# MOD p POINCARÉ DUALITY FOR P-ADIC PERIOD DOMAINS

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ABSTRACT. In this article, we introduce a new class of partially proper smooth rigid analytic varieties over an complete algebraically closed p-adic field that satisfy Poincaré duality for étale cohomology with mod p-coefficients; satisfying a so-called "primitive comparison with compact support". We show that smooth proper varieties, their Zariski-open subsets, as well as p-adic (weakly admissible) period domains, in the sense of Rappoport-Zink, belong to this class. In particular, we recover Poincaré duality for almost proper varieties of [LRZ24], and we compute the  $\mathbb{F}_p$ -étale cohomology of period domains without supports, generalizing the result for Drinfeld's symmetric spaces in [CDN21].

This argument relies crucially on Mann's six functors formalism for solid  $\mathcal{O}^{+,a}/\pi$  coefficients developed in [Man22], and the computation for period domains relies on an explicit geometric construction due to Orlik.

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### 1. Introduction

## 1.1. Overview

Let K be a nonarchimedean field of mixed characteristic (0,p), C a complete algebraic closure of K, and  $\Lambda = \mathbb{Z}/n\mathbb{Z}$  a ring of coefficients. Let X be an "analytic" space over C, in an sense that we voluntarily keep imprecise for now.

There is a fully developed theory of étale cohomology for sheaves on  $\Lambda$ -modules in analytic geometry over K in many of its incarnations, as developed by Huber, Berkovich, or Scholze. When n is coprime to p, they admit the same properties satisfied by étale cohomology of schemes in characteristic zero (as developed by Grothendieck and Verdier in [GV72]): The category  $\mathcal{D}(X_{\acute{e}t}, \Lambda)$  admits a six functor formalism; in particular, it admits a Poincaré duality. However, the Poincaré duality for rigid analytic spaces fails badly when  $p \mid n$  - famously, it does not even hold for the open unit ball  $\mathring{\mathbb{B}}_C$  (see, for example, [LRZ24], Remark 6.4.11.).

Nonetheless, there have been recent advances in proving Poincaré duality with  $\mathbb{F}_p$ —coefficients for some class of *nice* rigid-analytic that are proper, or *almost proper*. It has been established for smooth proper rigid-analytic varieties over C, simultaneously by Mann and Zavyalov, in [Man22] and [Zav24b], respectively. In [LRZ24], Li, Reinecke, and Zavyalov show that Poincaré duality also holds for smooth Zariski open subsets of proper rigid analytic varieties.

The approach of Lucas Mann consists in constructing a six functors formalism in another category,  $\mathcal{D}^a_{\square}(X, \mathcal{O}^{+,a}_X/p)$ , consisting of complexes of solid quasi-coherent almost  $\mathcal{O}^+/p$  sheaves. This category is defined by descent from strictly totally disconnected perfectoid spaces, so that objects in this category are hard to construct, but very nice to work with, due to the presence said six functors formalism. This category is related to the more standard derived category of étale  $\mathbb{F}_p$ -sheaves  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)$  via a Riemann-Hilbert functor  $(-)\otimes\mathcal{O}^+/p$ . Lucas Mann's proof of Poincaré duality for  $\mathbb{F}_p$ -coefficients crucially relies on the six functor formalism constructed above, as well as an adequate version of Scholze's primitive comparison theorem [Sch12, Theorem 1.3], comparing étale cohomology with  $\mathcal{O}^+/p$ -coefficients with  $\mathbb{F}_p$ -coefficients, for smooth proper rigid-analytic varieties.

In this paper, we expand on this idea. We prove that smooth and partially proper analytic adic spaces over C satisfying an analogue of Scholze's primitive comparison theorem for compactly supported cohomology satisfy Poincaré duality. The proof of such a result relies heavily on Mann's six functor formalism.

Examples of such spaces naturally include the smooth proper varieties, and the smooth Zariski open subsets of proper varieties, so that we recover the result of [LRZ24]. We also consider the case of p-adic period domains (under some extra technical hypothesis), as defined by Rappoport-Zink in [RZ96], and for which étale  $\mathbb{F}_p$ -cohomology with compact support was computed in [CDHN21], for which we establish primitive comparison with compact support, and hence Poincaré duality. The most famous example of such period space is surely Drinfeld's upper half plane (and its generalization in higher dimension), with  $\mathbb{F}_p$ -coefficients and without support was computed in [CDN21].

# 1.2. The main results

Let us first introduce our notion of "primitive comparison with compact support".

**Definition 1.2.1.** Let X be a smooth and taut locally noetherian analytic adic space over C, and  $X_{C,\acute{e}t}$  denote the associated étale site. Let  $H^{\bullet}_{\acute{e}t}$  (resp.  $H^{\bullet}_{\acute{e}t,c}$ ) denote the étale cohomology groups (resp. with compact support) as defined by Huber in [Hub96]. Let  $\mathcal{L}$  be an  $\mathbb{F}_p$ -local system on  $X_{\acute{e}t}$ .

We say that X satisfies **primitive comparison with compact support** with respect to  $\mathcal{L}$ , if, for all  $i \in \mathbb{N}$ , the natural morphism (that will be constructed in 5.2) of  $\mathcal{O}_C/p$ -modules

(1) 
$$H^{i}_{\acute{e}t,c}(X,\mathcal{L})\otimes\mathcal{O}_{C}/p\to H^{i}_{c}(X_{\acute{e}t},\mathcal{L}\otimes\mathcal{O}^{+}/p)$$

is an almost isomorphism (with respect to the almost system  $(\mathcal{O}_C/p, \mathfrak{m}_C/p)$ ).

Note that, if we assume X to be additionally proper, cohomology with compact support coincides with étale cohomology, and the above morphism is always isomorphism by Scholze's primitive comparison theorem [Sch12, Thm 5.1]. When X is not necessarily proper, primitive comparison with compact support does not hold in general - for example, one can prove that that the open unit disk over C does not satisfy it, even for the trivial local system  $\mathcal{L} = \mathbb{F}_p$ .

The main result of this article is the following:

**Theorem 1.2.2.** (cf. 5.2.2) Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a locally noetherian and partially proper analytic adic space over a complete algebraically closed extension of  $\mathbb{Q}_p$ , and  $\mathcal{L}$  be an étale local system of  $\mathbb{F}_p$ -modules on X.

Assume furthermore that X satisfies primitive comparison with compact support with respect to  $\mathcal{L}$ , and that f is smooth of equidimension d.

Then, for any  $0 \le k \le 2d$ , there exists a natural isomorphism :

$$H_{\acute{e}t}^k(X,\mathcal{L}^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X,\mathcal{L}),\mathbb{F}_p)$$

where  $\mathcal{L}^{\vee}$  denotes the dual local system to  $\mathcal{L}$ .

Note that, the isomorphism needs not come from a perfect pairing  $H^i_{\acute{e}t}(X,\mathcal{L}^{\vee}) \otimes_{\mathbb{F}_p} H^{2d-i}_{\acute{e}t}(X,\mathcal{L}) \to \mathbb{F}_p(-d)$ , as cohomology groups may not be finite dimensional. If they are not (which happens when X is, say, Drinfeld's upper half plane), the duality is not an involution, and such a perfect pairing cannot exist. In particular, one cannot, a priori, recover the compactly supported cohomology from the standard one.

In what follows, we explain our strategy for the proof of the above theorem, before looking at applications of the above theorem to almost proper varieties and to period domains. The proof relies heavily on Lucas Mann's six functor formalism.

In his thesis [Man22], he attaches, to every space X, a category  $\mathcal{D}_{\square}^{a}(X, \mathcal{O}^{+}/p)$ , which admits a six functor formalism. Informally, such a category consists of *complexes of solid étale quasi-coherent almost*  $\mathcal{O}_{X}^{+}/p$ -sheaves on X. Whenever  $X = \operatorname{Spa}(C, C^{+})$  is an affinoid perfectoid space of weakly perfectly finite type,  $\mathcal{D}_{\square}^{a}(X, \mathcal{O}^{+}/p) = \mathcal{D}_{\square}^{a}(C^{+}/p)$  is the infinity derived category of solid almost  $C^{+}/p$ -modules. For other spaces, this is constructed by descent from covers by perfectoid spaces.

Crucially, there exists a fully faithful "Riemann-Hilbert" functor  $(-)\otimes \mathcal{O}_X^+/p: \mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)^{oc} \to \mathcal{D}_\square^a(X,\mathcal{O}^+/p)$ , relating Mann's category to the "standard" derived category of overconvergent  $\mathbb{F}_p$  sheaves on X. Using the available six-functor formalism, the following computation is somewhat straightforward:

**Proposition 1.2.3.** (cf. 2.4.1) Let  $f: X \to \operatorname{Spa}(C)$  be a locally noetherian smooth adic space, and  $\mathcal{L}$  be an  $\mathbb{F}_p$ -local system on  $X_{\acute{e}t}$ . Suppose that there exists an isomorphism in  $\in \mathcal{D}^a_{\square}(Spa(C, C^+), \mathcal{O}^+_{C}/p)$ :

(2) 
$$f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^+/p) \cong R\Gamma_{\acute{e}t,c}(X,\mathcal{L}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p$$

Then, for any  $0 \le k \le 2d$ , there exists a natural isomorphism:

$$H_{\acute{e}t}^k(X,\mathcal{L}^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X,\mathcal{L}),\mathbb{F}_p)$$

Since the category of Mann is obtained by descent, it is a priori difficult to compute  $f_!^{Mann}$ , and test the criterion 2 in examples. The key statement is then, for partially proper f,  $f_!^{Mann}(\mathcal{L}\otimes\mathcal{O}^+/p)$  computes the actual cohomology groups  $H^k(X,\mathcal{L}\otimes\mathcal{O}^+_X/p)$  up to an almost isomorphism, so that it suffices to verify

primitive comparison with compact support in the sense of 1.2.1. Let us give a precise statement, which proof occupies most of sections 3 and 4 of this article.

**Proposition 1.2.4.** (cf. 4.0.1) Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a smooth and partially proper locally noetherian analytic adic spaces, and  $\mathcal{L}$  be an étale  $\mathbb{F}_p$ -local system on X.

Then  $f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}^+/p)$  is a discrete object of  $\mathcal{D}^a_{\square}(\operatorname{Spa}(C, \mathcal{O}_C), \mathcal{O}^+/p)$ , and there is an natural isomorphism of almost modules

$$f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^+/p) \cong R\Gamma_{\acute{e}t,c}(X,\mathcal{L} \otimes \mathcal{O}_X^+/p)^a$$

Where we implicitly use the identification of [Man22, Prop 3.3.16].

Hence the theorem 1.2.2 directly follows from the above results 1.2.3 and 1.2.4.

Now, the question becomes to understand which spaces satisfy primitive comparison with compact support (and, in turn, Poincaré duality for mod p coefficients). This holds for smooth proper varieties, by Scholze's primitive comparison theorem [Sch12, TODO], but also for "nice" open subsets of such varieties. We recover a version of [LRZ24, Corro 1.1.2], albeit for a less general class of local systems.

**Corollary 1.2.5.** (cf. 5.3.1). Let  $j: X \hookrightarrow \overline{X}$  be a Zariski-open subset of a smooth proper rigid analytic variety of equidimension d over C, and  $\mathcal{L}$  be an étale  $\mathbb{F}_p$ -local system on X.

Assume that there exists an  $\mathbb{F}_p$ -local system  $\overline{\mathcal{L}}$  on  $\overline{X}$  such that  $\mathcal{L} = j^*\overline{\mathcal{L}}$ .

Then X satisfies primitive comparison with compact support with respect to  $\mathcal{L}$ , and hence, for any  $k \in \mathbb{N}$ , there is a natural isomorphism:

$$H_{\acute{e}t}^k(X,\mathcal{L}^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X,\mathcal{L}),\mathbb{F}_p)$$

In the second part of the article, we prove that p-adic period domains, in the sense of [RZ96] satisfy the primitive comparison theorem with compact support, in the following sense.

Let  $(G, [b], \{\mu\})$  be a local Shimura datum. Assume furthermore that G is quasi-split, and that b is basic and s-decent. Let E be the Shimura field of  $\{\mu\}$ ,  $\check{E} = E \cdot \check{\mathbb{Q}}_p$ , and  $J_b$  be the inner form of G constructed as the automorphism group of the isocrystal  $N_b$  associated to b.

Let  $\mathscr{F}^{wa}(G,[b],\{\mu\})$  denote the admissible open subset defined as the weakly admissible locus of the flag variety  $\mathscr{F}(G,\{\mu\}) \otimes_E \check{E}$ , and which admits a canonical model over  $E_s = E \cdot \mathbb{Q}_{p^s}$ .

Let  $\mathscr{F}_C^{wa} := \mathscr{F}^{wa}(G, [b], \{\mu\}) \times_{\operatorname{Spa}(E_s, \mathcal{O}_{E_s})} \operatorname{Spa}(C, \mathcal{O}_C)$  denote the base change to C.

**Theorem 1.2.6.** (cf. 7.0.1) In the above setup, the variety  $\mathscr{F}^{wa}(G,[b],\{\mu\})$  satisfies primitive comparison with compact support with  $\mathbb{F}_p$ -coefficients, i.e, for all  $i \in \mathbb{N}$ , the natural morphism :

$$H^i_{\acute{e}t,c}(\mathscr{F}^{wa}_C,\mathcal{O}^+_{\mathscr{F}^{wa}_C}/p) \to H^i_{\acute{e}t,c}(\mathscr{F}^{wa}_C,\mathbb{F}_p) \otimes \mathcal{O}_C/p$$

is an almost isomorphism.

In particular, for any  $k \in \mathbb{N}$ , there exists a  $\operatorname{Gal}(\overline{Q_p}/E_s) \times J_b(\mathbb{Q}_p)$ -isomorphism:

$$H^k_{\acute{e}t}(\mathscr{F}^{wa}_C,\mathbb{F}_p(d))\cong \mathrm{Hom}_{\mathbb{F}_p}(H^{2d-k}_{\acute{e}t,c}(\mathscr{F}^{wa}_C,\mathbb{F}_p),\mathbb{F}_p)$$

where d denotes the dimension of  $\mathscr{F}(G,\{\mu\})$  (that can be explicited in terms of  $(G,\{\mu\})$ ).

In particular, using the computation of the compactly supported cohomology of [CDHN21, Thm 1.2] (for  $p \geq 5$ ) allows us to compute the étale cohomology groups without support. When  $\mathscr{F}^{wa}$  is Drinfeld's upper half plane, we recover the dual Steinberg representation, as computed in [CDN21]. Note that the approach of loc.cit does not step from a Poincaré duality result, but rather an explicit computation using formal models  $A_{inf}$ —cohomology, that does not generalize to general period domains.

In order to prove the above result, we use the following geometric composition, due to Orlik:

**Proposition 1.2.7.** The complement of  $\mathscr{F}^{wa}(G,[b],\{\mu\}) \subset \mathscr{F}(G,[b],\{\mu\})$  can be stratified as

$$\partial \mathscr{F}^{wa} = \mathscr{F} \setminus \bigcup_{i \in I} J_b(\mathbb{Q}_p) \cdot Z_i$$

where the  $Z_i$  is an explicit finite set of Schubert varieties, which are the analytification of smooth proper algebraic varieties

*Proof.* This is due to [Orl05a].

Using a localization sequence, we can reduce the computation to a computation of the cohomology of the complement, for which we use a variant of Mayer-Vietoris sequence, already constructed in [Orl05a]. We check that it can be used to compute  $\mathcal{O}^+/p$ -cohomology in the same way that it computes  $\mathbb{F}_p$ -cohomology, so that the comparison result ensues.

Note that, while in [Orl05a], the complement is considered as a pseudo-adic space, here, we work with locally spatial diamonds throughout.

# 1.3. Outline of the paper

In section 2, after a recollection about the sheaf  $\mathcal{O}^+/p$  on locally spatial diamonds, we briefly go through Mann's six functors formalism from [Man22], and introduce all of the tools that will be useful to us later. We also extract from the formalism a version of Poincaré duality on the essential image of the Riemann-Hilbert functor 2.4.1.

In section 3, we provide technical topological results for partially proper analytic adic spaces, that will be useful in section 4. We also establish a localisation sequence in classical étale cohomology.

Section 4 is the technical heart of the thesis, and is dedicated to the proof of theorem . Using the results layed out in sections 2 and 3, we compare Mann's lower shriek functor with the classical étale cohomology with compact support for partially proper spaces.

In section 5, we deduce Poincaré duality for almost proper spaces.

The section 6 is an introduction to the theory of period domains, and their geometry, and, in section 7, we prove primitive comparison with compact support for such spaces.

# 1.4. Conventions and notations

In what follows, we fix p and C defines a complete algebraically closed p-adic field, of mixed characteristic (0, p).

In this paper, we use the upright D to denote a 1—derived category, and  $\mathcal{D}$  to denote an  $\infty$ -derived one. Recall that, inside a stable infinity category, a square :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

is a cartesian if and only if it is cocartesian. When this is the case, we say that  $X \to Y \to Z$  forms an exact sequence. Note that this is quite different from the concept of exact sequence in underived homological algebra, but should rather be thought of as a distinguish triangle (and, in fact, exact sequence in stable infinity categories correspond to distinguished triangle on the homotopy category).

Using the natural fully faithful embeddings, we canonically identify rigid analytic varieties with the associated analytic adic space, analytic adic spaces with the associated locally spatial diamond, and locally spatial diamonds with the associated small v-stack.

We denote using the subscript of superscript "Mann" the functors  $f^*, f_*, \ldots$ , as constructed by L.Mann in [Man22], and "Hub" to denote functors defined by Huber in [Hub96], or their improvements to the setup of diamonds in [Sch17]. Note that Mann's functors are always implicitly derived, whereas functors  $f^*_{Hub}, f^{Hub}_{!}, \ldots$  are classical functors, and we denote  $Rf^*_{Hub}, Rf^{Hub}_{!}, \ldots$  their derived version.

#### 1.5. ACKNOWLEDGEMENTS

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#### 2. A Crash Course in Mann's six functor formalism

In his PhD thesis [Man22], Lucas Mann developed a six functor formalism for some version of modp étale cohomology in analytic geometry, which interacts in a highly subtle way with "standard"  $\mathbb{F}_{p}$ étale cohomology. While the main result of this article 1.2.2 is purely stated in terms of standard étale cohomology (as defined by Huber), the proof crucially goes through Mann's six functor formalism, for which the available formalism makes some key computation work.

In this section, we recapitulate the main ideas and results of Mann's construction, focusing on the comparison results with more standard objects, and we prove our key technical lemma 2.4.1.

In this section, all categories considered will be  $\infty$ -categories. We'll to ensure our notation distinguishes the functors  $f_*, f^*, \dots$  from standard étale cohomology and the one constructed by Mann, so that we denote them  $f_*^{Hub}$  of  $f_*^{Mann}$ , respectively. Note that the notations attributes to Huber some constructions that are due to Scholze in [Sch17], whenever the only difference is the level of generality of the spaces (v-stacks or locally spatial diamonds versus locally noetherian analytic adic spaces), see Remark 2.3.2.

# 2.1. The structure sheaf $\mathcal{O}^+/\pi$ for locally spatial diamonds

Recall that any locally spatial diamond X admits both an étale site  $X_{\acute{e}t}$  and a quasi pro-étale site  $X_{apro\acute{e}t}$ , as defined in [Sch17, Def 14.1]. There is a natural morphism of sites  $\nu_X: X_{qpro\acute{e}t} \to X_{\acute{e}t}$ , which induces a geometric morphism of topos given by the pushforward  $\nu_*^X: X_{qpro\acute{e}t}^{\sim} \to X_{\acute{e}t}^{\sim}$  and, more importantly for us, a pullback  $\nu_X^*: X_{\acute{e}t}^\sim \to X_{qpro\acute{e}t}^\sim.$ 

Recall that any analytic adic space X over  $\mathbb{Z}_p$  can naturally be viewed as a locally spatial diamond, and that the natural map objects induces an isomorphism on the underlying topological spaces as well as the associated étale sites (cf. [Sch17, Lemma 15.6]) such that  $|X| \cong |X^{\diamond}|$ , and  $X_{\acute{e}t} \cong X_{\acute{e}t}^{\diamond}$ .

A process described in [MW20, Section 2] endows every diamond X over  $\mathrm{Spd}(\mathbb{Z}_p)$  with quasi-pro-étale structure sheaves,  $\widehat{\mathcal{O}}_X$  and  $\widehat{\mathcal{O}}_X^+$ , and, hence, a pro-étale  $\widehat{\mathcal{O}}_X^+/p$  - which coincides with the  $\nu^*\mathcal{O}^+/p$  for locally noetherian adic spaces (cf. [MW20, Lemma 2.7])<sup>1</sup>.

Here, we show that, more generally, if  $X \to \operatorname{Spd}(\mathbb{Z}_p)$  is a locally spatial diamond, the pro-étale sheaf  $\hat{\mathcal{O}}_X^+/p$  comes from the pullback of an étale sheaf, that we may reasonably call  $\mathcal{O}_X^+/p$  (note that this is implicately claimed in the proof of [Sch17, Theorem 25.1], but we make the argument explicit here).

Recall the following lemma:

**Lemma 2.1.1.** Let  $f: X \to Y$  be a quasi-pro-étale morphism of locally spatial diamonds, and  $\mathcal{F} \in Y_{apro\acute{e}t}$ . Then, the pullback  $f^*\mathcal{F}$  is given by the restriction of  $\mathcal{F}$  to  $X_{qpro\acute{e}t}$ , i.e, for any quasi-pro-étale  $U \to Y$ ,  $f^*\mathcal{F}(U) = \mathcal{F}(U \to X)$ , where we consider the quasi-pro-étale composition  $U \to Y \to X$ .

The same holds by replacing "quasi-pro-étale" by "étale" everywhere.

*Proof.* By general formalism (cf. [Sta23, Section 00VC], or [Sta23, Section 03PZ] in the étale case), the inverse image  $f^*\mathcal{F}$  is given by the sheafification of the following:

$$U/X \mapsto \varinjlim_{(V,\varphi)} \mathcal{F}(V/Y)$$

Where the limit is taken over  $(I_U)^{opp}$ , where  $I_U$  denotes the category of pairs  $(V,\varphi)$  where:

- $\operatorname{Ob}(I_U) = \{(V, \varphi) : V \in Y_{qpro\acute{e}t}, \varphi : U \to V \times_X Y \}$   $\operatorname{Hom}((V, \varphi), (V', \varphi')) = \{g : V \to V' \text{ quasi-pro-\'etale }, g' \circ \varphi = \varphi' \}$  where g' denotes the base change  $V \times_X Y \to V' \times_X Y$  of g.

Since f is quasi-pro-étale, we may view any U/X as quasi-pro-étale over Y, so that we may view  $(U, \varphi_U)$ :  $U \to U \times_X Y$ ) as an object of  $I_U$ . Moreover, for any  $(V, \varphi_V) \in I_U$ , there is a natural quasi-pro-étale morphism  $U \to V$  given by the composition  $U \xrightarrow{\varphi_V} V \times_X Y \to V$ , that is pro-étale by the two-out-of-three property [Sch17, Prop 10.4].

<sup>&</sup>lt;sup>1</sup>Note that the data of the morphism to  $\mathbb{Z}_p$  is crucial in defining the structure sheaf; as an abstract diamond does not even know over which characteristic it lives! The good notion is the one of an untilted diamond, as in [Man22]

It is straightforward to check that this induces a morphism  $(U, \varphi_U) \to (V, \varphi_V)$  in  $I_U$ , making  $(U, \varphi_U)$  an initial object of  $I_U$ , and hence a final object of  $I_U^{op}$ . Hence, the colimit above simply computes  $\mathcal{F}(U \to Y)$ , which clearly defines a sheaf, so that there is no need to sheafify. This concludes.

We may now prove that the sheaf  $\mathcal{O}^+/p$  is, in fact, an étale sheaf.

**Proposition 2.1.2.** Let X a locally spatial diamond over  $\operatorname{Spd}(\mathbb{Z}_p)$ .

Then, there exists a unique étale sheaf on  $X_{\text{\'et}}^{\sim}$ , denoted  $\mathcal{O}_X^+/p$ , such that  $\widehat{\mathcal{O}}_X^+/p \cong \nu_X^* \mathcal{O}_X^+/p$ .

*Proof.* Since this property is local over X, we may assume that X is spatial, so that there exists a surjective quasi-pro-étale morphism  $f:Y\to X$  from a strictly totally disconnected perfectoid space, by [Sch17, Pro 11.24]. Recall that  $\hat{O}_X^+$  is defined such that, for any morphism  $Z\to X$  from a totally disconnected affinoid perfectoid,  $\hat{O}_X^+(Z)=\mathcal{O}_{Z^\sharp}^+(Z^\sharp)$ , where  $Z^\sharp$  is the until of Z viewed through the composition  $Z\to X\to \operatorname{Spd}(\mathbb{Z}_p)$ .

Hence, by the above lemma,  $f^*\hat{\mathcal{O}}_X^+$  is simply  $\hat{\mathcal{O}}_Y^+$ , and, since pullbacks commute with colimits, the same holds for  $\hat{\mathcal{O}}/p$ . Now, it follows from the proof of [MW20, Lemma 2.7] that the pro-étale sheaf  $\hat{\mathcal{O}}_X^+$  actually comes from the pullback of an étale sheaf. Hence, by [Sch17, Thm 14.12], the sheaf  $\hat{\mathcal{O}}_X^+/p$  on X comes from the pullback of an étale sheaf, that we denote  $\mathcal{O}_X^+/p$ 

Moreover, by [Sch17, Prop 14.8], we have  $\mathcal{O}_X^+/p \cong \nu_{X*}\nu_X^*\mathcal{O}_X^+/p \cong \nu_{X*}\hat{\mathcal{O}}_X^+/p$ , so that it is uniquely defined.

We'll also use the following lemma.

**Lemma 2.1.3.** Let  $f: X \to Y$  be a quasi-pro-étale morphism of locally spatial diamonds over  $\operatorname{Spd}(\mathbb{Z}_p)$ . Then  $\mathcal{O}_Y^+/p \cong f_{\acute{e}t}^*\mathcal{O}_X^+/p$ , where  $f_{\acute{e}t}^*$  denotes the pullback of étale sheaves along f.

*Proof.* We start by establishing the commutativity of the following diagram :

$$X_{\acute{e}t}^{\sim} \leftarrow Y_{\acute{e}t}^{*} - Y_{\acute{e}t}^{\sim}$$

$$\nu_{X}^{*} \downarrow \qquad \qquad \downarrow \nu_{Y}^{*}$$

$$X_{qpro\acute{e}t}^{\sim} \leftarrow Y_{qpro\acute{e}t}^{\sim}$$

All of the four arrows in this diagram are induced from morphisms of sites, which itself are induced by continuous functors satisfying the conditions of [Sta23, Proposition 00X6]. Hence, if suffices to consider the following diagram (Here, the arrows are continuous functors, and the associated morphisms of site are contravariant):

$$X_{\acute{e}t} \xleftarrow{Z \mapsto Z \times_X Y} Y_{\acute{e}t}$$

$$id \downarrow \qquad \qquad \downarrow id$$

$$X_{qpro\acute{e}t} \xleftarrow{Z \mapsto Z \times_X Y} Y_{qpro\acute{e}t}$$

Here, the vertical arrows are obtained by viewing any étale-cover as a pro-étale one, so that the diagram obviously commutes. It follows from functoriality of morphisms of sites [Sta23, Lemma 03CB] that both compositions  $\nu_Y^* \circ f_{\acute{e}t}^*$  and  $f_{qpro\acute{e}t}^* \circ \nu_X^*$  are equal to the pullback along the morphism of site induced by the diagonal  $X_{\acute{e}t} \to Y_{qpro\acute{e}t}$ , so that the first diagram commutes.

Let us now prove that  $f_{\acute{e}t}^* \mathcal{O}_Y^+/p \cong \mathcal{O}_X^+/p$ . By commutation of the above diagram and by 2.1.2, we have

$$\hat{\mathcal{O}}_X^+/p \cong f_{qpro\acute{e}t}^* \nu_Y^* \hat{\mathcal{O}}_Y^+/p \cong \nu_X^* f_{\acute{e}t}^* \mathcal{O}_Y^+/p$$

By the uniqueness clause of 2.1.2, we deduce that  $f_{\acute{e}t}^* \mathcal{O}_Y^+/p \cong \mathcal{O}_X^+/p$ .

Let us now say a few word about this étale sheaf  $\mathcal{O}^+/p$ , and how it relates to Mann's category.

2.2. Mann's functors 
$$\mathcal{D}_{\Box}^a(-,\mathcal{O}^+/\pi)$$
 and  $\mathcal{D}_{\Box}^a(-,\mathcal{O}^+/\pi)^{\varphi}$ 

Mann associates to any analytic adic space X over a p-adic field with pseudo-uniformizer  $\pi$  (or, more generally, any untilted small v-stack, with a fixed pseudouniformizer), an  $\infty$ -category  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  consisting of *complexes* of *almost*, *quasi-coherent*, *solid* sheaves of  $\mathcal{O}^+_X/\pi$ -modules on X.

For an analytic adic space X, objects of  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  are not explicit - rather, this category is constructed by gluing from an adequate basis of the pro-étale topology, given by (an loosely improved class of) totally disconnected perfectoid spaces, as follows:

**Definition 2.2.1.** Let  $X = \operatorname{Spa}(A, A^+)$  be an affinoid perfectoid space that is of weakly perfectly finite type over some totally disconnected space (for example: the spectrum of a perfectoid field).

Then, let  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi) := \mathcal{D}^a_{\square}(A^+/\pi)$  be the derived  $\infty$ -category of solid almost  $A^+/\pi$ -modules.

Let us explain very briefly the adjectives of the last sentence:

- The "almost" is to be taken in the sense of Faltings ([Fal88], see also [GR03]). Denoting  $m_A$  the maximal ideal of  $A^+$ , an  $A^+$ -module over is said to be almost zero if it is killed by multiplication by any element of  $m_A$ ; and two modules are almost isomorphic if they differ by an almost zero module. An almost  $A^*$ -module is an object of the localization of the category of  $A^+$ -modules with respect to almost isomorphisms.
- The "solid" is to be taken in the sense of condensed mathematics, as introduced by Clausen and Scholze (see [Sch19]). Informally, solid modules correspond to *complete topological* modules, in a modified topological setting that is constructed to be compatible with standard categorical constructions. The precise interplay between almost and condensed mathematics is constructed in [Man22, Section 2].

From there, the category of *quasi-coherent* (almost, solid) sheaves over an arbitrary space can be defined by gluing from totally disconnected affinoid perfectoid spaces - as one may do when defining quasi-coherent schemes by gluing from affine schemes (cf. [GR19, Chap 3, I.1.4.4]).

The notion of space that we'll use is the one of small v-stacks, defined in [Sch17]. Recall that they form a generalizations of analytic adic spaces, using a stacky approach - so that they are best used in descent problems. In order to properly endow a small v-stack X with a sheaf  $\mathcal{O}_X^+/\pi$ , we need to chose an untilt<sup>3</sup>  $X^{\sharp}$  of X, and a pseudo-uniformizer  $\pi$ . Let vStack $_{\pi}^{\sharp}$  denote the associated category.

**Proposition 2.2.2.** ([Man22, Thm 1.2.1]) There exists a unique hypercomplete v-sheaf of  $\infty$ -categories:

$$(X,X^{\sharp},\pi) \in (\mathrm{vStack}_{\pi}^{\sharp})^{op} \mapsto \mathcal{D}_{\square}^{a}(X,\mathcal{O}_{X}^{+}/\pi) \in \mathrm{Cat}_{\infty}$$

That extends the map defined in 2.2.1. This construction admits a six-functor formalism.

We will simply denote  $(X, X^{\sharp}, \pi)$  as X for simplicity. While we do not precisely define what a six functors formalism  $is^4$ , let us present the main characteristic properties, as in [Man22, Theorem 1.2.4].

**Proposition 2.2.3.** The mapping  $X \mapsto \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  admits the following properties :

(1) For any X, the category  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  is closed symmetric monoidal, i.e. admits a tensor product  $\_\otimes\_$ , which admits a right adjoint, given by an internal Hom functor  $\underline{\operatorname{Hom}}$ . For any X, the unit of the tensor product over X is  $\mathcal{O}^+_X/\pi$ , denoted  $\mathbb{1}_X$  for simplicity.

<sup>&</sup>lt;sup>2</sup>More generally, Mann defines the category  $\mathcal{D}^a_{\square}(A^+/\pi)$  when  $A^+$  is an animated ring, and one replaces the derived category by the category of "modules" over the animated ring, in the sense of Lurie's higher algebra [Lur17, Def 7.1.1.2]. In our exposition, we always consider static (i.e. not animated) rings, so that this is simply a derived category by [Lur17, Remark 7.1.1.6]. Note that the gluing step crucially relies on animated techniques, and, more generally, on Scholze's notion of analytic rings and spaces - even though they will not appear explicitly in this paper.

<sup>&</sup>lt;sup>3</sup>Note that, contrary to what the notation might suggest, the sheaf  $\mathcal{O}_X^+/\pi$  depends on the choice of the until  $X^{\sharp}$ , and should maybe rather be denoted  $\mathcal{O}_{X^{\sharp}}^+/\pi$ . In most of our applications, X will be either an affinoid perfectoid space or an analytic adic space, the choice of an until is not important

<sup>&</sup>lt;sup>4</sup>A precise definition was coined in [Man22, Definition A.5.7]. See also these lectures notes by Scholze [Sch22]

- (2) Any morphism  $f: X \to Y$  induces adjoint functors :  $f^*: \mathcal{D}^a_{\square}(Y, \mathcal{O}^+_Y/\pi) \rightleftarrows \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi) : f_*$ , where the left adjoint  $f^*$  denotes a pullback functor, and  $f_*$  denotes a pushforward. Both are compatible with composition.
- (3) Any  $bdcs^5$  morphism  $f: X \to Y$  induces an adjoint pair  $: f_! : \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi) \rightleftharpoons \mathcal{D}^a_{\square}(Y, \mathcal{O}^+_Y/\pi) : f_!$ , where  $f_!$  denotes a lower shriek functor, and  $f^!$  an upper shriek functor. Both are compatible with composition of bdcs morphisms.
- (4) Let  $f: X \to Y$  be a bdcs morphism. If f is proper, then  $f_! = f_*$ . If f is étale, then  $f^! = f^*$ .
- (5) (Projection formula) Let  $f: X \to Y$  be a bdcs morphism,  $\mathcal{M} \in \mathcal{D}^a_{\square}(Y, \mathcal{O}_Y/\pi)$  and  $\mathcal{N} \in \mathcal{D}^a_{\square}(X, \mathcal{O}_X/\pi)$ . Then there exists a natural isomorphism:

$$f_!(\mathcal{N} \otimes f^*(\mathcal{M})) \cong f_!\mathcal{N} \otimes \mathcal{M}$$

(6) (Proper base change) Consider the following cartesian square:

$$X' \xrightarrow{f'} Y$$

$$\downarrow^{g'} \qquad \downarrow^{g}$$

$$X \xrightarrow{f} Y'$$

Assume that f is bdcs. Then, there is a natural isomorphism of functors:

$$g^* \circ f_! \cong f'_! \circ g'^*$$

(7) Let  $f: X \to Y$  be a smooth morphism of locally noetherian analytic spaces adic over  $\mathbb{Q}_p$  of equidimension d. Then  $f^!\mathcal{O}_Y^+/\pi \cong f^*(d)[2d]\mathcal{O}_Y^+/\pi$ .

As is classical whenever we have a six functor formalism, we may define a notion of "in the sense of Mann" (resp. cohomology with compact support) as the pushforward  $f_*$  (resp  $f_!$ ) of the unit object  $\mathbb{1}_X$ , where  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  is a morphism to a point. This should be interpreted as a form of "almost  $\mathcal{O}^+/\pi$ -cohomology", that should, at least in some cases, coincide, up to an almost isomorphism, with the standard cohomology groups of the étale sheaf  $\mathcal{O}^+/\pi$  (cf. [Man22, Ex. 3.3.17] and 4.0.1 for precise such statements). From the properties above, we may formally deduce some nice properties for this version of cohomology, such as the Künneth formula (for cohomology with compact support) and Poincaré duality (see [HM24, Ex. 1.0.1 and 1.0.2]).

In order to descend from  $\mathcal{O}^+/\pi$ -cohomology groups to some  $\mathbb{F}_p$ -cohomology groups, we need to take some form of Frobenius invariants. Assuming that  $\pi \mid p$ , a reasonable approach is to start from the Artin-Schreier exact sequence for sheaves on X:

$$0 \to \mathbb{F}_p \to \mathcal{O}_X^+/\pi \xrightarrow{x \mapsto x^p - x} \mathcal{O}_X^+/\pi \to 0$$

And consider some form of associated long exact sequence in cohomology (or, rather, a derived refinement).

The key technical difficulty is Mann's category only captures the information of the  $\mathcal{O}^+/\pi$ -cohomology up to an *almost* isomorphism, which need not commute with taking Frobenius invariants. This is not an artifact of Mann's formalism, as, in practice, most computations of  $\mathcal{O}^+$  or  $\mathcal{O}^+/\pi$ -cohomology rely on the fundamental result of *almost* vanishing of cohomology affinoid perfectoid spaces [Sch17, Lemma 8.8], which only holds in the almost sense.

However, we can still adapt this idea, in a less naive way. Whenever  $\pi \mid p$ , the unit object  $\mathcal{O}_X^+/\pi \in \mathcal{D}_X^a(X, \mathcal{O}_X^+/\pi)$  is naturally endowed with a Frobenius endomorphism  $\varphi$ . This endows the whole category  $\mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)$  with a Frobenius endomorphism  $\varphi^*$ , similarly as how the Frobenius endomorphism on  $\mathbb{F}_q$  induces a mapping  $R \mapsto \varphi^*R := R \otimes_{x \mapsto x^p} \mathbb{F}_q$  on any  $\mathbb{F}_q$ -module.

<sup>&</sup>lt;sup>5</sup>Bdcs maps are the locally compactifiable ones, i.e. the maps that can locally be written as the composition of an étale and a proper map. This is a technical assumption (somewhat similar to the taut one in Huber's formalism), that is satisfied by every morphism of rigid analytic varieties, and by any morphism of analytic adic spaces that is locally of weakly finite type, finite type, cf. [Man22, Prop 3.5.14]

Recall that, in Fontaine's classical theory of  $(\varphi, \Gamma)$ —modules, a  $\varphi$ -module is the data of a module equipped with a fixed bijective and Frobenius-semilinear endomorphism. Such an endomorphism is equivalent to the data of an isomorphism  $R \cong \varphi^*R$ . This interpretation motivates the following definition:

**Definition 2.2.4.** A  $\varphi$ -module inside  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  is the data of  $M \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$ , together with a (specified) isomorphism  $\varphi^*M \simeq M$ . We let:

$$\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)^{\varphi} = eq \left( \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi) \xrightarrow{\varphi^*} \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi) \right)$$

denote the category of  $\varphi$ -modules inside  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)^{\varphi}$ .

**Remark 2.2.5.** Note that, contrary to what the notation might suggest,  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)^{\varphi}$  is very far from being a full subcategory of  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$ . Indeed, morphisms in  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)^{\varphi}$  should be thought of as *Frobenius-equivariant* morphisms.

There is a canonical "forgetful" functor  $\mathcal{D}_{\square}^{a}(X,\mathcal{O}_{X}^{+}/\pi)^{\varphi} \to \mathcal{D}_{\square}^{a}(X,\mathcal{O}_{X}^{+}/\pi)$ , that forgets the  $\varphi$ -module structure.

Assume that  $\pi =$ . We may then expect the following two functors:

- (1) A functor of " $\varphi$ -invariants" :  $(-)^{\varphi}: \mathcal{D}^{a}_{\square}(X, \mathcal{O}^{+}_{X}/p)^{\varphi} \to \mathcal{D}^{?}(X, \mathbb{F}_{p})$  to some category of "standard"  $\mathbb{F}_{p}$ -étale sheaves.
- (2) An extension of scalar functor :  $-\otimes \mathcal{O}_X^{+a}/p: D^?(X,\mathbb{F}_p) \to \mathcal{D}_\square^a(X,\mathcal{O}_X^+/p).$

This will indeed be the case, as we will see in the next subsection. Moreover, the assignment  $X \mapsto \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)^{\varphi}$  is particularly well behaved, since it also satisfies a six-functors formalism, and all properties of 2.2.3 are satisfied with  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)^{\varphi}$  in place of  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)$ .

Before introducing the two functors above, we'll need to recall some facts in standard étale cohomology.

# 2.3. Classical $\mathbb{F}_p$ -étale sheaves and the Riemann-Hilbert adjunction

Let us start by recalling some facts about the classical theory of étale sheaves in analytic geometry, as considered by Huber in [Hub96], and refined by Scholze in [Sch17].

Fix S a locally noetherian analytic adic space, and some  $n \in \mathbb{N}$ . To any locally noetherian adic space X over S, Huber associates in [Hub96] a 1-category  $D_{\acute{e}t}(X,\mathbb{Z}/n\mathbb{Z})$ , as the derived category of étale  $\mathbb{Z}/n\mathbb{Z}$ -sheaves on X. He shows that, whenever n is invertible in  $\mathcal{O}_S^+$  (and when restricting to spaces of finite type over S, the setup  $\mathcal{D}(X_{\acute{e}t},\mathbb{Z}/n\mathbb{Z})$  admits a six functor formalism.

Remark 2.3.1. While Huber works 1-categorically, and the notion of "six functor formalism" was not properly formalized yet, the six functor formalism can be translated in an  $\infty$ -categorical setup, provided we take  $\infty$ -derived constructions instead of 1-categorical ones, and be shown to be an *actual* six-functor formalism, as defined by [Man22]. This has been checked thoroughly by Zavyalov in [Zav23a, Section 8] (see also [Zav23b, Section 6.1]). In particular, it satisfies all the properties of 2.2.3.

When n is not invertible in  $\mathcal{O}_S^+$ , most of the constructions still apply, but some key properties are lacking. In particular, we do not get general proper base change (see [Man22, p. 4] for a crucial counter-example), nor a general projection formula, and and, in turn, the Poincaré duality fails. For example, the open and closed unit ball over  $\mathbb{C}_p$  does not satisfy Poincaré Duality with  $\mathbb{F}_p$  coefficients (cf. [LRZ24, Remark 6.4.11]).

**Remark 2.3.2.** Most of Huber's theory (at least in the  $\ell \neq p$  setup) has been generalized by Scholze in [Sch17] in the generality of small v-stacks, and the constructions coincide for locally noetherian analytic adic spaces. We'll use both theories, usually preferring the more modern notations of [Sch17].

<sup>&</sup>lt;sup>6</sup>Throughout the whole book, Huber works under the local noetherian assumption (cf. [Hub96, Conditions 1.1.1]). This, for the most part, can be replaced by an assumption guarantying of sheafyness (as in [SW20]), but the details of the theory have never been thoroughly checked so far

<sup>&</sup>lt;sup>7</sup>All hope is not lost, as some weak versions of proper base change and projection formula still hold; they will be extensively used in this paper.

Let us now precisely define the "standard" setup best suited for comparisons with Mann's six functor formalism. Following Mann's spirit, we will mostly work with infinity categories. In this exposition, we restrict to the setup of locally spatial diamonds, as they admit an étale site, which will simplify the constructions below.

Let X be a locally spatial diamond, and  $X_{\acute{e}t}$  the associated étale site. Following [Sch17, Prop 14.15], we let  $\mathcal{D}(X_{\acute{e}t},\mathbb{F}_p)$  be the  $\infty$ -derived category of sheaves of  $\mathbb{F}_p$ -modules over  $X_{\acute{e}t}$ . This category needs not be left-complete in general<sup>8</sup>, and we denote its left completion  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)$ .

This is functorial, in the sense that for any morphism  $f: X \to Y$  of locally spatial diamonds, there is an adjoint pair of functors (that we attribute to Huber):

$$Rf_*^{Hub}: \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p) \leftrightarrows \mathcal{D}_{\acute{e}t}(Y, \mathbb{F}_p): Rf_{Hub}^*$$

Moreover, for any X, the category  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)$  admits a well behaved derived tensor product  $\otimes_{\mathbb{F}_n}^{\mathbb{L}}$ , which admits a right adjoint, given and a derived internal Hom, denoted  $\frac{9}{RHom_{\mathbb{F}_n}}$ .

**Remark 2.3.3.** If X is of p-cohomologically finite dimension, the category  $\mathcal{D}(X_{\acute{e}t}, \mathbb{F}_p)$  is already leftcomplete by [Sch17], and there is no need to take left completion.

**Remark 2.3.4.** Note that one can also define a category  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)$  for any small v-stack X, which needs not admit an étale site, as the subcategory of the derived category of v-sheaves that pullback to étale sheaves from strictly totally perfected spaces [Sch17, Def 14.13]. We stick to the locally spatial diamond setup for our introduction.

For technical reasons, the Riemann-Hilbert functor identifies overconvergent sheaves, that we define below.

**Definition 2.3.5.** An étale sheaf  $\mathcal{F} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)$  is said to be **overconvergent** if, for any quasi-pro-étale  $morphism \ \eta : \operatorname{Spa}(C, C^+) \to X \ from \ an \ algebraically \ closed \ perfect oid \ field \ C \ with \ open \ bounded \ valuation$ subring  $C^+$ , and  $x^{\circ}$  denotes the composition  $\operatorname{Spa}(C, \mathcal{O}_C)$  where C is a complete algebraically closed local field and  $C^+$  is an open bounded valuation subring, the value of the pullback  $\eta^*\mathcal{F}$  does not depend on  $C^+$ . We let  $\mathcal{D}(X_{\acute{e}t}, \mathbb{F}_n)^{oc} \subset \mathcal{D}(X_{\acute{e}t}, \mathbb{F}_n)$  denote the full subcategory of overconvergent étale  $\mathbb{F}_n$ -sheaves on X.

This category does not admit very good stability properties; it is stable under general pullbacks and (derived) pushforward by quasi-compact morphisms, but not under general pushforwards.

We may now introduce precisely the Riemann-Hilbert and  $\varphi$ -invariants functor, as well as a few of their properties.

**Definition 2.3.6.** For X an untilted locally spatial diamond over a complete extension K of  $\mathbb{Q}_p$  with pseudo-uniformizer  $\pi$ , Mann defines :

- (1) A "Riemann-Hilbert" functor ⊗ O<sub>X</sub><sup>+</sup>/π : D<sub>ét</sub>(X, F<sub>p</sub>)<sup>oc</sup> → D<sub>□</sub><sup>a</sup>(X, O<sub>X</sub><sup>+a</sup>/π)<sup>φ</sup>
  (2) Its left adjoint, a functor of φ-invariants : (-)<sup>φ</sup> : D<sub>□</sub><sup>a</sup>(X, O<sub>X</sub><sup>+</sup>/π)<sup>φ</sup> → D<sub>ét</sub>(X, F<sub>p</sub>)<sup>oc</sup>

It is straightforward to check that the Riemann-Hilbert functor commutes with pullbacks, as it is defined by descent.

The key fact for our purposes is as follows:

Proposition 2.3.7. ([Man22, Thm 3.9.23]) The Riemann-Hilbert functor is fully faithful.

Moreover, when restricting both sides of the equivalence to the full subcategories of dualizable objects on both sides, L.Mann proves that the Riemann-Hilbert functor induces an equivalence of category. In [Man22], this is the result used by Mann to establish his form of Poincaré duality. Dualizable objects of  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)^{oc}$  can be seen to correspond to perfect complexes, i.e. the ones that are locally quasi-isomorphic to bounded complexes of finite dimensional.

<sup>&</sup>lt;sup>8</sup>Recall that a stable  $\infty$ -category  $\mathcal C$  with a t-structure is said to be left-complete if, for all objects A of  $\mathcal C$ , the natural morphism  $A \to \varprojlim_{n \in \mathbb Z} \tau^{\geq n} A$  is an isomorphism, where  $\tau^{\geq n}$  denotes the truncation. The left completion of  $\mathcal C$  is then defined by  $\hat{\mathcal{C}} = \varprojlim_{n \in \mathbb{Z}} \tau^{\geq n} \mathcal{C}$ , which remains a stable infinity category.

<sup>&</sup>lt;sup>9</sup>This is neither the notation of [Hub96] nor [Sch17], but mimics the one of Mann.

In our setup, we're interested in the étale cohomology of non-proper spaces, for which cohomology groups may be infinite dimensional, and, hence, non dualizable. We'll nonetheless prove that the Poincaré duality holds on the essential image of the Riemann-Hilbert functor.

## 2.4. Poincaré Duality on the essential image of Riemann-Hilbert

For any sheaf  $\mathcal{F} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)$ , we may define its dual  $\mathcal{F}^{\vee}$  as  $\mathcal{F}^{\vee} = \underline{\mathrm{RHom}}_{\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)}(\mathcal{F}, (\mathbb{F}_p)_X)$ , where  $(\mathbb{F}_p)_X$  denotes the constant sheaf valuing  $\mathbb{F}_p$  on  $X_{\acute{e}t}$ . Our technical lemma can then be stated as follows:

**Lemma 2.4.1.** Let K be a complete extension of  $\mathbb{Q}_p$ ,  $\mathcal{O}_K$  be its ring of integers, and  $\pi$  be a pseudo-uniformizer.

Let  $f: X \to Y$  be a smooth morphism of locally noetherian analytic adic spaces over  $\operatorname{Spa}(K, \mathcal{O}_K)$ , that is pure of dimension d. Let  $\mathcal{L} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}$ .

Assume that there exists some  $\mathcal{F}_{\mathcal{L}} \in \mathcal{D}_{\acute{e}t}(Y,\mathbb{F}_p)^{oc}$  and an isomorphism in  $\mathcal{D}_{\square}^a(Y,\mathcal{O}_X^+/\pi)$ :

$$f_1^{Mann}(\mathcal{L}\otimes\mathcal{O}_{\mathbf{Y}}^{+a}/\pi)\simeq\mathcal{F}_{\mathcal{L}}\otimes\mathcal{O}_{\mathbf{Y}}^{+a}/\pi$$

Then, there is an isomorphism in  $\mathcal{D}_{\acute{e}t}(Y,\mathbb{F}_p)$ :

$$Rf_*^{Hub}\mathcal{L}^{\vee}(d)[2d] \simeq \mathcal{F}_{\mathcal{L}}^{\vee}$$

Note that, if f is additionally proper and  $\mathcal{L}$  is a perfect (e.g. a local system), the primitive comparison theorem [Man22, Corro 3.9.24] applies, so that the assumption is automatically verified with  $\mathcal{F}_{\mathcal{L}} = Rf_*^{Mann}\mathbb{L}$  (that remains overconvergent since f is quasi-compact). In that setup, the above lemma recovers [Man22, Corro 3.10.22].

*Proof.* Recall that the Riemann-Hilbert functor is symmetric monoidal, and commutes with pullbacks. Hence, for any  $A \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)$ , we have :

$$f_{!}^{Mann}\left(\left(\mathcal{L}\otimes\mathbb{F}_{p}^{\mathbb{L}}Rf_{Hub}^{*}A\right)\otimes\mathcal{O}_{X}^{+a}/\pi\right)\cong f_{!}^{Mann}\left(\left(\mathcal{L}\otimes\mathcal{O}_{X}^{+a}/\pi\right)\otimes\left(Rf_{Hub}^{*}A\otimes\mathcal{O}_{X}^{+a}/\pi\right)\right)$$

$$\cong f_{!}^{Mann}\left(\left(\mathcal{L}\otimes\mathcal{O}_{X}^{+a}/\pi\right)\otimes f_{Mann}^{*}(A\otimes\mathcal{O}_{Y}^{+a}/\pi)\right)$$

$$\cong f_{!}^{Mann}\left(\mathcal{L}\otimes\mathcal{O}_{X}^{+a}/\pi\right)\otimes\left(A\otimes\mathcal{O}_{Y}^{+a}/\pi\right)$$

$$\simeq\left(\mathcal{F}_{\mathcal{L}}\otimes\mathcal{O}_{Y}^{+a}/\pi\right)\otimes\left(A\otimes\mathcal{O}_{Y}^{+a}/\pi\right)$$

$$\cong\left(\mathcal{F}_{\mathcal{L}}\otimes\mathbb{F}_{x}^{\mathbb{L}}A\right)\otimes\mathcal{O}_{Y}^{+a}/\pi$$

Where the third isomorphism follows from the projection formula. From there, we compute:

$$\operatorname{Hom}_{\mathcal{D}_{\acute{e}t}(Y,\mathbb{F}_p)} \left( A, Rf_*^{Hub} \mathcal{L}^{\vee}(d)[2d] \right)$$
 By adjunction 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)} \left( Rf_{Hub}^* A, \operatorname{\underline{R}Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)}(\mathcal{L}, \mathbb{F}_p) (d)[2d] \right)$$
 By adjunction 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)} \left( Rf_{Hub}^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}, \mathbb{F}_p (d)[2d] \right)$$
 By adjunction 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\Box}^a(\mathcal{O}_X^{+a}/\pi)^{\varphi}} \left( (Rf_{Hub}^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi, \mathcal{O}_X^{+a}/\pi (d)[2d] \right)$$
 By full faithfulness 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\Box}^a(\mathcal{O}_X^{+a}/\pi)^{\varphi}} \left( (Rf_{Hub}^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi, f_{Mann}^{\dagger} \mathcal{O}_Y^{+a}/\pi \right)$$
 By [Man22, Thm 3.10.20] 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\Box}^a(\mathcal{O}_X^{+a}/\pi)^{\varphi}} \left( f_!^{Mann} \left( (Rf_{Hub}^* A \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{L}) \otimes \mathcal{O}_X^{+a}/\pi \right), \mathcal{O}_Y^{+a}/\pi \right)$$
 By adjunction 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\Box}^a(\mathcal{O}_X^{+a}/\pi)^{\varphi}} \left( (\mathcal{F}_{\mathcal{L}} \otimes_{\mathbb{F}_p}^{\mathbb{L}} A) \otimes \mathcal{O}_Y^{+a}, \mathcal{O}_Y^{+a}/\pi \right)$$
 By the above 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)^{\varphi}} \left( \mathcal{F}_{\mathcal{L}} \otimes_{\mathbb{F}_p}^{\mathbb{L}} A, \mathbb{F}_p \right)$$
 By [Man22, Thm 3.9.23] 
$$\cong \operatorname{Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)} \left( A, \operatorname{\underline{R}Hom}_{\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)}(\mathcal{F}_{\mathcal{L}},\mathbb{F}_p) \right)$$
 By adjunction

We conclude using Yoneda's lemma.

We then deduce the following corollary, that will be useful in practice. By taking global sections, we may freely identify  $\mathcal{D}_{\acute{e}t}(\operatorname{Spa}(C,\mathcal{O}_C),\mathbb{F}_p)$  with the derived infinity category of  $\mathbb{F}_p$ -vector spaces, denoted  $\mathcal{D}_{\acute{e}t}(\mathbb{F}_p)$ , with its natural monoidal structure.

**Corollary 2.4.2.** Let C be a complete algebraically closed extension of  $\mathbb{Q}_p$ , and  $\mathcal{O}_C$  be its ring of integers. Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a smooth analytic adic space that is pure of dimension d.

Let  $\mathcal{L}$  be an étale local system on X. Assume that there exists some  $\mathcal{F}_{\mathcal{L}} \in \mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathbb{F}_p)$  and an isomorphism in  $\mathcal{D}^a_{\square}(Y,\mathcal{O}^+_C/\pi)$ :

$$f_!(\mathcal{L}\otimes\mathcal{O}_X^+/\pi)\simeq\mathcal{F}_{\mathcal{L}}\otimes\mathcal{O}_C^{+a}/\pi$$

Then, for all  $0 \le i \le 2d$ , there exists an isomorphism:

$$H^i_{\acute{e}t}(X,\mathcal{L}^{\vee})(d) \simeq \operatorname{Hom}_{\mathbb{F}_n}(\pi_{i-2d}(\mathcal{F}_{\mathcal{L}}),\mathbb{F}_p)$$

*Proof.* Recall that the sheaf  $\mathcal{F}_{\mathbb{L}}$  is necessarily overconvergent, so that the lemma above 2.4.1 applies, and we get an isomorphism  $Rf_*^{Hub}\mathcal{L}^{\vee}(d)[2d] \simeq \mathcal{F}_{\mathcal{L}}^{\vee}$ .

Let us take cohomology groups  $\pi_{i-2d}$  on both sides. Since  $\mathcal{L}$  is concentrated in degree zero, so is  $\mathcal{L}^{\vee}$ , and we compute :

$$\pi_{2d-i}\left(Rf_*^{Hub}(\mathcal{L}^{\vee})(d)[2d]\right) \cong R^i f_*^{Hub} \mathcal{L}^{\vee}(d) \cong H^i(X, \mathcal{L}^{\vee})(d)$$

Moreover,  $\mathbb{F}_p$  is an injective object of  $\mathcal{D}(\mathbb{F}_p)$ , so that we get :

$$\pi_{2d-i}(\mathrm{RHom}_{\mathcal{D}_{\acute{e}t}(\mathbb{F}_p)}(\mathcal{F}_{\mathcal{L}},\mathbb{F}_p)) = \mathrm{Hom}_{\mathbb{F}_p}(\pi_{i-2d}(\mathcal{F}_{\mathcal{L}}),\mathbb{F}_p)$$

This yields the desired result.

**Remark 2.4.3.** Note that this is clearly functorial in  $\mathcal{L}$ , provided that  $\mathcal{F}_{\mathcal{L}}$  can be chosen to be functorial in  $\mathcal{L}$ .

Checking whether an element of  $\mathcal{D}^a_{\square}(Y,\mathcal{O}^+_Y/\pi)$  lies in the essential image of the Riemann-Hilbert functor is hard to check a priori. Following the usual statements of Poincaré Duality, we would like  $\mathcal{F}_{\mathcal{L}}$  to be some form of " $Rf_!^{Hub}\mathcal{L}$ ". Assuming temporarily that K=C, the hypothesis becomes of the form:

$$f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^{+a}/\pi) \cong R f_!^{Hub} \mathcal{L} \otimes \mathcal{O}_C/\pi$$

Providing that we can construct a map, proving that it is an isomorphism can be checked on cohomology groups, and is a form of "primitive comparison theorem with compact support" - provided that one can identify  $\pi_k f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^{+a}/\pi)$  with standard étale cohomology groups with compact support  $H^k_{\acute{e}t,c}(X,\mathcal{L} \otimes \mathcal{O}_X^{+}/\pi)$ , up to an almost isomorphism.

On the one hand  $f_!^{Mann}(\mathcal{F} \otimes \mathcal{O}_X^{+a}/\pi)$  lives inside  $\mathcal{D}_{\square}^a(Y, \mathcal{O}^{+a}/\pi)$ , while, on the other,  $Rf_!^{Hub}(\mathcal{F} \otimes \mathcal{O}_X^{+a}/\pi)$  lives a priori in  $\mathcal{D}(\operatorname{Spa}(C, \mathcal{O}_C)_{\acute{e}t}, \mathbb{F}_p)$ , but should naturally admit a structure of an  $\mathcal{O}_C/\pi$ -module.

For an arbitrary X, Mann provides a way to embed (overconvergent objects inside) this category of standard sheaves of  $\mathcal{O}_X^+/\pi$ -modules as a full subcategory of  $\mathcal{D}_{\square}^a(X,\mathcal{O}^{+a}/\pi)$ , formed by discrete objects, i.e. the one that have trivial condensed structure. We present this result in the following section.

2.5. Standard almost  $\mathcal{O}^+/\pi$ -étale sheaves and discrete objects in  $\mathcal{D}^a_\square(X,\mathcal{O}^+/\pi)$ 

This section follows [Man22, Section 3.3].

2.5.1. Standard almost  $\mathcal{O}^+/\pi$ -étale sheaves. Let X be an untilted locally spatial diamond over a complete extension K of  $\mathbb{Q}_p$ . We will construct a left-complete category  $\mathcal{O}^{10}_{\mathcal{E}}(X, \mathcal{O}_X^{+a}/\pi)$  of almost quasi-coherent étale sheaves of  $\mathcal{O}_X^+/\pi$ -modules over  $X_{\acute{e}t}$ , that compatible with pushforwards and pullbacks.

While the pushforward of an  $\mathcal{O}^+/\pi$ -module is naturally an  $\mathcal{O}^+/\pi$ -module, the pullback should really be of a pullback of *quasi-coherent* sheaves, and does not coincide with the pullback of étale sheaves  $f_{Hub}^*$  in general. Hence, we really need to look at the  $\mathcal{O}_X^+/\pi$ -modules, over the ringed object  $\mathcal{O}_X^+/\pi$ .

Let us first recall the setup in the non-almost case, following [Sta23, Chapter 03A4].

 $<sup>^{10}\</sup>mathrm{That}$  it is denoted  $\mathrm{Shv}^{\wedge}(X_{\acute{e}t},\mathcal{O}_X^{+a}/\pi)$  in [Man22]

**Definition 2.5.1.** Let X be an untilted locally spatial diamond over a complete extension K of  $\mathbb{Q}_p$ , with pseudouniformizer  $\pi$ .

We may view the sheaf  $\mathcal{O}_X^+/\pi$  as a sheaf of rings on  $X_{\acute{e}t}$ , and we consider the ringed topos  $(\operatorname{Sh}(X_{\acute{e}t}), \mathcal{O}_X^+/\pi)$  associated to the ringed site  $(X_{\acute{e}t}, \mathcal{O}_X^+/\pi)$ .

Every morphism of analytic adic spaces  $X \to Y$  induces a morphism  $\mathcal{O}_Y \to f_*\mathcal{O}_X$  compatible with the valuations, which induces a morphism  $\mathcal{O}_Y^+/\pi \to f_*\mathcal{O}_X^+/\pi$ , and hence defines a morphism of ringed sites. This defines a pullback functor

$$f^{*,qcoh}\mathcal{F} = \mathcal{O}_X^+/\pi \otimes_{f^*\mathcal{O}_Y^+/\pi} f^*\mathcal{G}$$

Where  $f^*$  denotes the pullback of étale sheaves. <sup>11</sup>

We let  $\operatorname{Mod}(\mathcal{O}_X^+/\pi)$  denote the abelian category of sheaves of  $\mathcal{O}_X^+/\pi$ -modules in  $X_{\acute{e}t}$ , and  $\mathcal{D}(X_{\acute{e}t},\mathcal{O}_X^+/\pi)$  be its derived infinity category. Any functor  $f:Y\to X$  induces an adjoint pair:

$$Rf_{Hub}^{*,qcoh}: \mathcal{D}_{\acute{e}t}(X,\mathcal{O}_{X}^{+}/\pi) \leftrightarrows \mathcal{D}_{\acute{e}t}(Y,\mathcal{O}_{Y}^{+}/\pi): Rf_{*}^{Hub}$$

Defining the almost version  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)$ , it is a bit more tricky, as the object " $\mathcal{O}_X^{+a}/\pi$ " does not exist as a sheaf of classical rings inside  $\operatorname{Sh}(X_{\acute{e}t},\mathbb{F}_p)$ . Instead, we may define a classical sheaf  $\mathcal{F}$  of  $\mathcal{O}_X^{+a}/\pi$ -modules on X as a sheaf on  $X_{\acute{e}t}$ , valued in the category of almost  $\mathcal{O}_X^+/\pi(X)$ -modules, together with the compatible structure of an  $\mathcal{O}^+/\pi(U)$ -module on  $\mathcal{F}(U)$  for all  $U \in X_{\acute{e}t}$ . We then let  $\mathcal{D}(X_{\acute{e}t},\mathcal{O}_X^{+a}/\pi)$  be the derived infinity category of the above category, and  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}^{+a}/\pi)$  be its left completion. See also [GR03, Paragraph 5.5.1] for the classical theory of almost étale sheaves over schemes.

Here are the main properties that will be useful to us.

**Proposition 2.5.2.** Let  $f: X \to Y$  be a morphism of analytic adic spaces over a complete nonarchimedean field K, with pseudo-uniformizer  $\pi$ . Then, there exists an adjoint pairs of functors

$$Rf_{Hub}^{*,qcoh}: \mathcal{D}_{\acute{e}t}(X,\mathcal{O}_{X}^{+a}/\pi) \leftrightarrows \mathcal{D}_{\acute{e}t}(Y,\mathcal{O}_{Y}^{+a}/\pi): Rf_{*}^{Hub}$$

Moreover, for any analytic adic space X, there is a natural t-exact almostification functor

$$(-)^a: \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^+/\pi) \to \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+a}/\pi)$$

It is computed as  $\mathcal{F}^a(U) = \mathcal{F}(U)^a$ , where the second a denotes the almostification for derived  $\mathcal{O}_X^+/p(X)$ modules, and is compatible with the formation of pushforward and pullbacks. This functor admits both a
right and a left adjoint.

Remark 2.5.3. Note that there is a natural morphism  $-\otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p : \mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p) \to \mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^+/\pi)$ , obtained by left-completing the derived tensor product. Composing this with the almostification functor  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^+/\pi) \to \mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)$  defines a functor that it closely related to the Riemann-Hilbert functor (at least under some extra hypothesis), cf. 2.5.8.

The same definition as in 2.3.5 allows us to consider overconvergent objects inside  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)$ , and we denote the full subcategory as  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)^{oc}$  (cf. [Man22, Defi 3.3.14] for more detail).

Note that this is really a notion of overconvergence for almost sheaves, in particular, the sheaf  $\mathcal{O}^+/p$  itself is not overconvergent when viewed as an étale sheaf of  $\mathbb{F}_p$ -modules. It is, however, almost overconvergent. Let us now identify this category with the subcategory of discrete objects inside  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$ .

2.5.2. Identification of discrete objects. Recall that there is a fully faithful inclusion of rings into condensed rings (viewed as sheaves of rings on the category of profinite spaces), given by constant sheaves, which factors through solid objects. The essential image of this functor is the full subcategory of discrete rings. For any totally disconnected affinoid perfectoid  $\operatorname{Spa}(A, A^+)$ , we may consider the subcategory  $\mathcal{D}^a_{\square}(A^+/\pi)_{\omega} \subset \mathcal{D}^a_{\square}(A^+/\pi)$  of discrete (derived, almost)  $A^+/\pi$ -modules. This construction descends to a subfunctor of discrete objects  $\mathcal{D}^a_{\square}(\mathcal{O}^+_X/\pi)_{\omega} \subset \mathcal{D}^a_{\square}(\mathcal{O}^+_X/\pi)$  for any untilted v-stack X, defined as follows.

<sup>&</sup>lt;sup>11</sup>Note that, what we denote  $f^*$  here is usually denoted  $f^{-1}$ , while what we denote  $f^{*,qcoh}$  is usually just referred to as  $f^*$ 

**Definition 2.5.4.** (cf. [Man22, Def 3.2.17])

Let X be a small untilted v-stack, with pseudo-uniformizer  $\pi$ .

We let  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)_{\omega} \subset \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)$  be the full subcategory formed by objects  $\mathcal{F}$  such that, for any map  $f: Y \to X$  from a totally disconnected affinoid perfectoid space, the pullback  $f^*_{Mann}\mathcal{F}$  is discrete.

We have the following properties:

**Proposition 2.5.5.** For any untilted small v-stack X, the subcategory  $\mathcal{D}^a_{\square}(\mathcal{O}_X^+/\pi)_{\omega}$  is stable under colimits, and the inclusion  $\mathcal{D}^a_{\square}(\mathcal{O}_X^+/\pi)_{\omega} \subset \mathcal{D}^a_{\square}(\mathcal{O}_X^+/\pi)$  admits a right adjoint, called the discretization and denoted  $(-)_{\omega}$ .

The final result crucial result is that discrete objects identify with our almost quasi-coherent category defined above, under an additional technical hypothesis.

**Proposition 2.5.6.** Let X be an untilted locally spatial diamond, with pseudo-uniformizer  $\pi$ . Assume moreover that X admits a map to an affinoid perfectoid space.

Then, there is an equivalence of  $\infty$ -categories:

$$J_X: \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega} \cong \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+a}/\pi)^{oc}$$

Moreover, for all morphism  $f: X \to Y$  of untilted locally spatial diamonds over an affinoid perfectoid space, there is a canonical isomorphism of functors  $\mathcal{D}_{\acute{e}t}(Y,\mathcal{O}_{Y}^{+,a}/\pi)^{oc} \to \mathcal{D}_{\square}^{a}(X,\mathcal{O}_{X}^{+}/\pi)_{\omega}$ :

$$f_{Mann}^* \circ J_Y \cong J_X \circ Rf_{Hub}^{qcoh,*}$$

*Proof.* This is [Man22, Prop 3.3.16], which is stated in loc. cit. under the additional assumption that  $X \in X_v^{\Lambda}$  and likewise for Y (with the notations of loc.cit.). This is satisfied whenever X and Y admit a morphism to an affinoid perfectoid space, using [Man22, Remark 3.2.4].

If  $X = \operatorname{Spa}(C, \mathcal{O}_C)$ , we denote for simplicity  $J_C := J_{\operatorname{Spa}(C, \mathcal{O}_C)}$ .

**Remark 2.5.7.** Note that the hypothesis that X admits a map to an affinoid perfectoid space allows implies that X admits "enough pseudouniformizers", in the sense of [Man22, Defi 3.2.6], which is crucially used in the proof.

At this point, we constructed tools for two natural functors  $\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)^{oc} \to \mathcal{D}^a_{\square}(X,\mathcal{O}^+/\pi)$ ; the Riemann-Hilbert functor, and the almostification of the tensor product of classical étale sheaves. We show that they actually coincide (this was informally remarked in [Zav23b, Paragraph 6.2]).

**Proposition 2.5.8.** Let X be an untilted locally spatial diamond, with pseudo-uniformizer  $\pi$ . Suppose that X admits a map to an affinoid perfectoid space.

Let  $\mathcal{F} \in \mathcal{D}_{\acute{e}t}(X, \mathbb{F}_p)^{oc}$ . Then, the following square commutes:

$$\mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p)^{oc} \xrightarrow{-\otimes \mathcal{O}_X^{+a}/\pi} \mathcal{D}_{\square}^a(X,\mathcal{O}_X^+/\pi)_{\omega}^{\varphi}$$

$$\left(-\otimes_{\mathbb{F}_p}^{\mathbb{L}}\mathcal{O}_X^+/\pi\right)^a \downarrow \qquad \qquad \downarrow$$

$$\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)^{oc} \xrightarrow{J_X^{-1}} \mathcal{D}_{\square}^a(X,\mathcal{O}_X^+/\pi)_{\omega}$$

Here, the top horizontal arrow is the Riemann-Hilbert functor of 2.3.6, the rightmost vertical arrow is the forgetful functor described in 2.2.5, the leftmost vertical arrow is the functor described in 2.5.3 (it preserves overconvergent objects, since taking stalks commutes with tensor products), and the lower horizontal arrow is the identification of 2.5.6.

*Proof.* It follows from [Man22, Lemma 3.9.18.(i)], the proof of [Man22, Prop 3.3.16], [Man22, Remark 3.9.11] and [Man22, Lemma 3.2.19.(i)] that the four categories satisfy pro-étale descent, so that it suffices to check the commutation on a basis of strictly totally disconnected affinoid perfectoid spaces, for which, using the identifications from [Man22, Lemma 3.3.15.(ii)] [Man22, Lemma 3.9.18.(iii)], this is simply the definition of the Riemann-Hilbert functor [Man22, Prop 3.9.8]. □

Moreover, since the equivalence is stable under pullback, it is also stable by the right adjoints of such maps. We already defined a "discretization" functor, but we'll need the following lemma:

**Lemma 2.5.9.** Let X be an untilted locally spatial diamond, with pseudo-uniformizer  $\pi$ . Then the inclusion  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)^{oc} \to \mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^{+a}/\pi)$  admits a right adjoint, denoted  $(-)^{oc}$ .

Proof. The category of "real" (i.e. non almost) sheaves  $\mathcal{D}(X_{\acute{e}t}, \mathcal{O}_X^+/\pi)$  is presentable, since it is the derived infinity category of a Grothendieck abelian category by [Lur17, Prop 1.3.5.21] and [Sta23, Section 07A5]. The category of almost sheaves  $\mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+a}/\pi)$  remains presentable, since it is constructed by an colimits-preserving localization [Lur08, Remark 5.5.16]. Moreover, the category  $\mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+a}/\pi)^{oc}$  is presentable, by the identification above and [Man22, Lemma 3.2.19], so that we may apply the adjoint functor theorem [Lur08, Corro 5.5.2.9].

Hence, it suffices to prove that a colimit of overconvergent sheaves remains overconvergent, but that is clear, since overconvergence can be read on stalks, which commute with colimits.  $\Box$ 

We'll deduce of the above the following compatibility statement between pushforwards:

Corollary 2.5.10. Let  $f: X \to Y$  be a morphism of untilted locally spatial diamonds, and  $\pi$  be a pseudo-uniformizer of X.

Then, there is a natural isomorphism of functors  $\mathcal{D}^a_{\square}(X,\mathcal{O}^+_X/\pi)_{\omega} \to \mathcal{D}_{\acute{e}t}(Y,\mathcal{O}^{+,a}_Y/\pi)^{oc}$ :

$$J_Y^{-1} \circ (f_*^{Mann} -)_\omega \cong \left( R f_*^{Hub} J_X^{-1}(-) \right)^{oc}$$

Proof. Recall that there is a natural isomorphism of functors  $\mathcal{D}^a_{\square}(\mathcal{D}_{\acute{e}t}(Y,\mathcal{O}_Y^{+,a}/\pi)^{oc} \to \mathcal{D}^a_{\square}(X,\mathcal{O}_X^+/\pi)_{\omega})$  defined by  $J_X \circ f^*_{Mann} \cong Rf^{qcoh,*}_{Hub} \circ J_Y$ , as in 2.5.6. We'll deduce the result above by passing to right adjoints. Since  $J_X$  and  $J_Y$  are equivalences of categories, they are adjoint to their inverse.

Recall that  $f_{Mann}^*$  preserves discrete objects, by definition. By composition of right adjoints, the pull-back  $f_{Mann}^*$ :  $\mathcal{D}_{\square}^a(Y, \mathcal{O}_Y^+/\pi)_{\omega} \to \mathcal{D}_{\square}^a(X, \mathcal{O}_X^+/\pi)_{\omega}$  restricted to discrete objects admits the right adjoint  $(f_*^{Mann}-)_{\omega}$ , where  $(-)_{\omega}$  is the discretization functor described in 2.5.5.

Likewise, since pullback preserves overconvergent objects, the functor  $(Rf_*^{Hub}-)^{oc}$  is right adjoint to  $Rf_{Hub}^*: \mathcal{D}_{\acute{e}t}(Y, \mathcal{O}_Y^{+a}/\pi)^{oc} \to \mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^{+a}/\pi)^{oc}$ .

Hence,  $J_Y^{-1} \circ (f_*^{Mann} -)_\omega$  and  $(Rf_*^{Hub}J_X^{-1}(-))^{oc}$  are left adjoint to  $Rf_{Hub}^{qcoh,*} \circ J_Y$  and  $J_X \circ f_{Mann}^*$ , respectively. Since naturally isomorphic functors admit naturally isomorphic (right) adjoints, this concludes.

In particular, we deduce the following practical version:

Corollary 2.5.11. Let C be a complete algebraically closed extension of  $\mathbb{Q}_p$ , with integral ring  $\mathcal{O}_C$ . Let  $f: X \to \operatorname{Spd}(\mathcal{O}_C)$  be a morphism from an locally spatial diamonds, and  $\pi$  be a pseudo-uniformizer of C. Let  $\mathcal{F}$  be a sheaf of abelian groups over  $X_{\acute{e}t}$ .

Then, there is a natural isomorphism :  $(f_*^{Mann}(\mathcal{F} \otimes \mathcal{O}_X^{+a}/\pi))_{\omega} \cong J_C \circ (Rf_*^{Hub}\mathcal{F} \otimes \mathcal{O}_X^+/\pi)^a$ 

*Proof.* Recall that all étale sheaves on  $\operatorname{Spa}(C, \mathcal{O}_C)$  are overconvergent, so that  $(-)^{oc}$  is simply the identity. Then, this directly follows from 2.5.8 and 2.5.10, together with the fact that  $Rf_*^{Hub}$  commutes with almositification.

**Remark 2.5.12.** In the setup of the corollary above, assume furthermore that f is quasi-compact, so that  $f_*^{Mann}$  preserves discrete objects by [Man22, Lemma 3.3.10.(ii)].

Letting  $\mathcal{F} = \mathbb{F}_p$ , we get  $f_*^{Mann}\mathcal{O}_X^{+,a}/\pi \cong J_C \circ \left(Rf_*^{Hub}\mathcal{O}_X^+/\pi\right)^a$ , which recovers a variant of [Man22, Example 3.3.17]. Informally, this means that  $f_*^{Mann}\mathcal{O}_X^{+,a}/p$  computes, up to an almost isomorphism, the étale cohomology of the classical étale sheaf  $\mathcal{O}_X^+/p$ .

#### 3. About the topology of locally spatial diamonds

This section contains many technical results that will be useful in what follows. We work with Huber's notion of étale cohomology developed in [Hub96] - or, rather, its refinement to diamonds by Scholze [Sch17].

## 3.1. The closed complement of an open sub variety

A standard problem in the theory of analytic geometry is that, unlike for schemes, the complement of an open subset of a rigid analytic variety needs not admit a structure of an rigid analytic variety, nor of an analytic adic space. Indeed, every morphism of analytic adic space is generalizing (in a topological sense, cf. [Sta23, Section 0060]), so that the complement of a non-specializing open subset never admits an analytic structure.

However, this is not the only problem. Fix  $C = \mathbb{C}_p$ , and consider the inclusion of the open unit disk  $\mathbb{B}_C$  inside the rational subset  $\{|T| < 1\}$  of the closed unit disk  $\mathbb{B}_C = \operatorname{Spa}(C\langle T \rangle, \mathcal{O}_C\langle T \rangle)$ . The topological complement contains a single point of rank 2, and hence needs not admit an analytic structure.

In the realm of rigid-analytic geometry, one often uses the following:

**Lemma 3.1.1.** Let U be an quasi-compact admissible open subset of a quasi-separated rigid analytic variety X. Then, the complement  $X \setminus U$  is an admissible open subset of X.

Proof. This is [Hub96, Prop 5.1.5] or [Van92, Lemma 1.5].

Recall that the considered topology is a Grothendieck topology, so we can may (and, usually, can) decompose a connected rigid-analytic variety as the disjoint union of two disjoint admissible open subsets, as long as the considered covering is not admissible. The situation is different in the world of adic spaces, where the analytic topology is not a Grothendieck topology.

Let r be the standard functor from the category of rigid-analytic varieties over a fixed complete non-archimedean extension K of  $\mathbb{Q}_p$  to the category of analytic adic spaces over  $\mathrm{Spa}(K, \mathcal{O}_K)$ , as developed in [Hub96, Paragraph 1.11].

Let U and X be as in the previous lemma, and assume moreover that X is connected, and U is neither empty, nor equal to X. The functor r preserves open immersions, so that both r(U) and  $r(X \setminus U)$  are open subsets of the connected topological space r(X), so that, necessarily  $r(X \setminus U) \neq r(X) \setminus r(U)$ . However, we have the following:

**Lemma 3.1.2.** Let X be a quasi-separated rigid-analytic variety over a complete non-archimedean field K, and  $U \subset X$  a quasi-compact admissible open subset.

Then,  $r(X \setminus U) = (r(X) \setminus r(U))^{\circ}$ , where  $T^{\circ}$  denotes the interior of a topological space T.

*Proof.* Recall that, since X is quasi-separated, by [Hub96, Paragraph 1.1.11], r induces an increasing bijection between quasi-compact admissible open subsets of X and quasi-compact open subsets of r(X). Moreover, for any open U of X,  $r(U) \cap X = U$ , so that r is a isomorphism of posets.

Since r is order-preserving,  $r(X) = r(X \setminus U \sqcup U) \subset r(X \setminus U) \cup r(U)$ , so that  $r(U \setminus X) \subset r(X) \setminus r(U)$ . Since  $U \setminus X$  is an admissible open of X,  $r(X \setminus U)$  is open in r(X), so that  $r(X \setminus U) \subset (r(X) \setminus r(U))^{\circ}$ .

Since r(X) admits a basis of quasi-compact open subsets, we may write  $(r(X) \setminus r(U))^{\circ}$  as the union of all quasicompact open subsets contained in  $r(X) \setminus r(U)$ . Let V be such a quasicompact open. By the properties of r, we may write V = r(W) for some quasicompact admissible open W of X. We have  $r(W) \subset r(X) \setminus r(U)$ , so that  $r(W) \cup r(X) \subset r(U)$  so that, since W respects the ordering,  $W \subset X \setminus U$ . Since r is order-preserving, we have  $r(W \cap U) \subset r(W) \cap r(U) = \emptyset$ , so that  $W \cap U = \emptyset$ , and  $W \subset X \setminus U$ . Finally,  $V = r(W) \subset r(X \setminus U)$ , which concludes.

Huber, in [Hub96], defines a theory of *pseudo*-adic spaces, so that any topologically reasonable (convex, pro-constructible) subset of an analytic adic space admits a pseudo-analytic structure, and, in turn, an étale site. However, this language is quite outdated, and is rarely used in the litterature.

In this paper, we'll bypass the use of pseudo-adic spaces, using locally spatial diamonds instead.

Note that by [Sch17, Prop 11.15], open subsets of locally spatial diamonds naturally admit a structure of a diamond, which is then locally spatial diamond by [Sch17, Prop 11.20]. For closed subsets, we'll use the following lemma:

**Lemma 3.1.3.** Let X be a diamond, and  $F \subset X$  be a generalizing closed subset of X. Then F is the underlying space of a sub-locally spatial diamond, still denoted F, given by the fiber product of v-sheaves  $F = \underline{F} \times_{|X|} X$ .

Here, we use the notations from [Sch17]. In particular, for X a diamond, we denote |X| its underlying topological space, and, for T a topological space, we let T denote the v-sheaf given by T(X) := Cont(|X|, T).

*Proof.* By [Ans+22, Lemma 2.7], the above defines a closed sub v-stack of underlying topological space F. It is also a v-sheaf, since the fiber product can be seen as v-sheaves.

By [Sch17, Prop 11.10], it is naturally a diamond, and, by [Sch17, Prop 11.20], it is locally spatial.

Hence, the complement of a generalizing open subset of an analytic adic space needs not admit a structure of an analytic adic space, but will always admit the structure of a locally spatial diamond. I that setting, we furthermore have an triangle for étale sheaves, as developed in the following.

#### 3.2. Localization for locally spatial diamonds

In this section, we work with the left completed derived category of étale sheaves on a locally spatial diamond, as considered in 2.3.

In the proposition below, for any open immersion j of locally spatial diamonds, the proper pushforward  $Rj_{!}^{Hub}$  is taken in the sense of [Sch17, Defi 19.1], as the left adjoint to  $Rj_{Hub}^{*}$ . It coincides with the functor defined by Huber in [Hub96], when j is defined between analytic adic spaces.

**Proposition 3.2.1.** Let X be a locally spatial diamond whose underlying topological space decomposes as the union  $|X| = U \sqcup F$  of an open and a closed subset respectively, such that F is generalizing. Let  $j: U \to X$  and  $i: F \to X$  be the associated open (resp. closed) immersion of diamonds.

For any sheaf  $\mathcal{F}$  of abelian groups on  $X_{\acute{e}t}$ , there is an exact sequence in  $\mathcal{D}_{\acute{e}t}(X,\mathbb{Z})$ :

$$Rj_{!}^{Hub}j_{Hub}^{*}\mathcal{F} \to \mathcal{F} \to Ri_{*}^{Hub}i_{Hub}^{*}\mathcal{F}$$

This construction is moreover functorial in  $\mathcal{F}$ .

*Proof.* To simplify the notations, we consider implicitly every functor to be in the sense of Huber.

The morphisms of the triangle come from (co)unit of the adjunctions. The étale site of X has enough points, cf. [Sch17, Prop 14.3], to that it suffices to check the exactness on stalks. Let  $\overline{x} : \operatorname{Spa}(C, C^+) \to X$  be a quasi-pro-étale geometric point of X, where C is algebraically closed,  $C^+ \subset C$  is an open and bounded valuation subring, and  $\overline{x}$  maps the unique closed point of  $\operatorname{Spa}(C, C^+)$  to some  $x \in X$ .

We compute the stalks using the following lemma, which immediately concludes:

**Lemma 3.2.2.** Under the hypothesis of 3.2.1, the stalks behave as expected, i.e.:

$$(Rj_! \circ j^*\mathcal{F})_{\overline{x}} \cong \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in |U| \\ 0 & \text{otherwise} \end{cases}$$
 and  $(Ri_* \circ i^*\mathcal{F})_{\overline{x}} \cong \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in |F| \\ 0 & \text{otherwise} \end{cases}$ 

We apply base change ([Sch17, Prop 19.1], resp. [Sch17, Corro 16.10.(ii)]) to the diagrams:

$$U \times_X \operatorname{Spa}(C, C^+) \xrightarrow{j'} \operatorname{Spa}(C, C^+) \qquad F \times_X \operatorname{Spa}(C, C^+) \xrightarrow{i'} \operatorname{Spa}(C, C^+)$$

$$\downarrow^{\overline{x}'} \qquad \downarrow^{\overline{x}} \qquad \downarrow^{\overline{x}} \qquad \downarrow^{\overline{x}'} \qquad \downarrow^{\overline{x}}$$

$$U \xrightarrow{j} X \qquad F \xrightarrow{i} X$$

Hence,  $(Rj_!j^*\mathcal{F})_{\overline{x}} = \Gamma(Rj'_!\circ x'^*j^*\mathcal{F}) = \Gamma(Rj'_!j'^*\overline{x}^*\mathcal{F})$  and  $(Ri_*i^*\mathcal{F})_{\overline{x}} = \Gamma(Ri'_*\overline{x}'^*i^*\mathcal{F}) = \Gamma(Ri'_*i'^*\overline{x}^*\mathcal{F})$ . From [Sch17, Prop 12.10], we know that there are surjections:

$$|U \times_X \operatorname{Spa}(C, C^+)| \twoheadrightarrow |U| \times_{|X|} |\operatorname{Spa}(C, C^+)| \text{ and } |F \times_X \operatorname{Spa}(C, C^+)| \twoheadrightarrow |F| \times_{|X|} |\operatorname{Spa}(C, C^+)|$$

Here, the fiber product on the right hand side is taken as topological spaces. Topologically  $|\operatorname{Spa}(C,C^+)|$  is a totally ordered chain of specializations. It admits a unique closed point, which specializes to every other points. Hence, since continuous maps preserves specializations (since the image of the closure is always contained in the closure of the image), the image of  $|\operatorname{Spa}(C,C^+)|$  in |X| is contained in the set of generalizations of x. Assume that  $x \notin |U|$ . Since  $|F| = |X| \setminus |U|$  is a generalizing subset of |X| by hypothesis, the image of  $\overline{x}$  is contained inside |F|, and hence disjoint from |U|, so that the fiber product  $|U| \times_{|X|} |\operatorname{Spa}(C,C^+)|$  is empty. Hence,  $U \times_X \operatorname{Spa}(C,C^+)$  maps surjectively to the empty set, hence is empty, so that  $x'^*$  is zero, and hence  $(Rj_!j^*\mathcal{F})_{\overline{x}} = 0$ , as desired.

The exact same argument holds for  $x \notin |F|$ , since |U| is generalizing (as it is open), so that, for  $x \notin F$ ,  $(j_*j^*\mathcal{F})_{\overline{x}} = \emptyset$ .

Now, assume that  $x \in |U|$ . We'll show that  $Rj'_!j'^*\mathcal{F} \cong \mathcal{F}$ . The topological space  $|U \times_X \tilde{X}|$  identifies with an open subset of  $|\tilde{X}|$  containing the unique closed point  $x \in \tilde{X}$ , so that,  $|U \times_X \tilde{X}| = |\tilde{X}|$ . Since j' is an open immersion, it's an injection of perfectoid spaces, so that the combined [Sch17, Prop 5.3 and Prop 5.4] prove that j' is an isomorphism of diamonds, so that  $Rj'_!j'^*\cong id$ , which concludes.

Finally, if  $x \in |F|$ , which is generalizing, the image of  $|\operatorname{Spa}(C, C^+)|$  is included in |F|. Since  $F = \underline{F} \times_{|X|} X$ , the morphism  $x : \operatorname{Spa}(C, C^+) \to X$  then factors through F, so that there exists a diagonal map making the diagram commute :

$$F \times_X \operatorname{Spa}(C, C^+) \xrightarrow{i'} \operatorname{Spa}(C, C^+)$$

$$\downarrow^{\overline{x'}} \qquad \downarrow^{\overline{x}}$$

$$F \xrightarrow{i'} X$$

By the universal property of fiber products, this diagonal map  $x_F$  induces a splitting of i', which is hence surjective. Finally, a surjective closed immersion of perfectoid spaces is an isomorphism (cf. [Sch17, Lemma 5.3 and 5.4]), so that i' induces an isomorphism of diamonds. Hence  $Ri'_*i'^* \cong id$ , which concludes.

**Remark 3.2.3.** In the above setup, we may rewrite the *fake* base change result [Sch17, Thm 19.2 and Rk 19.3] as a real base change result, without needing to go through pseudo-adic spaces.

Note that this also holds when we consider the "quasicoherent pullbacks", as defined in 2.5.1.

**Corollary 3.2.4.** Let X be an untilted locally spatial diamond with pseudo-uniformizer  $\pi$ , whose underlying topological space decomposes as the union  $|X| = U \sqcup F$  of an open and a closed subset respectively, such that F is generalizing. Let  $j: U \to X$  and  $i: F \to X$  be the associated open (resp. closed) immersion of diamonds.

For any sheaf  $\mathcal{F}$  of  $\mathcal{O}_X^+/\pi$ -modules on  $X_{\acute{e}t}$ , there is an exact sequence in  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^+/\pi)$ :

$$Rj_{!}^{Hub}j_{Hub}^{*,qcoh}\mathcal{F} \to \mathcal{F} \to Ri_{*}^{Hub}i_{Hub}^{*,qcoh}\mathcal{F}$$

This construction is moreover functorial in  $\mathcal{F}$ .

*Proof.* We will simply show that  $j^{qcoh,*} = j^*$  and  $i^{qcoh,*} = i^*$ . Since the inclusion  $\mathcal{D}_{\acute{e}t}(X, \mathcal{O}_X^+/\pi) \to \mathcal{D}_{\acute{e}t}(X, \mathbb{Z})$  preserves exact sequence, this will automatically conclude.

Recall that, for any sheaf  $\mathcal{F}$  of  $\mathcal{O}_X^+/\pi$ -modules, the quasicoherent pullback  $i^{qcoh,*}$  is defined as the tensor product  $i^*\mathcal{F}\otimes_{i^*(\mathcal{O}_X^+/\pi)}\mathcal{O}_F^+/\pi$ . However, since i is a generalizing closed immersion, it follows from [Ans+22, Lemma 2.7] and [Sch17, Corro 10.6] that i is quasi-pro-étale, so that, by 2.1.3,  $i^*\mathcal{O}_X^+/\pi\cong\mathcal{O}_F^+/\pi$ . Likewise, j is an open immersion, hence (quasi-)pro-étale, which concludes.

# 3.3. A CONVENIENT OPEN COVERING

In this section, we construct two intertwined covering of any partially proper analytic adic space : one considering of qcqs open, and the other one consisting of partially proper open subsets. Such a covering will play an important technical role in the following section.

Before anything else, we'll need two quick topological lemmas.

**Lemma 3.3.1.** Let X be a topological space that admits a basis of qcqs open subsets and such that, for all quasi-compact open  $U \subset X$ , the closure  $\overline{U}$  is quasi-compact.

Then, there exists a filtered poset  $(I, \leq)$  and a decomposition of X as an increasing union of quasi-compact open subspaces  $(X_i)_{i \in I}$ , such that :

- (1) For any  $i \in I$ , there is finitely many  $j \in I$  such that  $j \leq i$ .
- (2) For any i < i',  $\overline{X_i} \subset X_{i'}$ .

*Proof.* Since X admits a basis of qcqs open subsets, we may chose a covering  $X = \bigcup_{\lambda \in \Lambda} X_{\lambda}$  by some qcqs open  $X_{\lambda}$ .

For any  $J \subset \Lambda$  finite, let  $X_J = \bigcup_{\lambda \in J} X_\lambda$ , so that the  $(X_J)_{J \subset \Lambda}$  finite form a filtered covering of X by qcqs open subsets. For any finite  $J \subset \Lambda$ , consider the covering  $\overline{X_J} \subset \bigcup_{J \subset J'} X_{J'}$ . Since  $\overline{X_J}$  is quasi-compact, we can extract a finite subcover, and, since the covering is filtered, there exists some minimal element J'' such that  $J \subset J''$  and  $\overline{X_J} \subset X_{J''}$ .

Then, up to recursively removing all intermediary subsets  $J \subsetneq J' \subsetneq J''$ , the covering  $(X_J)_{J \subset \Lambda}$  finite satisfies the desired properties.

**Lemma 3.3.2.** Let X be a topological space which admits a basis of quasi-compact open subsets.

Let F be a quasi-compact closed subset a X, and U an open subset such that  $F \subset U$ . Then there exists a qcqs open V of X such that  $F \subset V \subset U$ .

*Proof.* Since the quasi-compact open form a basis of the topology, every  $x \in F$  admits a quasi-compact neighborhood contained in U, denoted  $U_x$ . Then,  $F \subset \bigcup_{x \in F} U_x$ . Since F is quasi-compact, one may extract a finite sub-covering  $F \subset \bigcup_{i=1}^n U_{x_i}$ . Then  $V = \bigcup_{i=1}^n U_{x_i}$  satisfies the desired property.  $\square$ 

We may now state our result. Note that it can be seen as a more general variant of the covering used in the proof of [ABM24, Prop 6.2.5]. Compare also with the covering constructed in [AGN25, Corro 2.29], or with [Hub96, Lemma 5.3.3].

**Proposition 3.3.3.** Let  $f: X \to Y$  be a partially proper morphism of analytic adic spaces, with taut Y. Then, there exists a filtered poset  $(I, \leq)$ , together with two open coverings of X:

- An increasing open cover  $X = \bigcup_{i \in I} X_i$  by qcqs open subsets
- An increasing open cover  $X = \bigcup_{i \in I}^{\infty} U_i$  by partially proper open subsets

such that:

- (1) For all  $i \in I$ ,  $X_i \subset U_i$  and for any i < j,  $U_i \subset X_j$ .
- (2) For any  $i \in I$ , there is only finitely many i' < i.
- (3) For any i < j,  $\overline{X_i} \subset U_j$ , and the morphism  $X_i \to X_j$  factors through the universal adic compactification  $\overline{X_i}^Y$  of  $X_i$  over Y, as constructed in [Hub96, Thm 5.1.5].

*Proof.* Since f is a partially proper, it is taut, so that, since Y is taut, X is also taut. Moreover, X admits a basis of affinoid open subsets, so that the assumptions of 3.3.1 apply. Let  $(I, \leq)$  and  $(X_i)_{i \in I}$  be a filtered open cover of X by qcqs open subsets, as given by 3.3.1.

For any  $i \in I$ , let  $Y_i = \overline{X_i}$  be the closure of  $X_i$  inside X. Since X is taut,  $Y_i$  is quasi-compact inside X. Let  $Z_i = Y_i \setminus X_i$ , which is a quasi-compact closed subset of  $Y_i$ . The above lemma 3.3.2, applied to  $Z_i \subset (Y_i \setminus \bigcup_{j < i} Y_j)$  inside  $Y_i$ , yields a quasi-compact open subset  $Y_i$  of  $Y_i$  such that  $Z_i \subset Y_i \setminus \bigcup_{j < i} Y_j$ .

By construction,  $X_i \cap V_i = \emptyset$ , and  $X_i$  is open, so that  $X_i \cap \overline{V_i} = \emptyset$ . Let  $U_i = Y_i \setminus \overline{V_i}$ , so that, for all i < j;  $X_i \subset U_i \subset X_j$ . We claim that the map  $U_i \to X$  is partially proper. It suffices to show that  $U_i$  is specializing, hence that  $\overline{V_i}$  is generalizing.  $V_i$  is quasi-compact, hence retrocompact, so, by [FK17, Ch 0, Corro 2.2.27],  $\overline{V_i}$  is simply the closure under specializations of  $V_i$ . Since  $X_i$  is an analytic adic space, all specializations are vertical, and, by [Hub96, Lemma 1.1.10.(i)], the set of generalizations (or specializations) of any given point forms a chain, so that  $\overline{V_i}$  is stable under generalization. Hence the  $U_i$  are generalizing open subsets of X, which are then partially proper.

It suffices to prove the last property. For any i < j, there is a natural inclusion  $X_i \subset U_i \subset X_j$ . Since the composition  $U_i \to Y$  is partially proper, it defines a compactification of  $X_i$  over  $U_i$ , so that, by [Hub96, Thm 5.1.5], the morphism  $X_i \to U_i$  factors through the universal compactification  $\overline{X_i}^Y$ . Hence, it is also the case for  $X_i \subset X_j$ .

In [ABM24, Section 6.2], it is crucial for some arguments that a covering as constructed above can be made to be *countable*. This is clearly possible whenever the covering from 3.3.1 is countable. While this is not needed in our setup, we'll show that this condition is equivalent to a the more standard paracompacity assumption.

Let's say that a rigid-analytic variety is *paracompact* if it admits a locally finite covering by admissible affinoids, as in  $[JP96, Definition 2.5.6]^{12}$ .

**Proposition 3.3.4.** Let X be a connected partially proper rigid-analytic variety over  $Spa(C)^{13}$ . Then, the following are equivalent:

- (1) X is paracompact
- (2) X is countable at infinity in the sense of [ABM24, Defi 6.2.5], i.e. we may write may write  $X = \bigcup_{n \in \mathbb{N}} X_n$  as a countable increasing union of qcqs open subspaces  $X_n$  such that, for each n, the inclusion  $X_n \subset X_{n+1}$  factors through the adic compactification  $Y_n$  of  $X_n$ .

Note that the "connected" hypothesis is purely technical - otherwise it is possible that X admits uncountably many connected components, and one may not expect to find a countable covering.

*Proof.* (i)  $\implies$  (ii) Since X is connected, we may choose a *countable* admissible locally finite affinoid covering in 3.3.1, using [JP96, Lemma 2.5.7]. Then, the construction of 3.3.3 produces such a result.

(ii)  $\Longrightarrow$  (i). Write  $X = \bigcup X_n$  as a strictly increasing countable union of qcqs open subsets, as in the definition of countable at infinity, and let  $Y_n$  be the adic compactification of  $X_n$  over  $\operatorname{Spa}(C)$ . Consider  $Y_n \setminus X_{n-1}$  as a closed subset of  $X_{n+1} \setminus X_{n-2}$ . By the separation lemma 3.3.2, for all  $n \in \mathbb{N}$ , there exists qcqs open subsets  $U_n$  inside  $X_{n+1}$  such that  $Y_n \setminus X_{n-1} \subset U_n \subset X_{n+1} \setminus Y_{n-2}$ . By construction, each  $U_n$  might intersect  $U_{n\pm 1}$ , but not any  $U_k$  for |k-n| > 1.

Each  $U_n$  is a quasi-compact open subset of X, which can hence be written as a finite union of affinoids. Such collection of affinoids is locally finite by construction, so that X is paracompact.

**Remark 3.3.5.** Note that the above applies to period domains, as they are paracompact by [Har13, Lemma A.3].

# 4. Compatibility between Mann and Huber's formalisms

The goal of this section is to identify Mann's  $f_!^{Mann}$  and Huber's  $Rf_!^{Hub}$  (whose construction will be shortly recalled in paragraph 4.1) for partially proper morphisms f. The precise statement is as follows:

**Proposition 4.0.1.** Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a partially proper analytic adic space over a complete algebraically closed extension of  $\mathbb{Q}_p$ , with pseudo-uniformizer  $\pi$ . Let  $\mathcal{L}$  be an overconvergent étale sheaf of  $\mathbb{F}_p$ -modules on X.

 $<sup>^{12}</sup>$ see also [Bos23, section 5.2], for a nice concise overview of the concept

<sup>&</sup>lt;sup>13</sup>The argument would work mutatis mutandis for analytic adic spaces over a more general base, but we stick to the generality used in the litterature

Then, there is a natural isomorphism in  $\mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathcal{O}^{+,a}/\pi)^{oc}$ :

$$f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/\pi) \cong J^{-1}_{\operatorname{Spa}(C,\mathcal{O}_C)} \left( R f_!^{Hub}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/\pi) \right)^a$$

where J denotes the identification functor from 2.5.6.

Here is a sketch of the proof.

**Foreshadowing 4.0.2.** Fix an open covering of X by partially proper open subsets  $(U_i)_{i \in I}$ , as constructed in 3.3.3. We'll proceed in four steps, as follows:

(1) Prove that, for any standard étale sheaf  $\mathcal{F}$  of  $\mathcal{O}_X^+/\pi$ -modules on X, there is an exact sequence in  $\mathcal{D}_{\acute{e}t}(Y,\mathcal{O}^+/\pi)$ :

$$Rf_{!}^{Hub}\mathcal{F} o Rf_{*}^{Hub}\mathcal{F} o \varinjlim_{i \in I} Rf_{*}^{Hub} \circ i_{*,Hub}^{X \setminus U_{i}} \circ i_{X \setminus U_{i}}^{*,Hub}\mathcal{F}$$

Here,  $i^{X\setminus U_i}$  denotes the immersion of the generalizing closed subset  $X\setminus U_i\to X$ , viewed as a locally spatial diamond.

- (2) Prove that  $f_1^{Mann}$  preserves discrete objects.
- (3) Establish, for any  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)_{\omega}$ , the following exact sequence in  $\mathcal{D}^a_{\square}(\operatorname{Spa}(C, \mathcal{O}_C), \mathcal{O}^+/\pi)$ :

$$f_{!}^{Mann}\mathcal{F} \to \left(f_{*}^{Mann}(\mathcal{F})\right)_{\omega} \to \varinjlim_{i \in I} \left(f_{*}^{Mann} \circ i_{*,Mann}^{X \setminus U_{i}} \circ i_{X \setminus U_{i}}^{*,Mann} \mathcal{F}\right)_{\omega}$$

Where the notations are the same as in the first point.

(4) Identify the second and third terms of the two exact sequences above, and deduce an isomorphism between the first terms.

We'll do so in order. The first point is an essentially standard result (cf. [AGN25, Corro 2.17]), but is written in loc.cit. for rigid analytic varieties, where the colimit is taken along a covering by quasi-compact open subsets, and the complement  $X \setminus U$  taken in the rigid-analytic sense, as in 3.1.1. We'll recover this result in 4.1.5.

**Remark 4.0.3.** Since we expect some technical lemmas of this section to be useful in other setups, we try and work at the highest level of generality, oftentimes higher than what is actually needed in that paper.

4.1. Proper pushforward for almost quasi-coherent classical étale sheaves

First, let us recall Huber's definition of direct image with compact support  $Rf_{Hub,!}$  [Hub96, Defi 5.2.1], when the morphism f is partially proper.

For any separated morphism of locally noetherian adic space  $f: X \to Y$ , we may define a left exact functor  $f_!^{Hub}: X_{\acute{e}t}^{\sim} \to Y_{\acute{e}t}^{\sim}$ , defined, for  $U \in Y_{\acute{e}t}$ , by :

$$f_{!}^{Hub}\mathcal{F}(U) := \{ s \in \Gamma(U \times_{Y} X, \mathcal{F}), supp(s) \text{ is proper over } U \}$$

Whenever f is étale,  $f_!^{Hub}$  is exact, and, when f is proper,  $f_!^{Hub} = f_*$ . For a general morphism f, we should not expect the derived functors of  $f_!^{Hub}$  to have good properties - rather, for morphisms f that can be written as a composition  $f = p \circ i$  of a proper morphism p and an open immersion i, we should define the derived pushforward  $Rf_!^{Hub}$  as  $Rp_*^{Hub} \circ Ri_!^{Hub}$ .

However, as we will see, for some partially proper morphism f of analytic adic spaces, it is reasonable to consider the derived functors of  $f_!^{Hub}$ , as they admit reasonable properties. In [Hub96], Huber considers a right derived functor  $R^+f_!^{Hub}:D^+(X_{\acute{e}t},\mathbb{Z})\to D^+(Y_{\acute{e}t},\mathbb{Z})$  at the level of the bounded below derived 1-categories, and additionally constructs a "fully derived functor"  $Rf_!^{Hub}:D(X_{\acute{e}t},\mathbb{Z})\to D(Y_{\acute{e}t},\mathbb{Z})$  under some finite dimensionality assumption on f. Using more modern categorical methods (cf. [Sta23, Lemma 070K] and the discussion at the beginning of [Sta23, Section 07A5]), since the categories of modules on a site forms a Grothendieck abelian categories (cf. [Sta23, Section 07A5]). Hence, we may define the

right derived functor  $Rf_{!,Hub}: D(X_{\acute{e}t},\mathbb{F}_p) \to D(Y_{\acute{e}t},\mathbb{F}_p)$  for any partially proper morphism f, as the (unbounded) derived functors of  $f_!$ .

This construction naturally admits an infinity categorical enhancement (cf. [Lur17, Section 1.3.5]), i.e. there exists a canonical morphism of infinity categories  $Rf_{!,Huber}: \mathcal{D}(X_{\acute{e}t},\mathbb{F}_p) \to \mathcal{D}(Y_{\acute{e}t},\mathbb{F}_p)$ , that induces the morphism above at the level of homotopy categories, and it passes to left completions to a morphism  $Rf_{!,Huber}: \mathcal{D}_{\acute{e}t}(X,\mathbb{F}_p) \to \mathcal{D}_{\acute{e}t}(Y,\mathbb{F}_p)$ 

Likewise, we can check that, for some sheaf any partially proper morphism  $f: X \to Y$  and any sheaf  $\mathcal{F}$  of  $\mathcal{O}_X/\pi$ -modules on X,  $f_!^{Hub}\mathcal{F}$  admits a natural structure of an  $\mathcal{O}_Y/\pi$ -module, viewing every  $f_!^{Hub}\mathcal{F}(U)$  as a submodule of  $\Gamma(U \times_Y X)$ .

Hence, the derived functor  $Rf_!^{Hub}$  induces a functor  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^+/p) \to \mathcal{D}_{\acute{e}t}(Y,\mathcal{O}_V^+/p)$ .

**Remark 4.1.1.** Note that this approach is not the one of [Sch17]. In loc. cit. the functor  $Rj_!^{Scholze}$  is defined as the left adjoint of  $Rj_{Hub}^*$  for open immersions, and hence coincides with the one of Huber. For a partially proper morphism  $f: X \to Y$ , Scholze defines [Sch17, Defi 22.13]):

$$Rf_{!}^{Scholze} = \varinjlim_{U \subset X} Rf_{*}^{Hub} \circ Rj_{U\,!}^{Hub} \circ Rj_{U}^{*,Hub}$$

where the limit is taken alongside qcqs open subsets of X, and  $j_U$  denotes the open immersion  $U \subset X$ .

The next proposition will show that the result above holds for  $Rf_{!,Huber}$  as define here. This can be seen as a compatibility statement between Scholze's formalism for diamonds and Huber's formalisms for analytic adic spaces, at least for coefficients concentrated in degree zero.

**Lemma 4.1.2.** Let  $f: X \to Y$  be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y. Let  $(U_i)_{i \in I}$  be an increasing covering of X by partially proper open subsets as in 3.3.3.

Then, for any sheaf  $\mathcal{F}$  of abelian groups over  $X_{\acute{e}t}$ , there is a natural isomorphism in  $\mathcal{D}_{\acute{e}t}(X,\mathbb{Z})$ :

$$\varinjlim_{i \in I} Rf_*^{Hub} \circ j_{U_i!}^{Hub} \circ j_{U_i}^{*,Hub} \mathcal{F} \cong Rf_!^{Hub} \mathcal{F}$$

For technical reasons, it's easier to compute the colimit alongside partially proper open subsets rather than qcqs opens, but the colimit is the same, as one can take intertwining cofinal coverings as in 3.3.3.

*Proof.* For any  $i \in I$ , there is a natural morphism  $Rj_{U_i}^{Hub}Rj_{U_i}^{*,Hub}\mathcal{F} \to \mathcal{F}$  induced by the unit of the adjunction. Everything lives inside the bounded below subcategory  $\mathcal{D}_{\acute{e}t}^+(X,\mathbb{F}_p) \cong \mathcal{D}^+(X_{\acute{e}t},\mathbb{F}_p)$ , so that we may forget about the left completions. In order to check that it is an isomorphism in the derived category, it suffices to show that it is a quasi-isomorphism, i.e. to check that it is an isomorphism on (co)homology. By the proof of [Lur17, Prop 1.3.5.21], since we're working over the derived category of a Grothendieck abelian category, taking (co)homology commutes with filtered colimits, so that:

$$\pi_{-n}\left(\varinjlim_{i\in I} Rf_*^{Hub}\circ j_{U_i!}^{Hub}\circ j_{U_i}^{*,Hub}\mathcal{F}\right)\cong \varinjlim_{i\in I} R^n f_!^{Hub}\circ j_{U_i!}^{Hub}\circ j_{U_i}^{*,Hub}\mathcal{F}$$

We easily check that the covering  $(U_i)_{i\in I}$  satisfies the properties [Hub96, Lemma 5.3.3.(ii)], so that the result then follows from [Hub96, Lemma 5.3.3.(iii)].

Let us now establish the announced exact sequence.

**Proposition 4.1.3.** Let  $f: X \to Y$  be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y. Let  $(U_i)_{i \in I}$  be an increasing covering of X by partially proper open subsets as in 3.3.3.

For any  $i \in I$ , we let  $Z_i$  be the complement of the partially proper  $U_i$ , viewed as a closed locally spatial subdiamond of  $X^{\diamond}$  by 3.1.3. Let  $i_{Z_i}: Z_i \to X^{\diamond}$  be the associated closed immersion. Then, for any sheaf  $\mathcal{F}$ 

 $<sup>^{14}</sup>$ Note that the definition is only made with additional hypothesis on f, as well as an hypothesis on the ring of coefficients, but those do not impact the definition

of abelian groups over  $X_{\acute{e}t}$ , there is a natural exact sequence in  $\mathcal{D}_{\acute{e}t}(Y,\mathbb{Z})$ :

$$Rf_{!}^{Hub}\mathcal{F} \to Rf_{*}^{Hub}\mathcal{F} \to \varinjlim_{i \in I} Rf_{*}^{Hub} \circ i_{Z_{i}}^{Hub} \circ i_{Z_{i}}^{*,Hub}\mathcal{F}$$

Note that we implicitly identify the étale site of X with the étale site of  $X^{\diamond}$ , so that the composition  $Rf_*^{Hub} \circ i_*^{Z_i}$  makes sense, even though  $Z_i$  only admits the structure of a locally spatial diamond.

*Proof.* To lighten the notations, we drop the 'Hub' superscripts from the proof.

By excision in that setup 3.2.1, there is a short exact sequence of sheaves of abelian groups on  $Y_{\acute{e}t}$ :

$$0 \to j_{U_i}! j_{U_i}^* \mathcal{F} \to \mathcal{F} \to i_{Z_i} * i_{Z_i}^* \mathcal{F} \to 0$$

This induces an exact sequence in  $\mathcal{D}_{\acute{e}t}(X,\mathbb{Z})$ . Applying the exact functor  $Rf_*$  yields the exact sequence:

$$Rf_* \circ j_{U_i}! \circ j_{U_i}^* \mathcal{F} \to Rf_* \mathcal{F} \to Rf_* \circ i_{Z_i *} \circ i_{Z_i}^* \mathcal{F}$$

One can form the colimit over I. Since the morphisms from passing from i to some j > i are induced by (co)units of the adjunctions, the diagram is homotopy coherent, and, since colimits of exact sequences remain exact (as they can be viewed as colimit diagrams), we have :

$$\varinjlim_{i \in I} Rf_* \circ Rj_{U_i}! \circ j_{U_i}^* \mathcal{F} \to \varinjlim_{i \in I} Rf_* \mathcal{F} \to \varinjlim_{i \in I} Rf_* \circ i_{Z_i} * \circ i_{Z_i}^* \mathcal{F}$$

By the previous lemma, the first term identifies to  $Rf_1\mathcal{F}$ , which concludes.

Now, we'll like a version for sheaves of  $\mathcal{O}^+/\pi$ -modules.

Corollary 4.1.4. In the same setup as 4.1.3, let  $\mathcal{F}$  be an étale sheaf of  $\mathcal{O}_X^+/\pi$ -modules on X, where  $\pi$  is a pseudo-uniformizer of a field of definition of Y. Then, there is an exact sequence :

$$Rf_{!}^{Hub}\mathcal{F} \to Rf_{*}^{Hub}\mathcal{F} \to \varinjlim_{i \in I} Rf_{*}^{Hub} \circ i_{*,Hub}^{Z_{i}} \circ i_{Z_{i},Hub}^{*,qcoh}\mathcal{F}$$

*Proof.* By 2.1.3, the quasi-coherent pullback  $i_{Z_i,Hub}^{*,qcoh}$  simply coincides with the pullback  $i_{Z_i,Hub}^{*}$  as étale sheaves, so that this directly follows from the above result, since the natural forgetful functor  $\mathcal{D}_{\acute{e}t}(X,\mathcal{O}_X^+/\pi) \to \mathcal{D}_{\acute{e}t}(X,\mathbb{Z})$  respects exact sequences.

We conclude this paragraph by noting that we can deduce an alternative writing of 4.1.3, that remains in the realm of analytic adic spaces, in the spirit of [AGN25, Corro 2.17]. Following 3.1.2, we'd like to replace the  $Z_i$  by the interior of the complement of some qcqs open subspaces. This is what is done in the following.

**Corollary 4.1.5.** Let  $f: X \to Y$  be a partially proper morphism of locally noetherian analytic adic spaces, with taut Y. Let  $(X_i)_{i \in I}$  be an increasing covering of X by quasicompact open subsets.

For any  $i \in I$ , let  $Y_i = (X \setminus X_i)^{\circ} = X \setminus \overline{X_i}$ , which admits an analytic structure as an open subset of X. Let  $j_{Y_i}: Y_i \to X$  denote the associated open immersion. Then, for any étale sheaf  $\mathcal{F}$  of abelian groups on  $X_{\acute{e}t}$ , there is an exact sequence:

$$Rf_{!}^{Hub}\mathcal{F} \to Rf_{*}^{Hub}\mathcal{F} \to \varinjlim_{i \in I} Rf_{*}^{Hub} \circ j_{Y_{i} *}^{Hub} \circ j_{Y_{i}}^{*,Hub}\mathcal{F}$$

*Proof.* By the proof of 3.3.3, we may construct partially proper open subsets  $(U_i)_{i \in I}$  such that  $X_i \subset U_i \subset X_j$  for all i < j, satisfying the properties of 3.3.3.

We reuse the notations from the previous proposition. Hence, it suffices to check that there is a canonical isomorphism  $\lim_{i \in I} Rf_* \circ i_{F_i *} \circ i_{F_i *}^* \mathcal{F} \to \lim_{i \in I} Rf_* \circ j_{Z_i *} \circ j_{Z_i *}^* \mathcal{F}$ .

From the properties of the covering, for any i < j in I, we have  $Y_j \subset Z_j \subset Y_i$ , so that both families of subsets are cofinal in one another. Since the category  $\mathcal{D}(X_{\acute{e}t},\mathbb{Z})$  admits all colimits, the colimit alongside alternating Z and Y's exists, and can can be computed alongside one family or the other, so that both two colimits coincide.

# 4.2. Mann's proper pushforward preserves discrete objects

To avoid overly busy notations, in this paragraph, all functors are implicitly in the sense of Mann Let us proceed with the established program. This subsection is dedicated to the proof of the following:

**Proposition 4.2.1.** Let  $f: X \to Y$  be a partially proper morphism of analytic adic spaces over a complete extension K of  $\mathbb{Q}_p$ , with pseudo-uniformizer  $\pi$ .

Then, for any  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/\pi)_{\omega}$ ,  $f_!^{Mann}\mathcal{F}$  is discrete inside  $\mathcal{D}^a_{\square}(Y, \mathcal{O}^+_Y/\pi)_{\omega}$ .

We postpone the proof at the end of this paragraph. We'll reduce the the case where f factors as the composition of a partially proper open immersion and a proper morphism. Note that, the proper pushforward  $p_!$  by a proper morphism preserves discrete objects by [Man22, Lemma 3.3.10.(ii)], so that it will then suffice to deal with open immersions.

In order to reduce to open immersions, we'll need the following lemma:

**Lemma 4.2.2.** Let  $(I, \leq)$  be a filtered partial order, and  $(U_i)_{i \in I}$  be an increasing open covering of an untilted locally spatial diamond X, with pseudo-uniformizer  $\pi$ . For any  $i \in I$ , let  $g_i : U_i \to X$  denote the open immersion.

Then, there is a natural isomorphism  $\mathcal{F} \cong \varinjlim_{i \in I} g_i! g_i^* \mathcal{F}$  for any  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)$ .

Note that the argument works more generally for untilted v-stacks.

*Proof.* For any  $j \in I$ ,  $g_j^* = g_j^!$  admits the left adjoint  $g_{j!}$ . The counit of the adjunctions defines a morphism  $\varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F} \to \mathcal{F}$ .

By (hyper)v-descent [Man22, Thm 1.2.1], there is an isomorphism:

$$\mathcal{D}^a_{\square}(X,\mathcal{O}^+/p) \cong \varprojlim_{n \in \Delta} \mathcal{D}^a_{\square}(Y_n,\mathcal{O}^+/p), \text{ where } Y = \bigsqcup_{i \in I} U_i \text{ and } Y_n = Y^{\times n} = \bigsqcup_{i_1,\dots,i_n \in I} U_{i_1} \times_X \dots \times_X U_{i_k}$$

Hence, it suffices to check the isomorphisms after pulling back to all such  $U_{i_1} \times_X \cdots \times_X U_{i_n}$ , for fixed  $n \in \mathbb{N}$  and  $(i_1, \ldots, i_n) \in I$ . Since all the  $U_i$  are open subsets of X, the fiber product is simply the intersection, and, since I is filtered, every such morphism  $U_{i_1} \times_X \cdots \times_X U_{i_n} \to X$  factors through  $U_j$  for some large enough  $j \in I$ ; so that if suffices to prove that, for all  $j \in J$ , the map:

$$g_j^* \left( \varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F} \right) \to g_j^* \mathcal{F}$$

is an isomorphism. Since  $g_j^*$  admits both a left and a right adjoint, it commutes with colimits, so that it suffices to prove that :

$$\varinjlim_{i\in I} g_j^* g_{i!} g_i^* \mathcal{F} \cong g_j^* \mathcal{F}$$

Since the covering is increasing, the fiber product  $U_i \times_X U_j$  is simply  $U_{\min(i,j)}$ . Since I is filtered, the subset  $J = \{i \in I, i \geq j\}$  is cofinal inside I, so that we may compute the limit alongside J. For any  $i \in J$ , consider the following cartesian diagram:

$$\begin{array}{ccc}
U_j & \longrightarrow U_i \\
\downarrow & & \downarrow \\
U_j & \longrightarrow X
\end{array}$$

By proper base change (cf. [Man22, Thm 1.2.4]), we see that, for i > j,  $g_j^* g_{i!} g_i^* \mathcal{F} = g_j^* \mathcal{F}$ . Hence, the colimit along J is trivial, and this concludes.

Remark 4.2.3. A similar argument shows that and  $\mathcal{F} \cong \varprojlim_{i \in I} g_{i*}^{Mann} g_{i}^{*,Mann} \mathcal{F}$  for  $\mathcal{F} \in \mathcal{D}_{\square}^{a}(X, \mathcal{O}_{X}^{+}/\pi)$ . Using a technical lemma on  $(\infty, 2)$ -categories [HM24, p. D.4.7], we may improve the adjunction triangle  $g_{i!}^{Mann} \dashv g_{i*}^{*,Mann} \dashv g_{i*}^{Mann} \dashv g_{i*}^{Mann}$  to the expected adjunctions between the following three functors:

- The map  $(\mathcal{F}_i)_{i \in I} \in \varprojlim \mathcal{D}^a_{\square}(U_i, \mathcal{O}^+_{U_i}/p) \mapsto \varinjlim_{i \in I} g_{i!} \mathcal{F}_i \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+/p)$
- A right adjoint  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+/p) \mapsto (\mathcal{F} \mapsto (g_i^*\mathcal{F}))_{i \in I} \in \varprojlim_{i \in I} \mathcal{D}^a_{\square}(U_i, \mathcal{O}_U^+/p)$
- A right adjoint of the above map  $(\mathcal{F}_i)_{i \in I} \in \varprojlim_{i \in I} \mathcal{D}^a_{\square}(U_i, \mathcal{O}^+_{U_i}/p) \mapsto \varprojlim_{i \in I} g_{i*} \mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+/p)$

Here, the limits  $\varprojlim_{i\in I} \mathcal{D}^a_{\square}(U_i, \mathcal{O}^+_{U_i}/p)$  is taken along pullbacks. The above proposition shows that the counit of the first adjunction and the unit of the second one are isomorphisms, both these statements being equivalent, by formal nonsense ([Lur25, Corollary 02FF]), to the full faithfulness of the middle arrow.

Let us now prove that partially proper open immersions preserve discrete objects. We start by doing so for open immersions of totally disconnected perfectoid spaces.

Note that the proof resembles [Zav23b, Lemma 6.2.1 and 6.2.2], which moreover assumes U to be a Zariski open subset.

**Lemma 4.2.4.** Let X be a totally disconnected perfectoid space and  $j: U \to X$  be a partially proper open immersion. Then  $j_!$  preserves discrete objects.

*Proof.* The map j is partially proper, hence specializing (by the valuative criterion), in the sense that the image j(X) is stable under specialization. By abuse of notation, we let U = |j(U)|.

Consider the natural quotient map  $\pi: |X| \to \pi_0(X)$ . Since X is a totally disconnected perfectoid, by definition, every connected component of |X| admits a unique closed point, to which any other point in the connected component specializes. Since U open and stable under specializations, U contains the connected component of every  $x \in U$ . Hence,  $U \cong \pi^{-1}(\pi(U))$ . Since  $\pi$  is a quotient map, it follows that  $\pi(U)$  is open in  $\pi(X)$ .

By [Sta23, Lemma 0906],  $\pi_0(X)$  is profinite, so that it admits a basis of clopen subsets. Thus,  $\pi(U)$  can be written as a filtered union of quasi-compact clopen sets; so that the same holds for  $U = \pi^{-1}(\pi(U))$ . Hence, there exists an filtered poset I such that  $U = \bigcup_{i \in I} U_i$  is a filtered union of clopen subsets of |X|. For all  $i \in I$ , let  $U'_i = j^{-1}(U_i) \subset U$ , which admits an analytic structure as an open subset of U.

Consider the map  $j_i: U_i' \to X$ , which is the inclusion of a clopen subset of X. Let  $\mathcal{F} \in \mathcal{D}^a_{\square}(U, \mathcal{O}^+_U/\pi)_{\omega}$ .

By the lemma 4.2.2,  $\mathcal{F} \cong \varinjlim_{i} j_{i!} j_{i}^* \mathcal{F}$ , so that  $j_! \mathcal{F} \cong j_! \varinjlim_{i} j_{i!} j_{i}^* \mathcal{F} \cong \varinjlim_{i} (j \circ j_i)_! j_{i}^* \mathcal{F}$ , since  $j_!$  commutes with colimits (as it admits a right adjoint).

By definition, pullback preserve discrete objects, and  $j \circ j_i$  is the closed immersion of an open subset of Y, so that  $(j \circ j_i)_! = (j \circ j_i)_*$ , which, by [Man22, 3.3.10.(ii)], preserves discrete objects. Finally, by [Man22, lemma 3.2.19.(ii)], (small) colimits of discrete objects remain discrete, so that  $j_!\mathcal{F}$  is discrete.

Let us now establish the result for general partially proper open immersions.

**Lemma 4.2.5.** Let  $j: U \to X$  be a partially proper open immersion of untilted small v-sheaves, with pseudouniformizer  $\pi$ .

Then j! preserves discrete objects.

*Proof.* Fix a discrete object  $\mathcal{F} \in \mathcal{D}^a_{\square}(U, \mathcal{O}^+_U/\pi)$ . We'll reduce to the situation where X is a totally disconnected perfectoid space.

As discreteness is defined by descent (cf [Man22, Def 3.2.17]), we need to show that, for every morphism  $f: \tilde{X} \to X$  from a totally disconnected perfectoid space  $\tilde{X}$ , the sheaf  $f^*j_!\mathcal{F}$  is discrete in  $\mathcal{D}^a_{\square}(\tilde{X}, \mathcal{O}^+_{\tilde{X}}/\pi)$ . Consider the cartesian square below:

$$\tilde{U} \xrightarrow{j'} \tilde{X} \\
f' \downarrow \qquad \qquad \downarrow f \\
U \xrightarrow{j} X$$

<sup>&</sup>lt;sup>15</sup>Note that, by unraveling the proof of [Man22, Lemma 3.6.2], this is actually the definition of  $j_!$  for arbitrary open immersions

By proper base change [Man22, Thm 1.2.4.(iv)],  $f^*j_!\mathcal{F} = j_!'f'^*\mathcal{F}$ .

By definition [Sch17, Defi 10.7], j' is a separated open immersion. Note that the valuative criteria from [Sch17, Defi 18.4] is automatically stable under base change, so that j' remains partially proper, and, by the above lemma 4.2.4,  $j'_1$  preserves discrete objects.

Finally, pullbacks preserve discrete objects by definition, so that  $j'_!f'^*\mathcal{F}$  is discrete, which concludes.  $\square$ 

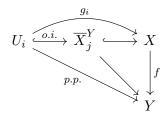
We may now prove the main result 4.2.1.

*Proof.* (of 4.2.1) By 3.3.3, one can write  $X = \bigcup_{i \in I} X_i$  as an increasing filtered cover by qcqs open subsets, such that, for any i < j, the inclusion  $X_i \subset X_j$  factors through a partially proper  $U_i$ , and such that the map  $X_i \to X_j$  factors through Huber's universal compactification  $X_i^Y$  of  $X_i$  over Y. Denote  $f_i : X_i \to X$  and  $g_i : U_i \to X$  the associated open immersions.

By lemma 4.2.2, there is an isomorphism  $\mathcal{F} \cong \varinjlim_{i \in I} g_{i!} g_{i}^{*} \mathcal{F}$ ,

Hence,  $f_! \mathcal{F} = f_! \circ \varinjlim_{i \in I} g_{i!} g_i^* \mathcal{F} = \varinjlim_{i \in I} (f \circ g_i)_! (f \circ g_i)^* \mathcal{F}$ , since left adjoints preserve colimits. By [Man22], Lemma 3.2.19.(ii), a (small) colimit of discrete objects remains discrete, so that, as pullbacks preserve discrete objects, it suffices to show that every  $(f \circ g_i)_!$  preserves discrete objects.

Here,  $g_i: U_i \to X$  is the inclusion of a partially proper open subset inside X. Moreover, the partially proper map  $f \circ g_i$  factors through  $X_i$ , then, for any i < j, through the adic compactification  $\overline{X}_j^Y$  of  $X_j$  over Y. We then have the following diagram:



By [Hub96, Corro 5.1.6], the map  $\overline{X}_j^Y \to Y$  is proper, so that, by the two-out-of three property for partially proper morphisms [Hub96, Lemma 1.10.17.(vi)], the open immersion  $U_i \to \overline{X}_j^Y$  is additionally partially proper.

Hence, the composition  $g_i \circ f$  factors as the composition of a partially proper open immersion and a proper morphism, and both  $g_i$  and f preserve discrete objects by the lemma 4.2.5 and [Man22, Lemma 3.3.10.(ii)] respectively. This concludes.

We completed the second step of our program. In order to establish the second one, we'll need some excision result, similarly to the one obtained in 3.2.1 for standard étale cohomology.

4.3. Excision in 
$$\mathcal{D}^a_{\square}(\_,\mathcal{O}^+/p)$$

We will use the following result, which is the direct analogue, in Mann's formalism, of the excision result proven in 3.2.1.

**Proposition 4.3.1.** Let X be an untilted locally spatial diamond, with pseudo-uniformizer  $\pi$ , together with a partially proper open immersion  $j: U \to X$ . By 3.1.3, the complement  $F = |X| \setminus j(|U|)$  admits a structure of a locally spatial diamond, and we let  $i: F \to X$  be the corresponding closed immersion.

Let  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+/\pi)$ . Then, there is a natural exact sequence in  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+/p)$ :

$$j_!^{Mann}j_{Mann}^*\mathcal{F}\to\mathcal{F}\to i_*^{Mann}i_{Mann}^*\mathcal{F}$$

Note that this is the analogue to [Zav23b, Lemma 6.2.2], which only considers Zariski open subsets.

*Proof.* There are natural morphisms induced by adjunction, so that it suffices to prove the exactness of the sequence. It suffices to check on a basis, i.e. after pullback to any totally disconnected perfectoid

space. By proper base change, we may assume that X is a strictly totally disconnected affinoid perfectoid space.

Since  $\mathcal{O}^{+a}/\pi$  is the unit of the tensor product, we may write  $\mathcal{F} = \mathcal{F} \otimes \mathcal{O}^{+a}/\pi$  and, using the projection formula, the sequence reduces to:

$$j_!^{Mann}j_{Mann}^*\mathcal{O}_X^+/\pi\otimes\mathcal{O}_X^+/\pi\otimes\mathcal{F}\to\mathcal{F}\to j_!^{Mann}j_{Mann}^*\mathcal{O}_X^+/\pi\otimes\mathcal{F}$$

Since tensoring with  $\mathcal{F}$  is exact, we reduced to the case  $\mathcal{F} = \mathcal{O}^+/\pi$ , which is discrete. By the above proposition 4.2.1 and [Man22, Lemma 3.3.10], all objects appearing in the triangle are discrete. Recall that there is a functorial equivalence of category  $\mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)_{\omega} \cong \mathcal{D}_{\acute{e}t}(X_{\acute{e}t}, \mathcal{O}^{+,a}/p)^{oc}$ , as recalled in 2.5.6, so that we can check that it is exact as a sequence of almost étale sheaves, which are simply étale sheaves, valued in some abelian category.

Since the étale site of X admits enough points by [Sch17, Prop 14.3], it suffices to check the isomorphism after pullback to geometric points of the form  $\operatorname{Spa}(C, C^+)$ , for some complete algebraically closed C, with open bounded valuation subring  $C^+$ . Here, the pullback is to be taken in the sense of the morphism of ringed topos, which coincide with  $f_{Mann}^*$  by 2.5.6. Let  $f: \operatorname{Spa}(C, C^+) \to X$  be such a morphism.

By proper base change applied to the same diagram as in the proof of 3.2.1, we reduce to the case where X is a totally disconnected perfectoid space, and U, F are a partially proper open and a closed subset of X, respectively. By the same argument, U is either empty or U = X, so that, in both cases, the exactness is trivial.

Let us now construct the announced exact sequence.

**Lemma 4.3.2.** Let  $f: X \to \operatorname{Spa}(C)$  be a partially proper morphism of analytic adic spaces, and  $\pi$  a pseudo-uniformizer of C. Let  $(U_i)_{i \in I}$  be an open covering of X by partially proper open subsets, as in 3.3.3.

For any  $i \in I$ , we let  $Z_i$  be the complement of the partially proper  $U_i$ , viewed as a closed locally spatial subdiamond of  $X^{\diamond}$  by 3.1.3. Let  $i_{Z_i}: Z_i \to X^{\diamond}$  be the associated closed immersion. Then, for  $\mathcal{F} \in \mathcal{D}^a_{\square}(X, \mathcal{O}^+_X/p)$ , there is a natural exact sequence:

$$f_!^{Mann}\mathcal{F} \to f_*^{Mann}\mathcal{F} \to \varinjlim_{i \in I} f_*^{Mann} \, i_{Z_i,*}^{Mann} \, i_{Z_i}^{*,Mann}\mathcal{F}$$

For improved readability of the following proof, everything is implicitely taken in the sense of Mann.

*Proof.* Let  $i_{U_i}: U_i \to X$  denote the associated open immersions. Starting from the excision triangle from the above proposition 4.3.1, we then apply  $f_!^{Mann}$  and take colimits, as in the proof of the proof of 4.1.3 applies. This yields the exact sequence (where everything is taken in the sense of Mann):

$$\varinjlim_{i\in I} f_*\circ j_{U_i\,!}\circ j_{U_i}^*\mathcal{F}\to f_*\mathcal{F}\to \varinjlim_{i\in I} f_*\circ i_{Z_i\,*}\circ i_{Z_i}^*\mathcal{F}$$

Let  $(X_i)_{i\in I}$  be an open covering of X by quasi-compact opens subsets, such that  $((U_i),(X_i))$  satisfy the properties of 3.3.3.

We'll now establish that  $f_* \circ j_{U_i!} \cong f_! \circ j_{U_i!}$ . Consider the following diagram, for any  $i < j \in I$ :

$$U_{i} \xrightarrow{j_{U_{i}}} X \xrightarrow{f} Y$$

$$X_{j} \xrightarrow{\pi_{j}} Y$$

By [Hub96, Corro 5.1.6], since  $X_i$  is quasi-compact,  $p_i$  is proper, so that

$$f_! \circ j_{U_i!} = p_{j!} \circ k_{i,j!} = p_{j*} \circ k_{i,j!} = f_* \circ \pi_{j!} \circ k_{i,j!} = f_* \circ j_{U_i!}$$

Hence, using 4.2.2 and the fact that  $f_!$  commutes with colimits (as it admits a right adjoint), we have

$$\varinjlim_{i\in I} f_*\circ j_{U_i}!\circ j_{U_i}^*\mathcal{F}\cong \varinjlim_{i\in I} f_!\circ j_{U_i}!\circ j_{U_i}^*\mathcal{F}\circ f_!\circ \varinjlim_{i\in I} j_{U_i}!\circ j_{U_i}^*\mathcal{F}\circ f_!\mathcal{F}$$

So that the first term of the above exact sequence identifies with  $f_!\mathcal{F}$ , which concludes.

Let us now apply the discretization functor, to get the sequence announced in 4.0.2.

Corollary 4.3.3. In the situation above, there is a natural exact sequence:

$$f_!^{Mann}\mathcal{F} \to \left(f_*^{Mann}\mathcal{F}\right)_\omega \to \varinjlim_{i \in I} \left(f_*^{Mann}i_{Z_i,*}^{Mann}i_{Z_i}^{*,Mann}\mathcal{F}\right)_\omega$$

*Proof.* Apply the exact functor  $(-)_{\omega}$  from 2.5.5 to previous result, which commutes with colimits (as it is a right adjoint). The object  $f_!^{Mann}\mathcal{F}$  is discrete by 4.2.1, which concludes.

**Remark 4.3.4.** As in 4.1.5, we may rewrite the right hand side as a colimit alongside any complement of a reasonable covering, using a cofinality argument.

## 4.4. Identification of Mann and Huber's proper pushforwards

Let us finally prove 4.0.1. Recall the statement:

**Proposition.** Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a partially proper analytic adic spaces over a complete algebraically closed extension of  $\mathbb{Q}_p$ . Let  $\mathcal{L}$  be an overconvergent sheaf of  $\mathbb{F}_p$ -vector spaces on X.

Then, there is a natural isomorphism

$$f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \cong \left(Rf_!^{Hub}(\mathcal{L} \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p)\right)^a$$

in  $\mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathcal{O}_C^{+,a}/p)$ , where we implicitly use the identification from 2.5.6.

*Proof.* Let  $X_i$  be a covering of X by partially proper open subsets, as in 3.3.3. For any i, let  $Z_i$  be the complement of  $X_i$  in X, viewed as a locally spatial diamond. By 4.3.3, there is an exact sequence :

$$f_!^{Mann}(\mathcal{F}\otimes\mathcal{O}_X^{+,a}/p)\to f_*^{Mann}(\mathcal{F}\otimes\mathcal{O}_X^{+,a}/p)\to \varinjlim_{i\in I}f_*^{Mann}\circ i_{Z_i*}^{Mann}\circ i_{Z_i}^{*,Mann}(\mathcal{F}\otimes\mathcal{O}_X^{+,a}/p)$$

Applying the functor  $(-)^a: \mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathcal{O}_C^+/\pi) \to \mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathcal{O}_C^{+,a}/\pi)$  to the exact sequence constructed in 4.1.4, we get the following exact sequence:

$$\left(Rf_!^{Hub}(\mathcal{F}\otimes\mathcal{O}_X^+/p)\right)^a\to Rf_*^{Hub}(\mathcal{F}\otimes\mathcal{O}_X^+/p)^a\to \varinjlim_{i\in I}Rf_*^{Hub}\circ i_{Z_i}^{Hub}\circ i_{Z_i}^{*,Hub}(\mathcal{F}\otimes\mathcal{O}_X^+/p)^a$$

By 2.5.11, there is a natural isomorphism between the second terms of both sequences.

By 2.5.6 and 2.5.11, there are compatible isomorphism  $i_{Z_i}^{*,Mann}(\mathcal{F}\otimes\mathcal{O}_X^{+,a}/\pi) \stackrel{\cdot}{\cong} i_{Z_i}^{*,qcoh,Hub}(\mathcal{F}\otimes\mathcal{O}_X^+/\pi)^a$  for all  $i\in I$ , and, by 2.5.11 applied to  $f\circ i_{Z_i}$ , there are natural isomorphisms

$$Rf_*^{Hub} \circ i_{Z_i *}^{Hub} \circ i_{Z_i}^{*,qcoh,Hub} (\mathcal{F} \otimes \mathcal{O}_X^+/p)^a \cong f_{Mann}^* \circ i_{Z_i *}^{Mann} \circ i_{Z_i *}^{*,Mann} (\mathcal{F} \otimes \mathcal{O}_X^{+,a}/p)$$

By the proof of 3.2.1, we moreover have  $i_{Z_i}^{*,qcoh,Hub} \cong i_{Z_i}^{*,Hub}$ .

They are compatible with the colimit alongside  $i \in I$ , so that the third term appearing in the exact sequences are isomorphic, in a way that is compatible with the identification of the second terms.

This allows us to construct a natural map  $f_!^{Mann}(\mathcal{F} \otimes \mathcal{O}_X^{+,a}/p) \to (Rf_!^{Hub}(\mathcal{F} \otimes \mathcal{O}_X^+/p))^a$ , which is an isomorphism.

# 5. PRIMITIVE COMPARISON WITH COMPACT SUPPORT

In this section, we may finally define our notion of primitive comparison with compact support, and prove that they imply Poincaré duality. We start this section by

#### 5.1. Survey around the primitive comparison theorem

In the following, we'll need the following Galois-equivariant version of the primitive comparison theorem:

Corollary 5.1.1. Let K be a complete extension of  $\mathbb{Q}_p$ ,  $\overline{K}$  an algebraic closure of K, and C be the completion of  $\overline{K}$ . Let  $f: X \to \operatorname{Spa}(K, \mathcal{O}_K)$  be either a proper rigid analytic space or proper analytic adic space, and  $\mathcal{F} \in D^b_{zc}(X, \mathbb{F}_p)$  a system of Zariski-constructible coefficients.

Then, the natural morphism:

$$H^i(X_C, \mathcal{F}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \to H^i(X_C, \mathcal{O}_{X_C}^+/p \otimes_{\mathbb{F}_p} \mathcal{F})$$

is a  $Gal(\overline{K}/K)$ -equivariant almost isomorphism, where the Galois action on the left hand side is diagonal.

*Proof.* This is a special case of [Heu24, Thm 4.3].

This is an improved version compared to the "standard" one ([Sch12, Theorem 5.1]) in the following ways :

- (1) The smoothness assumption is removed.
- (2) The almost isomorphism is specified and Galois-equivariant.
- (3) It is stated for Zariski-constructible coefficients, instead of local systems.

Before proving the above theorem, let us say a few words about the (many) version of the primitive comparison theorem. While most of them are directly deduced from [Sch12, Theorem 5.1], many results coexist in the literature.

Let us investigate the different improvements described above.

- (1) The smoothness hypothesis has been removed in [Sch13, Thm 3.17], in the setup of rigid-analytic varieties. Note that we should not expect primitive comparison to hold for general proper morphisms of analytic adic spaces see [ABM24, Example 5.2.5.(b)] for a counter-example.
- (2) The Galois-equivariance is easily obtained from a *relative* version, obtained by working over K instead of C. Such a statement was already present in [Sch12, Corro 5.11].
- (3) While the initial statement allowed for coefficients in étale  $\mathbb{F}_p$ -local systems, Zavyalov [Zav24a, Lemma 7.3.7] uses a clever argument from in Bhatt-Hansen [BH21, Prop 3.6] to improve the result for Zariski-constructible coefficients (see also [Sch13, Thm 3.13] for an earlier, weaker version). See also [Man22, Corro 3.9.24.(ii)] for a relative version with more general coefficients, albeit in a more sophisticated setup.

An interesting discussion related to the one above, as well as a very general version combining most of the above, can be found in [Heu24, Section 4, Thm 4.3]. In this article, B.Heuer provides an *integral* version over  $\mathbb{Z}_p$ , that can also be adapted to study spaces of positive characteristic, or even mixed spaces.

Let us define our notion of primitive comparison with compact support. Recall that, for any finite dimensional finite dimensional partially proper morphism of analytic adic spaces  $f: X \to Y$  and sheaf of  $\mathbb{F}_p$ -modules  $\mathcal{F}$  on X, there is a natural morphism

$$Rf_!^{Hub}\mathcal{F}\otimes\mathcal{O}_Y^+/p\cong Rf_!^{Hub}(\mathcal{F}\otimes f_{Hub}^*\mathcal{O}_Y^+/p)\to Rf_!^{Hub}(\mathcal{F}\otimes\mathcal{O}_X^+/p)$$

Where the isomorphism is the projection formula of [Hub96, Prop 5.5.1.(iv)]. We may wonder when this map is an almost isomorphism. We make the following definition.

**Definition 5.2.1.** Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a partially proper analytic adic space of finite dimension over a complete algebraically closed extension of  $\mathbb{Q}_p$ , and  $\mathcal{L}$  be an étale local system of  $\mathbb{F}_p$ -modules on X. We say that X satisfies primitive comparison with compact support with respect to  $\mathcal{L}$  if, for all  $n \in \mathbb{N}$ , the natural morphism:

$$H^n_{\acute{e}t,c}(X,\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \to H^n_{\acute{e}t,c}(X,\mathcal{L} \otimes \mathcal{O}_X^+/p)$$

is an almost isomorphism of  $\mathcal{O}_C/p$ -modules.

The main result is now as follows:

**Theorem 5.2.2.** Let  $f: X \to \operatorname{Spa}(C, \mathcal{O}_C)$  be a partially proper analytic adic space over a complete algebraically closed extension of  $\mathbb{Q}_p$ , and  $\mathcal{L}$  be an étale local system of  $\mathbb{F}_p$ -modules on X.

Assume furthermore that X satisfies primitive comparison with compact support with respect to  $\mathcal{L}$ , and that f is smooth of pure of dimension d.

Then, for any  $0 \le k \le 2d$ , there exists a natural isomorphism:

$$H_{\acute{e}t}^k(X, \mathcal{L}^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(X, \mathcal{L}), \mathbb{F}_p)$$

*Proof.* By the primitive comparison with compact support hypothesis, the map:

$$\left(Rf_!^{Hub}(\mathcal{L} \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_X^+/p)\right)^a \to \left(Rf_!^{Hub}\mathcal{L} \otimes \mathcal{O}_C/p\right)^a$$

is an isomorphism in  $\mathcal{D}_{\acute{e}t}(\mathrm{Spa}(C,\mathcal{O}_C),\mathcal{O}_C^{+a}/p).$ 

By Prop 4.0.1, the first term canonically identifies with  $J_C\left(f_!^{Mann}(\mathcal{L}\otimes\mathcal{O}_X^{+,a}/p)\right)$ .

By 2.5.8, there is an isomorphism  $J_C^{-1}\left(((Rf_!^{Hub}\mathcal{L})\otimes\mathcal{O}_C/p)^a\right)\cong \left(J_C^{-1}(Rf_!^{Hub}\mathcal{L})\otimes\mathcal{O}_C/p\right)^a$ .

Combining the above, there is an isomorphism  $f_!^{Mann}(\mathcal{L} \otimes \mathcal{O}_X^{+,a}/p) \cong (J_C^{-1}(Rf_!^{Hub}\mathcal{L}) \otimes \mathcal{O}_C/p)^a$ .

We may then apply Corollary 2.4.2 with  $\mathcal{F}_{\mathcal{L}} = J_C^{-1}(Rf_!^{Hub}\mathcal{L})$ , which yields isomorphisms:

$$H^i_{\acute{e}t}(X,\mathcal{L}^\vee)(d) \simeq \operatorname{Hom}_{\mathbb{F}_p} \left( \pi_{i-2d} \left( J_C^{-1}(Rf_!^{Hub}\mathcal{L}) \right), \mathbb{F}_p \right)$$

Since  $J_C$  is t-exact,  $\pi_{i-2d}\left(J_C^{-1}(Rf_!^{Hub}\mathcal{L})\right) \cong R^{2d-i}f_!^{Hub}\mathcal{L} \cong H^{2d-i}_{\acute{e}t,c}(X,\mathcal{L})$ , which concludes.

Since this is a criterion on étale cohomology with compact support, it is best checked for open subsets of proper varieties, using localization sequences. In particular, we get the following.

**Lemma 5.2.3.** Let X be a proper rigid-analytic variety or a smooth proper adic space over a complete extension K of  $\mathbb{Q}_p$ , and let C be a complete algebraically closed extension of K. Let F be a generalizing closed subset of X, viewed with the induced structure of a locally spatial diamond.

Denote  $i: F \to X$  be the closed immersion, let  $U = X \setminus F$  with the induced analytic structure, and  $j: U \to X$  the associated open immersion. Denote by  $X_C, U_C$ , and  $F_C$  the base change of X, U and F to C, and  $i_C, j_C$  the respective morphisms. Let  $f_C: X_C \to C$  be the structural morphism.

Let  $\mathcal{L}'$  be a local system of  $\mathbb{F}_p$ -vector spaces on X, and  $\mathcal{L}$  denote the pullback to  $X_C$ . Assume that, for all  $n \in \mathbb{N}$ , the natural morphism

$$H_{\acute{e}t}^n(F_C, i_C^*\mathcal{L} \otimes \mathcal{O}_{F_C}^+/p) \to H_{\acute{e}t}^n(F_C, i_C^*\mathcal{L}) \otimes \mathcal{O}_C/p$$

is an almost isomorphism.

Then, for any  $0 \le k \le 2d$ , there exists a natural  $Gal(\overline{K}/K)$ -equivariant isomorphism:

$$H_{\acute{e}t}^k(U_C, \mathcal{L}^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(U_C, \mathcal{L}), \mathbb{F}_p)$$

*Proof.* Everything is in the sense of Huber, and we drop the "Hub" superscripts to alleviate notations. From the localization sequence of 3.2.1 applied to  $\mathcal{L}$ , we get the following exact triangle in  $\mathcal{D}_{\acute{e}t}(X_C, \mathbb{F}_p)$ :

$$0 \to j_C! j_C^* \mathcal{L} \to \mathcal{L} \to i_{C,*} i_C^* \mathcal{L}$$

Likewise, by applying the localisation sequence to  $\mathcal{L}\otimes\mathcal{O}_{X_C}^+/p$ , we get the following exact sequence in  $\mathcal{D}_{\acute{e}t}(X_C,\mathcal{O}_{X_C}^+/p)$ :

$$0 \to Rj_{C,!}^{Hub}j_{C,Hub}^*(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) \to (\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) \to Ri_{C,*}^{Hub}i_{C,Hub}^*(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{X_C}^+/p) \to 0$$

We may then apply the exact functor  $Rf_C!$  to both sequences, and extend the scalars of the first one to  $\mathcal{O}_C/p$ . By functoriality of localization sequences, we obtain the following diagram with exact rows:

$$0 \longrightarrow R(f_{C} \circ j_{C})! j_{C}^{*}(\mathcal{L} \otimes_{\mathbb{F}_{p}} \mathcal{O}_{X_{C}}^{+}/p) \longrightarrow Rf_{C*}(\mathcal{L} \otimes_{\mathbb{F}_{p}} \mathcal{O}_{X_{C}}^{+}/p) \longrightarrow R(f_{C} \circ i_{C})_{*} i_{C}^{*}(\mathcal{L} \otimes_{\mathbb{F}_{p}} \mathcal{O}_{X_{C}}^{+}/p) \longrightarrow 0$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow (R(f_{C} \circ j_{C})! j_{C}^{*}\mathcal{L}) \otimes_{\mathbb{F}_{p}}^{\mathbb{L}} \mathcal{O}_{C}/p \longrightarrow Rf_{C*}\mathcal{L} \otimes_{\mathbb{F}_{p}}^{\mathbb{L}} \mathcal{O}_{C}/p \longrightarrow (R(f_{C} \circ i_{C})_{*} i_{C}^{*}\mathcal{L}) \otimes_{\mathbb{F}_{p}}^{\mathbb{L}} \mathcal{O}_{C}/p \longrightarrow 0$$

Since f is proper, by the primitive comparison theorem, the middle arrow induces an almost isomorphism. We'll prove that the rightmost vertical arrow is also an almost isomorphism. Since  $f_C \circ i_C$  is the structure morphism of  $F_C$  and  $\mathcal{O}_C/p$  is flat over  $\mathbb{F}_p$ , the bottom term  $(R(f_C \circ i_C)_* i_C^* \mathcal{L}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} \mathcal{O}_C/p$  clearly computes the cohomology groups  $H_{\acute{e}t}^*(F_C, i_C^* \mathcal{L}) \otimes \mathcal{O}_C/p$ .

Moreover, pullbacks commute with tensor products, and, by 2.1.3 (together same argument as in the proof of 3.2.4), we obtain:

$$R(f_c \circ i_C)_* i_C^* (\mathcal{L} \otimes \mathcal{O}_{X_C}^+/p) \cong R(f_c \circ i_C)_* \left( i_C^* \mathcal{L} \otimes \mathcal{O}_{X_C}^+/p \right)$$

Hence, the top right term computes the cohomology groups  $H_{\acute{e}t}^*\left(F_C, i_C^*\mathcal{L}\otimes\mathcal{O}_{X_C}^+/p\right)$ . By our hypothesis, the rightmost vertical arrow induces an isomorphism in the almost category.

Hence the leftmost vertical arrow is an almost isomorphism. By the same arguments as the right hand side, using once again 2.1.3, this corresponds to the fact that the following morphisms:

$$H^k_{\acute{e}t,c}(U_C,j^*\mathcal{L}\otimes\mathcal{O}^+_{X_C}/p)\to H^k_{\acute{e}t,c}(U_C,j^*\mathcal{L}\otimes\mathcal{O}^+_{X_C}/p)$$

are almost isomorphisms. We may then conclude using Theorem 5.2.2.

Since  $\mathcal{L}$  comes from a local system on X,  $j^*\mathcal{L}$  is necessarily Galois-equivariant as a vector bundle on  $U_C$ . The Galois equivariance then follows from the functoriality of the construction, using 2.4.3.

Now, let us apply this result to deduce Poincaré duality for some concrete spaces.

#### 5.3. Poincaré duality for almost proper varieties

In this paragraph, we deduce Poincaré Duality for *almost proper* varieties, in the sense of ?? (see statement of Theorem 8.1.). Doing so, we recover a variant of a result by Zavyalov-Li-Reinecke [LRZ24, Corro 1.1.2], establishing Poincaré duality for *Zariski* open subspaces of smooth proper rigid analytic spaces.

Note that the approach of Zavyalov-Li-Reinecke uses different tools, as it barely uses perfected methods, and, rather, relies on a study of formal models, and of trace formalisms.

Our result is as follows.

**Corollary 5.3.1.** Let X be a smooth proper rigid analytic variety over K, and  $F \subset X$  a closed rigid-analytic sub variety. Let  $U = X \setminus F$  be the complementary open. Let  $j : U \to X$  denote the associated open immersion.

Then, for any étale  $\mathbb{F}_p$ -local system  $\mathcal{L}$  on X, U satisfies Poincaré duality with respect to  $j^*\mathcal{L}$ , i.e there is a natural Galois-equivariant isomorphism :

$$H_{\acute{e}t}^k(U_C,(j^*\mathcal{L})^{\vee}(d)) \cong \operatorname{Hom}_{\mathbb{F}_p}(H_{\acute{e}t,c}^{2d-k}(U_C,j^*\mathcal{L}),\mathbb{F}_p)$$

*Proof.* We'll use the criterion from 5.2.3, so that it suffices to check that the map  $H^n_{\acute{e}t}(F_C, i^*_C \mathcal{L} \otimes \mathcal{O}^+_{F_C}/p) \to H^n_{\acute{e}t}(F_C, i^*_C \mathcal{L}) \otimes \mathcal{O}_C/p$  is an almost isomorphism. This simply follows from the primitive comparison theorem, since  $F_C$  is a proper rigid-analytic variety, and pullbacks of local system are local systems.  $\square$ 

In particular, this is the case whenever U is Zariski-open in X.

**Remark 5.3.2.** In particular, the above recovers Corollary 1.1.2 from [LRZ24], but only for local systems on U obtained by pulling back local systems on X.

**Remark 5.3.3.** The above property works whenever for which the complement F admits the structure of a smooth analytic closed adic subspace, as the primitive comparison theorem also holds in that setup. This is, however, not very useful in practice.

However, the criterion 5.2.3 also allows us to consider other setups, where the complement is, say, a profinite topological space.

**Proposition 5.3.4.** Let X be a smooth proper analytic adic space over a complete algebraically closed extension C of  $\mathbb{Q}_p$ , and S be a profinite topological space, viewed as a locally spatial diamond over K, together with a closed immersion  $S \to K$ . Let  $U = X \setminus S$ .

Then, U satisfies primitive comparison with compact support with respect to the constant local system  $\mathbb{F}_p$ .

*Proof.* Recall that, by definition of the locally spatial diamond associated to a profinite space, S admits is an affinoid perfectoid given by  $S = \text{Spa}(C^0(S, C), C^0(S, \mathcal{O}_C))$ .

Let us write S as a projective limit of  $S = \varprojlim_{i \in I} S_i$  of finite discrete spaces. By [Sch17, Prop 14.9], there is a natural isomorphism  $H^j(S, \mathcal{F}) \cong \varinjlim_{i \in I} H^j(S_i, \mathcal{F}_i)$  for any étale sheaf  $\mathcal{F}$  on S; where  $\mathcal{F}_i$  denotes the pullback of  $\mathcal{F}$  so  $S_i$ .

Now, for any  $i \in I$ ,  $S_i$  is a discrete topological space, étale cohomology is simply sheaf cohomology, and the pullback  $((\mathcal{O}^+/p)_S)_i$  is simply the constant sheaf  $\mathcal{O}_C/p$  on  $S_i$ . Since spaces are discrete, there is no higher cohomology, and  $H^0$  is simply given by global sections.

We then deduce  $H^0(S, \mathbb{F}_p) \cong \varinjlim_{i \in I} \operatorname{Cst}(S_i, \mathbb{F}_p) \cong \mathcal{C}^0(S, \mathbb{F}_p)$  and  $H^0(S, \mathcal{O}_S^+/p) \cong \varinjlim_{i \in I} \operatorname{Cst}(S_i, \mathcal{O}_C/p) \cong \mathcal{C}^0(S, \mathcal{O}_C/p)$  since  $\mathcal{O}_C/p$  is a discrete topological space, where Cst denotes the ring of constant functions. The natural map  $\mathcal{C}^0(S, \mathbb{F}_p) \otimes \mathcal{O}_C/p \to \mathcal{C}(S, \mathcal{O}_C/p)$  is naturally an isomorphism, which concludes.  $\square$ 

Note that, in that setup, the étale cohomology groups of  $U_C$  (with or without support) are not As a corollary, we easily Poincaré duality for Drinfeld's upper half plane.

Corollary 5.3.5. Let C be a complete algebraically closed extension of  $\mathbb{Q}_p$ , and  $\Omega_C = \mathbb{P}^1_C \setminus \mathbb{P}^1(\mathbb{Q}_p)$  denote Drinfeld's upper half plane.

Then  $\Omega_C$  satisfies Poincaré duality, and for  $0 \le i \le 2$ , there is a natural  $\operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -equivariant isomorphism:

$$H^{i}_{\acute{e}t}(\Omega_{C}, \mathbb{F}_{p}) \cong \operatorname{Hom}(H^{2-i}_{\acute{e}t,c}(\Omega_{C}, \mathbb{F}_{p}), \mathbb{F}_{p})$$

*Proof.* This follows directly from the above proposition.

Note that one can easily check that the above morphism is also  $\mathbb{GL}_2(\mathbb{Q}_p)$ -equivariant, since every morphism appearing in the primitive comparison theorem is  $\mathbb{GL}_2(\mathbb{Q}_p)$ -equivariant, and using the functoriality established in 2.4.3.

In the rest of this paper, we prove that such a Poincaré duality result also holds for Drinfeld's symmetric spaces of higher dimension, as well as period domains in general.

# 6. Period domains

In this section, we give a precise definition of period domains, for which we claim no originality. We start by specializing the theory for  $GL_n$ . Such a context is sufficient to define Drinfeld's spaces, is necessary for the general theory, and can give most of the necessary intuition.

The original construction was developed in [RZ96], see also a short introduction by the first author in [Rap97]. Concise introductions, somewhat similar to the one below, can be found in [Orl05a], [CDHN21], or [CFS17], the latter also exploring the point of view of the Fargues-Fontaine curve. The book [DOR10] presents an extensive version of the theory, whose structure we broadly follow. Some classical motivation is nicely presented in the Master's thesis of H.Wang, cf. [Ha009].

In comparison to the references above, we take meticulous care in being as explicit as possible regarding the "easy case"  $G = GL_n$ , as well as the construction of Drinfeld's spaces.

# 6.1. Filtered isocrystals, and period domains for $G = \mathbb{GL}_n$

In this section, we defined a category of filtered isocrystals. Period domains for  $\mathbb{GL}_n$  will be moduli spaces of weakly admissible isocrystals (in the sense of Fontaine), which we view as semi-stable objects of slope zero inside an adequate Harder-Narasimhan formalism on the category of filtered isocrystals. In what follows, we fix a field K.

In the following, we assume that the reader is somewhat knowledgeable with Tannakian categories. These concepts can be found in [DM82] or [DOR10, Section 4].

6.1.1. Filtered vector spaces. Let V be a finite dimensional K-vector space. A  $\mathbb{Q}$ -filtration  $\mathcal{F}V$  on V is the data of a map  $\mathcal{F}: x \in \mathbb{Q} \mapsto \mathcal{F}^x V \in \{K - \text{linear subspaces of } V\}$  such that :

- $\mathcal{F}$  is decreasing, i.e, for all x < y,  $\mathcal{F}^x \subset \mathcal{F}^y$ .
- There exists x < y such that  $\mathcal{F}^x V = V$  and  $\mathcal{F}^y V = \{0\}$ .
- For all x,  $\mathcal{F}^x = \bigcap_{y < x} \mathcal{F}^y$

We define the **type** of the filtration  $\mathcal{F}$  by  $\nu(\mathcal{F}) = \left(x_1^{n(1)}, \dots, x_n^{n(i)}\right)^{16}$ , where  $x_1 > \dots > x_n$  are the jumps of the filtration, i.e.  $\{x_1, \dots, x_n\} = \{x \in \mathbb{R}, gr_{\mathbb{F}}^{x_i} \neq \{0\}\}$ , and  $n_i = \dim_V(gr_{\mathcal{F}}^{x_i})$  is the multiplicity at  $x_i$ .

Likewise, the data of  $(x_1^{n(1)}, \ldots, x_n^{n(i)})$ , where  $x_1 > \cdots > x_n$  and  $\sum n_i = dim(V)$  is called the possible type of a filtration on V.

**Definition 6.1.1.** Let  $\nu$  be the possible type of a filtration on V.

We construct the flag variety (cf. [RZ96, p. 1.31])  $\mathscr{F}(\mathbb{GL}(V), \nu)^{17}$  as the variety parameterizing filtrations of type  $\nu$  over K, i.e. the projective K-scheme such that, for any field extension K' over K:

$$\mathscr{F}(\mathbb{GL}(V),\nu)(K')\cong\{\text{filtrations on }V\otimes_K K'\text{ of type }\nu\}$$

It can be constructed as  $\mathscr{F}(\mathbb{GL}(V), \nu) = \mathbb{GL}(V)/P(\nu)$ , where  $\mathbb{GL}(V)$  is the reductive group defined by  $R \mapsto \mathrm{GL}(V \otimes_K R)$  for all K-algebras R, and  $P(\nu)$  is the parabolic subgroup of  $\mathbb{GL}(V)$  stabilizing a given filtration of type  $\nu$  on V.

It is naturally endowed with an action of the linear group  $GL(V) = \mathbb{GL}(V)(K)$ , since composing with a linear transformation does not change the type of the filtration.

For K/k a field extension, we let  $\mathbb{Q} - \mathrm{Fil}_k^K$  denote the category of pairs  $(V, \mathcal{F}_K)$ , where V is a k-vector space, and  $\mathcal{F}_K$  is a  $\mathbb{Q}$ -filtration on the K-vector space  $V \otimes_k K$ . A morphism in  $\mathbb{Q} - \mathrm{Fil}_k^K$  is a k-linear map  $f: V \to W$  such that, for all  $x \in \mathbb{Q}$ ,  $f \otimes id_K(V^x) \subset W^x$ .

The category  $\operatorname{Fil}_k^K$  is an additive k-linear  $\otimes$ -category. It is not abelian (as a change of the filtration cannot be read on the kernel nor cokernel), but it is quasi-abelian, and, in particular, an exact category.

We could define a Harder-Narasimhan formalism in that setup on  $\operatorname{Fil}_k^K$ , and a notion of semi-stability. We'll do so only in the context of filtered *isocrystals*, with a slope morphism decomposing a the sum of a "filtration" part and an "isocrystal" part. When the isocrystal structure is trivial (which is for example the case when defining Drinfeld's symmetric spaces), this is equivalent in working inside  $\operatorname{Fil}_k^K$ . We refer to [DOR10] for a detailed overview of this setup.

6.1.2. Isocrystals. Let L be a perfect field of characteristic p > 0,  $K_0 = \text{Frac}(W(L))$ , and  $\sigma$  be the Frobenius endomorphism on  $K_0$ . In most applications, we'll fix  $L = \overline{\mathbb{F}_p}$ , so that  $K_0 = \mathbb{Q}_p$  is the completion of the maximal unramified extension of  $\mathbb{Q}_p$ .

<sup>&</sup>lt;sup>16</sup>Here, the exposant  $n_i$  indicates that the corresponding  $\nu(i)$  is repeated  $n_i$  times.

<sup>&</sup>lt;sup>17</sup>The more standard notation (cf. [DOR10]) is  $\mathscr{F}(V,\nu)$ . We prefer denoting  $\mathbb{GL}(V)$  so that our notation coincides with the general case, where  $\mathbb{GL}(V)$  will be replaced by a reductive algebraic group.

An isocrystal<sup>18</sup> over  $K_0$  is a pair  $N=(V,\varphi)$ , where V is a finite-dimensional vector space over  $K_0$ , and  $\varphi:V\to V$  is a  $\sigma$ -linear<sup>19</sup> bijective automorphism.

We let  $\text{Isoc}(K_0)$  denote the category of such isocrystals, with morphisms given by  $K_0$ -linear maps  $f: V \to V'$  such that  $f \circ \varphi_V = \varphi_{V'} \circ f$ . It is clearly an abelian category, and, moreover, we have the following result (cf. [Man63]):

**Theorem 6.1.2.** (Dieudonné-Manin classification)

Assume that L is algebraically closed. Then,  $\operatorname{Isoc}(K_0)$  is semi-simple, and the simple objects are of the form:

$$V_{r/s} = \begin{pmatrix} K_0^s, \begin{bmatrix} 0 & 1 \\ & \ddots & 1 \\ p^r & & 0 \end{bmatrix} \cdot \sigma \end{pmatrix}$$

for any coprime  $(r,s) \in \mathbb{Z} \times \mathbb{N}_{\geq 1}$ .

More explicitly, any  $N \in \text{Isoc}(K_0)$  can be uniquely written as  $V = \bigoplus_{x \in E} V_x$  for some finite  $E \subset \mathbb{Q}$ . Using the terminology described above, the decomposition as a sum of simple objects yields defines a grading functor  $\omega : \text{Isoc}(K_0) \to \mathbb{Q} - \text{Grad}_{K_0}$ .

We may now combine the two notions defined above.

6.1.3. Filtered isocrystals. Let, as before L be a perfect field of characteristic p > 0,  $K_0 = \text{Frac}(W(L))$ , and  $\sigma$  be the Frobenius morphism on  $K_0$ . Let K be a field extension of  $K_0$ .

A **filtered** isocrystal  $(V, \varphi, \mathcal{F}V_K)$  over  $K_0$  is the data of an isocrystal  $(V, \varphi)$  over  $K_0$ , together with a  $\mathbb{Q}$ -filtration  $\mathcal{F}V_K$  on the vector space  $V_K = V \otimes_{K_0} K$ .

We let  $\operatorname{FilIsoc}_{K_0}^K$  denote the category of such filtered isocrystals. The category  $\operatorname{FilIsoc}_{K_0}^K$  is endowed with a Harder-Narasimhan formalism, defined by :

- The rank of  $(V, \varphi, \mathcal{F}V_K)$  is  $dim_{K_0}V$
- The **degree** of  $(V, \varphi, \mathcal{F}V_K)$  is  $deg(V, \varphi, \mathcal{F}V_K) = \sum_{i \in \mathbb{Z}} i \cdot dim_K(gr_{\mathcal{F}}^i(V_K)) v_F(det(\varphi))$
- The slope of  $(V, \varphi, \mathcal{F}V_K)$  is  $\nu(V, \varphi, \mathcal{F}V_K) = \frac{deg(V, \varphi, \mathcal{F}V_K)}{rk(V, \varphi, \mathcal{F}V_K)}$
- If  $(V, \varphi, \mathcal{F}_K)$  is a filtered isocrystal, a **sub**-isocrystal is a triplet  $(V', \varphi', \mathcal{F}'_K)$  where V' is a subspace of V stable under  $\varphi$ , endowed with the induced structure, i.e.  $\varphi' = \varphi_{V'}$ , and  $\mathcal{F}'_K = \mathcal{F}_K|_{V' \otimes_{K_0} K'}$ .

We say that a filtered isocrystal is **weakly admissible** if it is semi-stable<sup>20</sup> and of slope zero. A nontrivial result of Faltings and Totaro (cf. [Fal95] and [Tot96]) states the following:

**Theorem 6.1.3.** The tensor product of semi-stable filtered isocrystals is semi-stable

**Remark 6.1.4.** This degree function (and, in turn, the slope function) is the difference of two natural and classical degree functions: one associated to the isocrystal, and the other to the filtration:

- (1) For  $(V, \mathcal{F}_K V) \in \operatorname{Fil}_k^K$  a filtered vector space, the degree  $deg(V, \mathcal{F}_K V) = \sum_i i \cdot dim_K (gr_{\mathcal{F}}^i V_K)$  is the y-axis value of the terminal point of the Hodge polygonal associated to the filtration  $\mathcal{F}V_K$ .
- (2) For an isocrystal  $(V, \varphi)$  over  $K_0$ , we consider its degree  $-v_p(det(A))$ , where A is a matrix representing the action of  $\varphi$  in any given base. It is the y-axis value of the terminal point of the Newton polygonal associated with the isocrystal  $(V, \varphi)$ . For example, for a simple object  $V = V_{r/s}$ ,  $-v_p(det(A_{r/s})) = r$ .

In other terms, a filtered isocrystal is weakly admissible if and only if its Newton and Hodge polygonals terminate at the same point, and the Newton polygonal of any sub-isocrystal lies above its Hodge polygonal. This is the original interpretation of Fontaine [Fon94, Defi 4.4.3].

<sup>&</sup>lt;sup>18</sup>This is sometimes referred to as F-isocrystals, where the "F" reminds us of the presence of a Frobenius morphism. We simply call them Isocrystals for clarity of notation. Dear reader, please remind yourself at all times that isocrystals come endowed with a Frobenius morphism.

<sup>&</sup>lt;sup>19</sup>i.e.  $\forall \lambda \in K_0, \forall x \in V, \varphi(\lambda \cdot x) = \sigma(\lambda) \cdot \varphi(x)$ .

 $<sup>^{20}</sup>$ Recall that an object is said to be semi-stable if every sub-object has a non-greater slope.

We may now define our p-adic period domains, in the  $\mathbb{GL}_n$  case, as moduli spaces of weakly admissible representations.

**Proposition–Definition 6.1.5.** Let  $(V, \varphi) \in \text{Isoc}(K_0)$ , and  $\nu$  be the possible type of a filtration on V.

There exists an open subset defining a smooth, partially proper rigid-analytic variety over  $K_0$ :

$$\check{\mathscr{F}}^{wa}(\mathbb{GL}(V), \varphi, \nu) \subset \mathscr{F}(V, \nu) \otimes_{\mathbb{Q}_n} K_0$$

Such that, for any complete field extension K of  $K_0$ :

$$\check{\mathscr{F}}^{wa}(\mathbb{GL}(V),\varphi,\nu)(K) = \{ \mathcal{F} \in \mathscr{F}(\mathbb{GL}(V),\nu)(K), (V \otimes_{K_0} K,\varphi \otimes_{K_0} id_K,\mathcal{F}) \text{ is weakly admissible} \}$$

We call it the **period domain** associated to  $(GL(V), \nu)$ .

It admits an action of the linear group  $J(K_0) := \{g \in \mathbb{GL}(V)(K_0), g\varphi = \varphi g\}$ 

Moreover,  $\check{\mathscr{F}}^{wa}(\mathbb{GL}(V), \varphi, \nu)$  can be obtained as the analytification of a Berkovich space over  $K_0$ . At some point, denote it  $\mathscr{F}(\mathbb{GL}_n, N, \nu)$  ?

*Proof.* This is done in [RZ96, Prop 1.36] in the rigid analytic setup, and in [DOR10, Prop 8.2.1 & 8.2.4] in the Berkovich setup.

As we will see in ??, this period domain can actually be defined over a finite extension of  $\mathbb{Q}_p$ . We'll do so directly in the general case.

Let us conclude this section by some examples of such period domains.

6.1.4. Examples of period domains. In the same setup as below, let us define Drinfeld's symmetric spaces as a period domain. We first view the projective space as a flag variety.

Fix  $n \in \mathbb{N}_{\geq 1}$ ,  $K_0 = \mathbb{Q}_p$ ,  $V = K_0^n$ ,  $\sigma \in \operatorname{Aut}(K_0/\mathbb{Q}_p)$  be the Frobenius endomorphism. Let  $\nu = (n-1, -1, \dots, -1)$ .

For any field extension  $K/\mathbb{Q}_p$ , the flag variety  $\mathscr{F}(\mathbb{GL}(V),\nu)(K)$  parametrizes filtrations on  $V\otimes_{\mathbb{Q}_p} K$  with only two jumps: a 1-dimensional jump at n-1, and a (n-1)-dimensional jump at (-1).

Any such filtration  $\mathcal{F}_K \in \mathscr{F}(\mathbb{GL}(V), \nu)(K)$  is of the following form, for some K-line  $\mathcal{L}_{\mathcal{F}} \subset V \otimes_{\mathbb{Q}_p} K$ :

$$\mathcal{F}_{K,x} = \begin{cases} V \otimes_{\mathbb{Q}_p} K & \text{if} & x \leq -1\\ \mathcal{L}_{\mathcal{F}} & \text{if} & -1 < x \leq n-1\\ \{0\} & \text{if} & n-1 < x \end{cases}$$

Clearly, such filtrations correspond one-to-one to lines  $\mathcal{L}_{\mathcal{F}}$  of  $V \otimes_{\mathbb{O}_n} K$ .

Hence,  $\mathscr{F}(\mathbb{GL}(V), \nu) = \mathbb{P}(V)$  is the  $\mathbb{Q}_p$ -variety whose K-points are  $\mathbb{P}(V)(K) := \mathbb{P}(V \otimes_{\mathbb{Q}_p} K)$ .

We may now define Drinfeld's symmetric spaces as open subsets of those flag varieties. We start by endowing  $V \otimes_{\mathbb{Q}_p} K_0$  with the trivial isocrystal structure  $\varphi = \mathrm{id}_V \otimes_{\mathbb{Q}_p} \sigma$ .

For any  $K/K_0$  and  $\mathcal{F}_K \in \mathscr{F}(\mathbb{GL}(V), \nu)(K)$  corresponding to a line  $\mathcal{L}_{\mathcal{F}} \subset V \otimes_{\mathbb{Q}_p} K$  as above, the slope of the isocrystal  $(V, \mathrm{id}_V \otimes \sigma, \mathcal{F}_K)$  is simply induced by the filtration (the "Hodge polynomial" part):

$$\nu(V \otimes_{\mathbb{Q}_p} K_0, id_V \otimes \sigma, \mathcal{F}_K) = \frac{1}{n} \sum_{i \in \mathbb{Z}} i \cdot dim_K(gr_{\mathcal{F}}^i(V_{K'})) = \frac{1}{n} (n - 1 - (n - 1)) = 0$$

Sub-objects of  $(V \otimes_{\mathbb{Q}_p} K_0, id_V \otimes \sigma, \mathcal{F}'_K)$  are sub- $K_0$ -vector spaces of  $V \otimes_{\mathbb{Q}_p} K_0$ , together with the induced filtered isocrystal structure. Let  $W \subset V \otimes_{\mathbb{Q}_p} K_0$  be such a sub-vector space,  $d = dim_{K_0} W$ , so that :

$$\mathcal{F}'_{K,x} = \begin{cases} W \otimes_{K_0} K & \text{if} & x \leq -1\\ \mathcal{L}_{\mathcal{F}} \cap (W \otimes_{K_0} K) & \text{if} & -1 < x \leq n-1\\ \{0\} & \text{if} & n-1 < x \end{cases}$$

 $\mathcal{F}'_{K,x} = \begin{cases} W \otimes_{K_0} K & \text{if} & x \leq -1 \\ \mathcal{L}_{\mathcal{F}} \cap (W \otimes_{K_0} K) & \text{if} & -1 < x \leq n-1 \\ \{0\} & \text{if} & n-1 < x \end{cases}$  One easily computes :  $deg(W, id_W \otimes \sigma, \mathcal{F}'_K|_W) = \begin{cases} -d & \text{if } \mathcal{L}_{\mathcal{F}} \not\subset W \otimes_{K_0} K \\ n-d & \text{if } \mathcal{L}_{\mathcal{F}} \subset W \otimes_{K_0} K \end{cases}$ 

It follows that  $\mathcal{F}$  is semi-stable if and only if the line  $\mathcal{L}_{\mathcal{F}}$  is not contained in any  $K_0$ -rational hyperplane. Hence,

$$\breve{\mathscr{F}}^{wa}\left(\mathbb{GL}(V),\varphi,(n-1,-1,\ldots,-1)\right)(K) = \mathbb{P}_{n-1}(K) \setminus \bigcup_{H \in \mathcal{H}_{K_0}} H$$

where  $\mathcal{H}$  describes all  $K_0$ -rational hyperplanes of V. Such a space will be denoted  $\Omega_{K_0}^{n-1}$ , and is called Drinfeld's symmetric space of dimension n-1 - over  $K_0$ .

It is naturally the base change to  $K_0$  of the Drinfeld space over  $\mathbb{Q}_p$ , defined similarly as:

$$\Omega_{\mathbb{Q}_p} = \mathbb{P}^n_{\mathbb{Q}_p} \setminus \bigcup_{H \in \mathcal{H}_{\mathbb{Q}_p}} H$$

where  $\mathcal{H}$  describes all  $K_0$ -rational hyperplanes of  $\mathbb{Q}_p^n$ . This is **Drinfeld's symmetric space** of dimension n-1 over  $\mathbb{Q}_p$ .

**Remark 6.1.6.** Note that the fact that the period domain over  $K_0$  descends to a variety defined over a finite extension of  $\mathbb{Q}_p$  will be a general fact under a decency hypothesis, cf. ??.

**Remark 6.1.7.** Note that, as in [DOR10], there is a natural analogue of Drinfeld's symmetric spaces over finite fields. If that setup, the union is finite, so that Drinfeld space is a Zariski open. This setup is somewhat easier to study.

**Remark 6.1.8.** The above construction admits a natural dual, given by the trivial isocrystal structure on  $K_0^n$ , and  $\nu = (-1, \dots, -1, n-1)$ . Here, filtrations correspond to hyperplanes in  $K_0^n$ , and weakly admissible filtrations correspond to the hyperplane not containing a rational line.

The associated period domain is known as the dual Drinfeld space.

We may also view the affine and projective spaces as examples of period domains.

## 6.2. Link with crystalline representations and the admissible locus

In the spirit of Fontaine's famous "weakly admissible implies admissible" conjecture, such period domains are deeply related to Galois representations.

**Definition 6.2.1.** Let K be a discretely valued extension of  $\mathbb{Q}_p$ , with residue field k, and let  $G_K$  denote the absolute Galois group of K. Let  $K_0 = W(k)[1/p]$  be the maximal unramified subextension of K.

Then, there is a fully faithful functor from the category of finite-dimensional crystalline representations of  $G_K$  to the category of filtered isocrystals  $\operatorname{FilIsoc}_K^{K_0}$ , given by

$$V \mapsto (\mathbb{B}_{crys} \otimes_K V)^{G_K}$$

where the filtration on  $(\mathbb{B}_{crys} \otimes_K V)^{G_K} \otimes_{\mathbb{Q}_p} K_0 \cong (V \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR})^{G_K}$  is induced from the natural filtration  $\mathrm{Fil}^i \mathbb{B}_{dR} = t^i \mathbb{B}_{dR}^+$  on the de Rham period ring.

We say that a filtered isocrystal is admissible if it lies in the essential image of the above functor.

Fontaine's famous conjecture, which is now a theorem, is as follows:

**Theorem 6.2.2.** Let  $(V, \varphi, \mathcal{F})$  be a filtered isocrystal in the above context. Assume furthermore that K is a finite extension of  $\mathbb{Q}_p$ .

Then  $(V, \varphi, \mathcal{F})$  is weakly admissible if and only if it is admissible.

*Proof.* "Admissible implies weakly admissible" is straightforward (as the terminology suggests), and was proven in [FW79, Proposition 5.4.2.(i)].

The reciprocal is one of the most important result in p-adic Hodge theory. It was conjectured in the above article of Fontaine ([FW79, Conjecture 5.6.9]), and first proven in [CF00]. The modern and more insightful viewpoint, using the Fargues-Fontaine curve, is developed in [FF18].

One can then be tempted to think of period domains as moduli spaces of admissible crystalline representations, but one needs to be careful. Crucially, the above equivalence does not hold when K is an infinite

extension. It is unclear how we can associate Galois representations to K-points of  $\tilde{\mathscr{F}}^{wa}$ , whenever K is infinite. Rappoport and Zink conjectured the existence of an **admissible** locus  $\mathscr{F}^a \subset \tilde{\mathscr{F}}^{wa}$ , which is better suited for the study of crystalline representations, as it forms the essential image of the crystalline period morphism.

How one should precisely define an admissible locus was unclear for a long time, but was finally settled thanks to works of Fargues-Scholze regarding the geometry of the Fargues-Fontaine curve (cf. [CFS17]. See also [Sch15, Appendix]). The "weakly admissible implies admissible", for any finite  $K/K_0$ ,  $\mathscr{F}^a$  and  $\check{\mathscr{F}}^{wa}$  have the same K-points - but they do not coincide as geometric spaces. In some space, the space of interest really should be  $\mathscr{F}^a$ , and  $\check{\mathscr{F}}^{wa}$  can be seen as an approximation of it. It is however much trickier to compute the cohomology of the admissible locus, as there is no known explicit geometric stratification of the complement.

The results of [CFS17] establish an explicit criterion (see also [Har13] for the  $GL_n$  case) under which the weakly admissible and admissible loci coincide. The reader only interested in the admissible locus can assume that we work under that hypothesis (Hodge-Newton decomposability of  $B(G, \mu)$ ) throughout.

## 6.3. Period domains for arbitrary reductive groups

We'll define period domains relative to more general reductive algebraic groups. In the case  $G = \mathbb{GL}_n$ , we will recover the above. The construction will be mostly the same, except that we replace :

- Isocrystals by isocrystals with G-structure.
- Filtrations by 1-parameter subgroups.
- Type of filtrations with conjugacy classes of 1-parameter subgroups.

6.3.1. Gradings and filtrations. Let k be any field, K/k a field extension.

Starting from ??, we'll have  $k = \mathbb{Q}_p$ , and  $K_0 = \check{\mathbb{Q}}_p$ . For now, we work in full generality.

**Definition 6.3.1.** We let  $Rep_k(G)$  be the category of finite dimensional representations of G over k.

An object of  $\operatorname{Rep}_k(G)$  is of the form  $(V, \rho)$  where V is a finite dimensional k-vector space, and  $\rho$  is a morphism of affine group schemes  $G \to \mathbb{GL}_V$ , i.e. contains the compatible data of morphisms  $\rho_R : G(R) \to GL_n(V \otimes_k R)$  for any k-algebra R. The category  $\operatorname{Rep}_k(G)$  is naturally equipped with a tensor product, and the fiber functor obtained by forgetting :  $w^G : (V, \rho) \in \operatorname{Rep}_k(G) \mapsto V \in \operatorname{Vect}_k$ .

This makes it a Tannakian category over  $\mathbb{Q}_p$ .

Our goal is to understand filtrations of the fiber functor  $\omega^G$ . Let us first introduce graduations.

For M an abelian group, we let  $M-\operatorname{Grad}_k$  denote the category of finite dimensional k-vector spaces, equipped with a grading indexed by M:

$$V = \bigoplus_{m \in M} V_m$$

Morphisms of such graded vector spaces are linear maps respecting the gradation.

Consider the functor  $\omega: M-\operatorname{Grad}_k \to \operatorname{Vect}_k$  obtained by forgetting the grading :  $\omega\left(V = \bigoplus_{m \in M} V_m\right) \mapsto V$ .

**Remark 6.3.2.** The category of  $\mathbb{Q}$ -grading is related to the category of  $\mathbb{Q}$ -filtrations, introduced in 6.1.1. Indeed, there exists a functor fil:  $\mathbb{Q} - \operatorname{Grad}_k \to \mathbb{Q} - \operatorname{Fil}_k^K$ , obtained by

fil: 
$$V = \bigoplus_{x \in \mathbb{Q}} V_x \mapsto \left( V_K, \mathcal{F}^x V_K = \sum_{y \ge x} V_y \otimes K \right)$$

More generally, if we let  $\mathbb{Q} - \operatorname{Grad}_k^K$  denote the category of vector spaces V over k together with a  $\mathbb{Q}$ -gradation on  $V_K$ , the above definition extends to a functor fil:  $\mathbb{Q} - \operatorname{Grad}_k^K \to \operatorname{Fil}_k^K$ .

In the other direction, there exists a functor  $\operatorname{gr}:\mathbb{Q}-\operatorname{Fil}_k^K\to\mathbb{Q}-\operatorname{Grad}_K$  obtained by  $\mathcal{F}\mapsto\bigoplus_{k}\operatorname{gr}_{\mathcal{F}}^xV_K$ .

Note that, while fil respects the underlying vector space (i.e. commutes with fiber functors), gr does not, even if K = k.

In order to study the category of  $\mathbb{Q}$ -gradings, we introduce the slope torus  $\mathbb{D}_k = \varprojlim_{n \in \mathbb{N}} \mathbb{G}_{m,k}$ , as studied in [DOR10, p. IV.2]. We recapitulate its major properties in the following.

**Proposition–Definition 6.3.3.** The functor  $\mathcal{D}_k : R \in \operatorname{Ab}^{op} \mapsto \operatorname{Spec}(k[R]) \in \operatorname{AffSch}_k$  is the left adjoint to  $\operatorname{Spec}(R) \mapsto R^*$ . We have  $D(\mathbb{Z}) = \mathbb{G}_m$ , so that :

$$D(\mathbb{Q}) = D\left(\underset{n \in \mathbb{N}^*}{\underline{\lim}} \frac{1}{n} \cdot \mathbb{Z}\right) \cong \underset{n \in \mathbb{N}^*}{\underline{\lim}} D(\mathbb{Z}) = \underset{n \in \mathbb{N}^*}{\underline{\lim}} \mathbb{G}_m$$

We denote  $\mathcal{D}_k = D_k(\mathbb{Q})$ , known as tge the slope torus. The multiplication by any  $x \in \mathbb{Q}$  induces a map  $\chi_x : D(\mathbb{Q}) \to D(\mathbb{Z})$ , viewed as a character of  $D(\mathbb{Q})$ .

**Proposition 6.3.4.** The category  $\mathbb{Q}$  –  $\operatorname{Grad}_k$  is a neutral Tannakian category over k, and  $\omega$  is a fiber functor over k. Moreover, there is a natural equivalence of category:

$$\operatorname{Rep}_{k}(\mathbb{D}_{k}) \to \mathbb{Q} - \operatorname{Grad}_{k}$$
$$(V, \rho) \mapsto V = \bigoplus_{x \in \mathbb{Q}} V_{x}$$

Where  $V_x = \{v \in V, \rho(g)v = \chi_x(g) \cdot v, \forall g \in D(\mathbb{Q})\}$  is the weight space of V associated to  $x \in \mathbb{Q}$ .

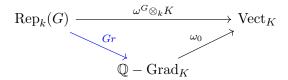
Proof. cf. [Riv72], IV.1.2.1. 
$$\Box$$

Note that, similarly, for any abelian group M, the category  $\operatorname{Rep}_k D(M)$  is equivalent to  $M - \operatorname{Grad}_k$ .

6.3.2. Relative gradings and filtrations. Fix k, K as above. Let G be a connected algebraic group over k. We let  $G_k = G \times_{\operatorname{Spec}(k)} \operatorname{Spec}(K)$  denote the base change of G to K.

Now, let us define categories of filtrations and gradings relative to G.

**Definition 6.3.5.** A  $\mathbb{Q}$ -grading of  $\omega^G$  over K is a tensor functor  $Gr : \operatorname{Rep}_k(G) \to \mathbb{Q} - \operatorname{Grad}_K$  such that the diagram below commutes.



Note that such functors are automatically exact and faithful.

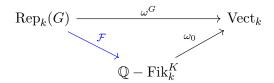
*Proof.* The faithfulness is clear. The exactness follows from the exactness of  $\omega^G$ , hence of  $\omega^G \otimes_k K$ , and the fact that  $\omega_0$  both preserves and reflects exact sequences.

**Lemma 6.3.6.** There is a one-to-one correspondence between  $\mathbb{Q}$ -gradings of  $\omega^G$  over K and homomorphisms  $\mathbb{D}_K \to G$ .

*Proof.* This follows from Prop. 6.3.4 and a form of Tannakian duality, [DOR10, Corro 4.1.19].

Note that such a grading can be seen as the compatible data of grading on  $V \otimes_k K$ , for each representation  $(V, \rho)$  of G (cf. [DOR10, Remark 4.2.5.(ii)] for a more precise formulation).

**Definition 6.3.7.** A  $\mathbb{Q}$ -filtration of  $\omega^G$  over K is a tensor functor  $F : \operatorname{Rep}_k(G) \to \mathbb{Q} - \operatorname{Fil}_k^K$  such that the diagram below commutes:



And such that the composition  $\operatorname{Rep}_k(G) \xrightarrow{\mathcal{F}} \mathbb{Q} - \operatorname{Fil}_k^K \xrightarrow{gr} \mathbb{Q} - \operatorname{Grad}_K$  (where gr is from 6.3.2) is exact.

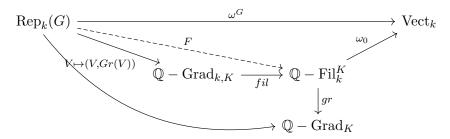
**Remark 6.3.8.** Note that, by definition, such a filtration  $\mathcal{F}$  defined over K induces, for all representation  $(V, \rho)$  of G, a filtration  $\mathcal{F}^{\rho}$  on  $V \otimes_k K_0$ .

One can reformulate the above definition as the *compatible data* of filtrations  $\mathcal{F}^{\rho}$  on  $V \otimes_k K$ , cf. [DOR10, Defi 4.2.6].

The exactness condition ensures we can recover the filtration from the grading.

**Proposition 6.3.9.** Every  $\mathbb{Q}$ -grading of  $\omega^G$  over K naturally induces a  $\mathbb{Q}$ -filtration of  $\omega^G$  over K.

*Proof.* Let Gr be such a grading. The morphism fil from 6.3.2 commutes with fiber functors, so that the forgetful functor  $\omega_0: \mathbb{Q} - \operatorname{Grad}_{k,K} \to \operatorname{Vect}_K$  factors through  $\mathbb{Q} - \operatorname{Fil}_K^K$ . We have:



The dashed arrow will be our filtration. It suffices to show that the long curved composition  $\operatorname{Rep}_k(G) \to \mathbb{Q} - \operatorname{Grad}_K$  is exact. G is exact by the remark 6.3.5, and fil and gr can be checked to be exact by an easy linear algebra argument.

We conclude this paragraph by showing that gradings and filtrations relative to  $\mathbb{GL}(V)$  correspond to "standard" grading and filtrations on V (see also [DOR10, Remark 4.2.11])

**Example 6.3.10.** Let  $G = \mathbb{GL}(V)$ . Then,  $\mathbb{Q}$ -gradings of  $\omega^G$  over K correspond to graduations on  $V \otimes_k K$ , and  $\mathbb{Q}$ -filtrations of  $\omega^G$  over K correspond to filtrations on  $V \otimes_k K$ .

Indeed, any  $\mathbb{Q}$ -grading of  $\omega^G$  over K induces a  $\mathbb{Q}$ -grading on V through the standard representation of  $\mathbb{GL}(V)$  acting on V. Reciprocally, a  $\mathbb{Q}$ -grading  $V = \bigoplus_{x \in \mathbb{Q}} V_x$  induces a morphism

$$D(\mathbb{Q}) \to \mathbb{GL}(V)$$
 given by  $g \mapsto \sum_{x \in \mathbb{Q}} i \circ \chi_x(g) \cdot id_{V_x}$ 

where  $i: \mathbb{G}_m \to \mathbb{GL}(V)$  is the diagonal embedding, and  $\chi_x: D(\mathbb{Q}) \to \mathbb{G}_m$  is the map from 6.3.4.

Likewise, filtrations on  $\omega^G$  induce filtrations on any representations, and one can consider the standard representation on  $V \otimes_k K$ . Reciprocally, consider a filtration on  $V \otimes_k K$ . As a filtration on a finite dimensional vector space, it is necessarily split (as in the image of the morphism fil from 6.3.2). Hence, by the discussion above, it defines a grading on  $\omega^G$  over K. Then the filtration given by 6.3.9 works.

6.3.3. Cocharacters and the flag variety. We still let K/k be an arbitrary field extension, and G a connected reductive algebraic group over k. G may furthermore be assumed to be quasi-split when needed. We fix  $k^{sep}$  a fixed separable closure of k containing K.

**Definition 6.3.11.** A  $\mathbb{Q}$ -one-parameter subgroup of G defined over K (also known as a  $\mathbb{Q}$ -cocharacter, or  $\mathbb{Q}-1-PS$  for short), is a morphism of algebraic groups  $\mathcal{D}_K \to G_K$ .

We let  $X_*(G)^K_{\mathbb{Q}}$  denote the set of such cocharacters.

We call  $X_*(G)^k_{\mathbb{Q}}$  the set of **rational** cocharacters, and  $X_*(G)^{k^{sep}}_{\mathbb{Q}}$  the set of **geometric** cocharacters.

Remark 6.3.12. The literature is quite inconsistent on whether a "cocharacter" should denote a rational cocharacter, or a geometric one.<sup>21</sup> We try to always specify the field of definition.

<sup>&</sup>lt;sup>21</sup>This is partly due to the regrettable fact that most literature of algebraic groups was written in the somewhat outdated language of algebraic varieties - over algebraically closed fields.

By 6.3.4, elements of  $X_*(G)_{\mathbb{Q}}^K$  are equivalent to gradings of  $\omega^G$  over K. In particular, they and hence induce filtrations of  $\omega^G$ , by 6.3.2.

**Definition 6.3.13.** We say that two 1-parameter subgroups  $\lambda, \lambda' \in X_*(G)_{\mathbb{Q}}^K$  are **par-equivalent** if they define the same filtration on  $\omega^G$  over K.

Equivalently, they define the same filtration for any representation  $(V, \rho)$  of G.

For  $\lambda \in X_*(G)_{\mathbb{Q}}^K$ , we let  $P_{\lambda}$  be the subgroup of  $G_K$  defined, for any K-algebra, by :

$$P_{\lambda}(K) = \left\{ g \in G(K), \lim_{t \to 0} \lambda(t) \circ g \circ \lambda(t)^{-1} \text{ exists} \right\}$$

**Proposition 6.3.14.** Assume that G is reductive. Then:

- (1) All filtrations of  $\omega^G$  over K are splittable, i.e. come from a grading  $\omega^{22}$
- (2) For any  $\lambda \in X_*(G)^K_{\mathbb{Q}}$ ,  $P_{\lambda}$  is a parabolic subgroup of  $G_K$ .
- (3) Let  $\lambda, \lambda' \in X_*(G)_{\mathbb{Q}}^K$ , and  $\mathcal{F}_{\lambda}$  (resp.  $\mathcal{F}_{\lambda'}$ ) be the filtration on  $\omega^G$  associated to  $\lambda$  (resp.  $\lambda'$ ). The following are equivalent:
  - (a)  $\mathcal{F}_{\lambda} = \mathcal{F}_{\lambda'}$ , i.e  $\lambda$  and  $\lambda'$  are par-equivalent
  - (b) There exists  $g \in P_{\mathcal{F}_{\lambda}}(K)$  such that  $\lambda' = g \circ \lambda \circ g^{-1}$

*Proof.* cf. [MFK94, Def 2.3/Prop 2.6] and [DOR10, Thm 4.2.13].

Note that, in particular, two par-equivalent  $\lambda, \lambda' \in X_*(G)^K_{\mathbb{O}}$  belong in the same G(k)-conjugacy class.

**Example 6.3.15.** In the  $GL_n$  case, the parabolic subgroup  $P_{\mathcal{F}}$  associated to a filtration of  $\omega^G$  - hence of V - corresponds to the stabilizer of such filtration.

Finally, we turn our attention to conjugacy classes of such  $\mathbb{Q}$ -parameter subgroups, and the associated flag variety. There are two natural and compatible actions on  $X_*(G)^K_{\mathbb{Q}}$ :

• Assume K/k is Galois. The Galois group  $\Gamma_E = K = \operatorname{Gal}(K/k)$  acts, for  $\sigma \in \Gamma_k, \lambda \in X_*(G)_{\mathbb{O}}^K$ , by

$$\sigma \lambda = \sigma_{G_K} \circ \lambda \circ \sigma_{\mathbb{G}_{m,L}}^{-1}$$

where  $\sigma_G$  (resp  $\sigma_{\mathbb{G}_{m,K}}$ ) is the natural action of  $\Gamma_E$  on  $G_K$  (resp.  $\mathbb{G}_{m,K}$ ).

• The group G(K) acts by conjugation, via

$$g\lambda := g \cdot \lambda \cdot g^{-1}$$

where the right hand side uses the natural action of G(K) on  $G_K$ .

Note that there is a natural inclusion  $X_*(G)^k_{\mathbb{Q}} \subset X_*(G)^K_{\mathbb{Q}}$  that is compatible with the action of G. Actually, it follows from Galois descent (cf. [BLR90, Theorem 6.1.6.a and Paragraph 6.2.B]) that :  $(X_*(G)^K_{\mathbb{Q}})^{\mathrm{Gal}(K/k)} = X_*(G)^K_{\mathbb{Q}}$ .

For simplicity (and following the literature), we simply denote  $X_*(G)/G = X_*(G)_{\mathbb{Q}}^{k^{sep}}/G(k^{sep})$  the group of  $G(k^{sep})$ —conjugacy classes of geometric one parameter subgroups. It is a discrete  $\Gamma_k$ -module.

**Remark 6.3.16.** The quotient admits an easier formulation. By general theory (cf. [CGP15, Lemma C.3.5]<sup>23</sup>), we have :

$$X_*(G)/G \cong X_*(T)/W$$

Where T is a maximal torus of G, and  $W = N_G(T)(k^{sep})/Z_G(T)(k^{sep})$  is the absolute Weyl group of G with respect to T.

**Example 6.3.17.** Let  $G = GL_n$ , and T be the maximal torus consisting of diagonal matrices. All cocharacters of T are defined over k, and are of the form  $t \mapsto \operatorname{diag}(t^{a_1}, \ldots, t^{a_n})$  for some  $a_k \in \mathbb{Q}$ .

By the above remark,  $X_*(G)/G \cong X_*(T)/S_n \cong (\mathbb{Q}^n)_+$ , i.e. the set of n-tuples of decreasing rational numbers. This coincides with the possible types of a filtration on  $k^n$ .

 $<sup>^{22}</sup>$ This also holds for general G, provided that k is of characteristic zero.

 $<sup>^{23}</sup>$ Albeit for  $\mathbb{Z}$ -cocharacters, but the situation is similar.

We may now define our flag varieties of interest, that will classify filtrations of  $type \{\mu\}$ . By the above example, this will be replaced by a conjugacy class of rational cocharacters. Those flag varieties will be defined over a finite extension of  $\mathbb{Q}_p$ , as defined below.

Every conjugacy class  $\{\mu\} \in X_*(G)_{\mathbb{Q}}/G$ , is defined over a finite extension of k, called its reflex field:

$$E(G,\{\mu\}) = \overline{k}^{\Gamma_{\{\mu\}}} \text{ where } \Gamma_{\{\mu\}} = \operatorname{Stab}_{\operatorname{Gal}(\overline{k}/k)}(\{\mu\}) = \{g \in \operatorname{Gal}(\overline{k}/k), \forall \lambda \in \{\mu\}, {}^g\lambda(t) \in \{\mu\}\}$$

**Proposition–Definition 6.3.18.** Let G be a connected reductive group over k, and  $\{\mu\}$  be a conjugacy class of geometric cocharacters of G. Let E/k finite be the reflex field of  $\{\mu\}$ . Fix a separable closure  $\overline{E}$  of E.

There exists a unique projective reduced scheme of finite type over E, denoted  $\mathscr{F}(G, \{\mu\})$ , whose K-points, for any intermediary field extension  $\overline{E}/K/E$ , are :

$$\mathscr{F}(G, \{\mu\})(K) = \left\{\lambda \in X_*(G)_{\mathbb{Q}}^K, \lambda \in \{\mu\}\right\} / \text{ par-equivalence}$$
$$= \left\{\mathcal{F} \in \operatorname{Fil}_K(\omega^G), \exists \lambda \in \{\mu\}, \mathcal{F} = \mathcal{F}_{\lambda}\right\}$$

This is called the flag variety associated to  $(G, \{\mu\})$ .

We only give the construction under the assumption that G is quasi-split, which will be the case in our applications.

*Proof.* The equivalence of the two descriptions follows easily from the definition and first point of Prop 6.3.14.

Assume G is quasi-split. Then, a lemma of Kottwitz (cf. [Kot84, Lemma 1.1.3] (see also the proof of [Kot85, Prop 6.2])) tells us that there exists  $\mu \in \{\mu\}$  defined over E. Let  $P_{\mu,E}$  be the parabolic subgroup of  $G_E$  associated to  $\mu$  under 6.3.3. Then  $\mathscr{F}(G, \{\mu\}) = G_E/P_{\mu,E}$  works, by 6.3.14. The unicity follows from the by descent from an algebraically closed field, where reduced schemes of finite types can be viewed as algebraic varieties.

If G is not quasi-split, we refer to the argument of [DOR10].

In the  $\mathbb{GL}(V)$  setup, this coincides with the flag varieties we defined in 6.1.1.

**Example 6.3.19.** Let V be a k-vector space, and  $G = \mathbb{GL}(V)$ .

We already saw in 6.3.10 that  $\operatorname{Fil}_K(\omega^G)$  corresponds to filtrations of  $V \otimes_k K$  in 6.3.10. By 6.3.17, two cocharacters of G are conjugated if and only if they admit the same ordered weights, i.e. if the associated filtrations admit the same type.

Hence the above flag variety coincides with the one of 6.1.1.

We will then define period domains as analytic open subsets of the above flag varieties, corresponding to semi stable *isocrystals*. We first need to define a theory of isocrystals relative to an algebraic group.

6.3.4. Isocrystals with additional structure. Let G a connected reductive algebraic group over  $\mathbb{Q}_p$ . Let  $K_0 = \check{\mathbb{Q}}_p$ .

Recall that the category  $\operatorname{Isoc}(K_0)$  defined in 6.1.2 is naturally endowed with the tensor product  $(V_1, \varphi_1) \otimes (V_2, \varphi_2) := (V_1 \otimes V_2, \varphi_1 \otimes \varphi_2)$ , which makes it a (non neutral!)  $\mathbb{Q}_p$ -linear<sup>24</sup> Tannakian category, whose associated fiber functor is the forgetful functor  $\omega_{\operatorname{Isoc}} : \operatorname{Isoc}(K_0) \to \operatorname{Vect}_{K_0}$ .

**Definition 6.3.20.** An isocrystal with G-structure is a  $\otimes$ -exact functor  $\operatorname{Rep}_{\mathbb{Q}_n}(G) \to \operatorname{Isoc}(K_0)$ .

It follows from the definition that such a functor commutes with fiber functors, since G is connected (cf. [DOR10, Remark 9.1.6]).

<sup>&</sup>lt;sup>24</sup>It is **not**  $K_0$  linear! Endomorphisms of the unit object  $(K_0, id_{K_0})$ , are endomorphisms  $K_0 \to K_0$  commuting with the Frobenius, so that they correspond with elements of  $K_0^{\sigma} = \mathbb{Q}_p$ .

**Example 6.3.21.** Let V be a  $\mathbb{Q}_p$ -vector space of dimension n, and  $G = \mathbb{GL}(V)$  be the group scheme defined by  $R \mapsto GL(V \otimes_{\mathbb{Q}_p} R)$  (so that there is a non canonical isomorphism  $\mathbb{GL}(V) \cong \mathbb{GL}_n$ ).

Through the canonical representation of  $\mathbb{GL}(V)$  acting on V, we can attach to any isocrystal with  $\mathbb{GL}(V)$ structure a (standard) isocrystal over  $V \otimes_{\mathbb{Q}_p} K_0$ .

It follows from the classification of irreducible representations of  $\mathbb{GL}(V)$  via Schur functors that, reciprocally, the structure of an  $\mathbb{GL}(V)$  – isocrystal is fully determined by its value on the standard representation (cf. [RR96, Remark 3.4.(ii)]) - so that  $\mathbb{GL}(V)$ —isocrystals correspond one-to-one with isocrystals structure on  $V \otimes_{\mathbb{O}_n} K_0$ .

Our goal is to construct moduli spaces of such isocrystals with G-structure, which seems hard a priori. Luckily for us, the available Tannakian formalism makes everything explicit.

Any  $b \in G(K_0)$  induces the isocrystal with G-structure  $N_b : (V, \rho) \mapsto (V \otimes_{\mathbb{Q}_p} K_0, \rho_{K_0}(b)(id \otimes \varphi))$ .

**Proposition 6.3.22.** All isocrystals with G-structure are of the form  $N_b$ , for some  $b \in G(K_0)$ . Moreover, there is an isomorphism of exact  $\otimes$ -functors  $N_b \cong N_{b'}$  if and only if there exists  $g \in G(K_0)$  such that  $b' = gb\varphi(g)^{-1}$ .

*Proof.* This relies on Tannakian duality, cf. [DOR10], lemma 9.1.4, and remark 9.1.6.

We define B(G), the *Kottwitz* set of G, as the quotient of  $G(K_0)$  by the above equivalence relation. Hence, B(G) classifies isocrystals with G-structure (by 6.3.21,  $B(\mathbb{GL}_n)$  classifies equivalence classes of standard isocrystals). For any  $b \in G(K_0)$ , we denote [b] its associated class in B(G).

We are now equipped to define period domains.

6.3.5. *Period domains*. Period domains will be admissible open subsets of the flag varieties defined in 6.3.18, corresponding to weakly admissible objects. Let us first define our relative notion of weak admissibility.

**Definition 6.3.23.** Let G be an algebraic group over  $\mathbb{Q}_p$ , and  $b \in G(K_0)$ , which defines the isocrystal with G-structure  $N_b$ . Recall that the data of a point  $\mathcal{F} \in \mathscr{F}(G, \{\mu\})(K) \subset \mathrm{Fil}_K(\omega^G)$  defines a filtration  $\mathcal{F}^\rho$  on  $V \otimes_k K$  for all representations  $(V, \rho)$  of G.

We say that (b, x) is **weakly admissible** if, for all representation  $(V, \rho)$  of G, the filtered isocrystal  $(N_b(V), \mathcal{F}^xV \otimes_k K)$  is weakly admissible, in the sense of 6.1.3.

**Proposition 6.3.24.** This is equivalent to asking that  $(N_b(V), \mathcal{F}^xV \otimes_k K)$  is weakly admissible for some faithful representation of G.

*Proof.* This essentially follows from the tensor product theorem 6.1.3. See also [RZ96, Defi 1.18].  $\Box$ 

We may finally define general period domains. A **local Shimura datum** is the data  $(G, [b], \{\mu\})$  of:

- (1) G, a connected reductive group over  $\mathbb{Q}_p$
- (2) [b] an element of the Kottwitz set B(G), i.e. a  $\sigma$ -conjugacy class inside  $G(K_0)$
- (3)  $\{\mu\}$  a conjugacy class of geometric cocharacters of G, where  $\lambda, \lambda'$  are conjugated if there exists  $g \in G(\overline{\mathcal{K}_0}), \lambda' = g\lambda g^{-1}$

**Definition 6.3.25.** Let  $(G, [b], \{\mu\})$  be a local Shimura datum. Let  $E = E(G, \{\mu\})$  be the associated reflex field, and  $E = E \cdot K_0$ .

There exists a unique partially proper open subset

$$\check{\mathscr{F}}^{wa}(G,b,\{\mu\})\subset\mathscr{F}(G,\{\mu\})\otimes_E\check{E}$$

Such that, for any field extension  $K/\check{E}$ 

$$\check{\mathscr{F}}^{wa}(G,b,\{\mu\})(K) = \{\mathcal{F} \in \mathscr{F}(G,\{\mu\})(K), \text{ the filtered isocrystal } (N_b,\mathcal{F}) \text{ is weakly admissible}\}$$

We call it the **period domain** associated to  $(G, b, \{\mu\})$ .

Moreover,  $\check{\mathscr{F}}^{wa}(G,b,\{\mu\})$  can be obtained as the analytification of a Berkovich space.

*Proof.* Cf. [RZ96, Prop 1.36] in the rigid analytic setup, and [DOR10, Prop 9.5.3] in the Berkovich setup.  $\Box$ 

Note that, up to isomorphism, the period domain only depends on the class  $[b] \in B(G)$ . Indeed, if  $b' = gb\sigma(g)^{-1}$  for some  $g \in G(K_0)$ , the map  $\mathcal{F} \mapsto Int(g) \cdot \mathcal{F}$  induces an isomorphism  $\check{\mathscr{F}}^{wa}(G, b, \{\mu\}) \cong \check{\mathscr{F}}^{wa}(G, b', \{\mu\})$ .

Let  $J_b$  denote the automorphims group of  $N_b$ . The group  $J_b(\mathbb{Q}_p)$  acts naturally on  $\check{\mathscr{F}}^{wa}(G, b, \{\mu\})$ , via  $J_b(\mathbb{Q}_p) \subset G(K_0)$  via conjugation;  $\mu \mapsto \gamma \mu \gamma^{-1}$  for  $\gamma \in J_b(\mathbb{Q}_p)$ , where  $\mu$  is a cocharacter

**Remark 6.3.26.** Note that such period domains are automatically paracompact by [Har13, Lemma A.3]. Now, as hinted earlier, we'll show that, under a very weak assumption, such period domains can actually be defined over a finite extension of  $\mathbb{Q}_n$ .

6.3.6. The slope morphism and decency. Let us say a bit about the additional structure on B(G). Let  $\mathbb{D}_{K_0}$  be the pro-torus defined over  $K_0$ . Recall that there is an equivalence  $\operatorname{Rep}_{K_0}(\mathbb{D}_{K_0}) \cong \mathbb{Q} - \operatorname{Grad}_{K_0}$ . For any  $b \in G(\mathbb{Q}_p)$ , Kottwitz defines a slope morphism  $\nu_b : \mathbb{D}_{K_0} \to G_{K_0}$  characterized by the following property (cf. [Kot85, sec 4.2]):

**Definition 6.3.27.** For any  $V \in \operatorname{Rep}_{\mathbb{Q}_p}(G)$ , the  $\mathbb{Q}$ -grading on the vector space  $V \otimes_{\mathbb{Q}_p} K_0$  associated with the representation of the pro-torus  $\mathbb{D}_{K_0} \xrightarrow{\nu_b} G_{K_0} \xrightarrow{\rho_{K_0}} \operatorname{GL}(V \otimes K_0)$  coincides with the (Dieudonné-Manin) slope decomposition of the isocrystal  $N_b = (V \otimes K_0, \rho_{K_0}(Id \otimes \varphi))$ .

Note that, if  $b' = gb\sigma(g)^{-1}$  (where  $\sigma(g)$  denotes the natural Frobenius on  $G(K_0)$ ), then  $\nu_{b'} = \operatorname{Int}(g) \circ \nu_b$ .

**Definition 6.3.28.** We say that an element  $b \in G(K_0)$  is **s-decent** for an integer  $s \ge 1$  if:

- The morphism  $s \cdot \nu_b : \mathbb{D}_{K_0} \to G_{K_0}$  factors through  $\mathbb{G}_{m,K_0}$
- The following equality holds in  $G(K_0) \rtimes \sigma^{\mathbb{Z}} : (b\sigma)^s = (s\nu_b)(p)\sigma^s$

Since we are working over  $\mathbb{Q}_p$  and G is connected, every class  $[b] \in B(G)$  contains an s-decent element, for some  $s \ge 1$  (cf. [DOR10, lemma 9.1.33, remark 9.1.34]).

By [DOR10, lemma 9.6.19], if G is furthermore assumed to be quasi-split, each  $[b] \in B(G)$  contains an s-decent element b such that  $\nu_b$  is defined over  $\mathbb{Q}_p$ .

**Remark 6.3.29.** One could define period domains over a finite extension F of  $\mathbb{Q}_p$ . Most of the theory still holds, but crucially, the above statements do not, as they require  $\check{F}$  to be of the form  $\operatorname{Frac}(W(L))$  for some algebraically closed L.

**Proposition 6.3.30.** Let  $(G, b, \{\mu\})$  be a local Shimura datum as above. Let  $s \ge 1$ , and assume that b is s-decent. Then, the period domain  $\check{\mathscr{F}}^{wa}(G, b, \{\mu\})$  admits a model over  $E \cdot \mathbb{Q}_{p^s}$ 

We still need one last technical hypothesis for Orlik's geometric decomposition to hold.

6.3.7. Basic elements.

**Definition 6.3.31.** We say that an element  $b \in B(K_0)$  is **basic** if the slope morphism  $\nu_b$  factors through  $Z_{K_0}$ , where Z denotes the center of G. We let  $B(G)_{basic}$  denote the subset of B(G) consisting of basic elements.

If  $G = \mathbb{GL}(V)$ , B(G) consists of equivalent classes of isocrystals (given by the Dieudonné-Manin theorem).  $B(G)_{basic}$  consists of such *isoclinic* isocrystals, i.e. the ones of the form  $E_{\lambda}^{\oplus n}$ 

**Proposition 6.3.32.** The following are equivalent:

• [b] is basic

 $<sup>^{25}</sup>$ A more modern formulation of B(G) presents it as a set of isomorphism classes of G-bundles on the Fargues-Fontaine curve. We do not develop this point of view, but refer the reader to [CDHN21, Sect 3.1.3] or [CFS17].

- The automorphism group  $J_b$  of the isocrystal  $N_b$ , defined by  $J_b(R) = \{g \in G(K_0 \otimes_{\mathbb{Q}_p} R), gb\sigma(g)^{-1} = b\}$ , is an inner form of G
- The associated vector bundle  $\mathcal{E}_b$  on the Fargues-Fontaine curve is semistable.

Let us now study their cohomology with compact support.

### 7. Primitive comparison theorem with compact support for period domains

The goal of this section is to show that period domains, as defined in the previous paragraph, satisfy primitive comparison with compact support (the main result is 7.0.1). This computation is based on standard sheaf techniques, together with a non-trivial geometric stratification of the complement  $\mathscr{F} \setminus \mathscr{F}^{wa}$  of period domains, constructed by S.Orlik in [Orl05a].

In the above article, Orlik uses his geometric decomposition to compute the étale cohomology with compact support of period domains with  $\mathbb{Z}/\ell^n\mathbb{Z}$  coefficients. In [CDHN21], the authors adapt the computation for  $\ell=p$ . While the geometric part of the argument does not really depend on the coefficients, working with p-adic coefficients adds a lot of representation-theoretic complexity, as one needs to compute some Ext<sup>1</sup> groups between Steinberg representations.

In this section, we compute the  $\mathcal{O}^+/p$  cohomology with compact support of period domains. Going from  $\mathbb{F}_p$  to  $\mathcal{O}^+/p$ -coefficients adds no representation theoretic complexity, as only the "geometric" part of the argument is needed.

In this article, we completely black-box the origin of the geometric decomposition (which uses beautiful ideas in geometric invariant theory, cf. [MFK94]), and directly study the consequences for period domains, as they will be recapitulated in Prop 7.1.1.

Compared to the previous expositions of the geometric decomposition and its consequence on cohomology appearing in the literature, we provide the following technical improvements:

- (1) Throughout most of the computation, we work with coefficients in a general abelian étale sheaf. Doing so, we remove a superfluous overconvergence hypothesis that appears in the literature.
- (2) We work directly over the field of definition of the period domains, rather than over a complete algebraic closure, which allows to properly keep track of the Galois action.
- (3) We work with the language of locally spatial diamonds throughout, rather than pseudo-adic spaces.

Let us say a few words about the last point. Our first step will be to use excision, and try to show primitive comparison theorem for the complement  $\mathscr{F} \setminus \mathscr{F}^{wa}$ . This complement needs not admit a rigid-analytic (nor analytic adic) structure over  $E_s$  - as can already been seen for Drinfeld's upper half plane. In order to circumvent this issue, [Hub96] develops a theory of so called pseudo-adic spaces, and shows that such a complement necessarily admits a pseudo-adic structure. He develops a theory of étale cohomology for pseudo-adic spaces, which is well behaved. Sadly, this theory has become somehow outdated, and has been overshadowed by the theory of diamonds and v-stacks, due to Scholze in [Sch17]. While general pseudo-adic spaces do not admit a structure of a diamond (not a v-stack), we'll show that we may view such complement  $\mathscr{F} \setminus \mathscr{F}^{wa}$  - as well as all closed subsets appearing in the stratification - as a locally spatial diamond, which also admit a well behaved étale site. Then, all the arguments can be directly translated in the setup of diamonds.

Let us now precisely state our result:

We fix a local Shimura datum  $(G, [b], \{\mu\})$ , where G is furthermore assumed to be quasi-split, b is basic and s-decent. For simplicity, we'll simply denote  $\mathscr{F} = \mathscr{F}(G, b, \{\mu\})$ , and  $\mathscr{F}^{wa} = \mathscr{F}^{wa}(G, b, \{\mu\})$ , and  $\mathscr{F}_{E_s} = \mathscr{F} \times_{\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)} \operatorname{Spa}(E_s, \mathcal{O}_{E_s})$ . We let  $Y = \mathscr{F} \setminus \mathscr{F}^{wa}$ , viewed as a locally spectral diamond over  $E_s$  by 3.1.3.

In the rest of this section, every pullback, pushforward, ect. is in the sense of Huber, and is underived, unless specified otherwise.

**Theorem 7.0.1.** In the above setup, let  $j: \mathscr{F}^{wa} \to \mathscr{F}_{E_s}$  be the open immersion, and  $g: \mathscr{F}^{wa} \to \operatorname{Spa}(E_s, \mathcal{O}_{E_s})$  the structure morphism.

Let  $\mathcal{L}$  be an étale  $\mathbb{F}_p$ -local system on  $\mathscr{F}_{E_s}$ . Then,  $\mathscr{F}^{wa}$  satisfies Poincaré duality with respect to  $j^*\mathcal{L}$ , i.e, for any  $0 \leq k \leq 2d$ , there is a  $J(\mathbb{Q}_p) \times \operatorname{Gal}(\overline{E_s}/E_s)$ -equivariant isomorphism:

$$H^{2d-k}(\mathscr{F}_C,j^*\mathcal{L})\cong \mathrm{Hom}_{\mathbb{F}_p}(H^{2d-k}_{\acute{e}t,c}(\mathscr{F}_C,j^*\mathcal{L}),\mathbb{F}_p)$$

Where d denotes the dimension of  $\mathscr{F}$ .

We'll prove this using 5.2.3, so that this reduces to a primitive comparison statement for the complementary. Let  $Y = \mathscr{F}^{wa} \setminus \mathscr{F}_{E_s}$  denote the closed complement, viewed as a locally spatial diamond. Throughout this section, we work relatively over  $E_s$ , since this adds virtually no difficulty.

**Proposition 7.0.2.** Let  $h: Y \to \operatorname{Spa}(E_s, \mathcal{O}_{E_s})$  denote the structure morphism. Let  $\mathcal{L}$  be an étale  $\mathbb{F}_p$ -local system on Y. Then, for all  $q \geq 0$ , the natural map:

$$R^q h_* \mathcal{L} \otimes \mathcal{O}_{E_s}^+ \to R^q h_* (\mathcal{O}_Y^+/p \otimes_{\mathbb{F}_p} \mathcal{L})$$

is an almost isomorphism in the category of étale  $\mathcal{O}_{E_s}^+/p$ -sheaves over  $\operatorname{Spa}(E_s, \mathcal{O}_{E_s})$ .

In order to prove this, we'll need a finer understanding of the geometric structure of Y.

#### 7.1. Orlik's geometric decomposition

In this section, we present the geometric decomposition of the complement of period domains, as is due to Orlik in [Orl05a] (see also [Orl99] for the simpler analogue over finite base fields).

We let  $\Delta$  be a basis of the relative root system of G, which is a finite set. All of the useful properties are recapitulated in the following proposition:

**Proposition 7.1.1.** There exists a finite set of rigid-analytic subvarieties  $Y_I \subset Y$  indexed by subsets  $I \subset \Delta$  of a basis of a root system of G, together with a family of parabolic subgroups  $P_I \subset J$  for  $I \subset \Delta$  such that there is a stratification:

$$|Y| = \bigcup_{|\Delta \setminus I| = 1} \bigcup_{g \in J(\mathbb{Q}_p)} g \cdot |Y_I|$$

Moreover, we have the following:

- (1) For any  $I \subset \Delta$ ,  $Y_I$  is the analytification of a proper algebraic variety over  $E_s$  (that is moreover a Schubert variety).
- (2) The family  $(P_I)_{I\subset\Delta}$  is increasing. Moreover,  $P_{\emptyset}$  is a minimal parabolic subgroup, and  $P_{\Delta}=J$ .
- (3) For any  $I, J \subset \Delta$ ,  $Y_I \cap Y_J = Y_{I \cap J}$ .
- $(4) \quad (J/P_I)(\mathbb{Q}_p) = J(\mathbb{Q}_p)/P_I(\mathbb{Q}_p).$
- (5) The action of  $J(\mathbb{Q}_p)$  on  $\mathscr{F}$  restricts to an action of  $P_I(\mathbb{Q}_p)$  on  $\mathscr{F}^{wa}$ .

*Proof.* Cf. [Orlo5a, Section 3] of [CDHN21, Section 5.4].

From the above stratification, we'll be able to construct a functorial resolution  $\mathcal{C}(\mathcal{F})$  of any sheaf of abelian group  $\mathcal{F}$  on  $Y_{\acute{e}t}$ , relating the cohomology of  $\mathcal{F}$  on Y to the cohomology of  $\mathcal{F}$  restricted to the  $Y_I$ , for varying  $I \subset \Delta$ . This exactness was only considered in [Orl05a] for overconvergent étale sheaves  $\mathcal{F}$ , but, as we'll see, this assumption is not needed, so that it also works for  $\mathcal{F} = \mathcal{O}_{\mathcal{F}}^+/p \otimes i^*\mathcal{L}$ .

In order to go deduce a resolution from the topological stratification considered above, we need the following lemma :

**Lemma 7.1.2.** For any compact open subset  $T \subset J/P_I(\mathbb{Q}_p)$ , the closed subspace  $Z_I^T = \bigcup_{g \in T} g \cdot Y_I$  of  $\mathscr{F}^{wa}$  admits a natural structure of a closed locally spatial sub-diamonds of Y.

*Proof.* Fix a compact open  $T \subset X_I$ . By 3.1.3, it suffices to show that the underlying topological space of  $Z_I^T$  is closed and generalizing. It's closed by the proof of [Orlo5a, Lemma 3.2]. All of the  $g \cdot Y_I$  are generalizing since they are Zariski closed subsets, and a union of generalizing spaces remains generalizing.

The strategy to prove primitive comparison with compact support for Y is now as follows:

- (1) From the functorial resolution  $C(\mathcal{F})$ , we construct a spectral sequence, consisting of spaces relating to the cohomology of  $\mathcal{F}$  on spaces relating to the  $Y_I$ , and converging to the cohomology of  $\mathcal{F}$  on the Y in a way that is functorial in the sheaf  $\mathcal{F}$ .
- (2) Since the  $Y_I$  are Zariski closed subsets, admit the structure of a proper rigid-analytic variety, the primitive comparison theorem applies. Hence, we can directly compare the  $\mathcal{O}^+/p$ -cohomology of all the  $Y_I$  with their  $\mathbb{F}_p$ -cohomology<sup>26</sup>.
- (3) We show that the almost isomorphism goes through the convergence of the spectral sequence.

In order to put this plan into action, let us first introduce some more notation.

# 7.2. A Family of Auxiliary Sheaves

For any  $I \subset \Delta$ , we let  $X_I = J(\mathbb{Q}_p)/P_I(\mathbb{Q}_p)$ . By the third point of 7.1.1, this is a profinite set - as it identifies to the  $\mathbb{Q}_p$ -points of a proper algebraic variety (which can then seen as  $\mathbb{Z}_p$ -points by the valuation criterion for properness).

For any compact open subset  $T \subset X_I$ , we let  $Z_I^T = \bigcup_{g \in T} g \cdot Y_I$ , which is a closed generalizing subspace of  $Y_C$  by 7.1.2. We denote  $i_T : Z_I^T \to Y$  the natural closed immersion.

**Definition 7.2.1.** We let  $\operatorname{Disj}_{X_I}$  be the category of disjoint nonempty open covers of  $X_I$ , with morphisms given by refinements. Such disjoint covers are necessarily finite, as  $X_I$  is compact.

Objects of  $\operatorname{Disj}_{X_I}$  will be denoted  $c = \{T_k\}_{1 \leq k \leq n(c)}$ .

For any 
$$c \in \operatorname{Disj}_{X_I}$$
 and any étale sheaf  $\mathcal F$  on  $Y$ , we let  $\mathcal F_c := \bigoplus_{k=1}^{n(c)} i_{T_k*} i_{T_k}^* \mathcal F$ , and  $\mathcal F_I := \varinjlim_{c \in \operatorname{Disj}_{X_I}} \mathcal F_c$ .

Note that, if  $\mathcal{F}$  is  $J_I(\mathbb{Q}_p)$ -equivariant, then the sheaves  $\mathcal{F}_I$  are  $J_I(\mathbb{Q}_p)$ -equivariant.

The sheaf  $\mathcal{F}_c$  admits a natural interpretation as the étale sheaf on  $X_I$  whose sections are locally constant sections of  $\mathcal{F}$ , trivialized on the disjoint covering c. Likewise,  $\mathcal{F}_I$  can be thought of as the sheaf of *locally constant* sections of  $\prod_{x \in X_I} \mathcal{F}_{xY_r^{ad}}$ . This is made somewhat more explicit in [Orlo5a, p.538].

Remark 7.2.2. Let us briefly explicit the transition morphisms appearing in the colimit.

Let  $c = \{T_k\}_{1 \le k \le n(c)}$  be an object of  $\mathrm{Disj}_{X_I}$ , and  $c' = \{S_{k,l}\}_{1 \le k \le n(c); 1 \le l \le m(k)}$  be a refinement of c, in the sense that, for any  $1 \le k \le n(c)$ , the  $(S_{k,l})_{1 \le l \le m(k)}$  form a disjoint compact open covering of  $T_k$ .<sup>27</sup>

The morphism  $\mathcal{F}_{c'} \to \mathcal{F}_c$  is then given as a direct sum of the natural morphisms  $i_{S_{k,l}*}i_{S_{k,l}}^*\mathcal{F} \to i_{T_k*}i_{T_k}^*\mathcal{F}$ , induced by the unit of the adjunction fact that, if  $S \subset T$ , are clopen subsets of  $X_I$ , there is a natural isomorphism

$$i_{S*}i_S^*\mathcal{F} \cong i_{S*}i_S^*i_{T*}i_T^*\mathcal{F}$$

Indeed, such a morphism is induced by the unity of the adjunction  $\mathcal{F} \to i_{T*}i_T^*\mathcal{F}$ , and one can check that it is an isomorphism on stalks, using 3.2.2.

Let us study the sheaves  $\mathcal{F}_I$ . We start with the following lemma (which is a version of [CDHN21, Prop 6.11.(a)] over an arbitrary base field, and in the language of diamonds).

**Lemma 7.2.3.** Let  $\overline{x}$  be a geometric point of  $Y_{\acute{e}t}$ , with underlying topological point x, and  $\mathcal{F}$  be an étale sheaf of abelian groups over Y.

The stalk of  $\mathcal{F}_I$  at  $\overline{x}$  identifies with

$$(\mathcal{F}_I)_{\overline{x}} \cong LC(X_I(x), \mathcal{F}_{\overline{x}})$$

where  $X_I(x) = \{g \in X_I, x \in g \cdot Y_I\}$  is a closed subset of  $X_I$ , and LC denotes the set of locally constant functions on  $X_I(x)$  (with the topology induced by  $X_I$ ), with value in the stalk  $\mathcal{F}_{\overline{x}}$ , with the discrete topology.

<sup>&</sup>lt;sup>26</sup>Here, the primitive comparison theorem will suffice, but we can really explicit  $\mathbb{F}_p$ -cohomology groups of the  $Y_I$  (at least after base change to an algebraically closed base); cf. [CDHN21, Corrolary 5.11]

<sup>&</sup>lt;sup>27</sup>Note that, since coverings are disjoint, if such a refinement exists, there is no choice for the refinement map.

*Proof.* Taking stalks commutes with colimits, so that:

$$(\mathcal{F}_I)_{\overline{x}} \cong \varinjlim_{c \in \operatorname{Disj}_{X_I}} \bigoplus_{k=1}^{n(c)} (i_{T_k,*}i_{T_k}^*\mathcal{F})_{\overline{x}}$$

Moreover, by 3.2.2, since  $i_{T_k}$  is a generalizing closed immersion :

$$(i_{T_k*}i_{T_k}^*\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & \text{if } x \in T_k \cdot Y_I, \text{ i.e. if } T_k \cap X_I(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We may drop from all the zero terms from the direct sum.

If  $c = \{T_k\}_{1 \le k \le n(c)} \in \mathrm{Disj}_{X_I}$ , the  $T_k \cap X_I(x)$  form a disjoint compact open cover of  $X_I(x)$ . Reciprocally, any  $c' \in \mathrm{Disj}_{X_I(x)}$  can be refined by a covering of the above form, by the topological lemma below.

Hence, such coverings form a cofinal system in  $\mathrm{Disj}_{X_I(x)}$ , so that :

$$(\mathcal{F}_I)_{\overline{x}} = \varinjlim_{c \in \overrightarrow{\mathrm{Disj}}_{X_I}} \bigoplus_{k=1}^{n(c)} \left( i_{T_k \cap X_I(x)*} i_{T_k \cap X_I(x)}^* \mathcal{F} \right)_{\overline{x}} \cong \varinjlim_{c' \in \overrightarrow{\mathrm{Disj}}_{X_I(x)}} \bigoplus_{k=1}^{n(c')} \mathcal{F}_{\overline{x}} \cong LC(X_I(x), \mathcal{F}_{\overline{x}})$$

Here, the last isomorphism follows from identifying a locally constant function to the set of its value over any disjoint trivializing (cl)open cover (that is necessarily finite since  $X_I(x)$  is compact).

In the proof above, we used the following topological lemma:

**Lemma 7.2.4.** Let X be a profinite space, and  $F \subset X$  be a closed subset.

Let  $c' = \{T_k\}_{1 \leq k \leq n(c')} \in \text{Disj}_F$ . Then, there exists  $c = \{S_{k,l}\}_{1 \leq k \leq n(c), 1 \leq l \leq m(k)} \in \text{Disj}_X$  such that, for all k, the  $\{S_{k,l} \cap T\}_{1 < l < m(k)}$  form a disjoint clopen union of  $T_k$ .

*Proof.* Every  $T_k$  is an open of F, so it can be written as  $T_k = U_k \cap F$  for some open  $U_k$  of X. Consider the open cover  $(X \setminus F) \cup \{U_k\}_{1 \leq k \leq n(c)}$  of X. By [Sta23, Lemma 08ZZ], we may find a refinement by a clopen disjoint cover  $S'_0 \sqcup \{S'_{k,l}\}_{1 \leq k \leq n(c), 1 \leq l \leq m(k)}$ , where the  $S'_{k,l} \subset U_k$  for  $k \geq 1$ , and  $S'_0 \cap F = \emptyset$ .

We then let  $S_{k,l} = S'_{k,l} \cap T \subset U_k \cap T = T_k$ , which are closed subsets of T. Moreover, we can write  $S_{k,l} = T \setminus \bigsqcup_{(k',l')\neq(k,l)} S_{k,l}$ , so that they also are open in T.

Pick your favorite  $(k_0, l_0)$ . After replacing  $S_{k_0, l_0}$  by  $S_{k_0, l_0} \sqcup S'_0$ , the family  $\{S_{k, l}\}$  satisfies all desired conditions.

# 7.3. The fundamental complex

We may now define the fundamental complex, promised in 7.1.1.

**Proposition 7.3.1.** Let  $\mathcal{F}$  be a sheaf of abelian groups on  $Y_{\acute{e}t}$ . The following complex of étale sheaves on  $Y_{\acute{e}t}$  is exact:

$$C(\mathcal{F}): 0 \to \mathcal{F} \to \bigoplus_{|\Delta \setminus I|=1} \mathcal{F}_I \to \bigoplus_{|\Delta \setminus I|=2} \mathcal{F}_I \to \cdots \to \bigoplus_{|\Delta \setminus I|=|\Delta|-1} \mathcal{F}_I \to F_\emptyset \to 0$$

Let us now define the morphisms. Fix  $I, I' \subset \Delta$  with  $|I' \setminus I| = 1$ , The inclusion  $P_I \subset P_{I'}$  induces a quotient map  $p_{I,I'}: X_I \to X_{I'}$ . For any  $x \in X_{I'}$  and  $y \in X_I$ , we consider a map  $F_{xY_{I'}} \to F_{yY_I}$ , simply being the zero map whenever  $p_{I',I}(y) = x$ , and induced by the natural inclusion  $xY_I \subset p_{I,I'}(x) \cdot Y_{I'} y \in X_I$  otherwise. This induces a map  $f_{I',I}: \mathcal{F}_{I'} \to \mathcal{F}_I$ .

Finally, recall that the root system  $\Delta$  is naturally ordered, so that we may write  $I' = \{\alpha_1 < \cdots < \alpha_n\}$ , and  $I = I' \setminus \{\alpha_i\}$ . We then let  $d_{I,I'}I = (-1)^i p_{I,I'} : \mathcal{F}_{I'} \to \mathcal{F}_{I}$ , and consider the direct sums.

When  $\mathcal{F}$  is a sheaf of  $\mathbb{Z}/n\mathbb{Z}$ —modules for some n prime to p, this is due to [Orl05a, Thm 3.3]. This has been generalized in [Orl05b, Thm 2.1] (see also [CDHN21, Thm 6.13]) for  $\mathcal{F}$  an overconvergent étale sheaf of abelian groups. We'll show that a similar argument works for arbitrary étale sheaves.

*Proof.* The étale topos of Y has enough points by [Sch17, Prop 14.3], so that it suffices to check the exactness on stalks. Let  $\eta : \overline{x} \to Y$  be a geometric point, and  $x \in Y$  the image of the unique closed point. By 7.2.3, the localization at  $\overline{x}$  is:

$$C(\mathcal{F}_{\overline{x}}): 0 \to \mathcal{F}_{\overline{x}} \to \bigoplus_{|\Delta \setminus I|=1} LC(X_I(x), \mathcal{F}_{\overline{x}}) \to \cdots \to \bigoplus_{|\Delta \setminus I|=|\Delta|-1} LC(X_I(x), \mathcal{F}_{\overline{x}}) \to LC(X_{\emptyset}(x), F_{\overline{x}}) \to 0$$

However, by [Sch19, Thm 5.4] (result attributed to Bergman), all locally constant modules are free over  $\mathcal{F}_{\overline{x}}$ , so that  $C(\mathcal{F}_{\overline{x}}) = C(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{F}_{\overline{x}}$ . Since free  $\mathbb{Z}$ -modules are flat, it suffices to show that  $C(\mathbb{Z})$  is exact. This is done in [Orl05b, Thm 2.1].

**Remark 7.3.2.** Note that the above proof also shows that, for any two sheaves  $\mathcal{F}$ ,  $\mathcal{G}$  of abelian groups on  $Y_{\acute{e}t}$ , we have  $\mathcal{C}(\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}) \cong \mathcal{C}(\mathcal{F}) \otimes_{\mathbb{Z}} \mathcal{G}$ .

Remark 7.3.3. The proof of [Orl05a, Thm 3.3] can in fact easily be adapted to work in that setup. Indeed, Orlik shows that the above localized complex  $C(\mathcal{F}_{\overline{x}})$  computes the cohomology of the constant sheaf<sup>28</sup> associated to  $\mathcal{F}_{\overline{x}}$  on some topological space, that is obtained as the geometric realization of some simplicial subcomplex of the combinatorial Tits building of J, which can be shown to be contractible and hence has trivial cohomology for arbitrary constant coefficients.

From the resolution  $\mathcal{C}(\mathcal{F})$  of  $\mathcal{F}$  on the étale site  $Y_{\acute{e}t}$ , we'll construct a spectral sequence, by applying  $Rh_*\mathcal{C}(\mathcal{F})$ , where  $h:Y\to \operatorname{Spa}(E_s,\mathcal{O}_{E_s})$  is the structure morphism of Y. Considering the filtration on  $\mathcal{C}(\mathcal{F})$  (minus the  $\mathcal{F}$  term) obtained by truncations, we deduce a spectral sequence:

$$E_1^{p,q}(\mathcal{F}) := R^q h_* \left( \bigoplus_{|\Delta \setminus I| = p+1} \mathcal{F}_I \right) \implies R^{p+q} h_* \mathcal{F} =: E_{\infty}^{p+q}(\mathcal{F})$$

The above spectral sequence is clearly functorial in  $\mathcal{F}$ . We want to compare the cohomology of  $\mathcal{F}$ , which is computed by the abutment of the above spectral sequence. In order to do so, we'll show an identification of the first page, as below:

**Proposition 7.3.4.** Let  $p, q \in \mathbb{N}$ , and  $\mathcal{L}$  be a local system on Y. Then, the natural morphisms:

$$R^q h_* \left( \bigoplus_{|\Delta \setminus I| = p+1} (\mathcal{L})_I \right) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}/p \to R^q h_* \left( \bigoplus_{|\Delta \setminus I| = p+1} \left( \mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p \right)_I \right)$$

are almost isomorphisms.

Before anything else, let us prove that it implies our main result.

**Proposition 7.3.5.** Assume that the result of the above proposition holds. Then, for all  $q \ge 0$ , the natural morphism

$$R^q h_* \mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \to R^q h_* (\mathcal{L} \otimes \mathcal{O}_Y^+/p)$$

is an almost isomorphism. Hence 7.0.2 holds (which itself implies our main theorem 7.0.1).

*Proof.* There is a natural morphism of complexes of sheaves on  $Y_{\acute{e}t}$ :

$$\mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} h^* \mathcal{O}_{E_s}^+ / p \to \mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p \cong \mathcal{C}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+ / p)$$

Where the isomorphism follows from 7.3.2. This induces a morphism of complexes on  $\operatorname{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}$ :

$$R^q h_* \mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \to R^q h_* \left( \mathcal{C}(\mathcal{L}) \otimes_{\mathbb{F}_p} h^* \mathcal{O}_{E_s}^+/p \right) \to R^q h_* \mathcal{C}(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p)$$

Where the first map is obtained by adjunctions (and is the morphism from the projection formula). This induces a morphism of spectral sequences  $E_k^{p,q}(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_e}^+/p \to E_k^{p,q}(\mathcal{O}_Y^+/p)$ .

Recall that, a morphism of spectral sequences valued in any abelian category and inducing an isomorphism at a given page induces an isomorphism at the level of abutments. All morphisms above commute with

 $<sup>^{28}</sup>$ Note that, in that setup, constant sheaf cohomology needs not coincide with singular cohomology

the projection to the almost category of sheaves of almost  $\mathcal{O}_{E_s}^+/p$ -modules on  $E_s$ , which is a an abelian category (as it is a Serre quotient). This concludes.

Let us rewrite the terms appearing in the proposition.

Let  $\mathcal{F}$  be an arbitrary étale sheaf of abelian groups on Y. Since Y is proper, it necessarily qcqs, so that the morphism  $Y \to \operatorname{Spa}(E_s, \mathcal{O}_{E_s})$  is a coherent morphism of algebraic topoi (cf. [Sch17, Section 8]) . Then, by [GV72, Thm 5.1],  $R^q f_*$  commutes with filtered inductive limits.

Direct sums and the colimit along  $\mathrm{Disj}_{X_I}$  are filtered, so that, for all  $p,q\in\mathbb{N}$ :

$$R^{q} f_{*} \left( \bigoplus_{|\Delta \setminus I| = p+1} \mathcal{F}_{I} \right) \cong \bigoplus_{|\Delta \setminus I| = p+1} R^{q} f_{*} \mathcal{F}_{I} \cong \bigoplus_{|\Delta \setminus I| = p+1} \varinjlim_{c \in \operatorname{Disj}_{X_{I}}} \bigoplus_{i=1}^{n(c)} R^{q} f_{*} \left( i_{T_{k}} i_{T_{k}}^{*} \mathcal{F} \right)$$

The map  $i_{T_k}$  is a closed immersion, hence, by [Sch17, Lemma 21.13],  $R^q f_* \circ i_{T_k *} = R^q (f \circ i_{T_k})_*$ . Hence, we reduced to prove the following:

**Lemma 7.4.1.** For all  $p, q \in \mathbb{N}$ , the morphism

$$\varinjlim_{c \in \overrightarrow{\mathrm{Disj}}_{X_I}} \bigoplus_{k=1}^{n(c)} R^q f_*(i_{T_k*}i_{T_k}^*\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \to \varinjlim_{c \in \overrightarrow{\mathrm{Disj}}_{X_I}} \bigoplus_{k=1}^{n(c)} R^q f_*(i_{T_k*}i_{T_k}^*(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p))$$

is an almost isomorphism

The key idea will be to view the  $U \mapsto R^q f_*(i_{U*}i_U^*\mathcal{F})$  as presheaves on the profinite space  $X_I$ , and  $\lim_{C \in \operatorname{Disj}_{X_I}} \bigoplus_{i=1}^{n(c)} R^q f_*\left(i_{T_k*}i_{T_k}^*\mathcal{F}\right)$  as a form of sheafification of the above presheaves. Proving such an isomorphism of sheaves on  $X_I$  reduced to proving an isomorphism on stalks, which will compute the cohomology of  $Z_I$ , which is proper and admits a rigid-analytic structure - so that the primitive comparison theorem 5.1.1 applies.

**Definition 7.4.2.** We let, for all  $T \subset X_I$  and  $q \geq 0$ :

$$\mathcal{S}^q(\mathcal{F})(T) := R^q f_*(i_{T*}i_T^*\mathcal{F}) \in \operatorname{Sh}(\operatorname{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}, \operatorname{Ab})$$

We let  $\mathcal{B}$  denote the set of all clopen subsets of  $X_I$ . Since  $X_I$  is profinite, this forms a basis of the topology on  $X_I$  by [Sta23, Lemma 08ZZ].

Recall the notion of (pre)sheaves on a basis, as in [Sta23, Section 009H]. In this formalism, the  $S^{j}(\mathcal{F})$  define a presheaf on  $\mathcal{B}$  valued in the abelian category  $\operatorname{Sh}(\operatorname{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}, \operatorname{Ab})$ . Note that we may also view  $\mathcal{O}_{E_s}^+/p$  as the constant presheaf on  $\mathcal{B}$  on  $X_I$ , valuing  $\mathcal{O}_{E_s}^+/p \in \operatorname{Sh}(\operatorname{Spa}(E_s, \mathcal{O}_{E_s})_{\acute{e}t}, \operatorname{Ab})$ .

Let us start by two quick lemmas regarding sheaves on profinite spaces.

**Lemma 7.4.3.** Let X be a profinite topological space, and  $\mathcal{B}$  be the basis formed by clopen subsets of  $X_I$ . Let  $\mathcal{A}$  be an abelian category, and  $\mathcal{G}$  be an  $\mathcal{A}$ -valued presheaf on  $\mathcal{B}$ .

Then, the presheaf on  $\mathcal{B}$  defined, for any  $U \in \mathcal{B}$  by  $\mathcal{G}^+(U) = \varinjlim_{c \in \text{Disj}_U} \bigoplus_{k=1}^{n(c)} \mathcal{G}(T_k)$  defines a  $\mathcal{B}$ -sheaf on X, valued in  $\mathcal{A}$  (in the sense of [Sta23, Definition 009J]).

*Proof.* We'll apply the criterion [Sta23, Lemma 009L]. For any  $U \in \mathcal{B}$ , the set of disjoint clopen coverings is cofinal<sup>29</sup> inside the coverings of U by elements of  $\mathcal{B}$ .

Hence, it suffices to check the sheaf condition relative to a disjoint clopen covering  $U = \bigsqcup_{i \in I} U_i$ . Since the covering is disjoint, the cocycle condition is trivial, so that it suffices to check that  $\mathcal{G}^+(U) = \bigoplus_{i \in I} \mathcal{G}^+(U_i)$ .

<sup>&</sup>lt;sup>29</sup>Here, we use cofinality in the sense of the lemma of loc. cit. We do not claim cofinality in the sense of [Sta23, Definition 04E6], as the second condition needs not hold.

For all  $c = (T_k)_{1 \le k \le n(c)} \in \operatorname{Disj}_U$ , the covering  $\{T_k \cap U_i\}_{(k,i) \in A \times I}$  remains in  $\operatorname{Disj}_U$  and forms a refinement of  $(U_i)$ , so that the subcategory  $\operatorname{Disj}_{U|U_i} := \{\{T_k\}_{1 \le k \le n(c)} \in \operatorname{Disj}_U, \forall i, \exists k, T_k \subset U_i\}$  is cofinal inside  $\operatorname{Disj}_U$ . Since the choices of refinements inside  $\operatorname{Disj}_U$  are canonical, the second condition of [Sta23, Definition 04E6] is automatically satisfied. Hence, we can compute the colimit alongside  $\operatorname{Disj}_{U|U_i}$ , so that:

$$\bigoplus_{i \in I} \mathcal{G}^{+}(U_{i}) = \bigoplus_{i \in I} \varinjlim_{c_{i} \in \operatorname{Disj}_{U_{i}}} \bigoplus_{k=1}^{n(c_{i})} \mathcal{G}(T_{k,i}) = \varinjlim_{c \in \operatorname{Disj}_{U|U_{i}}} \bigoplus_{k=1}^{n(c)} \mathcal{G}(V_{\alpha}) \cong \varinjlim_{c \in \operatorname{Disj}_{U}} \bigoplus_{k=1}^{n(c)} \mathcal{G}(V_{\alpha}) = \mathcal{G}(U)$$

This concludes.  $\Box$ 

Remark 7.4.4.  $\mathcal{G}^+$  can be thought of as the  $\mathcal{B}$ -sheafification of  $\mathcal{G}$ . The construction above is the analogue, with respect to the basis  $\mathcal{B}$ , of the standard "plus" construction (as in [Sta23, Section 00W1]), that usually needs to be applied twice to sheafify. Here, once suffices, even for non separated presheaves.

Recall that, for  $\mathcal{G}$  a  $\mathcal{B}$ -presheaf on  $X_I$ , one can compute its stalk at some  $x \in X_I$  by  $\mathcal{F}_x = \varinjlim_{U \in \mathcal{B}, x \in U} \mathcal{F}(U)$ . We have the following:

**Lemma 7.4.5.** Let X be a profinite topological space, and  $\mathcal{F}$ ,  $\mathcal{G}$  be two  $\mathcal{B}$ -presheaves on X, together with a morphism  $f: \mathcal{F} \to \mathcal{G}$  of  $\mathcal{B}$ -presheaves. Assume that f induces isomorphisms on stalks  $\mathcal{F}_x \cong \mathcal{G}_x$  at all  $x \in X$ .

Then, the morphism  $f^+: \mathcal{F}^+ \to \mathcal{G}^+$  is an isomorphism of  $\mathcal{B}$ -sheaves.

*Proof.* By [Sta23, Lemma 009R], there is a functorial equivalence of category between  $\mathcal{B}$ -sheaves on X and sheaves on X, denoted  $\mathcal{F} \mapsto \mathcal{F}^{ext}$ , such that, for all  $U \in \mathcal{B}$ ,  $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ , and, for all  $x \in X$ ,  $\mathcal{F}^{ext}_x = \mathcal{F}_x$ .

Hence, the morphism  $f^+$  induces a morphism of sheaves  $(\mathcal{F}^+)^{ext} \to (\mathcal{G}^+)^{ext}$ , that induces an isomorphism of stalks, so that it is an isomorphism of sheaves.

Hence, so that, for all  $U \in \mathcal{B}$  the map  $(\mathcal{F}^+)^{ext}(U) = (\mathcal{F}^+)(U) \to (\mathcal{G}^+)(U) = (\mathcal{G}^+)^{ext}(U)$  is an isomorphism. This concludes.

Let us now come back to the proof of 7.4.1. Using the notations above, it is equivalent to:

$$\mathcal{S}^q(\mathcal{L})^+ \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \stackrel{a}{\cong} \mathcal{S}^q(\mathcal{L} \otimes \mathcal{O}_Y^+/p)^+$$

Hence, it suffices to show that  $\mathcal{S}^q(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \to \mathcal{S}^q(\mathcal{L} \otimes_{\mathbb{F}_p} \mathcal{O}_Y^+/p)$  induces an almost isomorphism on stalks. The stalks are easy to compute, as shown by the next lemma.

**Lemma 7.4.6.** For any  $x \in X_I$ , let Neigh<sub>x</sub> be the category of compact open neighborhoods of  $x \in X_I$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $Y_{\acute{e}t}$ . Then, there is a natural isomorphism

$$\mathcal{S}^q(\mathcal{F})_x \cong R^q(f \circ i_x)_* i_x^* \mathcal{F}$$

Where, for  $x \in X_I$ ,  $i_x$  denotes the inclusion  $x \cdot Y_I \subset X$ .

Proof. Let  $x \in X_I$ . By [Orl05a, Lemma 4.4], for any family  $(W_s)_{s \in \mathbb{N}}$  of compact open neighborhoods of the point  $1 \cdot P_I(\mathbb{Q}_p) \in X_I$  such that  $\bigcap_{s \in \mathbb{N}} W_s = \{1 \cdot P_I(\mathbb{Q}_p)\}$ , we have  $\bigcap_{s \in \mathbb{N}} Z_I^{W_s} = Y_I$ . Fix such a family  $W_s$ . Hence, for all  $x \in X$  the  $\{x \cdot W_s\}$  form a family of compact open neighborhoods of x such that  $\bigcap_{s \in \mathbb{N}} x \cdot W_s = \{x\}$ , and  $\bigcap_{s \in \mathbb{N}} Z_I^{x \cdot W_s} = x \cdot Y_I$ .

In what follows, we let  $W_s(x) = x \cdot W_s$ . Then, the  $W_s(x)$  form a cofinal family inside Neigh<sub>x</sub>.

Moreover, since  $f_*$  is qcqs, by [Sch17, Prop 14.9],  $R^q f_*$  commutes with filtered colimits, so that :

$$\mathcal{S}^q(\mathcal{F})_x = \varinjlim_{W \in \operatorname{Neigh}_x} R^q(f \circ i_W)_* i_W^* \mathcal{F} \cong R^q f_* \varinjlim_{W \in \operatorname{Neigh}_x} i_{W*} i_W^* \mathcal{F}$$

Since  $x \cdot Y_I \subset Z_I^{W(x)}$  for all compact open neighborhood W(x) of x, there is a natural morphism (as in the discussion in 7.2.2):

$$\underset{W \in \overrightarrow{\text{Neigh}}_x}{\varinjlim} i_{W*} i_W^* \mathcal{F} \to i_{x*} i_x^* \mathcal{F}$$

We'll prove that it is an isomorphism of étale sheaves, so it suffices to prove it on fibers. Let  $\eta$ : Spa $(C, C^+) \to Y$  be a geometric point, and  $y \in Y$  be the image of the unique closed point.

Then, by 3.2.2 we have, for any  $W \in \text{Neigh}_x$ :

$$(i_{x*} i_x^* \mathcal{F})_{\overline{y}} = \begin{cases} \mathcal{F}_{\overline{y}} & \text{if } y \in x \cdot Y_I \\ 0 & \text{otherwise} \end{cases} \text{ and } (i_{W*} i_W^* \mathcal{F})_{\overline{y}} = \begin{cases} \mathcal{F}_{\overline{y}} & \text{if } x \in Z_I^W \\ 0 & \text{otherwise} \end{cases}$$

Since stalks commute with colimits, it follows that

$$\left( \varinjlim_{W \in \operatorname{Neigh}_x} i_{W*} i_W^* \mathcal{F} \right)_{\overline{y}} \cong \begin{cases} \mathcal{F}_{\overline{y}} & \text{if } y \in \bigcap_{W \in \operatorname{Neigh}_x} Z_I^W \\ 0 & \text{otherwise} \end{cases}$$

This concludes since  $\bigcap_{W \in \text{Neigh}_{-}} Z_{I}^{W} = x \cdot Y_{I}$ , as we explicited a cofinal system  $W_{s}(x)$ .

We are now ready to prove 7.4.1.

*Proof.* of 7.4.1. With the notation of 7.4.2 and 7.4.3, we want to prove that the natural morphism :

$$\mathcal{S}^q(\mathcal{L})^+ \otimes_{\mathbb{F}_n} \mathcal{O}_{E_-}^+ \to \mathcal{S}^q(\mathcal{L} \otimes_{\mathbb{F}_n} \mathcal{O}_V^+/p)^+$$

is an almost isomorphism. By the combined lemmas 7.4.5 and 7.4.6, this reduces to proving that, for all  $x \in X_I$ , the morphism:

$$R^q(f \circ i_x)_* i_x^*(\mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_{E_s}^+/p \to R^q(f \circ i_x)_* i_x^*(\mathcal{L} \otimes \mathcal{O}_Y^+/p)$$

is an almost isomorphism. Since the  $i_x: x\cdot Y_I \to Y$  is a generalizing closed immersion, 3.2.2 shows that  $i_x^*\mathcal{O}_Y^+/p \cong \mathcal{O}_{x\cdot Y_I}^+/p$ ; and pullback commutes with tensor products. Moreover, the pullback  $i_x^*\mathcal{L}$  defines an  $\mathbb{F}_p$ -local system on  $Y_I$ , and  $f \circ i_x$  is the structure morphism of  $x \cdot Y_I$ , which admits a structure of a smooth proper rigid analytic variety, by 7.1.1.

Hence, the primitive comparison theorem 5.1.1 applies. This is the desired result.

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