition 6.4.7. $|\mathcal{T}_\bullet^{(k)}|^{pro}$ and $|Y_\bullet^{(k)}|^{pro}$ have isomorphic sheaf cohomology with $\mathbb{F}_p$ -coefficients (as a representation of $GL_{d+1}(K)$).

Proof. This is Prop 5 of section 3 of [SS91] for $\mathbb{Z}$ -coefficients, and follows for $\mathbb{F}_p$ coefficients using the universal coefficient theorem. $\square$

In order to describe their cohomology groups, we need to introduce generalized Steinberg representations.

Definition 6.4.8. Let T and B be the standard maximal torus and Borel subgroups of the algebraic group GL_{d+1} . Let $\Delta = \{1, \dots, d\}$, be a basis of the associated relative root system.

For each subset $J \subset \Delta$, we let P_J denote the corresponding parabolic subgroup of GL_{d+1} .

For M an abelian group and $I \subset \Delta$, we let

$$\mathrm{Sp}_I(M) := \mathrm{LC}(G/P_I, M) / \sum_{i \in \Delta \setminus I} \mathrm{LC}(G/P_{I \sqcup \{i\}}, M)$$

be the associated **generalized Steinberg representation** with coefficients in M , with natural G -action induced by left multiplication.

For $0 \leq r \leq d$, we let $\mathrm{Sp}_r(M) := \mathrm{Sp}_{\{1, 2, \dots, d-r\}}(M)$.

Note that Sp_d is the usual Steinberg representation, while $\mathrm{Sp}_0(M) \cong M$.

Lemma 6.4.9. For any $\mathbb{F}_p$ -vector space M , $\mathrm{Sp}_I(M) \cong \mathrm{Sp}_I(\mathbb{F}_p) \otimes M$.

Proof. This follows from [lemma B.1.11](#), together with the commutation of tensor product with quotients. $\square$

We may now state the following result.

Proposition 6.4.10. There is a natural $GL_{d+1}(K)$ -equivariant isomorphism :

$$H_{\mathrm{Shf}}^r(|\mathcal{T}_\bullet^{(k)}|^{pro}, \mathbb{F}_p) \cong \begin{cases} 0 & \text{if } r \neq 0, k-1; \\ \mathbb{F}_p & \text{if } r = 0 \text{ and } k \neq 1; \\ \mathrm{LC}(\mathbb{P}((K^{d+1})^*), \mathbb{F}_p) & \text{if } r = 0 \text{ and } k = 1; \\ \mathrm{Sp}_k(\mathbb{F}_p) & \text{if } r = k-1 \text{ and } k \neq 1. \end{cases}$$

Proof. This is [SS91, Section 3, Prop 6], and [SS91, Section 4, Lemma 1] for $\mathbb{Z}$ -coefficients. This passes to $\mathbb{F}_p$ -coefficients using the universal coefficient theorem (One can check that there is no nontrivial Ext^1 as the cohomology groups appearing are free), together with lemma B.1.11 and lemma 6.4.9. $\square$

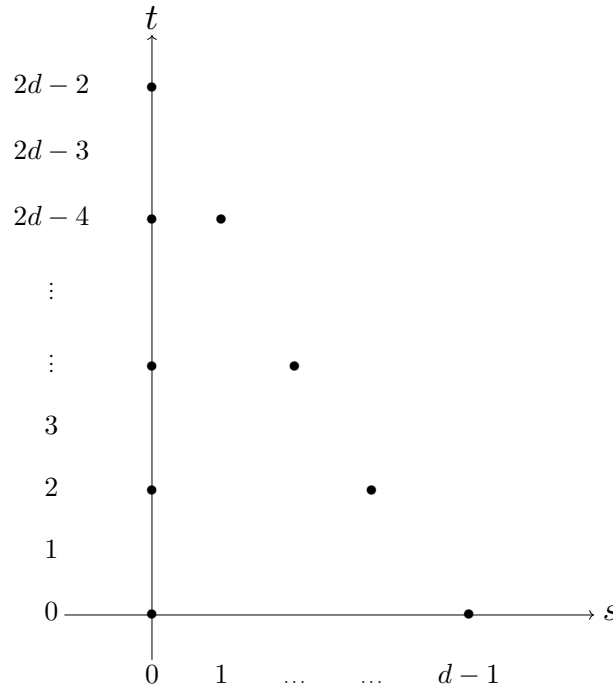
6.5 Convergence and splitting of the spectral sequence

It follows from proposition 6.4.4 that the second page of the spectral sequence has the following form :

Proposition 6.5.1. *We have natural $\text{GL}_{d+1}(K)$ -equivariant isomorphisms :*

$$F_2^{s,t}(\infty) \stackrel{a}{\cong} \begin{cases} \mathcal{O}_C/p & \text{if } s = 0, t \geq 0 \text{ is even and } d - t/2 > 1; \\ \text{LC} \left(\mathbb{P}(K^{d+1})^*, \mathbb{F}_p \right) \otimes \mathcal{O}_C/p & \text{if } s = 0, t = 2d - 2; \\ \text{Sp}_{d-t/2}(\mathbb{F}_p) \otimes \mathcal{O}_C/p & \text{if } t \geq 0 \text{ is even and } s = d - t/2 - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Here is a diagram of the nonzero terms of $F_2^{s,t}(\infty)$:



Note that all arrows in the subsequent pages $F_r, r \geq 2$ are trivial (as they all have either trivial domain or codomain), so that $F_2^{s,t}(\infty) = F_\infty^{s,t}(\infty)$ for all s, t .

Recall furthermore, that $F_2^{s,t}(\infty) \implies \varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{s+t}(\Omega_n, \mathcal{O}^+/p)^a$.

We have hence established the following.

Proposition 6.5.2. *The cohomology groups $\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^*(\Omega_n, \mathcal{O}^+/p)$ admit the following description, with $\text{GL}_{d+1}(K)$ -action :*

- For t even, $0 \leq t < d - 1$, $\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^t(\Omega_n, \mathcal{O}^+/p) \stackrel{a}{\cong} \mathcal{O}_C/p$
- For t odd, $0 \leq t < d - 1$, $\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^t(\Omega_n, \mathcal{O}^+/p) \stackrel{a}{\cong} 0$.

- For t even, $d - 1 \leq t < 2d - 2$, there is an exact sequence of almost $\mathcal{O}_C/p[GL_{d+1}(K)]$ -modules :

$$0 \rightarrow (\mathcal{O}_C/p)^a \rightarrow \left(\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^t(\Omega_n, \mathcal{O}^+/p) \right)^a \rightarrow (\mathrm{Sp}_{2d-1-t}(\mathbb{F}_p) \otimes \mathcal{O}_C/p)^a \rightarrow 0. \quad (6.7)$$

- For t odd, $d - 1 \leq t < 2d - 2$, there is, $GL_{d+1}(K)$ -equivariantly :

$$\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^t(\Omega_n, \mathcal{O}^+/p) \stackrel{a}{\cong} \mathrm{Sp}_{2d-1-t}(\mathbb{F}_p) \otimes \mathcal{O}_C/p.$$

- There is a $GL_{d+1}(K)$ -equivariant isomorphism :

$$\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2d-2}(\Omega_n, \mathcal{O}^+/p) \stackrel{a}{\cong} \mathrm{LC}\left(\mathbb{P}(K^{d+1})^*, \mathbb{F}_p\right) \otimes \mathcal{O}_C/p.$$

- For $t > 2d - 2$, $\varinjlim_{n \in \mathbb{N}} H_{\text{ét}}^t(\Omega_n, \mathcal{O}^+/p) \stackrel{a}{\cong} 0$.

The only missing part for a complete description is following lemma :

Lemma 6.5.3. *The short exact sequence of [equation \(6.7\)](#) admits a $GL_{d+1}(K)$ -equivariant splitting, as almost $\mathcal{O}_C/p$ -modules.*

Proof. Apply the functor $(-)_!$, which is exact by [\[GR03, Cor 2.2.22\]](#). By definition [\[GR03, Def 2.2.19\]](#), for any $\mathcal{O}_C/p$ -module M , $(M^a)_! \cong M^a \otimes_{\mathcal{O}_C} \mathfrak{m}_C$. Indeed, a computation yields $(M^a)_! = (M^a)_* \otimes_{\mathcal{O}_C} \mathfrak{m}_C$. Since $(M^a)_*$ is almost isomorphic to M , they become isomorphic after tensoring with $\mathfrak{m}_C$ (by [lemma 2.5.2](#)).

This yields a $GL_{d+1}(K)$ -equivariant short exact sequence of classical modules :

$$0 \rightarrow \mathfrak{m}_C/p \rightarrow \varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2d-2}(\Omega_n, \mathcal{O}^+/p) \otimes_{\mathcal{O}_C/p} \mathfrak{m}_C/p \rightarrow \mathrm{Sp}_{2d-1-t}(\mathbb{F}_p) \otimes \mathfrak{m}_C/p \rightarrow 0. \quad (6.8)$$

We will show that this is split. By the $\mathfrak{m}_C/p$ -analogue of [\[CDHN21, Lemma 6.19\]](#), the middle term is a smooth representation.

By [\[CDHN21, Thm 2.1\]](#), $\mathrm{Ext}_{\mathbb{F}_p[G]}^1(\mathrm{Sp}_k(\mathbb{F}_p), \mathbb{F}_p) = 0$ (see [\[CDHN21, Rk 1.9\]](#) for the case $p = 2$), where the Ext group is taken in the category of smooth representations. Moreover, by [\[CDHN21, Prop 6.1\]](#), there is natural resolution as smooth $\mathbb{F}_p[GL_{d+1}(K)]$ -modules :

$$\cdots \rightarrow \mathrm{LC}(X_2, \mathbb{F}_p) \rightarrow \mathrm{LC}(X_1, \mathbb{F}_p) \rightarrow \mathrm{LC}(X_0, \mathbb{F}_p) \rightarrow \mathrm{Sp}_k(\mathbb{F}_p)$$

for some explicit profinite spaces X_n , endowed with a $GL_{d+1}(K)$ -action.

The above is a projective resolution in the category of smooth representations (cf. [\[Sch15a\]](#)). Since $\mathfrak{m}_C$ is flat over $\mathbb{F}_p$ (as $\mathbb{F}_p$ is a field), we have a projective resolution :

$$\cdots \rightarrow \mathrm{LC}(X_2, \mathbb{F}_p) \otimes \mathfrak{m}_C \rightarrow \mathrm{LC}(X_1, \mathbb{F}_p) \otimes \mathfrak{m}_C \rightarrow \mathrm{LC}(X_0, \mathbb{F}_p) \otimes \mathfrak{m}_C \rightarrow \mathrm{Sp}_k(\mathbb{F}_p) \otimes \mathfrak{m}_C$$

Hence, $\mathrm{Ext}_{\mathfrak{m}_C[G]}^1(\mathrm{Sp}_k(\mathbb{F}_p) \otimes \mathfrak{m}_C, \mathfrak{m}_C)$ can be computed by the complex :

$$\mathrm{Hom}_{\mathfrak{m}_C[G]}(\mathrm{LC}(X_2, \mathfrak{m}_C), \mathfrak{m}_C) \rightarrow \mathrm{Hom}_{\mathfrak{m}_C[G]}(\mathrm{LC}(X_1, \mathfrak{m}_C), \mathfrak{m}_C) \rightarrow \mathrm{Hom}_{\mathfrak{m}_C[G]}(\mathrm{LC}(X_0, \mathfrak{m}_C), \mathfrak{m}_C) \quad (6.9)$$

But, for all i , $\mathrm{Hom}_{\mathfrak{m}_C[G]}(\mathrm{LC}(X_i, \mathfrak{m}_C), \mathfrak{m}_C) \cong \mathrm{Hom}_{\mathbb{F}_p[G]}(\mathrm{LC}(X_i, \mathbb{F}_p), \mathfrak{m}_C)$.

Since $\text{Ext}_{\mathbb{F}_p[G]}^1(\mathbb{F}_p, \text{Sp}_k(\mathbb{F}_p)) = 0$, the complex :

$$\text{Hom}_{\mathbb{F}_p[G]}(\text{LC}(X_2, \mathbb{F}_p), \mathbb{F}_p) \rightarrow \text{Hom}_{\mathbb{F}_p[G]}(\text{LC}(X_1, \mathbb{F}_p), \mathbb{F}_p) \rightarrow \text{Hom}_{\mathbb{F}_p[G]}(\text{LC}(X_0, \mathbb{F}_p), \mathbb{F}_p)$$

is exact. By the universal coefficient theorem for cohomology, so is the complex of [equation \(6.9\)](#).

Hence, the short exact sequence [equation \(6.8\)](#) splits equivariantly. We may then apply the additive functor $(-)^a$, which satisfies $(-)^a \circ (-)_! = \text{id}$. This concludes. $\square$

We may now go back to the study of $R\Gamma_{\text{ét},c}(\Omega^d, \mathcal{O}^+/p)$.

Recall the excision [equation \(6.1\)](#), and the cofinal covering established in [equation \(6.3\)](#). We have the exact sequence

$$R\Gamma_{\text{ét},c}(\Omega_C^d, \mathcal{O}^+/p) \rightarrow R\Gamma_{\text{ét}}(\mathbb{P}_C^d, \mathcal{O}^+/p) \rightarrow \varinjlim_{n \in \mathbb{N}} R\Gamma_{\text{ét}}(\Omega_n, \mathcal{O}^+/p).$$

Recalling the cohomology of the projective space computed in [proposition 6.1.3](#), we obtain the following long exact sequence in the almost category, for any $k \geq 0$ (where cohomology is implicitly taken with $\mathcal{O}^+/p$ -coefficients) :

$$0 \rightarrow \varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2k-1}(\Omega_n) \rightarrow H_{\text{ét},c}^{2k}(\Omega_C^d) \xrightarrow{f} \mathcal{O}_C/p \rightarrow \varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2k}(\Omega_n) \rightarrow H_{\text{ét},c}^{2k+1}(\Omega_C^d) \rightarrow 0. \quad (6.10)$$

For all $k \neq d$, it follows from [lemma 6.3.6](#) that f is almost injective, which yields :

1. An almost isomorphism $\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2k-1}(\Omega_n, \mathcal{O}^+/p) \xrightarrow{a} H_{\text{ét},c}^{2k}(\Omega_C^d, \mathcal{O}^+/p)$.
2. A short exact sequence :

$$0 \rightarrow (\mathcal{O}_C/p)^a \rightarrow \left(\varinjlim_{n \rightarrow \infty} H_{\text{ét}}^{2k}(\Omega_n, \mathcal{O}^+/p) \right)^a \rightarrow H_{\text{ét},c}^{2k+1}(\Omega_C^d, \mathcal{O}^+/p)^a \rightarrow 0,$$

where the first term identifies to $H_{\text{ét}}^{2k}(\mathbb{P}_C^d, \mathcal{O}^+/p)^a$.

It then follows from [proposition 6.5.2](#) that :

- For $0 \leq t \leq d-1$, $H_{\text{ét},c}^t(\Omega_C^d, \mathcal{O}^+/p) \xrightarrow{a} 0$.
- For any even $d \leq t \leq 2d-2$, $H_{\text{ét},c}^t(\Omega_C^d, \mathcal{O}^+/p) \xrightarrow{a} \text{Sp}_{2d-t}(\mathbb{F}_p) \otimes \mathcal{O}_C/p$.
- For any odd $d \leq t \leq 2d-2$, there is an exact sequence :

$$0 \rightarrow (\mathcal{O}_C/p)^a \xrightarrow{g} (\mathcal{O}_C/p \oplus \text{Sp}_{d-t}(\mathbb{F}_p) \otimes \mathcal{O}_C/p)^a \rightarrow H_{\text{ét},c}^t(\Omega_C^d, \mathcal{O}^+/p)^a \rightarrow 0.$$

Where g is induced by the inclusions $\Omega_n \subset \mathbb{P}_C^d$. By construction, g identifies with the first morphism of [equation \(6.7\)](#) (i.e. the inclusion of $\mathcal{O}_C/p$), through the identification of [proposition 6.5.1](#). Hence

$$H_{\text{ét},c}^t(\Omega_C^d, \mathcal{O}^+/p) \xrightarrow{a} \text{Sp}_{d-t}(\mathbb{F}_p) \otimes \mathcal{O}_C/p.$$

- There is a short exact sequence :

$$0 \rightarrow (\mathcal{O}_C/p)^a \xrightarrow{g} \text{LC}(\mathbb{P}(K^{d+1})^*, \mathbb{F}_p) \otimes \mathcal{O}_C/p^a \rightarrow H_{\text{ét},c}^{2d-1}(\Omega_C^d, \mathcal{O}^+/p)^a \rightarrow 0.$$

It follows from the definition of E_1 in [lemma 6.1.11](#) that the natural map :

$$H_{\text{ét}}^{2d-2}(\mathbb{P}_C^d, \mathcal{O}^+/p) \rightarrow E_1^{0,2d-2}(n) \cong \bigoplus_{H \in \mathcal{H}_0} H_{\text{ét}}^{2d-2}(H, \mathcal{O}^+/p)$$

is induced by the diagonal embedding of the inclusions $H \subset \mathbb{P}_C^d$.

This passes to $F_1(\infty)$, and, through the identification of [lemma 6.4.3](#), the morphism $\mathcal{O}_C/p \rightarrow \text{LC}(Y_0^{(1)}, \mathcal{O}_C/p)$ maps any x to the constant function valuing x .

This then passes to cohomology in $F_2(\infty)$ (which corresponds to connected components of the simplicial object), so that the image of g is given by constant functions.

Hence, we have :

$$H_{\text{ét},c}^{2d-1}(\Omega_C^d, \mathcal{O}_C/p) \xrightarrow{\cong} (\text{LC}(\mathbb{P}((K^{d+1})^*), \mathbb{F}_p)/\mathbb{F}_p) \otimes \mathcal{O}_C/p \xrightarrow{\cong} \text{Sp}_1(\mathbb{F}_p) \otimes \mathcal{O}_C/p.$$

- For $k = d$, the long exact sequence [equation \(6.10\)](#) yields :

$$H_{\text{ét},c}^{2d}(\Omega_C^d, \mathcal{O}_C/p) \xrightarrow{\cong} \mathcal{O}_C/p.$$

- For $t > 2d$, $H_{\text{ét},c}^t(\Omega_C^d, \mathcal{O}_C/p) \xrightarrow{\cong} 0$.

This finally concludes the proof of [proposition 6.0.1](#).

Before passing to more general period domains, it is interesting to wonder exactly which properties of $\mathcal{O}^+/p$ -cohomology were used in this computation.

Recall the four axioms used in [\[SS91\]](#) : The $\mathbb{D}_C$ -invariance of cohomology, the cup product structure, the triviality of the cohomology of the point, and the form of the cohomology of the projective spaces.

Here, we used the following :

1. The triviality of the cohomology of the point (implicitly) ;
2. A weak excision property, as in [equation \(6.1\)](#);
3. The homotopy invariance property with respect to $\varprojlim_{n \rightarrow \infty} \mathbb{D}(|p|^n)$, in the form of [proposition 6.3.2](#);
4. The cohomology of the projective space [proposition 6.1.3](#), as well as the relative version in [lemma 6.3.6](#);
5. Some representation theoretic argument to show the splitting of the short exact sequence [equation \(6.8\)](#).

While working with almost coefficients everywhere is technically cumbersome, it poses no major difficulty. The story is the same regarding the condensed formalism associated to $\mathcal{O}^b$ -coefficients.

The notion of p-adic period domains

Content of this chapter

In this chapter, we give a precise definition of period domains, for which we claim no originality. The original construction was developed in [RZ96], see also a short introduction by the first author in [Rap97]. Concise introductions, somewhat similar to the one below, can be found in [Orl05a], [CDHN21]. The book [DOR10] presents an extensive version of the theory, whose structure we broadly follow.

We start by specializing the theory for GL_n as such a context is sufficient to define Drinfeld's symmetric spaces, is necessary for the general theory, and contains most of the necessary intuition, before passing to the general case.

Compared to the rest of the cited literature, we make sure to precisely lay out how the GL_n case embeds into the general setup.

We briefly touch on the modern interpretation using the Fargues-Fontaine curve, as laid out in [CFS17].

7.1 Filtered isocrystals, and period domains for GL_n

Period domains for GL_n are classifying spaces of *weakly admissible* isocrystals in the sense of Fontaine, which are viewed as *semi-stable objects of slope zero* in an adequate Harder-Narasimhan formalism on the category of filtered isocrystals. In what follows, we fix a field K .

In the following, we assume some familiarity with Tannakian formalism, and refer the reader to [DM82] or [DOR10, Section 4].

7.1.1 Filtered vector spaces and isocrystals

Let V be a finite dimensional K -vector space. A $\mathbb{Q}$ -filtration $\mathcal{F}V$ on V is the datum of a map $\mathcal{F} : x \in \mathbb{Q} \mapsto \mathcal{F}^x V \in \{K\text{-linear subspaces of } V\}$ such that :

- $\mathcal{F}$ is decreasing, i.e, for all $x < y$, $\mathcal{F}^y \subset \mathcal{F}^x$.
- There exists $x < y$ such that $\mathcal{F}^x V = V$ and $\mathcal{F}^y V = \{0\}$.
- For all x , $\mathcal{F}^x = \bigcap_{y < x} \mathcal{F}^y$.

We define the **type** of the filtration $\mathcal{F}$ to be $\nu(\mathcal{F}) = (x_1^{n(1)}, \dots, x_n^{n(i)})^1$, where $x_1 > \dots > x_n$ are the jumps of the filtration, i.e. $\{x_1, \dots, x_n\} = \{x \in \mathbb{R}, \mathrm{gr}_{\mathcal{F}}^{x_i} \neq \{0\}\}$, and $n_i = \dim_V(\mathrm{gr}_{\mathcal{F}}^{x_i})$ is the multiplicity at x_i .

The datum of $(x_1^{n(1)}, \dots, x_n^{n(i)})$, where $x_1 > \dots > x_n$ and $\sum n_i = \dim(V)$ is called the possible type of a filtration on V .

Definition 7.1.1. Let V be a finite dimensional K -vector space, and let ν be the possible type of a filtration on V . The flag variety (cf. [RZ96, p. 1.31]) $\mathcal{F}(\mathrm{GL}(V), \nu)^2$ is the scheme parameterizing filtrations of type ν over K . It is the projective K -scheme such that, for any field extension K' over K :

$$\mathcal{F}(\mathrm{GL}(V), \nu)(K') \cong \{\text{filtrations on } V \otimes_K K' \text{ of type } \nu\}.$$

It can be constructed as $\mathcal{F}(\mathrm{GL}(V), \nu) = \mathbb{G}\mathrm{L}(V)/P(\nu)$, where $\mathrm{GL}(V)$ is the reductive group defined by $R \mapsto \mathrm{GL}(V \otimes_K R)$ for all K -algebras R , and $P(\nu)$ is the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing a given filtration of type ν on V .

It is naturally endowed with an action of the linear group $GL(V) = \mathrm{GL}(V)(K)$.

For K/k a field extension, we let $\mathbb{Q} - \mathrm{Fil}_k^K$ denote the category of pairs $(V, \mathcal{F}_K)$, where V is a k -vector space, and $\mathcal{F}_K$ is a $\mathbb{Q}$ -filtration on the K -vector space $V \otimes_k K$. A morphism in $\mathbb{Q} - \mathrm{Fil}_k^K$ is a k -linear map $f : V \rightarrow W$ such that, for all $x \in \mathbb{Q}$, $f \otimes \mathrm{id}_K(V^x) \subset W^x$.

The category Fil_k^K is an additive k -linear $\otimes$ -category. It is not abelian (as a change of the filtration cannot be read on the kernel nor cokernel), but it is *quasi-abelian*, and is in particular an exact category.

Let L be a perfect field of characteristic $p > 0$, $K_0 = \mathrm{Frac}(W(L))$, and σ be the Frobenius endomorphism on K_0 .

¹Here, the exponent n_i indicates that the corresponding $\nu(i)$ is repeated n_i times.

²The more standard notation (cf. [DOR10]) is $\mathcal{F}(V, \nu)$. We prefer denoting $\mathrm{GL}(V)$ so that our notation coincides with the general case, where $\mathrm{GL}(V)$ will be replaced by a reductive algebraic group.

An isocrystal³ over K_0 is a pair $N = (V, \varphi)$, where V is a finite-dimensional vector space over K_0 , and $\varphi : V \rightarrow V$ is a σ -linear⁴ bijective automorphism.

We let $\text{Isoc}(K_0)$ denote the category of such isocrystals, with morphisms given by K_0 -linear maps $f : V \rightarrow V'$ such that $f \circ \varphi_V = \varphi_{V'} \circ f$. It is an abelian category, and, moreover, we have the following (cf. [Man63]) :

Theorem 7.1.2. (Dieudonné-Manin classification)

Assume that L is algebraically closed. Then, $\text{Isoc}(K_0)$ is semi-simple, and the simple objects are of the form :

$$V_{r/s} = \left(K_0^s, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ p^r & & 0 & \end{bmatrix} \cdot \sigma \right)$$

for any coprime $(r, s) \in \mathbb{Z} \times \mathbb{N}_{\geq 1}$.

More explicitly, any $N \in \text{Isoc}(K_0)$ can be uniquely written as $V = \bigoplus_{x \in E} V_x$ for some finite $E \subset \mathbb{Q}$. Using the terminology described above, the decomposition as a sum of simple objects yields defines a grading functor $\omega : \text{Isoc}(K_0) \rightarrow \mathbb{Q} - \text{Grad}_{K_0}$, where $\mathbb{Q} - \text{Grad}_{K_0}$ denotes the category of finite dimensional K_0 -vector spaces equipped with a grading indexed by $\mathbb{Q}$.

7.1.2 Filtered isocrystals

We keep the notation of subsection 7.1.1, and let K be a field extension of K_0 . A **filtered** isocrystal $(V, \varphi, \mathcal{F}V_K)$ over K_0 is the datum of an isocrystal (V, φ) over K_0 , together with a $\mathbb{Q}$ -filtration $\mathcal{F}V_K$ on the vector space $V_K = V \otimes_{K_0} K$.

We let $\text{FilIsoc}_{K_0}^K$ denote the category of such filtered isocrystals. The category $\text{FilIsoc}_{K_0}^K$ is endowed with a Harder-Narasimhan formalism, defined by :

- The **rank** of $(V, \varphi, \mathcal{F}V_K)$ is $\dim_{K_0}(V)$
- The **degree** of $(V, \varphi, \mathcal{F}, V_K)$ is :

$$\deg(V, \varphi, \mathcal{F}V_K) = \sum_{i \in \mathbb{Z}} i \cdot \dim_K(\text{gr}_{\mathcal{F}}^i(V_K)) - v_{K_0}(\det(\varphi))$$

- The **slope** of $(V, \varphi, \mathcal{F}V_K)$ is $\nu(V, \varphi, \mathcal{F}V_K) = \frac{\deg(V, \varphi, \mathcal{F}V_K)}{\text{rk}(V, \varphi, \mathcal{F}V_K)}$
- If $(V, \varphi, \mathcal{F}_K)$ is a filtered isocrystal, a **sub-isocrystal** is a triplet $(V', \varphi', \mathcal{F}'_K)$ where V' is a subspace of V stable under φ , endowed with the induced structure, i.e. $\varphi' = \varphi_{V'}$, and $\mathcal{F}'_K = \mathcal{F}_K|_{V' \otimes_{K_0} K}$.

We say that a filtered isocrystal is **weakly admissible** if it is semi-stable⁵ and of slope zero. A nontrivial result of Faltings and Totaro (cf. [Fal95] and [Tot96]) states the following :

Theorem 7.1.3. The tensor product of semi-stable filtered isocrystals is semi-stable.

Remark 7.1.4. This degree function (and, in turn, the slope function) is the difference of two natural and classical degree functions : one associated to the isocrystal, and the other to the filtration :

³This is sometimes referred to as F-isocrystals, where the "F" reminds us of the presence of a Frobenius morphism. We simply call them Isocrystals for clarity of notation. Dear reader, please remind yourself at all times that isocrystals come endowed with a Frobenius morphism.

⁴i.e. $\forall \lambda \in K_0, \forall x \in V, \varphi(\lambda \cdot x) = \sigma(\lambda) \cdot \varphi(x)$.

⁵Recall that, in a Harder-Narasimhan formalism, an object is said to be semi-stable if every sub-object has a non-greater slope.

1. For any filtered vector space $(V, \mathcal{F}_K V) \in \text{Fil}_K^K$ a filtered vector space, the degree

$$\deg(V, \mathcal{F}_K V) = \sum_i i \cdot \dim_K(\text{gr}_{\mathcal{F}}^i V_K)$$

is the y-axis value of the terminal point of the Hodge polygonal associated to the filtration $\mathcal{F}V_K$.

2. For an isocrystal (V, φ) over K_0 , we consider its degree $-v_p(\det(A))$, where A is a matrix representing the action of φ in any given base. It is the y-axis value of the terminal point of the Newton polygonal associated with the isocrystal (V, φ) .

In other terms, a filtered isocrystal is weakly admissible if and only if its Newton and Hodge polygonals terminate at the same point, and the Newton polygonal of any sub-isocrystal lies above its Hodge polygonal. This recovers the original interpretation of Fontaine [Fon94, Def 4.4.3].

We may now define p -adic period domains for GL_n (cf. [RZ96, Prop 1.36]).

Proposition–Definition 7.1.5. Let $(V, \varphi) \in \text{Isoc}(K_0)$, and ν be the possible type of a filtration on V . There exists an open subset defining a smooth, partially proper rigid-analytic variety over K_0 :

$$\check{\mathcal{F}}^{wa}(\text{GL}(V), \varphi, \nu) \subset \mathcal{F}(V, \nu)$$

such that, for any complete field extension K of K_0 ,

$$\check{\mathcal{F}}^{wa}(\text{GL}(V), \varphi, \nu)(K) = \left\{ \mathcal{F} \in \mathcal{F}(\text{GL}(V), \nu)(K), \right. \\ \left. (V \otimes_{K_0} K, \varphi \otimes_{K_0} \text{id}_K, \mathcal{F}) \text{ is weakly admissible} \right\}.$$

We call it the **period domain** associated to $(\text{GL}(V), \varphi, \nu)$.

Let us conclude this section by viewing Drinfeld's symmetric spaces, as defined in chapter 6, as a period domain.

7.1.3 Drinfeld's symmetric spaces

Fix $n \in \mathbb{N}^*$, L a perfect field of characteristic p , $K_0 = \text{Frac}(W(L))$, $V = K_0^n$, and $\sigma \in \text{Aut}(K_0/\mathbb{Q}_p)$ be the Frobenius endomorphism. Let $\nu = (n-1, -1, \dots, -1)$.

For any field extension K/K_0 , $\mathcal{F}(\text{GL}(V), \nu)(K)$ parametrizes filtrations on $V \otimes_{K_0} K$ with only two jumps : a 1-dimensional jump at $n-1$, and a $(n-1)$ -dimensional jump at (-1) .

Any such filtration $\mathcal{F}_K \in \mathcal{F}(\text{GL}(V), \nu)(K)$ is of the following form, for some K -line $\mathcal{L}_{\mathcal{F}} \subset V \otimes_{K_0} K$:

$$\mathcal{F}_{K,x} = \begin{cases} V \otimes_{K_0} K & \text{if } x \leq -1 \\ \mathcal{L}_{\mathcal{F}} & \text{if } -1 < x \leq n-1 \\ \{0\} & \text{if } n-1 < x \end{cases}.$$

Such filtrations hence correspond one-to-one to lines $\mathcal{L}_{\mathcal{F}}$ of $V \otimes_{K_0} K$.

Hence, $\mathcal{F}(\text{GL}(V), \nu) = \mathbb{P}(V)$ is the K_0 -variety whose K -points are $\mathbb{P}(V)(K) := \mathbb{P}(V \otimes_{K_0} K)$.

We may now define Drinfeld's symmetric spaces as open subsets of those flag varieties. Endow V with the trivial isocrystal structure $\sigma^{\times n}$. One can check that a filtration $\mathcal{F}$ is semi-stable if and only if the line $\mathcal{L}_{\mathcal{F}}$ is not contained in any K_0 -rational hyperplane. Hence,

$$\check{\mathcal{F}}^{wa}(\mathrm{GL}(V), \varphi, \nu)(K) = \mathbb{P}^n(K) \setminus \bigcup_{H \in \mathcal{H}_{K_0}} H$$

where $\mathcal{H}$ describes all K_0 -rational hyperplanes of V . Such a space will be denoted $\Omega_{K_0}^n$, and is called Drinfeld's symmetric space of dimension n over K_0 .

It is naturally the base change to K_0 of the Drinfeld space over $\mathbb{Q}_p$, given by :

$$\Omega_{\mathbb{Q}_p}^n = \mathbb{P}_{\mathbb{Q}_p}^n \setminus \bigcup_{H \in \mathcal{H}_{\mathbb{Q}_p}} H$$

where $\mathcal{H}$ describes all K_0 -rational hyperplanes of $\mathbb{Q}_p^n$.

7.2 Link with crystalline representations and the admissible locus

In the spirit of Fontaine's famous "weakly admissible implies admissible" conjecture, such period domains are deeply related to Galois representations.

Definition 7.2.1. Let K be a discretely valued extension of $\mathbb{Q}_p$, with residue field k , and let G_K denote the absolute Galois group of K . Let $K_0 = W(k)[1/p]$ be the maximal unramified subextension of K .

Then, there is a fully faithful functor from the category of finite-dimensional crystalline representations of G_K to the category of filtered isocrystals $\mathrm{FilIsoc}_K^{K_0}$, given by

$$V \mapsto (\mathbb{B}_{\mathrm{crys}} \otimes_K V)^{G_K}$$

where the filtration on $(\mathbb{B}_{\mathrm{crys}} \otimes_K V)^{G_K} \otimes_{\mathbb{Q}_p} K_0 \cong (V \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR})^{G_K}$ is induced from the natural filtration $\mathrm{Fil}^i \mathbb{B}_{dR} = t^i \mathbb{B}_{dR}^+$ on the de Rham period ring.

We say that a filtered isocrystal is **admissible** if it lies in the essential image of the above functor.

Fontaine's famous conjecture, which is now a theorem, is as follows :

Theorem 7.2.2. Let $(V, \varphi, \mathcal{F})$ be a filtered isocrystal in the above context. Assume furthermore that K is a finite extension of $\mathbb{Q}_p$.

Then $(V, \varphi, \mathcal{F})$ is weakly admissible if and only if it is admissible.

Proof. "Admissible implies weakly admissible" is straightforward (as the terminology suggests), and was proven in [FW79, Proposition 5.4.2.(i)].

The reciprocal is one of the most important result in p -adic Hodge theory. It was conjectured in the above article of Fontaine ([FW79, Conjecture 5.6.9]), and first proven in [CF00]. The modern and more insightful viewpoint, using the Fargues-Fontaine curve, is developed in [FF18]. $\square$

One can then be tempted to think of period domains as moduli spaces of *admissible crystalline representations*, but one needs to be careful. Crucially, the above equivalence does *not* hold when K is an infinite extension. It is unclear how we can associate Galois representations to K -points of $\check{\mathcal{F}}^{wa}$, whenever K is infinite. Rappoport and Zink conjectured the existence of an **admissible** locus $\mathcal{F}^a \subset \check{\mathcal{F}}^{wa}$, which is better suited for the study of crystalline representations, as it forms the essential image of the crystalline period morphism.

How one should precisely define an admissible locus was unclear for a long time, but was finally settled thanks to works of Fargues-Scholze regarding the geometry of the Fargues-Fontaine curve (cf. [CFS17]. See also [Sch15b, Appendix]). The "weakly admissible implies admissible", for any finite K/K_0 , $\mathcal{F}^a$ and $\check{\mathcal{F}}^{wa}$ have the same K -points - but they do not coincide as geometric spaces. In some space, the space of interest really should be $\mathcal{F}^a$, and $\check{\mathcal{F}}^{wa}$ can be seen as an *approximation* of it. It is however much trickier to compute the cohomology of the admissible locus, as there is no known explicit geometric stratification of the complement.

The results of [CFS17] establish an explicit criterion (see also [Har13] for the GL_n case) under which the weakly admissible and admissible loci coincide. The reader only interested in the admissible locus can assume that we work under that hypothesis (Hodge-Newton decomposability of $B(G, \mu)$) throughout.

7.3 Period domains for arbitrary reductive groups

We will define period domains relative to more general reductive algebraic groups. In the case $G = GL_n$, we will recover the above. The construction will be mostly the same, except that we replace :

- Isocrystals by isocrystals with G -structure.
- Filtrations by 1-parameter subgroups.
- Type of filtrations with conjugacy classes of 1-parameter subgroups.

7.3.1 Gradings and filtrations

Let k be any field, and K/k be a fixed field extension. Let G be a connected algebraic group over k .

Definition 7.3.1. We let $\text{Rep}_k(G)$ be the category of finite dimensional representations of G over k .

An object of $\text{Rep}_k(G)$ is of the form (V, ρ) where V is a finite dimensional k -vector space, and ρ is a morphism of affine group schemes $G \rightarrow \mathbb{G}_m^V$, i.e. contains the compatible data of morphisms $\rho_R : G(R) \rightarrow GL_n(V \otimes_k R)$ for any k -algebra R . The category $\text{Rep}_k(G)$ is naturally equipped with a tensor product, and the fiber functor obtained by forgetting the representation $w^G : (V, \rho) \in \text{Rep}_k(G) \mapsto V \in \text{Vect}_k$.

This makes it a Tannakian category over k .

Our goal is to understand filtrations of the fiber functor ω^G . Let us first introduce gradations. We let $\mathbb{Q} - \text{Grad}_k$ denote the category of finite dimensional k -vector spaces, equipped with a grading indexed by $\mathbb{Q}$, as $V = \bigoplus_{x \in \mathbb{Q}} V_x$. Morphisms of such graded vector spaces are linear maps respecting the gradation.

Consider the functor $\omega_0 : \mathbb{Q} - \text{Grad}_k \rightarrow \text{Vect}_k$ obtained by forgetting the grading : $\omega_0 \left(V = \bigoplus_{x \in \mathbb{Q}} V_x \right) \mapsto V$.

Remark 7.3.2. This category of $\mathbb{Q}$ -grading is related to the category of $\mathbb{Q}$ -filtrations introduced in subsection 7.1.1. Indeed, there exists a functor $\text{fil} : \mathbb{Q} - \text{Grad}_k \rightarrow \mathbb{Q} - \text{Fil}_k^K$, obtained by

$$\text{fil} : V = \bigoplus_{x \in \mathbb{Q}} V_x \mapsto \left(V_K, \mathcal{F}^x V_K = \sum_{y \geq x} V_y \otimes K \right).$$

More generally, if we let $\mathbb{Q} - \text{Grad}_k^K$ denote the category of vector spaces V over k together with a $\mathbb{Q}$ -gradation on V_K , the above definition extends to a functor $\text{fil} : \mathbb{Q} - \text{Grad}_k^K \rightarrow \text{Fil}_k^K$.

In the other direction, there exists a functor $\text{gr} : \mathbb{Q} - \text{Fil}_k^K \rightarrow \mathbb{Q} - \text{Grad}_K$ obtained by $\mathcal{F} \mapsto \bigoplus_{x \in \mathbb{Q}} \text{gr}_{\mathcal{F}}^x \mathcal{F}_K$.

Note that, while fil respects the underlying vector space (i.e. commutes with fiber functors), gr does not, even if $K = k$.

Let us now introduce the slope torus, as studied in [DOR10, p. IV.2].

Proposition–Definition 7.3.3. We define a functor $D_k : R \in \text{Ab}^{op} \mapsto \text{Spec}(k[R]) \in \text{AffSch}_k$. It is left adjoint to $\text{Spec}(R) \mapsto R^*$. By construction, $D_k(\mathbb{Z}) = \mathbb{G}_{m,k}$, so that :

$$D_k(\mathbb{Q}) = D_k \left(\varinjlim_{n \in \mathbb{N}^*} \frac{1}{n} \cdot \mathbb{Z} \right) \cong \varprojlim_{n \in \mathbb{N}^*} D_k(\mathbb{Z}) = \varprojlim_{n \in \mathbb{N}^*} \mathbb{G}_{m,k}.$$

We let $\mathbb{D}_k = D_k(\mathbb{Q}) \cong \varprojlim_{n \in \mathbb{N}^*} \mathbb{G}_{m,k}$ be the slope torus, where transition maps are given by the Frobenius morphism. The multiplication by any $x \in \mathbb{Q}$ induces a map $\chi_x : \mathcal{D}_k \rightarrow D_k(\mathbb{Z}) \cong \mathbb{G}_{m,k}$, viewed as a character of $\mathbb{D}_k$.

Proposition 7.3.4. (cf. [Riv72, p. IV.1.2.1.]). The category $\mathbb{Q} - \text{Grad}_k$ is a neutral Tannakian category over k , and ω is a fiber functor over k . Moreover, there is a natural equivalence of category :

$$\begin{aligned} \text{Rep}_k(\mathbb{D}_k) &\rightarrow \mathbb{Q} - \text{Grad}_k \\ (V, \rho) &\mapsto V = \bigoplus_{x \in \mathbb{Q}} V_x \end{aligned}$$

where $V_x = \{v \in V, \rho(g)v = \chi_x(g) \cdot v, \forall g \in D(\mathbb{Q})\}$ is the weight space of V associated to $x \in \mathbb{Q}$.

Let $G_K = G \times_{\text{Spec}(k)} \text{Spec}(K)$ denote the base change of G to K .

Definition 7.3.5. A $\mathbb{Q}$ -grading of ω^G over K is a tensor functor $\text{Gr} : \text{Rep}_k(G) \rightarrow \mathbb{Q} - \text{Grad}_K$ such that the diagram below commutes :

$$\begin{array}{ccc} \text{Rep}_k(G) & \xrightarrow{\omega^G \otimes_k K} & \text{Vect}_K \\ & \searrow \text{Gr} \quad \nearrow \omega_0 & \\ & \mathbb{Q} - \text{Grad}_K & \end{array}.$$

Note that such functors are automatically exact and faithful.

Proof. The faithfulness is clear. The exactness follows from the exactness of ω^G , hence of $\omega^G \otimes_k K$, and the fact that ω_0 both preserves and reflects exact sequences. $\square$

Lemma 7.3.6. There is a one-to-one correspondence between $\mathbb{Q}$ -gradings of ω^G over K and homomorphisms $\mathbb{D}_K \rightarrow G$.

Proof. This follows from [proposition 7.3.4](#) and a form of Tannakian duality, cf. [\[DOR10, Cor 4.1.19\]](#). $\square$

Note that such a grading can be seen as the compatible data of a grading on $V \otimes_k K$ for each representation (V, ρ) of G (cf. [\[DOR10, Remark 4.2.5.\(ii\)\]](#) for a more precise formulation).

Definition 7.3.7. A $\mathbb{Q}$ -filtration of ω^G over K is a tensor functor $\mathcal{F} : \text{Rep}_k(G) \rightarrow \mathbb{Q} - \text{Fil}_k^K$ such that the diagram below commutes :

$$\begin{array}{ccc} \text{Rep}_k(G) & \xrightarrow{\omega^G} & \text{Vect}_k \\ & \searrow \mathcal{F} & \nearrow \omega_0 \\ & \mathbb{Q} - \text{Fil}_k^K & \end{array}$$

and such that the composition $\text{Rep}_k(G) \xrightarrow{\mathcal{F}} \mathbb{Q} - \text{Fil}_k^K \xrightarrow{gr} \mathbb{Q} - \text{Grad}_K$ (where gr is from [remark 7.3.2](#)) is exact.

Remark 7.3.8. Note that, by definition, such a filtration $\mathcal{F}$ defined over K induces, for all representation (V, ρ) of G , a filtration $\mathcal{F}^\rho$ on $V \otimes_k K_0$. One can reformulate the above definition as the compatible data of filtrations $\mathcal{F}^\rho$ on $V \otimes_k K$, cf. [\[DOR10, Def 4.2.6\]](#).

The exactness condition ensures we can recover the filtration from the grading.

Proposition 7.3.9. Every $\mathbb{Q}$ -grading of ω^G over K naturally induces a $\mathbb{Q}$ -filtration of ω^G over K .

Proof. Let Gr be such a grading. The morphism fil from [remark 7.3.2](#) commutes with fiber functors, so that the forgetful functor $\omega_0 : \mathbb{Q} - \text{Grad}_{k,K} \rightarrow \text{Vect}_K$ factors through $\mathbb{Q} - \text{Fil}_K^K$. We have :

$$\begin{array}{ccccc} \text{Rep}_k(G) & \xrightarrow{\omega^G} & & & \text{Vect}_k \\ & \searrow \text{Gr} & \nearrow \omega_0 & & \\ & \mathbb{Q} - \text{Grad}_{k,K} & \xrightarrow{fil} & \mathbb{Q} - \text{Fil}_k^K & \\ & \searrow & & \downarrow gr & \\ & & & \mathbb{Q} - \text{Grad}_K & \end{array}$$

(A dashed arrow from $\text{Rep}_k(G)$ to $\mathbb{Q} - \text{Fil}_k^K$ represents the filtration $\mathcal{F}$.)

The dashed arrow will be our filtration.

It suffices to show that the long curved composition $\text{Rep}_k(G) \rightarrow \mathbb{Q} - \text{Grad}_K$ is exact. The map $(V \mapsto (V, \text{Gr}(V)))$ is exact by the [definition 7.3.5](#), and fil and gr can be checked to be exact by an easy linear algebra argument. $\square$

We conclude this paragraph by showing that gradings and filtrations relative to $\text{GL}(V)$ correspond to "standard" grading and filtrations on V (see also [\[DOR10, Remark 4.2.11\]](#)).

Example 7.3.10. Let $G = \text{GL}(V)$. Then, $\mathbb{Q}$ -gradings of ω^G over K correspond to gradations on $V \otimes_k K$, and $\mathbb{Q}$ -filtrations of ω^G over K correspond to filtrations on $V \otimes_k K$.

Indeed, any $\mathbb{Q}$ -grading of ω^G over K induces a $\mathbb{Q}$ -grading on V through the standard representation of $\text{GL}(V)$ acting on V . Reciprocally, a $\mathbb{Q}$ -grading $V = \bigoplus_{x \in \mathbb{Q}} V_x$ induces a morphism $D(\mathbb{Q}) \rightarrow \text{GL}(V)$ as :

$$g \mapsto \sum_{x \in \mathbb{Q}} i \circ \chi_x(g) \cdot id_{V_x}$$

where $i : \mathbb{G}_m \rightarrow \mathrm{GL}(V)$ denotes the diagonal embedding, and $\chi_x : D(\mathbb{Q}) \rightarrow \mathbb{G}_m$ is the map from [proposition 7.3.4](#).

Likewise, filtrations on ω^G induce filtrations on any representations, and one can consider the standard representation on $V \otimes_k K$. Reciprocally, consider a filtration on $V \otimes_k K$. As a filtration on a finite dimensional vector space, it is necessarily split (i.e. lies in the image of the morphism fil from [remark 7.3.2](#)). Hence, by the discussion above, it defines a grading on ω^G over K . Then the filtration given by [proposition 7.3.9](#) works.

7.3.2 Cocharacters and flag varieties

We keep the notations of [subsection 7.3.1](#). We fix k^{sep} a separable closure of k containing K .

Definition 7.3.11. A $\mathbb{Q}$ -one-parameter subgroup (or $\mathbb{Q}$ -cocharacter) of G defined over K is a morphism of algebraic groups $\mathcal{D}_K \rightarrow G_K$. We let $X_*(G)_{\mathbb{Q}}^K$ denote the set of such cocharacters.

We call $X_*(G)_{\mathbb{Q}}^k$ the set of **rational** cocharacters, and $X_*(G)_{\mathbb{Q}}^{k^{\mathrm{sep}}}$ the set of **geometric** cocharacters.

Remark 7.3.12. The literature is quite inconsistent on whether a cocharacter denotes a rational cocharacter, or a geometric one.⁶ In this thesis, we always specify the field of definition.

By [proposition 7.3.4](#), elements of $X_*(G)_{\mathbb{Q}}^K$ are equivalent to gradings of ω^G over K . In particular, they induce filtrations of ω^G over K , by [remark 7.3.2](#).

Definition 7.3.13. We say that two 1-parameter subgroups $\lambda, \lambda' \in X_*(G)_{\mathbb{Q}}^K$ are **par-equivalent** if they define the same filtration on ω^G over K .

For $\lambda \in X_*(G)_{\mathbb{Q}}^K$, we let P_λ be the subgroup of G_K defined, for any K -algebra R , by :

$$P_\lambda(R) = \left\{ g \in G(R), \lim_{t \rightarrow 0} \lambda(t) \circ g \circ \lambda(t)^{-1} \text{ exists} \right\}.$$

Proposition 7.3.14. Assume that G is reductive. Then :

1. All filtrations of ω^G over K are splittable, i.e. come from a grading by [proposition 7.3.9](#).
2. For any $\lambda \in X_*(G)_{\mathbb{Q}}^K$, P_λ is a parabolic subgroup of G_K .
3. Let $\lambda, \lambda' \in X_*(G)_{\mathbb{Q}}^K$, and $\mathcal{F}_\lambda$ (resp. $\mathcal{F}_{\lambda'}$) be the filtration on ω^G associated to λ (resp. λ'). The following are equivalent :
 - a) $\mathcal{F}_\lambda = \mathcal{F}_{\lambda'}$, i.e λ and λ' are par-equivalent.
 - b) There exists $g \in P_\lambda(K)$ such that $\lambda' = g \circ \lambda \circ g^{-1}$.

Proof. cf. [MFK94, Def 2.3/Prop 2.6] and [DOR10, Thm 4.2.13]. □

Finally, we turn our attention to conjugacy classes of such $\mathbb{Q}$ -parameter subgroups, and the associated flag variety. There are two natural and compatible actions on $X_*(G)_{\mathbb{Q}}^K$:

- Assume K/k is Galois. The Galois group $\mathrm{Gal}(K/k)$ acts, for $\sigma \in \mathrm{Gal}(K/k)$ and $\lambda \in X_*(G)_{\mathbb{Q}}^K$, by :

$$\sigma \lambda = \sigma_{G_K} \circ \lambda \circ \sigma_{\mathbb{D}_K}^{-1},$$

where σ_G (resp $\sigma_{\mathbb{D}_K}$) is the natural action of Γ_E on G_K (resp. $\mathbb{D}_K$).

⁶This is partly due to the fact that most literature of algebraic groups was written in the language of algebraic varieties over algebraically closed fields.

- The group $G(K)$ acts by conjugation, via :

$${}^g\lambda := g \cdot \lambda \cdot g^{-1}$$

where the right hand side uses the natural action of $G(K)$ on G_K .

There is a natural inclusion $X_*(G)_{\mathbb{Q}}^k \subset X_*(G)_{\mathbb{Q}}^K$ compatible with the action of G , and it follows from Galois descent (cf. [BLR90, Theorem 6.1.6.a and Paragraph 6.2.B]) that $(X_*(G)_{\mathbb{Q}}^K)^{\text{Gal}(K/k)} \cong X_*(G)_{\mathbb{Q}}^k$.

For simplicity (and following the literature), we simply denote $X_*(G)/G = X_*(G)_{\mathbb{Q}}^{k^{sep}}/G(k^{sep})$ the group of conjugacy classes of geometric cocharacters. It is a discrete $\text{Gal}(k^{sep}/k)$ -module.

We may now define our flag varieties of interest, that will classify filtrations of *fixed type*. This will be replaced by a conjugacy class of geometric cocharacters.

Every conjugacy class $\{\mu\} \in X_*(G)_{\mathbb{Q}}/G$, is defined over a finite extension of k , called its *reflex field* :

$$E(G, \{\mu\}) = (k^{sep})^{\Gamma_{\{\mu\}}} \text{ where } \Gamma_{\{\mu\}} = \text{Stab}_{\text{Gal}(k^{sep}/k)}(\{\mu\}) = \{g \in \text{Gal}(k^{sep}/k), \forall \lambda \in \{\mu\}, {}^g\lambda(t) \in \{\mu\}\}$$

Proposition–Definition 7.3.15. *Let G be a connected reductive group over k , and $\{\mu\}$ be a conjugacy class of geometric cocharacters of G , with reflex field E . Fix a separable closure E^{sep} of E .*

There exists a unique projective reduced scheme of finite type over E , denoted $\mathcal{F}(G, \{\mu\})$, whose K -points, for any intermediary field extension $E^{sep}/K/E$, are :

$$\begin{aligned} \mathcal{F}(G, \{\mu\})(K) &= \{\lambda \in X_*(G)_{\mathbb{Q}}^K, \lambda \in \{\mu\}\} / \text{par-equivalence} \\ &= \{\mathcal{F} \in \text{Fil}_K(\omega^G), \exists \lambda \in \{\mu\}, \mathcal{F} = \mathcal{F}_{\lambda}\}. \end{aligned}$$

This is called the flag variety associated to $(G, \{\mu\})$.

Proof. The equivalence of the descriptions follows from the definition and first point of [proposition 7.3.14](#).

Assume G is quasi-split. Then, a lemma of Kottwitz (cf. [Kot84, Lemma 1.1.3] (see also the proof of [Kot85, Prop 6.2])) shows that there exists an element $\mu \in \{\mu\}$ defined over E . Let $P_{\mu,E}$ be the parabolic subgroup of G_E associated to μ under [definition 7.3.2](#). Then $\mathcal{F}(G, \{\mu\}) = G_E/P_{\mu,E}$ satisfies the above property, by [proposition 7.3.14](#). The unicity follows by descent from an algebraically closed field, where reduced schemes of finite types can be viewed as algebraic varieties.

If G is not quasi-split, we refer to the argument of [DOR10]. □

In the $\text{GL}(V)$ setup, this coincides with the flag varieties defined in [definition 7.1.1](#). We will then define period domains as analytic open subsets of the above flag varieties, corresponding to semi stable isocrystals.

7.3.3 Isocrystals with additional structure

Let G a connected reductive algebraic group over $\mathbb{Q}_p$, and $K_0 = \check{\mathbb{Q}}_p$. Recall that the category $\text{Isoc}(K_0)$ defined in [definition 7.1.1](#) is naturally endowed with the tensor product $(V_1, \varphi_1) \otimes (V_2, \varphi_2) := (V_1 \otimes V_2, \varphi_1 \otimes \varphi_2)$, which makes it a (non neutral !) $\mathbb{Q}_p$ -linear⁷ Tannakian category, whose associated fiber functor is the forgetful functor $\omega_{\text{Isoc}} : \text{Isoc}(K_0) \rightarrow \text{Vect}_{K_0}$.

⁷It is **not** K_0 linear, as endomorphisms of the unit object (K_0, id_{K_0}) , are endomorphisms $K_0 \rightarrow K_0$ commuting with the Frobenius, so that they correspond with elements of $K_0^G = \mathbb{Q}_p$.

Definition 7.3.16. An *isocrystal with G -structure* is a $\otimes$ -exact functor $\text{Rep}_{\mathbb{Q}_p}(G) \rightarrow \text{Isoc}(K_0)$.

It follows from the definition that such a functor commutes with fiber functors, since G is connected (cf. [DOR10, Remark 9.1.6]).

Example 7.3.17. Let V be a $\mathbb{Q}_p$ -vector space of dimension n , and $G = \text{GL}(V)$ be the group scheme defined by $R \mapsto \text{GL}(V \otimes_{\mathbb{Q}_p} R)$ (so that there is a non canonical isomorphism $\text{GL}(V) \cong \text{GL}_n$).

Through the canonical representation of $\text{GL}(V)$ acting on V , we can attach to any isocrystal with $\text{GL}(V)$ -structure a (standard) isocrystal over $V \otimes_{\mathbb{Q}_p} K_0$.

It follows from the classification of irreducible representations of $\text{GL}(V)$ via Schur functors that, reciprocally, the structure of an $\text{GL}(V)$ -isocrystal is fully determined by its value on the standard representation (cf. [RR96, Remark 3.4.(ii)]) - so that $\text{GL}(V)$ -isocrystals correspond one-to-one with structure of isocrystals on the vector space $V \otimes_{\mathbb{Q}_p} K_0$.

Any $b \in G(K_0)$ induces the isocrystal with G -structure $N_b : (V, \rho) \mapsto (V \otimes_{\mathbb{Q}_p} K_0, \rho_{K_0}(b)(\text{id} \otimes \varphi))$.

Proposition 7.3.18. All isocrystals with G -structure are of the form N_b , for some $b \in G(K_0)$.

Moreover, there is an isomorphism of exact $\otimes$ -functors $N_b \cong N_{b'}$ if and only if there exists $g \in G(K_0)$ such that $b' = gb\varphi(g)^{-1}$.

Proof. This relies on Tannakian duality, cf. [DOR10, Lemma 9.1.4 and Remark 9.1.6]. $\square$

We define $B(G)$, the Kottwitz set of G , as the quotient of $G(K_0)$ by the above equivalence relation. Hence, $B(G)$ classifies isocrystals with G -structure. For any $b \in G(K_0)$, we denote $[b]$ its associated class in $B(G)$.

7.3.4 Period domains

We reuse the notations from subsection 7.3.3.

Definition 7.3.19. Let G be an algebraic group over $\mathbb{Q}_p$, and $b \in G(K_0)$, with associated isocrystal with G -structure N_b . Recall that the datum of a point $\mathcal{F} \in \mathcal{F}(G, \{\mu\})(K) \subset \text{Fil}_K(\omega^G)$ defines a filtration $\mathcal{F}^p$ on $V \otimes_k K$ for all representations (V, ρ) of G by remark 7.3.8.

We say that (b, x) is **weakly admissible** if, for all representation (V, ρ) of G , the filtered isocrystal $(N_b(V), \mathcal{F}^x V \otimes_k K)$ is weakly admissible in the sense of subsection 7.1.2.

By the tensor product theorem theorem 7.1.3, this is equivalent to asking that $(N_b(V), \mathcal{F}^x V \otimes_k K)$ is weakly admissible for some faithful representation of G (see also [RZ96, Def 1.18]).

We may finally define general period domains. Fix the datum $(G, [b], \{\mu\})$ of :

1. A connected quasi-split group G over $\mathbb{Q}_p$.
2. An element $[b]$ of the Kottwitz set $B(G)$, i.e. a σ -conjugacy class in $G(K_0)$.
3. A conjugacy class $\{\mu\} \in X_*(G)/G$ of geometric cocharacters of G .

Definition 7.3.20. Let $(G, [b], \{\mu\})$ be a datum as above. Let $E = E(G, \{\mu\})$ be the associated reflex field, and $\check{E} = E \cdot K_0$. There exists a unique partially proper open subset :

$$\check{\mathcal{F}}^{wa}(G, b, \{\mu\}) \subset \mathcal{F}(G, \{\mu\}) \otimes_E \check{E}$$

such that, for any field extension $K/\check{E}$,

$$\check{\mathcal{F}}^{wa}(G, b, \{\mu\})(K) = \{\mathcal{F} \in \mathcal{F}(G, \{\mu\})(K), \text{ the filtered isocrystal } (N_b, \mathcal{F}) \text{ is weakly admissible}\}.$$

This is called the **period domain** associated to $(G, b, \{\mu\})$. Moreover, it can be obtained as the analytification of a Berkovich space.

Proof. cf. [RZ96, Prop 1.36] in the rigid analytic setup and [DOR10, Prop 9.5.3] in the Berkovich setup. $\square$

Note that, up to an isomorphism, the period domain only depends on the class $[b] \in B(G)$. Indeed, if $b' = gb\sigma(g)^{-1}$ for some $g \in G(K_0)$, the map $\lambda \mapsto g\lambda$ induces an isomorphism $\check{\mathcal{F}}^{wa}(G, b, \{\mu\}) \cong \check{\mathcal{F}}^{wa}(G, b', \{\mu\})$.

Let J_b denote the automorphisms group of N_b , defined, for any $\mathbb{Q}_p$ -algebra R , by :

$$J_b(R) = \{g \in G(\check{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} R), gb\sigma(g)^{-1} = b\}.$$

The group $J_b(\mathbb{Q}_p)$ acts naturally on $\check{\mathcal{F}}^{wa}(G, b, \{\mu\})$, via $J_b(\mathbb{Q}_p) \subset G(K_0)$ via conjugation $\lambda \mapsto g\lambda$.

We conclude this chapter with the definition of basic and decent elements. We reuse the notations from [subsection 7.3.3](#). For any $b \in G(\check{\mathbb{Q}}_p)$, Kottwitz [Kot85, sec 4.2]) defines a *slope morphism* $\nu_b : \mathbb{D}_{K_0} \rightarrow G_{K_0}$ as follows :

Definition 7.3.21. For any $V \in \text{Rep}_{\mathbb{Q}_p}(G)$, the $\mathbb{Q}$ -grading on the vector space $V \otimes_{\mathbb{Q}_p} K_0$ associated with the representation of the pro-torus $\mathbb{D}_{K_0} \xrightarrow{\nu_b} G_{K_0} \xrightarrow{\rho_{K_0}} \text{GL}(V \otimes K_0)$ coincides with the Dieudonné-Manin slope decomposition of the isocrystal N_b , as recalled in [theorem 7.1.2](#).

Note that, if $b' = gb\sigma(g)^{-1}$, then $\nu_{b'} = \text{Int}(g) \circ \nu_b$.

Definition 7.3.22. We say that an element $b \in G(K_0)$ is ***s-decent*** for an integer $s \geq 1$ if :

- The morphism $s \cdot \nu_b : \mathbb{D}_{K_0} \rightarrow G_{K_0}$ factors through $\mathbb{G}_{m, K_0}$,
- The following equality holds in $G(K_0) \rtimes \sigma^{\mathbb{Z}} : (b\sigma)^s = (s\nu_b)(p)\sigma^s$.

Since G is connected, every class $[b] \in B(G)$ contains an s -decent element, for some $s \geq 1$ (cf. [DOR10, lemma 9.1.33, remark 9.1.34]). By [DOR10, lemma 9.6.19], if G is furthermore assumed to be quasi-split, each $[b] \in B(G)$ contains an s -decent element b such that ν_b is defined over $\mathbb{Q}_p$.

Proposition 7.3.23. (cf. [DOR10, Prop 1.36.(ii)]). Let $(G, [b], \{\mu\})$ be a datum as above. Let $s \geq 1$, and assume that b is s -decent, with reflex field E . Then, the period domain $\check{\mathcal{F}}^{wa}(G, b, \{\mu\})$ admits a model over $E_s := E \cdot \mathbb{Q}_{p^s}$.

Definition 7.3.24. We say that an element $b \in B(K_0)$ is **basic** if the slope morphism ν_b factors through Z_{K_0} , where Z denotes the center of G . We let $B(G)_{\text{basic}}$ denote the subset of $B(G)$ consisting of basic elements.

Note that an element b is basic if and only if J_b is an inner form of G .

The cohomology of p-adic period domains

Content of this chapter

The goal of this chapter is to compute the $\mathcal{O}^+/p$ -cohomology of period domains, as defined in the previous paragraph, and show that they satisfy primitive comparison with compact support, as follows.

Theorem 8.0.1. *Fix a datum $(G, [b], \{\mu\})$ as in [definition 7.3.20](#), where b is to be basic and s -decent for some integer $s \geq 1$.*

For simplicity, we denote $\mathcal{F} = \mathcal{F}(G, b, \{\mu\})$ and $\mathcal{F}^{wa} = \mathcal{F}^{wa}(G, b, \{\mu\})$, with field of definition E_s , and $\mathcal{F}_{E_s}$ be the base change of $\mathcal{F}$ to E_s . Let $Y = \mathcal{F}_{E_s} \setminus \mathcal{F}^{wa}$, and $J := J_b$ be the inner form of G associated to the basic element b .

We fix $\overline{E_s}$ an algebraic closure of E_s , C be a completion of $\overline{E_s}$, and $\mathcal{F}_C, \mathcal{F}_C^{wa}$ and Y_C be the respective base change to C . Let $j_C : \mathcal{F}_C^{wa} \rightarrow \mathcal{F}_C$ denote the open immersion. Let $\mathcal{L}$ be an étale $\mathbb{F}_p$ -local system on $\mathcal{F}_{E_s}$, and $\mathcal{L}_C$ be its pullback to $\mathcal{F}_C$.

Then, $\mathcal{F}_C^{wa}$ satisfies Poincaré duality with respect to $j_C^* \mathcal{L}$, i.e, for any $0 \leq k \leq 2d$, there is a $J(\mathbb{Q}_p) \times \text{Gal}(\overline{E_s}/E_s)$ -equivariant isomorphism :

$$H_{\text{ét}}^k(\mathcal{F}_C, j_C^* \mathcal{L}_C) \simeq \text{Hom}_{\mathbb{F}_p}(H_{\text{ét},c}^{2d-k}(\mathcal{F}_C, j_C^* \mathcal{L}_C), \mathbb{F}_p)$$

where d denotes the dimension of $\mathcal{F}_C$.

From there, they satisfy Poincaré duality for $\mathbb{F}_p$ -coefficients by our main result [theorem 5.2.4](#), so that we may compute their $\mathbb{F}_p$ -étale cohomology as the dual of their compactly supported étale cohomology, computed in [\[CDHN21, Thm 1.2\]](#), for $p \neq 2$. We do so in [corollary 8.4.1](#).

This computation is based on a geometric stratification of the complement $\mathcal{F} \setminus \mathcal{F}^{wa}$ of period domains, constructed by S.Orlik in [\[Orl05a\]](#), together with standard sheaf-theoretic techniques.

In loc.cit., Orlik uses his geometric decomposition to compute the étale cohomology with compact support of period domains with $\mathbb{Z}/\ell^n\mathbb{Z}$ coefficients. In [\[CDHN21\]](#), the authors adapt the computation for $\ell = p$. While the geometric part of the argument does not depend on the coefficients, working with p -adic coefficients adds a lot of representation-theoretic complexity, as one needs to compute some Ext^1 groups between Steinberg representations.

Going from $\mathbb{F}_p$ to $\mathcal{O}^+/p$ -coefficients adds no representation theoretic complexity, as only the "geometric" part of the argument is needed, via a spectral sequence argument.

Compared to the previous expositions of this computation appearing in the literature, we provide the following technical improvements :

1. Throughout most of the computation, we work with coefficients in a general abelian étale sheaf. Doing so, we remove a superfluous overconvergence hypothesis from [\[Orl05a\]](#).

2. We work directly over E_s , rather than over a complete algebraic closure. This allows to properly keep track of the Galois action.
3. We work with the language of locally spatial diamonds throughout, rather than pseudo-adic spaces.

Moreover, we justify two imprecise steps from [Orl05a].

The first one relies on a punctual-to-global result on equivariant étale sheaves, which reduces to an explicit computation of sheafification of presheaves on profinite spaces. This is handled in [section 8.3](#).

The second one is a statement regarding the vanishing of cohomology of convex subcomplexes of the topological Tits building. As the necessary tools are quite remote from the main topic of this thesis, we postpone it to [proposition B.4.4](#).

Using the criterion [proposition 5.3.1](#), the statement [theorem 8.0.1](#) reduces to a establishing primitive comparison on $\mathcal{F}_C$ and on Y_C . As $\mathcal{F}_C$ is smooth and proper, the primitive comparison on $\mathcal{F}_C$ follows directly from the primitive comparison theorem [theorem 5.1.3](#), so that it suffices to establish the following (applied to the pullback of a local system on $\mathcal{F}_{E_s}$, and base changing to C using [Sch17, Prop 16.10]) :

Proposition 8.0.2. *Let $h : Y \rightarrow \mathrm{Spa}(E_s, \mathcal{O}_{E_s})$ denote the structure morphism of Y . Let $\mathcal{L}$ be an étale $\mathbb{F}_p$ -local system on Y . Then, for all $q \in \mathbb{N}$, the natural map :*

$$R^q h_* \mathcal{L} \otimes_{\mathcal{O}_{E_s}^+}^L / p \rightarrow R^q h_* (\mathcal{L} \otimes_{\mathbb{F}_p}^L \mathcal{O}_Y^+ / p)$$

is an almost isomorphism in the category of étale $\mathcal{O}_{E_s}^+ / p$ -sheaves over $\mathrm{Spa}(E_s, \mathcal{O}_{E_s})$.

Note that this morphism (as well as the primitive comparison morphism on $\mathcal{F}_C$) is naturally $J(\mathbb{Q}_p)$ -invariant, so that the $J(\mathbb{Q}_p)$ -invariance in [theorem 8.0.1](#) follows from [remark 5.2.5](#).

8.1 Orlik's geometric decomposition

Let us fix an étale $\mathbb{F}_p$ -local system $\mathcal{L}$ on Y . In this section, we present a geometric decomposition of Y , due to Orlik in [Orl05a] (see also [Orl99] for the simpler analogue over finite fields).

Fix S a maximal $\mathbb{Q}_p$ -split torus of J , and let Δ be a basis of the relative root system of G , which is a finite set. The following proposition regroups the properties that we will need :

Proposition 8.1.1. *Let P_0 be a fixed minimal parabolic subgroup of J containing S . For any $I \subset \Delta$, let P_I be the associated standard parabolic subgroup of J .*

There exists a finite set of rigid-analytic subvarieties $Y_I \subset Y$ indexed by subsets $I \subset \Delta$ together with a stratification :

$$|Y| = \bigcup_{|\Delta \setminus I|=1} \bigcup_{g \in J(\mathbb{Q}_p)} g \cdot |Y_I|,$$

such that:

1. *For any $I \subset \Delta$, Y_I is the analytification of a proper algebraic variety over E_s (that is moreover a Schubert variety).*
2. *For any $I, J \subset \Delta$, $Y_I \cap Y_J = Y_{I \cap J}$.*
3. *$(J/P_I)(\mathbb{Q}_p) = J(\mathbb{Q}_p)/P_I(\mathbb{Q}_p)$.*
4. *The action of $J(\mathbb{Q}_p)$ on $\mathcal{F}_{E_s}$ restricts to an action of $P_I(\mathbb{Q}_p)$ on Y_I .*

Proof. Cf. [Orl05a, Section 3] or [CDHN21, Section 5.4]. □

From the above stratification, we will construct a functorial resolution $\mathcal{C}(\mathcal{F})$ of any sheaf of abelian group $\mathcal{F}$ on $Y_{\text{ét}}$, relating the cohomology of $\mathcal{F}$ on Y to the cohomology of $\mathcal{F}$ restricted to the Y_I , for varying $I \subset \Delta$. This exactness was only considered in [Orl05a] for *overconvergent* étale sheaves $\mathcal{F}$, but, as we will see, this assumption is not needed, so that it also works for $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_Y^+/p$.

The strategy to prove primitive comparison with compact support for Y is as follows :

1. From the functorial resolution $\mathcal{C}(\mathcal{F})$, construct a spectral sequence, whose first page computes the cohomology of $\mathcal{F}$ on the Y_I , and which converges to the cohomology of $\mathcal{F}$ on the Y - in a way that is functorial in the sheaf $\mathcal{F}$.
2. Use the primitive comparison theorem to compare the first page of the spectral sequence with coefficients in $\mathcal{L}$ or in $\mathcal{L} \otimes \mathcal{O}_Y^+/p$ ¹.
3. Prove that this comparison goes through the convergence of the spectral sequence in order to deduce a primitive comparison result for the cohomology of Y .

For any $I \subset \Delta$, we let $X_I = J(\mathbb{Q}_p)/P_I(\mathbb{Q}_p)$. By the third point of [proposition 8.1.1](#), this is a profinite space - as it identifies to the $\mathbb{Q}_p$ -points of a proper algebraic variety (which can then be seen as $\mathbb{Z}_p$ -points using the valuation criterion). We will need the following lemma.

Lemma 8.1.2. *For any compact open subset $T \subset J/P_I(\mathbb{Q}_p)$, the space $Z_I^T = \bigcup_{g \in T} g \cdot Y_I$ admits a natural structure of a closed locally spatial sub-diamond of Y .*

¹Here, the primitive comparison theorem will suffice, but one can explicitly compute the $\mathbb{F}_p$ -cohomology groups of the Y_I (at least after base change to an algebraically closed base) ; cf. [CDHN21, Cor 5.11]

Proof. Note, by that [proposition 8.1.1](#), $g \cdot Y_I$ is well defined for $g \in J/P_I(\mathbb{Q}_p) = J(\mathbb{Q}_p)/P_I(\mathbb{Q}_p)$ as the action of $P_I(\mathbb{Q}_p)$ stabilizes Y_I .

Fix a compact open $T \subset X_I$. By [lemma A.1.3](#), it suffices to show that the underlying topological space of Z_I^T is closed and generalizing. It is closed by the proof of [\[Orl05a, Lemma 3.2\]](#). All of the $g \cdot |Y_I|$ are generalizing since they are Zariski closed subsets, and a union of generalizing subspaces remains generalizing. $\square$

For any compact open subset $T \subset X_I$, we let Z_I^T be as above and $i_T : Z_I^T \rightarrow Y$ be the associated closed immersion.

Definition 8.1.3. We let Disj_{X_I} be the category of covers of X_I by disjoint nonempty open subsets, with morphisms given by refinements. Such disjoint covers are necessarily finite, as X_I is compact.

Objects of Disj_{X_I} will be denoted $c = \{T_k\}_{1 \leq k \leq n(c)}$.

For any $c \in \text{Disj}_{X_I}$ and any étale sheaf $\mathcal{F}$ on Y , we let $\mathcal{F}_c := \bigoplus_{k=1}^{n(c)} i_{T_k*} i_{T_k}^* \mathcal{F}$, and $\mathcal{F}_I := \varinjlim_{c \in \text{Disj}_{X_I}} \mathcal{F}_c$.

Note that, if $\mathcal{F}$ is $J(\mathbb{Q}_p)$ -equivariant, then $\mathcal{F}_I$ is also $J(\mathbb{Q}_p)$ -equivariant.

For a fixed c , $\mathcal{F}_c$ admits a natural interpretation as the étale sheaf on X_I whose sections are locally constant sections of $\mathcal{F}$, trivialized on the disjoint covering c . Likewise, $\mathcal{F}_I$ can be thought of as the sheaf of locally constant sections of $\prod_{x \in X_I} \mathcal{F}_{xY_I^{\text{ad}}}$. This is made more explicit in [\[Orl05a, p.538\]](#).

Remark 8.1.4. Let us briefly explicit the transition morphisms appearing in the colimit. Let $c = \{T_k\}_{1 \leq k \leq n(c)}$ be an object of Disj_{X_I} , and $c' = \{S_{k,l}\}_{1 \leq k \leq n(c); 1 \leq l \leq m(k)}$ be a refinement of c , in the sense that, for any $1 \leq k \leq n(c)$, the $(S_{k,l})_{1 \leq l \leq m(k)}$ form a disjoint compact open covering of T_k .²

The morphism $\mathcal{F}_{c'} \rightarrow \mathcal{F}_c$ is then given as a direct sum of the natural morphisms $i_{S_{k,l}*} i_{S_{k,l}}^* \mathcal{F} \rightarrow i_{T_k*} i_{T_k}^* \mathcal{F}$ induced by the unit of the adjunction $\mathcal{F} \rightarrow i_{S*} i_S^* \mathcal{F}$ together with the natural isomorphism :

$$i_{S_{k,l}*} i_{S_{k,l}}^* \mathcal{F} \cong i_{S_{k,l}*} i_{S_{k,l}}^* i_{T_k*} i_{T_k}^* \mathcal{F}$$

induced by the unit of the adjunction $\mathcal{F} \rightarrow i_{T_k*} i_{T_k}^* \mathcal{F}$. The fact that it is indeed an isomorphism can be checked on stalks using [lemma A.2.2](#).

Let us study the sheaves $\mathcal{F}_I$. We start with the following lemma (which is a version of [\[CDHN21, Prop 6.11.\(a\)\]](#) over an arbitrary base field, and in the language of diamonds) :

Lemma 8.1.5. Let $\bar{x} = \text{Spa}(C, C^+) \rightarrow Y$ be a geometric point of $Y_{\text{ét}}$, with underlying topological point x , and $\mathcal{F}$ be an étale sheaf of abelian groups over Y . The stalk of $\mathcal{F}_I$ at $\bar{x}$ identifies with :

$$(\mathcal{F}_I)_{\bar{x}} \cong LC(X_I(x), \mathcal{F}_{\bar{x}}),$$

where $X_I(x) = \{g \in X_I, x \in g \cdot Y_I\}$ is a closed subspace of X_I , and LC denotes the set of locally constant functions on $X_I(x)$ (with the topology induced by X_I), with value in the stalk $\mathcal{F}_{\bar{x}}$.

²Note that, since coverings are disjoint, if such a refinement exists, the refinement map is uniquely determined.

Proof. Taking stalks commutes with colimits, so that :

$$(\mathcal{F}_I)_{\overline{x}} \cong \varinjlim_{c \in \text{Disj}_{X_I}} \bigoplus_{k=1}^{n(c)} (i_{T_k,*} i_{T_k}^* \mathcal{F})_{\overline{x}}.$$

Using [lemma A.2.2](#), since i_{T_k} is a generalizing closed immersion, it follows that, for all $1 \leq k \leq n(c)$:

$$(i_{T_k,*} i_{T_k}^* \mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in T_k \cdot Y_I, \text{ i.e. if } T_k \cap X_I(x) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We may then replace all terms in the direct sum by $i_{T_k \cap X_I(x),*} i_{T_k \cap X_I(x)}^* \mathcal{F}$.

If $c = \{T_k\}_{1 \leq k \leq n(c)} \in \text{Disj}_{X_I}$, the $T_k \cap X_I(x)$ form a disjoint compact open cover of $X_I(x)$. Reciprocally, any $c' \in \text{Disj}_{X_I(x)}$ can be refined by a covering of the above form, by the [lemma 8.1.6](#) below.

Hence, coverings of the form $\{T_k \cap X_I(x)\}_{1 \leq k \leq n(c)}$ for $\{T_k\}_{1 \leq k \leq n(c)} \in \text{Disj}_{X_I}$ form a cofinal system in $\text{Disj}_{X_I(x)}$, so that :

$$(\mathcal{F}_I)_{\overline{x}} = \varinjlim_{c \in \text{Disj}_{X_I}} \bigoplus_{k=1}^{n(c)} (i_{T_k \cap X_I(x),*} i_{T_k \cap X_I(x)}^* \mathcal{F})_{\overline{x}} \cong \varinjlim_{c' \in \text{Disj}_{X_I(x)}} \bigoplus_{k=1}^{n(c')} \mathcal{F}_{\overline{x}} \cong LC(X_I(x), \mathcal{F}_{\overline{x}}).$$

The last isomorphism follows from identifying a locally constant function to the set of its values over any disjoint trivializing cover (that is necessarily finite since $X_I(x)$ is compact). $\square$

The following lemma was used in the proof :

Lemma 8.1.6. *Let X be a profinite topological space, and $F \subset X$ be a closed subset.*

Let $c = \{T_k\}_{1 \leq k \leq n(c')} \in \text{Disj}_F$. Then, there exists $\{S_{k,l}\}_{1 \leq k \leq n(c), 1 \leq l \leq m(k)} \in \text{Disj}_X$ such that, for all k , the $\{S_{k,l} \cap T\}_{1 \leq l \leq m(k)}$ form a disjoint clopen union of T_k .

Proof. For any $1 \leq k \leq n(c)$, T_k is an open subset of F , so it can be written as $T_k = U_k \cap F$ for some open subset U_k of X . Consider the open cover $X = (X \setminus F) \cup \bigcup_{1 \leq k \leq n(c)} U_k$ of X . By [[Sta23](#), [Lemma 08ZZ](#)], we may find a refinement by a disjoint cover formed by clopen subsets of X :

$$X = S'_0 \sqcup \bigsqcup_{1 \leq k \leq n(c), 1 \leq l \leq m(k)} S'_{k,l},$$

where $S'_0 \subset X \setminus F$ and, for all k, l in range, $S'_{k,l} \subset U_k$.

We then let $S_{k,l} := S'_{k,l} \cap T$, which are closed and open subsets of T . Pick some arbitrary (k_0, l_0) in range. After replacing S_{k_0, l_0} by $S_{k_0, l_0} \sqcup S'_0$, the family $\{S_{k,l}\}$ satisfies the desired properties. $\square$

8.2 Orlik's fundamental complex

We may now define the fundamental complex, promised in [proposition 8.1.1](#).

Proposition 8.2.1. *Let $\mathcal{F}$ be a sheaf of abelian groups on $Y_{\text{ét}}$. The following is complex of étale sheaves on $Y_{\text{ét}}$ is exact :*

$$\mathcal{C}(\mathcal{F}) : 0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{|\Delta \setminus I|=1} \mathcal{F}_I \rightarrow \bigoplus_{|\Delta \setminus I|=2} \mathcal{F}_I \rightarrow \cdots \rightarrow \bigoplus_{|\Delta \setminus I|=|\Delta|-1} \mathcal{F}_I \rightarrow \mathcal{F}_\emptyset \rightarrow 0$$

where the morphisms are defined as follows : fix $I \subset I' \subset \Delta$ with $|I' \setminus I| = 1$, The inclusion $P_I \subset P_{I'}$ induces a quotient map $p_{I,I'} : X_I \rightarrow X_{I'}$. For any $x \in X_{I'}$, we consider a map $\mathcal{F}_{x \cdot Y_{I'}} \rightarrow \mathcal{F}_{p_{I,I'}(x) \cdot Y_I}$ induced by the natural inclusion $x \cdot Y_I \subset p_{I,I'}(x) \cdot Y_{I'}$. The collection of such maps defines a map $f_{I',I} : \mathcal{F}_{I'} \rightarrow \mathcal{F}_I$.

Finally, recall that the root system Δ is naturally ordered, so that we may write $I' = \{\alpha_1 < \cdots < \alpha_n\}$, and $I = I' \setminus \{\alpha_i\}$. We then let $d_{I,I'} = (-1)^i p_{I,I'} : \mathcal{F}_{I'} \rightarrow \mathcal{F}_I$, and consider the direct sum along I, I' .

When $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n\mathbb{Z}$ -modules for some n prime to p , the exactness is claimed in [Orl05a, Thm 3.3]. This has been generalized in [Orl05b, Thm 2.1] (see also [CDHN21, Thm 6.13]) for overconvergent étale sheaves of abelian groups.

While Orlik's original proof is incomplete, we give a precise argument, relying on results obtained in [Appendix B](#).

Proof. The étale topos of Y has enough points by [Sch17, Prop 14.3], so that it we may check exactness on stalks. Let $\eta : \bar{x} \rightarrow Y$ be a geometric point, with $\bar{x} = \text{Spa}(C', C' +)$ for some complete algebraically closed field C , and let $x \in Y$ be the image of its unique closed point.

We let $|\Delta(J)|$ be the geometric realization of the Tits complex of J , as introduced in [section B.2](#).

By [lemma 8.1.5](#), the stalk at $\bar{x}$ is :

$$C(\mathcal{F})_{\bar{x}} : 0 \rightarrow \mathcal{F}_{\bar{x}} \rightarrow \bigoplus_{|\Delta \setminus I|=1} LC(X_I(x), \mathcal{F}_{\bar{x}}) \rightarrow \bigoplus_{|\Delta \setminus I|=|\Delta|-1} LC(X_I(x), \mathcal{F}_{\bar{x}}) \rightarrow \cdots \quad (8.1)$$

By [proposition B.1.3](#), the complex [equation \(8.1\)](#) computes the reduced sheaf cohomology groups, valued in $\mathcal{F}_{\bar{x}}$, of the topological space obtained as the geometric realization of a subcomplex of $|\Delta(J)|$, whose vertices are exactly :

$$F = \{gP_{\Delta \setminus \{i\}}g^{-1}, g \in J(\mathbb{Q}_p), x \in g \cdot Y_{\Delta \setminus \{i\}}(C'), i \in \Delta\},$$

interpreted as a simplicial profinite topological space, via the topology induced from $|\Delta(J)|^{pro}$, as defined in [definition B.2.5](#). Since Y_I is a proper variety and the group action is continuous, F is closed subset of the set of vertices (for the profinite topology).

Moreover, by construction, (cf. [Orl99, p.10]), the varieties Y_I can be defined as the locus of strict negativity with respect to a slope function, so that F rewrites as $\{\lambda \in |\Delta(J)|, \mu(x, \lambda) < 0\}$.

It is shown in [Orl99, Lemma 3.3] (this in in the setup of finite fields, but the argument goes through similarly in the local field case - see [CDHN21, Par 5.3.5]) that the function $\mu(x, -)$ naturally extends on the whole of $|\Delta(G)|$ in a way that is affine on each chamber, and that the space $\{\lambda \in |\Delta(G)|, \mu(x, \lambda) < 0\}$ is convex, in the sense of [definition B.4.1](#) (this relies on an argument of Mumford [MFK94, Cor 2.16]).

It now follows from [proposition B.4.4](#) that the subspace of $|\Delta(J)|^{pro}$ whose vertices lie in F is admits trivial sheaf cohomology, with coefficients in $\mathbb{Z}$. By the universal coefficient theorem, it follows that the complex [equation \(8.1\)](#) is acyclic. $\square$

Remark 8.2.2. Note that the above proof also shows that, for any two sheaves $\mathcal{F}, \mathcal{G}$ of abelian groups on $Y_{\text{ét}}$, we have $\mathcal{C}(\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}) \cong \mathcal{C}(\mathcal{F}) \otimes_{\mathbb{Z}} \mathcal{C}(\mathcal{G})$.

We may now consider the spectral sequence obtained by applying Rh_* to $\mathcal{C}(\mathcal{F})$, (recall that $h : Y \rightarrow \text{Spa}(E_s, \mathcal{O}_{E_s})$ is the structure morphism of Y). Considering the filtration on $\mathcal{C}(\mathcal{F})$ (minus the $\mathcal{F}$ term) obtained by truncations, we deduce a spectral sequence :

$$E_1^{p,q}(\mathcal{F}) := R^q h_* \left(\bigoplus_{|\Delta \setminus I| = p+1} \mathcal{F}_I \right) \implies R^{p+q} h_* \mathcal{F} =: E_{\infty}^{p+q}(\mathcal{F}).$$

The above spectral sequence is clearly functorial in $\mathcal{F}$. We will show the following :

Proposition 8.2.3. Let $\mathcal{L}$ be a local system of $\mathbb{F}_p$ -vector spaces on Y . Then, for all $p, q \in \mathbb{N}$, the natural morphism :

$$R^q h_* \left(\bigoplus_{|\Delta \setminus I| = p+1} \mathcal{L}_I \right) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ / p \rightarrow R^q h_* \left(\bigoplus_{|\Delta \setminus I| = p+1} (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p)_I \right)$$

is an almost isomorphism.

Before proving this proposition, let us show that it implies [proposition 8.0.2](#).

Proposition 8.2.4. Assume that the result of [proposition 8.2.3](#) holds. Then, for all $q \geq 0$, the natural morphism :

$$(R^q h_* \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ / p \rightarrow R^q h_* (\mathcal{L} \otimes \mathcal{O}_Y^+ / p)$$

is an almost isomorphism, so that [proposition 8.0.2](#) holds (and, in turn, so does [theorem 8.0.1](#)).

Proof. There is a natural morphism of complexes of sheaves on $Y_{\text{ét}}$:

$$\mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} h^* \mathcal{O}_{E_s}^+ / p \rightarrow \mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p \cong \mathcal{C}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p)$$

using the isomorphism from [remark 8.2.2](#). This induces a morphism of complexes on $\text{Spa}(E_s, \mathcal{O}_{E_s})_{\text{ét}}$:

$$R^q h_* \mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ / p \rightarrow R^q h_* (\mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} h^* \mathcal{O}_{E_s}^+ / p) \rightarrow R^q h_* \mathcal{C}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p),$$

where the first map is the one of [remark 5.1.2](#), and the second one is induced by the map $h^* \mathcal{O}_{E_s}^+ / p \rightarrow \mathcal{O}_Y^+ / p$ using [remark 8.2.2](#). This induces a morphism of spectral sequences $E_k^{p,q}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ / p \rightarrow E_k^{p,q}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p)$.

All morphisms above commute with the localization to the category of sheaves of almost $\mathcal{O}_{E_s}^+ / p$ -modules, which is an abelian category. By [proposition 8.2.3](#), the considered morphism of spectral sequences induces an almost isomorphism on E_1 , so that it also induces an almost isomorphism on E_{∞} . This is the desired result by construction of the spectral sequence. $\square$

8.3 A primitive comparison result

Let us rewrite the terms appearing in the proposition. Fix an étale $\mathbb{F}_p$ -local system $\mathcal{L}$ on Y , and let $\mathcal{F}$ be an arbitrary étale sheaf of abelian groups on Y .

Since Y is proper, it is necessarily qcqs, so that the morphism $Y \rightarrow \mathrm{Spa}(E_s, \mathcal{O}_{E_s})$ is a coherent morphism of algebraic topoi (cf. [Sch17, Section 8]). Then, by [SGA4, Thm 5.1], $R^q h_*$ commutes with filtered inductive limits.

Direct sums and the colimit along Disj_{X_I} are filtered, so that, for all $p, q \in \mathbb{N}$:

$$R^q h_* \left(\bigoplus_{|\Delta \setminus I| = p+1} \mathcal{F}_I \right) \cong \bigoplus_{|\Delta \setminus I| = p+1} R^q h_* \mathcal{F}_I \cong \bigoplus_{|\Delta \setminus I| = p+1} \varinjlim_{c \in \mathrm{Disj}_{X_I}} \bigoplus_{i=1}^{n(c)} R^q h_* (i_{T_k*} i_{T_k}^* \mathcal{F}).$$

The map i_{T_k} is a closed immersion, hence, by [Sch17, Lemma 21.13], $R^q h_* \circ i_{T_k*} = R^q (f \circ i_{T_k})_*$.

Hence, we reduced to the following :

Lemma 8.3.1. *For any $q \in \mathbb{N}$, the morphism :*

$$\varinjlim_{c \in \mathrm{Disj}_{X_I}} \bigoplus_{k=1}^{n(c)} R^q h_* (i_{T_k*} i_{T_k}^* \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ / p \rightarrow \varinjlim_{c \in \mathrm{Disj}_{X_I}} \bigoplus_{k=1}^{n(c)} R^q h_* (i_{T_k*} i_{T_k}^* (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p))$$

is an almost isomorphism.

The key idea will be to view the $U \mapsto R^q h_* (i_{U*} i_U^* \mathcal{F})$ as presheaves on the profinite space X_I , and $\varinjlim_{c \in \mathrm{Disj}_{X_I}} \bigoplus_{i=1}^{n(c)} R^q h_* (i_{T_k*} i_{T_k}^* \mathcal{F})$ as a form of *sheafification* of the above presheaves. Proving such an isomorphism of sheaves reduced to proving an isomorphism on stalks, which will compute the étale cohomology of the Y_I , for which the primitive comparison theorem [theorem 5.1.3](#) applies.

Definition 8.3.2. *We let, for all compact open $T \subset X_I$ and $q \geq 0$:*

$$S^q(\mathcal{F})(T) := R^q h_* (i_{T*} i_T^* \mathcal{F}) \in \mathrm{Sh}(\mathrm{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}, \mathrm{Ab}).$$

We let $\mathcal{B}$ denote the set of all clopen subsets of X_I . Since X_I is profinite, this forms a basis of the topology on X_I by [lemma B.1.12](#).

Recall the notion of (pre)sheaves on a basis, as in [Sta23, Section 009H]. In this formalism, the $S^q(\mathcal{F})$ are presheaves on $\mathcal{B}$ valued in the abelian category $\mathrm{Sh}(\mathrm{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}, \mathrm{Ab})$.

Let us start by two quick lemmas regarding sheaves on profinite spaces.

Lemma 8.3.3. *Let X be a profinite topological space, and $\mathcal{B}$ be the basis formed by clopen subsets of X . Let $\mathcal{A}$ be an abelian category, and let $\mathcal{G}$ be an $\mathcal{A}$ -valued presheaf on $\mathcal{B}$.*

Then, the presheaf $\mathcal{G}^\sharp$ on $\mathcal{B}$ defined, for any $U \in \mathcal{B}$ by $\mathcal{G}^\sharp(U) = \varinjlim_{c \in \mathrm{Disj}_U} \bigoplus_{k=1}^{n(c)} \mathcal{G}(T_k)$, defines a $\mathcal{B}$ -sheaf on X , valued in $\mathcal{A}$ (in the sense of [Sta23, Definition 009J]).

Proof. We will apply the criterion of [Sta23, Lemma 009L]. For any $U \in \mathcal{B}$, the set of disjoint clopen coverings is cofinal³ in the coverings of U by elements of $\mathcal{B}$.

Hence, it suffices to check the sheaf condition relative to a disjoint clopen covering $U = \bigsqcup_{i \in I} U_i$, where I is finite. Since the covering is disjoint, the cocycle condition is trivial, so that it suffices to check that $\mathcal{G}^\sharp(U) \cong \bigoplus_{i \in I} \mathcal{G}^\sharp(U_i)$.

³Here, we use cofinality in the sense of the lemma of loc. cit. We do not claim cofinality in the sense of [Sta23, Definition 04E6], as the second condition need not hold.

For all $c = \{T_k\}_{1 \leq k \leq n(c)} \in \text{Disj}_U$, the covering $\{T_k \cap U_i\}_{(k,i) \in A \times I}$ remains in Disj_U and forms a refinement of the covering $\{U_i\}_{i \in I}$, so that the subcategory whose objects are :

$$\text{Ob}(\text{Disj}_{U|U_i}) := \{\{T_k\}_{1 \leq k \leq n(c)} \in \text{Disj}_U, \forall i \in I, \exists 1 \leq k \leq n(c), T_k \subset U_i\}$$

and whose morphisms are given by refinements, is cofinal in Disj_U . Since the choices of refinements in Disj_U are canonical, the second condition of [Sta23, Definition 04E6] is automatically satisfied. Hence, we can compute the colimit defining $\mathcal{G}^\#$ alongside $\text{Disj}_{U|U_i}$, so that :

$$\bigoplus_{i \in I} \mathcal{G}^\#(U_i) = \bigoplus_{i \in I} \varinjlim_{c_i \in \text{Disj}_{U_i}} \bigoplus_{k=1}^{n(c_i)} \mathcal{G}(T_{k,i}) = \varinjlim_{c \in \text{Disj}_{U|U_i}} \bigoplus_{k=1}^{n(c)} \mathcal{G}(V_\alpha) \cong \varinjlim_{c \in \text{Disj}_U} \bigoplus_{k=1}^{n(c)} \mathcal{G}(V_\alpha) = \mathcal{G}^\#(U).$$

This concludes. $\square$

Remark 8.3.4. $\mathcal{G}^\#$ can be thought of as the $\mathcal{B}$ -sheafification of $\mathcal{G}$. The construction above is the analogue, with respect to the basis $\mathcal{B}$, of the standard "plus" construction (as in [Sta23, Section 00W1]), that usually needs to be applied twice to sheafify. Here, once suffices, even for non separated presheaves.

Recall that, for $\mathcal{G}$ a $\mathcal{B}$ -presheaf on X , one can compute its stalk at some $x \in X$ by $\mathcal{F}_x = \varinjlim_{U \in \mathcal{B}, x \in U} \mathcal{F}(U)$. Note that the plus construction of the above lemma is functorial. We have the following :

Lemma 8.3.5. Let X be a profinite topological space, with basis $\mathcal{B}$ of clopen subsets as above.

Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{B}$ -presheaves on X , together with a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of $\mathcal{B}$ -presheaves. Assume that f induces isomorphisms on stalks $\mathcal{F}_x \cong \mathcal{G}_x$ for all $x \in X$.

Then, the induced morphism $f^\# : \mathcal{F}^\# \rightarrow \mathcal{G}^\#$ is an isomorphism of $\mathcal{B}$ -sheaves.

Proof. By [Sta23, Lemma 009R], there is a functorial equivalence of categories between $\mathcal{B}$ -sheaves on X and sheaves on X , denoted $\mathcal{F} \mapsto \mathcal{F}^{ext}$, such that, for all $U \in \mathcal{B}$, $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$, and, for all $x \in X$, $\mathcal{F}_x^{ext} = \mathcal{F}_x$.

Hence, the morphism $f^\#$ induces a morphism of sheaves $(\mathcal{F}^\#)^{ext} \rightarrow (\mathcal{G}^\#)^{ext}$, that induces an isomorphism of stalks, so that it is an isomorphism of sheaves.

Hence, so that, for all $U \in \mathcal{B}$ the map $(\mathcal{F}^\#)(U) \cong (\mathcal{F}^\#)^{ext}(U) \rightarrow (\mathcal{G}^\#)^{ext}(U) \cong (\mathcal{G}^\#)(U)$ is an isomorphism. This concludes. $\square$

Let us now come back to the proof of [proposition 8.2.3](#). Using the notations above, it is equivalent to the fact that :

$$\mathcal{S}^q(\mathcal{L})^\# \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \xrightarrow{a} \mathcal{S}^q(\mathcal{L} \otimes \mathcal{O}_Y^+/p)^\#.$$

Hence, by [lemma 8.3.5](#), it suffices to show that the morphism $\mathcal{S}^q(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \rightarrow \mathcal{S}^q(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p)$ of $\mathcal{B}$ -sheaves on X_I induces an almost isomorphism on stalks.

Lemma 8.3.6. For any $x \in X_I$, let Neigh_x be the category of compact open neighborhoods of $x \in X_I$. Let $\mathcal{F}$ be a sheaf of abelian groups on $Y_{\acute{e}t}$. Then, there is a natural isomorphism

$$\mathcal{S}^q(\mathcal{F})_x \cong R^q(h \circ i_x)_* i_x^* \mathcal{F}$$

where, for $x \in X_I$, i_x denotes the closed immersion $x \cdot Y_I \subset X$.

Proof. Fix $x \in X_I$. By [Orl05a, Lemma 4.4], for any family $(W_s)_{s \in \mathbb{N}}$ of compact open neighborhoods of the point $1 \cdot P_I(\mathbb{Q}_p) \in X_I$ such that $\bigcap_{s \in \mathbb{N}} W_s = \{1 \cdot P_I(\mathbb{Q}_p)\}$, we have $\bigcap_{s \in \mathbb{N}} Z_I^{W_s} = Y_I$. Fix such a family W_s . Hence, for all $x \in X$ the family $\{x \cdot W_s\}_{s \in \mathbb{N}}$ forms a family of compact open neighborhoods of x such that $\bigcap_{s \in \mathbb{N}} x \cdot W_s = \{x\}$, and $\bigcap_{s \in \mathbb{N}} Z_I^{x \cdot W_s} = x \cdot Y_I$.

We let $W_s(x) = x \cdot W_s$. Then $\{W_s(x)\}_{s \in \mathbb{N}}$ forms a cofinal family in Neigh_x .

Moreover, since h is qcqs, by [Sch17, Prop 14.9], $R^q h_*$ commutes with filtered colimits, so that :

$$\mathcal{S}^q(\mathcal{F})_x = \varinjlim_{W \in \text{Neigh}_x} R^q(h \circ i_W)_* i_W^* \mathcal{F} \cong R^q h_* \varinjlim_{W \in \text{Neigh}_x} i_{W*} i_W^* \mathcal{F}$$

Since $x \cdot Y_I \subset Z_I^{W(x)}$ for all compact open neighborhood $W(x)$ of x , there is a natural morphism (as in the discussion in [remark 8.1.4](#)) :

$$\varinjlim_{W \in \text{Neigh}_x} i_{W*} i_W^* \mathcal{F} \rightarrow i_{x*} i_x^* \mathcal{F}$$

It suffices to prove that it is an isomorphism of étale sheaves on Y , and we will check this on fibers. Let $\eta : \text{Spa}(C, C^+) \rightarrow Y$ be a geometric point, and $y \in Y$ be the image of the unique closed point.

Then, by [lemma A.2.2](#) we have, for any $W \in \text{Neigh}_x$:

$$(i_{x*} i_x^* \mathcal{F})_{\bar{y}} = \begin{cases} \mathcal{F}_{\bar{y}} & \text{if } y \in x \cdot Y_I \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (i_{W*} i_W^* \mathcal{F})_{\bar{y}} = \begin{cases} \mathcal{F}_{\bar{y}} & \text{if } x \in Z_I^W \\ 0 & \text{otherwise.} \end{cases}$$

Since stalks commute with colimits, it follows that

$$\left(\varinjlim_{W \in \text{Neigh}_x} i_{W*} i_W^* \mathcal{F} \right)_{\bar{y}} \cong \begin{cases} \mathcal{F}_{\bar{y}} & \text{if } y \in \bigcap_{W \in \text{Neigh}_x} Z_I^W \\ 0 & \text{otherwise.} \end{cases}$$

This concludes since $\bigcap_{W \in \text{Neigh}_x} Z_I^W = x \cdot Y_I$, as we explicited a cofinal system $W_s(x)$. □

We are now ready to prove [lemma 8.3.1](#), hence [proposition 8.2.3](#).

Proof. of lemma 8.3.1. Using the reformulation above, we want to prove that the natural morphism :

$$\mathcal{S}^q(\mathcal{L})^\sharp \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+ \rightarrow \mathcal{S}^q(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p)^\sharp$$

is an almost isomorphism. By the combined lemmas [lemma 8.3.5](#) and [lemma 8.3.6](#), this reduces to proving that, for all $x \in X_I$, the morphism :

$$R^q(h \circ i_x)_* i_x^*(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \rightarrow R^q(h \circ i_x)_* i_x^*(\mathcal{L} \otimes \mathcal{O}_Y^+/p)$$

is an almost isomorphism. Pullbacks commutes with tensor products, and, since $i_x : x \cdot Y_I \rightarrow Y$ is a generalizing closed immersion, [lemma 2.4.1](#) shows that $i_x^* \mathcal{O}_Y^+/p \cong \mathcal{O}_{x \cdot Y_I}^+/p$. Moreover, the pullback $i_x^* \mathcal{L}$ defines an $\mathbb{F}_p$ -local system on $x \cdot Y_I$, and $h \circ i_x$ is the structure morphism of $x \cdot Y_I$, which admits a structure of a proper rigid analytic variety, by [proposition 8.1.1](#).

The result now follows from the primitive comparison theorem [theorem 5.1.3](#). □

8.4 $\mathbb{F}_p$ -étale cohomology of period domains

In this section, we compute the $\mathbb{F}_p$ -étale cohomology groups of p -adic period domains. As we have established primitive comparison for $\mathbb{F}_p$ -coefficients, Poincaré duality holds, so these étale cohomology groups are simply dual of the compactly supported étale cohomology groups with $\mathbb{F}_p$ -coefficients, computed by [CDHN21, Thm 1.2].

Let us state the result.

Let $\mathcal{G}_{E_s} := \text{Gal}(\overline{E_s}/E_s)$ denote the absolute Galois group of E_s .

Let T be a maximal torus of G such that μ factors through T , and $W = N_G(T)/Z_G(T)$ be the absolute Weyl group of G with respect to T , which acts naturally on $X_*(T)$.

Let W^μ be the set of Kostant representatives with respect to $W/\text{Stab}(\mu)$. The action of $\mathcal{G}_{E_s}$ on W preserves W^μ . To each $\mathcal{G}_{E_s}$ -orbit $[w] \in W^\mu/\mathcal{G}_{E_s}$, we associate :

- Its length, an integer denoted by $l_{[w]}$, defined as the length of any element of the orbit.
- A $\mathbb{F}_p$ -module $\rho_{[w]}(\mathbb{F}_p)$ of $\mathbb{F}_p$ -valued functions on $[w]$, equipped with the induced $\mathcal{G}_{E_s}$ -action twisted by $(-l_{[w]})$, and $\rho_{[w]}(\mathbb{F}_p)^*$ denotes its $\mathbb{F}_p$ -linear dual.

Consider a maximal $\mathbb{Q}_p$ -split torus S of the derived group $J_{b,der}$ of J_b contained in T , and a maximal $\mathbb{Q}_p$ -parabolic subgroup P_0 of J containing S . Let $\Delta \subset X^*(S)$ be the associated set of relative simple roots.

For any $I \subset \Delta$, we let P_I be the corresponding standard parabolic subgroup of J_b , and $X_I = J_b(\mathbb{Q}_p)/P_I(\mathbb{Q}_p)$, viewed as a profinite topological space. We let :

$$v_{P_I}^{J_b}(\mathbb{F}_p) = \text{LC}(X_I, \mathbb{F}_p) / \sum_{I' \subsetneq I} \text{LC}(X_{I'}, \mathbb{F}_p)$$

be the corresponding generalized Steinberg representation of $J_b(\mathbb{Q}_p)$, where $\text{LC}(-, \mathbb{F}_p)$ denotes the module of locally constant $\mathbb{F}_p$ -valued functions.

We let $v_{P_I}^{J_b}(\mathbb{F}_p)^* = \text{Hom}_{\mathbb{F}_p}(v_{P_I}^{J_b}(\mathbb{F}_p), \mathbb{F}_p)$ be the dual Steinberg representation, equipped with trivial Galois action.

Fix an invariant inner product $(-, -)$ on G . For each $\mathcal{G}_{E_s}$ -orbit $[w]$ of some $w \in W^\mu$, we define

$$I_{[w]} = \{\alpha \in \Delta, (w\mu - v, \omega_\alpha) \leq 0\}, \quad P_{[w]} := P_{I_{[w]}},$$

where $(\omega_\alpha)_{\alpha \in \Delta} \in X_*(S) \otimes \mathbb{Q}$ forms the dual basis of Δ .

We may now state the result.

Corollary 8.4.1. *Let $\mathcal{F}_C^{wa}$ be a period domain as in the above setup, and let C be the completion of $\overline{E_s}$. Let $\mathcal{F}_C^{wa}$ be the dimension⁴ of $\mathcal{F}_C^{wa}$.*

Assume furthermore that $p \neq 2$ and that $\mathcal{F}_C^{wa}$ is nonempty.⁵

Then, there is a $J_b(\mathbb{Q}_p) \times \mathcal{G}_{E_s}$ -equivariant isomorphism :

$$H_{\text{ét}}^\bullet(\mathcal{F}_C^{wa}, \mathbb{F}_p(d)) \simeq \bigoplus_{[w] \in W^\mu/\mathcal{G}_{E_s}} v_{P_{[w]}}^{J_b}(\mathbb{F}_p)^* \otimes \rho_{[w]}(\mathbb{F}_p)^* [-2d + n_{[w]}].$$

⁴Note that d can be explicited as $d = (2\rho, \nu_b)$ where ρ denotes the half sum of the positive roots of Δ , ν_b is the slope morphism associated to b as defined in definition 7.3.21, and $\langle -, - \rangle$ is the natural pairing between characters and cocharacters.

⁵By [DOR10, Thm 9.5.10], the non-emptiness is equivalent to b being acceptable for $\{\mu\}$, in the sense of [RV14, Section 2.2].

Proof. This follows directly from Poincaré duality [theorem 5.2.3](#) and the computation of compactly supported cohomology in [\[CDHN21, Thm 1.2\]](#). $\square$

The condition " $p \neq 2$ " in the above theorem stems from a technical statement [\[CDHN21, Thm 1.8\]](#) where it is needed to establish the vanishing on some Ext^1 groups between generalized Steinberg representations. It is expected from the authors of loc. cit. that this hypothesis is likely not needed.

Appendix - On the topology of locally spatial diamonds

Content of this chapter

In this chapter, we establish a few technical results regarding the topology of locally spatial diamonds, and their étale cohomology, to be used mainly, but not exclusively, in [chapter 4](#).

In the first section, we explore the structure of locally spatial diamond on the complement of open subvarieties, and how it relates to the rigid-analytic notion of complement.

In the second section, we obtain the following localization triangle for open-closed decompositions of locally spatial diamonds (as in the étale case), which is used throughout this thesis.

Proposition A.0.1. (cf. [proposition A.2.1](#)).

Let X be a locally spatial diamond whose underlying topological space decomposes as the union $|X| = |U| \sqcup |F|$ of an open and a closed subset respectively, such that $|F|$ is generalizing. Let $j : U \rightarrow X$ and $i : F \rightarrow X$ be the associated open (resp. closed) immersion of diamonds.

For any sheaf $\mathcal{F}$ of abelian groups on $X_{\text{ét}}$, there is an exact sequence in $\mathcal{D}_{\text{ét}}(X, \mathbb{Z})$, functorial in $\mathcal{F}$:

$$Rj_!^S j_S^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow Ri_*^S i_S^* \mathcal{F},$$

where the functors are taken in the sense of [\[Sch17\]](#).

In the third and final section, we construct cofinal open coverings of rigid analytic varieties. More precisely, we show that partially proper open spaces can admit two intertwining covers, comprised of quasicompact and partially proper opens respectively ; a construction that is used in [chapter 4](#) (and very useful when studying compactly supported étale cohomology), and generalizes the countable case used in the proof of [\[ABM24, Prop 6.2.5\]](#).

A.1 The closed complement of an open sub variety

As exposed in the end of [subsection 2.1.1](#), a standard problem in p -adic geometry is the fact that, unlike for schemes, the complement of an open subset of a rigid analytic variety need not admit a structure of an rigid analytic variety, nor of an analytic adic space. Indeed, every morphism of analytic adic space is generalizing, so that the complement of a non-specializing open subset never admits an analytic structure.

This is not the only problem. Indeed, consider the natural inclusion of the open unit disk in the closed unit disk over C , viewed as adic spaces. The topological complement contains a single point of rank 2, and hence does not admit an analytic structure.

In the realm of rigid-analytic varieties, one often uses the following :

Lemma A.1.1. *Let U be an quasi-compact admissible open subset of a quasi-separated rigid analytic variety X . Then $X \setminus U$ is an admissible open subset of X .*

Proof. This is [[Hub96](#), Prop 5.6.5] or [[Put92](#), Lemma 1.5]. □

Recall that the topology on rigid-analytic spaces is a Grothendieck topology, so we may decompose a connected rigid-analytic variety as the disjoint union of two admissible open subsets, as long as the covering is not admissible.

Let r be the functor from the category of rigid-analytic varieties over K to the category of analytic adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ constructed in [[Hub96](#), Paragraph 1.11]. It can be seen that a family $(r(U_i))_{i \in I}$ covers $r(X)$ if and only if the covering $(U_i)_{i \in I}$ is an admissible covering of X .

Hence, if X is connected, $r(U) \sqcup r(X \setminus U)$ cannot be a covering of $r(X)$. We however get the following.

Lemma A.1.2. *Let X be a quasi-separated rigid-analytic variety over a complete non-archimedean field K , and $U \subset X$ a quasi-compact admissible open subset.*

Then, $r(X \setminus U) = (r(X) \setminus r(U))^\circ$, where T° denotes the interior of T .

Proof. Recall that, since X is quasi-separated, by [[Hub96](#), Paragraph 1.1.11], r induces an increasing bijection between quasi-compact admissible open subsets of X and quasi-compact open subsets of $r(X)$. Moreover, for any open U of X , $r(U) \cap X = U$, so that r is a isomorphism of posets.

Since r is order-preserving, $r(X) = r((X \setminus U) \sqcup U) \subset r(X \setminus U) \cup r(U)$, so that $r(U \setminus X) \subset r(X) \setminus r(U)$. By [lemma A.1.1](#), the complement $U \setminus X$ is an admissible open of X , so that $r(X \setminus U)$ is open in $r(X)$, and we have $r(X \setminus U) \subset (r(X) \setminus r(U))^\circ$.

Since $r(X)$ admits a basis of quasi-compact open subsets, $(r(X) \setminus r(U))^\circ$ can be written as the union of all quasicompact open subsets contained in $r(X) \setminus r(U)$. Let V be such a quasicompact open. By the properties of r , we may write $V = r(W)$ for some quasicompact admissible open W of X , so that $r(W) \cap r(U) = \emptyset$. Since r is order-preserving, we have $r(W \cap U) \subset r(W) \cap r(U) = \emptyset$, so that $W \cap U = \emptyset$ since r is injective, and $W \subset X \setminus U$, so that $V = r(W) \subset r(X \setminus U)$.

Hence $(r(X) \setminus r(U))^\circ \subset r(X \setminus U)$, which concludes. □

Huber, in [[Hub96](#)], defines a theory of *pseudo-adic spaces*, so that any "nice" (convex and pro-constructible) subset of an analytic adic space admits a pseudo-analytic structure, and, in turn, an étale site. In this thesis, we will bypass the use of pseudo-adic spaces, using locally spatial diamonds instead.

Note that by [Sch17, Prop 11.15], open subsets of locally spatial diamonds naturally admit a structure of a diamond, which is then locally spatial by [Sch17, Prop 11.20].

We use some notations from [Sch17]. In particular, for X a diamond, we denote $|X|$ its underlying topological space, and, for T a topological space, we let $\underline{T}$ denote the v-sheaf given by $\underline{T}(X) := \text{Cont}(|X|, T)$. We will use the following lemma :

Lemma A.1.3. *Let X be a locally spatial diamond, and $F \subset X$ be a generalizing closed subset of X . Then F is the underlying space of a sub-locally spatial diamond, still denoted F , given by the fiber product of v-sheaves $F = \underline{F} \times_{|X|} X$.*

Proof. By [Ans+22, Lemma 2.7], the above defines a closed sub v-stack of underlying topological space F . It is also a v-sheaf, since the fiber product can be taken in the category of v-sheaves.

By [Sch17, Prop 11.10], it is naturally a diamond, and, by [Sch17, Prop 11.20], it is locally spatial. $\square$

Hence, the complement of a specializing¹ open subset of an analytic adic space need not admit a structure of an analytic adic space, yet always admit the structure of a locally spatial diamond.

In that setting, we furthermore have a localization (or excision) sequence for étale sheaves, as developed in the following.

A.2 Localization for locally spatial diamonds

In this section, we work with the left completed derived category of étale sheaves on a locally spatial diamond, as considered in [section 3.2](#).

For a partially proper open immersion j of locally spatial diamonds, we consider the proper pushforward $Rj_!^S$ as taken in the sense of [Sch17, Def 19.1], as the left adjoint to Rj_S^* . It coincides with the functor defined by Huber in [Hub96], when j is defined between locally noetherian analytic adic spaces by [remark 4.2.1](#).

Proposition A.2.1. *Let X be a locally spatial diamond whose underlying topological space decomposes as the union $|X| = |U| \sqcup |F|$ of an open and a closed subset respectively, such that $|F|$ is generalizing. Let $j : U \rightarrow X$ and $i : F \rightarrow X$ be the associated open (resp. closed) immersion of diamonds.*

For any sheaf $\mathcal{F}$ of abelian groups on $X_{\text{ét}}$, there is an exact sequence in $\mathcal{D}_{\text{ét}}(X, \mathbb{Z})$, functorial in $\mathcal{F}$:

$$Rj_!^S j_S^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow Ri_*^S i_S^* \mathcal{F}$$

This generalizes the version of [Hub96, Rk. 5.5.11.iv].

Proof. To simplify the notations, we consider implicitly every functor to be in the sense of Scholze in [Sch17].

The morphisms come from (co)unit of the adjunctions. The étale site of X has enough points, cf. [Sch17, Prop 14.3], so that it suffices to check the exactness on stalks. Let $\bar{x} : \text{Spa}(C, C^+) \rightarrow X$ be a quasi-pro-étale geometric point of X , where C is algebraically closed, $C^+ \subset C$ is an open and bounded valuation subring, and $\bar{x}$ maps the unique closed point of $\text{Spa}(C, C^+)$ to some $x \in X$.

¹It would be hopeless to expect a more general statement, as, even in the language of v-stacks, arbitrary morphisms are (weakly) generalizing, cf. [Ans+22, Lemma 2.4].

We compute the stalks using the following lemma, which immediately concludes :

Lemma A.2.2. *Under the hypothesis of [proposition A.2.1](#), we have :*

$$(Rj_! \circ j^* \mathcal{F})_{\bar{x}} \cong \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in |U| \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (Ri_* \circ i^* \mathcal{F})_{\bar{x}} \cong \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in |F| \\ 0 & \text{otherwise} \end{cases}$$

We apply base change ([[Sch17](#), Prop 19.1], resp. [[Sch17](#), Cor 16.10.(ii)]) to the diagrams :

$$\begin{array}{ccc} U \times_X \mathrm{Spa}(C, C^+) & \xrightarrow{j'} & \mathrm{Spa}(C, C^+) \\ \downarrow \bar{x}' & & \downarrow \bar{x} \\ U & \xrightarrow{j} & X \end{array} \quad \begin{array}{ccc} F \times_X \mathrm{Spa}(C, C^+) & \xrightarrow{i'} & \mathrm{Spa}(C, C^+) \\ \downarrow \bar{x}' & & \downarrow \bar{x} \\ F & \xrightarrow{i} & X. \end{array}$$

It follows that :

$$\begin{aligned} (Rj_! j^* \mathcal{F})_{\bar{x}} &= R\Gamma(Rj'_! \circ \bar{x}'^* j^* \mathcal{F}) = R\Gamma(Rj'_! j'^* \bar{x}^* \mathcal{F}) \\ (Ri_* i^* \mathcal{F})_{\bar{x}} &= R\Gamma(Ri'_* \bar{x}'^* i^* \mathcal{F}) = R\Gamma(Ri'_* i'^* \bar{x}^* \mathcal{F}), \end{aligned}$$

where $R\Gamma$ denotes the derived global section functor on $\mathrm{Spa}(C, C^+)$.

From [[Sch17](#), Prop 12.10], we know that there are surjections :

$$\begin{aligned} |U \times_X \mathrm{Spa}(C, C^+)| &\twoheadrightarrow |U| \times_{|X|} |\mathrm{Spa}(C, C^+)| \\ |F \times_X \mathrm{Spa}(C, C^+)| &\twoheadrightarrow |F| \times_{|X|} |\mathrm{Spa}(C, C^+)|. \end{aligned}$$

Here, the fiber product on the right hand side is taken as topological spaces. Topologically $|\mathrm{Spa}(C, C^+)|$ is a totally ordered chain of specializations. It admits a unique closed point, which is a specialization of every other points. Since continuous maps preserves specializations, the image of $|\mathrm{Spa}(C, C^+)|$ in $|X|$ is contained in the set of generalizations of x .

Assume that $x \notin |U|$. Since $|F| = |X| \setminus |U|$ is a generalizing subset of $|X|$ by hypothesis, the image of $\bar{x}$ is contained in $|F|$, and hence disjoint from $|U|$, so that the fiber product $|U| \times_{|X|} |\mathrm{Spa}(C, C^+)|$ is empty. Hence, $U \times_X \mathrm{Spa}(C, C^+)$ maps surjectively to the empty set, hence is empty, so that x'^* is zero, and hence $(Rj_! j^* \mathcal{F})_{\bar{x}} = 0$, as desired.

The exact same argument holds for $x \notin |F|$, since $|U|$ is generalizing (as it is open), so that, for $x \notin F$, $(Ri_* i^* \mathcal{F})_{\bar{x}} = \emptyset$.

Now, assume that $x \in |U|$. We will show that $Rj'_! j'^* \mathcal{F} \cong \mathcal{F}$.

The topological space $|U \times_X \mathrm{Spa}(C, C^+)|$ identifies with an open subset of $|\mathrm{Spa}(C, C^+)|$ containing the unique closed point $x \in \mathrm{Spa}(C, C^+)$, so that $|U \times_X \mathrm{Spa}(C, C^+)| = |\mathrm{Spa}(C, C^+)|$. Since j' is an open immersion, it's an injection of perfectoid spaces, so that the combined [[Sch17](#), Prop 5.3] and [[Sch17](#), Prop 5.4] prove that j is an isomorphism, hence $Rj'_! j'^* \cong id$, which gives the desired stalk.

Finally, if $x \in |F|$, which is generalizing, the image of $|\mathrm{Spa}(C, C^+)|$ is included in $|F|$. Since, by [lemma A.1.3](#), $F = \underline{F} \times_{|X|} X$, the morphism $x : \mathrm{Spa}(C, C^+) \rightarrow X$ then factors through F , so that there exists a diagonal map x_F making the diagram commute :

$$\begin{array}{ccc}
F \times_X \mathrm{Spa}(C, C^+) & \xrightarrow{i'} & \mathrm{Spa}(C, C^+) \\
\downarrow \bar{x}' & \nearrow x_F & \downarrow \bar{x} \\
F & \xrightarrow{i} & X
\end{array}$$

By the universal property of fiber products, this diagonal map x_F induces a splitting of i' , which is hence surjective. Finally, a surjective closed immersion of perfectoid spaces is an isomorphism (cf. [Sch17, Lemma 5.3 and 5.4]), so that i' induces an isomorphism of diamonds.

Hence $Ri'_* i'^* \cong id$, which concludes. $\square$

Remark A.2.3. In the above setup, we may rewrite the fake base change results [Sch17, Thm 19.2 and Rk 19.3] as a real base change result, without needing to go through pseudo-adic spaces.

This result also holds for quasicohherent pullbacks, as defined in definition 3.4.1.

Corollary A.2.4. Let X be an untilted locally spatial diamond with pseudo-uniformizer π , whose underlying topological space decomposes as the union $|X| = U \sqcup F$ of an open and a closed subset respectively, such that F is generalizing. Let $j : U \rightarrow X$ and $i : F \rightarrow X$ be the associated open (resp. closed) immersion of diamonds. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X^+/\pi$ -modules on $X_{\text{ét}}$, there is an exact sequence in $\mathcal{D}_{\text{ét}}(X, \mathcal{O}_X^+/\pi)$:

$$Rj_!^S j_S^{*, qcoh} \mathcal{F} \rightarrow \mathcal{F} \rightarrow Ri_*^S i_S^{*, qcoh} \mathcal{F}.$$

This construction is moreover functorial in $\mathcal{F}$.

This directly follows from the following lemma :

Lemma A.2.5. Let $f : X \rightarrow Y$ be a quasi-pro-étale morphism of locally spatial diamonds.

Then there is a natural isomorphism of functors $f_S^{qcoh,*} \cong f_S^*$.

This is in particular the case if f is an open immersion or a generalizing closed immersion.

Proof. Recall that, for any sheaf $\mathcal{F}$ of $\mathcal{O}_Y^+/\pi$ -modules, the quasicohherent pullback $f_S^{qcoh,*}$ is defined as the tensor product $f_S^* \mathcal{F} \otimes_{f_S^*(\mathcal{O}_Y^+/\pi)} \mathcal{O}_X^+/\pi$. But, for a quasi-pro-étale f , it follows from lemma 2.4.3 that $f_S^* \mathcal{O}_Y^+/\pi \cong \mathcal{O}_X^+/\pi$, which yields the desired result.

Finally, open immersions are quasi-pro-étale by definition, and generalizing closed immersions are quasi-pro-étale by [Sch17, Cor 10.6]. $\square$

A.3 A convenient open covering

In this section, we construct two intertwined covering of any partially proper analytic adic space : one consisting of qcqs open, and the other one consisting of partially proper open subsets. Such a covering will play an important technical role in the following section.

Before anything else, let us first establish two topological lemmas.

Lemma A.3.1. Let X be a topological space that admits a basis of quasi-compact quasi-separated open subsets and such that, for any quasi-compact open $U \subset X$, the closure $\overline{U}$ is quasi-compact.

Then, there exists a filtered poset $(I, \leq)$ and a decomposition of X as an increasing union of qcqs open subspaces $(X_i)_{i \in I}$, such that :

1. For any $i \in I$, there are only finitely many $j \in I$ such that $j \leq i$.
2. For any $i < i'$, $\overline{X_i} \subset X_{i'}$.

Proof. Since X admits a basis of qcqs open subsets, we may choose a covering $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ by some qcqs open X_λ . Let us denote $\mathcal{P}_\omega(\Lambda)$ the set of finite subsets of Λ .

For any $J \in \mathcal{P}_\omega(\Lambda)$, let $X_J = \bigcup_{\lambda \in J} X_\lambda$, so that the $(X_J)_{J \in \mathcal{P}_\omega(\Lambda)}$ form a filtered covering of X by qcqs open subsets. For any $J \in \mathcal{P}_\omega(\Lambda)$, consider the open cover $\overline{X_J} \subset \bigcup_{J' \subset J'} X_{J'}$. Since $\overline{X_J}$ is quasi-compact, we can extract a finite subcover, so that there exists some $m(J) \in \mathcal{P}_\omega(\Lambda)$ such that $J \subset m(J)$ and $\overline{X_J} \subset X_{m(J)}$.

By construction, for any $J \in \mathcal{P}_\omega(\Lambda)$, $J \subset m(J)$ and $\overline{X_J} \subset X_{m(J)}$.

Now, define a partial order on $\mathcal{P}_\omega(\Lambda)$, denoted $\leq'$, by :

$$A \leq' B \Leftrightarrow A = B \text{ or } m(A) \subset B.$$

By construction, it satisfies both desired properties.

It is moreover filtering since, for any $A, B \in \mathcal{P}_\omega(\Lambda)$, both $A \leq' m(A) \cup m(B)$ and $B \leq' m(A) \cup m(B)$. $\square$

Lemma A.3.2. *Let X be a topological space that admits a basis of quasi-compact open subsets.*

Let F be a quasi-compact closed subset of X , and U an open subset such that $F \subset U$. Then there exists a quasicompact open V of X such that $F \subset V \subset U$.

Proof. Since the quasi-compact opens form a basis of the topology, every $x \in F$ admits a quasi-compact neighborhood contained in U , denoted U_x .

Then, $F \subset \bigcup_{x \in F} U_x$. Since F is quasi-compact, one may extract a finite sub-covering $F \subset \bigcup_{i=1}^n U_{x_i}$. Then $V = \bigcup_{i=1}^n U_{x_i}$ satisfies the desired property. $\square$

We may now state our result. Note that it can be seen as a more general variant of the covering used in the proof of [ABM24, Prop 6.2.5]. Compare also with the covering constructed in [AGN25, Cor 2.29], or with [Hub96, Lemma 5.3.3].

Proposition A.3.3. *Let $f : X \rightarrow Y$ be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y . Then, there exists a filtered poset $(I, \leq)$, together with two open coverings of X :*

- An increasing open cover $X = \bigcup_{i \in I} X_i$ by qcqs open subsets.
- An increasing open cover $X = \bigcup_{i \in I} U_i$ by partially proper open subsets.

such that :

1. For any $i \in I$, there are only finitely many $i' < i$.
2. For all $i \in I$, $X_i \subset U_i$ and for any $i < j$, $U_i \subset X_j$.
3. For any $i < j$, $\overline{X_i} \subset U_j$, and the morphism $X_i \rightarrow X_j$ factors through the universal adic compactification $\overline{X_i}^Y$ of X_i over Y , as constructed in [Hub96, Thm 5.1.5].

Proof. Since f is a partially proper, it is taut, so that, since Y is taut, X is also taut. Moreover, X admits a basis of affinoid open subsets, so that the assumptions of lemma A.3.1 apply. Let $(I, \leq)$ and $(X_i)_{i \in I}$ be a filtered open cover of X by qcqs open subsets, satisfying the conditions of lemma A.3.1.

For any $i \in I$, let $Y_i = \overline{X_i}$ be the closure of X_i in X . Since X is taut, Y_i is quasi-compact in X . Let $Z_i = Y_i \setminus X_i$, which is a quasi-compact closed subset of Y_i . The above [lemma A.3.2](#), applied to $Z_i \subset (Y_i \setminus \bigcup_{j < i} Y_j)$ in Y_i , yields a quasi-compact open subset V_i of Y_i such that $Z_i \subset V_i \subset Y_i \setminus \bigcup_{j < i} Y_j$.

Since $Z_i \subset V_i$, $X_i \cap V_i = \emptyset$, and X_i is open, so that $X_i \cap \overline{V_i} = \emptyset$. Let $U_i = Y_i \setminus \overline{V_i}$, so that, for all $i < j$; $X_i \subset U_i \subset X_j$ using the second hypothesis from [lemma A.3.1](#). We claim that the map $U_i \rightarrow X$ is partially proper.

It suffices to show that U_i is specializing, hence that $\overline{V_i}$ is generalizing. V_i is quasi-compact, hence retrocompact, so, by [\[Fo17, Ch 0, Cor 2.2.27\]](#), $\overline{V_i}$ is simply the closure under specializations of V_i . Since X_i is an analytic adic space, all specializations are vertical, and, by [\[Hub96, Lemma 1.1.10.\(i\)\]](#), the set of generalizations (or specializations) of any given point forms a chain, so that $\overline{V_i}$ is stable under generalization. Hence the U_i are specializing open subsets of X , hence partially proper.

It now suffices to prove the last property. For any $i < j$, there is a natural inclusion $X_i \subset U_i \subset X_j$. Since $X_i \rightarrow U_i \rightarrow Y$ is the composition of an open immersion and a partially proper morphism, it defines a compactification of X_i over U_i , so that, by [\[Hub96, Thm 5.1.1\]](#), the morphism $X_i \rightarrow U_i$ factors through the universal compactification $\overline{X_i}^Y$. Hence, it is also the case for the morphism $X_i \rightarrow X_j$. $\square$

In [\[ABM24, Section 6.2\]](#), it is crucial for some arguments that a covering as constructed above can be made to be *countable*. We show that this is equivalent to a more standard paracompactness assumption.

Let's say that a rigid-analytic variety is *paracompact* if it admits a locally finite covering by admissible affinoids, cf. [\[JJP96, Definition 2.5.6\]](#).

Proposition A.3.4. *Let X be a connected partially proper rigid-analytic variety over $\mathrm{Spa}(C, \mathcal{O}_C)^2$, where C is a complete algebraically closed extension of $\mathbb{Q}_p$. Then, the following are equivalent :*

1. X is paracompact.
2. X is countable at infinity in the sense of [\[ABM24, Def 6.2.5\]](#), i.e. we may write $X = \bigcup_{n \in \mathbb{N}} X_n$ as a countable increasing union of qcqs open subspaces X_n such that, for each n , the inclusion $X_n \subset X_{n+1}$ factors through the adic compactification Y_n of X_n over C .

Note that the "connected" hypothesis is purely technical - otherwise it is possible that X admits uncountably many connected components, and one may not expect to find a reasonable countable covering.

Proof. (i) $\implies$ (ii) Since X is connected, we may choose a *countable* admissible locally finite affinoid covering in [lemma A.3.1](#), using [\[JJP96, Lemma 2.5.7\]](#). Then, the construction of [proposition A.3.3](#) produces such a result.

(ii) $\implies$ (i). Write $X = \bigcup_{n \in \mathbb{N}} X_n$ as a strictly increasing countable union of qcqs open subsets, as in the definition of countability at infinity, and let Y_n be the adic compactification of X_n over $\mathrm{Spa}(C, \mathcal{O}_C)$. Consider $Y_n \setminus X_{n-1}$ as a closed subset of $X_{n+1} \setminus Y_{n-2}$. By the separation [lemma A.3.2](#), for all $n \in \mathbb{N}$, there exists qcqs open subsets U_n in X_{n+1} such that $Y_n \setminus X_{n-1} \subset U_n \subset X_{n+1} \setminus Y_{n-2}$. By construction, each U_n might intersect $U_{n \pm 1}$, but not any U_k for $|k - n| > 1$.

Each U_n is a quasi-compact open subset of X , which can hence be written as a finite union of affinoids. Such collection of affinoids is locally finite by construction, so that X is paracompact. $\square$

Remark A.3.5. *The above applies to p -adic period domains as they are paracompact by [\[Har13, Lemma A.3\]](#).*

²The argument would work mutatis mutandis for locally noetherian analytic adic spaces over a more general base, but we stick to the generality used in the literature.

Appendix - On topological Tits buildings

Content of this chapter

The goal of this chapter is to recall the construction and classical properties of Tits buildings over p -adic fields, equipped with a topology introduced in [BS76], mixing a profinite and a Euclidean topology. We moreover establish a few novel results, that are used in [chapter 8](#) (and, to a lesser extent, in [chapter 6](#)).

In particular, we prove in [proposition B.2.9](#) that the topology introduced by Borel-Serre in [BS76] can be reformulated as the geometric realization of a simplicial profinite set, as appeared in [SS91], making fully explicit an idea that was implicit in the literature (cf. [SS91, Rk, p.84]).

Our main result is [proposition B.4.4](#), which establishes a vanishing result of cohomology groups of some specific *convex* subsets of topological Tits buildings. This is obtained by carefully adapting the proof of [BS76, Thm. 2.6].

These convex subsets appear in the proof of the acyclicity of Orlik's fundamental complex, which was introduced in [proposition 8.2.1](#). The [proposition B.4.4](#) is then used to fill in an incomplete argument of Orlik in [Orl05a] (which also appeared in [Orl05b], [DOR10] and [CDHN21]).

Along the way, we make a few mildly interesting remarks about the topology of simplicial profinite spaces, contractibility, and their cohomology.

B.1 The realization of simplicial profinite spaces

A **simplicial space** is defined as a simplicial object of the category $\mathbf{Top}$, i.e. a functor $X : \Delta^{op} \rightarrow \mathbf{Top}$, where Δ denotes the simplex category (of nonempty finite ordinals and increasing maps).

In particular, a **simplicial profinite space** is as simplicial object of the category $\mathbf{Prof}$ of profinite spaces.

For a simplicial profinite space X , we denote $X_n := X[n]$, and, for all increasing map $\alpha : [m] \rightarrow [n]$, we let $\alpha^* : X_n \rightarrow X_m$ be the associated continuous map. Such a data is equivalent to a sequence of profinite spaces $(X_n)_{n \in \mathbb{N}}$, together with continuous face and degeneracy maps, satisfying the usual commutation relations.

Any simplicial space X admits a (thin)¹ geometric realization (introduced in [Seg68]), defined as follows :

Definition B.1.1. For any $n \in \mathbb{N}$, we let $\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}_+^{n+1}, \sum x_k = 1\}$ denote the standard topological n -simplex. Any $\alpha : [n] \rightarrow [m] \in \Delta$ induces natural affine maps $\alpha_* : \Delta^n \rightarrow \Delta^m$, endowing $\Delta^\bullet$ with a cosimplicial structure.

Let X be a simplicial profinite space. Its thin **geometric realization**, denoted² $|X|^{pro}$, is defined as the quotient :

$$|X|^{pro} = \bigsqcup_{n \in \mathbb{N}} X_n \times \Delta^n / \sim$$

by the equivalence relation identifying

$$(x, \alpha_* p) \in X_m \times \Delta^m \sim (\alpha^* x, p) \in X_n \times \Delta^n$$

for all $\alpha : [n] \rightarrow [m]$.

Such geometric realizations are poorly behaved topological spaces. Indeed, they are not locally contractible (outside of trivial cases), and need not have the homotopy type of a CW-complex. As a consequence, their constant sheaf cohomology need not coincide with their singular cohomology.

However, for any reasonable cohomology theory, the cohomology of $|X|^{pro}$ can be computed from the cohomology of the X_n . We start with a useful notation.

Definition B.1.2. Let A be a cosimplicial object of an abelian category $\mathcal{A}$, i.e. a functor $A : \Delta \rightarrow \mathcal{A}$. We let $A^\sim$ denote the associated cochain complex, with $A_n^\sim = A[n]$, and morphisms $A_n^\sim \rightarrow A_{n+1}^\sim$ given by the alternate sums of the coface maps.

In the following, we will mostly be interested in constant sheaf cohomology, for which we have the following.

Proposition B.1.3. Let X be a simplicial profinite space. For any $n \in \mathbb{N}$, we let (X_n, A) denote the set of locally constant functions $X_n \rightarrow A$. The simplicial structure on X induces a natural cosimplicial structure on $\mathbf{LC}(X_\bullet, A)$.

The associated cochain complex $(\mathbf{LC}(X, A))^\sim$ computes the cohomology $H_{\mathbf{Shf}}^*(|X|^{pro}, \underline{A})$ of the constant sheaf $\underline{A}$ on $|X|^{pro}$.

¹This is different from the fat geometric realization, where one does not quotient by the degeneracy maps. For so-called good simplicial spaces, both construction are homotopy equivalent ; this is a priori not the case for simplicial profinite spaces. Indeed, simplicial profinite spaces are never good, as one can check that a pair (A, X) for a profinite X is a (strong) neighborhood deformation retract if and only $A = X$.

²This notation is not standard. We try to avoid ambiguity with the realization of X viewed as a simplicial set, forgetting the levelwise topology, that will also appear.

Proof. By [Seg68, Prop 5.1], for any generalized cohomology theory $k^\bullet$, there is a spectral sequence :

$$E_2^{p,q} = H^p(k^q(X)^\sim) \implies k^{p+q}(|X|^{pro}).$$

We apply it with $k^q(X) := H_{\text{Shf}}^q(X, \underline{A})$. A classical computation [Sta23, Lemma 0A3F] shows that, for a profinite space S :

$$H_{\text{Shf}}^i(S, \underline{A}) = \begin{cases} \text{LC}(S, A) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Hence, the second page of the above spectral sequence only has one nontrivial line, so that $E_\infty = E_2$, and one can read the limit on E_∞ . This concludes. $\square$

Remark B.1.4. The same argument shows that the cochain complex $(\text{Fun}(X, A))^\sim$ of non-necessarily continuous functions computes singular cohomology $H_{\text{Sing}}^*(|X|^{pro}, A)$.

Indeed, for S a profinite space, $H_{\text{Sing}}^0(S, A)$ is given by all functions $S \rightarrow A$, and there is no higher cohomology, as continuous maps $\Delta^n \rightarrow S$ are constant.

Hence, singular cohomology of $|X|^{pro}$ does not see the topology on the X_n .

To any simplicial profinite space X , we consider its underlying simplicial set $\tilde{X}$, defined as the composition $\Delta^{op} \rightarrow \text{Top} \rightarrow \text{Set}$ with the forgetful functor. Let $|\tilde{X}|$ be its geometric realization. The spaces $|\tilde{X}|$ and $|X|^{pro}$ admit the same underlying set, that we simply denote $|X|$, but not the same topology.

Lemma B.1.5. The topology $|\tilde{X}|$ is finer than $|X|^{pro}$.

Proof. It follows from definition B.1.1 that $|\tilde{X}| \xrightarrow{id} |X|^{pro}$ is continuous. $\square$

By remark B.1.4 $|\tilde{X}|$ and $|X|^{pro}$ have the same singular cohomology groups.³

In general, the singular and sheaf cohomology of $|X|^{pro}$ do not coincide, as shown by the example below. This example also shows that it is possible for $|\tilde{X}|$ to be contractible, while $|X|^{pro}$ is not.⁴

Example B.1.6. Let $S = \{1/n, n \in \mathbb{N}^*\} \sqcup \{0\}$, endowed with its topology as a subset of $\mathbb{R}$, making it a profinite space (as it is compact, Hausdorff, and extremally disconnected).

Consider the simplicial profinite space X such that $X_0 = X \times \{0, 1\}$ and X_1 consists of edges linking :

- $(s, 0)$ with $(s, 1)$ for all $s \in S$,
- $\left(\frac{1}{n}, 1\right)$ and $\left(\frac{1}{n+1}, 0\right)$ for all $n \in \mathbb{N}^*$,
- $(0, 0)$ and $(1, 0)$.

and no nontrivial information in higher dimension. Its geometric realization $|X|^{pro}$ is homeomorphic to the following topological space, endowed with the topology induced from $\mathbb{R}^2$:

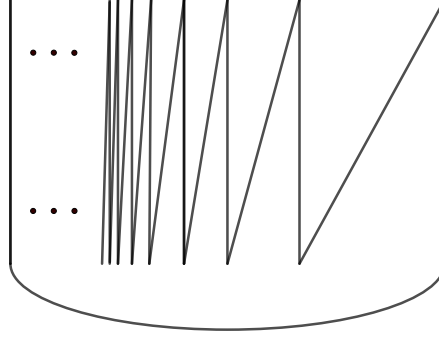
This is homeomorphic to the Warsaw circle, which is known (cf. [Bor75]) to satisfy :

$$H_{\text{Shf}}^1(|X|^{pro}, \mathbb{Z}) \cong \mathbb{Z},$$

as sheaf and Čech cohomology coincide for paracompact Hausdorff spaces. It is a fortiori not contractible.

³Moreover, the singular and constant sheaf cohomology of $|\tilde{X}|$ coincide, as $|\tilde{X}|$ is paracompact and locally contractible.

⁴I don't know whether the converse is possible.



However, the geometric realization $|\tilde{X}|$ is homeomorphic to a half line (as it "does not see" the vertical segment linking $(0,0)$ and $(0,1)$ as consisting of accumulation points). It is hence contractible, so that $H_{\text{Sing}}^1(|X|^{\text{pro}}, \mathbb{Z}) \cong H_{\text{Shf}}^1(|\tilde{X}|, \mathbb{Z}) \cong 0$.

Remark B.1.7. In [Mor96, p. 352], F.Morel asks whether the contractibility of $|\tilde{X}|$ implies that $|X|^{\text{pro}}$ is cohomologically trivial, when $\tilde{X}$ is additionally a Kan complex. This question has not been answered, but is related to [proposition B.4.4](#).

The following proposition will be needed in the rest of the appendix. It shows that the geometric realization can be recovered from the data of non-degenerate simplices.

Proposition B.1.8. Let X be a profinite simplicial space. For any $n \in \mathbb{N}$, we define the subset of **non-degenerate** simplices $X'_n \subset X_n$ as the subset of simplices who do not lie in the image of any degeneracy maps, equipped with the induced topology.

Assume that X is of dimension d for some $d \in \mathbb{N}$ (i.e., $X'_d \neq \emptyset$ and, for all $n > d$, $X'_n = \emptyset$). Then, there is a natural homeomorphism :

$$|X|^{\text{pro}} \cong \bigsqcup_{n=0}^d X'_n \times \Delta^n / \sim,$$

where $\sim$ is the restriction of the equivalence relation of [definition B.1.1](#).

For simplicial sets, this is [GJ09, Chap 1, Prop 2.3].

Proof. Denote temporarily $|X'|^{\text{pro}} = \bigsqcup_{n \in \mathbb{N}} X'_n \times \Delta^n / \sim$. There is a natural continuous injection $f : |X'|^{\text{pro}} \rightarrow |X|^{\text{pro}}$ induced by the inclusions $X'_n \rightarrow X_n$.

It follows from the Eilenberg-Zilber lemma (cf. [GZ67, p.26-27]) that every element $x \in X_n$ is of the form $\alpha^* y$ for some $y \in X'_k$ and $\alpha : [n] \rightarrow [k]$. Hence, by definition of the equivalence relation, f is surjective.

It is then a homeomorphism, as $|X'|^{\text{pro}}$ is compact (as a quotient from a compact set) and $|X|^{\text{pro}}$ is Hausdorff by [SP13, Thm 1.1]. $\square$

Remark B.1.9. In the above proposition, assume furthermore that all face maps induce surjections $X'_n \rightarrow X'_{n-1}$ for all $n < d$.

Then, the morphism $X'_n \times \Delta^n \rightarrow |X|^{\text{pro}}$ is surjective.

We conclude this section with a few lemmas, that are useful at various places of this thesis.

Lemma B.1.10. Let $S = \varprojlim_{n \in \mathbb{N}} S_n$ be a profinite space (with finite S_n), and T be discrete. Then $\text{Cont}(S, T) = \text{LC}(S, T) \cong \varinjlim_{n \in \mathbb{N}} \text{Fun}(S_n, T)$.

Proof. Since T is discrete, continuous functions $S \rightarrow T$ are necessarily locally constant. Moreover, any locally constant function $S \rightarrow T$ necessarily factors through a finite quotient of S , which itself factors through some S_n . $\square$

Lemma B.1.11. *Let S be a profinite space. Then, for all abelian group A , $\mathrm{LC}(S, A)$ is a free A -module. Moreover, $\mathrm{LC}(S, A) \cong \mathrm{LC}(S, \mathbb{Z}) \otimes A$.*

Proof. The first part is [Sch19, Thm 5.4] (attributed to Nöbeling [Nöb68]). The second part follows from lemma B.1.10, as tensor products commute with colimits. $\square$

Lemma B.1.12. *Any profinite set admits a basis formed by clopen subsets.*

Proof. This is [Sta23, Lemma 08ZZ]. $\square$

B.2 The topologized combinatorial Tits building

In this section, we view how to endow spherical Tits building with the topology of the realization of a simplicial profinite space. Let us start by recalling standard facts and notations.

We fix K a complete extension of $\mathbb{Q}_p$, and G a connected reductive algebraic group over K . Fix a maximal K -split torus $T \subset G$, a minimum parabolic subgroup B containing T , and let Δ be an associated relative root system, such that subsets $I \subset \Delta$ correspond to standard parabolic subgroups $(P_I)_{I \subset \Delta}$ of G , containing B . The map $I \mapsto P_I$ is increasing, with $P_\emptyset = B$, $P_\Delta = G$.

Let r be the semi-simple rank of G , i.e. the cardinal of Δ .

We will freely use the following standard facts :

- All subgroups of G containing B are of the form P_I for some $I \subset \Delta$ (cf. [Tit, Prop 3.2.2]).
- All parabolic subgroup of G are conjugated, under an element of $G(K)$, to a unique P_I (cf. [Bor91, Prop 21.12]).
- The quotient $G(K)/P_I(K) = (G/P_I)(K)$ is the K -points of some projective scheme over K (cf. [Bor91, Prop 20.25]). In particular, it is compact Hausdorff.
- The intersection of two parabolic subgroups contains a conjugate of T (cf. [Bou07, Ch 4, Sec 2.6, Prop 5]).
- If two parabolic subgroups are conjugated, their intersection is not parabolic (cf. [Bou07, Ch 4, Sec 2.6, Thm 4.(ii)]).
- The minimal parabolic subgroups containing T are conjugated under the action of the Weyl group $W = N_G(T)/Z_G(T)$, which is finite (cf. [Bor91, Prop 11.19]).

Let $\mathcal{P}$ denote the set of maximal proper parabolic subgroups of G ; i.e. the subgroups conjugated to some $P_{\Delta \setminus \{i\}}$ for some $i \in \Delta$.

The **combinatorial Tits building** associated to G , denoted $\Delta(G)$, is classically defined as the abstract simplicial complex whose simplices are the proper parabolic subgroups of G (cf. [Tit] for a good reference on this theory). In other words, its vertices are given by elements of $\mathcal{P}$, and its d -dimensional simplices are the $(P_0, \dots, P_d) \in \mathcal{P}$ such that $P_0 \cap \dots \cap P_d$ is parabolic.

We are then interested in the geometric realization of this abstract simplicial complex. In order to fit into the framework of [section B.1](#), and define an alternative topology, we would rather view this geometric realization as the geometric realization of a simplicial set.

The conjugacy action on $\mathcal{P}$ admits finitely many orbits, corresponding to the subgroups $P_{\Delta \setminus \{i\}}$ for $i \in \Delta$. We arbitrarily order these conjugacy classes, and extend this order⁵ to $\mathcal{P}$. We then define the simplicial set $\Delta_\bullet(G)$ by :

$$\Delta_n(G) = \{P_0 \leq \dots \leq P_n \in \mathcal{P}, P_0 \cap \dots \cap P_n \text{ is parabolic}\},$$

where degeneracy (resp. face) maps are given by repeating (resp. forgetting) a parabolic subgroup. Non-degenerate simplices are given by :

$$\Delta'_n(G) = \{P_0 < \dots < P_n \in \mathcal{P}, P_0 \cap \dots \cap P_n \text{ is parabolic}\}.$$

The sets $\Delta_n(G)$ (and $\Delta'_n(G)$) admit a natural $G(K)$ -action, given by conjugation. By definition, $\Delta(G)$ is of dimension $r - 1$, in the sense of [remark B.1.9](#).

It follows essentially by definition that the geometric realization of $\Delta(G)$, viewed as an abstract simplicial complex or as a simplicial set, are homeomorphic (cf. [[Cur71](#), Def 1.29]). We denote this topological space $|\widehat{\Delta(G)}|$, with underlying set $|\Delta(G)|$. This comes equipped with a $G(K)$ -action, such that any $x \in |\Delta(G)|$ is conjugated to an element of the (closed) simplex associated to B .

Lemma B.2.1. *Let P be a parabolic subgroup of G . Then there exists a unique $n \in \mathbb{N}$ and $P_0 < \dots < P_n \in \mathcal{P}$ such that $P = P_0 \cap \dots \cap P_n$.*

Proof. Up to conjugation, we may assume that P is a standard parabolic subgroup $P_I = \bigcap_{i \in I} P_{\Delta \setminus \{i\}}$. The subgroups it contains are exactly the P_J for $I \subset J$, so that the maximal parabolic subgroups are exactly the $P_{\Delta \setminus \{i\}}$, and there is only one way to order them. $\square$

Hence, the subset of non-degenerate simplices $|\Delta'_n(G)|$ is exactly the set of parabolic subgroups of G conjugated to some P_I with $|I| = r - 1 - n$. Let us introduce the most classical example.

Example B.2.2. *For $G = \text{GL}_n$, the set of parabolic subgroups of G is in bijection with the set of partial flags, i.e. family of K -linear subspaces $\{\{0\} \subsetneq F_0 \subset \dots \subset F_k \subset K^n\}$. Parabolic subgroups correspond to the stabilizer of such flags.*

Viewed as a simplicial set, we have :

$$\Delta_k^{\text{GL}_n} = \{\text{Flags } \{0\} \subsetneq F_0 \subset F_1 \subset \dots \subset F_k \subset K^n\}.$$

A simplex is non-degenerate if the corresponding flag is strict, i.e $F_i \subsetneq F_{i+1}$ for all i .

This is the simplicial set that appeared in [definition 6.4.5](#), where it was endowed with the structure of a simplicial profinite space.

Let us define this topology in the general setting.

Definition B.2.3. *For all n , the set $\Delta_n(G)$ contains the $G(K)$ -orbit of all parabolic subgroups P_I alongside the $I \subset \Delta$ with $|I| \leq r - 1 - n$.*

⁵We will never need to compare parabolic subgroups from the same conjugacy class.

We endow it with the finest⁶ topology whose restriction to each orbit :

$$G(K) \cdot P_I \cong (G/P_I)(K)$$

is a homeomorphism.

This definition mimics the one of [SS91, p. 66], and does not depend on the choice of B , nor S . An equivalent description is as follows.

Remark B.2.4. The set $\mathcal{P}$ can decomposes into $G(K)$ -orbits as :

$$\mathcal{P} = \bigsqcup_{i \in \Delta} (G/P_{\Delta \setminus \{i\}})(K) \cdot P_{\Delta \setminus \{i\}}.$$

This defines a profinite topology on $\mathcal{P}$. For any $n \geq 0$, there is a natural injection $\Delta'_n(G) \rightarrow \mathcal{P}^{n+1}$ defined by lemma B.2.1, which is $G(K)$ -equivariant by construction of the order.

This extends to an equivariant injection on $\Delta_n(G)$ by the following :⁷

$$\Delta_n(G) \simeq \bigsqcup_{0 \leq k \leq n} \Delta'_k(G) \hookrightarrow \bigsqcup_{0 \leq k \leq n} \mathcal{P}^k \xrightarrow{(\delta_{n-k} \times id_{\mathcal{P}^k})} \mathcal{P}^n,$$

where the leftmost isomorphism removes redundancies, and $\delta_{n-k} : \mathcal{P} \rightarrow \mathcal{P}^{n-k}$ is the diagonal morphism.

The topology on $\Delta_n(G)$ of definition B.2.3 makes each $\Delta_n(G)$ a closed subset of $\mathcal{P}^n$.

We may now define an alternate topology on $|\Delta(G)|$.

Definition B.2.5. We let $|\Delta(G)|^{pro}$ be the geometric realization of the simplicial profinite space $\Delta(G)$, as defined in definition B.1.1.

In the rest of this section, we show that the topology $|\Delta(G)|^{pro}$ coincides with a more classical (and, in some sense, more tractable) one, due to Borel-Serre [BS76, Sec 1.3]. Let us recall their definition.

Definition B.2.6. Let $|\Delta_B| \subset |\Delta(G)|$ be the geometric realization of the simplex corresponding to the B . There is a natural surjection :

$$G(K) \times |\Delta_B| \rightarrow \Delta'_r(G) \times |\Delta_B| \rightarrow |\Delta(G)|,$$

where the first morphism is given by $(g, v) \mapsto (gBg^{-1}, v)$, and the second one is from remark B.1.9. As elements of $B(K)$ stabilize the action, this morphism factors through $(G/B)(K) \times |\Delta_B|$.

We endow $|\Delta(G)|$ with the quotient topology from $(G/B)(K) \times |\Delta_B|$, denoted $|\Delta(G)|^{BS}$.

This topology does not depend on the choice of B .

Example B.2.7. If $G = \mathrm{GL}_2(K)$, with maximal torus given by diagonal matrices and minimal parabolic subgroup given by upper triangular matrices, Δ contains one element, and $|\Delta(G)| \cong \mathbb{P}^1(K)$.

While $|\widetilde{\Delta(G)}|$ is endowed with the discrete topology, $|\Delta(G)|^{pro}$ and $|\Delta(G)|^{BS}$ correspond to the topology on $\mathbb{P}^1(K)$ induced from the one on K .

The original motivation for $|\Delta(G)|^{BS}$ in [BS76] is the following.

⁶This admits the following explicit description : An subset $U \subset \Delta_n(G)$ is open if and only if its intersection with each orbit is open in $(G/P_I)(K)$.

⁷Outside of $\Delta_n(G)$, one needs to pick $G(K)$ -equivariantly a convention for repeating elements of $\mathcal{P}$. We arbitrarily chose to repeat the smallest one, for the order on $\mathcal{P}$.

Remark B.2.8. The set $|\Delta(G)|$ corresponds to the boundary at infinity of the Bruhat-Tits building of G , that we denote here $BT(G)$. The topology $|\Delta(G)|^{BS}$ was introduced so that $BT(G) \sqcup |\Delta(G)|^{BS}$ is a suitable compactification⁸ of $BT(G)$ (cf. [BS76, Thm 5.4]), used to compute the compactly supported cohomology of $BT(G)$.

For $G = \mathrm{GL}_2(\mathbb{Q}_p)$, the Bruhat-Tits building is the regular tree of degree $p + 1$, whose ends indeed correspond to $\mathbb{P}^1(\mathbb{Q}_p)$.

We now prove that, in general, the Borel-Serre topology coincides with $|\Delta(G)|^{pro}$.

Proposition B.2.9. The identity map induces an homeomorphism :

$$|\Delta(G)|^{pro} \cong |\Delta(G)|^{BS}.$$

Proof. We apply [proposition B.1.8](#), noting that $\Delta(G)$ is of dimension r , and that $\Delta'_r(G) \cong (G/B)(K)$. Hence :

$$|\Delta(G)|^{pro} \cong \left((G/B)(K) \times \Delta^r \sqcup \bigsqcup_{0 \leq n < r} |\Delta'_n(G)| \times \Delta^n \right) / \sim,$$

where the equivalence relation is the restriction from the one in [definition B.1.1](#). We have the following commutative diagram :

$$\begin{array}{ccc} (G/B)(K) \times \Delta^r & \xrightarrow{f} & \left((G/B)(K) \times \Delta^r \sqcup \bigsqcup_{0 \leq n < r} |\Delta'_n(G)| \times \Delta^n \right) \\ \downarrow & & \downarrow \\ |\Delta(G)|^{BS} & \xrightarrow{id} & |\Delta(G)|^{pro}, \end{array}$$

where vertical arrows are quotient maps. The map f is continuous, hence so is id . $|\Delta(G)|^{BS}$ is compact (as the quotient of a compact set), and $|\Delta(G)|^{pro}$ is Hausdorff by [\[SP13\]](#).

The morphism id is then a continuous closed bijection, hence a homeomorphism. $\square$

Remark B.2.10. The same argument shows that $|\widetilde{\Delta(G)}|$ admits the quotient topology from $(G/B)(K)^{disc} \times |\Delta^r|$, where $(G/B)(K)^{disc}$ is endowed with the discrete topology.

In [\[BS76, Corollary 3.3\]](#), the authors prove that the sheaf cohomology of $|\Delta(G)|^{BS}$ is computed by the complex of locally constant functions on the $\Delta_n(G)$. Using the identification of [proposition B.2.9](#), this reduces to [proposition B.1.3](#).

In the next section, we introduce a notion of geodesics on $|\Delta(G)|^{BS}$, yielding to a notion of convex subsets. We will need to introduce another point of view on $|\Delta(G)|$, focused on the notion of apartment.

B.3 The Curtis-Lehrer-Tits version of the building

We keep the notations of the above section. Let us review a variant of the building $|\Delta(G)|$, as constructed by C.Curtier, G.Lehrer and J.Tits in [\[CLT80\]](#). It is a *continuous* version of the building constructed by Mumford in [\[MFK94\]](#), and already appeared in the short note [\[Rou78\]](#).

⁸This is far from being the only reasonable one !

The Curtier-Lehrer-Tits building⁹ is a topological space is in bijection to the geometric realization $|\Delta(G)|$ of the combinatorial Tits building considered above. Rather than focusing on a simplicial structure, it is built from gluing together *apartments*, which are homeomorphic to Euclidean spheres.

For any maximal K -split torus S of G , we let :

$$\mathcal{B}(S) := (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}) / \mathbb{R}_+^*$$

be the set of half-lines in the space of real cocharacters on S (Here, $X_*(S)$ is the set of morphisms $\mathbb{G}_m \rightarrow S$ over K , as in [definition 7.3.11](#)).

$\mathcal{B}(S)$ is naturally homeomorphic to the Euclidean sphere of dimension $r - 1$.

For any maximal split torus S , the sphere $\mathcal{B}(S)$ can be decomposed as a *Coxeter complex*, with chambers corresponding to the parabolic subgroups of G containing S . Hence, to each $b \in \mathcal{B}(S)$, we may associate a parabolic subgroup $P(b)$ of G , corresponding to the chamber containing b (cf. [\[CLT80, Section 1\]](#)).

We may now define the Curtis-Lehrer-Tits building of G as :

$$\mathcal{B}(G) := \left(\bigsqcup_S \mathcal{B}(S) \right) / \sim, \quad (\text{B.1})$$

where the union is taken on all maximal K -split tori of G , and the equivalence relation is given by $b \sim b' \iff \exists g \in P(b)(K), b' = g \cdot b \cdot g^{-1}$. It admits a natural $G(K)$ -action by conjugation. We may extend the mapping $b \mapsto P(b)$ in an equivariant manner, to a surjective mapping $\mathcal{B}(G) \rightarrow \{\text{Parabolic subgroups of } G\}$.

For any such S , the natural mapping $\mathcal{B}(S) \rightarrow \mathcal{B}(G)$ is injective by [\[CLT80, Lemma 2.2.ii\]](#), and its image is referred to as an **apartment** of $\mathcal{B}(G)$.

The building $\mathcal{B}(G)$ is related to $|\Delta(G)|$ by the following ([\[CLT80, Prop 6.1\]](#)) :

Proposition B.3.1. *Assume that G is semi-simple. Then, there exists a (non canonical) $G(K)$ -equivariant bijection :*

$$\tau : |\Delta(G)| \simeq \mathcal{B}(G)$$

such that, for all $x \in |\Delta(G)|$, the parabolic subgroup $P(\tau(x))$ is the one associated to the unique simplex containing x in its interior. Moreover, for all maximal K -split torus S , the restriction of τ on $|\Delta_S|$ (which is the subcomplex of parabolic subgroups containing S) induces a triangulation of $\mathcal{B}(S)$.

For a general G , one has $|\Delta(G)| \simeq \mathcal{B}(G_{\text{der}})$, and $\mathcal{B}(G)$ is a suspension of $\mathcal{B}(G_{\text{der}})$, cf. [\[CLT80, Prop 7.2\]](#).

Proof. Let us recall the proof, as some elements will be used in the following.

For any parabolic subgroup P of G containing a maximal split torus S , it can be shown (cf. [\[CLT80, Sec. 1\]](#)) that the set $\mathcal{C}(P) := \{b \in \mathcal{B}(G), P \subset P(b)\}$ is a subset of $\mathcal{B}(S)$, corresponding to a simplex of a Coxeter complex, and is then homeomorphic (with the topology induced by the topology on the Euclidean sphere) to the simplex $|\Delta_P|$ of $|\widetilde{\Delta(G)}|$ associated to P .

Hence, fix a bijection $\tau_B : |\Delta_B| \cong \mathcal{C}(B)$.

We extend it $G(K)$ -equivariantly, in the following way : for all x in $|\Delta(G)|$, there exists some $g \in G(K)$ such that $gxg^{-1} \in |\Delta_B|$. We then let $\tau(x) = g^{-1} \cdot \tau_B(gxg^{-1}) \cdot g$. This is well defined (i.e. independent of the choice of g) by [\[CLT80, Prop 2.2.\(i\)\]](#). $\square$

⁹This terminology is non standard. Usually, both notions of buildings are identified, and apartments are freely defined as subsets of $|\Delta(G)|$. Here, since we are specifically interested in the topology $|\Delta(G)|^{BS}$, which is not the standard choice from the litterature, we try and keep the terminologies distinct.

The two following properties $\mathcal{B}(G)$ are crucial (cf. [Tit, Thm 5.2]).

- Proposition B.3.2.** 1. Any two points $x, y \in |\Delta(G)|$ lie in a common apartment (i.e. there exists a maximal K -split torus S such that $\tau(x), \tau(y) \in \mathcal{B}(S)$).
2. For any two x, y belonging in two apartments $\mathcal{B}(S)$ and $\mathcal{B}(S')$, there exists a $g \in G(K)$ such that $gSg^{-1} = S'$, stabilizing both x and x' .

Since $\mathcal{B}(G)$ is built together from Euclidean spheres, it is well-suited to define a notion of distance, and of convexity.

B.4 Convexity and contractibility

We present here are a spherical variant of [BT72, Section 2.5]. Some notions were present in [CLT80, Section 8].

We simply define the distance, and the notion of convexity, in any fixed apartment.

Definition B.4.1. Let $x, y \in \mathcal{B}(G)$. Let S such that $x, y \in \mathcal{B}(S)$.

We say that x, y are **antipodal** if they are antipodal in the sphere $\mathcal{B}(S)$.

If x, y are not antipodal, the **geodesic** linking x and y , denoted $[x, y] \in \mathcal{B}(G)$, is defined as the small arc of the unique great circle passing through x and y in $\mathcal{B}(S)$. This is independent of the choice of S , by [proposition B.3.2](#).

For $A \subset \mathcal{B}(G)$, we say that A is **convex**¹⁰ if, for all $x, y \in \mathcal{B}(G)$, x and y are not antipodal, and $[x, y] \subset A$.

We define the **angular distance** $d(x, y) \in [0, \pi]$ as the angle between x and y , taken in any apartment.

By construction, the mapping $(x, y) \mapsto d(x, y)$ is $G(K)$ -invariant.

Remark B.4.2. The topology induced on $\mathcal{B}(G)$ by the distance d is homotopy equivalent to the topology of $|\widetilde{\Delta(G)}|$ as the quotient of the disjoint union of Euclidean spheres as in [equation \(B.1\)](#) (cf. [CLT80, Section 8]).

The distance d is however not compatible with the topology induced by $|\Delta(G)|^{pro}$. Indeed, for $G = \mathrm{GL}_2$ over $\mathbb{Q}_p$, recall [example B.2.7](#). Any two distinct points are antipodal, hence of distance π .

We may now obtain contractibility results for convex subsets of $|\Delta(G)|$. For the topology $|\widetilde{\Delta(G)}|$, this is relatively straightforward.

Proposition B.4.3. Let $A \subset |\widetilde{\Delta(G)}|$ be a convex subset. Then A is contractible.

Proof. For any $s, s' \in \mathcal{B}(S)$ and $\lambda \in [0, 1]$, we let $\lambda \cdot s + (1 - \lambda) \cdot s'$ be the unique point of $[s, s']$ such that $d(s, \lambda \cdot s + (1 - \lambda) \cdot s') = \lambda \cdot d(s, s')$. It is well defined by [proposition B.3.2](#).

We will show that A is starred. Let $y \in A$. Define the contracting map :

$$H : (x, \lambda) \in A \times [0, 1] \mapsto \tau^{-1}(\lambda\tau(x) + (1 - \lambda)\tau(y)) \in A,$$

This is well defined by convexity of A . It defines a homotopy between id_A and the constant map valuing y , so that it suffices to prove that H is continuous.

¹⁰This notion is adapted from [MFK94, Ch 3, Def 2.10].

This follows from a spherical variant of [BT72, Lemma 2.5.15] (see also [CLT80, Lemma 8.2]). By projecting everything to an apartment, may assume that we are working inside a fixed $\mathcal{B}(S)$, for which the continuity is clear. $\square$

It is unclear whether contractibility holds for the topology $|\Delta(G)|^{pro}$; we saw in [example B.1.6](#) that this does not follow for free from the above. In particular, the contracting map H of the above proof need not be continuous for the topology induced by $|\Delta(G)|^{BS}$. The problem is that any open neighborhood of $1 \in (G/B)(K)$ intersects all cells of the Bruhat decomposition.

We however establish the following weaker version, which is enough for the application we have in mind (namely, [proposition 8.2.1](#)). This is the main result of this appendix.

Proposition B.4.4. *Let $\mu : \mathcal{P} \rightarrow \mathbb{R}$ be a function such that $F := \{P \in \mathcal{P}, \mu(P) < 0\}$ is closed¹¹ inside $\mathcal{P}$, for the topology of [remark B.2.4](#). We extend μ to a function $\tilde{\mu} : |\Delta(G)| \rightarrow \mathbb{R}$, that is affine on every simplex of $|\Delta(G)|$.*

Assume that $A := \{x \in |\Delta(G)|, \tilde{\mu}(x) < 0\}$ is convex, in the sense of [definition B.4.1](#).

Let X be the simplicial profinite subset of $\Delta(G)$ whose simplices are exactly the simplices of $|\Delta(G)|$ whose vertices lie in A .

Then, the reduced sheaf cohomology groups $\tilde{H}_{\text{Shf}}^*(|X|^{pro}, \mathbb{Z})$ are zero.

Proof. In order to simplify the notations, cohomology is implicitly sheaf cohomology. We let H_c^* denote compactly supported sheaf cohomology groups.

We adapt the proof strategy of [BS76, Thm 2.6], which computes the cohomology of the whole space $|\Delta(G)|^{pro}$ (It is zero outside of degree $r - 1$, where it is huge). Let us recall the notations used in loc. cit..

Let C be a chamber of $|\Delta(G)|$ such that $C \cap |X|^{pro} \neq \emptyset$, corresponding to a minimal parabolic subgroup B . Let T be a maximal K -split torus inside B . We view $|\Delta(G)|^{BS}$ as the quotient of $(G/B)(K) \times C$, and let π denote the quotient map. For $I \subset \Delta$, we let P_I be the associated family of standard parabolic subgroups, and, for any $I \subset \Delta$, we let C_I denote the face of C whose vertices correspond to the $P_{\{i\}}$, for $i \in I$.

Let Σ be the set of reflections alongside elements of Δ , generating W . We identify implicitly Σ and Δ , by associating to $\alpha \in \Delta$ the reflection $s_\alpha \in \Sigma$.

We let N be the cardinal of W , and order the elements of the Weyl group with nondecreasing length¹² (with respect to T), i.e., we let $W = \{w_1, \dots, w_N\}$ such that $i \leq j \implies l(w_i) \leq l(w_j)$.

For any $1 \leq m \leq N$, we define $I_m := \{s \in \Sigma, l(w_m \cdot s) > l(w_m)\} \subset \Delta$.

It satisfies $I_1 = \Sigma$, $I_N = \emptyset$ and $\emptyset \subsetneq I_k \subsetneq \Sigma$ whenever $1 \leq k \leq N - 1$ (cf. [Bou07, Sec. 1, Ex. 3, p. 37]).

For any $1 \leq m \leq N$, we let $L_m := \bigcup_{I \subset I_m} \overset{\circ}{C}_I$, such that $L_1 = C$ and $L_N = \emptyset$.

They can be written as the complement of some faces of codimension 1 in C :

$$L_m = C \setminus \bigcup_{s \in \Sigma \setminus I_m} C_{\Delta \setminus \{s\}}.$$

¹¹This hypothesis is probably not needed. If F is not closed, then, X , defined below, is not necessarily a simplicial profinite space, but one can still consider its geometric realization as a simplicial space.

¹²The length of an element $w \in W$ is defined as the minimal $q \in \mathbb{N}$ such that w writes as $w = s_1 \circ \dots \circ s_q$, for $s_1, \dots, s_q \in \Sigma$ (cf. [Bou07, Sec. 1, Defi. 1.1]).

Recall the Bruhat decomposition (cf. [Bor91, p. 14.12]) :

$$G(K) = \bigsqcup_{w \in W} B(K)wB(K), \quad \text{hence} \quad (G/B)(K) = \bigsqcup_{w \in W} (B(K)wB(K))/B(K).$$

For $w \in W$, we denote $C(w) := (B(K)wB(K))/B(K)$. For $1 \leq m \leq N$, we let :

$$E_m := \bigcup_{1 \leq k \leq m} C(w_k) \subset (G/B)(K) \text{ and } Y_m := \pi(E_m \times C).$$

The family E_m forms an increasing filtration of $(G/B)(K)$ by closed subsets, so that the Y_m form an increasing filtration of $|\Delta(G)|^{pro}$ by closed subsets. Moreover, every $C(w_k)$ is open inside E_k , so that every $C(w_k)$ is Hausdorff and locally compact.

From [BS76, Lemma 2.4], there is a homeomorphism $Y_m \setminus Y_{m-1} \simeq C(w_m) \times L_m$, where the reciprocal maps $(g, v) \in C(w_m) \times L_m$ to $gv g^{-1} \in Y_m \setminus Y_{m-1}$.

For $J \subset \Delta$, we let $D_J \subset (G/B)(K)$ defined by :

$$D_J(w) := \{g \in C(w) \text{ such that, } \forall i \in \Delta, gP_i g^{-1} \in F \iff i \notin J\},$$

endowed with the induced topology. It is closed inside an open subset of $C(w)$.

By construction, $Y_1 \cap |X|^{pro} = C \cap |X|^{pro}$, and, for all $2 \leq m \leq N$, there is a homeomorphism :

$$(Y_m \setminus Y_{m-1}) \cap |X|^{pro} \simeq \bigsqcup_{J \subset \Delta} D_J(w_m) \times (L_m \cap C_J). \quad (\text{B.2})$$

Moreover, we have :

$$L_m \cap C_J = C_J \setminus \left(\bigcup_{s \in \Sigma \setminus I_m} (C_{\Delta \setminus \{s\}} \cap C_J) \right) = C_J \setminus \left(\bigcup_{s \in \Sigma \setminus I_m} C_{J \cap (\Delta \setminus \{s\})} \right).$$

The topological space $\bigcup_{s \in \Sigma \setminus I_m} C_{J \cap (\Delta \setminus \{s\})}$ is contractible in all but the following cases :

1. $I_m = \emptyset$, i.e. $m = 1$.
2. $I_m = \Sigma$, i.e. $m = N$.
3. There exists $s \in \Sigma$ such that $J = \{s\}$ and $I_m = \Sigma \setminus \{s\}$.

Let us start by handling the third case.

Lemma B.4.5. *Let $2 \leq m \leq N - 1$, and assume that there exists $s \in \Sigma$ such that $J = \{s\}$ and $I_m = \Sigma \setminus \{s\}$.*

Then $D_J(w_m) = \emptyset$.

Proof. Assume by contradiction that there exists $g \in D_{\{s\}}(w_m)$ and write $g = bw_m b'$ for $b, b' \in B(K)$. Up to conjugating everything by b , we may assume that $b = 1$.

Let S be a maximal torus contained in both B and gBg^{-1} , such that $\mathcal{B}(S)$ contains all chambers $C_w = wCw^{-1}$ for $w \in W$. We freely identify $\mathcal{B}(S)$ with the union of these chambers, by the morphism τ from [proposition B.3.1](#).

Fix an arbitrary point $x \in C \cap |X|$. By definition of $D_{\{s\}}(w_n)$, the only vertex of the face $w_m \cdot C \cdot w_m^{-1}$ belonging in A is the vertex corresponding to $w_m \cdot P_{\Delta \setminus \{s\}} \cdot w_m^{-1}$. Since A is convex, the geodesic in $\mathcal{B}(T)$ joining x to the point corresponding to $w_m \cdot P_{\Delta \setminus \{s\}} \cdot w_m^{-1}$ is contained in A .

By the spherical version of [BT72, Prop 2.5.7], this geodesic is contained in a minimal gallery of $|\Delta(G)|$, so that :

- (i) It only visits chambers wCw^{-1} for $l(w) \leq l(w_m)$.
- (ii) It exits C_{w_m} by going to a neighboring chamber $C_{w \cdot s'}$ for some $s' \in \Sigma$ such that $l(w_m \cdot s') < l(w_m)$.

By (i), this geodesic is contained in Y_m . By construction, $s' \notin I_m$, so that, by hypothesis, $s' = s$.

In particular, this geodesic passes through the face corresponding to $w_m P_{\{s\}} w_m^{-1}$, and is contained in A . By definition of A , at least one of the vertices of the face $w_m P_{\{s\}} w_m^{-1}$ belongs to F .

This contradicts the hypothesis $g \in D_{\{s\}}(w_m)$. □

We may now establish the following.

Lemma B.4.6. *For any $2 \leq m \leq N - 1$ and $J \subset \Delta$:*

$$H_c^*(D_J(w_m) \times (L_m \cap C_J), \mathbb{Z}) = 0,$$

where H_c^* denotes compactly supported sheaf cohomology groups.

Proof. If $D_J(w_m) = \emptyset$, the result is trivial, so that, by lemma B.4.5, we may assume that $\bigcup_{s \in \Sigma \setminus I_m} C_{J \cap (\Delta \setminus \{s\})}$ is contractible.

Both $D_J(w_n)$ and $(L_m \cap C_J)$ are Hausdorff and locally compact topological spaces. Hence, by [KS90, Exercise II.18], we have a Künneth formula for compactly supported cohomology of sheaves¹³ :

$$R\Gamma_c(D_J(w_n) \times (L_m \cap C_J), \mathbb{Z}) = R\Gamma_c(D_J(w_n), \mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_c(L_m \cap C_J, \mathbb{Z}). \quad (\text{B.3})$$

As $L_m \cap C_J$ is open in C_J , we have a (localization) exact sequence, which follows from proper base change [KS90, Prop 2.6.7] (as in proposition A.2.1) :

$$R\Gamma_c(L_m \cap C_J, \mathbb{Z}) \rightarrow R\Gamma(C_J, \mathbb{Z}) \rightarrow R\Gamma\left(\bigcup_{s \in \Sigma \setminus I_m} (C_{\Delta \setminus \{s\}} \cap C_J), \mathbb{Z}\right). \quad (\text{B.4})$$

The space C_J is contractible, and so is $\left(\bigcup_{s \in \Sigma \setminus I_m} (C_{\Delta \setminus \{s\}} \cap C_J)\right)$ by lemma B.4.5.

They have trivial sheaf cohomology, so that $R\Gamma_c(L_m \cap C_J, \mathbb{Z})$ is trivial. The result follows from equation (B.3). □

It follows from lemma B.4.6 and equation (B.2) that, whenever $2 \leq m \leq N - 1$,

$$H_c^*((Y_m \setminus Y_{m-1}) \cap |X|^{pro}, \mathbb{Z}) = 0.$$

¹³Both this Künneth formula and the localization sequence below follow from a six-functor formalism for compact, locally Hausdorff spaces, cf. [Vol25] (see Prop 6.13, as well as Cor. 4.5, together with Lemma 6.5).

For any $2 \leq m \leq N-1$, the localization sequence from [equation \(B.4\)](#) associated to the pair $(Y_m \cap |X|^{pro}, Y_{m-1} \cap |X|^{pro})$, yields isomorphisms in reduced cohomology :

$$\tilde{H}^*(Y_m \cap |X|^{pro}, \mathbb{Z}) \simeq \tilde{H}^*(Y_{m-1} \cap |X|^{pro}, \mathbb{Z}).$$

The space $Y_1 \cap |X|^{pro}$ is contractible, so that its reduced cohomology is trivial. Hence, a quick induction on m shows that $\tilde{H}^*(Y_{N-1} \cap |X|^{pro}, \mathbb{Z}) = 0$.

The localization sequence of the pair $(Y_N \cap |X|^{pro}, Y_{N-1} \cap |X|^{pro})$ yields :

$$\tilde{H}^k(|X|^{pro}, \mathbb{Z}) \simeq \tilde{H}_c^k((Y_N \cap Y_{N-1}) \cap |X|^{pro}, \mathbb{Z}).$$

This reduces the proof of [proposition B.4.4](#) to the following :

Proposition B.4.7. $\tilde{H}_c^*((Y_N \setminus Y_{N-1}) \cap |X|^{pro}, \mathbb{Z}) = 0$.

Proof. By [equation \(B.2\)](#), we have $(Y_N \setminus Y_{N-1}) \cap |X|^{pro} \simeq \bigsqcup_{J \subset \Delta} D_J(w_N) \times (L_N \cap C_J)$.

Since $L_N = \overset{\circ}{C}$, the intersection $L_N \cap C_J$ is empty whenever $J \neq \emptyset$. Hence :

$$(Y_N \setminus Y_{N-1}) \cap |X|^{pro} \simeq D_{\emptyset}(w_N) \times \overset{\circ}{C}.$$

We will prove that $D_{\emptyset}(w_N) = \emptyset$, which concludes.

By definition, $D_{\emptyset}(w_N) = \{g \in C(w_N), \forall i \in \Delta, gP_i g^{-1} \in F\}$, which is non-empty if and only if the whole chamber $w_n C w_n^{-1}$ is contained in $|X|^{pro}$, hence in A .

However, the chamber $w_n C w_n^{-1}$ is the chamber opposed to C , and $C \cap F \neq \emptyset$. Since A is convex, it contains no pair of opposite points. Hence $D_{\emptyset}(w_N) = \emptyset$. □

This concludes the proof of [proposition B.4.4](#). □

Remark B.4.8. *It is unclear to me whether or not, in the above setup, $|X|^{pro}$ is contractible. It may be possible to iteratively construct a deformation retract between $Y_{m-1} \cap |X|^{pro}$ and $Y_m \cap |X|^{pro}$.*

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