Extension groups of period sheaves

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Motivation

The Fargues-Fontaine curve, introduced in [FF18], is a fundamental and ubiquitous object in p-adic Hodge theory. For example, it provides a nice framework to study some standard period rings [FF18], allows for a reformulation of local class field theory [LCF], and paves the way to a *geometrization* of the local Langlands correspondence [FS21].

In the absolute case, the Fargues-Fontaine curve is a (nice) scheme of dimension 1, whose closed points classify untilts of a perfectoid field. While it is not of finite type, it essentially behaves like a standard curve. In the relative case, the Fargues-Fontaine curve is not a scheme, but rather a p-adic analytic space. Nevertheless, it is expected to behave like a curve.

In order to explore this idea, one might try and establish analogues of standard duality results for vector bundles on the Fargues-Fontaine curve - which admit a very convenient Harder–Narasimhan filtration. In [ALB21], Arthur-César Le Bras and Johannes Anschütz develop a version of Serre's duality.

One key piece of the argument is a vanishing result of higher extension groups of some standard period sheaves, as introduced in [Sch13], on perfectoid sites in fixed finite characteristic, endowed with the v-topology.

The goal of this document is to explain how one can compute such extensions groups. Following [ALB21], we view this computation as a *analytification* of an old result by Lawrence Breen in [Bre81], of extension groups of étale sheaves in a fully algebraic setup.

In this document, we introduce the necessary technical tools to study extension groups on arbitrary sites, before specifying to an étale, then an analytic setup. We prove as precisely as possible, and using little prerequisite knowledge, the announced results.

Structure

In this document, we do not assume prior knowledge about period sheaves, adic geometry nor perfectoid objects. We, however, assume familiarity with scheme theory, elementary étale cohomology, homological algebra, some algebraic topology and category theory.

In section 1, we introduce standard notions that will be used throughout the paper. We assume that the reader is at least somewhat familiar with most of them.

In section 2, we introduce technical tools regarding cohomology in topoi, in order to prepare the reader for section 3 (and 5).

The section 3 is dedicated to the computation by L.Breen and J.Anschütz of the extension groups of the additive group scheme on the perfect étale site over a fixed affine base of characteristic p.

The section 4 aims to motivate and introduce the main concepts of adic and perfectoid geometry, as well as some useful period sheaves, motivated by the classifications of untilts.

In section 5, we explain, following [ALB21], how to deduce some results about extensions of period sheaves from the result by L.Breen and J.Anschütz. We then briefly discuss the importance of such results.

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Notations and conventions

Throughout the paper, we fix p a prime number and $q = p^f$ a power of p.

The terms *chain complex* and *cochain complex* will denote complexes with differentials of degree $+1$ and -1 respectively. Chain complexes are usually noted $C_\bullet$, while cochain complexes are noted $C^\bullet$.

Unless specified otherwise, rings are unitary and commutative.

1 Preliminaries

1.1 Sites, Sheaves and Topoi

Throughout this section, we assume familiarity with (1-)categories and sheaves on topological spaces.

1.1.1 Sites

Our main reference is the Stacks project [Sta22, Ch.7].

Informally, a "site" is a category endowed with a topology, which allows for the definition of an adequate notion of sheaves.

Definition 1.1.1. *A **site**[1] is a pair $(\mathcal{C}, J)$ consisting of a category $\mathcal{C}$ with fiber products equipped with a **Grothendieck pretopology** J which consists of, for each object U of $\mathcal{C}$, a collection of **covering families** $\{f_i : U_i \to U\}_{i \in I}$ such that :*

- *For all isomorphisms $f : V \cong U$ in $\mathcal{C}$, $\{f : V \to U\}$ is a covering family.*
- *If $\{f_i : U_i \to U\}_{i \in I}$ is a covering family and, for each i, so is $\{h_{i,j} : U_{i,j} \to U_i\}_{j \in J_i}$, then the family $\{f_i \circ h_{i,j} : U_{i,j} \to U\}_{i \in I, j \in U_i}$ is a covering family.*
- *If $\{f_i : U_i \to U\}_{i \in I}$ is a covering family and $g : V \to U$ is a morphism, then the family of pullbacks $\{g^* f_i : U_i \times_U V \to V\}_{i \in I}$ given by the diagram below is a covering family.*

$$\begin{array}{ccc} U_i \times_U V & \xrightarrow{g^* f_i} & V \\ \big\downarrow{\scriptstyle \pi} & \lrcorner & \big\downarrow{\scriptstyle g} \\ U_i & \xrightarrow{f_i} & U \end{array}$$

Historically, the notion of site was introduced to construct a proper theory of étale cohomology.

Example 1.1.2. *If X is a topological space, the category of its open sets $\mathrm{Op}(X)$, where morphisms reflect inclusions, is naturally endowed with a structure of a site in which covers are jointly surjective families, i.e. a family $\{f_i : U_i \to U\}_{i \in I}$ is a cover if $\mathrm{U} = \bigcup f_i(U_i)$.*

If S is a scheme, the big étale site of S is the category $(Sch/S)_{ét}$ of all $S-$schemes, in which covers are jointly surjective families of étale maps.[2]

As is the case on topological spaces, there is a natural notion of refinement.

Definition 1.1.3. *Let $(\mathcal{C}, J)$ be a site and $\{f_i : U_i \to U\}_{i \in I}$ and $\{g_j : V_j \to U\}_{j \in J}$ be covering families. We say that the first one **refines** the second one if there exists a map $\alpha : I \to J$ as well as morphisms $h_i : U_i \to V_{\alpha(i)}$ for every $i \in I$ such that the diagrams :*

$$\begin{array}{ccc} U_i & \xrightarrow{h_i} & V_{\alpha(i)} \\ & {\scriptstyle f_i}\searrow \quad \swarrow{\scriptstyle g_{\alpha(i)}} & \\ & U & \end{array}$$

commute for any $i \in I$.

[1] This terminology somewhat depends on the author. This definition is the *most restrictive* one, but every concrete site considered in this paper respects these conditions.

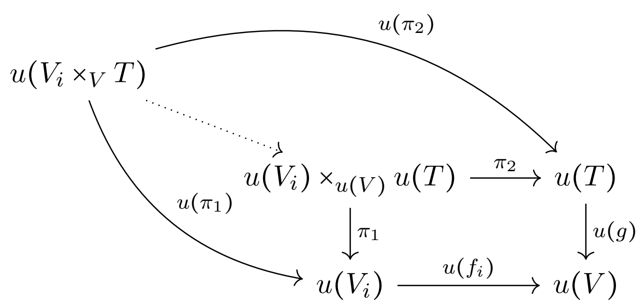
[2] Note that, if we replace "étale" with "open immersion", the obtained site is simply $\mathrm{Op}(S)$, with the Zariski topology

Remark 1.1.4. *Any two coverings* $\{f_i : U_i \to U\}_{i\in I}$ *and* $\{g_j : V_j \to U\}_{j\in J}$ *of some* U *admit a common refinement, given by the pullbacks* $\{h_{i,j} : U_i \times_U V_j \to U\}_{i,j}$.

We willingly do not define morphisms of site in order to avoid confusion. Indeed, the standard notion of a morphism of site $(\mathcal{C}, J) \to (\mathcal{D}, J')$ is a functor $\mathcal{D} \to \mathcal{C}$ satisfying certain conditions. We prefer talking about covariant functors between the underlying categories.

Definition 1.1.5. *Let* $(\mathcal{C}, J)$ *and* $(\mathcal{D}, J')$ *be sites.*
A functor $u : \mathcal{C} \to \mathcal{D}$ *is* ***continuous*** *if, for every covering family* $\{f_i : V_i \to V\}_{i\in I}$ *in* $\mathcal{C}$ *:*

1. *The family* $\{u(f_i) : u(V_i) \to u(V)\}_{i\in I}$ *is a covering family in* $\mathcal{D}$.
2. *For any morphism* $g : T \to V$ *in* $\mathcal{C}$, *the morphism* $u(V_i \times_V T) \to u(V_i) \times_{u(V)} u(T)$ *induced by the universal property of the fiber product :*



is an isomorphism.

The terminology is justified by the fact that a map between topological spaces $f : X \to Y$ is continuous if and only if the associated pre-image functor $f^* : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ is continuous.

We conclude this section with a more technical notion :

Definition 1.1.6. *Let* $(\mathcal{C}, J)$ *and* $(\mathcal{D}, J')$ *be sites.*
A functor $\mathcal{C} \to \mathcal{D}$ *is* ***cocontinuous*** *if, for every* $U \in Ob(\mathcal{C})$ *and every covering family* $\{g_j : V_j \to u(U)\}_{j\in J}$ *in* $\mathcal{D}$, *there exists a covering family* $\{f_i : U_i \to U\}_{i\in I}$ *of* $\mathcal{C}$ *such that the family*[3] $\{u(f_i) : u(U_i) \to u(U)\}_{i\in I}$ *refines* $\{g_j : V_j \to u(U)\}_{j\in J}$.

1.1.2 Sheaves

Our main reference is still the Stacks project [Sta22, Ch.7].
In this section, we define sheaves on sites.

Definition 1.1.7. *Let* $\mathcal{C}$, $\mathcal{D}$ *be two categories.*
A ***presheaf*** *on* $\mathcal{C}$ *valued in* $\mathcal{D}$ *is a functor* $\mathcal{C}^{op} \to \mathcal{D}$, *where* $\mathcal{C}^{op}$ *denotes the opposite category of* $\mathcal{C}$.
We let $\mathrm{Psh}(\mathcal{C}, \mathcal{D})$ *denote the category of such presheaves, with morphisms given by natural transformations.*
Until specified otherwise, the term "presheaf" denotes Set*-valued presheaves. We let* $\mathrm{Psh}(\mathcal{C}) \coloneqq \mathrm{Psh}(\mathcal{C}, \mathrm{Set})$.

If X is a topological space, presheaves on $\mathrm{Op}(X)$ are exactly presheaves over X, in the usual sense.

An important class of presheaves is given by the representable ones :

Definition 1.1.8. *Any object* $X \in \mathcal{C}$ *induces a presheaf, given by* $h^X = \mathrm{Hom}_{\mathcal{C}}(\_, X)$. *Such a presheaf is said to be* ***representable****.*

[3] It needs not be an covering family

For example, the functor of points of an S−scheme X defines a presheaf on the étale site of S.

At this point, it is a legal obligation to mention Yoneda's lemma.

Lemma 1.1.9. *(Yoneda) If G is a presheaf on a category $\mathcal{C}$, and A an object of $\mathcal{C}$, there is a canonical isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h^A, G) \cong G(A)$.*
In particular, the embedding $\mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is fully faithful.

We may now define the gluing condition for presheaves to be sheaves. This mimics the topological case, except that intersections are replaced by fiber products.

Definition 1.1.10. *Let $(\mathcal{C}, J)$ be a site and $\mathcal{D}$ be a category with products.*[4]
*A **sheaf** on $\mathcal{C}$ valued in $\mathcal{D}$ is a presheaf $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ such that, for every covering family $\{f_i : U_i \to U\}_{i\in I}$ in $\mathcal{C}$, the diagram :*

$$\mathcal{F}(U) \xrightarrow{\prod \mathcal{F}(f_i)} \prod_{i\in I} \mathcal{F}(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

identifies the image of first arrow with the equalizer of pr_0^* and pr_1^*.
Here, pr_0^* is induced by the inclusions $U_i \to U_i \times_U U_j$ and pr_1^* is induced by the inclusions $U_i \to U_k \times_U U_i$.
We let $\mathrm{Sh}((\mathcal{C}, J), \mathcal{D})$ be the category of $\mathcal{D}$-valued sheaves on $\mathcal{C}$, seen as a full subcategory of $\mathrm{PSh}(C, \mathcal{D})$.
When the coverage is clear, we simply write $\mathrm{Sh}(\mathcal{C}, \mathcal{D})$.

If X is a topological space, we note $\mathrm{Sh}(X, \mathcal{D}) := \mathrm{Sh}(\mathrm{Op}(X), \mathcal{D})$. This is the usual notion of sheaves.
For short, we let $\mathrm{Sh}(\mathcal{C}) := \mathrm{Sh}(\mathcal{C}, \mathrm{Set})$.

Remark 1.1.11. *If $\mathcal{A}$ is an abelian category with products, the sheaf condition for functors $\mathcal{C}^{op} \to \mathcal{A}$ can be rewritten by asking the sequence :*

$$\mathcal{F}(U) \xrightarrow{\prod \mathcal{F}(f_i)} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{pr_0^* - pr_1^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

to be exact.

Definition 1.1.12. *A site (C, J) is said to be **subcanonical** if every representable presheaf is a sheaf.*

While every site considered in this paper will be subcanonical, this is far from being an automatic condition.

As in the topological case, there is a canonical process of "sheafification". In practice, in order to define a sheaf on a site, it will be enough to define a presheaf.

Proposition 1.1.13. *Let $(\mathcal{C}, J)$ be a site, and $\mathcal{D}$ be a category with products.*
*The forgetful functor $\mathrm{Sh}(C, \mathcal{D}) \to \mathrm{Psh}(C, \mathcal{D})$ admits a left adjoint, the **sheafification**, noted $\mathcal{F} \mapsto \mathcal{F}^\sharp$.*

Proof. cf. [Sta22, Proposition 7.10.12] □

1.1.3 Topos

This sections is mainly based on the Stacks project [Sta22, Ch.7] and SGA IV [GV72], Chapter 4.

When considering topoi, set-theoretic issues appear quite quickly. For our purposes, we will ignore them, and treat every category as if it were small (which is somewhat true if one accepts the existence of a strongly inaccessible cardinal[5]).

[4] One can define presheaves valued in category without products - see [Sta22, Definition 00VR]
[5] Such a statement is independant of ZFC

Definition 1.1.14. *A* ***topos*** *is a category $\mathcal{T}$ that is isomorphic to a category of* Set*-valued sheaves* $\mathrm{Sh}(\mathcal{C}, J)$ *on some site $(\mathcal{C}, J)$, called the site of definition.*

Note that a topos admits many good categorical properties, including the existence of all finite limits and colimits. The simplest topos is Set, seen as the category of sheaves on a point.

There is an intrinsic definition of a topos via categorical axioms, due to Giraud. Since the site defining a topos is not canonically defined, we will to avoid referencing it whenever possible.

Definition 1.1.15. *Let $\mathcal{T}$ and $\mathcal{R}'$ be topoi.*
A ***geometric morphism of topos*** *$\mathcal{T} \to \mathcal{T}'$ is a pair of functors (u^*, u_*), such that $u^* : \mathcal{T}' \to \mathcal{T}$*[6] *and $u_* : \mathcal{T} \to \mathcal{T}'$, satisfy :*

- *u^* commute with finite limits.*
- *u^* is left-adjoint to u_*.*

We call u_* the ***push-forward****, and u^* the* ***pullback****.*

The notation (u^*, u_*) is somewhat unfortunate, since the first morphism u^* is contravariant.
For psychological reasons, we prefer writing the morphisms in order of the adjunction - the convention depends on the authors.

Example 1.1.16. *Every continuous map between topological spaces $f : X \to Y$ induces a morphism of topos $(f^*, f_*) : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, where :*

- *$f_*\mathcal{F} = \mathcal{F} \circ f^{-1}$*
- *$f^*\mathcal{G}$ is the sheafification of the presheaf $V \mapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V)$*

We would like to define cohomology of sheaves on a site. In order to do so, we need to consider sheaves valued in modules, which *a priori* might depend on the choice of a site of definition. However, there is a way to internalize the definition in the topos, using ring and module objects.

Definition 1.1.17. *Let $\mathcal{C}$ be a category with finite products, and an initial object $0_{\mathcal{C}}$.*
A ***ring object*** *in $\mathcal{C}$ is the data of an object R of $\mathcal{C}$ together with morphisms $(+_R, -_R, \times_R, 1_R)$, where $+_R : R \times R \to R$, $-_R : R \to R$, $\times_R : R \times R \to R$, $0_R : 0_{\mathcal{C}} \to R$ and $1_R : 0_{\mathcal{C}} \to R$ such that the diagrams corresponding to the usual axioms of a ring commute.*

As one might expect, there are similar definitions of group objects, module objects over a ring object, ...
For topoi, such a construction coincides with the expected definition.

Proposition 1.1.18. *If $\mathcal{T}$ is the topos associated to a site $(\mathcal{C}, J)$, the category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Ring})$ *is isomorphic to the category of ring objects on $\mathcal{T}$.*

We conclude this section by providing two necessary conditions for functors between sites to induce a geometric morphism of topos. Both are pretty technical ; the first one will be used in 5.1.6, and the second will be used in 3.4.3.

Definition 1.1.19. *If (C, J) is a site, the topos $Sh(C, J)$ is naturally endowed with a site structure, called the* ***canonical coverage****, which is the largest subcanonical one.*

In this setup, it can be explicitely described, since coverings are exactly finite jointly surjective families.

[6]Some author reserve the notation u^* when rings are involved (such as morphisms of $\mathcal{O}_X$−modules on sheaves). We do not take such precautions

Lemma 1.1.20. *Let (C, J) and (D, J') be sites such that finite limits are representable in C, with a left-exact functor $F : C \to Sh(D, J')$ that maps covering families to covering families.*
Then F comes from a geometric morphism of topos $(f^*, f_*) : Sh(\mathcal{D}) \to Sh(\mathcal{C})$, i.e. $f^* \circ h^C = F$ where $h^C : C \to Sh(\mathcal{C})$ maps an object to the (sheafification of) the represented functor.

Proof. This follows from [GV72], proposition 4.9.4. □

Lemma 1.1.21. *Let $(\mathcal{C}, J)$ and $(\mathcal{D}, J')$ be sites, together with continuous functors $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to C$ such that u is cocontinuous and right adjoint to v.*
The functors $u^p : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ and $v^p : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ defined by $u^p(\mathcal{G})(U) = \mathcal{G}(u(V))$ and $v^p(\mathcal{F})(V) = \mathcal{F}(v(V))$ define a geometric morphism of topos $(v^p, u^p) : \mathrm{Sh}(\mathcal{D}, J') \to \mathrm{Sh}(\mathcal{C}, J)$.
Moreover, u^p is exact (i.e. commutes with finite projective and inductive limits).

Proof. By [Sta22, Lemma 7.13.2], since u, v are continuous, u^p and v^p map sheaves to sheaves.
By [Sta22, Lemma 7.19.3][7], v^p coincides with the functor ${}_pu$ defined at the beginning of [Sta22, Section 7.19], which is left adjoint to u^p by [Sta22, Lemma 7.19.2].
Moreover, by cocontinuity of u and [Sta22, Lemma 7.20.3], the functor $\mathcal{G} \mapsto (u^p)^\# \mathcal{G}$ is exact. But by what was said above, this is simply u^p. □

1.1.4 Cohomology, points and stalks

The main reference for this section is SGA IV [GV72], Chapters 4-5.

In order to define cohomology, we will fix a ring and work with modules over it.

Definition 1.1.22. *A **ringed topos** is a pair $(\mathcal{T}, \mathcal{R})$ where $\mathcal{T}$ is a topos and $\mathcal{R}$ is a ring object in $\mathcal{T}$.*
If $(\mathcal{T}, \mathcal{R})$ is a ringed topos, we let $\mathrm{Mod}_\mathcal{R}$ denote the category of $\mathcal{R}$-modules objects in $\mathcal{T}$.[8]
We let $\mathrm{Hom}_\mathcal{R}$ denote morphisms in that category.

Throughout the paper, we will use curved letters to denote rings in topoi.

As in the standard case, $\mathrm{Mod}_\mathcal{R}$ is an abelian category with enough injective objects. Moreover, the functors $\mathrm{Hom}_\mathcal{R}(A, \_)$ are left-exact as $\mathrm{Mod}_\mathcal{R} \longrightarrow \mathrm{Set}$. We note $\mathrm{Ext}^i_\mathcal{R}(A, \_)$ their right derived functors.

The construction of a free module can easily be generalized in that setup.

Definition 1.1.23. *Let $(\mathcal{T}, \mathcal{R})$ denote a ringed topos.*
The forgetful functor $\mathrm{Mod}_\mathcal{R} \to \mathcal{T}$ admits a left adjoint, noted $X \mapsto \mathcal{R}[X]$.
*We call $\mathcal{R}[X]$ the **free $\mathcal{R}$-module** generated by X.*

Proof. cf. [GV72], 4.11.3.3. □

We may finally define cohomology in ringed topoi. The following notation will justified when looking at the étale topos (or any topos of schemes over a fixed base), cf 3.3.4.

Definition 1.1.24. *Let $(\mathcal{T}, \mathcal{R})$ be a ringed topos, X an object of $\mathcal{T}$, and N an $\mathcal{R}-$module.*
*We define the **cohomology groups** as : $\mathrm{H}^q(X, N) = \mathrm{Ext}_\mathcal{R}(\mathcal{R}[X], N)$.*

The notation does not make explicit the base ring $\mathcal{R}$. One can show ([GV72], 5.3.5) that the result does not depend on the choice of such a $\mathcal{R}$.

Let us now study how ringed objects and cohomology behave with geometric morphisms of topoi.

Proposition 1.1.25. *If $(f^*, f_*) : \mathcal{T} \to \mathcal{T}'$ is a morphism of topos, then f^* commutes with finite projective limits and arbitrary inductive limits, while f_* commutes with arbitrary projective limits.*

[7] The morphisms u and v defined there are swapped

[8] This corresponds to sheaves of R-module on $(\mathcal{C}, J)$ for any choice of a site of definition

Proof. cf. [GV72], 4.3.1.2. □

The definitions of group, ring and module objects only involve products, which are finite projective limits. If one wanted to define some more exotic structures such as comodules, one would need to additionally use finite inductive limits. Hence, the pullback preserves all usual algebraic structures, while the push-forward preserves some of them.

This remark will be especially useful when we consider the notion of stalks of a topos.

Remark 1.1.26. *For any topos $\mathcal{T}$, there is an unique geometric morphism $(f_{\mathcal{T}}^*, f_{\mathcal{T},*}) : \mathcal{T} \to \mathrm{Set}$. For this reason,* Set *is often referred to as "the final topos".*

Proof. This relies on the fact that functors $\mathrm{Set} \to \mathcal{T}$ commuting with arbitrary colimits correspond to objects of $\mathcal{T}$, since such a functor is determined by its value on a singleton, and can be extended by colimits. We refer to [GV72], 4.4.3 for more details. □

Definition 1.1.27. *If $\mathcal{T}$ is a topos and I a set, the* ***constant object*** *$\underline{I}$ in $\mathcal{T}$ is the object $f_{\mathcal{T}}^*(I)$*
If I is a group (resp. ring, module, ...), then $\underline{I}$ is a group (resp. ring, module, ...) object in $\mathcal{T}$.

If $\mathcal{T}$ admits (C, J) as a site of definition, the constant object $\underline{I}$ corresponds to the sheafification of the constant presheaf valuing I.

Definition 1.1.28. *For $\mathcal{T}$ a topos, a* ***point*** *of $\mathcal{T}$ is a geometric morphism $\mathrm{Set} \to \mathcal{T}$.*
If $p = (p^*, p_*) : \mathrm{Set} \to \mathcal{T}$ is a point of $\mathcal{T}$, and X is any object of $\mathcal{T}$, we note $X_p := p^*(X)$ the ***stalk*** *of X at the point p.*

This notion extends the notion of stalks of sheaves.

Example 1.1.29. *If X is a sober topological space[9], then the points of $\mathrm{Sh}(X)$ are exactly points of X. The stalks of objects in $\mathrm{Sh}(X)$ correspond with the usual notion of a stalk of a sheaf.*

The stalk is *a priori* only a set, but, if X is endowed with a structure of group/ring/module object, the stalk becomes a group/ring/module.

Definition 1.1.30. *A topos $\mathcal{T}$ has* ***enough points*** *if isomorphisms can be tested on stalks, i.e, if morphisms $f : X \to Y$ in $\mathcal{T}$ are isomorphisms iff, for any point $p : \mathrm{Set} \to E$, the map induced on the stalks $f_p : E_p \to F_p$ is an isomorphism.*

While every topos encountered in this paper has enough points, this is not true in general. The following example will be crucial.

Example 1.1.31. *Let X be a scheme. The étale topos $\mathrm{Sh}((\mathrm{Sch}/X), J_{\acute{e}t})$ has enough points, and points are given by morphisms $Spec(k) \to X$, for k an algebraically closed field.*

1.2 Homological algebra

Throughout this section, we assume that the reader is familiar with the construction of derived functors in abelian categories via injective or projective resolutions. We do not assume familiarity with the vocabulary of model categories.

We fix $\mathcal{A}$ an abelian category. Unless specified otherwise, objects and morphisms live in $\mathcal{A}$.
We develop the theory for cochain complexes.

[9] This is the case whenever X is Haussdorf

1.2.1 Spectral sequences

This section is based on [Hat00] *and* [McC00].

Spectral sequences are a very useful technical tool for computing (co)homology groups, based on the (co)homology of simpler yet related objects. We define the Grothendieck spectral sequence as well as the spectral sequence associated to a double complex.

We will only need spectral sequence in nice cases, where everything lives in positive degree, and hence converges. The definition given below is therefore quite restrictive.

Definition 1.2.1. *A (cohomological, bigraded, first quadrant)* ***spectral sequence*** *is a sequence* $\{E_r^{p,q}, d_r^{p,q}\}_{p,q,r\geq 0}$ *of objects* $E_r^{p,q}$ *and morphisms* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+r-1}$ *in* $\mathcal{A}$ *such that, for every* $r \geq 0$ *:*

1. $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0.$
2. $E_{r+1}^{p,q} = Ker(d_r^{p,q}) \big/ Im(d_r^{p-r,q-r+1})$

where, outside of the desired range, objects and morphisms are chosen to be zero by convention.
The elements $(E^{p,q})_{p,q\geq 0}$ *for any fixed* r *form the* ***r-th page****.*

The page E_{r+1} is hence computed as the cohomology of the r-th page with respect to the differentials d_r.

Definition 1.2.2. *Given a spectral sequence* $E_r^{p,q}$*, the sequence* $(E_r^{p,q})_{r\geq 0}$ *is stationary*[10] *as* $r \to \infty$*.*
We note $E_\infty^{p,q}$ *its limit.*
We say that a spectral sequence ***converges*** *to a family* $(F^k E^l)_{k,l\geq 0}$ *if*

1. *For every* $l \geq 0$*, the family* $(F^k E^l)_{k\geq 0}$ *is a decreasing family of subsets of* E^l *such that* $\bigcap_k F^k E^l = 0$ *and* $F^0 E^l = E^l$
2. *There are isomorphisms* $E_\infty^{p,q} \cong F^p E^{p+q} / F^{p+1} E^{p+q}$ *for any* $p, q \geq 0$*.*

When this is the case, we write $E_r^{p,q} \implies E^{p+q}$*. The filtration is often implicit, and the choice of an appropriate filtration does not matter in most cases.*
We say that a spectral sequence ***degenerates*** *at some degree* r_0 *if, for any* $r \geq r_0$ *and* $p, q \geq 0$*,* $E_r^{p,q} = E_{r_0}^{p,q}$*.*
A ***morphism*** *of spectral sequences* $f : (E_r^{p,q}, d_r^{p,q})_{p,q,r\geq 0} \to (F_r^{p,q}, d_r'^{p,q})_{p,q,r\geq 0}$ *is a family of morphisms* $f_r^{p,q} : E_r^{p,q} \to F_r^{p,q}$ *commuting with the differentials.*

Sadly, when a spectral sequence converges, there is no easy general way to compute its limit using E_∞. In special cases, for example when E_∞ is concentrated in only a few (1 or 2) columns or lines, one can compute the limit - at least up to extensions. We refer the reader to [McC00], 5.2 for more detail.
We now turn our attention to some classical ways to construct spectral sequences.

Definition 1.2.3. *A* ***double (cochain) complex*** *is a family of objects* $(C^{p,q})_{p,q\geq 0}$ *together with differentials* $d_1^{p,q} : C^{p,q} \to C^{p+1,q}$ *and* $d_2^{p,q} : C^{p,q} \to C^{p,q+1}$ *such that, for any* $p, q \geq 0$ *:*[11]

$$d_1^{p+1,q} \circ d_1^{p,q} = 0,\ d_2^{p,q+1} \circ d_2^{p,q} = 0 \text{ and } d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$$

The associated ***total complex*** *is the cochain complex* $Tot^\bullet(C)$ *given by*

$$Tot^n(C) = \bigoplus_{p+q=n} C^{p,q}$$

with differentials $d^n = \sum_{p+q=n} (d_1^{p,q} + (-1)^p d_2^{p,q}) : Tot^n(C) \to Tot^{n+1}(C)$*.*

[10]This is only true because the sequences live in the first quadrant - and is essentially trivial
[11]Some authors require anti-commutation instead. In the case, one should replace the definition of the total differential by $d^n = \sum_{p+q=n} d_1^{p,q} - (-1)^p d_2^{p,q}$

Proposition 1.2.4. *Let $C^{\bullet,\bullet}$ be a double complex as above. Let H_I^* denote the "horizontal cohomology" (i.e. differentials are given by d_1), and H_{II}^* the vertical one (i.e. differentials are given by d_2).*
There are two spectral sequences :

1. *${}_IE$, given by ${}_IE_1^{p,q} = \mathrm{H}^q_{II}(C^{p,\bullet})$ and ${}_IE_2^{p,q} = \mathrm{H}^p_I(H^q_{II}C^{\bullet,\bullet})$*
2. *${}_{II}E$, given by ${}_{II}E_1^{p,q} = \mathrm{H}^q_I(C^{\bullet,p})$ and ${}_{II}E_2^{p,q} = \mathrm{H}^p_{II}(H^q_I C^{\bullet,\bullet})$*

Both converge to $\mathrm{H}^\bullet(Tot(C))$.

Proof. cf. [McC00], Theorem 2.15. □

Theorem 1.2.5. *(Grothendieck's spectral sequence) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, G is left exact and F maps injective objects to G-acyclic objects.*[12]
Then there is a spectral sequence

$$E_2^{p,q} = (R^pG)(R^qF(A)) \implies E_\infty^{p+q} = R^{p+q}(G \circ F)(A)$$

In particular, the above spectral sequences exists whenever F maps injective objects of $\mathcal{A}$ to injective objects of $\mathcal{D}$.

Remark 1.2.6. *We will mention the concept of transgression of a spectral sequence, which we do not want to expand on. We refer to [McC00], section 6.2 for an overview of the concept.*

1.2.2 Derived categories

This section is inspired by [Wei94] *and* [Mat18b].
It will be useful to define the general formalism of localization in categories, which mimics the usual localization in modules.

Definition 1.2.7. *If $\mathcal{C}$ is a category and W a class of morphisms in $\mathcal{C}$, the* ***localized category*** *of $\mathcal{C}$ with respect to W is, if it exists, the category $\mathcal{C}[W^{-1}]$ equipped with a canonical functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ such that functors $\mathcal{C}[W^{-1}] \to D$ are exactly functors $\mathcal{C} \to D$ sending every morphism of W to isomorphisms in $\mathcal{D}$.*

Up to set-theoretic issues, localized categories can be constructed by adjoining formal inverses to every morphism in W. In reality, localized category need not exist in general - we refer to [GZ67] for details. We will ignore such issues and pretend that localized categories always exist. In the contexts encountered below, there is, indeed, no problem.

If $\mathcal{A}$ is an abelian category, we will define its derived category $\mathcal{D}(A)$ as a nice technical framework to work with derived functors. Let us start with a few preliminary standard notions :

Definition 1.2.8. *Let $\mathcal{A}$ be an abelian category. We let $\mathrm{Ch}(\mathcal{A})$ be the category of cochain*[13] *complexes valued in $\mathcal{A}$, and $\mathrm{Ch}^+(\mathcal{A})$ be the category of cochain complex that vanish in negative degree.*

A morphism of complexes $f^* : C^\bullet \to D^\bullet$ consists of a family of morphisms $f^n : C^n \to D^n$ commuting with the differentials.

Two morphisms of cochain complexes $f^*, g^* : C^\bullet \to D^\bullet$ are ***chain homotopic*** *if there exist morphisms $h^n : C^n \to D^{n-1}$ such that for all n, $f^n - g^n = d_D^{n+1}h^n + h^{n+1}d_C^n$.*[14]

[12] i.e. $R^nG(F(A)) = 0$ for any object $A \in \mathcal{A}$ and $n > 0$

[13] The notation $\mathrm{CoCh}(\mathcal{A})$ is more standard. Since we will mostly use cochain complexes in this paper, it is more convenient to directly write $\mathrm{Ch}(\mathcal{A})$. The theory for chain complexes is similar

[14] We do not require the (h^n) to form a morphism of complex.

A morphism of cochain complexes $f^*: C^\bullet \to D^\bullet$ is a ***homotopy equivalence*** *if there exists $g^*: D^\bullet \to C^\bullet$ such that $f^* \circ g^*$ and $g^* \circ f^*$ are respectively chain homotopic to the identity maps $id_{D^\bullet}$ and $id_{C^\bullet}$.*

A map of cochain complexes $f^*: C^* \to D^*$ is a ***quasi-isomorphism*** *if all the induced maps in cohomology $H^n(f): H^n(C^\bullet) \to H^n(D^\bullet)$ are isomorphisms.*

For $k \in \mathbb{Z}$, we let the ***shift operator*** *$[k]: \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ be defined as $([k] \cdot C^\bullet)^n = C^{n+k}$, endowed with the alternating shifted differentials $d^n_{X[k]} = (-1)^k d^{n+k}$.*[15]

There is a natural embedding $\mathcal{A} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ that maps an object X to the complex valuing X concentrated in degree 0. This identifies $\mathcal{A}$ as a full subcategory of $\mathrm{Ch}(\mathcal{A})$. If we want to emphasise on this identification, we let $X[0]$ denote the associated complex.
We define $X[k] := X[0][k]$ the complex concentrated in degree $-k$ and valuing X.

Note that additive functors $\mathcal{A} \to \mathcal{B}$ can naturally be extended to functors $\mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$.

Remark 1.2.9. *By definition, every resolution $X \to A^\bullet$ induces a quasi-isomorphism between the complexes $X[0]$ and $A^\bullet$.*

We may now construct the derived category.

Definition 1.2.10. *Let $\mathcal{A}$ an abelian category.*
Its ***derived category****, noted $\mathcal{D}(\mathcal{A})$, is the localisation of $\mathrm{Ch}(\mathcal{A})$ with respect to quasi-isomorphisms.*
Likewise, we define $\mathcal{D}^+(\mathcal{A})$ as the localisation of $Ch^+(\mathcal{A})$ with respect to quasi-isomorphisms.

Ideally, we would like additive functors $F: \mathcal{A} \to \mathcal{B}$ to extend to $\widetilde{F}: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, in which the functors $\mathcal{A} \to \mathcal{B}$ associated with $\left(X \mapsto \widetilde{F}(X[k])\right)_{k \in \mathbb{Z}}$ coincide with the derived functors of F.
They are two difficulties with this approach :

1. It is not clear how functors $\mathcal{A} \to \mathcal{B}$ can be extended at the derived level $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ since they do not preserve quasi-isomorphisms in general.[16]
2. It is not clear how to describe morphisms in the derived category.

Both these problems can be somewhat tackled by looking at injective resolutions. Starting from now, we will assume that all our abelian categories have enough injective objects. We note $\mathrm{Inj}_{\mathcal{A}}$ the full subcategories formed by injective objects of $\mathcal{A}$.

Definition 1.2.11. *We define $K(\mathcal{A})$ the* ***quotient category***[17]*, whose objects are cochain complexes and whose morphisms are chain homotopy classes of morphisms of complexes.*
Likewise, let $K^+(\mathcal{A})$ be the quotient category of non-negative complexes.

By definition, isomorphisms in $K(A)$ are exactly homotopy equivalences.

The quotient category is more convenient then the derived category, since morphisms are explicit and functors can easily be extended.

Proposition 1.2.12. *Any additive functor $F: \mathcal{A} \to \mathcal{B}$ preserves chain homotopies, and thus can be extended to $K(F): K(\mathcal{A}) \to K(\mathcal{B})$.*

The quotient and derived categories are closely related, as illustrated by the next three results.

Proposition 1.2.13. *Let $X, Y \in \mathrm{Ch}(\mathcal{A})$ such that Y is bounded below and composed of injective objects. Then $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{K(\mathcal{A})}(X, Y)$.*

[15] The alternating facilitates the definition of cones and distinguished triangles. It will not play a major role in this paper.
[16] They do if they are exact, in which their derived functors are of little interest
[17] The term "quotient category" is not standard. It is sometimes called the "homotopy category of chain complexes", but this can sometimes be confusing

Proof. This is quite technical since it requires to dive into the construction of the localization. This is done in [Wei94], corrolary 10.4.7 □

Recall that the functors Ext are the derived functors of Hom. As announced, they admit a straightforward reformulation in the language of derived categories.

Corollary 1.2.14. *If X, Y are objects of $\mathcal{A}$ (and $\mathcal{A}$ has enough injectives), then*

$$\mathrm{Ext}^n(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[0], Y[n])$$

Proof. Fix $Y \to I^*$ an injective resolution, such that, via 1.2.9 :

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[0], Y[n]) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[0], I^\bullet[n]) \cong \mathrm{Hom}_{K(\mathcal{A})}(X[0], I^\bullet[n])$$

By definition, morphisms of complexes $X[0] \to I^\bullet[n]$ are exactly elements of $\mathrm{Hom}_{\mathcal{A}}(X, I^n)$ that are in the kernel of $\mathrm{Hom}_{\mathcal{A}}(X, I^n) \to \mathrm{Hom}_{\mathcal{A}}(X, I^{n+1})$. Moreover, one can check that chain homotopies of such morphisms of complexes are exactly given by the differentials of morphisms $X \to I^{n-1}$.

Hence we can identify $\mathrm{Hom}_{K(\mathcal{A})}(X[0], I^\bullet[n])$ with the cohomology of the complex $\mathrm{Hom}(X, I^\bullet)$, which is exactly $\mathrm{Ext}^n_{\mathcal{A}}(X,Y)$, by the standard theory of derived functors. □

Let us now give an alternative definition of arbitrary derived functors.

Proposition 1.2.15. *The natural functor $\iota : K^+(\mathrm{Inj}_{\mathcal{A}}) \to \mathcal{D}^+(\mathcal{A})$ exists and is an equivalence of categories.*

Proof. cf [Wei94], Theorem 10.4.8. □

Since every morphism of complexes that is chain homotopic to zero is a quasi-isomorphism, $\mathcal{D}(\mathcal{A})$ can also be realised as the localisation of $K(\mathcal{A})$ with respect to quasi-isomorphisms. It hence comes equipped with a structural functor $\pi_{\mathcal{A}} : K(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

The discussion above allows for a very convenient definition of derived functors.

Definition 1.2.16. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories. Assume that $\mathcal{A}$ has enough injective objects.*
Then F induces a ***total derived functor*** *$R^+F : \mathcal{D}(\mathcal{A})^+ \to \mathcal{D}^+(\mathcal{B})$ defined by*

$$R^+F = \pi_{\mathcal{B}} \circ K(F) \circ i^{-1}$$

The functors $R^iF(A^\bullet) := H^i(RF(A^\bullet))$ are the ***i-th right derived functors*** *of F.*

This construction coincides with the standard one, essentially by construction. However, it exhibits the fact that all the right derived functors can be *glued together* into a global object.

Notation 1.2.17. *We note* $\mathrm{RHom}(A,B) := \mathrm{RHom}(A, \_)(B)$.

While we will avoid using the RHom functor, it serves two purposes, as far as this paper is concerned. Firstly, it is a more modern and convenient way to rephrase the results of section 3 and 5 (and is overwhelmingly used in [ALB21]). Secondly, its construction has to be kept in mind when constructing the derived simplicial category in 2.3.5, and the hyper-ext functors in the paragraph below.

1.2.3 Hyper-derived functors

This section is inspired by [Ill71] *and* [Wei94].

We will extend the definition of $Ext_{\mathcal{A}}$ to an *hyperext* functor $\mathrm{Ext} : \mathrm{Ch}(\mathcal{A}) \times \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$, which coincides with $\mathrm{Ext}_{\mathcal{A}}$ when complexes are concentrated in degree 0.

Based on 1.2.14, it is natural to define :

Definition 1.2.18. *For $A, B \in \mathrm{Ch}(\mathcal{A})$, we let the n-th hyperext group* $\mathbf{Ext}^n(A,B) = Hom_{D(\mathcal{A})}(A, B[n])$.

As we will see, hyperext groups can be interpreted as derived functors of an *internal Hom object.*

Definition 1.2.19. *For $A_\bullet$ and $B^\bullet$ complexes*[18] *in $\mathcal{A}$, we let $\underline{\mathrm{Hom}}_{\mathcal{A}}(A,B)^\bullet$ be the complex defined by :*

- $\underline{\mathrm{Hom}}_{\mathcal{A}}(A,B)^n = \bigoplus_{n=p+q} \mathrm{Hom}(A_p, B^q) = \mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}})(A_\bullet, B^\bullet))^n$
- *The differentials are given by* $(\partial f)(v) = \partial(f(v)) - (-1)^{deg(f)} f(\partial(v))$

Remark 1.2.20. *It is an internal Hom object, i.e. there exists an adjunction*

$$\mathrm{Hom}_{Ch(\mathcal{A})}(A \otimes B, C) \cong \mathrm{Hom}_{Ch(\mathcal{A})}(A, \underline{\mathrm{Hom}}(B,C))$$

when $\otimes$ denotes the tensor product of complexes $(A^\bullet \otimes B_\bullet)^n = \bigoplus_{p+q=n} A^p \otimes B_{-q}$.

A pretty straightforward computation, somewhat similar to the one done in 1.2.2, yields

Proposition 1.2.21. $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, B[n]) = \mathrm{H}^n \underline{\mathrm{Hom}}_{\mathcal{A}}(A,B)$

Using injective resolutions, this allows us to identify hyper-ext functors as right derived functors of $\underline{\mathrm{Hom}}$.

Proposition 1.2.22. *When $A^\bullet$ is fixed, the construction above descends at the derived level to a functor $\underline{\mathrm{RHom}}(A^\bullet, \_) : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(Ab)$. For any $B \in Ch^+(\mathcal{A})$, we have :*

$$\mathbf{Ext}_{\mathcal{A}}(A, B[n]) = \mathrm{H}^n(\underline{\mathrm{RHom}}(A, \_)(B))$$

Proof. Pick an injective resolution of B (using a Cartan-Eilenberg resolution), and apply 1.2.13.
We refer to [Wei94], theorem 10.7.4. for details. □

When B is concentrated in degree 0, this yields an explicit way to compute hyper-ext.

Corollary 1.2.23. *Let $A_\bullet$ a complex in $\mathcal{A}$, B an object of $\mathcal{A}$ and $B \to I^\bullet$ an injective resolution. Then $\mathbf{Ext}(A_\bullet, B)$ can be computed as the homology of the complex $\underline{\mathrm{Hom}}(A, I)^\bullet$.*

Proof. Recall that $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, I^\bullet)$ by 1.2.9.
The result now follows from 1.2.21 and 1.2.13. □

One good way to compute hyper-ext groups is to use the following spectral sequence :

Proposition 1.2.24. *(Universal coefficient spectral sequence)*
Let, $X_\bullet$ be a chain complex in $\mathcal{A}$ and B an object of $\mathcal{A}$. There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{A}}(H_q(X_\bullet), B) \implies \mathbf{Ext}^{p+q}_{\mathcal{A}}(X_\bullet, B)$$

Proof. Choose an injective resolution $B \to I^*$. Consider the double complex defined by

$$K^{p,q} = \mathrm{Hom}(X_p, I^q)$$

with differentials given by :

[18] As the notation suggests, $A_\bullet$ is a chain complex and $B^\bullet$ is a cochain complex

- $d_1^{p,q} : \mathrm{Hom}(X_p, I^q) \to \mathrm{Hom}_{\mathcal{A}}(X_{p+1}, I^q)$ induced by $d_{p+1} : X_{p+1} \to X_p$
- $d_2^{p,q} : \mathrm{Hom}_{\mathcal{A}}(X_p, I^q) \to \mathrm{Hom}_{\mathcal{A}}(X_p, I^{q+1})$ induced by $\partial^q : I^q \to I^{q+1}$

such that $\mathrm{Tot}_{\mathcal{A}}(K^{p,q})^\bullet = \underline{\mathrm{Hom}}_{\mathcal{A}}(X, I)^\bullet$.

Note that $H_I^p(K^{\bullet,q}) = \mathrm{Hom}(H_p(X), I^q)$ and $H_{II}^q(H_I^p K^{\bullet,\bullet}) = \mathrm{Ext}^q(H_q(X), B)$.

The second spectral sequence given by 1.2.4 writes :

$$_{II}E_2^{p,q} = \mathrm{Ext}^p(\mathrm{H}_q(X), B) \implies \mathrm{H}^{p+q}\left(K^{tot}\right)$$

which yields the announced result. □

The name "universal coefficient spectral sequence" is justified by the following.

Remark 1.2.25. *In what follows, we'll define hypercohomology in a ringed topos $(\mathcal{T}, \mathcal{R})$ by a variant of $\mathbb{H}^i(X_\bullet, Y) = \mathrm{Ext}_{\mathcal{R}}(\mathcal{R}[X]_\bullet, Y)$.*[19]
If $\mathrm{H}^p(\mathcal{R}[X_i], Y) = 0$ for all $p > 0$, the spectral sequence above yields an hyper variant of the universal coefficient theorem :

$$\mathrm{Hom}_{\mathcal{R}}(H_q(X_\bullet, \mathcal{R}), B) \cong \mathbb{H}^q(X_\bullet, B)$$

for B an $\mathcal{R}-$module.

1.3 Commutative algebra

Recall that we assumed every ring to be commutative and unitary.

1.3.1 Perfect rings, perfect schemes

This section is based on [BGA16] *and* [BS15].

In what follows, we fix p a prime number, and R a ring of characteristic p. It admits a Frobenius endomorphism given by $F(x) = x^p$.

Definition 1.3.1. *We say that R is **perfect** if the Frobenius $F : R \to R$ is an isomorphism.*

Every ring admits a natural perfection.

Definition 1.3.2. *The **perfection** of R is defined by $R^{perf} = \varinjlim_{x \mapsto x^p} R$, where the limit is indexed by nonnegative integers. This construction is functorial, and $R \mapsto R^{perf}$ defines a functor*

$$\{ \text{ Rings of characteristic } p \ \} \to \{ \text{ Perfect rings of characteristic } p \ \}$$

that is left adjoint to the forgetful functor.

In R^{perf}, the translation map $(i, s) \mapsto (i+1, s)$ gives a *p-th root.*

Remark 1.3.3. *There is another notion of perfection, given by $R_{perf} = \varprojlim_{x \mapsto x^p} R$, which yields a right adjoint to the forgetful functor. This notion will not be used here.*
It is, however, closely related to the tilting functor, which will play a key role in sections 4 and 5.

Remark 1.3.4. *If R is an integral domain, we can explicitely write : $R^{perf} = R[\{r^{1/p^\infty}\}_{r \in R}]$, obtained by formally adjoining to R all p^n-th roots of every element $r \in R$ in some fixed algebraic closure $\overline{\mathrm{Frac}(R)}$.*

An important example is given by the perfection of polynomial rings.

[19] X will need to be a simplicial object

Example 1.3.5. *If R is a perfect ring,*

$$R[X_1, \ldots, X_n]^{perf} \cong R[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}] := \bigcup_{k \geq 0} R[X_1^{1/p^k}, \ldots, X_n^{1/p^k}]$$

We can define exact sequences in the category of rings by only looking at the underlying abelian groups. In that sense, we may say that a functor Ring → Ring is exact if it preserves exact sequences.

Lemma 1.3.6. *The perfection functor is exact.*

Proof. It is a general fact that filtered colimits are exact in categories of modules (this is Grothendieck's axiom Ab5). See for example [Kie06] for an elementary proof. □

Remark 1.3.7. *Since the perfection functor is exact and a left adjoint, it commutes with finite limits and arbitrary colimits. Note that it does not commute with arbitrary limits, since, in general,*

$$(R^{\mathbb{N}})^{perf} \ncong (R^{perf})^{\mathbb{N}}$$

For example, take $R = \mathbb{F}_p[x_1, x_2, \ldots]/\left(x_1^p, x_2^{p^2}, x_3^{p^3}, \ldots\right)$ (cf. [Ach20]).

We will now globalize this construction to schemes.

Definition 1.3.8. *Let X be a scheme over $\operatorname{Spec}(\mathbb{F}_p)$.*[20]
*The **absolute Frobenius** morphism, noted $F : X \to X$, is defined as :*

- *The identity $id_{|X|}$ on the underlying topological spaces.*
- *The map of sheaf is induced by the Frobenius $\mathcal{O}_X \xrightarrow{x \mapsto x^p} \mathcal{O}_X$.*

*The scheme X is said to be **perfect** if the absolute Frobenius map is an isomorphism.*
*The **perfection** of X is defined as $X_{perf} = \varprojlim_F X$. It comes equipped with a morphism $X_{perf} \to X$.*
The perfection defines a functor that is right-adjoint to the forgetful one.

Note that this we use a limit rather than a colimit, since the *Spec* functor is contravariant.

Remark 1.3.9. *By construction,* $\operatorname{Spec}(A)_{perf} = \operatorname{Spec}(A^{perf})$

Remark 1.3.10. *The historic definition, as introduced in [Gre65], seems to be flawed. The author claimed that the presheaf of rings $\mathcal{O}_X^{perf}(U) := \mathcal{O}_X(U)^{perf}$ is a sheaf - this does not hold on a non quasi-compact base since the perfection functor does not commute with arbitrary products.*
See the MathOverflow post [Ach20] for a discussion on that topic.

We will use the fact that the perfection functor preserves some algebraic properties. We refer the reader to [BS15], lemma 3.4. for an extensive list.

Remark 1.3.11. *Let X, Y be schemes of characteristic p. In particular,*

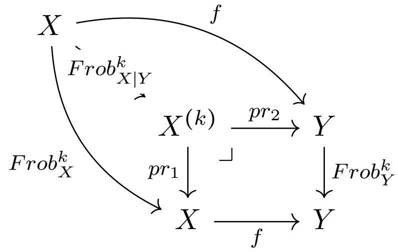
$$(X \times_S Y)_{perf} = X_{perf} \times_{S_{perf}} Y_{perf}$$

Proof. Right adjoints commute with projective limits. □

Proposition 1.3.12. *Let $f : X \to Y$ be an étale map between $\mathbb{F}_p$−schemes. Then the induced map $g : X_{perf} \to X \times_Y Y_{perf}$ is an isomorphism.*
In particular, the induced map $f_{perf} : X_{perf} \to Y_{perf}$ is étale.

[20] Also called a scheme of characteristic p

Proof. The morphism $g : X_{perf} \to X \times_Y Y_{perf} \cong \varprojlim_k X^{(k)}$ appears as the projective limit over k of the following diagram, where $X^{(k)}$ appears as the fiber product of the square below.



Hence, it suffices to show that the relative Frobenius $\mathrm{Frob}^k_{X|Y}$ is an isomorphism for any $k \geq 0$.
It is a universal homeomorphism, since its extension after any base change $Z \to X^{(k)}$ still is a relative Frobenius ; and hence induces the identity on the underlying topological spaces. In particular, it is affine and proper, hence finite, by [Sta22, Lemma 01WM].
It then suffices to show that it is an open immersion, which is local on target, so that we can assume X and $X^{(k)}$ to be affine. Note $X = \mathrm{Spec}(A)$ and $X^{(k)} = \mathrm{Spec}(B)$. Let $\varphi : B \to A$ be the étale integral ring morphism inducing $Frob^k_{X|Y}$. By [Sta22, Lemma 00NX], it makes B a locally free A−algebra . By locality, we can assume B to be a free A−algebra. Note $B \cong A^r$.
We pass the morphism through a geometric point $s = \mathrm{Spec}(\overline{\mathbb{F}_p})$, such that $\mathrm{Frob}^k_{X|Y}$ induces an homeomorphism $\mathrm{Spec}(\overline{\mathbb{F}_p}^r) \to \mathrm{Spec}(\overline{\mathbb{F}_p})$, so that $r = 1$. Hence the relative Frobenius is an isomorphism. This concludes. □

1.3.2 Witt vector rings

This section is inspired by the lecture notes [Mé19], §24.3, [RK22], §1.3.3, [Lur18], §3. *and* [Bej17], §2.

The theory of Witt vector rings plays a key role in this paper. They are, for example, involved in the classifications of untilts of a perfectoid field, and appears in the low-dimensional extensions of $\mathbb{G}_a$.

Let us start this section by a reminder of the notion of adic completion.

Definition 1.3.13. *Let I an ideal of a ring R. The I−adic topology on R is the unique structure of a topological ring on R such that the $(I^n)_{n\geq 1}$ form a basis of neighborhood of zero.*

In many algebraic contexts, it is the most reasonable topology one can consider.

Example 1.3.14.

1. *The usual topology on $\mathbb{Z}_p$ is the (p)−adic topology.*
2. *For R a topological ring, the (T)-adic topology on $R[\![T]\!]$ coincides with the product topology.*
3. *The (p,T)−adic topology on $\mathbb{Z}_p[\![T]\!]$ is the topology such that $(p^n)_{n\geq 0}$ and $(T^n)_{n\geq 0}$ converge to 0 at the same speed.*

The completion with respect to the I−adic topology can be described algebraically.

Definition 1.3.15. *Let I be an ideal of a ring R.*
The ***I-adic completion*** *of R with respect to I, noted $R^{\wedge I}$, is defined as :*

$$R^{\wedge I} = \varprojlim R/I^n$$

where transition morphisms are induced by the inclusion $I^{n+1} \subset I^n$.
The ring R is said to be ***I-adically complete*** *if the natural map $R \to R^{\wedge I}$.*
If I is generated by some $\pi \in R$, we usually note $R^{\wedge\pi} := R^{\wedge I}$.
Similarly, if M is an R−module, we let $M^{\wedge I} = \varprojlim M/I^n M$, and say that M is I-adically complete if the natural map $M \to M^{\wedge I}$ is an isomorphism of R-modules.

As announced, both notions of completion coincide.

Proposition 1.3.16. *$\hat{R}_I$ is the (topological) completion of R with respect to the $I-$adic topology.*

Let us give a few important examples.

Example 1.3.17.
1. *$\mathbb{Z}_p$ is $(p)-$adically complete, and $\mathbb{C}[\![T]\!]$ is $(T)-$adically complete.*
2. *The $(p,T)-$adic completion of $\mathbb{Z}_p[T]$ is $\mathbb{Z}_p\langle T\rangle \cong \{\sum_n x_n T^n \in \mathbb{Z}_p[\![T]\!], x_n \underset{n\to\infty}{\longrightarrow} 0\}$, called the ring of restricted power series over $\mathbb{Z}_p$.*

Let us now define the rings of Witt vectors, both unbounded and of finite length.
The idea of the construction is to mimic the construction of $\mathbb{Z}_p$, starting from $\mathbb{F}_p$. While there is a somewhat canonical set-theoretic bijection $\mathbb{F}_p^{\mathbb{N}} \cong \mathbb{Z}_p$, the ring structures are very different.
While the construction is quite obscure, over a perfect base, important properties are captured by 1.3.19.

Proposition–Definition 1.3.18. *Fix R a ring of characteristic p. Denote $W(R)$ the set $R^{\mathbb{N}}$.*
Define the map $\omega : W(R) \to W(R)$ by $\omega(x)_n = \sum_{k=0}^n p^k x_k^{p^{n-k}}$.
There exists unique polynomials $S, P \in \mathbb{Z}[(X_k)_{k\in\mathbb{N}}, (Y_k)_{k\in\mathbb{N}}]$ such that, for any $n \in \mathbb{N}$:

$$\omega_n(S(x,y)) = \omega_n(x) + \omega_n(y) \text{ and } \omega_n(P(x,y)) = \omega_n(x)\cdot\omega_n(y)$$

We let $S_k(X,Y) = S(X,Y)_k$ and $P_k(X,Y) = P(X,Y)_k \in \mathbb{Z}[X_1,\dots,X_k,Y_1,\dots,Y_k]$.
We endow $W(R)$ with the ring structure given by $x\hat{+}y = S(x,y)$ and $x\hat{\cdot}y = P(x,y)$, such that the map $\omega : W(R) \to R^{\mathbb{N}}$ is a ring morphism.
*We call $W(R)$ the **ring of Witt vectors**. It comes equipped with a multiplicative, but not additive map $x \in R \mapsto (x,0,0,\dots) \in W(R)$, called the **Teichmüller lift** and noted $x \mapsto [x]$.*

Note that the construction of Witt vectors can be made functorial by applying a ring morphism coordinate by coordinate.
Over a perfect ring, the ring of Witt vectors can be characterised as follows :

Theorem 1.3.19. *Let R be a perfect ring of characteristic p. $W(R)$ is uniquely characterized by the following properties :*

1. *There is an isomorphism $W(R)/pW(R) \cong R$*
2. *The element p is not a zero divisor in $W(R)$*
3. *$W(R)$ is (p)-adically complete.*

and we can write $W_n(A) \cong W(A)/(p^n)$

A technical, yet useful result is the following :

Proposition 1.3.20. *Every element $x \in W(R)$ can be uniquely written as*

$$x = [c_0] + [c_1]p + [c_2]p^2 + \dots$$

for some $(c_n)_{n\in\mathbb{N}} \in R$.

Proof. By construction, for $r \in R, \pi([r]) = r$ where π denotes the projection $W(R) \to W(R)/pW(R) \cong R$. Hence, if $x \in W(R)$, x and $[\pi(x)]$ are congruent modulo p, so one can write $x = [c_0] + p\cdot y$ where $c_0 = \pi(x)$ and for some $y \in W(R)$.
We then iterate the construction by induction. Unicity follows from the injectivity of $[\cdot]$. ☐

Let us now characterize the ring of Witt vectors via a universal property.

Fix R a perfect ring of characteristic p, A a p−adically complete ring and a map $h : R \to A/pA$.
One may extend it to a map $W(R) \to A$ by mapping $x = (x_0, x_1, \dots)$ to $\sum_{i=0}^{\infty} p^i [h(x_i)]^{1/p^i}$, where the choice of a p^i−th root exists and is unique since R is perfect. The sum converges since A is p-adically complete.

Proposition 1.3.21. *For any p-adically complete ring A, the reduction modulo p as well as the construction above induce a bijection :*

$$\mathrm{Hom}_{\mathrm{Ring}}(W(R), A) \cong \mathrm{Hom}_{\mathrm{Ring}}(R, A/pA)$$

Proof. cf. [Lur18], Proposition 3.8. □

We now turn our attention to Witt vectors of finite length.

Definition 1.3.22. *For $n \geq 0$, we define $W_n(R) := \pi_n(W(R))$ where $\pi_n : R^{\mathbb{N}} \to R^n$ is the projection on the first n coordinates, endowed with the induced ring structure.*
*This the ring of **Witt vectors of length n** over R.*

Example 1.3.23. *We have $W(\mathbb{F}_p) = \mathbb{Z}_p$ and, for $n \geq 0, W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$.*
By definition, for any ring R, $W_1(R) \cong R$.

Proposition 1.3.24. *The projections $W(R) \to W_n(R)$ induce an isomorphism of rings*

$$W(R) \cong \varprojlim W_n(R)$$

where transition morphisms are induced by the projections on the first coordinates.

The Witt vectors of finite length are equipped with :

1. The **Verschiebung** is the additive (but not multiplicative) map

$$(x_0, \dots, x_n) \in W_n(R) \mapsto (0, x_0, \dots, x_n) \in W_{n+1}(R)$$

2. The **Frobenius** is the unique ring morphism $F : W_n(R) \to W_{n-1}(R)$ such that the diagram below :

$$\begin{array}{ccc} W_n(R) & \xrightarrow{F} & W_{n-1}(R) \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ R^n & \xrightarrow{\pi_{n-1}} & R^{n-1} \end{array}$$

commutes (here, π_{n-1} denotes the projection on the first n-1 coordinates).

The construction of the Witt vectors rings can be globalized into group schemes. We will use very little about those objects, and will not define more than we need. We refer the reader to [Oor66], section 9 for an overview of the theory.

Definition 1.3.25. *Fix R a ring.*
*The **n-th Witt group scheme** over R is the group scheme over* $\mathrm{Spec}(R)$ *noted $\mathbb{W}_{R,n}$ and defined by*

$$\mathbb{W}_{R,n}(X) = W_n(\mathcal{O}_X(X))$$

endowed with the addition, for any scheme $X \to \mathrm{Spec}(R)$.
The morphisms defined above induce a Frobenius $F : \mathbb{W}_{n,R} \to \mathbb{W}_{n,R}$, a Verschiebung $V : \mathbb{W}_{n,R} \to \mathbb{W}_{n+1,R}$ and a restriction $R : \mathbb{W}_{n+1,R} \to \mathbb{W}_{n,R}$

Note that, by construction, $\mathbb{W}_{R,1}$ is simply the additive group $\mathbb{G}_a$, defined as follows.

Definition 1.3.26. *Let S be a scheme. The **additive group** over S is the sheaf $T \mapsto \mathcal{O}_T(T)$, seen as an étale sheaf on S.*

2 Stable homology theory

The goal of sections 2 and 3 is to establish the following, due to Lawrence Breen in [Bre81].

Theorem 2.0.1. *Let $S = \mathrm{Spec}(R)$ be an affine scheme of characteristic p, and $\mathbb{G}_a$ the additive group scheme over S, seen as a sheaf of $\mathbb{F}_p$-vector spaces over the site S^{perf} of all perfect $S-$schemes, endowed with the étale topology. The extension groups, computed in the associated topos, are :*

$$\mathrm{Ext}^n_{S^{perf},\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \begin{cases} R[T,T^{-1}]^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

where $R[T,T^{-1}]^{nc}$ denotes the non-commutative ring of Laurent polynomials, in which the commutation relation is induced by $T \cdot x := x^p \cdot T$ for $x \in R$.
The formal element T corresponds to the action of the Frobenius morphism on $\mathbb{G}_a$.

In this paper, we follow the historic proof by Breen. It is stems from a generalization of standard computations in algebraic topology, generalized to arbitrary topoi. Hence, we need to define some quite heavy machinery about simplicial objects and stable homology in arbitrary ringed topoi.

This section mostly acts as a toolbox for section 3, in which the computation will be detailed. Some concepts will also be needed in section 5

Remark 2.0.2. *Note that they are some more modern proofs that are more algebraic in nature. We refer the reader to the appendix of [Mat22] for alternative methods and further references.*

2.1 Elements of algebraic topology

This section presents some standard concepts and results in algebraic topology. While they will not be used directly, they act as a guideline ; as we will establish analogues of them when replacing topological spaces by simplicial sheaves in 2.3.

2.1.1 Eilenberg-MacLane spaces and suspension

Our main reference for this section is Hatcher's [Hat00].

In this paragraph, we recall various notions of algebraic topology. We will assume that the reader is familiar with CW-complexes, singular homology and higher homotopy groups.

Unless specified otherwise, (co)homology denotes singular (co)homology. If the coefficient ring is not specified, it is implicitely chosen to be $\mathbb{Z}$.

Let us start by defining two important notions of "weak isomorphisms" between topological spaces.[21]

Definition 2.1.1. *Let $f : X \to Y$ be a continuous map between topological spaces.*

1. *It is a* ***weak homotopy equivalence*** *if induces a bijection on the set of connected components $\pi_0(f) : \pi_0(X) \simeq \pi_0(Y)$, and isomorphisms in higher homotopy $\pi_n(f) : \pi_n(X,x) \simeq \pi_n(X, f(x))$ for any base point $x \in X$ and $n \geq 1$.*
2. *It is a* ***homotopy equivalence*** *if there exists a continuous map $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are respectively homotopic to id_X and id_Y.*

Lemma 2.1.2. *Weak homotopy equivalences induce isomorphisms in homology and cohomology.*

Proof. cf. [Hat00], proposition 4.21 □

[21] This definition should be read with 1.2.8 in mind

Clearly, every homotopy equivalence is a weak homotopy equivalence. A very important theorem by Whitehead tells us that the converse holds for CW-complexes.

Theorem 2.1.3. (*Whitehead*) *Any weak homotopy equivalence between two CW-complexes is a homotopy equivalence.*

Proof. cf. [Hat00], corollary 4.33 □

Let us now define the Eilenberg-MacLane spaces.

Proposition 2.1.4. *Let G be a group, and $n \geq 1$. We require G to be abelian whenever $n \geq 2$. There exists a connected pointed CW-complex (X, x) such that :*

$$\pi_k(X, x) = \begin{cases} G & \text{if } k = n \\ 0 & \text{if } k \notin \{0, n\} \end{cases}$$

Moreover, such a space is unique up to weak homotopy equivalence.
We note it $K(G, n)$, and call it the ***Eilenberg-MacLane space*** *associated with (G, n).*

A key property of Eilenberg-MacLane spaces is that they represent cohomology :

Proposition 2.1.5. *For any CW-complex X, there is a canonical isomorphism*

$$[X, K(G, n)] \cong \mathrm{H}^n(X, G)$$

where $[X, Y]$ denotes the set of homotopy classes of morphisms $X \to Y$.

Proof. cf. [Hat00], thm 4.57. The proof relies on the successive extension of maps on higher-dimensional cells vie obstruction theory. □

We conclude this section by a discussion on suspension of topological spaces.

Definition 2.1.6. *Let (X, x_0) and (Y, y_0) be two pointed topological spaces.*
The ***smash product*** *of X and Y, denoted $X \wedge Y$, is defined as*

$$X \wedge Y = (X \times Y) \big/ (X \vee Y)$$

where $X \vee Y$ denotes the wedge sum of X and Y based in (x_0, y_0), defined as $(X \sqcup Y) \big/ x_0 \sim y_0$.
The ***suspension*** *of X is the topological space :*

$$\Sigma X := X \wedge S^1 = X \times [0,1] \big/ \left((x, 0) \sim \{*\} \text{ and } (x, 1) \sim \{*\}\right)$$

The suspension is functorial, and respects the subcategory of CW-complexes.
If X and Y are path-connected, the suspension does not depend on the choice of the base points, and we need not explicit them.

Example 2.1.7. *The smash product of two spheres $S^n \wedge S^k$ is homeomorphic to S^{n+k}.*
The suspension of S^n is hence $\Sigma S^n \simeq S^{n+1}$.

The suspension operation appears somewhat naturally as an adjoint to the loop space construction.

Proposition 2.1.8. *The suspension functor is left adjoint to the* ***loop space*** *functor $(X, x) \mapsto \Omega X$, where ΩX is the space of loops in X based on x, endowed with the compact open topology.*
Note that $\pi_n(\Omega X) = \pi_{n-1}(X)$[22], so that, by unicity, $\Omega K(G, n) \cong K(G, n-1)$.

[22] This can be seen using the fibration $\Omega X \hookrightarrow PX \to X$ where PX is the space of paths starting from X, which is contractile.

The homology of a suspension can easily be computed :

Proposition 2.1.9. *There are isomorphisms in reduced homology* $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_{n+1}(\Sigma X)$.

Proof. This follows from the Mayer-Vietoris exact sequence. □

The homotopy groups can sometimes be computed.

Theorem 2.1.10. (*Freudenthal suspension theorem*)
Let X be a path-connected space such that $\pi_k(X) = 0$ whenever $1 \leq k \leq n$.
The suspension map $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i \leq 2n$ and a surjection when $i = 2n+1$.

Proof. This follows from properties of the long exact sequence of a pair in homology. □

Let us conclude this section by mentioning the Hurewicz isomorphism :

Theorem 2.1.11. *(Hurewicz) Let $n \geq 2$ and X be a path-connected topological space such that $\pi_k(X) = 0$ whenever $1 \leq k < n$. Then $\pi_n(X) \cong \mathrm{H}_n(X, \mathbb{Z})$.*

Proof. cf. [Hat00], theorem 4.32 □

2.1.2 The homotopy category

This section is intended to be read with 1.2.2 in mind. A good part of the redaction was inspired by the MathOverflow question [Pst11], and we thank Thomas Nikolaus and Karol Szumiło for their answers.

The goal of this section is to construct a *topological version* of the derived category and other concepts introduced in 1.2.2.
As a guideline, we want Eilenberg-MacLane spaces to represent cohomology at the derived level.

Based on 2.1.5, it is natural to define :

Definition 2.1.12. *The **quotient category**, noted $K(\mathrm{Top})$[23], is the category whose objects are topological spaces and morphisms are homotopy classes of continuous maps*

$$\mathrm{Hom}_{K(\mathrm{Top})}(X,Y) = [X,Y] := \mathrm{Hom}_{\mathrm{Top}}(X,Y)\big/ homotopy$$

As in the setup of chain complexes, while the quotient category is useful, it is not the 'right' notion of a derived category. The following definition is due to Quillen.

Definition 2.1.13. *The **true homotopy category**, noted* hTop*, is the localization of* Top *with respect to weak homotopy equivalences.*

Likewise, we'll let hCW denote the localization of the category of CW-complexes with respect to (weak) homotopy equivalences.

We will now show that Eilenberg-MacLane spaces represent cohomology in the homotopy category. Let us first establish two lemmas :

Lemma 2.1.14. *Two homotopic maps in* Top *are equal in the category* Ho(Top) *obtained by localizing Top with respect to homotopy equivalences.*

Proof. We let $F : \mathrm{Top} \to \mathrm{Ho(Top)}$ be the localization functor. Let $f, g : X \to Y$ be two homotopic maps. We note $H : X \times [0,1] \to Y$ the homotopy, define the inclusions on both ends $i_0 : X \cong X \times \{0\} \to I \times X$ and $i_1 : X \cong X \times \{1\} \to X \times [0,1]$ such that $f = H \circ i_1$, $g = H \circ i_2$, and the projection $\pi : X \times [0,1] \to X$. We know that $\pi \circ i_0 = \pi \circ i_1 = id_X$. Hence, $F(\pi) \circ F(i_0) = F(\pi) \circ F(i_1) = id_{F(X)}$. Moreover, since π is an homotopy equivalence, $F(\pi)$ is an isomorphism, so that $F(i_0) = F(i_1)$.
Finally $F(f) = F(H) \circ F(\pi_1) = F(H) \circ F(\pi_2) \cong F(g)$. □

[23] This is sometimes called the *naïve homotopy category* and noted hTop. We prefer reserving these terms for the true homotopy category, defined below

Likewise, we note Ho(CW) the localization of the category of CW-complexes with respect to homotopy equivalences.

Lemma 2.1.15. *The categories* $K(\mathrm{Top})$ *and* $\mathrm{Ho}(\mathrm{Top})$ *are equivalent.*

Proof. We check that $K(\mathrm{Top})$ satisfies the universal property of the localization.
To a functor $F : K(\mathrm{Top}) \to \mathcal{D}$, we can associate a functor $\widetilde{F} : \mathrm{Top} \to \mathcal{D}$ by declaring $\widetilde{F}(X) = F(X)$ and $\widetilde{F}(f) = F([f])$ where $[\cdot]$ denotes homotopy classes. This clearly maps homotopy equivalences to isomorphisms. By Yoneda's lemma, this defines a functor $\mathrm{Ho}(\mathrm{Top}) \to K(\mathrm{Top})$.
We construct a quasi-inverse $G : K(\mathrm{Top}) \to \mathrm{Ho}(\mathrm{Top})$ by choosing arbitrary representatives of homotopy classes. This is well defined by the above lemma.
We easily check those functors induce an equivalence of category. □

All the work above leads to the following reformulation of 2.1.5 :

Theorem 2.1.16. *Let X be a topological space, G an abelian group and $n \geq 2$.*
There is an isomorphism :

$$\mathrm{Hom}_{\mathrm{hTop}}(X, K(G,n)) \cong \mathrm{H}^n(X, G)$$

Proof. By the CW approximation theorem (cf. [Hat00], proposition 4.13), we may fix a CW-complex $\widetilde{X}$ that is weakly equivalent to X.
By Whitehead's theorem, quasi-isomorphisms between $CW-$complexes are exactly homotopy equivalences, so that morphisms in hTop between CW-complexes are exactly morphisms in Ho(CW).
Moreover, the embedding $\mathrm{Ho}(\mathrm{CW}) \to \mathrm{Ho}(\mathrm{Top})$ is clearly fully faithful. We now compute :

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{hTop}}(X, K(G,n)) &= \mathrm{Hom}_{\mathrm{hTop}}(\widetilde{X}, K(G,n)) && \text{since } X,\ \widetilde{X} \text{ are weakly equivalent} \\
&= \mathrm{Hom}_{\mathrm{Ho(CW)}}(\widetilde{X}, K(G,n)) && \text{since both spaces are CW-complexes} \\
&= \mathrm{Hom}_{K(\mathrm{Top})}(\widetilde{X}, K(G,n)) && \text{by 2.1.15} \\
&= [\widetilde{X}, K(G,n)] && \text{by definition of } K(\mathrm{Top}) \\
&\cong \mathrm{H}^n(\widetilde{X}, G) && \text{by 2.1.5} \\
&\cong \mathrm{H}^n(X, G) && \text{by 2.1.2}
\end{aligned}$$

Hence the result. □

This result is especially surprising since the proof of 2.1.5 makes heavy use of the cell structure. To the best of my knowledge, there is no direct proof of the result above.

2.1.3 Stable cohomology operations

This section is inspired by [FF16] *and* [Hat00].

The goal of this section is to define operations in singular cohomology, and their link with cohomology of Eilenberg-MacLane spaces. We then introduce a stabilized version.

For any ring R, the cup-product, defined at the level of singular cochains via $(c \smile d)(\sigma) = c(\sigma \circ \iota_{0,1,\dots,p}) \cdot d(\sigma \circ \iota_{p,p+1,\dots,p+q})$[24], induces a bilinear mapping

$$\smile : \mathrm{H}^p(X, R) \times \mathrm{H}^q(X, R) \to \mathrm{H}^{p+q}(X, R)$$

which endows $\mathrm{H}^\bullet(X, R)$ with the structure of a graded ring.
For example, this yields a map

$$x \in \mathrm{H}^p(X, R) \mapsto x \smile x \in \mathrm{H}^{2p}(X, R)$$

Such an operation is functorial in X, and forms an example of a cohomology operation.

[24] Here, ι denotes the inclusion of the standard simplexes

Definition 2.1.17. *Let $m, n \geq 0$ and R, S be rings.*
The additive group of ***cohomology operations*** *of type (R, m, S, n) is defined as*

$$\mathrm{Op}(R, m; S, n) = \{\textit{Natural transformations } \mathrm{H}^m(\_, R) \implies \mathrm{H}^n(\_, S)\}$$

where cohomology are seen as functors $(CW - cplx) \to \mathrm{Mod}_R$ and $(CW - cplx) \to \mathrm{Mod}_S$.

Because of the representability results established above, cohomology operations groups can be identified with cohomology of the appropriate Eilenberg-MacLane spaces.

Proposition 2.1.18. *There is a canonical isomorphism*

$$\mathrm{Op}(R, m; S, n) \cong \mathrm{H}^n(K(R, m), S)$$

given by $\Phi \in \mathrm{Op}(R, m; S, n) \mapsto \Phi(\iota)$, where $\iota \in \mathrm{H}^m(K(R, m), R)$ is the fundamental class corresponding to the identity map $id_{K(R,m)} \in [K(R, m), K(R, m)]$ via 2.1.5.

Proof. Since cohomology is invariant up to quasi-isomorphisms 2.1.2, the operations groups can be computed with functors $\mathrm{hCW} \to \mathrm{Mod}$. By Whitehead's theorem 2.1.3, $\mathrm{hCW} \cong \mathrm{Ho(CW)}$. By the proof of 2.1.16, $\mathrm{Ho(CW)} \cong \mathrm{hTop}$. The result now follows from 2.1.16 and Yoneda's lemma.
For a more geometric proof starting from 2.1.5, see [Hat00], proposition 4L.1. □

A fundamental class of cohomology operations are given by the stabilisation $\Sigma : \mathrm{H}^n(X, R) \to \mathrm{H}^{n+1}(\Sigma X, R)$. Cohomology operations commuting with the stabilization will be called *stable*.

Definition 2.1.19. *A* ***stable*** *cohomology operation of type $(k, R; S)$ is a family $(\varphi_n)_{n\geq 0} \in \mathrm{Op}(R, n; S, n+k)$ such that, for all X, $(\varphi_n)_X \circ \Sigma = \Sigma \circ (\varphi_n)_{\Sigma X}$.*
The group of such operations is noted $\mathrm{Op}^{st}(k, R; S)$.

Stable cohomology operations are well-behaved. For example, they are compatible with the cohomology of a pair (cf. [FF16], 28.4.B)
Using 2.1.18, a stable cohomology operation corresponds to a sequence $\psi_n \in \mathrm{H}^{n+r}(K(R, n), S)$.
The stability condition indicates that $f_n(\psi_n) = \psi_{n-1}$ where f_n is defined as the composition

$$\mathrm{H}^{r+n}(K(R, n), S) \xrightarrow{i_n^*} \mathrm{H}^{r+n}(\Sigma K(R, n-1), S) \xrightarrow{\Sigma^{-1}} \mathrm{H}^{r+n-1}(K(R, n-1), S)$$

where Σ^{-1} is induced by 2.1.9 and i_n^* is the pullback along any map of the class $[\Sigma K(R, n-1), K(R, n)]$ corresponding to $id_{K(R,n-1)}$ via the isomorphism given by 2.1.8 :

$$[\Sigma K(R, n-1), K(R, n)] \cong [K(R, n-1), \Omega K(R, n)] \cong [K(R, n-1), K(R, n-1)]$$

At last, we have proven :

Proposition 2.1.20. *There is a canonical isomorphism :*

$$\mathrm{Op}^{st}(k, R; S) \cong \varprojlim_n \mathrm{H}^{n+k}(K(R, n), S)$$

where transitions are given by the f_n.
These are also called ***stable cohomology groups*** *and noted $H^k_{st}(R, S) := \varprojlim_n \mathrm{H}^{n+k}(K(R, n), S)$.*

In this setup, the limit is stationary (cf. [FF16], p.394). This gives some form of hope that stable cohomology operation groups can actually be computed, at least in simple cases.
Let us see what happens when $R = S = \mathbb{Z}/p\mathbb{Z}$.

2.1.4 The Steenrod algebra

This section is mainly based on [FF16].
In this section, we define the stable cohomology groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$, then $\mathbb{Z}/p\mathbb{Z}$ for p a prime number.
While the operation $x \mapsto x \smile x$, is not stable, when $R = \mathbb{Z}/2\mathbb{Z}$, it appears as a special value of some important stable operations, called the Steenrod squares.
In order to lighten the notation, we briefly let $\mathbb{Z}_p \coloneqq \mathbb{Z}/p\mathbb{Z}$ for any prime number p.

Proposition–Definition 2.1.21. *Fix $i > 0$.*
The ***i-th Steenrod square*** *on a topological space X is a natural family of additive morphisms*

$$(\mathrm{Sq}^i(X))_{n\geq 0} : \mathrm{H}^n(X, \mathbb{Z}_2) \to \mathrm{H}^{n+i}(X, \mathbb{Z}_2)$$

They form a stable cohomology operation of type $(i, \mathbb{Z}_2; \mathbb{Z}_2)$ such that :

- $\mathrm{Sq}^i(x) = x \smile x$ *when $dim(x) = i$*
- $\mathrm{Sq}^i(x) = 0$ *when $deg(x) < i$.*
- $\mathrm{Sq}^i(x \smile y) = \sum_{p+q=i} \mathrm{Sq}^p(x) \smile \mathrm{Sq}^q(y)$.

Moreover, $\mathrm{Sq}^0 = id$.

The Steenrod squares are uniquely characterized by the properties above. The construction of those square is very technical, and we refer to [FF16], section 29, for details. We will briefly mention the idea behind the construction in the appendix 6.1.

Definition 2.1.22. *For p a prime number, we let $\mathcal{A}_p = \bigoplus_k \mathrm{Op}^{st}(k, \mathbb{Z}_p, \mathbb{Z}_p) \cong \bigoplus_k \mathrm{H}^k_{st}(K(\mathbb{Z}_p), \mathbb{Z}_p)$. The composition of natural transformations*

$$\mathrm{Op}^{st}(k, \mathbb{Z}_p, \mathbb{Z}_p) \times \mathrm{Op}^{st}(l, \mathbb{Z}_p, \mathbb{Z}_p) \to \mathrm{Op}^{st}(k+l, \mathbb{Z}_p, \mathbb{Z}_p)$$

induces a structure of a graded algebra on $\mathcal{A}_p$. We call it the ***Steenrod algebra*** *of rank p. Note that it is not commutative*[25]*.*

The algebra $\mathcal{A}_2$ is essentially generated by the Sq^i. The following structure theorem is due to Serre [Ser53].

Theorem 2.1.23. *The algebra $\mathcal{A}_2$ is isomorphic to the graded algebra generated by the Sq^a such that :*

- $\mathrm{Sq}^0 = id$
- *For any $0 < a < 2b$,* $\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$

The second condition defines the commutation relations.

Moreover, this structure theorem respects the structure of the cohomology rings

$$\mathrm{H}^*(K(\mathbb{Z}_2, n), \mathbb{Z}_2) = \bigoplus_{k\geq 0} \mathrm{H}^k(K(\mathbb{Z}_2, n), \mathbb{Z}_2) \hookrightarrow \mathcal{A}_p$$

Definition 2.1.24. *A sequence of positive integers $I = (i_1, \dots, i_k)$ is said to be* ***admissible*** *if*

$$i_1 \geq 2i_2 \geq 4i_3 \geq \cdots \geq 2^{k-1} i_k$$

If $I = (i_1, \dots, i_k)$ is an admissible sequence, its ***excess*** *is defined as $exc(I) \coloneqq i_1 - (i_2 + \cdots + i_k)$.*[26]
If I is an admissible sequence, we let $\mathrm{Sq}^I \coloneqq \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \dots \mathrm{Sq}^{i_k}$.

[25]It is actually commutative up to homotopy
[26]This is nonnegative by definition

The structure now theorem writes as follows :

Theorem 2.1.25. *For any* $k \geq 0$, $\mathrm{H}^*(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$ *is additively generated by the* $\mathrm{Sq}^I \cdot e_n$ *for all admissible* $I = (i_1, \dots, i_k)$ *such that* $exc(I) < n$,

where $e_n \in \mathrm{H}^n(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$ is the fundamental class corresponding to the identity as a cohomological operation via 2.1.18.

When $p \neq 2$, the situation is slightly more complicated.

Proposition 2.1.26. *We may define the following natural additive morphisms :*[27]

- *The* ***reduced p-th power operation*** $P^i : \mathrm{H}^n(X, \mathbb{Z}_p) \to \mathrm{H}^{n+2i(p-1)}(X, \mathbb{Z}_p)$ *for any* $i \geq 0$.
- *The* ***Bockstein map*** $\beta : \mathrm{H}^i(X, \mathbb{Z}_p) \to \mathrm{H}^{i+1}(X, \mathbb{Z}_p)$

They satisfy, and are characterised by the fact that :

1. $P^n : \mathrm{H}^{2n}(X, \mathbb{Z}_p) \to \mathrm{H}^{2np}(X, \mathbb{Z}_p)$ *is given by the p-th cup power* $x \mapsto x \smile \dots \smile x$.
2. *If* $2n > deg(x)$, $P^n(x) = 0$
3. $P^n(x \smile y) = \sum_{p+q=n} P^n(x) \smile P^n(y)$

Moreover, $P^0 = Id$

The algebra $\mathcal{A}_p$ is generated by the two operations above.

Theorem 2.1.27. $\mathcal{A}_p$ *is isomorphic to the graded algebra generated by* β *and the* P^a *such that :*

- $P^0 = 1$
- *If* $a < p \cdot b$, $P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$
- *If* $a \leq p \cdot b$, $P^a \beta P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i + \sum_i (-1)^{a+i+1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$

As is the case for $p = 2$, one can see that this decomposition respects the structure of the $\mathrm{H}^*(K(\mathbb{Z}_p, n), \mathbb{Z}_p)$. The key feature of the Steenrod algebra is that it admits a structure of a Hopf algebra, which endows its dual with an algebra structure, that is often easier to describe. We will develop this notion in full generality in 2.3.4.

2.2 Simplicial sets

In this section, we study simplicial sets, as a combinatorial *model* of topological spaces. We will define, in this setup, combinatorial analogues of the constructions of the paragraph above. They will then be further generalized to simplicial objects of any topos.

Good general references include [Bre78], [GJ09] and [Lur22].

[27] When $p = 2$, $\beta = Sq^1$ and $P^i = Sq^{2i}$

2.2.1 Simplicial objects

This section is based on [Lur22].
Let us start by some very standard definitions :

Definition 2.2.1. *Let Δ be the* ***simplicial category****, where :*

- *Objects are $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$*
- *Morphisms are increasing functions $[n] \to [m]$*

For any $n \geq 0$, $0 \leq i \leq n$ and $0 \leq j \leq n-1$, we define the $i-th$ face maps $\sigma_i^n : [n-1] \hookrightarrow [n]$ as the only injective increasing map whose image avoids i, and the $j-th$ face maps $\delta_j^n : [n] \twoheadrightarrow [n-1]$ as the only surjective increasing map that hits j twice.

Note that every morphism in Δ can be written as a composition of degeneracy and face maps.

Definition 2.2.2. *Let $\mathcal{C}$ be a category. A* ***simplicial object*** *in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$.*
The ***simplicial category*** *associated to $\mathcal{C}$ is the category $\mathrm{Simp}(\mathcal{C})$ whose objects are simplicial objects in $\mathcal{C}$ and whose morphisms are given by natural transformations.*
If $F : \Delta^{op} \to \mathcal{C}$ is a simplicial object, we will usually note $F_n := F([n])$.
A simplicial object of Set will be called a ***simplicial set****.*

Elements S_* of $\mathrm{Simp}(\mathcal{C})$ are given by diagrams in $\mathcal{C}$:

$$S_0 \Leftarrow\!\!\Rightarrow S_1 \Lleftarrow\!\!\Rrightarrow S_2 \;\overset{\longleftarrow}{\underset{\longleftarrow}{\vdots}}\; \cdots$$

where the arrows denote the image of the degeneracy and face maps.
Let us now define some standard simplicial sets.

Example 2.2.3. *To a topological space X, one can associate its* ***singular simplicial set*** $\mathrm{Sing}(X)$ *:*

$$\mathrm{Sing}(X)([n]) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$$

where $|\Delta^n|$ denotes the topological standard n-simplex $|\Delta^n| = \{(t_0, \dots, t_n) \in [0,1], \sum t_i = 1\}$, that contains enough information to compute singular homology. [28] *This defines a functor* $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Simp}(\mathrm{Set})$.

We refer to [Lur22, Subsection 001Q] for more details about this construction.

Definition 2.2.4. *The* ***standard n-simplex*** *is the simplicial set : $\Delta^n : [m] \mapsto \mathrm{Hom}_\Delta([m], [n])$.*
Its ***boundary*** *is the simplicial set defined as : $(\partial\Delta^n)[m] = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]), \alpha \text{ not surjective}\}$.*
If $0 \leq i \leq n$, the ***horn*** *Λ_i^n is the simplicial set defined by :*

$$(\Lambda_i^n)([m]) = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]) : [n] \notin \alpha([m]) \cup \{i\}\}$$

Remark 2.2.5. *By Yoneda's lemma, for S_* a simplicial set, $\mathrm{Hom}_{\mathrm{Simp(Set)}}(\Delta^n, S_*) = S_n$.*
For instance, functors $s : \Delta^0 \to S_*$ are exactly points $s \in S_0$.

Simplicial sets admits a geometric realisation, which is a (topological) simplicial complex. When considering this very restricted class of topological spaces, usual notions in topology can be made purely combinatorial.

Proposition–Definition 2.2.6. *The functor Simp admits a left adjoint* $\mathrm{Simp(Set)} \to \mathrm{Top}$*, called the* ***geometrical realization functor*** *and noted $X \mapsto |X|$.*

Proof. cf. [Lur22, Corollary 0022] □

[28] As we will later see, it also contains enough information to compute homotopy groups

For example, $|\Delta|^n$ is homeomorphic to the standard n-simplex, $|\partial\Delta^n|$ to its boundary, and $|\Lambda_i^n|$ to the subset of $|\partial\Delta^n|$ obtained after removing the face opposite to the $i-th$ vertex.

Let us now study homotopy groups. First, we will need a pointed version.

Definition 2.2.7. *A* ***pointed simplicial set*** *is a pair* (S_*, s) *for* S_* *a simplicial set and* $s : \Delta^0 \to S_*$. *A* ***morphism*** *between pointed simplicial sets* $(S_*, s) \to (T_*, t)$ *is a simplicial morphism* $f : S_* \to T_*$ *such that* $f(s) = t$.

2.2.2 Simplicial homotopy groups

This section is based on [Lur22] and [GJ09].

The goal of this section is to define homotopy groups of simplicial sets.

In order to define such groups, we'll need the notion of Kan complexes. While it is a key notion in the theory of infinity category, we'll try to avoid extensive use of this concept. We will simply use Kan complexes as a framework where homotopies are well behaved, and we refer to [Lur22, Chapter 00SY] for many more detail.

Definition 2.2.8. *A* ***Kan complex*** *is a simplicial set* X[29] *such that every morphism from every horn* $\Lambda_i^n \to X$ *can be extended to* $\Delta^n \to X$ *for* $0 \le i \le n$.

An important technical result is as follows :[30]

Proposition 2.2.9. *Any simplicial group is a Kan complex.*

Proof. cf. [Lur22, Proposition 00MG]. □

We will try and mimic the standard definition of higher homotopy groups :

Definition 2.2.10. *Let* (X, x) *be a pointed Kan complex.*
Let $f_0, f_1 : \Delta^n \to Y$ *such that* $f_i(\partial\Delta^n) = \{y\}$. *They are said to be* ***homotopic with respect to the boundary*** *if there exists* $h : \Delta^1 \times \Delta^n \to Y$ *such that*

- $h|_{\{0\}\times\Delta^n} = f_0$
- $h|_{\{1\}\times\Delta^n} = f_1$
- $h|_{\Delta^1\times\partial\Delta^n}$ *corresponds to the constant map valuing* x.

Proposition 2.2.11. *The relation defined above is an equivalence relation.*

Proof. This is non trivial, and makes use of the Horn extension property of Kan complexes. This is a consequence of [Lur22, Proposition 00HC] (where Lurie used an alternative definition of homotopy as pointed maps from the quotient $\Delta^n/\partial\Delta^n$). □

We may then define the homotopy groups :

Definition 2.2.12. *Let* (X, x) *be a pointed Kan complex. Let*

$$\pi_n(X, x) = \{f : \Delta^n \to X, f|_{\partial\Delta^n} = \{x\}\} \big/ \textit{homotopy}$$

Proposition 2.2.13. *There is a natural group structure on* $\pi_n(X, x)$ *for* $n \ge 1$.
It is abelian whenever $n \ge 2$.

[29] From the modern point of view, ∞-categories, form a good notion of a *space*. Since Kan complexes are a fundamental class of ∞-categories, they will be noted X - as for topological spaces

[30] One some vaguely related note, Lie groups are better behaved then varieties, since the neighborhood of every point may be canonically identified with the neighborhood of the identity element

Proof. cf. [Lur22, Subsection 00W3]. □

Fortunately, this is compatible with standard homotopy groups.

Proposition 2.2.14. *Let X be a Kan complex. The groups $\pi_n(X,x)$ and $\pi_n(|X|,x)$ are isomorphic.*

Proof. One needs to show that both groups behave well with (Kan) fibrations, use the path-loop fibration $\Omega X \to PX \to X$ and proceed inductively. We refer to [GJ09], proposition 11.1. □

The definition above can be extended to arbitrary simplicial sets X by picking a weak equivalences with a Kan complex $X \cong Y$ and computing the homotopy of Y. We will not need this construction.

However, there is no *a priori* good way to extend this definition onto $\mathrm{Simp}(\mathcal{C})$, for $\mathcal{C}$ an arbitrary category.

2.2.3 The Dold-Kan correspondence

This section is based on [Bre78] *and* [GJ09].

We fix $\mathcal{A}$ an abelian category. In this section (and this section only), we let $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ denote the category of bounded below **chain** complexes in $\mathcal{A}$.[31] Clearly, simplicial objects resemble chain complexes.

We present a construction, due to Dold and Puppe in [DP61], of an adjoint equivalence $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \leftrightarrow \mathrm{Simp}(\mathcal{A})$.

Let us give a first way of associating chain complexes to simplicial objects.

Proposition 2.2.15. *If X is a simplicial object of A, we define its **alternating face map complex** or **Moore complex** $X^{\sim}$ as the chain complex in $\mathcal{A}$ defined by :*

- $(X^{\sim})_n = X_n$
- $d^n = \sum_{i=0}^{n}(-1)^i d_i^n$

This construction is somewhat natural since is the one used to define singular homology from the singular simplex defined in 2.2.3. Likewise, once a base ring is chosen, we may define homology of simplicial sets.

Definition 2.2.16. *Let (S_*, s) be a pointed simplicial set, and R a ring.*
We let $R[S_*]$ be the freely generated simplicial R- module generated by S_*, as the simplicial R-module obtained by applying the (functorial) free module construction at each level.
This defines a left adjoint to the forgetful functor $\mathrm{Simp}(\mathrm{Mod}_R) \to \mathrm{Simp}(\mathrm{Set})$.
We now define $\mathrm{H}_n(S_*, R)$ as the homology of the complex of R-modules $R[S_*]^{\sim}$.
We also define the reduced homology $\widetilde{\mathrm{H}}_n(S_*, \mathcal{R})$ as the homology of $R^+[S_*] := R[S_*]^{\sim}/R[0]$.[32]

In order to construct the Dold-Puppe correspondence, we'll need to consider a large subcomplex $NX \subset X^{\sim}$.

Proposition 2.2.17. *For any simplicial object X in $\mathcal{A}$, there is a decomposition $X^{\sim} \cong NX \oplus DX$, where*

- *The **normalized complex** NX is defined by :*
  - $(NX)_n = \bigcap_{i>0} \mathrm{Ker}(d_i^n : X_n \to X_{n-1})$.
  - *The differentials are induced by the $0-$face map : $\partial_n := d_0^n \restriction_{(NX)_n} : (NX)_n \to (NX)_{n-1}$.*
- *The **degenerated complex** is defined by :*
  - $(DX)_n = \bigcup_i \mathrm{Im}(s_i^{n-1} : X_{n-1} \to X_n)$.
  - *Morphisms are induced via the inclusion $(DX)_n \subset X_n$.*

Moreover, DX is homotopically trivial, such that the natural inclusion $NX \hookrightarrow X^{\sim}$ is a quasi-isomorphism.

[31] One can likewise construct a *dual* equivalence between cochain complexes and cosimplicial objects

[32] This is defined as the complex valuing $R[S_*]$ in every degree besides 0, in which case it values $R[S_0]/R[s]$

Proof. cf. [GJ09], theorem 2.4. □

Note that both these constructions can be made functorial $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$.

Let us now construct a morphism that will form, together with N, an equivalence of categories.

Definition 2.2.18. *The* ***Dold-Puppe transformation*** *is the functor* $K : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$ *defined on a chain complex* $Y_\bullet$ *as :*

- $KY_n = \bigoplus_{0 \leq p \leq n} \bigoplus_{f:[0,n] \twoheadrightarrow [0,p]} Y_{p,f}$, *where the* $Y_{p,f}$ *are just copies of* Y_p.
- *To* $\theta : [m] \to [n]$, *we construct a map* $\theta^* : \bigoplus_{0 \leq k \leq n} \bigoplus_{f:[n] \twoheadrightarrow [k]} Y_{k,f} \to \bigoplus_{0 \leq l \leq m} \bigoplus_{g:[m] \twoheadrightarrow [l]} Y_{r,g}$ *induced by* $Y_{k,f} \to Y_{s,h}$ *where* $[m] \xrightarrow{g} [l] \xrightarrow{*} [k]$ *is the epi-mono factorization of* $[m] \xrightarrow{\theta} [n] \xrightarrow{f} [k]$.

Let us now state the main result :

Theorem 2.2.19. (***Dold-Kan equivalence***)
The pair of functors (K, N) *induces an adjoint equivalence* $\mathrm{Ch}_+(\mathcal{A}) \leftrightarrow \mathrm{Simp}(\mathcal{A})$.[33]

Proof. The original proof is [DP61], 3.6. For a more modern text, see [GJ09], corollary 2.3.
We refer to [Wei94], exercise 8.4.2, to show that (K, N) is an adjunction. □

While this construction may seem ad hoc, discussions related to 2.4.6 indicate that this definition is actually fundamental.

2.2.4 Eilenberg Mac-Lane spectra

This section is based on [Bre78].
The goal of this section is to establish a simplicial analogue of the Eilenberg Mac-Lane spaces, defined in 2.1.2, and show that they also represent cohomology in that setup. $\mathcal{A}$ is still a fixed abelian category.

A very important technical remark is as follows :

Theorem 2.2.20. *Let* (X, x) *be a pointed simplicial abelian group. Then* $\pi_n(X, x) \cong H_n(NX, \mathbb{Z})$.

A fortiori, such homotopy groups are independent of the base point.

Proof. This essentially follows from the definitions. See [GJ09], corollary 2.5. □

Definition 2.2.21. *For* A *an object of* $\mathcal{A}$, *define* $K(A, n) := K(A[n])$.
This is known as the ***Eilenberg-Mac Lane*** *space*[34] *of degree* n.

Example 2.2.22. $K(A, 0)$ *is the simplicial complex concentrated in degree* 0, *valuing* A.

The naming is justified by the following property :

Proposition 2.2.23. *Let* A *be a simplicial abelian group. Then :*

$$\pi_k(K(A, n)) = \begin{cases} A & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Proof. By 2.2.20, whenever X a simplicial abelian group, $\pi_n(X, x) = H_n(NX)$ for any point $x \in X$.
Then $\pi_n(X) = H_n(N \circ K(A[n]), \mathbb{Z}) = H_n(A[n], \mathbb{Z})$ since $N \circ K \cong id$, and isomorphisms of chain complexes preserve homotopy. Hence the result. □

As we will later see, in that setup, Eilenberg Mac-Lane represent hypercohomology at a derived level, and thus deserve the name of a spectra. We do not elaborate on these notions further, and refer the interested reader to [Fra74].

[33] i.e. K is left adjoint to N, such that $N \circ K$ and $K \circ N$ are natural isomorphisms

[34] They deserve the name "space" since, if $\mathcal{A} = \mathrm{Ab}$, they are Kan complexes (as simplicial abelian groups)

2.3 Stable homology in ringed topoi

This section is inspired by [Bre78] *and* [Ill71].

The goal of this section is to generalize the constructions of the previous one to simplicial objects valued in any topos. We will reuse the notations introduced in 1.1.3.

Let us fix $(\mathcal{T}, \mathcal{R})$ a ringed topos, realized as sheaves on a site $(\mathcal{C}, J)$.
We will generalize the standard case, where $(\mathcal{T}, \mathcal{R}) = (\mathrm{Set}, \mathbb{Z})$.

2.3.1 (Derived) simplicial sheaves

For psychological reasons, simplicial objects in $\mathcal{T}$ are called **simplicial sheaves**. Such objects are exactly sheaves on $(\mathcal{C}, J)$ valued in Simp(Set).
We let $\mathcal{A}$ denote the category of abelian groups in $\mathcal{T}$, and will use the functors N, K and the Eilenberg-MacLane spaces in this setup. Objects of Simp($\mathcal{A}$) is then called the simplicial abelian sheaves of $\mathcal{T}$.

A **pointed simplicial sheaf** is a pair (X, x) where X is a simplicial sheaf, and $x : e \to X$ where $e = \underline{\{*\}}$ is the final object of $\mathcal{T}$.

Our goal is to glue together the previous constructions on a sheaf.

Definition 2.3.1. *Let (X, x) be a pointed simplicial sheaf on $\mathcal{T}$.*
Define $\pi_n(X, x) \in \mathcal{T}$ as the sheafification of the presheaf on $(\mathcal{C}, J)$ defined by $U \mapsto \pi_n(X(U), x \upharpoonright U)$.
*It is a sheaf of abelian groups on (C, J), called the **n-th homotopy sheaf**.*

One can check that such a definition is independant of the site of definition. In fact, if X is a simplicial abelian group in $\mathcal{T}$, the homotopy sheaves admit the following reformulation.

Proposition 2.3.2. *If X is a simplicial abelian sheaf in $\mathcal{T}$, then $\pi_n(X, x) \cong \mathrm{H}_n(NX, \underline{\mathbb{Z}})$.*

Proof. This essentially follows from the definition. □

Therefore, the homotopy groups of Eilenberg-MacLane spaces are still as expected.

Remark 2.3.3. *If t is a point of $\mathcal{T}$ and X a simplicial sheaf, then $\pi_n(X, x)_t \cong \pi_n(X_t, x_t)$.*

The Hurewicz theorem still holds in that setup.

Theorem 2.3.4. *(Simplicial Hurewicz). Assume $\mathcal{T}$ has enough points.*[35]
Let (X, x) be a path-connected pointed simplicial abelian group and $n \geq 2$ such that $\pi_k(X, x) = 0$ whenever $0 < k < n$. Then there is an isomorphism $\pi_n(X, x) \cong \mathrm{H}_n(X, \underline{\mathbb{Z}})$.

Proof. Since $\mathcal{T}$ has enough points, we can assume $\mathcal{T} = \mathrm{Set}$. Then, we apply [GJ09], Theorem 3.7. □

In order to obtain representability results similar to 2.1.5, we'll need to construct a derived category of simplicial sheaves.

Definition 2.3.5. *A morphism between pointed simplicial sheaves $f : (X, x) \to (Y, f(x))$ in Simp($\mathcal{T}$) is a **quasi-isomorphism** if all the induced morphisms $\pi_n(f) : \pi_n(X, x) \to \pi_n(X, f(x))$ are isomorphisms of simplicial abelian sheaves.*
*The **derived category of simplicial sheaves in** $\mathcal{T}$ is the localisation of Simp($\mathcal{T}$) with respect to quasi-isomorphisms. We note it $\mathcal{D}(\mathcal{T})$.*

Note that, if $\mathcal{R}$ is a ring in $\mathcal{T}$, the category $\mathrm{Mod}_{\mathcal{R}}$ is an abelian category, so that we can define its derived category in the usual way.
Finally, let us remark that the stalk morphism can be extended at the derived level.

[35] I don't know if this assumption is necessary

Lemma 2.3.6. *Let X, Y be simplicial sheaves of $\mathcal{T}$, and t a point of $\mathcal{T}$.*
The point induces a morphism $\mathrm{Hom}_{\mathcal{D}(T)}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(\mathrm{Simp}(Set))}(X_t, Y_t)$

Proof. The morphism induced by the point at the level of simplicial objects descend at the derived level by 2.3.3. □

Proposition 2.3.7. *Let X, Y be $\mathcal{R}-$modules, and t a point of $\mathcal{T}$.*
The point induces a morphism $\mathrm{Hom}_{\mathcal{D}(\mathcal{R})}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{R}_t - Mod)}(X_t, Y_t)$

Proof. By definition, the pullback function of a geometric morphism of topos preserves finite limits and colimits, so it is exact, and hence commutes with homotopy.
The induced morphism at the level of complexes hence descends at the derived level. □

The morphism above can be reformulated as a morphism $\mathrm{Ext}^n_{\mathcal{R}-Mod}(X, Y) \to \mathrm{Ext}^n_{\mathcal{R}_t - mod}(X_t, Y_t)$.

2.3.2 Homology and (hyper)-cohomology

In this section, we define the adequate notion of homology and cohomology for simplicial sheaves.

Homology groups are naturally generalized from the definition for simplicial sets in 2.2.16.

Definition 2.3.8. *Let $\mathcal{P}$ be a $\mathcal{R}-$module, and X a simplicial sheaf. Recall the alternating face map construction from 2.2.15.*
The construction 1.1.23 can be applied at every rank of a simplex, and yields a functor

$$\mathrm{Simp}(\mathcal{T}) \to \mathrm{Simp}(\mathrm{Mod}_{\mathcal{R}})$$

that is left adjoint to forgetful functor. We denote it $\mathcal{R}[\_]$.
Likewise, we define $\mathcal{R}^+[\_]$ by $\mathcal{R}^+[X] = \mathcal{R}[X]/\mathcal{R}[0]$

We may then define the notion of homology of such a complex as follows :

Definition 2.3.9. *Let X be a simplicial sheaf in $\mathcal{T}$, and $\mathcal{P}$ be a $\mathcal{R}$-module.*
The homology of X with coefficients in $\mathcal{P}$ is the $\mathcal{P}$-module defined as $\mathrm{H}_n(X, \mathcal{P}) \coloneqq \mathrm{H}_n(\mathcal{P}[X]^{\sim})$
Likewise, we define the reduced homology as $\widetilde{\mathrm{H}}_n(X, \mathcal{P}) \coloneqq \mathrm{H}_n(\mathcal{P}^+[X]^{\sim})$

As usual, homology is compatible with stalks.

Remark 2.3.10. *In this context, one can construct a Künneth map*

$$\mathrm{H}_i(X, \mathcal{P}) \otimes_{\mathcal{P}} \mathrm{H}_j(Y, \mathcal{P}) \to \mathrm{H}_{i+j}(X \times Y, \mathcal{P})$$

Let us now define a notion of cohomology, which extends 1.1.24. To insist on the fact that this involves an hyper-ext functor (and following Illusie [Ill71]), we call it *hyper*cohomology.

Definition 2.3.11. *The* ***Hypercohomology*** *of a simplicial sheaf X with coefficients in $\mathcal{P}$ is defined as*

$$\mathbb{H}^n(X, \mathcal{P}) = \mathbf{Ext}^n_{\mathcal{R}}(\mathcal{R}[X]^{\sim}, \mathcal{P})$$

Moreover, we define a reduced version as $\widetilde{\mathbb{H}}^n(X, \mathcal{P}) = \mathbf{Ext}^n_{\mathcal{R}}(\mathcal{R}^+[X]^{\sim}, \mathcal{P})$
Here $\mathbf{Ext}_{\mathcal{R}}$ denotes the hyper-ext in the category of $\mathcal{R}-$modules.

One can check that such a definition is independant of the choice of $\mathcal{R}$. As in the standard case, hypercohomology is represented by Eilenberg-MacLane spaces.

Proposition 2.3.12. *Let X be a simplicial sheaf in $\mathcal{T}$, and $\mathcal{P}$ an $\mathcal{R}-$module. Then :*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(X, K(\mathcal{P}, n)) \cong \mathbb{H}^n(X, \mathcal{P})$$

Proof. We compute :

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(X, K(\mathcal{P}, n)) &= \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(X, K(\mathcal{P}[n])) && \text{by definition} \\
&\cong \mathrm{Hom}_{\mathcal{D}(Ch_{\geq 0}(\mathcal{A}))}(NX, \mathcal{P}[n]) && \text{by (derived) adjunction} \\
&\cong \mathrm{Hom}_{\mathcal{D}(Ch_{\geq 0}(\mathcal{A}))}(X^{\sim}, \mathcal{P}[n]) && \text{by 2.2.17} \\
&\cong \mathbf{Ext}^n_{\mathcal{A}}(X^{\sim}, \mathcal{P}) && \text{by definition} \\
&\cong \mathbf{Ext}^n_{\mathcal{R}}(\mathcal{R}[X^{\sim}], \mathcal{P}) && \text{by adjunction} \\
&\cong \mathbb{H}^n(X, \mathcal{P}) && \text{by definition}
\end{aligned}$$

The Dold-Kan adjunction descends at the derived level since the morphisms N and K transform simplicial quasi-isomorphisms into quasi-isomorphisms of complexes and vice-versa (this follows from 2.2.20). □

Note that a result that was highly non trivial in the case of topological spaces now follows straightforwardly from the formalism and the Dold-Kan correspondence. Such a result grants the Eilenberg Maclane spaces the status of a *spectra*. While we will not need to use such a concept, this is a very crucial concept in modern homotopy theory, and we refer the interested reader to [Fra74] for a nice introduction.

Based on this and the analogue result on the punctual topos, points of $\mathcal{T}$ induce morphisms in hypercohomology, and thus on the Eilenberg-MacLane spaces via Yoneda's lemma.

Definition 2.3.13. *For any point t of $\mathcal{T}$, we define a morphism $\theta_t : \mathbb{H}^n(X, M) \to \mathrm{H}^n(X_t, M_t)$ induced by the morphism defined in 2.3.6.*

As announced in 1.2.25, hypercohomology allows us to rephrase the universal coefficient spectral sequence in a more standard way. This spectral sequence will play a major role in the computation of section 3.

Proposition 2.3.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be $\mathcal{R}-$modules in $\mathcal{T}$, and $n \geq 0$. There is a spectral sequence :*

$$E_2^{p,q} = \mathrm{Ext}^p_V(\mathrm{H}_q(K(\mathcal{P}, n), \mathcal{R}), \mathcal{Q}) \implies E_\infty^{p+q} = \mathbb{H}^{p+q}(K(\mathcal{P}, n), \mathcal{Q})$$

Proof. We apply 1.2.24 for $X_* = \mathcal{R}[K(\mathcal{P}, n)]^{\sim}$.
This yields $E_2^{p,q} = \mathrm{Ext}^p_V(H_q(\mathcal{R}[K(\mathcal{P}, n)]^{\sim}), \mathcal{Q}) \implies \mathbf{Ext}^{p+q}_V(\mathcal{R}[K(\mathcal{P}, n)]^{\sim}, \mathcal{Q})$.
The result follows from the definition of cohomology and hypercohomology. □

As an example, if $\mathcal{T}$ has enough points and $\mathcal{Q}$ is a field object, the spectral sequence converges at page 2, and the standard universal coefficient theorem. Such a result can be proven by passing to stalks.

2.3.3 Suspension and stabilization

In this section, we will define a notion of smash product and stabilization for simplicial sheaves, which is a simplicial analogue of 2.1.6. This will lead to the definition of stable derived functors, which will be essential in the construction of the canonical resolution.

We fix A, B and C three $\mathcal{R}-$modules, and $\mu : A \otimes_{\mathcal{R}} B \to C$ a pairing of $\mathcal{R}-$modules.

Theorem 2.3.15. *It induces a "cup-product" : $\mathbb{H}^n(\_, A) \otimes_{\mathcal{R}} \mathbb{H}^m(\_, B) \to \mathbb{H}^{n+m}(\_, C)$*

Proof. We present an outline of the construction, following [Bre78].
Since hypercohomology is represented by Eilenberg-MacLane spaces, it is equivalent to construct a map

$$d_{m,n} : K(A, m) \times K(B, n) \to K(C, n+m) \tag{1}$$

where the cartesian product of two simplicial sheaves S, T is simply $(S \times T)[n] = S_n \times T_n$.
Such a map corresponds, by the Dold-Kan adjunction, to a map $N[K(A, m) \times K(B, n)] \to C[m+n]$.

Note that the pairing μ induces a map $A[m] \otimes B[n] \to C[m+n]$ since $A[m] \otimes B[n] \cong (A \otimes B)[m+n]$ by definition of the tensor product of chain complexes.
The only step left is to construct a map $N[K(A,m) \times K(B,n)] \to N[K(A,m)] \otimes N[K(B,n)]$.
We use the Alexander-Whitney map, as constructed for example in [ML95], corrolary 8.6. ☐

As in the topological case, there is a notion of smash product of pointed simplicial sheaves. This can be expressed as the the coproduct of the following diagram :

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge Y \end{array}$$

where $X \vee Y$ is defined by the coproduct of :

$$\begin{array}{ccc} * \sqcup * & \longrightarrow & X \sqcup Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \vee Y \end{array}$$

One can check that the cup product defined above descends to the smash product

$$d_{m,n} : K(A,m) \wedge K(B,n) \to K(C, m+n) \tag{2}$$

By analogy with the standard case where $\mathcal{T} = \mathrm{Set}$ and $\mathcal{R} = \mathbb{Z}$, we define :

Definition 2.3.16. *We define the* ***simplicial circle*** $S^1 = K(\mathcal{R}, 1)$.
For $n \geq 1$, let the ***simplicial n-sphere*** *be the iterated wedge product* $S^1 \wedge \cdots \wedge S^1$.
The ***suspension*** *of a simplicial sheaf is* $\Sigma X = X \wedge S^1$.

The Freudenthal suspension theorem still holds in that setup.

Theorem 2.3.17. *(Simplicial Freudenthal) Suppose that $\mathcal{T}$ has enough points.*[36]
Let X be a simplicial $\mathcal{R}$–module and $n \geq 2$ such that $\pi_k(X) = 0$ for $1 \leq k \leq n$.
Then the map $\pi_l(X) \to \pi_{l+1}(\Sigma X)$ is an isomorphism for any $l \leq 2n$.

Proof. Since $\mathcal{T}$ has enough points, this can be checked on stalks.
This then follows from a Freudenthal theorem in Simp(Set), cf. [GJ09], Theorem 3.10. ☐

As in the standard case, we may define (stable) cohomology operations, and can identify these groups with (stable) hypercohomology of the associated Eilenberg-MacLane spaces. Everything is analogue to 2.1.3.

Proposition–Definition 2.3.18. *Let $m, n \geq 0$ and $\mathcal{R}, \mathcal{S}$ be rings of $\mathcal{T}$.*
The group of ***cohomology operations*** *of type $(\mathcal{R}, m, \mathcal{S}, n)$ is*

$$\mathrm{Op}(\mathcal{R}, m; \mathcal{S}, n) = \{\textit{Natural transformations } \mathbb{H}^m(\_, \mathcal{R}) \implies \mathbb{H}^n(\_, \mathcal{S})\}$$

where $\mathbb{H}^m(\_, \mathcal{R})$ (resp $\mathbb{H}^n(\_, \mathcal{S})$) is functor $\mathrm{Simp}(\mathcal{T}) \to \mathrm{Mod}_{\mathcal{R}}$ (resp $\mathrm{Simp}(\mathcal{T}) \to \mathrm{Mod}_{\mathcal{S}}$).
By Yoneda's lemma, $\mathrm{Op}(\mathcal{R}, n; \mathcal{S}, n+k) \cong \mathbb{H}^{n+k}(K(\mathcal{R}, n), \mathcal{S})$.
A stable cohomology operations is a family of cohomology operations commuting with suspension.
The group of stable cohomology operations can be identified with

$$\mathrm{Op}^{st}(k, \mathcal{R}, \mathcal{S}) \cong \varprojlim_n \mathbb{H}^{n+k}(K(\mathcal{R}, n), \mathcal{R})$$

The ***generalized Steenrod algebra*** *is defined as $\mathfrak{A}_{\mathcal{R}} := \bigoplus_k \mathrm{Op}^{st}(k, \mathcal{R}, \mathcal{R})$.*

[36] I don't know if this hypothesis is necessary

Note that, contrary to the standard case, this limit needs not stabilize.

For p a prime number, we let $\mathfrak{A}_p := \mathfrak{A}_{\underline{\mathbb{F}}_p}$, where $\underline{\mathbb{F}}_p$ denotes the constant sheaf valuing $\mathbb{F}_p$ in $\mathcal{T}$. While this is formally different from the algebra $\mathcal{A}_p$ defined in 2.1.4, the structure is the same, since Steenrod algebras are purely combinatorial objects. We will allow ourselves to freely identify both them.

2.3.4 Stable homology

This section is inspired by [Bre78] *and* [FF16].

In this paragraph, we assume that $\mathcal{T}$ has enough points.

The structure of the generalized Steenrod algebra is not known on arbitrary topoi. However, we will see that it admits a natural structure of a co-algebra, which suggest the existence of a *dual Steenrod algebra*, and thus of *stable homology groups*. This will lead to the notion of stable derived functors. Moreover, these dual objects are, in general, easier to understand.

Let us recall the standard case, as introduced in 2.1.4. For simplicity, we start with the case $p = 2$.

The standard Steenrod algebra $\mathcal{A}_2$ is generated by the Steenrod squares Sq^i. On the standard Steenrod algebra $\mathcal{A}_2$, the operation $\mu^* : \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$ defined by :[37]

$$\mu^*(Sq^k) = \sum_{i=0}^{k} Sq^i \otimes Sq^{k-i}$$

endows $\mathcal{A}_2$ with a structure of a commutative *coalgebra*, which is even a Hopf algebra.

This endows the dual $\mathcal{A}_2^* := \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}_2, \mathbb{F}_2)$ with an algebra structure, called the *dual* Steenrod algebra. Since $\mathcal{A}_2$ can be identified with stable cohomology of Eilenberg-MacLane spaces, its dual corresponds to *stable homology* of such spaces - at least in setups where the universal coefficient theorem holds (which is the case whenever $\mathcal{Q}$ is a field, if we use the notations from 2.3.14).

This suggests the existence of maps $\mathrm{H}_{n+i}(K(A,n),\mathcal{R}) \to \mathrm{H}_{n+i+1}(K(A,n),\mathcal{R})$, and a notion of stable homology. Let us define such morphisms.

Let A, B be $\mathcal{R}-$modules. By the Künneth morphism and functoriality, there are maps

$$\mathrm{H}_{n+i}(K(A,n),\mathcal{R}) \otimes \mathrm{H}_1(K(B,1),\mathcal{R}) \to \widetilde{\mathrm{H}}_{n+i+1}(K(A,n) \times K(B,1),\mathcal{R}) \to \widetilde{\mathrm{H}}_{n+i+1}(K(A,n) \wedge K(B,1),\mathcal{R})$$

The morphism $d_{m,n}$ from 2, associated with the trivial pairing $A \otimes_{\mathcal{R}} \mathcal{R} \to A$ induces a morphism

$$\widetilde{\mathrm{H}}_{n+i+1}(K(A,n) \wedge K(\mathcal{R},1),\mathcal{R}) \to \widetilde{\mathrm{H}}_{n+i+1}(K(A,n+1),\mathcal{R})$$

The composition yields a suspension morphism $\widetilde{\mathrm{H}}_{n+i}(K(A,n),\mathcal{R}) \to \widetilde{\mathrm{H}}_{n+i+1}(K(A,n+1),\mathcal{R})$ Moreover, such a construction is functorial in A.

Definition 2.3.19. *Let A and $\mathcal{P}$ be $\mathcal{R}$-modules. We define the* ***stable homology*** *of A as :*

$$\mathrm{H}_i^{st}(A,\mathcal{P}) := \varinjlim_n \widetilde{\mathrm{H}}_{n+i}(K(A,n),\mathcal{P})$$

Let us now construct the dual Steenrod algebra.

Proposition 2.3.20. *Fix a pairing $\mu : A \otimes_{\mathcal{R}} B \to C$*
The morphisms induced by $d_{m,n}$ are compatible with suspension and induce morphisms

$$\widetilde{\mathrm{H}}_i^{st}(A;\mathcal{R}) \otimes \widetilde{\mathrm{H}}_j^{st}(B;\mathcal{R}) \to \widetilde{\mathrm{H}}_{i+j}^{st}(C;\mathcal{R})$$

[37] This definition is very ad hoc. A more general approach will be given

This defines a structure of a commutative algebra on $\widetilde{\mathrm{H}}^{st}_*(\mathcal{R};\mathcal{R})$, called the **dual Steenrod algebra**. This generalizes, through the universal coefficient theorem, the comultiplication μ defined above.

In general, this is easier to compute than the stabilized Steenrod algebra.

Lemma 2.3.21. *The sequence* $\left(\widetilde{\mathrm{H}}_{n+k}(K(A,n),\mathcal{P})\right)_{n\geq 0}$ *is stationary.*

Proof. Homotopy groups of a suspension can be computed as in the standard case. Hence, by the Freuhental suspension theorem 2.3.17, the map $\sigma : S^1 \wedge K(m,n) \to K(M,n+1)$ induces an isomorphism in homotopy $\pi_i(S^1 \wedge K(m,n)) \to \pi_i(K(M,n+1))$ for $i < 2n-1$.
By a corrolary of the (relative) Hurewicz theorem, it also induces an isomorphism in homology

$$\widetilde{H}_i(K(A,n),B) \cong \widetilde{H}_{i+1}(K(A,n+1),B)$$

for $i < 2n-1$. This concludes the proof. □

The structure of the dual algebra of $\mathfrak{A}_p$ is well understood, but will not be needed here. We refer the reader to [Bre78] for a full description.

2.4 The canonical resolution

This paragraph will contain less details than the rest. The notions developed here will not be used in the rest of the paper, only the results will. The two important results are Theorem 2.4.2 and Remark **??**.

The goal of this section is to present a sketch of the construction of the canonical resolution, essentially due to Saunders MacLane.
Let us present the result, as stated in [ALB21].

Theorem 2.4.1. *Let* $(\mathcal{T},\mathcal{R})$ *be a ringed topos, and* $\mathcal{P}$ *a* $\mathcal{R}-$*module. There exists a complex* $M'_{(\mathcal{T},\mathcal{R})}(\mathcal{P})$ *of* $\mathcal{R}-$*modules such that :*

1. $M'_{(\mathcal{T},\mathcal{R})}(\mathcal{P})_\bullet$ *is a resolution of* $\mathcal{P}$ *as* $\mathcal{R}-$*modules.*
2. *Every* $M'_{(\mathcal{T},R)}(P)_i$ *is of the form* $\mathcal{R}^+[\mathcal{P}^{r(i)} \times \mathcal{R}^{s(i)}]$ *for some* $r(i), s(i) \geq 0$.

Moreover, the construction is functorial in P.

While this construction is the one that appears from the standard construction, the use of $\mathcal{R}^+$ is quite unpractical. However, after multiplying every term of the resolution by $\mathcal{R}[0]$, one gets the following :

Theorem 2.4.2. *Let* $(\mathcal{T},\mathcal{R})$ *be a ringed topos, and* $\mathcal{P}$ *a* $\mathcal{R}-$*module. There exists a complex* $M_{(\mathcal{T},\mathcal{R})}(\mathcal{P})$ *of* $\mathcal{R}-$*modules such that :*

1. $M_{(\mathcal{T},\mathcal{R})}(\mathcal{P})_\bullet$ *is a resolution of* $\mathcal{P}$ *as* $\mathcal{R}-$*modules.*
2. *Every* $M_{(\mathcal{T},R)}(P)_i$ *is of the form* $\mathcal{R}[\mathcal{P}^{r(i)} \times \mathcal{R}^{s(i)}]$.

This resolution will be called the **canonical resolution**.

Some authors call it "MacLane resolution" instead, but it does not pay respect to the amount of various resolutions studied by MacLane. The literature on this topic is quite erratic.[38] While the idea result is often attributed to the article [Mac57], the "standard" resolution allowed for infinite sums. Some authors seem to believe that there exists a resolution with terms of the form $\mathcal{R}[\mathcal{P}^s(i)]$, but this seems to rely on an unpublished proof by Deligne and is not fully standard. Such an amelioration is not needed here.

Remark 2.4.3. *This resolution is closely related that the Breen-Deligne resolution, that recently rose to popularity due to its importance in Condensed Mathematics (cf. [Sch19], remark 4.6.). Note that, during a recent attempt at formalizing the Breen-Deligne resolution (as part of the Liquid Tensor Experiment), they found that it was sometimes possible to use MacLane's cubic construction instead, which has the merit of being somewhat explicit. This is not possible be possible in our setup.*

[38]See [Sch19], remark 4.6, for a discussion on the history of this result

2.4.1 Stable derived functors

This section is mainly based on [Bre78] *and* [Ill72].

In this sections, all complexes are chain complexes, and we use natural analogues of the notions introduced for cochain complexes in section 1.

The Dold-Kan construction allows us to define a derived functor of a non-additive functor.

Let R, S be (real) rings. If a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_S$ is not additive, its natural prolongation to chain complexes needs not preserve homotopies, and thus may not descend at the level of derived category.
It however induces a well-behaved functor $\mathrm{Simp}(R) \to \mathrm{Simp}(S)$. Let us use this idea.

Proposition–Definition 2.4.4. *Let $\mathcal{R}$ and $\mathcal{S}$ be rings of a topos $\mathcal{T}$, and $F : \mathrm{Mod}_{\mathcal{R}} \to \mathrm{Mod}_{\mathcal{S}}$ a functor such that $F(0) = 0$. For $X_\bullet \in Ch_{\geq 0}(\mathcal{R})$, we define :*

$$F(X) = N \circ F \circ K(X)$$

This functor preserves homotopies, and descends as $LF : \mathcal{D}(\mathrm{Mod}_{\mathcal{R}}) \to \mathcal{D}(\mathrm{Mod}_{\mathcal{S}})$
*The **derived functors** of F are $L_i F = H_i \circ LF : \mathcal{D}(\mathrm{Mod}_{\mathcal{R}}) \to \mathrm{Mod}_{\mathcal{S}}$.*

Note that, if F is additive, this coincides with the usual derived functors.

Somewhat surprisingly, homotopy groups of Eilenberg-MacLane spaces naturally appear as derived functors.

Proposition 2.4.5. *For $\mathcal{P}$ an $\mathcal{R}$–module, the i-th derived functor of $\mathcal{R}^+[\_] : \mathcal{T} \to \mathrm{Mod}_{\mathcal{R}}$ satisfies*

$$\mathrm{L}_i \mathcal{R}^+(\mathcal{P}[n]) = \mathrm{H}_i(K(\mathcal{P}, n), \mathcal{R})$$

Proof. The functor $\mathcal{R}^+[\cdot]$ extends as $\mathcal{R}^+[X] = N \circ \mathcal{R}^+ \circ K(X)$.
If $X = \mathcal{P}[n], \mathcal{R}^+[X] = N(\mathcal{R}^+[K(P,n)])$. We conclude since $NX \to X^\sim$ is a homotopy equivalence. ☐

The construction of stable homology may then be extended to arbitrary functors.

Definition 2.4.6. *Let $\mathcal{S}$ be a ring in $\mathcal{T}$, and $F : \mathrm{Mod}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{R}}$ be a functor such that $F(0) = 0$.*
*Its **stable derived functor** is*

$$\mathrm{L}_i^{st} F(M) = \varinjlim_n \mathrm{L}_{i+n} F(K(M, n))$$

where L_i denotes the i-th left derived functor of F, and M is a $\mathcal{S}$–module.

For example, stable homology groups appear as stable derived functor of $\mathcal{R}^+[\_]$.

As is the case with standard derived functors, one can regroup all of these objects under a single one, living at the derived level. This involves many technical difficulties, and we refer to [Ill72], 11.4.

Proposition 2.4.7. *For any F as above, there exists a functor $F^{st} : \mathrm{Mod}_{\mathcal{S}} \to \mathrm{Simp}(\mathrm{Mod}_{\mathcal{R}})$ such that*

$$\mathrm{H}_i(F^{st}(M)^\sim) \cong \mathrm{L}_i^{st} F(M)$$

Moreover, this construction is compatible with pairings.

If F is of the form $\mathcal{R}^+[\_]$, we let $\mathcal{R}^{st}(M)$ be the evaluation at M of the functor associated to the stabilization of F. It is a simplicial $\mathcal{R}$–module.
In particular, $\mathcal{R}^{st}(M)$ is a simplicial non-commutative ring[39]. Its homology satisfies :

$$\mathrm{H}_i(\mathcal{R}^{st}(M)^\sim) = \mathrm{H}_i^{st}(M, \mathcal{R})$$

[39] It is commutative up to homotopy

2.4.2 The Bar construction

This section is inspired by [Car54], [EL53] *and* [Bre78].

We present the bar resolution associated to a differential graded augmented algebra. Historically, this was introduced in order to explicitely compute some Tor groups. Our main motivation remains the definition the canonical resolution.

In this section, we fix R a commutative ring. Unless mentioned otherwise, tensor products are over R.

Definition 2.4.8. *Let A be an $R-$algebra with an augmentation morphism $A \xrightarrow{\eta} R$. The **bar resolution** is the resolution $B(A)_\bullet \xrightarrow{\varepsilon} R$ of R as $A-$modules given by :*

1. *$B(A)_n = A^{\otimes n+1}$, with structure of A-module induced by the right-most term.*
2. *The differential $d_n : B(A)_n \to B(A)_{n-1}$ is*

$$d_n(a_0 \otimes \cdots \otimes a_n) = (-1)^{n-1}\eta(a_0)a_1 \otimes \cdots \otimes a_n + \sum_{j=1}^{n-1}(-1)^{n-j-1}a_0 \otimes \cdots \otimes a_j \cdot a_{j+1} \otimes \cdots \otimes a_n$$

3. *The augmentation is $\epsilon = \eta$.*

Disclaimer. *The alternating signs throughout this section are notoriously tricky, and may be wrong.*

If A admits in addition a structure of a differential graded algebra, the complex $B(A)_n$ is naturally endowed with additional structure.

Definition 2.4.9. *A **differential graded augmented** R-algebra; or R-DGA for short, is an R-algebra A that admits :*

1. *A graded structure $A \cong \bigoplus_{n \geq 0} A_n$, with multiplication $A_p \times A_q \to A_{p+q}$.*
2. *Differentials $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$ and $d_n(xy) = d_k(x)\cdot y + (-1)^k x \cdot d_{n-k}(y)$ if $x \in A_k$. When the indices are clear from the context, we simply write d or d_A.*
3. *An morphism of $R-$algebras $A \xrightarrow{\nu} R$, called the **augmentation**, such that $\varepsilon \circ d = 0$ and $\epsilon(x) = 0$ whenever $x \in A_k, k \geq 1$*[40]

*An element $a \in A$ is said to be **homogeneous** if it belongs if A_k for some k. Such a k is known as its **degree**, noted $deg(a)$. If a is not homogeneous, we define its degree to be $deg(a) := max_i\,(deg(a_i))$ where $a = a_0 + \cdots + a_k$ is the homogeneous decomposition of a.*

If A and B are two $R-DGA$, the tensor product $A \otimes B$ is naturally endowed with an $R-DGA$ structure given by $A \otimes B = \bigoplus_{n=p+q} A_p \otimes B_q$.

Let us now define a $R-$DGA structure on $B(A) := \bigoplus_{n \in \mathbb{N}} B(A)_n$

Define on $B(A)$ the **total** degree :

$$deg(a_0 \otimes \cdots \otimes a_n) := n + \sum_k deg(a_k)$$

Let $B(A)_{n,s}$ be the homogeneous part of degree s of $B(A)$, such that $B(A) = \bigoplus_{n,s} B(A)_{n,s}$.

The differentials d_n defines above lower the degree by 1, since they lower the *length* by one.[41]

Define the differential $\partial_s : B(A)_{n,s} \to B(A)_{n,s-1}$ induced by the differential on A :

$$\partial_s(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n} a_0 \otimes \cdots \otimes d(a_j) \otimes \cdots \otimes a_n$$

The differentials ∂_s and d_n endow $B(A)$ with a structure of a double chain complex.

Moreover, if M is a differential graded $A-$module, $B(A) \otimes_A M$ is a bicomplex of $M-$modules.

[40] This is a morphism of DGA algebras $A \to R$, where we endow R with the trivial structure concentrated in degree 0.

[41] The left-most term of $d_n(a_0 \otimes \cdots \otimes a_n)$ is of degree 0 whenever $deg(a_0) \neq 0$.

Remark 2.4.10. *One of the main interest of the bar construction is that the total complex associated to $B(A)_\bullet \otimes_A M$ represents the derived tensor product $R \overset{L}{\otimes}_A M$.*
As a consequence, $\mathrm{H}_*(\mathrm{Tot}(B(A) \otimes_R M)) = \mathrm{Tor}_*^A(R, M)$, where $\mathrm{Tor}_k^A(R, \_)$ denotes the k-left derived functor of $R \otimes_A \_$, and R is viewed as an A-algebra via η.
The spectral sequences associated to the double complex, as described in 1.2.4, yield somewhat explicit ways to compute such a complex. We refer to [Bre78] for details and additional differences.

2.4.3 Definition of the canonical resolution

We will define the terms appearing in the canonical resolution. We will not check that they satisfy the desired properties, and refer to [Mac57] and [Ill72] for details on that matter.
Fix $(\mathcal{T}, \mathcal{R})$ a ringed topos, and let $\mathcal{P}$ be an $\mathcal{R}$−module.
Recall the definition of the simplicial $\mathcal{R}$−modules $\mathcal{R}^{st}(\mathcal{R})$ and $\mathcal{R}^{st}(\mathcal{P})$.
The associated alternating complex 2.2.15 $\mathcal{R}^{st}(\mathcal{R})^\sim$ is naturally a $\mathcal{R}$−DGA, and $\mathcal{R}^{st}(\mathcal{P})^\sim$ is a differential graded $\mathcal{R}^{st}(\mathcal{R})^\sim$-module.

Definition 2.4.11. *We define the bicomplex $\mathcal{M}_{\bullet,\bullet}(P) = B(\mathcal{R}^{st}(\mathcal{R})^\sim) \otimes_{\mathcal{R}^{st}(\mathcal{R})^\sim} \mathcal{R}^{st}(\mathcal{R})^\sim$*
Let $M'_\bullet(P)$ be the associated total complex

One can check that this satisfies the properties from 2.4.1.
Let us conclude with a more *philosophical* remark.

Remark 2.4.12. **??** *The complexity of the canonical resolution stems from the difficulty of the description of $\mathcal{R}^{st}$, which itself comes from the complexity of the homology of the Eilenberg-MacLane spaces.*
Note that this difficulty is purely combinatorial, and does not relate to the geometric properties of the topos, nor of the chosen ring.[42]
The coefficients $r(i)$ and $s(i)$ are hence in dependant of the topos.

This will be crucial when comparing an algebraic and an analytic topoi in section 5.

[42] The invariance by change of base ring will be more precisely stated in 3.3.7

3 Extension of the additive group on the (perfect) étale site

3.1 Introduction

In this section, we specify the notions from the previous sections to the additive group on the étale topos. Let us start by establishing some notations.

3.1.1 Notations and overview

We fix $S = \mathrm{Spec}(R)$ a perfect scheme of characteristic p.
We let S_{perf} be the **perfect étale site**, whose objects are perfect schemes over S, and covers are given by jointly surjective étale maps.[43] Let $S_{ét}$ be the (big) étale site over S.
Let $\underline{\mathrm{V}}_S^{perf}$ (resp. $\underline{\mathrm{V}}_S^{ét}$) be the category of sheaves of $\mathbb{F}_p$−vector spaces on S^{perf} (resp $S^{ét}$).
Let $\underline{\mathrm{Ab}}_S^{perf}$ (resp. $\underline{\mathrm{Ab}}_S^{ét}$) be the category of sheaves of abelian groups on S^{perf} (resp $S^{ét}$).
Let $\underline{\mathrm{T}}_S^{ét}$ be the topos of sheaves of sets over $S^{ét}$, and $\underline{\mathrm{T}}_S^{perf}$ the topos of sheaves of sets over S_{perf}.
Alternatively, $\underline{\mathrm{V}}_S^{perf}$ (resp $\underline{\mathrm{V}}_S^{ét}$) is the category of $\underline{\mathbb{F}}_p$-modules in $\underline{\mathrm{T}}_S^{perf}$ (resp $\underline{\mathrm{T}}_S^{ét}$).
The **additive group**, noted $\mathbb{G}_a$, is the sheaf defined by $X \mapsto \mathcal{O}_X(X)$. By abuse of notation, we see it as an object of $\underline{T}_S^{perf}, \underline{T}_S^{ét}, \underline{V}_S^{perf}$ and $\underline{T}_S^{perf}$. It is also a group scheme over S.

The goal of this section is to prove the following result :

Theorem 3.1.1. *The self-extension groups of $\mathbb{G}_a$ over S^{perf} as $\mathbb{F}_p$-vector spaces are :*

$$\mathrm{Ext}^n_{S^{perf},\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \begin{cases} R[T, T^{-1}]^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

where $R[T, T^{-1}]^{nc}$ denotes the non-commutative ring of Laurent polynomials, and T corresponds to the Frobenius morphism on R.

The self-extension groups of $\mathbb{G}_a$ on the étale site were heavily studied by Lawrence Breen in the beginning of his carreer. He achieved the following full computation in [Bre78], after a series of papers [Bre75], [Bre69a], [Bre69b].

Theorem 3.1.2. *([Bre78], theorem 1.3.) The self-extension groups of $\mathbb{G}_a$ over $S^{ét}$ as $\mathbb{F}_p$-vector spaces are :*

$$\mathrm{Ext}^n_{S^{ét},\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \begin{cases} R[T, T^{-1}]^{nc} / \left(T^{v_p(n/2)+1}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

where v_p denotes the p−adic valuation.

Note that the result in degree 1 was (essentially) already known by Serre [Ser88].
In the short article [Bre81], he cleans up his arguments and explains how to transfer his result on the étale site to the computation announced above on the *perfect* étale site. In fact, in order to deduce the theorem above, one only needs the following partial result :

Proposition 3.1.3. *For any $n > 0$, $\mathrm{Ext}^n_{S^{ét},\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a)$ is killed by some large power of the Frobenius.*

Let us conclude by saying a few words about the importance of the hypotheses.

Remark 3.1.4. *1. The result still stands with other usual topologies (Zariski, v, fpqc, ...).*

[43] Some variants appear in the literature. For example, in [BS15], Bhatt and Scholze consider the site of perfect quasi-compact and quasi-separated objects over X. The results still hold on such a site.

2. *An analogue result still holds when viewing $\mathbb{G}_a$ as sheaves of $\mathbb{F}_q$−vector spaces when S is a scheme over* $\operatorname{Spec}(\mathbb{F}_q)$*. This will be useful for section 5, and will be derived from the result above in 5.1.2.*
3. *The result does not hold when considering extension groups as abelian groups rather than sheaves of vector spaces. As we will see later (cf. 3.2.12), the extension group does not vanish in degree* 1.
4. *The method described here does not work over a non-affine base. The computation makes key use of the fact that higher cohomology of quasi-coherent sheaves on affine basis vanishes.*

We shall also note that Breen did not have a good *a priori* reason to study a perfect site, except for the fact that it allowed a convenient and nontrivial reformulation of his result on the étale site. The modern study of perfectoid spaces justifies the setup and allows for new perspectives on this result - see for example [Mat22].

Let us now outline the proof of 3.1.3.

3.1.2 Outline of the proof

Recall the notations from section 2.

We work in the étale topos $(\mathcal{T}, \mathcal{R}) = (\underline{\mathcal{T}}_S^{ét}, \underline{\mathbb{F}}_p)$.

The proof relies mostly on three ideas. The first one is the use of the canonical resolution, the second one is the use of the universal coefficient spectral sequence to the adequate simplicial object, and the third one is a study of the generalized Steenrod algebra in the étale topos.

The canonical resolution 2.4.2 writes as follows :

Definition 3.1.5. *There exists a resolution $M(\mathbb{G}_a)_\bullet \to \mathbb{G}_a$ where each $\mathrm{M}(\mathbb{G}_a)_i$ is of the form $\underline{\mathbb{F}}_p[X_i]$ for X_i an étale sheaf of the form $\mathbb{G}_a^{r(i)} \times \underline{\mathbb{F}}_p^{s(i)}$.*

One can check that the sheaves $\mathbb{G}_a^r \times \underline{\mathbb{F}}_p^s$ are represented by $\operatorname{Spec}(R^s[X_1, \dots, X_r])$. Hence, the extension groups $\operatorname{Ext}^n_{S^{ét}, \mathbb{F}_p}(M(\mathbb{G}_a)_j, \mathbb{G}_a)$ can be understood as étale cohomology groups of quasi-coherent sheaves over an affine base, and thus vanish whenever $i > 0$.

This allows us to compute $\operatorname{Ext}^i_{S^{ét}, \mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a)$ as the homology of the complex :[44]

$$0 \longrightarrow \operatorname{Hom}_{S^{ét}, \mathbb{F}_p}(M(\mathbb{G}_a)_0, \mathbb{G}_a) \longrightarrow \operatorname{Hom}_{S^{ét}, \mathbb{F}_p}(M(\mathbb{G}_a)_1, \mathbb{G}_a) \longrightarrow \dots$$

Unfortunately, since we cannot know the coefficients $r(i)$ and $s(i)$, this cannot be used for a direct computation. However, the fact that extension groups can be computed as the cohomology of such a simple complex admits many important consequences, such as the following :

Lemma 3.1.6. $\operatorname{Ext}^n_{S^{ét}, \mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) \cong \operatorname{Ext}^n_{\operatorname{Spec}(\mathbb{F}_p)^{ét}, \mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) \otimes_{\mathbb{F}_p} R$

Thus we may assume $S = \operatorname{Spec}(\mathbb{F}_p)$.

Then, one may note that $\mathbb{G}_a = \mathrm{H}_0^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p)$. The stabilized version of the universal coefficient spectral sequence described in 2.3.14 now yields :

$$E_2^{p,q} = \operatorname{Ext}^p_V(\mathrm{H}_q^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p), \mathbb{G}_a) \implies E_{p,q}^\infty = \mathbb{H}_{st}^{p+q}(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a)$$

The terms $E_2^{p,0}$ are then the desired terms. The limiting term can be described as a generalised Steenrod algebra. The terms $\operatorname{Hom}(\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p), \mathbb{G}_a)$ can be described in terms of the dual Steenrod algebra.

By a careful consideration of the structure of the spectral sequence and an explicit description of the terms and the morphisms, we will prove the desired result inductively on $n \geq 2$.

While the case $n = 0$ is essentially trivial, the case $n = 1$ is nontrivial and relies on very different tools. The proof that we present is due to Serre.

[44] Note that the resolution is not projective, even if it is 'free'.

3.2 Computation in low degree

In this section, we compute the extension groups in degree 0 and 1 on the étale site. In section 4, we will see how to deduce a computation over $\underline{T}_S^{perf}$.

3.2.1 In degree 0

In degree 0, since $\mathbb{G}_a$ is representable, we can compute directly $\mathrm{Hom}_{V_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a)$ using Yoneda's lemma.

Lemma 3.2.1. *The sheaf $\mathbb{G}_a$ is represented by $\mathbb{A}_S^1$ over $S^{\acute{e}t}$ and by $\mathbb{A}_S^{1,perf}$ over S^{perf}.*

Proof. We do the computation on the perfect étale site. If X is a perfect S−scheme,
$\mathrm{Hom}_{S-sch}(X, \mathbb{A}_S^{1,perf}) \cong \mathrm{Hom}_{S-sch}(X, \mathbb{A}_S^1) = \mathrm{Hom}_{S-sch}(X, S \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{A}_{\mathbb{Z}}^1)$
$\cong \mathrm{Hom}_{Sch}(X, \mathrm{Spec}(\mathbb{Z}[T])) \cong \mathrm{Hom}_{\mathbb{Z}-Alg}(\mathbb{Z}[T], \mathcal{O}_X(X)) \cong \mathcal{O}_X(X)$
For the first equality, we used the fact that the perfection functor is right adjoint to the forgetful one. □

Proposition 3.2.2. $\mathrm{Hom}_{V_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a) \cong R[T]^{nc}$*, which denotes the non-commutative polynomial ring, where the commutation relation is $T \cdot r = r^p \cdot T$.*
The element T will be called "the Frobenius morphism" in the following.

Proof. Since $\mathbb{G}_a$ is represented by $\mathbb{A}_S^1$, by Yoneda's lemmma, we get :

$$\mathrm{Hom}_{V_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a) \cong \mathrm{Hom}_{S-sch}(\mathbb{A}_S^1, \mathbb{A}_S^1) \cong \mathrm{Hom}_{R-alg}(R[T], R[T])$$

Where the ring structure on $\mathrm{Hom}_{R-alg}(R[T], R[T])$ is given by composition.
A morphism $f : R[T] \to R[T]$ are entirely determined by the polynomial $P = f(T)$, which satisfies $P(X+Y) = P(X) + P(Y)$. In characteristic p, such polynomials are exactly polynomials in X^p.
Hence $\mathrm{Hom}_{V_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a) \cong \{g(X^p), g \in R[X]\} \cong R[T]^{nc}$ where $T = X^p$. □

The computation in degree 1 is due to Serre, in [Ser88], VII-11. We give an outline of the proof below. Note that the techniques used here are very different from the ones used in higher dimension.

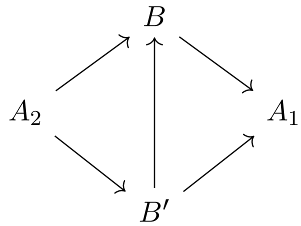
3.2.2 In degree 1 - Extensions groups, à la Serre

This section is based on [Ser88], [Oor66] *and* [Poo09].
We will establish the following computation of the first extension group on the étale site.

Proposition 3.2.3. *Let $S = \mathrm{Spec}(k)$ for some field k of characteristic p.*
The group $\mathrm{Ext}^1_{S^{\acute{e}t}, Ab}(\mathbb{G}_a, \mathbb{G}_a)$ is isomorphic to $\mathrm{Hom}_{S^{\acute{e}t}, Ab}(\mathbb{G}_a, \mathbb{G}_a) \cong R[T]^{nc}$, and a generator is given by the extension $0 \to \mathbb{G}_a \xrightarrow{V} \mathbb{W}_{k,2} \xrightarrow{R} \mathbb{G}_a \to 0$ (where we used the notations from 1.3.25).

When Serre studied algebraic groups in 1957, schemes did not exist, and sheaves were not completely standard yet. His notion of extension groups was of a more geometric nature.
Recall the reformulation of the first extension groups in an abelian category as a group of extensions :

Definition 3.2.4. *Let $\mathcal{A}$ be an abelian category, and A_1, A_2 be objects of $\mathcal{A}$.*
*An **extension** of A_1 by A_2 is a short exact sequence : $0 \to A_2 \to B \to A_1 \to 0$.*
*Two such sequences with middle terms B and B' are said to be **equivalent** if there exists an isomorphism $B \to B'$ such that the diagram below commutes.*



The ***extension group*** $\mathrm{Ext}(A_1, A_2)$ *is the equivalence class of such extensions.*
This is endowed with a group structure with the Baer sum - which is induced by the direct sum.

Proposition 3.2.5. *The extension group defined above is isomorphic to* $\mathrm{Ext}^1_{\mathcal{A}}(A_1, A_2)$*, the first derived morphism of* $\mathrm{Hom}_{\mathcal{A}}$.

Proof. This is a standard result - see for example [Wei94], Theorem 3.4.3. □

Serre considered such extensions of abelian group schemes over an algebraically closed field.

Note that, over an arbitrary base scheme, the category of abelian group schemes is not abelian (it is over a field). That said, one can generalize Serre's definition into an appropriate notion of exact sequence and extensions groups, and establish results analogue to the one above.

For simplicity, and following [Poo09] we'll assume everything to be fppf over the base. Over a field, the flatness condition is automatic, and fppf schemes are exactly the one of finite type.

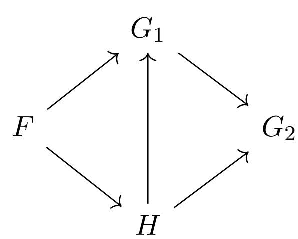
Definition 3.2.6. *Fix* S *a scheme, and* $f : G \to H$ *a morphism between commutative fppf group schemes over* S.
We let $\mathrm{Ker}(f)$ *be the group scheme over* S *defined by* $\mathrm{Ker}(f)(T) = \mathrm{Ker}(f(T) : G(T) \to H(T))$.
The morphism f *is* ***surjective*** *if for every* S*−scheme* T *and every* $h \in H(T)$*, there is a fppf morphism* $T' \to T$ *such that the image of* h *in* $H(T')$ *is the image of some* $g \in G(T')$.
A sequence $0 \to F \xrightarrow{f} G \xrightarrow{p} H \to 0$ *of fppf abelian group schemes over* S *is said to be* ***exact*** *if :*

1. $F = \mathrm{Ker}(p)$*,* $p \circ f = 0$ *and* p *is surjective.*
2. p *is faithfully flat and of finite presentation (fppf for short).*

Note that the second condition is very important, even if we didn't assume that F, G, H were fppf, since it allows the use of descent.

We may now define the adequate extension groups.

Definition 3.2.7. *Let* H, F *be commutative fppf group schemes over* S.
A (Serre) ***extension*** *of* H *by* F *is a short exact sequence* $0 \to F \xrightarrow{f} G \xrightarrow{p} H \to 0$ *in the above sense. The group of equivalence classes of such extensions will be noted* $\mathrm{Ext}_S(H, F)$*, where two classes* $G_1, G_2 \in \mathrm{Ext}_S(H, F)$ *are said to be* ***equivalent*** *if there exists an isomorphism of commutative group schemes* $G_1 \to G_2$ *such that the diagram below commutes.*



The appropriate topology to consider is the fpqc topology, since it allows for the use of some nice descent results.

Definition 3.2.8. *Let S be a scheme.*
The (big) ***fpqc***[45] *site on S, noted S_{fpqc}, is the category of all $S-$schemes, in which covers are exactly the $\{f_i : U_i \to U\}_{i\in I}$ where every f_i is flat and, for any affine $W \subset U$, there exists a finite $J \subset I$ and affine subsets $(V_j \subset U_j)_{j\in J}$ such that $\bigcup_{j\in J} f_j(V_j) = W$.*

We may now state the desired result :

Proposition 3.2.9. *Assume in addition that F is affine over S. There is an isomorphism :*

$$\mathrm{Ext}_S(H,F) \cong \mathrm{Ext}^1_{S^{fpqc},\mathrm{Ab}}(H,F)$$

where the right hand side is the Ext *functor computed in the category of sheaves of abelian groups over the fpqc site of S, and where we identified group schemes with the sheaf they represent.*

Proof. We follow [Oor66], 17.5. We omit some proofs, and refer to [Oor66] for detail. For clarity, we use curved letters to denote sheaves, and straight one to denote group schemes, seen as geometric objects.

By 3.2.4, the RHS term can be explicited as the group of extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, where objects here are abelian group schemes over S^{fpqc}. An exact sequence of group schemes clearly induces morphisms on the associated sheaves. The technical assumption that p is fpqc assures that this forms an exact sequence.
In order to construct an extension of group scheme from an extension of sheaves, one uses the following key lemma :

Lemma 3.2.10. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of sheaves, such that $\mathcal{F}, \mathcal{H}$ are representable by group schemes of finite type F, H over S, with $F \to S$ is affine.*
Then $\mathcal{G}$ is representable by a group scheme of finite type.

This is essentially a result about the effectivity of some fpqc descent data, that can be seen as a corollary of theorem 2 of [Gro60]. Moreover, under our technical hypothesis, the associated sequence of group schemes is exact.

Finally, one can easily check that both notions of equivalence coincide. ☐

There are two main differences between the Serre extension group and the one considered by Breen :

1. We consider the fpqc site rather than the étale site.
2. We consider sheaves of abelian groups rather than $\mathbb{F}_p-$vector spaces.

As will be discussed in 3.3.6, the extension groups are independant of the choice of a (reasonable) topology. The second point is more important. As we will see, there are non trivial extensions of $\mathbb{G}_a$ by itself as abelian groups, but they are not extensions as $\mathbb{F}_p-$vector spaces.

3.2.3 In degree 1 - Extensions and factor systems

In this section, we follow [Ser88].
We assume that $S = \mathrm{Spec}(k)$, where k is a field of characteristic $p > 0$. This will suffice based on 3.1.6. Recall that, in that setup, fppf reads as "of finite type".

For short, we note $\mathrm{Ext}_k := \mathrm{Ext}_{\mathrm{Spec}(k)}$. In this paragraph, every extension group is a Serre extension group, unless specified otherwise.

[45]This is french for "faithfully flat and quasi-compact"

Definition 3.2.11. *Fix $G \in \mathrm{Ext}_k(H,F)$ for H,F commutative group schemes of finite type. For $\varphi : F \to F'$ a morphism of group schemes, we define*

$$\varphi_*(G) := \{G \times F'\} / \langle \varphi(f) + f = 0, \forall f \in F \rangle$$

such that $0 \to F \to \varphi_*(G) \to H \to 0$ is an extension.
This defines a group morphism $\varphi \in \mathrm{Hom}_k(F,F') \mapsto \varphi_* G \in \mathrm{Ext}_k(H,F')$.

In our setup, we have the extension :

$$0 \to \mathbb{G}_a \xrightarrow{V} \mathbb{W}_{k,2} \xrightarrow{R} \mathbb{G}_a \to 0 \tag{3}$$

using the notation from 1.3.25.
We claim that this extension is essentially the only non-trivial one.

Proposition 3.2.12. *The map $\varphi \in \mathrm{Hom}_k(\mathbb{G}_a, \mathbb{G}_a) \mapsto \varphi^* W_2 \in \mathrm{Ext}_k(\mathbb{G}_a, \mathbb{G}_a)$ is a group isomorphism.*

In order to prove this statement, we need to study precisely the structure of $\mathrm{Ext}_k(\mathbb{G}_a, \mathbb{G}_a)$. Let us do this in a more general setup, in which the arguments remain the same.
Fix A, B two connected abelian group schemes over k. For simplicity, we let $+$ and $-$ denote the group operations in B.
We will see that $\mathrm{Ext}_k(A,B)$ is essentially controlled by a sheaf cohomology group $\mathrm{H}^1(A, B_A)$ (with the Zariski topology), and a group of symmetric 2-cocycles $\mathrm{H}^2_{reg}(A,B)$.

Definition 3.2.13. *A **symmetric factor system** is a morphism of schemes $f : A \times A \to B$ such that*

1. *$f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0$ for all $x,y,z \in A$.*
2. *$f(x,y) = f(y,x)$ for all $x,y \in A$.*

*For any morphism $g : A \to B$, the map $\delta g(x,y) := g(x+y) - g(x) - g(y)$ is a symmetric factor system, called **trivial**.*
Define $H^2(A,B)_s = \{$ symmetric factor systems $\} / \{\delta g \mid g : A \to B$ scheme morphism $\}$.

Proposition 3.2.14. *$H^2(A,B)_s$ is isomorphic to the subgroup $\mathrm{Ext}_k^{split}(A,B) \subset \mathrm{Ext}_k(A,B)$ of extensions $C \in \mathrm{Ext}_k(A,B)$ admitting a section $A \to C$.*

Proof. If $s : A \to C$ is such a section, define $f(x,y) = s(x+y) - s(x) - s(y) : A \to B \cong ker(C \to A)$. The difference between two sections can be lifted to a morphism $A \to B$, and induces a trivial factor system. This defines a map $\theta : \mathrm{Ext}_k^{split}(A,B) \to H^2(A,B)_s$. Let us show that θ is bijective.
If $\theta(C) = 0$, then there exists a section $s : A \to C$ that is linear, hence $C \cong A \oplus B$ as group schemes, and the extension is equivalent to the trivial one. Thus θ is injective.
For $f : A \times A \to B$ a symmetric factor system, the composition law

$$(a,b) \star (a',b') = (a +_A a', b + b' + f(a,a'))$$

on $A \times B$ forms an extension inducing f. Hence θ is surjective. ☐

Since $\mathbb{G}_a$ is representable by a very convenient scheme, one can explicitely compute $H^2_{reg}(\mathbb{G}_a, \mathbb{G}_a)_s$.

Proposition 3.2.15. *If k is of characteristic $p > 0$, $H^2(\mathbb{G}_a, \mathbb{G}_a)_s$ is a $k-$vector space of countable dimension. A basis is given by the classes of $(F^{p^i})_{i \geq 0}$, where $F(x,y) = \frac{1}{p}(x^p + y^p - (x+y)^p)$.*

Proof. Regular symmetric factor systems $\mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ coincide, by Yoneda's lemma, with functions $F : k[T] \to k[X,Y]$ such that the polynomial $f = F(T)$ satisfies $f(x,y) = f(y,x)$ and

$$f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0$$

It is clear that the F^{p^i} are such factors, and linear combinations of them are as well. One can check that those are the only one. This is for example done in [Laz55], section III. □

The only question left is to understand which extensions admit regular sections. A corollary of a result of Rosenlicht [Ros56] is as follows :

Lemma 3.2.16. *If $C \in \mathrm{Ext}_k(A,B)$ for A and B products of $\mathbb{G}_a$ and $\mathbb{G}_m$, then $C \to A$ admits a* ***rational*** *section. (i.e. there exists a covering $C = \bigcup_i U_i$ with morphisms $U_i \to A$ that form local sections, and coincide on the intersections).*

Proof. (Sketch) By the general theory of principal homogeneous spaces (cf. [LT58]), this is equivalent to the vanishing of the Galois cohomology groups $\mathrm{H}^1(\mathrm{Gal}(k^{sep}/k), k)$ and $\mathrm{H}^1(\mathrm{Gal}(k^{sep}/k), k^{sep})$.
This follows from the additive and multiplicative version of Hilbert 90's. □

The last step is showing that a rational section can be glued to a morphism. This is essentially a result of vanishing of Čech cohomology.

Proposition 3.2.17. *If $C \in \mathrm{Ext}_k(\mathbb{G}_a, \mathbb{G}_a)$, then $C \to \mathbb{G}_a$ admits a scheme-theoretic section.*

Proof. We apply the above lemma. Consider the covering $\mathbb{G}_a = \bigcup_i U_i$ together with sections $s_i : U_i \to C$. Each one induces a local trivialisation $s_i^{-1}(U_i) \cong \mathbb{G}_a \oplus U_i$, and thus, by the splitting lemma, we can view $s_i : U_i \to \mathbb{G}_a$.
Those section form a 1-cocycle on $\mathbb{A}^1_k$ valued in $\mathcal{O}_{\mathbb{G}_a}$. Since $\check{H}^1(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{G}_a}) = 0$ as higher cohomology of a quasi-coherent sheaf of an affine scheme is zero, the cocycle is a coboundary. Hence, they are induced by a global section $\mathbb{G}_a \to C$. □

Remark 3.2.18. *In general, if $C \in \mathrm{Ext}_k(A,B)$, one can define the sheaf $B_{\mathcal{A}}$ of germs of rational maps $B \to A$. The first cohomology group $H^1(B, B_{\mathcal{A}})$ can be interpreted as the group of classes of fiber spaces with base A and structure group B. We refer the reader to [Ser88] for more detail about this approach.*

We have proven that $\mathrm{Ext}_k(\mathbb{G}_a, \mathbb{G}_a) \cong H^2(\mathbb{G}_a, \mathbb{G}_a)_s \cong \mathrm{Vect}_k(F^i)_{i \geq 1}$. From there, we can prove 3.2.12.

Proof. The addition in vector Witt rings of length 2 is given by

$$(X_0, X_1) + (Y_0, Y_1) = \left(X_0 + X_0, X_1 + Y_1 - \frac{1}{p}((X_0+Y_0)^p - X_0^p - Y_0^p)\right)$$

Hence, by the reciprocal process described in the proof of 3.2.14, we see that the extension α described by 3 corresponds to the factor system F.
Every element $\varphi \in Hom_k(\mathbb{G}_a, \mathbb{G}_a)$ can be written as $\varphi(t) = \sum b_i t^{p^i}$, and every element $f \in \mathrm{Ext}_k(\mathbb{G}_a, \mathbb{G}_a)$ corresponds to a factor system of the form $\sum a_i F^{p^i}$
One can check that the map $\varphi \mapsto \varphi_* \mathbb{W}_{k,2}$ (as defined in 3.2.11) corresponds to $\sum b_i t^{p^i} \mapsto \sum b_i F^{p^i}$.
Hence the group morphism is a bijection, and thus a group isomorphism. □

We have shown that $\mathrm{Ext}^1_{\mathrm{Spec}(k)^{fpqc}, Ab}(\mathbb{G}_a, \mathbb{G}_a) \cong k[T]^{nc}$. As explained earlier, the result still holds when replacing the fpqc site by the étale site. However, there is no such extension as $\mathbb{F}_p$−vector spaces.

Proposition 3.2.19. *For any perfect field k of characteristic p,* $\mathrm{Ext}^1_{\mathrm{Spec}(k)^{\acute{e}t}, \underline{\mathbb{F}}_p}(\mathbb{G}_a, \mathbb{G}_a) = 0$

Proof. Any extension as sheaves $\mathbb{F}_p$−vector spaces is an extension sheaves of abelian groups. Since such extensions are generated by $\mathbb{W}_2$, it suffices to show that the extension $0 \to \mathbb{G}_a \to \mathbb{W}_2 \to \mathbb{G}_a \to 0$ defined in 3 is not an extension of sheaves of $\mathbb{F}_p$-vector spaces.
By general theory of sheaves, exact sequences of sheaves are sequences that are exact at every stalks. Points of the étale topos are given by geometric points, i.e. morphisms coming from the spectrum of algebraically closed fields.
Hence, the extension $0 \to \mathbb{G}_a \to \mathbb{W}_2 \to \mathbb{G}_a \to 0$ induces $0 \to \bar{k} \to W_2(\bar{k}) \to \bar{k} \to 0$.
This is not an exact sequence of $\mathbb{F}_p$−vector spaces, since since exact sequence of vector spaces are split, and the additive structure on $\bar{k} \oplus \bar{k}$ and $W_2(\bar{k})$ does not match. □

This establishes the desired result in degree 1.

3.3 In higher degree

In this section, we follow the structure of [Bre81] *and use many arguments from* [Bre78].
Let's go back to our previous setup. Ext groups are now computed in the category of sheaves of $\mathbb{F}_p$−vector spaces on the étale site of S, rather then as group schemes, and $S = \mathrm{Spec}(R)$ is a perfect affine scheme of characteristic p.

Starting from now, and when there is no confusion, we'll simply denote by $\mathbb{F}_p$ the constant sheaf $\underline{\mathbb{F}}_p$.

Let us start by discussing important consequences of the existence of the canonical resolution.

3.3.1 Consequences of the canonical resolution

Note the canonical resolution be $M(\mathbb{G}_a)_\bullet \to \mathbb{G}_a$, where each $M(\mathbb{G}_a)_i$ is of the form $\mathbb{F}_p[\mathbb{G}_a^{r(i)} \times \mathbb{F}_p^{s(i)}]$, as given in 2.4.2. Let us start with some crucial lemmas.

Definition 3.3.1. *For G a group, let G_S be the group scheme over S defined by*

$$G_S(T) = \{ \text{ locally constant } f : |T| \to G\}$$

for any T−scheme S. We call it the ***constant group scheme*** *associated to G.*

It can be seen as the sheafification of the constant presheaf valuing G on the topos of all S−schemes, endowed with the Zariski topology.

Lemma 3.3.2. *The sheaf $\mathbb{G}_a^r \times (\mathbb{F}_p^s)_S$ is representable by the affine S−scheme* $\mathrm{Spec}(R^{\mathbb{F}_p^s}[X_1, \dots, X_r])$.

Proof. Let T be an S−scheme. We compute :

$$\begin{aligned}
\mathrm{Hom}_{S\text{-}sch}(T, \mathrm{Spec}(R^s[X_1,\dots,X_r])) &\cong Hom_{R\text{-}alg}(R^{\mathbb{F}_p^s}[X_1,\dots,X_r], \mathcal{O}_T(T)) \\
&\cong \mathrm{Hom}_{R\text{-}alg}(R[X_1,\dots,X_r] \otimes_R R^{\mathbb{F}_p^s}, \mathcal{O}_T(T)) \\
&\cong \mathrm{Hom}_{R\text{-}alg}(R[X_1,\dots,X_r], \mathcal{O}_T(T)) \times \mathrm{Hom}_{R\text{-}alg}(R^{\mathbb{F}_p^s}, \mathcal{O}_T(T)) \\
&\cong \mathbb{G}_a^r(T) \times \mathrm{Hom}_{S\text{-}Sch}(T, \bigsqcup_{i \in \mathbb{F}_p^s} S)
\end{aligned}$$

Note $G = \mathbb{F}_p^s$, and let $(S_g)_{g \in G}$ be copies of S. We conclude using the following lemma :

Lemma 3.3.3. *Let X be a scheme and G a group. Then X_S is represented by $\bigsqcup_{g \in G} X_g$.*

Proof. Morphisms $T \to \bigsqcup_{g \in \mathbb{F}_p^s} X_g$ are determined by the reciprocal images $U_g = f^{-1}(S_g)$, which form an open disjoint cover of T. Clearly, $f \restriction U_i$ is necessarily the unique morphism of X−schemes $U_i \to X_g \cong X$. Such a function is then uniquely determined by the choice of the X_g. It determines and is determined by the map valuing g on X_g, which is a locally constant function $|T| \to G$. □

Hence $\mathrm{Hom}_{S-sch}(T, \bigsqcup_{i\in\mathbb{F}_p^s} S_i) \cong (\mathbb{F}_p^s)_S(T)$. ☐

The following observation is essentially trivial, but is absolutely crucial. It is due to L.Breen in [Bre69a].

Lemma 3.3.4. *Let X be an $S-$scheme, and $\mathcal{F}$ an element of $\underline{V}_{S^{\acute{e}t}}$. Then*

$$\mathrm{Ext}^n_{\underline{V}_S^{\acute{e}t}}(\mathbb{F}_p[h^X], \mathcal{F}) \cong \mathrm{H}^n_{\acute{e}t}(X, \mathcal{F})$$

where the right term denotes the étale cohomology of the sheaf $\mathcal{F}$ on X.

Proof. The left hand term is the n-th right derived functor of $\mathrm{Hom}_{V_S^{\acute{e}t}}\left(\mathbb{F}_p[h^X], \_\right)$ applied at $\mathcal{F}$.
The right hand term is the n-th right derived functor of $\Gamma(X, \_)$ applied at $\mathcal{F}$.
Hence, it suffices to show that the two functors coincide in degree 0.
By adjunction and Yoneda's lemma,

$$\mathrm{Hom}_{\underline{V}_S^{\acute{e}t}}(\mathbb{F}_p[h_X], \mathcal{F}) \cong \mathrm{Hom}_{\underline{T}_S^{\acute{e}t}}(h^X, \mathcal{F}) \cong \mathcal{F}(X) \cong \Gamma(X, \mathcal{F})$$

since morphisms of sheaves are simply natural transforms of the underlying presheaves. ☐

Note that the adequately modified result above holds in other topologies (fpqc, v, Zariski, ...) on S.

The combination of both lemmas above yields the announced way to compute extension groups.

Proposition 3.3.5. *The groups $\mathrm{Ext}^n_{V_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a)$ can be computed as the cohomology of the complex*

$$0 \longrightarrow \mathrm{Hom}_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_0, \mathbb{G}_a) \longrightarrow \mathrm{Hom}_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_1, \mathbb{G}_a) \longrightarrow \dots$$

Moreover, for all $i > 0$, $\mathrm{Hom}_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_0, \mathbb{G}_a) \cong \mathcal{O}_{X_i}(X_i)$.

Proof. By the reformulation of Ext groups as morphisms in the derived category, they are invariant by quasi-isomorphisms, so that $\mathrm{Ext}^n_{\underline{V}_S^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a) = \mathbf{Ext}^n_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_\bullet, \mathbb{G}_a)$.
Pick $I^\bullet$ an injective resolution of $\mathbb{G}_a$. The spectral sequence ${}_IE^{p,q}$ established in 1.2.4 associated to the double complex $\mathrm{Hom}(M(\mathbb{G}_a)_\bullet, I^\bullet)$ yields

$${}_IE_1^{p,q} = \mathrm{Ext}^q_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_p, \mathbb{G}_a) \implies {}_IE_\infty^{p,q} = \mathbf{Ext}^{p+q}_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_\bullet, \mathbb{G}_a)$$

By the lemma above, $\mathrm{Ext}^q_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_p, \mathbb{G}_a) = \mathrm{H}^q_{\acute{e}t}(X_i, \mathbb{G}_a)$.
We recall a fundamental result of étale cohomology ([Mil80], III.3.8) :

Proposition. *Let $\mathcal{G}$ a quasi-coherent $\mathcal{O}_X-$module on a scheme X. There is a canonical isomorphism $H^i_{Zar}(X, \mathcal{G}) \cong H^i_{\acute{e}t}(X, W(\mathcal{G}))$ where $W(\mathcal{G})$ is defined as $W(\mathcal{G})(U) \coloneqq \Gamma(U, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_U)$ for U an $X-$scheme.*

We compute $W(\mathcal{O}_X)(U) = \Gamma(U, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_U) = \mathcal{O}_U(U)$, hence $W(\mathcal{O}_X) = \mathbb{G}_a$.
Thus, $\mathrm{H}^q_{\acute{e}t}(X_i, \mathbb{G}_a) = \mathrm{H}^q_{Zar}(X_i, \mathcal{O}_{X_i})$.
It is well known (cf. [Sta22, Section 01X8]) that cohomology of quasi-coherent sheaves on the Zariski topology can be computed as Čech cohomology, where open covers are taken with affine subsets. Hence, the higher cohomology of quasi-coherent sheaves on affine schemes is zero (cf. [Gro61], Theorem I.3.1).
Finally, the only nonzero terms of the first page are ${}_IE_1^{p,0} = \mathrm{Hom}_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_p, \mathbb{G}_a)$, and the differentials of a cohomological spectral sequence in degree 1 match the horizontal ones of the complex. ☐

This result is not very usable in practice since the construction of the X_i is remarkably non constructive.
However, the existence of such a construction yields many important consequences, such as :

1. Invariance of the choice of the topology.

2. Invariance via flat base change.

Remark 3.3.6. *(Independence of the topology)*
We compute everything on the étale topos. However, the result still holds for any subcanonical topology on S satisfying the adequate analogue of the proposition used in the proof above.
For example, it works with the flat cohomology (cf. [Mil80], III.3.7), and justifies the computation in degree 1.
In the section 5, one will be interested in the algebraic v-topology, for which the result still holds.

Remark 3.3.7. *(Commutation with flat base change)*
Let $T \to S$ be a flat morphism from an affine scheme $T = \mathrm{Spec}(R')$. From the description of the schemes representing the sheaves appearing in the canonical resolution (cf. 3.3.2), it is clear that the complex introduced in 3.3.5 commutes with the flat base change.
Since homology commutes with flat rings extensions, it follows that, for all $n \geq 0$,

$$\mathrm{Ext}^n_{\underline{V}_T^{\acute{e}t}}(\mathbb{G}_a, \mathbb{G}_a) \cong \mathrm{Ext}^n_{\underline{V}_S^{\acute{e}t}}(M(\mathbb{G}_a)_i, \mathbb{G}_a) \otimes_R R'$$

In particular, since the morphism $S \to \mathrm{Spec}(\mathbb{F}_p)$ is flat[46]*, it suffices to show the vanishing result when $S = \mathrm{Spec}(\mathbb{F}_p)$.*

In what follows, we'll assume that $S = \mathrm{Spec}(\mathbb{F}_p)$.

3.3.2 The universal coefficient spectral sequence

To simplify the notations, we let $V := \underline{V}_S^{\acute{e}t}$. The proof of Breen relies on the following :

Lemma 3.3.8. *We have $\mathrm{H}_0^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p) = \mathbb{G}_a$*

Proof. Since the étale topos has enough points and $\underline{\mathbb{F}}_p$ is a field object, we can use the universal coefficient theorem to show that $\mathrm{H}_0^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p) = \mathrm{H}_0^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$.
Now, $\mathrm{H}_0^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{Z}) = \varinjlim \mathrm{H}_n(\mathrm{K}(\mathbb{G}_a, n), \mathbb{Z}) = \varinjlim \pi_n(\mathrm{K}(\mathbb{G}_a, n)) = \varinjlim \mathbb{G}_a = \mathbb{G}_a$ by Hurewicz 2.3.4.
We conclude since $\mathbb{G}_a \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{G}_a$. ∎

We may then apply the universal coefficient spectral sequence, as established in 2.3.14. The other terms in the spectral sequence will then be interpreted as a generalized Steenrod algebra and its dual.

The universal coefficient spectral sequence yields, for any $n \in \mathbb{N}$:

$$E_2^{p,q} = \mathrm{Ext}^p_V(\mathrm{H}_{q+n}(\mathrm{K}(\mathbb{G}_a, n), \mathbb{F}_p), \mathbb{G}_a) \implies E^\infty_{p,q} = \mathbb{H}^{p+q+n}(\mathrm{K}(\mathbb{G}_a, n), \mathbb{G}_a)$$

Every term in this sequence is a $\mathbb{F}_p$−vector space, and admits a natural left action by the Frobenius morphism $F \in \mathrm{Hom}_{\mathcal{T}}(\mathbb{G}_a, \mathbb{G}_a)$, by Yoneda's product. It is hence a sequence of left $\mathbb{F}_p[F]^{nc}$−modules.
Let $n \to \infty$, such that the spectral sequence becomes :[47]

$$E_2^{p,q} = \mathrm{Ext}^p_V(\mathrm{H}_q^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p), \mathbb{G}_a) \implies E^\infty_{p,q} = \mathbb{H}^{p+q}_{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a) \quad (4)$$

We see that $E_2^{p,0} = \mathrm{Ext}^p_V(\mathbb{G}_a, \mathbb{G}_a)$. Hence, the theorem 3.1.3 can be rewritten as :

Theorem 3.3.9. *For any $n > 0$, $E_2^{n,0}$ is killed by some power of the Frobenius.*

It is quite unpractical to deal with terms modulo powers of the Frobenius. Thankfully, using Serre classes allows a convenient reformulation.

[46]Everything is flat over a field
[47]Note that the notation is unfortunate, since the index p is used both as the index of spectral sequences, and as a fixed prime number. The notation should be clear from the context.

Definition 3.3.10. *Let $\mathcal{A}$ be an abelian category.*
*A **Serre subcategory** of $\mathcal{A}$ is a full subcategory $\mathcal{C} \subset \mathcal{A}$, such that for every exact sequence $A \to B \to C$ in $\mathcal{A}$ with A, C objects of $\mathcal{C}$, B is also an object of $\mathcal{A}$.*

One can easily check that the full subcategory of left $\mathbb{F}_p[F]^{nc}$–modules killed by some large enough power of the Frobenius is a Serre category.

Proposition 3.3.11. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ which is essentially surjective and whose kernel is C, such that exact functors $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ between abelian categories are exactly exact functors $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset \mathrm{Ker}(G)$.*

Proof. cf. [Sta22, Lemma 02MS] □

In what follows, we will implicitely work in the quotient category of left $\mathbb{F}_p[F]^{nc}$-modules, modulo large enough powers of the Frobenius.

We have already seen that 3.3.9 is true when $n = 1$. We will prove that the result follows by induction. The spectral sequence becomes especially useful because of the following observation :

Lemma 3.3.12. *If $E_2^{p,0}$ for some p, then $E_2^{p,q} = 0$ for any $q \geq 0$.*

Proof. Assume that $\mathrm{Ext}^p(\mathbb{G}_a, \mathbb{G}_a) = 0$, and pick any $q \geq 0$.
Let us first establish the following, which will be useful on its own.

Proposition 3.3.13. *We have $\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p) \cong \mathrm{H}_n^{st}(\mathrm{K}(\mathbb{F}_p), \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{G}_a$.*

Proof. For a (slightly) more general statement, see [Bre81], lemma 2.5.
Since the étale topos admits enough points, it suffices to show that the isomorphism stands at stalks. Let $t = \mathrm{Spec}(k)$ be a geometric point, where k is an algebraically closed field of characteristic p.
We know that stalks commutes with (stable) homology, such that :

$$\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{G}_a), \underline{\mathbb{F}}_p)_t = \mathrm{H}_n^{st}(\mathrm{K}((\mathbb{G}_a)_t), (\underline{\mathbb{F}}_p)_t) = \mathrm{H}_n^{st}(\mathrm{K}(k), \mathbb{F}_p)$$

The lemma clearly holds when replacing $\mathbb{G}_a$ by $\mathbb{F}_p$. Since the morphism $\mathrm{H}_n^{st}(\mathrm{K}(\_), \mathbb{F}_p)$ is additive, it holds for finite modules. We conclude by writing k as a projective limit of the finite subextensions. □

As we will see in 3.3.23, $\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{F}_p), \mathbb{F}_p)$ is a $\mathbb{F}_p$-vector space of finite dimension. Hence, by the proposition above, we may write $\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p) = \mathbb{G}_a^r$ for some $r \geq 0$.
Hence $\mathrm{E}_2^{p,q} = \mathrm{Ext}_V^p(\mathbb{G}_a^r, \mathbb{G}_a) = \bigoplus_{k=1}^r \mathrm{Ext}_V^p(\mathbb{G}_a, \mathbb{G}_a) = 0$. □

3.3.3 A fundamental long exact sequence

We fix $n > 1$ such that the result holds for any $1 \leq k \leq n$. Our goal is to establish the following :

Proposition 3.3.14. *There is a long exact sequence :*

$$0 \longrightarrow \mathbb{H}_{st}^n(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a) \xrightarrow{\alpha} \mathrm{Hom}_V(\mathrm{H}_n^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p), \mathbb{G}_a) \xrightarrow{d} \mathrm{Ext}_V^{n+1}(\mathbb{G}_a, \mathbb{G}_a) \xrightarrow{\beta} \mathbb{H}_{st}^{n+1}(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a) \to \mathrm{Hom}_V(\mathrm{H}_{n+1}^{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{F}_p), \mathbb{G}_a)$$

Remark 3.3.15. *Contrary to what the presentation somewhat suggests, at this stage of the induction, the exact sequence cannot be prolonged in the obvious way.*
It can be, if we already know that all the $\mathrm{Ext}_V^k(\mathbb{G}_a, \mathbb{G}_a)$ are trivial.

Proof. In what follows, the homology groups H_n are implicitely computed with coefficients in $\mathbb{F}_p$. The second page of the spectral sequence, reduced modulo $\mathcal{C}_F$, can then be written as :

$$
\begin{array}{c|cccccc}
q=n+1 & \mathrm{Hom}_V(\mathrm{H}^{st}_{n+1}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & 0 & \cdots & 0 & \mathrm{Ext}^{n+1}_V(\mathrm{H}^{st}_{n+1}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & \cdots \\
q=n & \mathrm{Hom}_V(\mathrm{H}^{st}_{n}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & 0 & \cdots & 0 & \mathrm{Ext}^{n+1}_V(\mathrm{H}^{st}_{n}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\
q=1 & \mathrm{Hom}_V(\mathrm{H}^{st}_{1}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & 0 & \cdots & 0 & \mathrm{Ext}^{n+1}_V(\mathrm{H}^{st}_{1}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & \cdots \\
q=0 & \mathrm{Hom}_V(\mathbb{G}_a,\mathbb{G}_a) & 0 & \cdots & 0 & \mathrm{Ext}^{n+1}_V(\mathbb{G}_a,\mathbb{G}_a) & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & \cdots & p=n & p=n+1 & \cdots
\end{array}
$$

Remember that the differentials at page r are given by $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q+r-1}$. Hence, every d_r starting from the first column or from the the term at coordinates $(p,q)=(n+1,0)$ vanishes for $2\le r\le n$. The differential d_{n+1} goes from $(0,q)$ to $(n+1,q-n)$. Therefore, the page E_{n+2} is of the form :

$$
\begin{array}{c|cccccc}
q=n+1 & \mathrm{Ker}(\mathrm{d}^{0,n+1}_{n+1}) & 0 & \cdots & 0 & * & \cdots \\
q=n & \mathrm{Ker}(\mathrm{d}^{0,n}_{n+1}) & 0 & \cdots & 0 & * & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\
q=1 & \mathrm{Hom}_V(\mathrm{H}^{st}_{1}(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) & 0 & \cdots & 0 & * & \cdots \\
q=0 & \mathrm{Hom}_V(\mathbb{G}_a,\mathbb{G}_a) & 0 & \cdots & 0 & \mathrm{Coker}(\mathrm{d}^{0,n}_{n+1}) & \cdots \\
\hline
E_{n+2}^{p,q} & p=0 & p=1 & \cdots & p=n & p=n+1 & \cdots
\end{array}
$$

where $*$ denotes arbitrary (a priori nonzero) terms.
Finally, all the terms explicited in the picture are unchanged in the subsequent page.
We will now use the process from **??** to compute the limiting term given the terms at finite rank.
By applying **??** to the $n-$th diagonal, we get :

$$\mathrm{Ker}(\mathrm{d}^{0,n}_{n+1}) \cong \mathbb{H}^n_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a) \tag{5}$$

Reading the $n+1-$th diagonal gives the short exact sequence :

$$0\to \mathrm{Coker}(\mathrm{d}^{0,n+1}_{n+1}) \to \mathbb{H}^{n+1}_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a) \to \mathrm{Ker}(\mathrm{d}^{0,n+1}_{n+1}) \to 0 \tag{6}$$

Since $\mathrm{d}^{0,n+1}_{n+1}$ is a morphism $\mathrm{Hom}_V(\mathrm{H}^{st}_{n+1}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)) \to \mathrm{Ext}^{n+1}_V(\mathbb{G}_a,\mathbb{G}_a)$, we get :

$$0\to \mathrm{Ker}(\mathrm{d}^{0,n}_{n+1}) \to \mathrm{Hom}_V(H_{n+1}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)) \to \mathrm{Ext}^{n+1}_V(\mathbb{G}_a,\mathbb{G}_a) \to \mathrm{Coker}(\mathrm{d}^{0,n}_{n+1}) \to 0 \tag{7}$$

The substitution of (7) into (6) yields :

$$0\to \mathbb{H}^n_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a) \to \mathrm{Hom}_V(\mathrm{H}^{st}_n(\mathrm{K}(\mathbb{G}_a)),\mathbb{G}_a) \to \mathrm{Ext}^{n+1}_V(\mathbb{G}_a,\mathbb{G}_a) \to \mathbb{H}^{n+1}_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a) \to \mathrm{Ker}(\mathrm{d}^{0,n+1}_{n+1}) \to 0$$

This concludes since $\mathrm{Ker}(\mathrm{d}^{0,n+1}_{n+1}) \subset \mathrm{Hom}_V(\mathrm{H}^{st}_{n+1}(K(\mathbb{G}_a),\mathbb{G}_a))$. ☐

The only step left is to understand the morphisms α and β. We will see that $\beta = 0$ and that α is surjective (modulo $\mathcal{C}_F$), which concludes the proof.

3.3.4 The morphism β and behaviour with stalks

Let us start with β.

Proposition 3.3.16. $\beta = 0$

Proof. The gist of the proof lies in the following lemma, which is a variant of [Bre81], lemma 2.12.

Lemma 3.3.17. *The stalk morphism defined in 2.3.13 and applied to the algebraic closure $\overline{\mathbb{F}_p}$:*

$$\mathbb{H}^n_{st}(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a) \to \mathbb{H}^n_{st}(\mathrm{K}(\overline{\mathbb{F}_p}), \overline{\mathbb{F}_p})$$

is an isomorphism.

Proof. The argument is a variation from the one in the proof of [Bre75], proposition 2

For any $k \geq 0$, hypercohomology can be computed as the limiting term of the spectral sequence :

$$\mathrm{E}_1^{p,q} = \mathrm{H}^p(\mathrm{K}(\mathbb{G}_a, k)_q, \mathbb{G}_a) \implies \mathrm{E}_\infty^{p+q} = \mathbb{H}^{p+q}(\mathrm{K}(\mathbb{G}_a, k), \mathbb{G}_a)$$

Indeed, fix $\mathbb{G}_a \to I^\bullet$ an injective resolution, and let $\mathrm{K}^{p,q} = \mathrm{Hom}(\mathbb{F}_p[\mathrm{K}(\mathbb{G}_a, k)_p], I^q)$. The first spectral sequence given in 1.2.4 gives

$${}_{\mathrm{I}}\mathrm{E}_1^{p,q} = \mathrm{Ext}^q_V(\mathbb{F}_p[\mathrm{K}(\mathbb{G}_a, k)_p], \mathbb{G}_a) \implies \mathrm{Ext}^{p+q}_V(\mathbb{F}_p[\mathrm{K}(\mathbb{G}_a, k)]^\sim, \mathbb{G}_a)$$

Note that $\mathrm{H}^p(\mathrm{K}(\mathbb{G}_a, k)_q, \mathbb{G}_a) = 0$ whenever $p > 0$. Indeed, every every $\mathrm{K}(\mathbb{G}_a, k)_q$ is finite power of $\mathbb{G}_a$, and hence representable by an affine scheme. Since the sheaf $\mathbb{G}_a$ is quasi-coherent for the Zariski topology, higher cohomology terms vanish, and we conclude by an argument similar to 3.3.1.

Hence, $\mathbb{H}^n(\mathrm{K}(\mathbb{G}_a, k), \mathbb{G}_a)$ can be computed as the homology of $(\mathbb{H}^0(\mathrm{K}(\mathbb{G}_a, k)_p, \mathbb{G}_a))_{p\geq 0}$.

By construction, $\mathrm{K}(\mathbb{G}_a, k)_p = \mathbb{G}_a^{r(p,k)}$ for some $r(p,k)$. Hence, we have

$$\mathrm{H}^0(\mathrm{K}(\mathbb{G}_a, k)_p, \mathbb{G}_a) = \mathrm{H}^0(\mathbb{G}_a^{r(p,k)}, \mathbb{G}_a) \cong \mathrm{H}^0_{ét}(\mathbb{A}^{r(p,k)}_{\mathbb{F}_p}, \mathbb{G}_a) \cong \mathbb{F}_p[X_1, \ldots, X_{r(p,k)}]$$

with differentials being morphisms of $\mathbb{F}_p$−vector spaces.

Likewise, $\mathbb{H}^n(\mathrm{K}(\overline{\mathbb{F}_p}, k), \overline{\mathbb{F}_p})$ can be computed as the limiting term of a spectral sequence :

$${}_{\mathrm{I}}\mathrm{E}_1^{p,q} = \mathbf{Ext}^q_{\mathbb{F}_p}\left(\mathbb{F}_p\left[\overline{\mathbb{F}_p}^{r(p,k)}\right], \overline{\mathbb{F}_p}\right) \implies \mathbb{H}^n\left(\mathrm{K}(\overline{\mathbb{F}_p}, k), \overline{\mathbb{F}_p}\right)$$

Clearly, all the higher extension terms Ext^q for $q > 0$, vanish ; hence the hypercohomology can be computed as the cohomology of a complex whose elements are of the form :

$$\begin{aligned}
\mathrm{Hom}_{\underline{V}_S^{ét}}\left(\mathbb{F}_p\left[\overline{\mathbb{F}_p}^{r(p,k)}\right], \overline{\mathbb{F}_p}\right) &\cong \mathrm{Hom}_{\underline{T}_S^{ét}}\left(\overline{\mathbb{F}_p}^{r(p,k)}, \overline{\mathbb{F}_p}\right) && \text{by adjunction}\\
&\cong \mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_p)}\left(\mathrm{Spec}\left(\mathbb{F}_p^{\overline{\mathbb{F}_p}^{r(p,k)}}\right), \mathrm{Spec}\left(\mathbb{F}_p^{\overline{\mathbb{F}_p}}\right)\right) && \text{by 3.3.3 and Yoneda}\\
&\cong \prod_{\overline{\mathbb{F}_p}^{r(p,k)}} \mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_p)}\left(\mathrm{Spec}(\mathbb{F}_p), \coprod_{\overline{\mathbb{F}_p}} \mathrm{Spec}(\mathbb{F}_p)\right) && \text{Since } \mathrm{Spec}(\textstyle\prod \cdot) = \coprod \mathrm{Spec}(\cdot)\\
&\cong \prod_{\overline{\mathbb{F}_p}^{r(p,k)}} \left\{\text{Locally constant } |\mathrm{Spec}(\mathbb{F}_p)| \to \overline{\mathbb{F}_p}\right\} && \text{by 3.3.3}\\
&\cong \prod_{\overline{\mathbb{F}_p}^{r(p,k)}} \overline{\mathbb{F}_p}
\end{aligned}$$

Moreover, one can check that, throught these identifications, the map $i_{\overline{\mathbb{F}_p}}$ can then be interpreted by the mapping that maps a polynomial to the associated polynomial function on $\overline{\mathbb{F}_p}$:

$$i_{\overline{\mathbb{F}_p}} : P \in \mathbb{F}_p[X_1, \dots, X_{r(p,k)}] \mapsto \prod_{(x_1,\dots,x_{r(p,k)}) \in \overline{\mathbb{F}_p}} P(x_1, \dots, x_{r(p,k)})$$

It is bijective since $\overline{\mathbb{F}_p}$ is infinite, and respects the differential of the complex.
Thus, the stalk morphism induced on the spectral sequences induces an isomorphism in the first page, and thus the limiting terms are isomorphic.
We have shown that, for every $n, k \geq 0$, the morphism $\mathbb{H}^n(\mathrm{K}(\mathbb{G}_a, k), \mathbb{G}_a) \to \mathbb{H}^n(\mathrm{K}(\overline{\mathbb{F}_p}, k), \overline{\mathbb{F}_p})$ is an isomorphism. The desired result is obtained by stabilization. □

Remark 3.3.18. *In [Bre81], Breen mentions an argument that is conceptual more difficult, but is too nice to be omitted.*
Since the computation of the spectral sequence $_{\mathrm{I}}\mathrm{E}$ *above does not depend on the choice of the, one can compute it on the chaotic topos, for which every presheaf is a sheaf. In that topos, the morphisms from* $\mathrm{Spec}(\mathbb{F}_q)$ *induce points for any* $q = p^f$.
If one chooses f *large enough, one can carry the same argument as above by replacing* $\overline{\mathbb{F}_p}$ *by* $\mathbb{F}_q$.

Now that we proved the lemma, let us go back to the main proof. One can check that the universal coefficient spectral sequence is compatible with stalks, in the sense that, for $t = \mathrm{Spec}(\overline{\mathbb{F}_p})$, the stalk morphisms defined in 2.3.7 and 2.3.13 can be inserted in a commutative diagram :

$$\begin{array}{ccc} \mathrm{Ext}_V^{n+1}(\mathbb{G}_a, \mathbb{G}_a) & \xrightarrow{\beta} & \mathbb{H}_{st}^{n+1}(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a) \\ \downarrow \theta_t & & \downarrow \theta_t \\ \mathrm{Ext}_{\mathbb{F}_p}^{n+1}(\overline{\mathbb{F}_p}, \overline{\mathbb{F}_p}) & \xrightarrow{\beta_t} & \mathbb{H}_{st}^{n+1}(\mathrm{K}(\overline{\mathbb{F}_p}), \overline{\mathbb{F}_p}) \end{array}$$

Clear, $\mathrm{Ext}_{\mathbb{F}_p}^{n+1}(\overline{\mathbb{F}_p}, \overline{\mathbb{F}_p}) = 0$, since the $\overline{\mathbb{F}_p}$ are free as $\mathbb{F}_p$-modules.
By the lemma above, $\beta = 0$. □

We now turn our attention to α. We will show the following :

Proposition 3.3.19. *α is surjective modulo $\mathcal{C}_F$.*

First, let us present some structure results regarding the Steenrod algebra in the étale topos.

3.3.5 The morphism α - Structure of the generalized Steenrod algebras

Let us explain the structure of the generalized Steenrod algebra $\mathbb{H}_{st}^n(K(\mathbb{G}_a), \mathbb{G}_a)$. This section should be read with 2.1.4 in mind.
This section only contains statements. We will give some elements of the proofs in the appendix 6. Note that these results are very specific to the sheaf of $\mathbb{G}_a$, and does not hold for other standard étale sheaves.
As in the topological case, the algebra $\mathbb{H}_{st}^n(K(\mathbb{G}_a), \mathbb{G}_a)$ can be understood as a ring of cohomology operations. The following is a variant of 2.1.23.

Proposition 3.3.20. *Let $p = 2$. As in the topological case, we might define Steenrod squares* $\mathrm{Sq}^i(X)$ *for any simplicial étale sheaf X as :*

$$(\mathrm{Sq}^i(X))_{n \geq 0} : \mathbb{H}^n(X, \mathbb{G}_a) \to \mathbb{H}^{n+i}(X, \mathbb{G}_a)$$

They form stable cohomology operations of type $(\mathbb{G}_a, n; \mathbb{G}_a, n+i)$*, which correspond to* $Sq^i \in \mathbb{H}^i(\mathrm{K}(\mathbb{G}_a), \mathbb{G}_a)$.

The algebra $\mathbb{H}^*(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)$ *is isomorphic to the free* $\mathbb{F}_p$*-algebra generated multiplicatively by the* Sq^i *such that, for any* $0 < a < 2b$,

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2\rfloor}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j$$

Remark 3.3.21. *When there is no ambiguity, we denote the same way the Steenrod squares in* $\mathbb{H}^*(K(\mathbb{G}_a),\mathbb{G}_a)$ *and in* $\mathbb{H}^*(K(\mathbb{F}_p),\mathbb{F}_p)$*. Note that these objects are conceptually very different.*

The only difference with the topological case is that $\mathrm{Sq}^0 \neq Id$. In fact, Sq^0 corresponds to the Frobenius endomorphism on $\mathbb{G}_a$.
When $p \neq 2$, the structure is a bit more complicated. Compare the following to 2.1.27.

Proposition 3.3.22. *Let* $p > 2$*. As in the topological case, we might define the p-th power operations and the Bockstein morphism as stable cohomological operations of type* $(\mathbb{G}_a, m; \mathbb{G}_a, m + 2i(p-1))$ *and* $(\mathbb{G}_a, m; \mathbb{G}_a, m+1)$ *respectively, which correspond to* $P^i \in \mathbb{H}^i_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)$ *and* $\beta \in \mathbb{H}^i_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)$
The algebra $\mathbb{H}^*_{st}(\mathrm{K}(\mathbb{G}_a),\mathbb{G}_a)$ *is isomorphic to the free* $\mathbb{F}_p$*-algebra multiplicatively generated by* β *and the* P^i *such that :*

- *If* $a < p\cdot b$, $P^aP^b = \sum_i (-1)^{a+i}\binom{(p-1)(b-i)-1}{a-pi}P^{a+b-i}P^i$
- *If* $a \le p\cdot b$, $P^a\beta P^b = \sum_i (-1)^{a+i}\binom{(p-1)(b-i)}{a-pi}\beta P^{a+b-i}P^i + \sum_i (-1)^{a+i+1}\binom{(p-1)(b-i)-1}{a-pi-1}P^{a+b-i}\beta P^i$

Note that we do not assume that $P^0 = 1$.

Let us now turn our attention to the dual algebra $\mathrm{H}^{st}_n(K(\mathbb{G}_a),\mathbb{F}_p)$. As shown in 3.3.13, it is enough to understand $\mathrm{H}^{st}_n(K(\mathbb{F}_p),\mathbb{F}_p)$.
The precise structure is well understood, but will be needed. We refer the reader to [Bre78] for a statement in the general case, and to [Mil58] for a proof of such results. We will simply need :

Proposition 3.3.23. *For any* $n \ge 0$, $\mathrm{H}^{st}_n(K(\mathbb{F}_p),\mathbb{F}_p)$ *is a free* $\mathbb{F}_p$*-algebra of finite type.*

which follows from the universal coefficient theorem (since $\underline{\mathbb{F}}_p$ is a field object and $\mathcal{T}$ has enough points).

3.3.6 The morphism α

Recall that $\alpha : \mathbb{H}^n(K(\mathbb{G}_a),\mathbb{G}_a) \to \mathrm{Hom}_V(\mathrm{H}^{st}_n(K(\mathbb{G}_a),\mathbb{F}_p),\mathbb{G}_a)$. Let us rewrite the image of α. We get :

$$\begin{aligned}
\mathrm{Hom}_V(\mathrm{H}^{st}_n(\mathrm{K}(\mathbb{G}_a),\mathbb{F}_p),\mathbb{G}_a) &\cong \mathrm{Hom}_V(\mathrm{H}^{st}_n(\mathrm{K}(\mathbb{F}_p),\mathbb{F}_p)\otimes \mathrm{H}^{st}_0(\mathrm{K}(\mathbb{G}_a),\mathbb{F}_p),\mathbb{G}_a) && \text{by 3.3.13}\\
&\cong \mathrm{Hom}_V(\mathrm{H}^{st}_n(\mathrm{K}(\mathbb{F}_p),\mathbb{F}_p))\otimes \mathrm{Hom}_V(\mathrm{H}^{st}_0(\mathrm{K}(\mathbb{G}_a),\mathbb{F}_p),\mathbb{G}_a) && \text{by 3.3.23}\\
&\cong \mathbb{H}^n_{st}(\mathrm{K}(\mathbb{F}_p),\mathbb{F}_p)\otimes R[F]^{nc}
\end{aligned}$$

where the last isomorphism relies on the universal coefficient theorem, 3.2.2 and 3.3.8.
Moreover, this identification maps the action of the Frobenius on $\mathrm{Hom}(\mathrm{H}^{st}_n(\mathrm{K}(\mathbb{G}_a),\mathbb{F}_p),\mathbb{G}_a)$ to the multiplication by $1\otimes F$ on $\mathbb{A}^n \otimes R[F]^{nc}$.
Hence α can be seen as a morphism $\mathbb{H}^n_{st}(K(\mathbb{G}_a),\mathbb{G}_a) \to \mathbb{H}^n_{st}(K(\mathbb{F}_p),\mathbb{F}_p)\otimes R[F]^{nc}$

Lemma 3.3.24. *Via those identifications, the morphism* α *is identified with the composition*

$$\mathbb{H}^n_{st}(K(\mathbb{G}_a),\mathbb{G}_a) \xrightarrow{\mu} (\mathrm{H}^*_{st}(K(\mathbb{F}_p),\mathbb{F}_p)\otimes \mathbb{H}^*_{st}(K(\mathbb{G}_a),\mathbb{G}_a))^n \xrightarrow{\pi} \mathrm{H}^n_{st}(K(\mathbb{F}_p),\mathbb{F}_p)\otimes \mathbb{H}^0_{st}(K(\mathbb{F}_p),\mathbb{F}_p)$$

where the second morphism is the projection in bidegree $(n,0)$ *and the first one is the restriction in degree* n *of the "coproduct" defined on the generators by :*[48]

[48]Recall that the notation does not distinguish between the "standard" Steenrod squares and the ones associated to $\mathbb{G}_a$

- *When $p = 2$, $\mu(Sq^i) = \sum_j Sq^j \otimes Sq^{i-j}$*
- *When $p \neq 2$,*
  1. *$\mu(P^i) = \sum_j P^j \otimes P^{i-j}$*
  2. *$\mu(Q^i) = \sum_j Q^j \otimes P^{i-j} + P^j \otimes Q^{i-j}$, where $Q^i := \beta P^i$.*

The morphism μ induced by the morphisms $\mathrm{K}(\mathbb{F}_p, m) \wedge \mathrm{K}(\mathbb{G}_a, n) \to \mathrm{K}(\mathbb{G}_a, m+n)$ associated to the trivial pairing $\mathbb{G}_a \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \mathbb{G}_a$ via 1.

Proof. By a careful rewriting of the objects, one can check that the morphism α factors through the map

$$\mathbb{H}^n_{st}(K(\mathbb{G}_a), \mathbb{G}_a) \xrightarrow{\mu^*} \mathbb{H}^n_{st}(K(\mathbb{F}_p) \wedge K(\mathbb{G}_a), \mathbb{G}_a) \to \mathrm{H}^n_{st}(K(\mathbb{F}_p), \mathbb{F}_p) \otimes \mathbb{H}^0(K(\mathbb{G}_a), \mathbb{G}_a)$$

where the second morphism is the projection in degree n of the Künneth isomorphism

$$\mathbb{H}^n_{st}(K(\mathbb{F}_p) \wedge K(\mathbb{G}_a), \mathbb{G}_a) \cong \bigoplus_{i+j=n} \mathrm{H}^i_{st}(K(\mathbb{F}_p), \mathbb{F}_p) \otimes \mathbb{H}^j_{st}(K(\mathbb{G}_a), \mathbb{G}_a)$$

This is essentially a formal verification - see [Bre81], lemma 3.4. for details. □

From there, one can fully explicit the morphism α. By 3.3.22, an additive basis of $\mathbb{H}^n_{st}(K(\mathbb{G}_a), \mathbb{G}_a)$ is given by elements of the form $\beta^{\varepsilon_1} P^{s_1} \beta^{\varepsilon_2} \dots \beta^{\varepsilon_{n-1}} P^{s_n} \beta^{\varepsilon_{n+1}}$ for some $\varepsilon_i \in \{0, 1\}$ and $s_i \geq 0$.

Proposition 3.3.25. *For any such sequence $I = (\varepsilon_1, s_1, \dots, s_n, \varepsilon_{n+1})$,*

$$\alpha(\beta^{\varepsilon_1} P^{s_1} \beta^{\varepsilon_2} \dots \beta^{\varepsilon_n} P^{s_n} \beta^{\varepsilon_{n+1}}) = \overline{\beta}^{\varepsilon_1} \overline{P}^{s_1} \beta^{\varepsilon_2} \dots \overline{\beta}^{\varepsilon_n} \overline{P}^{s_n} \overline{\beta}^{\varepsilon_{n+1}} \otimes F^{l(I)}$$

where the $\overline{\beta}$ and $\overline{P}$ are the corresponding elements in $\mathrm{H}^n_{st}(K(\mathbb{F}_p), \mathbb{F}_p)$ and $l(I) := n + \varepsilon_{n+1}$.

Proof. This is a straightforward computation. □

We may now conclude.

Proposition 3.3.26. $\mathrm{Coker}(\alpha)$ *is killed by large enough powers of the Frobenius.*

Proof. Pick a base element $y = \overline{\beta}^{\varepsilon_1} \overline{P}^{s_1} \beta^{\varepsilon_2} \dots \overline{\beta}^{\varepsilon_n} \overline{P}^{s_n} \overline{\beta}^{\varepsilon_{n+1}} \otimes F^m$ in $\mathrm{H}^n_{st}(K(\mathbb{F}_p), F_p) \otimes \mathrm{H}^0_{st}(K(\mathbb{F}_p), \mathbb{F}_p)$.
Recall that $P^0 \neq 0$, while $\overline{P}^0 = \mathrm{Id}$. Let $x_k = P^0 \cdot \beta \cdot P^0 \cdot \dots \cdot \beta$, with k iterations of $P^0 \cdot \beta$.
Then, if k is large enough,

$$\alpha\left(\beta^{\varepsilon_1} P^{s_1} \beta^{\varepsilon_2} \dots \beta^{\varepsilon_n} P^{s_n} \cdot x_k\right) \cong \overline{\beta}^{\varepsilon_1} \overline{P}^{s_1} \beta^{\varepsilon_2} \dots \overline{\beta}^{\varepsilon_n} \overline{P}^{s_n} \overline{\beta}^{\varepsilon_{n+1}} \otimes F^{n+k} \cong y \text{ modulo large powers of } F$$

Thus α is (essentially) surjective □

Remark 3.3.27. *Note that, before taking quotients, α is a priori not surjective.*

We finally established that $\mathrm{Ext}^{n+1}_V(\mathbb{G}_a, \mathbb{G}_a) = 0$ modulo large powers of the Frobenius.

3.4 From the étale site to the perfect site

This section follows [Bre81].
Recall the notations of 3.1.1. Now that we have established the partial computation on the étale site 3.1.3, we will deduce the result on the perfect étale site, as given in 3.1.1.
More precisely, we'll establish the following, for any $n \geq 0$.

Proposition 3.4.1. $\mathrm{Ext}^n_{\underline{V}_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a) = \varinjlim_F \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a)$, *where F denotes the Frobenius map.*

This relies essentially on the canonical resolution, and the fact that étale cohomology groups commutes with directed colimits over quasi-compact, quasi-separated schemes. To illustrate that this is the right setup, we will only assume that our base S is perfect and qcqs, but not necessarily affine.
Fix S a perfect quasi-compact, quasi-separated scheme, and $n \geq 0$.

3.4.1 The perfection morphism

Let us define a geometric morphism of topos between the étale and the perfect sites.

Definition 3.4.2. *Let i_*, i^* be the morphisms given by :*

- $i^* : \underline{T}_S \to \underline{T}_S^{perf}$ *is the restriction to the subcategory of perfect schemes.*
- $i_* : \underline{T}_S^{perf} \to \underline{T}_S$ *is defined by* $i_*\mathcal{F}(T) = \mathcal{F}(T^{perf})$

We will also denote by (i^*, i_*) the similarly defined maps $\underline{V}_S^{perf} \rightleftarrows \underline{V}_S^{ét}$ and $\underline{Ab}_S^{perf} \rightleftarrows \underline{Ab}_S^{ét}$.

Proposition 3.4.3. *The pair $i = (i^*, i_*)$ is a geometric morphism of topoi $\underline{T}_S^{perf} \to \underline{T}_S^{ét}$.*
Moreover, i_* is exact.

Proof. We will check every condition needed to apply 1.1.21.
We let the perfection $u : S^{ét} \to S^{parf}$, and the forgetful functor $v : S^{perf} \to S^{ét}$.
We will show that u is continuous and cocontinuous, and v is continuous.

Step 1. *u is continuous, in the sense of 1.1.5*

The first point in the definition of continuity was proved in 1.3.12, while the second one was follows from 1.3.11.

Step 2. *u is cocontinuous, in the sense of 1.1.6*

This essentially results on the following standard result. (cf. [Gro66], 8.8.2.ii.)

Proposition 3.4.4. *Let $(S_n)_{n\in\mathbb{N}}$, with morphisms $S_n \to S_{n+1}$ be an inductive system of schemes with affine transition maps such that S_0 is quasi-compact and quasi-separated, and whose limit is S.*
Let $X \to S$ be a scheme of finite presentation.
Then, there exists $n \in \mathbb{N}$ and a scheme $X_n \to S_n$ such that $X \cong X_n \times_{S_n} S$ as $S-$schemes.

Let U be an $S-$scheme and $\{f_j : V_j \to U^{perf}\}$ be an étale covering of U^{perf} by perfect $S-$schemes. We want to find an étale covering $\{W_j \to U\}$ whose perfection refines $\{V_j \to U^{perf}\}$. This property is local on the target, we might consider an affine covering U_i^{perf} of U^{perf} induced by an affine covering $(U_i)_{i\in I}$ of U. Up to replacing U^{perf} by U_i^{perf} and V_j by $f_j^{-1}(U_i^{perf})^{perf}$, we can assume U^{perf} to be affine.
By general theory of étale covers (cf. [Sta22, Lemma 03XA]), we can refine $V_j \to U^{perf}$ to a finite étale covering $W_j \to U^{perf}$ where each W_j is affine, and perfect by universal property. Those morphisms are of finite presentation since every morphism between affine schemes is qcqs, and they are locally of finite presentation since they are étale. Moreover, U is qcqs since it's affine.
By 3.4.4, the morphism $W_j \to U^{perf}$ comes from a *finite level*, i.e. there exists $W_j \xrightarrow{g_j} U$ of finite presentation such that $\forall j, V_j \cong W_j \times_U U^{perf}$ via g_j.
Such a morphism is étale by invariance by base change. Hence u is cocontinuous.

Step 3. *Conclude*

The morphism v is clearly continuous (as is every reasonable forgetful functor). We now conclude by applying 1.1.21. □

The announced result will follow from the following computation :

$$\mathrm{Ext}^n_{\underline{V}_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a) = \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, i_*\mathbb{G}_a) = \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \varinjlim_F \mathbb{G}_a) = \varinjlim_F \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a)$$

We will now prove that every equality holds, one lemma after another.

3.4.2 Commutation of Ext with directed colimits

The following is less trivial then it may seem.

Lemma 3.4.5. *Let $\{\mathbb{G}_a\}_{n\geq 0}$ be the system with all elements equal to $\mathbb{G}_a$ and the transitions being given by the Frobenius morphism. Then $i_*\mathbb{G}_a = \varinjlim_F \mathbb{G}_a$.*

Proof. If $U = \mathrm{Spec}(B)$ is affine, $i_*\mathbb{G}_a(U) = \mathbb{G}_a(\mathrm{Spec}(B)^{perf}) = B^{perf} = \varinjlim(\mathbb{G}_a(U))$.
The sheaf $\varinjlim_F \mathbb{G}_a$ is a priori defined as the sheafification of $Y \mapsto \varinjlim_F(\mathbb{G}_a(Y)) = \mathcal{O}_Y(Y)^{perf}$. We then need to show there is no need to sheafify, i.e. the presheaf is actually a sheaf.

Let X be a $S-$scheme, and consider an étale covering $\{U_i \xrightarrow{f_i} X\}$. Since perfection induces the identity on the underlying topological spaces, it behaves well with union, so we can assume X to be affine. By [Sta22, Lemma 03XA], it can be raffined to some afinite étale covering $\{V_i \xrightarrow{g_i} X\}_{i\in J}$ for some *finite* J. By [Sta22, Lemma 0G1L], it suffices to check the sheaf condition for V_i. Let $V_i = \mathrm{Spec}(B_i)$ and $U = \mathrm{Spec}(A)$. We need to show that the following sequence of $A-$modules is exact :

$$0 \to A^{perf} \xrightarrow{g_i} \prod_i B_i^{perf} \xrightarrow{(s_i)_i \mapsto (s_i\otimes 1 - 1\otimes s_i)} \prod_{i,j}(B_i \otimes_A B_j)^{perf}$$

as a map of $A-$algebras.

It follows from the fact that perfection is exact (cf. 1.3.6), that perfection commutes with finite products (cf 1.3.7) and that the structure presheaf is a sheaf on the étale topology. ☐

Let us now establish the following commutation result.

Lemma 3.4.6. $\varinjlim_F \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a) = \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \varinjlim_F \mathbb{G}_a)$

Note that, in general, there is no reason for an Ext functor to commute with colimits. This relies heavily on existence of canonical resolutions, and a standard commutation result for the cohomology of qcqs objects. Note that a similar argument will be used in section 5, in an analytic setup.

Let $M(\mathbb{G}_a)_\bullet \to \mathbb{G}_a$ be the canonical resolution given by 2.4.2, and write $M(\mathbb{G}_a)_i = \underline{\mathbb{F}_p}[h^{X_i}]$.
The lemma relies on the following key fact :

Proposition 3.4.7. *If X is a quasi-compact, quasi-separated scheme, then the functor $\mathrm{H}^q_{ét}(X, \_)$ commutes with directed colimits.*

Proof. cf. [GV72], Expose VI, Corollaire 5.2 ; or [Sta22, Lemma 073E] ☐

Note that the 'qcqs' condition, which can be understood as a *finiteness* condition, is essential.

Let us now deduce the proposition.

Proof. Let $\mathbb{G}_a \to I^\bullet$ be an injective resolution.
By general theory of injective resolutions (cf. [Wei94], 2.3.7), there exists maps $I^q \to I^q$ lifting the Frobenius morphism $\mathbb{G}_a \to \mathbb{G}_a$. They induce mappings $\mathrm{Hom}_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^p, I^q) \to \mathrm{Hom}_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^p, I^q)$
Note that Ext is invariant by quasi-isomorphism, so that $\mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathcal{F}) \cong \mathrm{Ext}^n_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)_\bullet, \mathcal{F})$. Consider the double complexes :

$$C^{\bullet,\bullet} = \varinjlim_F \mathrm{Hom}_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^\bullet, I^\bullet) \text{ and } D^{\bullet,\bullet} = \mathrm{Hom}_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^\bullet, (\varinjlim_F I)^\bullet)$$

Look at the induced spectral sequences of type ${}_IE$. We note them E and F.
Since homology commutes with filtered colimits (cf. [Sta22, Lemma 00DB]), the spectral sequences are :

- ${}_IE_1^{p,q} = \varinjlim \mathrm{Ext}^q_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^p, \mathbb{G}_a) \implies {}_IE_\infty^{p,q} = \varinjlim \mathrm{Ext}^{p+q}_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a)$

- ${}_I F_1^{p,q} = \mathrm{Ext}^q_{\underline{V}_S^{ét}}(M(\mathbb{G}_a)^p, \varinjlim \mathbb{G}_a) \implies {}_I E_\infty^{p,q} = \mathrm{Ext}^{p+q}_{\underline{V}_S^{ét}}(\mathbb{G}_a, \varinjlim \mathbb{G}_a)$

The morphisms $I^q \xrightarrow{f_i} \varinjlim I^q$ induce a morphism of double complexes $C^\bullet \to D^\bullet$, which in turn induce a morphism of spectral sequences $E \to F$.
By 3.3.2, the sheaves X_i are representable by qcqs schemes and, by 3.3.4, the extension groups correspond to cohomology groups on said schemes. Hence, by the commutation result above, this morphism induces an isomorphism on the first page. Hence (cf. [Wei94], 5.24), it induces an isomorphism on the limit.
This is the desired result. □

3.4.3 Computing on the perfect site

We may now conclude.

Proposition 3.4.8. $\mathrm{Ext}^n_{\underline{V}_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a) = \varinjlim_F \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a)$

Proof. We apply the Grothendieck spectral sequence (as described in 1.2.5), with $\mathcal{A} = V_S^{perf}$, $\mathcal{B} = V_S^{ét}$, $\mathcal{C} = \mathrm{Ab}$, $G = \mathrm{Hom}_{S^{ét}}(\mathbb{G}_a, \_)$ and $F = i_*$.
The morphism i_* maps injectives objects in $\mathcal{A}$ to injective (and hence acyclic) objects in $\mathcal{B}$ since it admits an exact left-adjoint i^* (by [Sta22, Lemma 015Z]).
This yields :

$$E_2^{p,q} = \mathrm{Ext}^p_{V_S}(\mathbb{G}_a, R^q i_* \mathbb{G}_a) \implies E_\infty^{p+q} = \mathrm{Ext}^{p+q}_{V_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a)$$

Since i_* is exact, then for all $q \neq 0, R^q i_* = 0$, and the sequence degenerates at page 2.
Hence

$$\varinjlim_F \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \mathbb{G}_a) = \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, \varinjlim \mathbb{G}_a) \cong \mathrm{Ext}^n_{\underline{V}_S^{ét}}(\mathbb{G}_a, i_* \mathbb{G}_a) \cong \mathrm{Ext}^n_{V_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a)$$

Where we used 3.4.6, 3.4.5 and the adjunction property (together with the notation $i^* \mathbb{G}_a = \mathbb{G}_a$). □

Note that this result also gives us the Ext group in degree 0 :

Corollary 3.4.9. *If* $V = \mathrm{Spec}(R)$ *for* R *a perfect ring of characteristic* p, $\mathrm{Hom}_{V_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a) = R[T, T^{-1}]$ *the non-commutative ring of polynomial, characterised by* $T^{\cdot} x = x^p T$.

Proof. We know that $\mathrm{Hom}_{V_S}(\mathbb{G}_a, \mathbb{G}_a) \cong R[F^{\pm 1}]^{nc}$. By the above result,

$$\mathrm{Hom}_{V_S^{perf}}(\mathbb{G}_a, \mathbb{G}_a) = \varinjlim \mathrm{Hom}_V(\mathbb{G}_a, \mathbb{G}_a) = \varinjlim R[F]^{nc} = R[F, F^{-1}]^{nc}$$

since the transition morphism is the multiplication by F. □

One could also compute this directly since $\mathbb{G}_a$ is represented by $\mathrm{Spec}(R[X]^{perf})$. This then relies on the fact that additive polynomials in $R[T^{1/p^\infty}]$ are exactly polynomials in T^p and $T^{1/p}$.

4 Perfectoid geometry and period sheaves

Good general references include [Wed15], [Sch12], [SW20] *and* [Mat18a].

As previously explained, the result obtained by Arthur-César Le Bras and Johannes Anschütz is a computation of extension groups of analytic sheaves, that ultimately allows for a deeper understanding of (some) sheaves on the Fargues-Fontaine curve - motivated by the classification of untilts of perfectoid objects.

First of all, we will explain the historical motivation between the study of perfectoid objects and untilts. We will then develop the necessary technical tools, and, in section 5, we directly tackle the computation.

4.1 Why study perfectoid spaces ?

This section is inspired by [Sch12] *and* [Mat18a].

The results of local class field theory showed that it was sometimes possible to study simultaneously finite extensions of complete valued fields of positive characteristic, such as $\mathbb{F}_p((t))$, and complete valued fields of mixed characteristic, such as finite extensions of $\mathbb{Q}_p$.

In fact, the situations are similar after taking wildly ramified extensions.

Theorem 4.1.1. *(Fontaine-Wittenberg, [FW79])*
The absolute Galois groups of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ and $\mathbb{F}_p((t))\left(t^{1/p^\infty}\right)$ are canonically isomorphic.

Scholze generalized this result by introducing a general operation of *tilting*, from the mixed characteristic world to the p-adic one, that respects finite extensions.

The right notion to study this operation is the one of a perfectoid field, of which the p-adic completion of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ and $\mathbb{F}_p((t))\left(t^{1/p^\infty}\right)$ are amongst the simplest examples.

Definition 4.1.2. *A **perfectoid field** is a complete topological field K whose topology is induced by a non-discrete valuation of rank 1, such that the Frobenius is surjective on K°/p, where K° denotes the ring of power-bounded elements.*

There is a natural functor $K \mapsto K^\flat$, called the **tilt** :

$$\{\text{Perfectoid fields of mixed characteristic } (0,p)\} \to \{\text{Perfectoid fields of characteristic } p\}$$

Scholze's result now writes :

Theorem 4.1.3. *(Scholze, 2011 [Sch12]) For K a perfectoid field, the absolute Galois groups of K and $K^\flat$ are canonically isomorphic.*

However, this notion appears to be more *geometric* than it is algebraic.

Recall that there is an interpretation of finite extensions of K as finite étale covers of $\mathrm{Spec}(K)$, and hence a reformulation of an absolute Galois group as an *étale fundamental group* of $\mathrm{Spec}(K)$, i.e. an information about étale cohomology in degree 1.

Scholze defined the notion of a perfectoid space, that is essentially constructed by gluing together the *spectrum* of perfectoid objects. In this setup, one can construct an étale topology, and establish the following :

Theorem 4.1.4. *(Scholze, 2011 [Sch12]) Let X be a perfectoid space, with tilt $X^\flat$.*
The tilting operation induces an equivalence of étale sites $X_{ét} \cong X^\flat_{ét}$.

The generalisation is twofold : It and explores spaces that are more complex than the spectrum of a point, and gives information about *higher-rank* étale cohomology.

We should mention at this point that the introduction of perfectoid spaces comes at the cost of some technical difficulties. Perfectoid objects carry some non-trivial topological information that we do not want to lose when taking the spectrum. Because of that, the study of prime ideals is not enough, and we need some additional *analytical* information. It turns out that the right formalism is Huber's notion of adic spectrum, in which one looks at continuous valuations instead.[49]

Moreover, in order to have an interesting notion of perfectoid spaces, one needs to glue spaces that do not come from fields, but rather more general classes of rings. Scholze's initial approach was to fix a perfectoid field K and study perfectoid *algebras* over K, but it was later deemed better to work without a base.

Given the importance of the tilting morphism, it should interesting to understand its behaviour. It is far from being injective, and the question of the classification of untilts of a given perfectoid field rose to the attention. This justifies the introduction of one of Fontaine's period ring $\mathbb{A}_{inf}$.
Moreover, the space of equivalence classes of untilts is endowed with an additional geometric structure, which is now known as the Fargues-Fontaine curve :

Theorem 4.1.5. (*Fargues-Fontaine, 2018, [FF18], theorem 6.5.20*).
Let K be an algebraically closed[50] perfectoid field with residual characteristic p.
There exists a Dedekind scheme of rank 1 over $\mathbb{Q}_p$, noted X_K, whose closed points are in bijection with isomorphism classes of untilts of K. The residual fields at a closed point is the desired untilt.

The result above can be generalized when K is not algebraically closed, in which case closed points of X_K characterize finite extensions of K.
This result can be extended when K is any perfectoid space. In this setup, the curve X_K is no longer a scheme, but rather an adic space, that can still be thought of as a *moduli space* of untilts.

4.2 Adic geometry

As explained earlier, we need to develop a theory of p-adic geometry that carries nontrivial topological information. The approach used is a p-adic variant of the construction of complex analytic geometry from complex algebraic geometry.
Let us give a few guidelines of what we should expect of a good theory of p-adic analytic geometry.

If k is an complete non-archimedean field of characteristic p (such as $\mathbb{Q}_p$ or $\mathbb{C}_p$), we should be able to construct a theory of *p-adic analytic varieties* over k, which are a p-adic version of complex analytic varieties. On such spaces, we should be able to work with converging power series in addition to simply polynomials, and such approach should allow the use of analytic and topological methods to understand algebraic objects.
For more concrete examples, in [Bos14], one can find a construction of a p-adic analogue of the equivalence between complex elliptic curves and complex tori. In his lectures notes, P.Achinger [Ach15] explains how to understand the genus of an elliptic curve over k as a number of "p-adic handles". Moreover, there is an analytification functor $X \to X^{an}$, that satisfies a p-adic GAGA theorem.

Historically, different definitions have been considered due, for example, to Tate [Tat71], Raynaud [Ray74], Berkovitch [Ber93], Fujiwara-Kato '[FK06] and Huber [Hub96]. They essentially allowed more and more general classes of objects to appear in their theory. For technical reasons - and following Scholze, we will use Huber's theory of adic spaces. We refer to [Ach15] for a short historical overview of the different points of view.

Let us now introduce some technical notions needed to properly define the objects.

[49] Since the support of any valuation is a prime ideal, this can be seen as a generalization of the usual spectrum
[50] This assumption only serves the simplicity of the present exposition

4.2.1 Valuations and Huber pairs

This section mostly follows [Ach15].

Huber's theory is based on continuous valuations on topological rings, rather than prime ideals.

Definition 4.2.1. *A* ***valuation*** *on a ring R is a surjective map $|\cdot| : R \to \Gamma \sqcup \{0\}$, where Γ is a totally ordered abelian group, written multiplicatively, such that :*

- $|0| = 0$, $|1| = 1$.
- $|a \cdot b| = |a| \cdot |b|$ *for any* $a, b \in R$.
- $|a + b| \leq max(|a|, |b|)$ *for any* $a, b \in R$.

If R is a topological ring, a valuation $|\cdot|$ is ***continuous*** *if, for every $\gamma \in \Gamma$, $\{x \in R, |x| < \gamma\}$ is open.*

A valuation is said to be ***trivial*** *if $\Gamma = \{1\}$, and* ***nontrivial*** *otherwise. It is* ***discrete*** *if Γ is isomorphic (as an ordered abelian group) to $(\mathbb{Z}, +)$, and* ***nondiscrete*** *otherwise. It is* ***of rank 1*** *if one can embed Γ into $\mathbb{R}_{\geq 0}$, and of* ***higher rank*** *otherwise.*

Two valuations $|\cdot|_1$, $|\cdot|_2$ are ***equivalent*** *if $\forall x, y \in A$, $|x|_1 \leq |y|_1 \iff |x|_2 \leq |y|_2$.*

The ***support*** *of a valuation $|\cdot|$ is $supp(|\cdot|) \coloneqq \{x \in R, |x| = 1\}$. It is a prime ideal of R.*

Remark 4.2.2. *The use of the term valuation is unfortunate, as valuations of rank one should more naturally be called seminorms. However, for higher ranks, which are a essential in Huber's theory, the term valuation is more appropriate.*
The vocabulary used here is standard in the literature.

Note that any valuation $|\cdot|$ on a ring R induces a topology spanned by the open balls $B(x, r)$, for $x \in R$ and $r \in \Gamma$, where $B(x, r) = \{y \in R, |x - y| < r\}$.

Definition 4.2.3. *A* ***complete nonarchimedean field*** *is a complete topological field k whose topology is induced by a nontrivial valuation $|\cdot|$ of rank 1. Its* ***ring of integers*** *is $\mathcal{O}_k \coloneqq \{x \in k, |x| \leq 1\}$.*
If K is a complete non-archimedean field, a ***pseudo-uniformizer*** *of K is an element $\varpi \in K$ such that $0 < |\varpi|_K < 1$ (equivalently, $\varpi^n \to 0$).*

Example 4.2.4. *$\mathbb{Q}_p$ is a complete nonarchimedean field with respect to the p-adic valuation $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ given by $|x| = p^{-v_p(x)}$. Its ring of integers is $\mathbb{Z}_p$.*
The element p^{-1} is a pseudo-uniformizer of $\mathbb{Q}_p$.

We now work on topological rings, with no fixed valuation.

Definition 4.2.5. *A subset S of a topological ring A is* ***bounded*** *if, for any neighboorhood U of 0, there exists a neighborhood V of 0 such that $V \cdot S \subset U$.*
An element $a \in A$ is ***power-bounded*** *if $\{a^n\}_{n\geq 0}$ is bounded, and* ***topologically nilpotent*** *if $a^n \to 0$*
We let A° be the set of power-bounded elements, and $A^{\circ\circ}$ the set of topologically nilpotent elements.

Definition 4.2.6. *A* ***Huber ring*** *is a topological ring A which admits an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such that the family $(I^n)_{n\geq 0}$ form a fondamental system of neighboorhoods of 0 in A_0*[51]*. The pair (A_0, I) is called a* ***pair of definition****.*
A ***Tate ring*** *is a Huber ring which admits a topologically nilpotent unit. Such an element is called a* ***pseudo-uniformizer****.*
A Huber ring is said to be ***uniform*** *if A° is bounded.*

Example 4.2.7.
- *Discrete rings are Huber (take $I = \{0\}$ and any A_0), but have no pseudo-uniformizer.*

[51] We say that A_0 is $I-adic$

- *If A is a Tate ring with ring of definition A_0 that admits a pseudo-uniformizer unit ϖ, then there exists $n \geq 1$ such that $\varpi^n \in A_0$ and $A = A_0[\varpi^{-n}]$. Moreover, a subset $S \subset A$ is bounded iff there exists $m \geq 1$ such that $S \subset \varpi^{-m} A_0$.*

Definition 4.2.8. *A **Huber pair** is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A^\circ$ is an integrally closed open subring.*
A morphism of Huber pair $\varphi : (A, A^+) \to (B, B^+)$ is a continuous ring homeomorphism $\varphi : A \to B$ such that $\varphi(A^+) \subset B^+$.
*We say that a Huber pair (A, A^+) is **complete** if both A and A^+ are complete as topological spaces.*

Very important classes of Huber pairs are given by quotients of Tate Algebras :

Example 4.2.9. *If A is a complete topological ring whose topology is induced by a valuation of rank 1 and ϖ is a pseudo-uniformizer of A, we define the ring of restricted power series over A as :*

$$A\langle x_1, \dots, x_n \rangle = \varprojlim_n A/(\varpi^n)[X_1, \dots, X_n]$$

If k is a complete nonarchimedean field, we define the Tate algebra :

$$T_{n,k} := k\langle X_1, \dots, X_n \rangle = \left\{ \sum_{I=(i_1,\dots,i_n)} a_I X_1^{i_1} \dots X_n^{i_n} \text{ such that } a_I \to 0 \text{ for } \sum_k i_k \to \infty \right\}$$

endowed with the multiplicative Gauss norm : $\|\sum a_I X_1^{i_1} \dots X_n^{i_n}\| := max_k |a_I|$. Geometrically, this corresponds to the set of power series converging on the closed unit disk of $\overline{k}$.
For any $n \geq 0$, the pair $(k\langle x_1, \dots, x_n \rangle, \mathcal{O}_k\langle x_1, \dots, x_n \rangle)$ is a Huber pair.
Moreover, if A is any quotient of a Tate algebra, then (A, A°) is a Huber pair.

Quotients of Tate algebra are the building block of Raynaud's theory of adic spaces. In Huber's theory, the building block is spaces of valuations on Huber pairs.

4.2.2 Adic spaces

This section is based on [SW20].
Let us now define (adic) spectras of Huber pair, and a way to glue them to geometric objects.

Definition 4.2.10. *The **adic spectrum** of a Huber pair (A, A^+) is defined as*

$$\mathrm{Spa}(A, A^+) = \{\textit{valuations } |\cdot| \textit{ on } A \textit{ such that } , \forall f \in A^+, |f| \leq 1\}/\sim$$

where $\sim$ denotes the equivalence of valuations, as defined in 4.2.1.

Notation 4.2.11. *For psychological reasons, if $x \in \mathrm{Spa}(A, A^+)$ is a valuation and f in A, the evaluation $x(f)$ is often noted $|f(x)|$.*

We endow $\mathrm{Spa}(A, A^+)$ with the topology generated by the **rational** open sets :

$$U\left(\frac{T}{g}\right) = \{x \in \mathrm{Spa}(A, A^+), \forall i, |f_i(x)| \leq g(x) \neq 0\}$$

for $g \in A$ and $T \subset A$ a finite subset such that $T \cdot A$ is open. Such a family is stable under finite intersections.
As topological spaces, adic spectra look like schemes :

Proposition 4.2.12. *Let (A, A^+) be an Huber pair. Then $\mathrm{Spa}(A, A^+)$ is **spectral**, i.e. there exists a ring B and a homeomorphism $\mathrm{Spa}(A, A^+) \cong \mathrm{Spec}(B)$.*

Proof. cf. [Hub93], Theorem 3.5. The proof is not constructive. □

We will now briefly explain how to endow some $X = \mathrm{Spa}(A, A^+)$ with two sheaves of complete topological rings : $\mathcal{O}_X, \mathcal{O}_X^+$. The construction is quite technical and can be skipped in first reading. We refer the interested reader to [Hub94].

Let (A, A^+) be a Huber pair. We will first define the presheaves on standard open sets, and then extend the definition to arbitrary open sets.
Fix $g \in A$ and $T = \{f_1, \dots, f_n\} \subset A$ such that $U\left(\frac{T}{g}\right)$ is a standard open set.
Choose any pair of definition (A_0, I) of A. Let $D = A_0\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right] \subset A[g^{-1}]$, endowed with the unique structure of a topological ring such that the $(I^n \cdot D)_{n \geq 1}$ form a basis of neighborhood of 0.
The ring B is a Huber ring, and we note $A\left\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\right\rangle$ its topological completion.
We then define $A\left\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\right\rangle^+$ as the completion of the integral closure of $A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$ in B.
We finally set : $\mathcal{O}_X\left(U\left(\frac{T}{g}\right)\right) = A\langle\frac{f_1}{g} \dots, \frac{f_n}{g}\rangle$; $\mathcal{O}_X^+\left(U\left(\frac{T}{g}\right)\right) = A\langle\frac{f_1}{g} \dots, \frac{f_n}{g}\rangle^+$.
This definition can be extended to arbitrary open subset by defining

$$\mathcal{F}(U) = \varprojlim_{U \subset V \text{rational}} \mathcal{F}(V)$$

for $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{F} = \mathcal{O}_X^+$.
The construction behaves as expected, as justified by the following results :

Proposition 4.2.13. *The definition above does not depend on the choice of the pair of definition.*

Proposition 4.2.14. *If (A, A^+) is a Huber pair and $X = \mathrm{Spa}(A, A^+)$, then $\mathcal{O}_X(X) = \widehat{A}$ and $\mathcal{O}_X^+(X) = \widehat{A^+}$, where $\widehat{A}$ denotes the topological completion of A.*

Proposition 4.2.15. *For any open $U \subset \mathrm{Spa}(A, A^+)$, $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U), \forall x \in U, |f(x)| \leq 1\}$.*

Beware that those presheaves need not be sheaves in general. However, it will be the case in particular case, such as perfectoid spaces.

Definition 4.2.16. *A Huber pair (A, A^+) is said to be **sheafy** if the structure presheaf $\mathcal{O}_X$ is a sheaf.*

As a corollary of 4.2.15, $\mathcal{O}_X^+$ is a sheaf whenever $\mathcal{O}_X$ is.

Proposition 4.2.17. *If (A, A^+) is a Huber pair, and $x \in X = \mathrm{Spa}(A, A^+)$, the stalk $\mathcal{O}_{X,x}$, taken in the category of sheaves of (non-topological) rings, is naturally endowed with a valuation $|\cdot|_x : \mathcal{O}_{X,x} \to \Gamma_x \sqcup \{0\}$. $\mathcal{O}_{X,x}$ is a local ring, whose maximal ideal is given by the support of $|\cdot|_x$.*

We may now define general adic spaces.

Definition 4.2.18. *An **affioid adic space** is some $Spa(A, A^+)$ for some sheafy Huber pair (A, A^+), seen as a topological space endowed with a sheaf of rings.*
*An **adic space** is the data of $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$, where*

- *X is topological space*
- *$\mathcal{O}_X$ is a sheaf of topological rings such that every stalk $\mathcal{O}_{X,x}$ is a local ring*
- *For each $x \in X$, $|\cdot|_x$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$ whose support is the maximal ideal*

that admits a covering by affinoid open sets $X = \bigcup_{i \in I} U_i$.
A morphism of adic spaces is a continuous map $f : X \to Y$ together with a sheaf morphism $\mathcal{O}_X \to f^*\mathcal{O}_Y$ preserving the above structure.

Proposition 4.2.19. *The functor* $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ *restricted to*

$$\{\textit{Sheafy complete Huber pairs}\} \mapsto \{\textit{Adic spaces}\}$$

is fully faithful.

Moreover, for any adic space Y *and sheafy Huber pair* (A, A^+)*, there is an isomorphism :*

$$\mathrm{Hom}_{\mathrm{AdSp}}(Y, \mathrm{Spa}(A, A^+)) \cong \mathrm{Hom}_{\mathrm{HuPr}}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X(X)^+))$$

Proof. cf. [Hub94], proposition 2.1.(ii). □

Example 4.2.20. *If we endow* $\mathbb{Z}$ *and* $\mathbb{Z}[T]$ *with the discrete topology, then*

$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z})) = \mathcal{O}_X(X) \text{ and } \mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z}[t])) = \mathcal{O}_X^+(X)$$

When first introduced, Huber's theory of analytic geometry aimed to generalize simultaneously the theory of Berkovich spaces and the one of formal schemes - which itself encompasses the theory of schemes. There are, in fact, many ways to *analytify* a scheme. The one we will care about is pretty naïve, and we refer to [MC17] for a discussion about more geometric ideas. Recall that, if R is a ring endowed with the discrete topology, (R, R) is automatically a Huber pair.

Definition 4.2.21. *The functor* $\mathrm{Spec}(R) \mapsto \mathrm{Spa}(R, R)$ *glues to a fully faithful functor*

$$\{\textit{Schemes}\} \to \{\textit{Adic Spaces}\}$$

*It is called the **analytification** functor.*

Remark 4.2.22. *Here are a few alternative notions:*

1. *In [Sch17], section 27, Scholze considers* $Spec(R) \mapsto Spa(R, cl_R(\mathbb{F}_p))$ *where* $cl(\mathbb{F}_p)$ *denotes the integral closure of* $\mathbb{F}_p$ *in* R*. This notions maps* $\mathbb{G}_a$ *to the sheaf* $\mathcal{O}$*, rather then* $\mathcal{O}^+$*.*
2. *The notions defined above do not behave well with pairs, since, if* $Spec(R) \to Spec(K)$ *is a scheme over a nonarchimedean field, there may not exist a map* $Spa(R, *) \to Spec(K, K)$*, since that would need a continuous map* $K \to R$ *(which does not exist if* R *is discrete).*
   The notion explored by [MC17] solves this problem.

4.3 Perfectoid geometry

As explained earlier, Scholze's motivation for introducing perfectoid spaces (and using adic geometry) was the study of perfectoid fields. Perfectoid spaces are a kind of adic spaces constructed by gluing the spectrum of perfectoid objects - they can be seen as some analogue of perfect schemes.

In order to define perfectoid spaces, one needs to glue the adic spectrum of perfectoid rings. The theory of perfectoid rings is essentially the same than the one of perfectoid rings. For simplicity of the exposition, we focus on perfectoid fields.

4.3.1 Perfectoid fields

This section is inspired by [SW20] *and* [Sch12].

Let us start with the definition and main properties of perfectoid fields.

Definition 4.3.1. *A **perfectoid field** of residual characteristic* p *is a field* K *equipped with a rank-one valuation* $|\cdot|_K : K \to \mathbb{R}_+$ *such that :*

1. K *is complete with respect to the topology induced by* $|\cdot|_K$

2. *$\mathcal{O}_K$ is a local ring, of maximal ideal $\mathfrak{m}_K$, and $\mathcal{O}_K/\mathfrak{m}_K$ is of characteristic p.*
3. *The Frobenius map $\varphi : O_K/pO_K \xrightarrow{x\mapsto x^p} O_K/pO_K$ is surjective*
4. *p does not generate m_K, i.e. there is x such that $|p|_K < |x|_K < 1$.*

Note that, by the last point, the valuation of a perfectoid field is necessarily non-discrete. In particular, $\mathbb{Q}_p$ and its finite extensions are not perfectoid fields.

Example 4.3.2. 1. *The (p)-adic completion of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$, $\overline{\mathbb{Q}}_p^{alg}$[52] and of $\mathbb{Q}_p(\mu_{p^\infty})$ are perfectoid fields of characteristic 0.*
2. *The (t)-adic completion of $\mathbb{F}_p((t))\left(t^{1/p^\infty}\right)$ and of $\mathbb{F}_p((t))^{sep}$ are perfectoid fields.*
3. *Perfectoid fields of characteristic p are exactly complete perfect valued field of characteristic p.*

Let us now define the tilt operation. As explained earlier, the tilt transforms objects of mixed characteristic $(0,p)$ into objects living in characteristic p.

Definition 4.3.3. *If K is a field of residual characteristic p, we define its **tilt** as :*

$$K^\flat = \varprojlim\left(\cdots \to K \xrightarrow{x\mapsto x^p} K \xrightarrow{x\mapsto x^p} K\right) = \left\{x_n \in K^{\mathbb{N}}, x_n = x_{n+1}^p\right\}$$

We endow it with the ring structure given by the following addition

$$\{x_n\}_{n\geq 0} + \{y_n\}_{n\geq 0} = \varinjlim\{x_{n+m} + y_{n+m}\}_{n\geq 0}^{p^m}$$

and the term by term multiplication.
It comes equipped with a multiplicative (but not additive) map $\sharp : (x_n)_n \in K^\flat \mapsto x_0 \in K$, which defines a norm $|x|_{K^\flat} := |x^\sharp|_K$.
If K is a perfectoid field, then $K^\flat$ is a perfectoid field of characteristic p.

Note that the definition above is very *ad hoc*, and may seem arbitrary, since the ring structure is not given by the limit in the category of rings. It has the advantage to clearly show that the tilt is automatically perfect. The more standard definition will me given in 4.3.17.

Example 4.3.4. *If K is a perfectoid field of characteristic p, then $K^\flat \cong K$.*
The tilt of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)^{\wedge p}$ is $\mathbb{F}_p((t))\left(t^{1/p^\infty}\right)^{\wedge t}$.

Many important algebraic properties are preserved under tilting. Here are some of them.

Proposition 4.3.5. *Let K be a perfectoid field with tilt $K^\flat$.*

1. *Continuous valuations coincide : $\mathrm{Spa}(K,K) \cong \mathrm{Spa}(K^\flat, K^\flat)$.*
2. *There is an equivalence of category between finite extensions and K and finite extensions of $K^\flat$.*

Note that perfectoid spaces have *a posteriori* proven to be useful in "standard" adic geometry. Indeed, every adic space X admits a pro-étale cover by a perfectoid space $\widetilde{X}$; and hence one apply results about perfectoid spaces to arbitrary adic spaces using pro-étale descent. Informally, perfectoid spaces live on top of towers.
We will not discuss this approach, and redirect the interested reader to [SW20], 7.2, and to the notion of diamond developed in [Sch17].

[52] i.e. $\mathbb{C}_p$

4.3.2 Classifying untilts

This section is based on the lecture notes by J.Lurie [Lur18]. and J.Anschütz [Ans20].
We fix $C^\flat$ a perfectoid field of characteristic p (that is not necessarily the tilt of some C).

Definition 4.3.6. *An **untilt** of $C^\flat$ is a pair (K, ι), where K is a perfectoid field and ι is a continuous isomorphism $\iota : C^\flat \cong K^\flat$.*

Our goal is to classify untilts of $C^\flat$. As we will see, they can be characterized as some distinguished elements of a *period ring* $\mathbb{A}_{inf}$.
Let us start by establishing a few properties.

Lemma 4.3.7. *Let K be a perfectoid field.*

1. *If $x \in O_K$, there is $x' \in K^\flat$ such that $x'^\sharp \equiv x \mod p$*
2. *If $x \in K$, there is $y \in K^\flat$ such that $|x|_K = |y|_{K^\flat}$*

Proof. We refer to [Lur18], 2-13 for a proof. 2-14 shows that the definition of a perfectoid fields is essentially constructed so that this lemma holds. □

Corollary 4.3.8. *Let K be a perfectoid field and $\pi \in K^\flat$ nonzero such that $|p|_K \le |\pi|_{K^\flat} < 1$.
Then $\sharp : K^\flat \to K$ induces an isomorphism $\mathcal{O}_K^\flat/(\pi) \cong \mathcal{O}_K/(\pi^\sharp)$*

Proof. Since $|p|_K \le |\pi|_{K^\flat} = |\pi^\sharp|_K$, $|p/\pi^\sharp|_K \le 1$, so that $p/\pi^\sharp \in \mathcal{O}_K$, and $\pi^\sharp$ divides p.
The surjectivity then follows from the first point of 4.3.7.
If $y \in (\pi^\sharp)$, $|y|_{K^\flat} = |y^\sharp|_K \le |\pi^\sharp|_K = |\pi|_{K^\flat}$, so π divides y. This shows injectivity. □

Proposition 4.3.9. *Fix $\iota : C^\flat \cong K^\flat$ an untilt of $C^\flat$, and $\pi \in K^\flat$, $\pi' \in C^\flat$ such that $\iota(\pi) = \pi'$ and $\pi'^\sharp = p$.
Then ι induces an isomorphism $\mathcal{O}_{C^\flat}/(\pi) \cong \mathcal{O}_K^\flat/(\pi') \cong \mathcal{O}_K/(p)$
Reciprocally, any isomorphism of such a quotient can be to lifted to $C^\flat \cong K$, and corresponds to an untilt.*

Proof. The first part follows from 4.3.2. The second part holds since

$$\mathcal{O}_C^\flat \cong \varprojlim \mathcal{O}_{C^\flat}/(\pi') \text{ and } \mathcal{O}_K^\flat \cong \varprojlim \mathcal{O}_K/(p)$$

with transitions given by the Frobenius. □

Let us now define our first period ring, due to Fontaine.

Definition 4.3.10. *Let $\mathbb{A}_{inf} = W(\mathcal{O}_C^\flat)$.*

Recall that, by 1.3.20, every element $x \in \mathbb{A}_{inf}$ writes uniquely as :

$$x = [c_0] + [c_1]p + [c_2]p^2 + \dots$$

where $[\cdot] : \mathcal{O}_{C^\flat} \to W(\mathcal{O}_{C^\flat})$ denotes the Teichmüller map. This writing is called the Teichmüller expansion.
Hence, $\mathbb{A}_{inf}$ can be seen as a ring of *formal power series* in the variable p. Heuristically, $\mathbb{A}_{inf} \approx \mathcal{O}_{C^\flat}[[p]]$.
For every untilt (K, i) of $C^\flat$, the map $\sharp : K^\flat \to K$ induces a morphism $\theta : \mathbb{A}_{inf} \to \mathcal{O}_K$ given by

$$\theta : \sum [c_n]p^n \mapsto \sum \iota(c_n)^\sharp p^n$$

The map θ evaluates the power series in a given untilt.

Lemma 4.3.11. *The morphism θ is surjective*

Proof. Apply recursively the second point of 4.3.7 □

The morphism θ hence induces an isomorphism $\mathbb{A}_{inf}/ker(\theta) \cong \mathcal{O}_K$.
Hence, every untilt can be exhibited as a quotient of $\mathbb{A}_{inf}$. We will that the reciprocal is also true.
The first step is to identify which ideals are of the form $ker(\theta)$.

Definition 4.3.12. *An element $\xi \in \mathbb{A}_{inf}$ is* ***distinguished*** *if, its Teichmüller expansion*

$$\xi = [c_0] + [c_1]p + [c_2]p^2 + \dots$$

satisfies $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$.

Proposition 4.3.13. *Let (K, ι) be an untilt and $\theta : \mathbb{A}_{inf} \to \mathcal{O}_K$ the associated morphism. Then $ker(\theta)$ contains a distinguished element ξ. Moreover, any such ξ generates the ideal $ker(\theta)$.*

Proof. By 4.3.7, there exists $\pi \in \mathcal{O}_C^\flat$ such that $|\pi|_{C^\flat} = |p|_K$. Hence there exists some invertible $\overline{u} \in \mathcal{O}_K$ such that $\pi^\sharp = \overline{u}p$. Write $\overline{u} = \theta(u)$.
Since $\pi^\sharp = \overline{u}p$, we have $\theta([\pi] - up) = 0$. One can check that $[\pi] - up$ is distinguished. □

Proposition 4.3.14. *For any distinguished element $\xi \in \mathbb{A}_{inf}$, the quotient $\mathbb{A}_{inf}/(\xi)$ can be identified with the valuation ring $\mathcal{O}_K$ of a perfectoid field K.*
Moreover, the natural map $\mathcal{O}_C^\flat = \mathbb{A}_{inf}/(p) \to \mathbb{A}_{inf}/(\xi) \cong \mathcal{O}_K/(p)$ exhibits K as an untilt of $C^\flat$ via 4.3.9.

Proof. We note $\mathcal{O}_K = \mathbb{A}_{inf}/(\xi)$, and $K = Frac(\mathcal{O}_K)$.
Define the norm $|\cdot|_K$ on K by $|y|_K := |x|_{C^\flat}$ for any $x \in \mathcal{O}_C^\flat$ such that $y = x^\sharp \cdot u$, for u a unit.
One can prove that this norm is well-defined, and satisfied the desired properties. We refer to [Lur18], 3.16 for details. □

We have hence proven :

Theorem 4.3.15. *There exists a natural bijection :*

$$\{Untilts\ of\ C^\flat\}/iso \cong \{Distinguished\ elements\ of\ \mathbb{A}_{inf}\}/unit$$

This set is noted $|Y_{FF}|$.

If (K, ι) is an untilt, the Frobenius automorphism $\phi : x \mapsto x^p$ of $C^\flat$ induces a family of other untilts $(K, \iota \circ \phi^n)_{n \in \mathbb{Z}}$, that are not very interesting. The space $|X_{FF}| := |Y_{FF}|/\varphi^{\mathbb{Z}}$ will be the space of closed points of the Fargues-Fontaine curve. We refer to [Mor18] for a very nice introduction.

4.3.3 Perfectoid rings and spaces

This section is inspired by [Bej17] and [SW20].

Definition 4.3.16. *A complete Tate ring R is* ***perfectoid*** *if it is uniform and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ in R°, and such that the p-th power Frobenius map $\Phi : R^0/\varpi \to R^0/\varpi^p$ is an isomorphism.*
A ***perfectoid field*** *is a perfectoid Tate ring that is also a field.*[53]
A ***perfectoid Huber pair*** *is a Huber pair (R, R^+) where R is perfectoid.*

The tilting process can be expressed similarily.

[53] This is not trivial, see [Ked16]

Definition 4.3.17. *Let R be a perfectoid ring. Its **tilt** is defined as :*

$$R^\flat = \varprojlim_{x \mapsto x^p} R/p$$

where the ring structure and the topology are given by the limit.
It comes equipped with a continuous multiplicative (but not additive) map $x \in R^\flat \mapsto x^\sharp \in R$ given by the projection on the first coordinate.

Example 4.3.18. *For every perfectoid field K, $\mathcal{O}_{K^\flat} \cong (\mathcal{O}_K)^\flat$*
If R is a perfectoid ring of characteristic p, then $R^\flat \cong R$.
Perfectoid rings of characteristic p are exactly complete perfect Tate rings.

A good algebraic motivation for the tilting functor is that it is adjoint to the Witt vector construction.

Proposition 4.3.19. *If R is a perfect ring of characteristic p, and S is a p-adically complete ring, there is an isomorphism*

$$\mathrm{Hom}_{Ring}(W(R), S) \cong \mathrm{Hom}_{Ring}(R, S^\flat)$$

Note that, if R is not perfectoid, the tilt tends to be small. This appears on the following examples.

Example 4.3.20. *Here are a few algebraically natural but geometrically useless computations :*

1. $\mathbb{F}_p^\flat \cong \mathbb{F}_p$ *and* $\mathbb{F}_p[T]^\flat \cong \mathbb{F}_p$
2. $\mathbb{Z}_p^\flat \cong \mathbb{F}_p$

As explained earlier, the main idea behind the tilting operation is that it preserves many algebraic properties of perfectoid rings, but changes the characteristic.

Proposition 4.3.21. *Fix R a perfectoid ring, and ϖ a pseudo-uniformizer of R such that $\varpi^p | p$. Let $\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \dots) \in R^\flat$.*

1. *The map $f \mapsto f^\sharp$ induces an isomorphism $R^{\flat\circ}/\varpi^\flat \cong R^\flat/\varpi$*
2. *$R^\flat$ is a perfect complete Tate ring (i.e. a perfectoid ring in characteristic p).*

As we will see, the tilting morphism can be made geometric. Conveniently, perfectoid rings induce well-behaved adic spaces.

Proposition 4.3.22. *Let (R, R^+) be a perfectoid Huber pair. Then for all rational $U \subset X = \mathrm{Spa}(R, R^+)$, $\mathcal{O}_X(U)$ is perfectoid. In particular, (R, R^+) is sheafy.*

Proof. cf [SW20], theorem 6.1.10. □

Thus, in what follows, we never need to assume that our local spaces are sheafy.

Definition 4.3.23. *A **perfectoid space** is an adic space X covered by affinoid adic spaces of the form $\mathrm{Spa}(R, R^+)$ where R is perfectoid.*

Note that, in the original definition [Sch12], Scholze fixed a perfectoid field K and worked with spectras of perfectoid K−algebras. The modern approach does not fix a base space. This is somewhat analogue to the conceptual difference between algebraic varieties and schemes.
The tilting equivalence can be made geometric.

Theorem 4.3.24. *([SW20], Theorem 7.1.4).*
Let (R, R^+) be a perfectoid Huber pair. Write $X = \mathrm{Spa}(R, R^+)$ and $X^\flat = \mathrm{Spa}(R^\flat, R^{+,\flat})$.
If $|\cdot|_x \in X$, we define $|\cdot|_x^\flat$ by declaring $|f|_x^\flat = |f^\sharp|_x$.
This induces a functorial morphism $\varphi : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\flat, R^{+,\flat})$, such that :

1. *φ is a homeomorphism on the underlying topological spaces. It preserves rational subsets.*
2. *If $U \subset X$ is a rational subset, then $\mathcal{O}_X(U)^\flat = \mathcal{O}_{X^\flat}(U^\flat)$.*

Fix S a perfectoid ring. The above morphism can be glued to a functor :

$$\{Perfectoid\ spaces\ /\ S\} \to \{Perfectoid\ spaces\ /\ S^\flat\}$$

This functor is an equivalence of categories.

4.3.4 Topologies

As is the case with schemes, we'd like to endow the category of perfectoid spaces (eventually over a fixed base) with a topology. As is the case with schemes, there are many interesting possibilities.

Since every perfectoid space is a topological space, one can consider this topology, called the *analytic* topology. As is the case for the Zariski topology, this carry little information, and we need other notions.

As previously announced, their is a notion of an étale site of perfectoid.

Definition 4.3.25. *A morphism of $f : Y \to X$ of perfectoid spaces is called :*

1. ***finite étale*** *if for all open affinoids $U = \mathrm{Spa}(A, A^+) \subset X$, $f^{-1}(U)$ is also affinoid, and one can write $f^{-1}(U) \cong \mathrm{Spa}(B, B^+)$ for some B finite étale over A and B^+ the integral closure of A^+ in B.*
2. ***étale*** *if for every $y \in Y$, there exists open neighborhoods $y \in U \subset Y$ and $f(x) \in V \subset f(U)$ such that $f|_U$ factorizes as $p \circ j$, where p is finite étale and j is an open embedding.*

*The **étale site** of a perfectoid space X is the category of perfectoid spaces étale over X, endowed with the analytic topology.*[54]

One can show that a morphism $f : X \to Y$ of perfectoid spaces is étale iff $f^\flat : X^\flat \to Y^\flat$ is étale.
An important result is as follows :

Theorem 4.3.26. *If X is a perfectoid space, the tilting morphism induces an equivalence of the small étale sites $f : X_{ét} \to X^\flat_{ét}$.*

Proof. cf [Sch12], theorem 7.12. □

As is the case with schemes, the étale cohomology of the adic spectrum of a point $\mathrm{Spa}(K, K^\circ)$ (for K is a perfectoid field) characterises the absolute Galois group of K.
We hence deduce the announced generalization of Fontaine-Wittenberg's theorem.

As explained earlier, perfectoid spaces often appear as projective limits of towers of spaces. For this reason, it is useful to work with the *pro-étale* topology, which is a recent refinement of the étale topology - introduced for schemes in [BS13].

Definition 4.3.27. *A morphism $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces is **affinoid pro-étale** if we can write $(B, B^+) = \widehat{\varinjlim_i (A_i, A_i^+)}$ as the completion of a filtered colimit of pairs (A_i, A_i^+), where A_i is perfectoid, such that $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale.*
A morphism $f : X \to Y$ of perfectoid spaces is pro-étale if it is locally (on the source and target) affinoid pro-étale.
*The **pro-étale** site of a perfectoid X is the category of all perfectoid spaces that are pro-étale over X, in which covers are of the form $\{f_i : Y_i \to Y\}_{i \in I}$ where all f_i are pro-étale, and for any quasi-compact $U \subset Y$, there exists a finite $J \subset I$ and quasi-compact $(V_j \subset Y_j)_{j \in J}$ such that $U = \bigcup_{j \in J} f_j(U_j)$.*

[54] Note the difference with the étale site of a scheme

Note that the 'finiteness' condition is the same as the 'qc' part of the fpqc site, cf. 3.2.8.

Surprisingely, in many setups (related to diamonds), it is very useful to consider the a topology that simply forgets the algebraic condition.

Definition 4.3.28. *Some $\{f_i : X_i \to Y\}_{i\in I}$ is a* ***v-cover*** *if and only for any quasi-compact $U \subset Y$, there exists a finite $J \subset I$ and quasi-compact $(V_j \subset Y_j)_{j\in J}$ such that $U = \bigcup_{j\in J} f_j(U_j)$.*

This implies the topology that will mainly be used below.

4.3.5 v-sheaves

In what follows, we let Perfd denote the category of perfectoid spaces, endowed with the v-topology.
We let Perf be the full subcategory formed by perfectoid spaces of characteristic p.
If S is a perfectoid space, we let Perf_S of S_v denote the site of all perfectoid spaces over S, where covers are given by $v-$covers.

Proposition 4.3.29. *The $v-$topology on the sites defined above is subcanonical.*

Note that this result is nontrivial and surprised many authors.
In fact, most of the theory regarding perfectoid spaces can be extended to the case of non-necessarily representable v-sheaves, under a smallness condition. Indeed, Scholze's theory of diamonds associates to any small v-sheaf a quotient of a perfectoid space by some equivalence relation. We will not use these notions, and refer the reader to [Sch17].

Definition 4.3.30. *A v-sheaf $F : \mathrm{Perf}^{op} \to \mathrm{Set}$ is said to be* ***small*** *if there exists a perfectoid space X together with a surjection $h^X \twoheadrightarrow F$.*

In particular, every adic space X admits a *functor of points* $\mathrm{h}^X : \mathrm{Perf}^{op} \to \mathrm{Set}$, which is automatically small. This allows to view arbitrary adic spaces as (weak forms of) perfectoid spaces, even if there is no good perfection functor in the analytic setup.

Remark 4.3.31. *The more modern approach is to generalize the notion of v-sheaf to the one of v-stack, which mimics the extension from schemes to algebraic stacks. This generality is not needed here, and we refer to [Sch17] for more information.*

The construction of Fontaine's period ring $\mathbb{A}_{inf}$ and a few variants can be globalized as v-sheaves. Since those sheaves are defined as the globalization of period rings, they are usually known as *period sheaves*. They were first introduced in [Sch13], with the pro-étale topology. As explained earlier, the goal of what follows will be to compute extension groups of such period sheaves over S_v

In what follows, we fix E a local field of residual characteristic $q = p^f$ and of uniformizer π.

Definition 4.3.32. *Let R be a perfect $\mathbb{F}_q-$algebra.*[55]
The ***ramified*** *Witt vectors of R over E is the $\mathcal{O}_E$-algebra $W_{\mathcal{O}_E}(R) = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.*

If $E = \mathbb{Q}_p$, this is simply the standard Witt vector ring.

Let us say a few words about this definition. It is known that $W(\mathbb{F}_q) = \mathcal{O}_K$, where $K = \mathbb{Q}_p(\mu_{q-1})$ is the unique ramified extension of degree n. Since E is of residual characteristic q, it can be seen as an extension of K (using the Galois correspondence). Hence there is a ring morphism $\mathcal{O}_K \hookrightarrow \mathcal{O}_E$, which endows $\mathcal{O}_E$ with a structure of $W(\mathbb{F}_q)$-algebra. The map $\mathbb{F}_q \to R$ induces a ring morphism $W(\mathbb{F}_q) \to W(R)$. Hence, the tensor product can be realised as $W(\mathbb{F}_q)$-algebras.
Moreover,

$$W_{\mathcal{O}_E}(R)/\pi W_{\mathcal{O}_E}(R) \cong W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E/\pi \cong W(R) \otimes_{W(\mathbb{F}_q)} \mathbb{F}_q \cong R$$

Let us globalize the construction of $\mathbb{A}_{inf}$ as a v-sheaf.

[55]For a more general definition, we refer to [FF18].

Definition 4.3.33. *Let S be an adic space of characteristic p. If (R, R^+) is a perfectoid Tate algebra of characteristic p with pseudo-uniformizer ϖ, we let :*

$$\mathbb{A}_{inf}(R, R^+) := W_{\mathcal{O}_E}(R^+)$$

We endow it with the $(\pi, [\varpi])$−adic topology.
This defines a v-sheaf of rings $\mathbb{A}_{inf}$ on S_v via $\mathbb{A}_{inf}(\mathrm{Spa}(R, R^+)) = W_{\mathcal{O}_E}(R^+)$.

One of the main purposes of the ring $\mathbb{A}_{inf}$ is that it allows for the choice of simultaneous untilts. Let us give a more precise statement.

Proposition 4.3.34. *Let S be an affinoid perfectoid space over $\mathbb{F}_q$.*
The choice of a primitive element $\xi \in \mathbb{A}_{inf}(S)$ corresponds to an untilt $S^\sharp$ of S, and induces, through the tilting equivalence $\mathrm{Perf}_S \cong \mathrm{Perf}_{S^\sharp}$ the simultaneous choice of an untilt of every perfectoid space T over S.

Note that the classification of untilts of perfectoid rings is very similar to the one of perfectoid fields
Before we conclude this section, we'll need some very basic elements in the theory of almost mathematics.

4.3.6 Interlude : Some almost mathematics

A short introduction is given in [Sch12]. A good extensive reference is [GR03].

In the next section, we will often need to say that, while a module is nonzero, it is *almost* zero, in the sense that it is killed by some ideal. The language of almost mathematics, introduced by G.Faltings, has become standard in perfectoid geometry.

Definition 4.3.35. *Let K be a perfectoid field, and $\mathfrak{m} = K^{\circ\circ} = \{x \in K, |x| < 1\}$ the subset of topologically nilpotent elements. Let M, N be K−modules.*
*We say that M is **almost zero** if, for every $m \in M$, there exists an element $x \in \mathfrak{m}$ such that $mx = 0$. We note $M \overset{a}{=} 0$ if this is the case.*
*If $f : M \to N$ is a morphism of K−modules, we say that it is an **almost isomorphism** if $ker(f) \overset{a}{=} 0$ and $coker(f) \overset{a}{=} 0$.*

More generally, these notions are interesting whenever V is a local integral domain, with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$, and $K = \mathrm{Frac}(V)$.

5 Extension groups of period sheaves

In this section, we derive the extension groups of some of the period sheaves introduced above. For example, we will establish the following :

Proposition 5.0.1. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbb{F}_q)$, and E a local field of residual characteristic q. The extension groups, computed as sheaves of $\mathcal{O}_E$−vector spaces over S_v, are (almost) :*

$$\mathrm{Ext}^n_{S_v, \mathcal{O}_E}(\mathbb{A}_{inf}, \mathbb{A}_{inf}) \overset{a}{=} \begin{cases} W_{\mathcal{O}_E}(R^+)\langle T^{\pm 1}\rangle^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

where $W_{\mathcal{O}_E}(R^+)\langle T^{\pm 1}\rangle^{nc}$ is the ϖ−adic completion of $W_{\mathcal{O}_E}(R^+)\left[T^{\pm 1}\right]^{nc}$, for ϖ a pseudo-uniformizer of R

As we will see, the first step is to transfer the result established by Breen to a result between analytic sheaves. The analytic analogue of the étale sheaf $\mathbb{G}_a$ will be the v-sheaf $\mathcal{O}^+$.
From there, we will establish results for other period sheaves using straightforward arguments, some of them looking a lot like the ones used in section 3.

5.1 Extension groups of analytic sheaves

Let us first establish the following :

Proposition 5.1.1. *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(\mathbb{F}_q)$.
Then extension groups of sheaves of $\mathbb{F}_q$−vector spaces on the v−site S_v of S are (almost) :*

$$\mathrm{Ext}^n_{S_v, \mathbb{F}_q}(\mathcal{O}^+, \mathcal{O}^+) \overset{a}{=} \begin{cases} R^+\langle T, T^{-1}\rangle^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

where $R\langle T, T^{-1}\rangle^{nc} := R \otimes_{R^+} R^+\langle T, T^{-1}\rangle^{nc}$, with $R^+\langle T, T^{-1}\rangle^{nc}$ denoting the ϖ−adic completion of $R^+[T, T^{-1}]^{nc}$, for ϖ any pseudo-uniformizer of R.

We already established a way to *analytify* schemes into adic spaces. However, the étale topology has no reason to correspond to the v-topology via this process. We will need to define a notion of an *algebraic v-topology* for schemes, on which Breen's result still holds, and whose analytification naturally leads to the v-topology.
Also note that this result is described over $\mathbb{F}_q$ rather than over $\mathbb{F}_p$. A straightforward but insightful computation will allows us to transfer Breen's result from $\mathbb{F}_p$ to $\mathbb{F}_q$.
Throughout this section, we fix $S = \mathrm{Spa}(R, R^+)$ an affinoid perfectoid space over $\mathbb{F}_q$, and ϖ a pseudo-uniformizer of R.

5.1.1 The algebraic v-site

The following is mostly based on [BS15] *and* [Gle20].
Let us start by defining the algebraic v-site of a scheme.

Definition 5.1.2. *A morphism $f : X \to Y$ between qcqs schemes is an (algebraic)* ***v-cover***[56] *if, for any map $\mathrm{Spec}(V) \to Y$ from a valuation ring V, there exists an extension of valuation rings $V \subset W$*[57] *and a map $\mathrm{Spec}(W) \to S$ such that the following diagram commutes :*

[56] This was previously known as a "universally subtrusive" morphism, as introduced and studied in [Ryd10] (the equivalent definition is established in 2.10)
[57] in the sense of [Sta22, Definition 0ASG]

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

A family $\{X_i \to Y\}_{i\in I}$ is an (algebraic) ***v-cover*** *if there is a finite subset $J \subset I$ such that $\bigsqcup_{j\in J} X_j \to Y$ is a v-cover.*

The letter v denotes "surjectivity at the level of valuations" (and the notion of v-covers is closely related with the notion of h-covers, defined by V.Voevodsky in [Voe96]), as justified by the following :

Proposition 5.1.3. *A morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes is a v-cover iff the map of topological spaces $f^\diamond : |\mathrm{Spa}(B,B)| \to |\mathrm{Spa}(A,A)|$ is surjective.*

The analytification functor $\diamond : \mathrm{Spec}(A) \mapsto \mathrm{Spa}(A,A)$ hence maps algebraic v-covers to analytic v-covers. Let us start by establishing clear notations.

Definition 5.1.4. *We let $\underline{T}_S^v$ denote the topos of sheaves on S_v, and $\widetilde{\mathrm{Perf}}_S$ denote the category of small v-sheaves on S.*
We let $\mathrm{PASch}_{\mathbb{F}_q}$ denote the category of perfect affine schemes over $\mathrm{Spec}(\mathbb{F}_q)$, and $\widetilde{\mathrm{PASch}}_{\mathbb{F}_q}$ the associated topos. If S is a scheme, we let S_v^{alg} denote the site of perfect S-schemes, endowed with the algebraic v-topology. We let $\underline{T}_S^{v,alg}$ denote the associated topos.
If $S = \mathrm{Spec}(A)$ is a perfect scheme over $\mathbb{F}_q$, we let associate a v−sheaf over $\mathrm{Spa}(\mathbb{F}_q)$, via :

$$(\mathrm{Spa}(R,R^+))^\diamond = \mathrm{Hom}_{\mathrm{Ring}}(A,R^+)$$

This induces a covariant functor $\diamond : \mathrm{PASch}_{\mathbb{F}_q} \to \widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$.

Note that the sheaf $\mathrm{Spec}(R)^\diamond$ is the sheaf represented by $\mathrm{Spa}(R,R)$ as an adic space, restricted to the category of perfectoid spaces.

Remark 5.1.5. *In reality, one cannot really consider $\widetilde{\mathrm{Perf}_{\mathbb{F}_q}}$ as a topos, since the definition given here quickly leads to set-theoretical issues. One should, for any cutoff cardinal κ, define the category $\widetilde{\mathrm{Perf}}_\kappa$ of κ−small v-sheaves[58], and pass to limits when κ increases.*
As usual, we'll blindly ignore such issues. We refer to [Gle20] for a more careful approach.

In order to define the desired geometric morphisms of topos, we will rely on 1.1.20. It then suffices to establish the following :

Proposition 5.1.6. *The functor $\diamond$ is left-exact and that maps covering families to covering families.*
It is also fully faithful.

Proof. The continuity follows from 5.1.3, and the commutation with finite limits is clear.
For the full faithfullness, we refer to [Gle20], theorem 2.29. □

Hence, by 1.1.20, we have defined a geometric morphism of topos

$$(f^*, f_*) : \widetilde{\mathrm{Perf}}_{\mathbb{F}_q} \to \widetilde{\mathrm{SchPerf}}_{\mathbb{F}_q}$$

which satisfies our purpose of analytification.

[58] i.e. sheaves that admit a surjection from a perfectoid space of size $< \kappa$

5.1.2 Extension groups of $\mathbb{G}_a$ over $\mathbb{F}_q$-schemes

As explained earlier, we take some time to discuss the extension of 3.1.1 when replacing $\mathbb{F}_p$ by $\mathbb{F}_q$. While this was not considered by Breen, this is used in [ALB21], and this generality comes at very little cost.

Theorem 5.1.7. *Let $S = \operatorname{Spec}(R)$ be an affine scheme over $\mathbb{F}_q$, and $\mathbb{G}_a$ the additive group scheme over S, seen as a sheaf of $\mathbb{F}_q$-vector spaces over S^{perf}. Then the extension groups are :*

$$\operatorname{Ext}^n_{S^{perf},\mathbb{F}_q}(\mathbb{G}_a,\mathbb{G}_a) = \begin{cases} R[T,T^{-1}]^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

In degree zero, the result can be derived directly, or one can check step by step that 3.4.1 still holds in our modified setup. In higher degree, the computation relies on the following lemma :

Lemma 5.1.8. *Let E be a $\mathbb{F}_q$−vector space of finite dimension.*
There are isomorphisms of $\mathbb{F}_q$-vector spaces :

$$\mathbb{F}_q \otimes_{\mathbb{F}_p} E \cong \bigoplus_{i=0}^{f-1} E \quad \text{and} \quad E \otimes_{\mathbb{F}_p} \mathbb{F}_q \cong \bigoplus_{i=0}^{f-1} E^{(i)}$$

where the tensor product is of $\mathbb{F}_p$-vector spaces, the $\mathbb{F}_q$-structure is induced by the term on the right, and, for $0 \le i < f$, $E^{(i)}$ denotes the $\mathbb{F}_q$-vector space that is isomorphic to E as an $\mathbb{F}_p$-vector space, but with the $\mathbb{F}_q$ action twisted by the i-th power of the Frobenius, i.e. given by $x \cdot e = x^{p^i} \cdot e$.

Proof. We define the morphisms :

$$x \otimes e \in \mathbb{F}_q \otimes_{\mathbb{F}_p} E \mapsto \bigoplus \left(x^{p^i} \cdot e\right) \quad \text{and} \quad e \otimes x \in E \otimes_{\mathbb{F}_p} \mathbb{F}_q \mapsto \bigoplus \left(x^{p^i} \cdot e\right)$$

We want to show that they are isomorphisms. They are morphisms of $\mathbb{F}_q$−vector spaces with the announced structure. Since both sides are of the same dimension, it suffices to show that the morphisms are injective. Let us do the verification for the second one.
Let $\sum_{k=0}^r e_k \otimes x_k \in E \otimes_{\mathbb{F}_p} \mathbb{F}_q$ such that, for all i, $\sum_{k=0}^r x_k^{p^i} \cdot e_k = 0$. We will show that the e_i are all zero.
Without loss of generality, we can assume that $r \le f - 1$, and that the x_k are $\mathbb{F}_p$-linearly independant.
Define the Moore matrix M by $M_{i,j} = x_j^{p^i}$, and X the column vector $X_i = s_i$, such that $M \cdot S = 0$.
It is a classical result that M is of full rank whenever the x_j are linearly independant. Hence $S = 0$. □

Remark 5.1.9. *This lemma is a very particular case of useful results in Galois descent theory.*[59]
For example, the isomorphisms also holds as $\mathbb{F}_q$−algebras when E admits a structure of an $\mathbb{F}_q$-algebra ; and can even be generalized when replacing $\mathbb{F}_q/\mathbb{F}_p$ by any finite Galois extension (the sum now ranges over the Galois group). We refer to [Sta22, Section 0CDQ] for a related discussion.

Note that the construction above is functorial, and can hence be globalized as a decomposition of sheaves. The tensor product $\mathbb{G}_a \otimes_{\mathbb{F}_p} \mathbb{F}_q$ can hence be decomposed as $\mathbb{G}_a \otimes_{\mathbb{F}_p} \mathbb{F}_q = \bigoplus_{i=0}^{f-1} \mathbb{G}_a^{(i)}$, where $\mathbb{G}_a$ is the sheaf induced by $X \mapsto \mathcal{O}_X(X)^{(i)}$.
We may now easily deduce the proposition 5.1.7.

Proof. In higher degree, the adjunction yields

$$\begin{aligned} 0 = \operatorname{Ext}^n_{S^{perf},\mathbb{F}_p}(\mathbb{G}_a,\mathbb{G}_a) &\cong \operatorname{Ext}^n_{S^{perf},\mathbb{F}_q}(\mathbb{G}_a \otimes_{\mathbb{F}_p} \mathbb{F}_q,\mathbb{G}_a) \\ &\cong \bigoplus_n \operatorname{Ext}^n_{S^{perf},\mathbb{F}_q}(\mathbb{G}_a^{(n)},\mathbb{G}_a) \end{aligned}$$

We conclude since $\mathbb{G}_a^{(0)} = \mathbb{G}_a$. □

[59] Thanks to Nataniel Marquis for suggesting this remark

5.1.3 Extension groups of analytic sheaves

The goal of this paragraph is to establish the following :

Proposition 5.1.10. *Then extension groups of sheaves of $\mathbb{F}_q$−vector spaces over S_v of $\mathcal{O}^+$ are :*

$$\mathrm{Ext}^n_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}) \overset{a}{\cong} \begin{cases} R^+\langle T,T^{-1}\rangle^{nc} & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

with $R^+\langle T,T^{-1}\rangle^{nc}$ denoting the ϖ−adic completion of $R^+[T,T^{-1}]^{nc}$.
This comes from a family of almost isomorphisms :

$$\left(\mathrm{Ext}^n_{Spec(\mathbb{F}_q)^{alg}_v,\mathbb{F}_q}(\mathbb{G}_a,\mathbb{G}_a)\otimes_{\mathbb{F}_q} R^+\right)^{\varpi} \to \mathrm{Ext}^n_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+)$$

Note that, by 3.3.7, $\mathrm{Ext}_{Spec(\mathbb{F}_q)^{alg}_v,\mathbb{F}_q}(\mathbb{G}_a,\mathbb{G}_a)\otimes_{\mathbb{F}_q} R^+ = \mathrm{Ext}_{\mathrm{Spec}(\mathrm{R}^+)^{alg}_v}(\mathbb{G}_a,\mathbb{G}_a)$
In 3.3.6, we established that Breen's computation does not depend on the chosen topology, given that :

1. The topology is subcanonical
2. The cohomology of sheaves on an affine base is concentrated in degree 0, and coincides with global sections.

Both of those results hold on the algebraic v-topology, as a corrolary of a result by Bhatt-Scholze (cf. [BS15], Theorem 4.1 (for $n=1$)).[60]
Let us now construct the almost-isomorphism.
By definition of a geometric morphism of topos, the functor $f^*:\widetilde{\mathrm{SchPerf}}_{\mathbb{F}_q}\to\widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$ is exact.
Hence, it descends at the derived level to a functor $\mathcal{D}(\widetilde{\mathrm{SchPerf}})\to\mathcal{D}(\widetilde{\mathrm{Perf}})$, and induces a morphism

$$\mathrm{Ext}^n_{Spec(\mathbb{F}_q)^{alg}_v,\mathbb{F}_q}(\mathbb{G}_a,\mathbb{G}_a)\to\mathrm{Ext}^n_{Spa(\mathbb{F}_q)_v,\mathbb{F}_q}(f^*\mathbb{G}_a,f^*\mathbb{G}_a)$$

for any $n\geq 0$. Firstly, let us make $f^*\mathbb{G}_a$ explicit.

Lemma 5.1.11. *$f^*\mathbb{G}_a\cong\mathcal{O}^+$.*

Proof. We know that $\mathbb{G}_a$ is represented over $\mathrm{PASch}_{\mathbb{F}_q}$ by the perfect affine line $\mathbb{A}^{1,perf}_{\mathbb{F}_q}=\mathrm{Spec}(\mathbb{F}_q[X]^{perf})$. Its image via $\diamond$ is the v-sheaf $\mathrm{Spa}(R,R^+)\mapsto\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{F}_p[X]^{perf},R^+)$. If (R,R^+) is a perfectoid Huber pair, R^+ is necessarily perfect since it is an integrally closed subring of a perfect ring, and hence admits all roots of X^p-a.
Hence, $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{F}_p[X]^{perf},R^+)=\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{F}_p[X],R^+)\cong R^+=\mathcal{O}^+(\mathrm{Spa}(R,R^+))$. Thus, $f^*\mathbb{G}_a=\mathcal{O}^+$. □

A similar computation yields the following.

Lemma 5.1.12. *In Perf_S, the sheaf $\mathcal{O}^{+,n}$ is represented by the perfectoid unit ball :*

$$\mathbb{B}^n_S=\mathrm{Spa}\left(R\langle T_1^{1/p^\infty}\dots,T_n^{1/p^\infty}\rangle,R^+\langle T_1^{1/p^\infty}\dots,T_n^{1/p^\infty}\rangle\right)$$

Note that fiber products of adic spaces do not exist in general, but fiber products of perfectoid spaces exist and are perfectoid, since the associated rings are Tate. We refer to the lecture notes [Mor19].
Here, we have

$$\mathbb{B}^n_S=\mathbb{B}^n_{\mathbb{F}_q}\times_{\mathrm{Spa}(\mathbb{Z},\mathbb{Z})}S=\mathrm{Spa}\left(\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}],\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]\right)\times_{\mathrm{Spa}(\mathbb{Z},\mathbb{Z})}S$$

[60] The algebraic v-site, as considered by Bhatt-Scholze, involves only perfect *qcqs* schemes. By [Sta22, Lemma 0ETM], the sheaf condition can be checked on finite affine coverings. Since every morphism between affine schemes is qcqs, the topoi associated to both sites are equivalent

Proof. Let X be a perfectoid space over S. We compute

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Perf}_S}(X,\mathbb{B}^n_S) &\cong \mathrm{Hom}_{\mathrm{Perf}_S}\left(X, \mathrm{Spa}\left(\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}],\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]\right)\times_{\mathrm{Spa}(\mathbb{F}_q,\mathbb{F}_q)} S\right)\\ &\cong \mathrm{Hom}_{\mathrm{AdSp}}\left(X,\mathrm{Spa}(\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}],\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}])\right)\\ &\cong \mathrm{Hom}_{\mathrm{TopRing}}\left(\mathbb{F}_q[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}],\mathcal{O}^+(X)\right)\cong \mathcal{O}^{+,n}(X)\end{aligned}$$

The second equality follows the universal property of fiber products, since the morphism $X\to S$ is fixed. The third one follows from 4.2.19. The last one follows from the fact that $\mathcal{O}^+(X)$ is perfect and 1.3.5. □

The fiber products of perfectoid spaces is compatible with the v-topology. One can easily check that the base extension and the forgetful functor induce a geometric morphism of topos $(f^*,f_*):\widetilde{\mathrm{Perf}}_S\to\widetilde{\mathrm{Perf}}_{\mathbb{F}_q}$. Since f^* is exact (as is every pullback in a geometric morphism), it induces a morphism :

$$\mathrm{Ext}^n_{\mathrm{Spa}(\mathbb{F}_q)_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+)\to \mathrm{Ext}^n_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+)$$

The composition of both morphisms defined above gives a map :

$$\mathrm{Ext}^n_{\mathrm{Spec}(\mathbb{F}_q)^{alg}_v,\mathbb{F}_q}(\mathbb{G}_a,\mathbb{G}_a)\to \mathrm{Ext}^n_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+)$$

Lemma 5.1.13. $\mathrm{Hom}_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+)=R^+\langle T^{\pm 1}\rangle^{nc}$

Proof. We proved that $\mathcal{O}^+$ is represented by $\mathrm{Spa}(R\langle T^{1/p^\infty}\rangle,R^+\langle T^{1/p^\infty}\rangle)$. By Yoneda's lemma,

$$\begin{aligned}\mathrm{Hom}_{S_v,\mathbb{F}_q}(\mathcal{O}^+,\mathcal{O}^+) &= \mathrm{Hom}_{\mathrm{Perf}_S}\left(\mathbb{B}^1_S,\mathbb{B}^1_{\mathbb{F}_q}\times_{\mathrm{Spa}(\mathbb{Z},\mathbb{Z})} S\right)\\ &= \mathrm{Hom}_{\mathrm{Perf}}\left(\mathbb{B}^1_S,\mathbb{B}^1_{\mathbb{F}_q}\right)\\ &= \mathrm{Hom}_{\mathrm{HuPr}}\left((\mathbb{F}_q[T^{1/p^\infty}],\mathbb{F}_q[T^{1/p^\infty}]),(R\langle T^{1/p^\infty}\rangle,R^+\langle T^{1/p^\infty}\rangle)\right)\\ &= \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{F}_q[T^{1/p^\infty}],R^+\langle T^{1/p^\infty}\rangle)\\ &= \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{F}_q[T],R^+\langle T^{1/p^\infty}\rangle)\end{aligned}$$

Such a morphism is determined by the image of T, which is some $P\in R^+\langle T^{1/p^\infty}\rangle$ such that $P(x+y)=P(x)+P(y)$. Such polynomials are exactly polynomials in T^p and $T^{1/p}$. Hence the result. □

This establishes the result in degree zero. Let us now go to higher degree.

Proof. We use the canonical resolution, as defined in 2.4.2, in the topoi $\widetilde{\mathrm{Perf}}_S$ and $\widetilde{\mathrm{SchPerf}}_{\mathbb{F}_q}$, with the ring $\underline{\mathbb{F}}_q$. This defines two canonical resolutions $M(\mathbb{G}_a)_\bullet\to\mathbb{G}_a$ and $M(\mathcal{O}^+)_\bullet\to\mathcal{O}^+$.
We write $M(\mathbb{G}_a)_i=\mathbb{F}_q\left[\mathbb{G}_a^{r(i)}\times\mathbb{F}_q^{s(i)}\right]$ and $M(\mathcal{O}^+)_i=\mathbb{F}_q\left[\mathcal{O}^{+,r(i)}\times\mathbb{F}_q^{s(i)}\right]$.
Since the construction of the canonical resolution is purely combinatorial and does not depend on properties of the topos, the coefficients $r(i)$ and $s(i)$ are the same on both sides.
We now use the same method as in 3.3.5. Let us briefly recall the important ideas.
We can compute both extension groups as the limiting term of the spectral sequence :

$${}_I\mathrm{E}_1^{p,q}=\mathrm{Ext}^q(M(\mathcal{F})_p,\mathcal{F})\implies \mathrm{Ext}^{p+q}(\mathcal{F},\mathcal{F})$$

The sheaf $\mathbb{G}_a^r$ is represented by the perfect affine space $\mathbb{A}^{r,perf}_{\mathbb{F}_p}$, while the sheaf $\mathcal{O}^{+,n}$ is represented by the perfectoid unit ball $\mathbb{B}^n_S$.
Moreover, the constant sheaf $\underline{\mathbb{F}}_p^s$ is represented by $\bigsqcup_{\mathbb{F}_p^s}\mathrm{Spec}(\mathbb{F}_q)$ on the étale site, and by $\bigsqcup_{\mathbb{F}_p^s}\mathrm{Spa}(R,R^+)$ on the analytic one (using similar arguments).
By Yoneda's lemma, the extension groups of a representable sheaf $\mathrm{Ext}(h^X,\mathcal{F})$ correspond to the v-cohomology groups $\mathrm{H}^n(X,\mathcal{F})$ with the corresponding topology.

By [Sch17], theorem 8.8 ; $H^i_v(\mathbb{B}^n_S \times \bigsqcup_{\mathbb{F}_q^s} \mathrm{Spa}(R,R^+), \mathcal{O}^+) \overset{a}{=} 0$ whenever $i > 0$, since the space is affinoid.
By [BS15], theorem 4.1 ; $H^i_v(\mathbb{A}^{n,perf}_{\mathbb{F}_q} \times \bigsqcup_{\mathbb{F}_q^s} \mathrm{Spec}(\mathbb{F}_q), \mathbb{G}_a) = 0$ whenever $i > 0$, since the scheme is affine.
Hence, the extension groups can (almost[61]) be computed as the cohomology of the associated cochain complex whose terms are of the form $\mathrm{Hom}(M(\mathcal{F})_p, \mathcal{F})$. We conclude with the following computation :

Lemma 5.1.14. *For any $n \geq 0$,*

$$\left(\mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_q)^{alg}_v, \mathbb{F}_q}(M(\mathbb{G}_a)_i, \mathbb{G}_a) \otimes R^+\right)^{\varpi} \cong \mathrm{Hom}_{\mathrm{Spa}(S_v),\mathbb{F}_q}(M(\mathcal{O}^+)_i, \mathcal{O}^+)$$

Proof. By Yoneda's lemma, those hom groups are sheaf cohomology in degree zero. We have :

$$\mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_q)_v,\mathbb{F}_q}(M(\mathbb{G}_a)_i, \mathbb{G}_a) \otimes R^+ = H^0_v\left(\mathbb{A}^{r(i),perf}_{\mathbb{F}_q} \times \bigsqcup_{\mathbb{F}_q^{s(i)}} \mathrm{Spec}(\mathbb{F}_q), \mathbb{G}_a\right) \otimes_{\mathbb{F}_q} R^+ = \mathbb{F}_q^{\mathbb{F}_q^s}[T_1^{1/p^\infty}, \dots, T_{r(i)}^{1/p^\infty}] \otimes R^+$$

$$\mathrm{Hom}_{\mathrm{Spa}(S_v),\mathbb{F}_q}(M(\mathcal{O}^+)_i, \mathcal{O}^+) = H^0_v(\mathbb{B}^{r(i)}_S \times \bigsqcup_{\mathbb{F}_q^{s(i)}} \mathrm{Spa}(R,R^+), \mathcal{O}^+) \cong (R^+)^{\mathbb{F}_q^{s(i)}} \langle T_1^{1/p^\infty}, \dots, T_{r(i)}^{1/p^\infty} \rangle$$

This yields the desired result ☐

This conclude the proof ☐

5.1.4 Extension groups of $\mathbb{A}_{inf}$

We may finally establish the announced result.

Proposition 5.1.15.

$$\mathrm{Ext}^n_{S_v,\mathcal{O}_E}(\mathbb{A}_{inf}, \mathbb{A}_{inf}) \overset{a}{=} \begin{cases} W_{\mathcal{O}_E}(R^+)\langle T^{\pm 1}\rangle^{nc} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Recall that, as was shown in 4.3.5, $\mathbb{A}_{inf}/\pi \cong \mathcal{O}^+$. We start with the following :

Lemma 5.1.16. *The $\mathcal{O}_E$-modules $\mathrm{Ext}^n_{S_v,\mathcal{O}_E}(\mathbb{A}_{inf}, \mathbb{A}_{inf})$ are π−adically complete.*

Proof. Recall that $\mathrm{Ext}^n_{S_v,\mathcal{O}_E}(\mathbb{A}_{inf}, \mathbb{A}_{inf}) = \mathrm{Hom}_{\mathcal{D}(S_v,\mathcal{O}_E)}(\mathbb{A}_{inf}, \mathbb{A}_{inf}[n])$ where $\mathcal{D}(S_v, \mathcal{O}_E)$ denotes the derived category of sheaves of $\mathcal{O}_E$-modules on S_v.
By construction of the Witt vector ring, for any affinoid perfectoid space $S = Spa(R,R^+)$, $\mathbb{A}_{inf}(R,R^+)$ is π-adically complete. Since the $\mathcal{O}_E$-linear structure is induced by the term on the right, this concludes. ☐

Likewise, $W_{\mathcal{O}_E}(R^+)$ is clearly π-adically complete. It then suffices to prove that the isomorphism holds modulo π. We compute :

$$\begin{aligned}
\mathrm{Ext}^n_{S_v,\mathcal{O}_E}(\mathbb{A}_{inf}, \mathbb{A}_{inf})/\pi &= \mathrm{Hom}_{\mathcal{D}(S_v,\mathcal{O}_E)}(\mathbb{A}_{inf}, \mathbb{A}_{inf}[n]) \otimes_{\mathcal{O}_E} \mathbb{F}_q && \text{since } \mathcal{O}_E/\mathfrak{m}_E = \mathbb{F}_q \\
&= \mathrm{Hom}_{\mathcal{D}(S_v,\mathcal{O}_E)}(\mathbb{A}_{inf}, \mathbb{A}_{inf}/(\pi)[n]) \\
&= \mathrm{Hom}_{\mathcal{D}(S_v,\mathcal{O}_E)}(\mathbb{A}_{inf}, \mathcal{O}^+[n]) && \text{since } \mathbb{A}_{inf}/(\pi) = \mathcal{O}^+ \\
&= \mathrm{Hom}_{\mathcal{D}(S_v,\mathbb{F}_q)}(\mathbb{A}_{inf} \otimes^L_{\mathcal{O}_E} \mathbb{F}_q, \mathcal{O}^+[n]) && \text{by (derived) adjunction} \\
&= \mathrm{Ext}^n_{S_v,\mathbb{F}_q}(\mathcal{O}^+, \mathcal{O}^+) && \text{since } \mathbb{A}_{inf} \otimes^L_{\mathcal{O}_E} \mathbb{F}_q = \mathcal{O}^+ \\
&\overset{a}{=} \begin{cases} R^+\langle T^{\pm 1}\rangle^{nc} \text{ if } n = 0 \\ 0 \qquad\qquad \text{if } n > 0 \end{cases} && \text{by 5.1.10}
\end{aligned}$$

And $W_{\mathcal{O}_E}(R^+)\langle T^{\pm 1}\rangle^{nc}/(\pi) = R^+\langle T^{\pm 1}\rangle$. This concludes.

[61] In the rigorous sense

5.2 To go further

We hope that this document inspired the reader to read the article [ALB21] *by Le Bras and Anschütz, of which we just proved the proposition 3.6. Let us briefly explain what happens in the following pages - and ultimately motivates this paper.*
This paragraph contains no proof and very little detail.

First of all, note that the authors present their statements using the functor $\mathcal{RH}om_{Spa(\mathbb{F}_q),\mathcal{O}_E}$, that is obtained by right derivation of the internal $\mathcal{H}om_{Spa(\mathbb{F}_q)_v,\mathcal{O}_E}$ functors in the category of sheaves of $\mathcal{O}_E$ vector spaces over $\mathrm{Spa}(\mathbb{F}_q)_v$. This simplifies the statements, and gives some information over non-affine bases, but really conveys the same information. For the purpose of this paper, we preferred to limit the technical complexity by staying with *standard* Ext groups.

In [ALB21], the authors are interested in the self-extensions groups of $\mathbb{B}_{inf} \coloneqq \mathbb{A}_{inf}\left[\frac{1}{\pi}\right]$. This follows from the result on $\mathbb{A}_{inf}$ by the same arguments than the ones used to go from the étale site to the perfect étale site, since, essentially, inverting π can be established as taking the colimit with respect to the multiplication by π.

From there, one can compute the extension groups of the sheaf $\mathcal{O}^\sharp$ defined by $\mathcal{O}^\sharp(T) \coloneqq \mathcal{O}(T^\sharp)$, where $T^\sharp$ is the untilt of T given by 4.3.34, once a primitive element of $\mathbb{A}_{inf}(S)$ is fixed. One can construct the following exact sequence :

$$0 \to \mathbb{B}_{inf} \xrightarrow{\times\xi} \mathbb{B}_{inf} \to \mathcal{O}^\sharp \to 0$$

which is essentially a rewriting of the tilting equivalence, as used in 4.3.34.[62] By a careful study the induced distinguished triangle at the derived level, one may compute the groups $\mathcal{RH}om^n_{S_v,\mathcal{O}_E}(\mathcal{O}^\sharp,\mathcal{O}^\sharp)$. Note that, for once, $\mathrm{Ext}^1_{S_v,\mathcal{O}_E}(\mathcal{O}^\sharp,\mathcal{O}^\sharp) \neq 0$.

Let $X_{S,E}$ denote the relative Fargues-Fontaine curve associated to S over E. In [ALB21], the authors are interested are about the groups $\mathcal{RH}om_{X_{S,E}}(\mathcal{E},\mathcal{F})$ for $\mathcal{E},\mathcal{F}$ two vector bundles on $X_{S,E}$.

Amongst many other things, the $X_{S,E}$ is an adic space over S. The structure map $\tau : X_{S,E} \to S$ induces a morphism over the associated sites, and then to the associated topoi. The authors prove the following :

Theorem 5.2.1. *([ALB21], 3.10). When restricted to a nice enough full subcategory, the functor :*

$$\mathcal{C} \subset \{\text{Complexes of vector bundles over } X_{S,E}\} \xrightarrow{R\tau_*} \mathcal{D}(\text{Sheaves of } \underline{E}\text{-vector spaces over } S_v)$$

is fully faithful.

By recent results due to Bhatt and Scholze ([FS21], Corollary II.2.20 and Proposition II.3.1), arbitrary vector bundles on the Fargues-Fontaine curve can essentially be presented by powers of line bundles.

It is well-known that line bundles over $X_{S,E}$ are exactly of the form $\mathcal{O}_{X_{S,E}}(\lambda)$ for some integer λ. By adequate variants of the Euler sequence, the sheaves $\mathcal{O}_{X_{S,E}}(\lambda)$ for any integer λ can understood in term of $\mathcal{O}_{X_{S,E}}(1)$ and other known sheaves. Finally, the the sheaf $\mathcal{O}_{X_{S,E}}(1)$ can be written as the extension :

$$0 \to \mathcal{O}_{X_{S,E}} \to \mathcal{O}_{X_{S,E}}(1) \to \mathcal{O}_{S^\sharp} \to 0$$

Thus, by combining all of the above, understanding the groups $\mathcal{RH}om_{X_{S,E}}(\mathcal{E},\mathcal{F})$ for $\mathcal{E},\mathcal{F} \in \left\{\mathcal{O}_{X_{S,E}},\mathcal{O}_{S^\sharp}\right\}$ allows to understand the extension groups of arbitrary sheaves on the Fargues-Fontaine curve.

The push forward of those sheaves via $R\tau_*$ are respectively $\underline{E}$ and $\mathcal{O}^\sharp$, seen as sheaves over S_v. On the right hand side of 5.2.1, it then suffices to compute $\mathcal{RH}om_{S_v,E}(\mathcal{E},\mathcal{F})$ for $\mathcal{E},\mathcal{F} \in \left\{\underline{E},\mathcal{O}^\sharp\right\}$

We explained how to compute $\mathcal{RH}om_{S,E}(\mathcal{O}^\sharp,\mathcal{O}^\sharp)$, and we claim that the other three groups can be computed similarly. In fact, we trivially have $\mathcal{RH}om_{S_v,\underline{E}}(\underline{E},\mathcal{G}) \cong \mathcal{G}$. Finally, $\mathcal{RH}om_{S_v,\underline{E}}(\mathcal{O}^\sharp,\underline{E})$ can be deduced from $\mathcal{RH}om_{S_v,E}(\mathbb{B}_{inf},\mathbb{B}_{inf})$ with some careful, but elementary analysis (cf. [ALB21], Theorem 3.8 and Lemma 3.9).

[62] We refer the reader to [Sch13], lemma 6.3. for more detail

6 Appendix : On the (generalized) Steenrod Algebra of $\mathbb{G}_a$

This section follows [Bre78] *and* [FF16].

In this appendix, we present an outline of the computation of the étale Steenrod algebra $\mathbb{H}^n_{st}(K(\mathbb{G}_a), \mathbb{G}_a)$. We present two approaches ; the first one relies on a theorem of Borel regarding transgression in spectral sequences, while the second one identifies $\mathbb{H}^n_{st}(K(\mathbb{G}_a), \mathbb{G}_a)$ with stable derived functors of the symmetric functor.

This section contains very little detail, and uses many notions not introduced before.

We work in the étale topos $(\mathcal{T}^{ét}_S, \underline{\mathbb{F}}_p)$ over a fixed scheme $S = \mathrm{Spec}(R)$ of characteristic p.

6.1 Via Borel's theorem

This is a direct generalization of the standard computation of the topological Steenrod algebra.

Proposition 6.1.1. *There exists a fibration*

$$K(\mathbb{G}_a, n) \to PK(\mathbb{G}_a, n+1) \to K(\mathbb{G}_a, n+1)$$

where $PK(\mathbb{G}_a, n+1)$ denotes the space of paths with fixed origin in $K(\mathbb{G}_a, n+1)$.

One then looks at the associated Leray-Serre spectral sequence, and apply the following theorem of Borel in [Bor53].

Theorem 6.1.2. *Let (E, B, F, p) be a fibration between simplicial objects of $\mathcal{T}$ such that B is simply connected and :*

1. $\widetilde{\mathbb{H}}^*(E, \mathcal{R}) = 0$
2. *The algebra $\mathbb{H}^*(F, \mathcal{R})$ has a system of additive generators a_i in the domain of the partially defined transgression morphism $\tau : \mathbb{H}^*(F, \mathcal{R}) \dashrightarrow \mathbb{H}^{*+1}(B, \mathcal{R})$.*

Then the choice of any family of representatives $b_i \in \mathbb{H}^{*+1}(B, \mathcal{R})$ of the $\tau(a_i)$ forms an additive generator of $\mathbb{H}^*(B, \mathcal{R})$.

The morphisms β and $P^I \in \mathbb{H}^*(K(\mathbb{G}_a, n), \mathbb{G}_a)$ are in the domain of the transgression. This way, one can explicitely construct generators of $\mathbb{H}^*(K(\mathbb{G}_a, n+1), \mathbb{G}_a)$ based on generators of $\mathbb{H}^*(K(\mathbb{G}_a, n), \mathbb{G}_a)$, whenever $n \geq 1$. This allows to compute the structure of the Steenrod algebra by induction.

It then suffices to understand $\mathbb{H}^n(K(\mathbb{G}_a, 1), \mathbb{G}_a)$; this was done independently by M.Lazard in [Law55].

6.2 The algebraic method

We'll show that there is a completely algebraic way of computing such objects. This approach has the advantage to show precisely *what* about the structure of $\mathbb{G}_a$ allows for a computation of the generalized Steenrod algebras.

The key properties are captured by the axioms $A1$ and $A2$ of [Bre78], which imply the following.

Theorem 6.2.1. *There is an isomorphism :*

$$\mathbb{H}^*(K(\mathbb{G}_a, k), \mathbb{G}_a) \cong R^* \,\mathrm{Sym}(R[-k])$$

where R^* denote right-derived functors of a non-additive functor (which are a dual variant of the theory developed in 2.4.4) ; and $\mathrm{Sym}(R[-k])$ *denotes the complex valuing the symmetric algebra*

$$\mathrm{Sym}(R) := \bigoplus_{n \geq 0} (R)^{\otimes n} \big/ x \otimes y = y \otimes x$$

in degree k, and zero elsewhere.

Proof. We start by proving that the axioms A1 and A2 hold.

Lemma 6.2.2. *For any $\mathbb{F}_p$-module M of finite type,*

$$\mathrm{Sym}_R(M^*) \cong \mathrm{Hom}_{\underline{\mathcal{T}}_S^{ét}}(M \otimes_{\underline{\mathbb{F}}_p} \mathbb{G}_a, \mathbb{G}_a)$$

where $\mathrm{Sym}_R(M^*)$ *is the symmetric algebra* $\mathrm{Sym}_R(M^*) = \bigoplus_{n\geq 0}(M^*)^{\otimes_R n}\big/ x\otimes y = y \otimes x$, *associated to the* R*–dual* $M^* = \mathrm{Hom}_{\mathbb{F}_p}(M, R)$.

Proof. Let $\widetilde{M}_R$ denote the Zariski sheaf associated to the R-module $R \otimes_{\mathbb{F}_p} M$, and $W(\widetilde{M}_R)$ be the associated étale sheaf via the process described in 3.3.1. By construction, $W(\widetilde{M}_R) \cong M \otimes_{\underline{\mathbb{F}}_p} \mathbb{G}_a$.

By [Dem70], Corrolaire 4.6.5, $W(\widetilde{M}_R)$ is represented by $\mathrm{Spec}(\mathrm{Sym}_R(M^*))$.

Then $\mathrm{Hom}_{\underline{\mathcal{T}}_S^{ét}}(M \otimes_{\underline{\mathbb{F}}_p} \mathbb{G}_a, \mathbb{G}_a) = \mathrm{Hom}_{\underline{\mathcal{T}}_S^{ét}}(W(\widetilde{M}_R), \mathbb{G}_a) \cong \Gamma(\mathrm{Spec}(\mathrm{Sym}_R(M^*)), \mathbb{G}_a) \cong \mathrm{Sym}_R(M^*)$. □

Lemma 6.2.3. *For any $\mathbb{F}_p$-module M of finite type and $X = M \otimes_{\mathbb{F}_p} \mathbb{G}_a$, $\mathbb{H}^q(X, \mathbb{G}_a) = 0$ whenever $q > 0$.*

Proof. By the proof of the above lemma, any such X is represented by an affine scheme, and we may apply 3.3.4. We conclude using 3.3.1 and the fact that higher cohomology of quasi-coherent sheaves on an affine basis vanishes. □

We may now go back to the main result. By definition, $\mathbb{H}^*(K(\mathbb{G}_a, k), \mathbb{G}_a) = \mathbf{Ext}^*(\mathbb{F}_p[K(\mathbb{G}_a, k)]^\sim, \mathbb{G}_a)$

We know that $\mathrm{Ext}^n_{\underline{\mathbb{F}}_p}(\mathbb{F}_p[K(\mathbb{G}_a, k)]_r, \mathbb{G}_a)$ vanishes for any $n > 0$, since every $K(\mathbb{G}_a, k)_r$ is of the form $\mathbb{G}_a^{s(r,k)}$, which is representable by an affine scheme.

Hence, the spectral sequence $\mathrm{Ext}^p_{\underline{\mathbb{F}}_p}(\mathbb{F}_p[K(\mathbb{G}_a, k)]_q, \mathbb{G}_a) \implies \mathbb{H}^{p+q}(K(\mathbb{G}_a, k), \mathbb{G}_a)$ is degenerated after the first page, so that the hypercohomology groups can be computed as the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[K(\mathbb{G}_a, k)]^\sim, \mathbb{G}_a)$.

By construction, $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[K(\mathbb{G}_a, k)]_q, \mathbb{G}_a)$ is an object of the form $\mathcal{D}(M \otimes \mathbb{G}_a)$ for $M = K(\mathbb{F}_p, k)_q$ a $\mathbb{F}_p$–module of finite type. Hence, by adjunction and 6.2.2, one has

$$\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[K(\mathbb{G}_a, k)]_q, \mathbb{G}_a) \cong \mathrm{Hom}_{\underline{\mathcal{T}}_S^{ét}}(K(\mathbb{F}_p, k)_q \otimes_{\mathbb{F}_p} \mathbb{G}_a, \mathbb{G}_a) \cong \mathrm{Sym}_R(K(\mathbb{F}_p, k)_q^*) \cong \mathrm{Sym}(K(R, -k)_q)$$

where $K(R, -k)$ is the cosimplicial object given by reversing the direction of every arrow in $K(R, k)$.

The extension via 2.4.4 of Sym_R to chain complexes is given by $\mathrm{Sym}_R(X) := N \circ \mathrm{Sym}_R \circ K(X)$. Hence, the i-th right derived functor of Sym, applied at $R[-k]$, is given by the i-th cohomology group of the complex $\mathrm{Sym}(K(\mathbb{F}_p, -k))$. Hence the result. □

Remark 6.2.4. *Note that $\mathbb{H}^*(K(\mathbb{F}_p, k), R)$ is realised as derived functors of $\nu(R) := \mathrm{Sym}(R)\big/ x^p = x$.*

Historically, the derived functors of Sym were understood before the geometric interpretation as a Steenrod algebra, by S.Priddy in [Pri73], in 1973.

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