A Fibred Approach to Algebraic Recognition

Internship report

Guillaume Rousseau
Univ Lyon, EnsL, UCBL, CNRS, LIP, F-69342, LYON Cedex 07, France
guillaume.rousseau@ens-lyon.fr
http://perso.ens-lyon.fr/guillaume.rousseau/

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Foreword Before we begin, I would like to thank the ENS Lyon, and particularly Colin Riba, for taking me in for an internship on such short notice. For the record, this internship took place during the covid-19 crisis, and as such was conducted almost entirely through telework. This report summarizes my internship under supervision of Colin Riba, and I will present our joint work, the main goal of which was to set a robust framework for future works.

1. Introduction

In [PR17] and [Rib20], the authors study a Curry-Howard approach to automaton theory, in the setting of infinite words and trees. The aim is to have a setting in which existential quantification has a computational content, in the sense that witnesses may be extracted from proofs of existential statements. This relies on constructions from categorical logic and semantics, presented for instance in Jacob's book [Jac01], which looks at logics and proofs in a categorical framework, where morphisms represent proofs or logical implications. When dealing with finite-state automata in the setting of [PR17, Rib20], we do not know how to define some operations on automata functorially, typically the simulation of alternating automata by non-deterministic ones, as in [MS95]. In order to extend the finite setting while still keeping some structure, a possibility is to consider profinite objects (see e.g. [Alm05]). It is natural to work with algebras, like semigroups, when dealing with profiniteness, and as such we shall take an algebraic point of view on automata and recognition. The goal of this report is to set up a framework which encompasses profiniteness, categorical logic and language recognition in order for future work to be done on the subject. Although the long-term goal is to work on structures such as infinite words and trees, we consider finite words as a first step in establishing a framework.

Language recognition by automata, and specifically the study of rational languages, is a field which is widely developed, and needs no presenting. A field which is generally less known is algebraic recognition, where languages are studied not through automata but through algebras such as monoids and semigroups. A known basic result is that the class of languages recognized by finite semigroups or monoids is precisely the class of rational languages. As such, algebraic recognition is related to universal algebra, which studies classes of algebraic structures rather than algebraic structures themselves. A fundamental result in universal algebra, Birkhoff's HSP theorem [BS81], states that classes of algebras defined by sets of identities, are precisely the classes which are stable by homomorphic image, subalgebra and product (hence the HSP name), called *varieties*.

Researchers in algebraic recognition looked for results similar to Birkhoff's in their field. Regular languages being recognized by finite semigroups and monoids, Birkhoff's results are not directely applicable. One of the first key theorems in that direction is Schützenberger's theorem (1965), which states a correspondence between a fragment of logic on words and rational languages. Finite and infinite words may be endowed with a logic, like Monadic Second-Order logic, where variables range over positions in the word or sets thereof. Straubing's book [Str94] gives a good idea of how logic and recognition interact with each other. Schutzenberger's theorem states that languages which may be defined using first order logic are precisely regular languages which are "star-free", and equivalently whose syntactic monoid is aperiodic. This result has motivated a more in-depth study of the links between algebraic recognition and logic. Some 10 years later, Eilenberg proved his Variety Theorem, which exhibits a correspondence between pseudo-varieties of monoids, which are classes of finite monoids stable by homomorphic image, submonoid, and finite product, and varieties of rational languages. This powerful result gave rise to many "Eilenberg-type theorems", whose purpose were to give similar correspondences between specific classes of (finite) algebras and specific classes of (rational) languages. An early reference on algebraic recognition was Eilenberg's book [ET76]. See e.g. Straubing's [Str94] and Pin's [Pin86] books for modern accounts.

Eilenberg's result was extended in 1982 when Reiterman showed that pseudovarieties are precisely classes of finite algebras which are defined by sets of profinite equations, that is to say equations with variables ranging over some profinite object related to the studied pseudovariety.

In [Alm05], J. Almeida gives a survey which recalls in particular a construction of profinite free semigroups for pseudo-varieties and studies recognition of finite and profinite languages. Profinite objects are naturally endowed with a very specific topology called Stone topology. Stone's duality theorem shows a one-to-one contravariant correspondence between Stone spaces and Boolean algebras which allows profinite algebras like monoids and semigroups to have a structure which is simple to use, while still being very rich mathematical objects. A good reference for the topological and profinite approach to automata and recognition theory is J-E. Pin's survey paper [Pin09].

Recent work on the subject was done by M. Gehrke, D. Petrisan and L. Reggio. In [Geh16], Gehrke shows results on an extended Stone duality, between topological algebras and Boolean algebras with additional operations, which extend the usual Eilenberg-Reiterman setting. In [GPR16], the three authors derive a notion of Schützenberger product, which reflects quantification for languages in algebraic recognition.

Report Outline In a first section, we give some background material on algebraic recognition. Then, we introduce the categorical concept of fibration, and use it to describe a structure on algebraic recognizers. In sections 4 and 5, we define profinite objects, as well as a monad on profinite sets which acts like the powerset does in the finite case. Finally, in section 6 we apply those results to the fibration introduced in section 3.

2. Preliminaries

On Category Theory A reader already familiar with category theory will recognize some concepts such as adjunctions and universal properties of limits in this report. By lack of space we do not include background on category theory, but we advise having a reference like [ML98] or [Awo06] at hand.

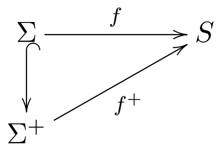
2.1. Algebraic Recognition

Let us begin by recalling the classical notions of finite semigroups and monoids, and language recognition.

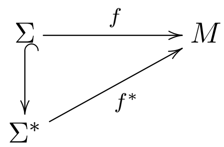
Definition 2.1. *Let S be a set, and $*$ a binary internal operation on S. If $*$ is associative then we say that $(S, *)$ is a semigroup, and when there is no ambiguity that S is a semigroup.*

If, moreover, S contains an element e such that for all $s \in S$, $s * e = e * s = s$, e is called the unit of S (it is necessarily unique), and we say that S is a monoid.

We consider alphabets, usually denoted Σ, Γ, which are non-empty sets. Elements of Σ are called 'letters', and (finite) sequences of letters are called (finite) words. Σ^* denotes the set of all words over Σ, including the empty word. The set Σ^* endowed with concatenation is a monoid with the empty word as a unit, it is the *free* monoid over Σ. Similarly, Σ^+, which denotes the set of all non-empty words over Σ, is a semigroup (a monoid without unit), and is the free semigroup over Σ. Given a semigroup S, a monoid M, an alphabet Σ and functions $f : \Sigma \to S$ $g : \Sigma \to M$, we write $f^+ : \Sigma^+ \to S$ and $f^* : \Sigma^* \to M$ for the unique morphisms such that



and



In the following, we often simply write f for f^+ and f^*. We work using semigroups, but our results are mainly also applicable to monoids.

Definition 2.2. *Consider*

$$\mathcal{A} \quad = \quad (\Sigma, S, I, f)$$

where Σ is an alphabet, S is a semigroup, $f : \Sigma \to S$ is a function and I is a subset of S. We say that $\mathcal{A}$ recognizes *a language $\mathcal{L} \subseteq \Sigma^+$ if $\mathcal{L} = {f^+}^{-1}(I)$.*

We may similarly define recognition by monoids, with the unit of Σ^* being the empty word ϵ.

As a short example, we may look at how to turn an automaton into a semigroup. Recall that a finite deterministic automaton is a tuple $\mathcal{A} = (\Sigma, Q, \delta, q_i, F)$ where Q is a finite set of states, q_i is the initial state, $F \subseteq Q$ the set of final states, and $\delta : \Sigma \times Q \to Q$ is a transition function. An automaton operates on finite words by reading each letter starting from the initial state, and following the transition function. δ extends to $\Sigma^+ \times Q$ in the obvious way, and a word $w \in \Sigma^+$ is accepted by $\mathcal{A}$ if $\delta(w, q_i) \in F$. The language of $\mathcal{A}$ is the set of all words accepted by $\mathcal{A}$. We can see δ as a function from Σ to $(Q \to Q)$, which is the semigroup of all functions from Q to itself. The language of $\mathcal{A}$ is then the inverse image of $\{f : Q \to Q \mid f(q_i) \in F\}$.

Conversely, one may easily turn an algebraic recognizer (Σ, S, I, f) into a finite deterministic automaton recognizing the same language [Str94]. Consequently, the class of languages recognized by finite semigroups is precisely the class of rational languages. As such it is a boolean subalgebra of $\mathcal{P}(\Sigma^+)$, and we may give explicit constructions of recognizers for boolean combinations.

Lemma 2.3. *Let*

$$\begin{array}{rcl} \mathcal{A} & = & (\Sigma, S, I, f) \\ \mathcal{B} & = & (\Sigma, T, J, g) \end{array}$$

If $\mathcal{A}$ recognizes $\mathcal{L}$ and $\mathcal{B}$ recognizes $\mathcal{L}'$, then

(a) $\mathcal{A} \otimes \mathcal{B}$ recognizes $\mathcal{L} \cap \mathcal{L}'$, where

$$\mathcal{A} \otimes \mathcal{B} \quad := \quad (\Sigma,\ S \times T,\ I \times J,\ \langle f, g \rangle)$$

(b) $\neg\mathcal{A}$ recognizes $\Sigma^+ \setminus \mathcal{L}$, where

$$\neg\mathcal{A} \quad := \quad (\Sigma,\ S,\ S \setminus I,\ f)$$

2.2. Quantifiers as Adjoints

We now recall some concepts of categorical logic which are expanded upon in this report. Jacobs' book [Jac01] is a good general reference for categorical logic and semantics. A key idea in this area, due to Lawvere, is that universal and existential quantifications come as respectively left and right adjoint to some weakening functor. On languages, existential and universal quantification correspond to operations called projection and coprojection.

Consider alphabets Σ and Γ. Write $\pi : \Sigma \times \Gamma \to \Sigma$ for the first projection. Let

$$\begin{array}{rcrcl} \pi^\bullet & : & \mathcal{P}(\Sigma^+) & \longrightarrow & \mathcal{P}((\Sigma \times \Gamma)^+) \\ & & \mathcal{L} & \longmapsto & \{\langle u, v \rangle \in (\Sigma \times \Gamma)^+ \mid u \in \mathcal{L}\} \end{array}$$

where $\langle u, v \rangle$ is the pairing of u and v, only defined when the two are of equal lengths, characterized by $\langle u, v \rangle_i = (u_i, v_i)$. Then $\pi^\bullet$ is a functor from $\mathcal{P}(\Sigma^+) \to \mathcal{P}((\Sigma \times \Gamma)^+)$ viewed as posets with regard to inclusion, called the weakening functor.

Definition 2.4 ((Co)Projection of a Language)**.** *Given alphabets Σ and Γ, the* projection $\exists_{(\Sigma,\Gamma)}$ *and the* coprojection $\forall_{(\Sigma,\Gamma)}$ *operations on languages are given by*

$$\begin{array}{rcrcl} \exists_{(\Sigma,\Gamma)} & : & \mathcal{P}((\Sigma \times \Gamma)^+) & \longrightarrow & \mathcal{P}(\Sigma^+) \\ & & \mathcal{L} & \longmapsto & \{u \in \Sigma^+ \mid \exists v \in \Sigma^{|u|},\ \langle u, v \rangle \in \mathcal{L}\} \end{array}$$

$$\begin{array}{rcrcl} \forall_{(\Sigma,\Gamma)} & : & \mathcal{P}((\Sigma \times \Gamma)^+) & \longrightarrow & \mathcal{P}(\Sigma^+) \\ & & \mathcal{L} & \longmapsto & \{u \in \Sigma^+ \mid \forall v \in \Sigma^{|u|},\ \langle u, v \rangle \in \mathcal{L}\} \end{array}$$

In the following we often write $\exists_\Gamma$ (resp. $\forall_\Gamma$) for $\exists_{(\Sigma,\Gamma)}$ (resp. for $\forall_{(\Sigma,\Gamma)}$).

Remark 2.5. *The word "projection" in this context unfortunately refers to two different objects: the projection of a language $\mathcal{L}$ on $\Sigma \times \Gamma$ is the language $\exists_\Gamma \mathcal{L}$, and the projection function π sends a word on $\Sigma \times \Gamma$ to a word on Σ, and induces the $\pi^\bullet$ functor.*

As expected with categorical logic, the operations $\exists_\Gamma$ and $\forall_\Gamma$ are respectively left and right adjoints to $\pi^\bullet$, as shown by the following property. Recall that two functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ form the respective left and right part of an adjunction if and only if for $C \in \mathbb{C}, D \in \mathbb{D}$ there is an isomorphism

$$\varphi : \mathrm{Hom}_{\mathbb{D}}(FC, D) \simeq \mathrm{Hom}_{\mathbb{C}}(C, GD)$$

that is natural in C and D [Awo06, Chap. 9]. Since $\mathcal{P}(\Sigma^+)$ and $\mathcal{P}((\Sigma\times\Gamma)^+)$ are posets, the situation is simple: a functor $\exists_\Gamma$ is left adjoint to $\pi^\bullet$ if and only if they form a Galois connection, *i.e.* if and only if for $\mathcal{L} \in \mathcal{P}((\Sigma\times\Gamma)^+)$ and $\mathcal{L}' \in \mathcal{P}(\Sigma^+)$, we have:

$$\mathcal{L} \subseteq \pi^\bullet(\mathcal{L}') \iff \exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$$

Lemma 2.6. *Consider alphabets Σ, Γ and write $\pi : \Sigma\times\Gamma \to \Sigma$ for the projection function. Then $\exists_{(\Sigma,\Gamma)}$ is left adjoint to $\pi^\bullet$, and $\forall_{(\Sigma,\Gamma)}$ is right adjoint to $\pi^\bullet$.*

Proof. We show the adjunction between $\exists_\Gamma$ and $\pi^\bullet$. The other adjunction may be proven from that and from the fact that $\exists_\Gamma\mathcal{L} = \neg\forall_\Gamma(\neg\mathcal{L})$.

Assume $\mathcal{L} \subseteq \pi^\bullet(\mathcal{L}')$. Let $u \in \exists_\Gamma(\mathcal{L})$. By definition, there is a $v \in \Gamma^+$ such that $\langle u, v\rangle \in \mathcal{L} \subseteq \pi^\bullet(\mathcal{L}')$. So u is in $\mathcal{L}'$. Hence $\exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$.

Now assume $\exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$. Let $\langle u, v\rangle \in \mathcal{L}$. Then $u \in \exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$, so $\langle u, v\rangle \in \pi^\bullet(\mathcal{L}')$. Hence $\mathcal{L} \subseteq \pi^\bullet(\mathcal{L}')$. □

2.3. Powersets and Quantifications

We now recall how, from a recognizer for a language $\mathcal{L}$ on $\Sigma\times\Gamma$ we may construct a recognizer for $\exists_\Gamma\mathcal{L}$. This operation reflects the usual powerset operation on automata.

Lemma 2.7. *If $(S, \cdot)$ is a semigroup then $\mathcal{P}(S)$ is a semigroup for the operation*

$$\begin{array}{rcl} \mathcal{P}(S)\times\mathcal{P}(S) & \longrightarrow & \mathcal{P}(S) \\ (X, Y) & \longmapsto & \{s\cdot t \mid s \in X \text{ and } t \in Y\} \end{array}$$

Moreover, if S is a monoid with unit $\mathbf{I}$, then $\mathcal{P}(S)$ is a monoid with unit $\{\mathbf{I}\}$.

Definition 2.8. *Consider*

$$\mathcal{A} \quad = \quad (\Sigma\times\Gamma, S, I, f)$$

where Σ, Γ are alphabets, S is a finite semigroup $f : \Sigma \to S$ is a function and I is a subset of S. Let

$$\begin{array}{rcl} \exists_{(\Sigma,\Gamma)}\mathcal{A} & := & (\Sigma,\, \mathcal{P}(S),\, \Diamond I,\, \pi(f)) \\ \forall_{(\Sigma,\Gamma)}\mathcal{A} & := & (\Sigma,\, \mathcal{P}(S),\, \Box I,\, \pi(f)) \end{array}$$

where

$$\begin{array}{rcl} \Diamond I & := & \{X \in \mathcal{P}(S) \mid X\cap I \neq \emptyset\} \\ \Box I & := & \{X \in \mathcal{P}(S) \mid X \subseteq I\} \end{array}$$

and

$$\begin{array}{rcccl} \pi(f) & : & \Sigma & \longrightarrow & \mathcal{P}(S) \\ & & \mathtt{a} & \longmapsto & \{f\langle\mathtt{a},\mathtt{b}\rangle \mid \mathtt{b} \in \Gamma\} \end{array}$$

Once again, we often write $\exists_\Gamma$ (resp. $\forall_\Gamma$) for $\exists_{(\Sigma,\Gamma)}$ (resp. for $\forall_{(\Sigma,\Gamma)}$).

Note that $\forall_{(\Sigma,\Gamma)}\mathcal{A} = \neg\exists_{(\Sigma,\Gamma)}\neg\mathcal{A}$ (see Lem. 2.3). Note also that

$$\pi(f)^+ \quad : \quad u \in \Sigma^+ \quad \longmapsto \quad \{f^+(w) \mid w \in \pi^{-1}(u)\} \in \mathcal{P}(S)$$

Lemma 2.9. *If $\mathcal{A}$ recognizes $\mathcal{L} \subseteq (\Sigma \times \Gamma)^+$ then $\exists_{(\Sigma,\Gamma)}\mathcal{A}$ recognizes $\exists_{(\Sigma,\Gamma)}\mathcal{L}$ and $\forall_{(\Sigma,\Gamma)}\mathcal{A}$ recognizes $\forall_{(\Sigma,\Gamma)}\mathcal{L}$.*

Proof. Write $\mathcal{A} = (\Sigma \times \Gamma, S, I, f)$. Given $u \in \Sigma^+$, we have:

$$\begin{array}{rcl} u \in \exists_\Gamma \mathcal{L} & \iff & \text{for some } w \in \pi^{-1}(u),\ w \in \mathcal{L} \\ & \iff & \text{for some } w \in \pi^{-1}(u),\ f^+(w) \in I \\ & \iff & \pi(f)^+(u) \cap I \neq \emptyset \\ & \iff & \pi(f)^+(u) \in \Diamond I \end{array}$$

Hence $\exists_{(\Sigma,\Gamma)}\mathcal{A}$ recognizes $\exists_{(\Sigma,\Gamma)}\mathcal{L}$. The result for $\forall_{(\Sigma,\Gamma)}$ follows from Lem. 2.3 and the fact that $\forall_{(\Sigma,\Gamma)}\mathcal{A} = \neg\exists_{(\Sigma,\Gamma)}\neg\mathcal{A}$. □

3. Fibrations for Recognizability

3.1. Fibrations

In this section we give an informal presentation of fibration theory and why it is used in categorical semantics.

Consider a category $\mathbb{B}$, which we call our *base* category. Think of $\mathbb{B}$ as of a category of typed terms. A fibration over $\mathbb{B}$ is a functor $p : \mathbb{E} \to \mathbb{B}$ verifying additional conditions which express that morphisms in $\mathbb{B}$ are reflected in $\mathbb{E}$ along p. For an object A in $\mathbb{B}$, we call *fibre of p over A* the category $\mathbb{E}_A$ of objects sent to A by p and such that morphisms are sent to id_A. The fibre $\mathbb{E}_A$ may be thought of as a logic of predicates over A, and the *total category* $\mathbb{E}$ as the collection of such logics. When A, B are objects of $\mathbb{B}$, and $u : A \to B$ a morphism in $\mathbb{B}$, there is a corresponding functor $u^\bullet : \mathbb{E}_B \to \mathbb{E}_A$ called *substitution functor*, which lifts objects over B onto A.

Fibred categories are used in categorical semantics. Essentially, an object $\mathcal{A} \in \mathbb{E}$ over an object $A \in \mathbb{B}$ may be thought of as a predicate over A, and a morphism $\mathcal{A} \to \mathcal{B}$ in $\mathbb{E}$ represents a logical implication $\mathcal{A} \Rightarrow \mathcal{B}$. Morphisms in $\mathbb{B}$ correspond to terms with possibly free variables, and the substitution functor they induce can indeed be thought of as a substitution (for example $b = f(a, a)$).

A related notion is that of indexed categories, which express that the $(-)^\bullet$ operator is itself functorial. An indexed category is a pseudo-functor $\Lambda : \mathbb{B}^{op} \to \mathbf{Cat}$ (notice that Λ is contravariant). In this report, we only focus on what are called *strict* indexed categories, where Λ is a functor, and when we write "indexed categories", it is implied that we are talking about strict indexed categories. Informally, $\Lambda(A)$ and $\mathbb{E}_A$ represent the same kind of information. In fact, there is a systematic way to turn an indexed category $\Lambda : \mathbb{B} \to \mathbf{Cat}$ into a fibration, called the *Grothendieck construction*. We consider the functor $p : \int \Lambda \to \mathbb{B}$ with $\int \Lambda$ having couples (A, X) with $A \in \mathbb{B}$ and $X \in \Lambda(A)$ as objects and where arrows $(A, X) \to (B, Y)$ are couples (u, f) with $u : A \to B$ in $\mathbb{B}$ and $f : X \to u^\bullet(Y) = \Lambda(u)(Y)$ in $\Lambda(A)$. The functor p itself is the first projection. The Grothendieck construction on Λ is $\int \Lambda$.

We give a more detailed and formal presentation in App. C.

3.2. Fibrations of Languages

We now give a simple example of fibration involving languages.

As stated in the introduction, an important result of algebraic recognition is Eilenberg's theorem. A pseudo-variety is a class of finite semigroups which is stable by homomorphic image, subsemigroup, and finite product. A variety of languages [ET76] is a collection of sets $\mathscr{V}(\Sigma)$ of languages for each alphabet Σ, which is stable by finite union, complementation and residuation, and such that $\mathscr{V}$ is stable by inverse homomorphic image. Eilenberg's result states that there is an exact correspondence between varieties of languages and pseudovarieties. See Pin's [Pin86] and Straubing's [Str94] books for more details.

Consider a variety $\mathscr{V}$ of regular languages. Then for each alphabet Σ we have a set $\mathscr{V}(\Sigma)$ of regular languages over Σ which is closed under Boolean operations and under quotients, and such that for each semigroup morphism $\varphi : \Sigma^+ \to \Gamma^+$ the inverse image $\varphi^{-1} : \mathcal{P}(\Gamma^+) \to \mathcal{P}(\Sigma^+)$ restricts to a function $\varphi^\bullet : \mathscr{V}(\Gamma) \to \mathscr{V}(\Sigma)$. Recall that φ^{-1} is automatically a map of Boolean algebras, and thus in particular preserves inclusion.

Recall that a poset P can be seen as a category with elements of P as objects, and at most one arrow from $p \in P$ to $p' \in P$, when $p \leq p'$. Keeping this in mind, $\mathscr{V}$ can be seen as a posetal fibration over the category **Alph** of finite non-empty sets and functions, with the fibre over Σ being the partial order $\mathscr{V}(\Sigma)$ seen as a category.

3.3. Fibration of Recognizers

We have seen in §3.2 a fibration of languages. In this section, we discuss several possibilities for a fibration of the recognizers defined in Def 2.1. The goal is to find a good setting for a category of recognizers on Σ where objects are triplets (S, I, f) where S is a semigroup, I is a subset of S and f is a function $\Sigma \to S$. Recall that the language recognized by such an object is the set $\mathcal{L} = f^{-1}I \subseteq \Sigma^+$. We want morphisms $(S, I, f) \to (T, J, g)$ in this category to reflect language inclusion. A first possibility is to choose semigroup morphisms $S \to T$ which send I to J. However this causes some issues with quantification (see App. C.4.1). Our notion of morphism is based on the following observation.

Lemma 3.1. *Let (S, I, f) and (T, J, g) be two recognizers on Σ. Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be the language recognized by (S, I, f) (resp. (T, J, g)). Then $\mathcal{L} \subseteq \mathcal{L}'$ if and only if $\mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*)$, where:*

- *$\mathsf{Im}\langle f, g\rangle$ is the subsemigroup of $S \times T$ equal to $\{(f(w), g(w) \mid w \in \Sigma^+\}$*
- *$I^* = I \times T$, $J^* = S \times J$*
- *for $A, B \subseteq C$, $A \Rightarrow B = \{x \in C \mid (x \in A \Rightarrow x \in B)\}$*

Proof. We have

$$\begin{aligned} \mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*) \quad &\iff \quad \text{for all } w \in \Sigma^+,\ f(w) \in I \Rightarrow g(w) \in J \\ &\iff \quad \text{for all } w,\ w \in \mathcal{L} \Rightarrow w \in \mathcal{L}' \\ &\iff \quad \mathcal{L} \subseteq \mathcal{L}' \end{aligned}$$

□

We may now define the indexed category Λ which we build upon in the following sections. For an alphabet Σ, let the category $\Lambda(\Sigma)$ be the following:

- Objects are triplets (S, I, f) where S is a semigroup, I is a subset of S and f is a function $\Sigma \to S$.
- There is an arrow ϕ from (S, I, f) to (T, J, g) iff $\mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*)$.

Notice that as an immediate corollary of Lem. 3.1, we have $\mathcal{L} \subseteq \mathcal{L}'$ iff there exists a subsemigroup $R \subseteq S \times T$, also called semigroup relation between S and T, such that $\mathsf{Im}\langle f, g\rangle \subseteq R \subseteq (I^* \Rightarrow J^*)$. Thus we could have extended arrows and considered all semigroup relations R satisfying this condition, but once again this causes some issues when doing quantification as seen in App. C.4.2.

We now check that Λ is an indexed category, and we then apply the Grothendieck construction.

Proposition 3.2. *The map* $\Lambda : \mathbf{Alph} \to \mathbf{Cat}$ *is a strict indexed category, with the function* $u : \Sigma \to \Gamma$ *yielding the functor* $\Lambda(u) = u^\bullet : \Lambda(\Gamma) \to \Lambda(\Sigma)$ *with* $u^\bullet(S, I, f) := (S, I, f \circ u)$.

Proof. It is clear that $u^\bullet$ is functorial, and that the action of Λ is strictly functorial, as long as it preserves arrows, namely:

$$\text{If} \quad \mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*) \quad \text{then} \quad \mathsf{Im}\langle f \circ u, g \circ u\rangle \subseteq (I^* \Rightarrow J^*)$$

Since $\mathsf{Im}\langle f \circ u, g \circ u\rangle \subseteq \mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*)$ this is verified. □

Applying the Grothendieck construction on this indexed category gives the following fibration $\mathbb{E} \to \mathbf{Alph}$:

- Objects of $\mathbb{E}$: tuples (Σ, S, I, f) with $\Sigma \in \mathbf{Alph}$, S a semigroup, $I \subseteq S$, and $f : \Sigma \to S$. We denote these objects with letters $\mathcal{A}, \mathcal{B}$, etc...
- Arrows from (Σ, S, I, f) to (Γ, T, J, g) are couples (u, φ) with $u : \Sigma \to \Gamma$ in $\mathbf{Alph}$, and $\varphi : (S, I, f) \to u^\bullet(T, J, g) = (T, J, g \circ u)$ in $\Lambda(\Sigma)$

In the rest of this report, we use the notation Λ for the category $\mathbb{E}$, and $\Lambda(\Sigma)$ for the fibre on Σ. This fibration has a quite simple structure, and in fact it seems that it may be too simple. We plan to look into a fibration like those that we tested, described in App. C.4.1, making use of semigroup actions, similar to what is done in [**?**].

3.4. Quantification for Finite Recognizers

In this section we study existential quantifications as left adjoints to weakening functors. We consider the fibration Λ introduced in the previous section. Let us look at the effect of the projection function $\pi : \Sigma \times \Gamma \to \Sigma$. This function yields a functor $\pi^\bullet : \Lambda(\Sigma) \to \Lambda(\Sigma \times \Gamma)$ called the *weakening* functor. Notice that if $\mathcal{L}$ is recognized by (Σ, S, I, f), then the language recognized by $\pi^\bullet(\Sigma, S, I, f)$ is precisely $\pi^\bullet(\mathcal{L})$ as in §2.2. As expected in categorical logic, we want to define an existential quantifier functor $\exists_\Gamma$ which is a left adjoint to $\pi^\bullet$. Following the notations introduced in Def. 2.8, we define $\exists_\Gamma$ on objects by $\exists_\Gamma(\Sigma \times \Gamma, S, I, f) = (\Sigma, \mathcal{P}(S), \Diamond I, \pi(f))$, with $\Diamond I = \{F \in \mathcal{P}(S) \mid I \cap F \neq \emptyset\}$ and $\pi(f)(u) = \{f(w) \mid w \in \pi^{-1}(u)\}$. The results of §2.3 ensure that $\exists_\Gamma(\Sigma \times \Gamma, S, I, f)$ recognizes exactly the language $\exists_\Gamma(\mathcal{L})$, where $\mathcal{L}$ is the language recognized by $(\Sigma \times \Gamma, S, I, f)$.

Following §2.2, since $\Lambda(\Sigma)$ and $\Lambda(\Sigma \times \Gamma)$ are posets, the required condition for an adjunction is that for $\mathcal{A} \in \Lambda(\Sigma \times \Gamma)$, $\mathcal{B} \in \Lambda(\Sigma)$, and $\mathcal{L}, \mathcal{L}'$ recognized respectively by $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{L} \subseteq \pi^\bullet(\mathcal{L}') \iff \exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$. The equivalence $\mathcal{L} \subseteq \pi^\bullet(\mathcal{L}') \iff \exists_\Gamma(\mathcal{L}) \subseteq \mathcal{L}'$ is the result of Lem. 2.6, and so we have proven the following:

Proposition 3.3. *The functor* $\exists_\Gamma : \Lambda(\Sigma \times \Gamma) \to \Lambda(\Sigma)$ *is left adjoint to* $\pi^\bullet : \Lambda(\Sigma) \to \Lambda(\Sigma \times \Gamma)$.

We may similarly define universal quantification $\forall_\Gamma$ using the $\Box$ operator, and this yields a right adjoint to $\pi^\bullet$.

In fibration theory, it is standard to require quantifications to satisfy an additional condition called the Beck-Chevalley Condition, or BCC. The BCC expresses the following: given a predicate $\mathcal{A}(\sigma, x)$ over $\Sigma' \times \Gamma$ and a function $u : \Sigma \to \Sigma'$, we may construct the predicate $\exists x, \mathcal{A}(u(w), x)$ by first quantifying on x and then reindexing along u, or first reindexing along $\langle u, \mathrm{id}_\Gamma \rangle$ and then quantifying on x. We want those two paths to meet up at the end, in other words, given the following pullback square:

$$\begin{array}{ccc} \Sigma \times \Gamma & \xrightarrow{\pi_\Sigma} & \Sigma \\ {\scriptstyle \langle u, \mathrm{id}_\Gamma \rangle} \downarrow & \lrcorner & \downarrow {\scriptstyle u} \\ \Sigma' \times X & \xrightarrow[\pi'_\Sigma]{} & \Sigma' \end{array}$$

we want the functors $\exists_{\Sigma,\Gamma} \circ \langle u, \mathrm{id}_\Gamma \rangle^\bullet$ and $u^\bullet \circ \exists_{\Sigma',\Gamma}$ to be isomorphic. We now look at the BCC in Λ. Let $\mathcal{A} = (\Sigma' \times \Gamma, S, I, f)$ be a recognizer. Then,

$$\begin{array}{rcl} \exists_{\Sigma,\Gamma} \circ \langle u, \mathrm{id}_\Gamma \rangle^\bullet(\mathcal{A}) & = & (\Sigma, \mathcal{P}(S), \Diamond(I), \pi_\Sigma(f \circ \langle u, id_X \rangle)) \\ & \text{and} & \\ u^\bullet \circ \exists_{\Sigma',\Gamma}(\mathcal{A}) & = & (\Sigma, \mathcal{P}(S), \Diamond(I), \pi_{\Sigma'}(f) \circ u) \end{array}$$

The BCC is thus satisfied as soon as the two functions in these recognizers are equal. This is easily verified: for $\sigma \in \Sigma^+$,

$$\begin{array}{rcl} \pi_\Sigma(f \circ \langle u, id_X \rangle)(\sigma) & = & \{ f(u(\sigma), x) \mid x \in \Gamma^{|\sigma|} \} \\ & = & \pi_{\Sigma'}(f) \circ u. \end{array}$$

3.5. Sum Fibration

The fibration Λ of §3.3 has universal and existential quantifiers, however it does not have witnesses of existential quantification, since arrows do not carry any information. One of the uses of the Sum construction [Hof11], besides giving free quantification, is to be able to reflect the computational content of quantification, as we will see in §3.6. The Sum construction is a generalization of the simple fibration construction, a standard concept of fibration theory.

The simple fibration is the first projection functor $S(\mathbb{B}) \to \mathbb{B}$ where $\mathbb{B}$ has finite products (see [Jac01] for more details) and $S(\mathbb{B})$ is the following category:

- Objects are couples (A, X) with $A, X \in \mathbb{B}$
- Arrows $(A, X) \to (B, Y)$ are pairs (h, h_0) with $h : A \to B$ and $h_0 : A \times X \to Y$, both h and h_0 being in $\mathbb{B}$.

Identity and composition are given in App. C.2.

The Sum construction [Hof11] turns a fibration $p : \mathbb{E} \to \mathbb{B}$ into a fibration $\mathsf{Sum}(p) : S(\mathbb{E}) \to \mathbb{B}$, as an extension of the simple fibration.

- Objects of $S(\mathbb{E})$ are triples $(A, X, \mathcal{A})$ with $A, X \in \mathbb{B}$ and $\mathcal{A} \in \mathbb{E}_{A \times X}$
- Arrows $(A, X, \mathcal{A}) \to (B, Y, \mathcal{B})$ are triples (h, h_0, ϕ) with $h : A \to B$ and $h_0 : A \times X \to Y$ in $\mathbb{B}$ and $\phi : \mathcal{A} \to (\langle h \circ \pi, h_0 \rangle)^\bullet(\mathcal{B})$ in $E_{A \times X}$.

Identity and composition are given in App. C.3.

In an object $(A, X, \mathcal{A})$ of $S(\mathbb{E})$, $\mathcal{A}$ is a predicate over $A \times X$, and $(A, X, \mathcal{A})$ is seen as the proposition $\exists x \in X, \mathcal{A}(a, x)$. The fibration $S(\mathbb{E})$ is equipped with an existential quantification functor $\exists_C : \mathbb{E}_{A\times C} \to \mathbb{E}_A$ which maps $(A\times C, X, \mathcal{A})$ to $(A, X\times C, \mathcal{A})$ and which is left adjoint to $\pi_A^\bullet : \mathbb{E}_A \to \mathbb{E}_{A\times C}$. Moreover, the Beck-Chevalley condition is always true in the Sum fibration (see [Hof11]).

3.6. Sum Fibration for Recognizers

In this section and the next, we apply the Sum construction to the fibration Λ that we described above, and exhibit its benefits regarding quantification and determinization. For an alphabet Σ, we denote π_Σ the projections onto Σ, and we make sure that it is always clear what the domain of the projection is. The category $S(\Lambda)$ is defined as follows:

- Objects are tuples (Σ, X, S, I, f) with Σ and X alphabets, S a semigroup, I a subset of S, and $f : \Sigma \times X \to S$ a function.
- Arrows from (Σ, X, S, I, f) to (Γ, Y, T, J, g) are triples (h, h_0, φ) with $h : \Sigma \to \Gamma$ a function, $h_0 : \Sigma \times X \to Y$ a function, and $\phi : (\Sigma \times X, S, I, f) \to (\Sigma \times X, T, J, g \circ \langle h \circ \pi, h_0\rangle)$ an arrow in $\Lambda_{\Sigma\times X}$.

We define the language $\mathcal{L} \subseteq \Sigma^+$ recognized by (Σ, X, S, I, f) by:

$$\sigma \in \mathcal{L} \iff \exists w \in \pi_\Sigma^{-1}(\sigma), f(w) \in I$$

Therefore, the object (Σ, X, S, I, f) acts like the object $\exists_X(\Sigma \times X, S, I, f)$ from the previous fibration.

Quantification As stated before, an advantage of the Sum construction is that it gives free existential quantification. Indeed, we may define a quantification operator $\exists_\Gamma^S : S(\Lambda)_{\Sigma\times\Gamma} \to S(\Lambda)_\Sigma$ as follows:

$$\exists_\Gamma^S(\Sigma \times \Gamma, X, S, I, f) = (\Sigma, \Gamma \times X, S, I, f)$$

and the definition of recognition in this category gives immediately that when $\mathcal{A}$ recognizes $\mathcal{L}$, $\exists_\Gamma^S \mathcal{A}$ recognizes $\exists_\Gamma^S \mathcal{L}$.

The effect of $\exists_\Gamma^S$ on arrow $(h_0, \phi) : (\Sigma\times\Gamma, X, S, I, f) \to (\Sigma\times\Gamma, Y, T, J, g)$ is $\exists_\Gamma^S(h_0) = \langle h_0, \pi_\Gamma\rangle : \Sigma\times\Gamma\times X \to Y\times\Gamma$. The resulting functor $\exists_\Gamma^S : S(\Lambda)_\Sigma \to S(\Lambda)_{\Sigma\times\Gamma}$ is left adjoint to the weakening functor.

Let us explain how the Sum construction encompasses witnesses with regards to existential quantification. The "True" formula on Σ, which describes the universal language Σ^+, is represented by the object $\mathbf{I} = (\Sigma, 1, \{1\}, \{1\}, 1)$, which is terminal in $S(\Lambda)_\Sigma$. A morphism from this object to $\mathcal{A} = (\Sigma, X, S, I, f)$ is a function $h_0 : \Sigma \to X$ along with the knowledge that the language recognized by $\mathcal{A}$ is Σ^+. This function gives a witness in the sense that for a given word $w \in \Sigma$, it gives a word $x \in X$ which is a witness for the formula $\exists x, f(\langle w, x\rangle) \in I$. In proof-theoretical terms, we may extract a witness from the proof of an existential formula.

3.7. Determinization in Sum

Informally, the X set in (Σ, X, S, I, f) represents some non-determinism introduced by existential quantification. In [PR17], the authors adopt a similar approach for a fibration of automata working on infinite words. In the automaton case, X is a set of "moves" which uniformly represent non-determinism: when reading a letter, a non-deterministic machine may follow several paths, which is what X quantifies. When $X = 1$ is a singleton, the object is deterministic. We now show how to determinize objects by constructing a powerset monad on $\mathsf{Sum}(\Lambda)$ which eliminates non-determinism by sending an object on Σ, X to an object on $\Sigma, 1$, which is deterministic. This operation is the analogue of determinization for automata.

Definition 3.4. *The powerset monad $\mathcal{P}$ on $\mathsf{Sum}(\Lambda)$ is the functor which acts on objects by sending (Σ, X, S, I, f) to $(\Sigma, 1, \mathcal{P}(S), \Diamond I, \pi_\Sigma(f))$. For an arrow $(h_0, \phi) : \mathcal{A} = (\Sigma, X, S, I, f) \to \mathcal{B} = (\Sigma, Y, T, J, g)$, the image by $\mathcal{P}$ is $(1, \phi') : (\Sigma, 1, \mathcal{P}(S), \Diamond I, \pi_\Sigma(f)) \to (\Sigma, 1, \widehat{\mathcal{P}}(T), \Diamond J, \pi_\Sigma(g))$*

To verify that this operation is well defined, we need to check that ϕ' is an arrow. In other terms, we need to show the following lemma:

Lemma 3.5. *Using the same notations as above, let $\mathcal{L}$ and $\mathcal{L}'$ be the languages respectively recognized by $\mathcal{A}$ and $\mathcal{B}$. Then $\exists_X \mathcal{L} \subseteq \exists_Y \mathcal{L}'$*

Proof. Let $\sigma \in \exists_X \mathcal{L}$. Then $\pi_\Sigma(f)(\sigma) \cap I \neq \emptyset$. Let $w \in (\Sigma \times \Gamma)^+$ such that $\pi_X(w) = \sigma$ and $f(w) \in I$. Then, since there is an arrow $\phi : (\Sigma \times X, S, I, f) \to (\Sigma \times X, T, J, g \circ \langle h \circ \pi, h_0 \rangle)$, $f(w) \in I$ implies that $g(\langle \pi_\Sigma(w), h_0(w) \rangle) = g(\langle \sigma, h_0(w) \rangle) \in J$. And since $\pi_\Sigma(\langle \sigma, h_0(w) \rangle) = \sigma$, then $g(\langle \sigma, h_0(w) \rangle)$ is in $\pi_\Sigma(g)(\sigma)$ as well. Therefore, $\pi_\Sigma(g)(\sigma) \in \Diamond J$, and thus $\sigma \in \exists_V \mathcal{L}'$. □

An important property of this determinization operator is that it preserves languages:

Lemma 3.6. *The languages recognized by $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$ are equal.*

Proof. Let $(\Sigma, X, S, I, f) = \mathcal{A}$. Then $\mathcal{P}(\mathcal{A}) = (\Sigma, 1, \widehat{\mathcal{P}}(S), \Diamond I, \pi_\Sigma(f)$. Let $\mathcal{L}, \mathcal{L}'$ be respectively recognized by $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$. Let $\sigma \in \Sigma^+$. Then:

$$\begin{aligned}
\sigma \in \mathcal{L} &\iff \exists x \in X, f(\sigma, x) \in I \\
&\iff \pi_\Sigma(f)(\sigma) \cap I \neq \emptyset \\
&\iff \pi_\Sigma(f)(\sigma) \in \Diamond I \\
&\iff \exists x \in 1, \pi_\Sigma(f)(\sigma) \in \Diamond I \\
&\iff \sigma \in \mathcal{L}'
\end{aligned}$$

□

It remains to show that $\mathcal{P}$ is a monad. Recall that a monad $T : \mathbb{C} \to \mathbb{C}$, is a functor equipped with two natural transformations, a unit $\eta : 1_\mathbb{C} \to T$ and a multiplication $\mu : T^2 \to T$, satisfying the two following conditions of commutativity:

$$\begin{aligned}
\mu \circ \mu_T &= \mu \circ T\mu \\
\mu \circ \eta_T &= \mu \circ T\eta = \mathrm{id}
\end{aligned}$$

In our case, η and μ exist due to Lem. 3.6, and they trivially satisfy the monad axioms due to the fact that $\mathcal{P}$ preserves languages and removes informations from arrows, in the sense that there is at most one arrow between two deterministic objects.

Since the powerset monad restricts to finite semigroups, we can consider the full subcategory of $S(\Lambda)$ of finite-state recognizers and in this subcategory we can still define quantification and determinization. We come back to these concepts in §6, once we have given some background on profinite theory.

4. Profiniteness

In this section, we recall the notions of profinite topological spaces and algebras, which are categorical limits of finite objects. Profinite objects are infinite, but, as *cofiltered* limits of finite objects they have a specific topology called Stone topology whose structure is highly compatible with algebraic recognition. See [DP02] for a modern approach to Stone topology.

4.1. Stone Topology

A topological space X is said to be a *Stone space*, or to have Stone topology, if it is compact Hausdorff and has a topological basis of clopens. For any topological space, the set of clopens is a Boolean algebra. But in the case of Stone spaces, this Boolean algebra entirely determines the space. The map taking a Stone space to its Boolean algebra of clopens induces a categorical duality between Stones spaces and Boolean algebras, in particular there is a one-to-one correspondence between the two. A continuous function $f : X \to Y$ between two Stone spaces contravariantly induces a Boolean morphism $f^{-1} : K\Omega(Y) \to K\Omega(X)$ between the clopens of those spaces, and conversely a morphism between two Boolean algebras contravariantly induces a continuous function between the corresponding Stone spaces.

The authors Gehrke, Petrisan and Reggio have done considerable work on Stone duality and algebraic recognition, like in [Geh16], where Gehrke gives an extension of Stone duality to general topological algebras and applies the result to recognition, or in [GPR16], where the authors study quantification in recognizers through Stone duality and derive generalized versions of Schützengerger and Reutenauer's theorems. The reader may refer to App. B.3 for some work that we did on the subject but did not include in the final version. In this appendix, we study the Stone spaces induced by Boolean algebras of languages and give a connection with free structures on pseudo-varieties.

4.2. Profinite Spaces

We now define profinite spaces, which have a natural Stone space structure. We refer to e.g. [Wil70] and [Run05] for results on general topology.

For the following notion, we refer to e.g. [Joh86, §I.3.9 & Chap. VI].

Definition 4.1 (Cofiltered Category)**.** *We say that a small category $\mathbb{J}$ is* cofiltered *if:*

- *$\mathbb{J}$ is non-empty (i.e. it has a least one object),*
- *for any object i, j of $\mathbb{J}$, there is an object k and arrows $k \to i$ and $k \to j$ in $\mathbb{J}$,*
- *for any pair of parallel arrows $f, g : i \to j$ in $\mathbb{J}$, there is an arrow $h : k \to i$ in $\mathbb{J}$ such that $fh = gh$.*

Let $F : \mathbb{J} \to \mathbf{Top}$ be a functor from a cofiltered category $\mathbb{J}$ to the category of topological spaces, with the $F(i)$ finite and discrete. We consider the product $\prod_{i\in\mathbb{J}} F(i)$, that we endow with the product topology. Recall that in such a product topology, basic opens are of the form $\prod_{i\in\mathbb{J}} S_i$, where $S_i \subseteq F(i)$ and $S_i \neq F(i)$ for at most finitely many i. The limit of F is the following subset of $\prod_{i\in\mathbb{J}} F(i)$, with the topology induced by the product topology that we just described.

$$\mathrm{Lim}(F) \quad = \quad \{(x_i)_{i\in\mathbb{J}} \in \textstyle\prod_{i\in\mathbb{J}} F(i) \mid \forall h : i \to_{\mathbb{J}} j,\ F(h)(x_i) = x_j\}$$

In the following, we write

$$\pi_i \quad : \quad \mathrm{Lim}(F) \quad \longrightarrow \quad F(i) \qquad\qquad (\text{for } i \in \mathbb{J})$$

for the projection, and given $x \in \mathrm{Lim}(F)$, we write either x_i or $x(i)$ for $\pi_i(x)$.

Definition 4.2. *A* profinite *space is a set which is the limit of a cofiltered diagram $F : \mathbb{J} \to \mathbf{Top}$ with the $F(i)$ finite and discrete.*

If we do not restrict the morphisms or the sets that are images by F, then $\pi_i : \mathrm{Lim}(F) \to F(i)$ may not reach all elements of $F(i)$. Conversely if we ask that for each $h : i \to_{\mathbb{J}} j$, $F(h)$ is surjective, then the projections π_i are as well. In fact this does not change the expressivity of limits:

Lemma 4.3. *Let X be a profinite space. Then X is the limit of a cofiltered diagram $F : \mathbb{J} \to$ **Top** with $F(i)$ finite and discrete for $i \in \mathbb{J}$ and $F(h)$ surjective for each morphism h in $\mathbb{J}$.*

Proof. We write $X = \mathrm{Lim}(F)$. For $i \in \mathbb{J}$, we consider $F'(i)$ the image of $\pi_i : \mathrm{Lim}(F) \to F(i)$. For i, j objects of $\mathbb{J}$ and $h, h' : i \to j$, the morphisms $F(h)$ and $F(h')$ are equal when restricted to $F'(i)$ since $\pi_j = h \circ \pi_i = h' \circ \pi_i$. We therefore define $F'(h)$ to be this restriction. $F'(h)$ takes values in $F'(j)$, and so F' is functorial, and it is easy to see that F and F' have the same limit. □

Remark 4.4. *In the case when F is such that morphisms $F(h)$ are all surjective, all morphisms from i to j in $\mathbb{J}$ are mapped to the same function $F(i) \to F(j)$.*

We now consider $X = \mathrm{Lim}(F)$ with $F : \mathbb{J} \to$ **FinSet** taking morphisms to surjective functions, and $\mathbb{J}$ cofiltered. We write $X_i = F(i)$, and $\varphi_{i,j}$ the unique morphism between X_i and X_j when it exists. X is a subset of the product $\prod_{i \in \mathbb{J}} X_i$, and as such we may endow it with the product topology, where each $F(i)$ is considered w.r.t. discrete topology. Explicitly, basic opens are sets of the form $X \cap \prod_{i \in \mathbb{J}} S_i$, where $S_i \subseteq X_i$ and $S_i \neq X_i$ for at most finitely many i. Let K be the subset of objects of $\mathbb{J}$ such that $S_i \neq X_i$. Then $X \cap \prod_{i \in \mathbb{J}} S_i = \bigcap_{i \in K} \pi_i^{-1}(S_i)$.

We may restrict ourselves to K singleton, in the following sense. Since $\mathbb{J}$ is cofiltered and K is finite, there is a $i_0 \in \mathbb{J}$ such that for all $i \in K$, there is an arrow $h_{i_0,i} : i_0 \to i$ in $\mathbb{J}$. We consider the set $S_{i_0} = \{\alpha \in F(i_0) \mid \forall i \in K, \phi_{i_0,i}(\alpha) \in S_i$. Then:

$$\bigcap_{i \in K} \pi_i^{-1}(S_i) = \pi_{i_0}^{-1}(S_{i_0})$$

Indeed:

$$\begin{aligned} x \in \textstyle\bigcap_{i \in K} \pi_i^{-1}(S_i) \quad &\iff \quad \forall i \in K, x_i \in S_i \\ &\iff \quad x_{i_0} \in S_{i_0} \\ &\iff \quad x \in \pi_{i_0}^{-1}(S_{i_0}) \end{aligned}$$

As the consequence, we have the following:

Lemma 4.5. *X has the family $\mathcal{B} = (\pi_i^{-1}(S))_{i \in \mathbb{J}, S \subseteq X_i}$ as a basis of opens. Moreover, $\mathcal{B}$ is a Boolean algebra of clopen subsets of X.*

Proof. Since the complement in X of a basic open of the form $\pi_i^{-1}(S)$ is $\pi_i^{-1}(F(i) \setminus S)$. □

Proposition 4.6. *The set $\mathrm{Lim}(F)$ equipped with the product topology is a Stone space.*

It is well-known [Joh86] that Stone topology and profinite topology are the same thing. We nonetheless give the proof that a profinite set has a Stone topology as it is quite simple.

Proof of Prop. 4.6. First, since the $F(i)$'s are compacts (as finite discrete) spaces, it follows from Tychonoff's Theorem (see e.g. [Wil70, Thm. 17.8]) that $\mathrm{Lim}(F)$ is compact. It is moreover clearly Hausdorff as if $x \neq y$, then $x_i \neq y_i$ for some $i \in \mathbb{J}$, while $x \in \pi_i^{-1}(\{x_i\})$, $y \in \pi_i^{-1}(\{y_i\})$, and the basic opens $\pi_i^{-1}(\{x_i\})$ and $\pi_i^{-1}(\{y_i\})$ are obviously disjoint. Finally, it follows from Lem. 4.5 that $\lim(F)$ has a basis of clopen sets. □

We state the following useful fact for the record.

Lemma 4.7. *If K is clopen in $\mathrm{Lim}(F)$ then $K \in \mathcal{B}$.*

Proof. Since K is open, it may be written as a union of members of $\mathcal{B}$:

$$K \quad = \quad \bigcup_{i \in I} C_i$$

But since K is a closed subset of a compact space, it is itself compact, so that we may take I to be finite. It then follows that $K \in \mathcal{B}$ since $\mathcal{B}$ is a Boolean algebra. □

4.3. Profinite Algebras

Consider now an algebraic variety $\mathbf{V}$ (*i.e.* a class of algebras over a fixed signature, and characterized by a given set of equations), One can show (see App. B.1) the following:

Lemma 4.8. *Let $F : \mathbb{J} \to \mathbf{V}$ be a diagram, with $\mathbb{J}$ cofiltered. Then $\mathrm{Lim}(F)$ is a topological $\mathbf{V}$-algebra with a Stone topology if the $F(i)$ are finite and discrete.*

Then we may use this result to define profinite semigroups, as follows:

Definition 4.9. *A profinite semigroup is a topological semigroup which is a cofiltered limit of finite semigroups each equipped with discrete topology.*

A known result due to Almeida is that a topological semigroup is profinite if, and only if, it has a Stone topology [Alm05, Th. 3.1].

For a given pseudo-variety $\mathbf{V}$ and an alphabet Σ, it is possible to construct the free pro-$\mathbf{V}$ semigroup over Σ [Alm05], $\overline{\Omega}_\Sigma(\mathbf{V})$, which satisfies the following universal property: There exists an injection $\iota : \Sigma \to \overline{\Omega}_\Sigma(\mathbf{V})$ such that for every pro-$\mathbf{V}$ semigroup, and every function $f : \Sigma \to S$, f factors as $\hat{f} \circ \iota$ uniquely, for a continuous semigroup morphism $\hat{f} : \overline{\Omega}_\Sigma(\mathbf{V}) \to S$. Free pro-$\mathbf{V}$ semigroups are important in the theory of recognition: Reiterman's theorem states that pseudo-varieties have equational descriptions in terms of free pro-$\mathbf{V}$ semigroups.

4.4. Open and Closed Languages

Recall that a recognizer is a tuple (Σ, S, I, f) with S a semigroup, $I \subseteq S$ and $f : \Sigma \to S$ a function. We consider profinite recognizers, *i.e.* recognizers where S is profinite. A known result (see [Alm05]) is that when restricting the set I to exclusively clopens of S, profinite recognizers have the same expressivity as finite ones, in the sense that they recognize exactly the class of regular languages. However we may also want to study other classes, like open and closed languages, which are recognized respectively by open and closed subsets I, and even languages recognized by arbitrary subsets.

5. Powersets of (Profinite) Semigroups

Now that we have introduced profinite semigroups, we need to describe the effect of the powerset operation on such objects, as we use this operation for quantification/determinization. Consider a semigroup S and its powerset $\mathcal{P}(S)$ as in Lem. 2.7. If S is finite, then so is $\mathcal{P}(S)$. But if S is profinite, then $\mathcal{P}(S)$ has in general no reason to be profinite. We discuss an operation $\widehat{\mathcal{P}}$ on profinite semigroups such that $\widehat{\mathcal{P}}(S)$ corresponds to the *closed* subsets of S. If S is a finite semigroup, then it is in particular profinite for the discrete topology and we will have $\widehat{\mathcal{P}}(S) = \mathcal{P}(S)$.

5.1. On Profinite Monads

In order to extend the *finite* powerset operation, which is a monad, from finite (discrete) Stone spaces to all Stone (*i.e.* profinite) spaces, we recall two concepts of category theory: Kan extensions and codensity monads.

Consider two small categories $\mathbb{B}$ and $\mathbb{C}$, and a complete (having all limits) category $\mathcal{E}$. Let $K : \mathbb{B} \to \mathcal{E}$ and $F : \mathbb{B} \to \mathbb{C}$ be two functors. Then the right Kan extension of K along F is a functor $\mathrm{Ran}_F(K) : \mathbb{C} \to \mathcal{E}$, which can be be computed on objects as the following limit (see e.g. [ML98, Thm. X.3.1]):

$$\mathrm{Ran}_F(K)(B) \quad := \quad \mathrm{Lim}\left((B \downarrow F) \xrightarrow{\pi} \mathbb{B} \xrightarrow{K} \mathcal{E}\right)$$

where the comma category $(B \downarrow F)$ is the following:

- Objects are tuples $(D, h : FB \to FD)$ with $D \in \mathbb{B}$.
- A morphism $m : (D, h) \to (D', h')$ is a $\mathbb{B}$-morphism $m : D \to D'$ such that $Fm \circ h = Fh'$

Intuitively, the right Kan extension operation extends the functor K, which was defined on $\mathbb{B}$, to the category $\mathbb{C}$. In the case when $\mathbb{B} = \mathbb{C}$ and $K = F$, it is possible to endow $\mathrm{Ran}_K(K)$ with a monad structure, called the codensity monad of K. Once again, it is possible to compute it as the following limit:

$$\mathrm{Ran}_K(K)(C) \quad := \quad \mathrm{Lim}\left((C \downarrow K) \xrightarrow{\pi} \mathbb{C} \xrightarrow{K} \mathcal{E}\right)$$

In [CAMU16], the authors introduce the notion of *profinite monad* which is based on *codensity* monads. It is defined as follows. Let (T, η, μ) be a monad in **Set**. Write $\mathbf{Set}^T$ for the category of (Eilenberg-Moore) T-algebras (see App. D.2), and $\mathbf{FinSet}^T$ for the category of *finite* algebras, *i.e.* the full subcategory of $\mathbf{Set}^T$ with objects the algebras $(A, \alpha : TA \to A)$ with A a finite set (note that TA is in general not finite). We write U for the composite

$$\mathbf{FinSet}^T \longrightarrow \mathbf{FinSet} \longrightarrow \mathbf{Stone}$$

where $\mathbf{FinSet}^T \to \mathbf{FinSet}$ is the forgetful functor and $\mathbf{FinSet} \to \mathbf{Stone}$ sees a finite set as a finite (discrete) Stone space. The *profinite* monad of (T, η, μ), notation $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$, is then defined as the codensity monad of

$$\mathsf{U} \quad : \quad \mathbf{FinSet}^T \quad \longrightarrow \quad \mathbf{Stone}$$

Let us unfold the definitions to get a good idea of the situation. $\mathsf{U} : \mathbf{FinSet}^T \to \mathbf{FinSet} \to \mathbf{Stone}$ forgets the $T-$algebra structure, and then embeds **FinSet** into **Stone** via the discrete topology. We write A again for $\mathsf{U}(A, \alpha)$. We consider a fixed Stone space X. The comma category $(X \downarrow \mathsf{U})$ has objects (A, α, h) with $(A, \alpha) \in \mathbf{FinSet}^T$ and $h : X \to A$ in **Stone** and morphisms $m : (A, \alpha, h) \to (B, \beta, k)$ $T-$algebra morphisms between (A, α) and (B, β) such that Fig. 1 (a) commutes. We consider the diagram:

$$\mathscr{D}_X : (X \downarrow \mathsf{U}) \xrightarrow{\mathsf{cod}} \mathbf{FinSet}^T \xrightarrow{\mathsf{U}} \mathbf{Stone}$$

where the cod functor maps (A, α, h) to (A, α). Then $\hat{T}(X)$ is the limit of $\mathscr{D}_X$.

Remark 5.1 (**Stone** is complete)**.** *It is well-known that the category* **Stone** *of Stone spaces and continuous functions is complete. See e.g. [BG01, Lem. 3.4.5 & Thm. 3.4.7] or [Joh86, Thm. VI.1.6, §VI.1.9 & Thm. VI.2.3].*

We are interested in giving a more concrete presentation of $\widehat{T}$, when T *restricts to* **FinSet**, that is, when TA is finite whenever A is finite.

Remark 5.2. *It is well-known that the monads of finite powerset, free semilattices, free distributive lattices, free Boolean algebras restrict to* **FinSet** *(see e.g. [Joh86, §I.4]). On the other hand, this is* not *the case of free lattices (see e.g. [Joh86, Cor. I.4.6]), free semigroups, free monoids.*

5.2. A Pointwise Presentation of $\widehat{T}(X)$

We fix a monad (T, η, μ) on **FinSet**. Consider a Stone space X. We can assume that X is presented as a cofiltered limit of finite (discrete) spaces as $X = \mathrm{Lim}F$ for $F : \mathbb{J} \to \mathbf{Stone}$ with $\mathbb{J}$ cofiltered and $F(i)$ a finite (discrete) space for each $i \in \mathsf{Obj}(\mathbb{J})$. For $i \in \mathsf{Obj}(\mathbb{J})$, let $X_i = F(i)$. We can assume wlog that each projection $\pi_i : X \to X_i$ is surjective. Note that this implies that there is at most one connection morphism $m_{i,j} : X_i \to X_j$ from X_i to X_j, which is moreover necessarily surjective. We shall see that

$$\widehat{T}(X) \quad \simeq \quad \mathrm{Lim}_i T X_i$$

One may show moreover that the unit $(\widehat{\eta}_X)$ and multiplication $(\widehat{\mu}_X)$ of the profinite monad $\widehat{T}$ at X are induced by the unit and multiplication of the monad T at the components $(X_i)_i$ of X.

As we will see in §5.3, the presentation of $\widehat{T}(X)$ as $\mathrm{Lim}_i T X_i$ is quite conventient to work with.

Proposition 5.3. *Let* (T, η, μ) *be a monad on* **FinSet** *and let* $X = \mathrm{Lim}F$ *with* $F : \mathbb{J} \to \mathbf{Stone}$ *where* $\mathbb{J}$ *is cofiltered, and where for each* $i \in \mathbb{J}$*,* $F(i)$ *is a finite (discrete) space and the projection* $\pi_i : X \to F(i)$ *is surjective. Then*

$$\widehat{T}X \quad \simeq \quad \mathrm{Lim}TF$$

The proof of Prop. 5.3 is based on the usual fact that limits are preserved by reindexing with *initial* functors (see e.g. [ML98, §IX.3] for the dual notion of *final* functor).

Definition 5.4. *A functor* $L : \mathbb{A} \to \mathbb{B}$ *is* initial *if for each object* $B \in \mathbb{B}$*, the comma category* $(L \downarrow B)$ *is nonempty and connected.*

The key property of final functors is that they do not affect limits, in the following sense. See [ML98, Thm. IX.3.1] for the dual result on final functors and colimits.

Theorem 5.5. *Let* $G : \mathbb{B} \to \mathbb{C}$ *be a functor, and let* $L : \mathbb{A} \to \mathbb{B}$ *be an initial functor. The canonical morphism* $\mathrm{Lim}(G) \to \mathrm{Lim}(G \circ L)$ *is an isomorphism.*

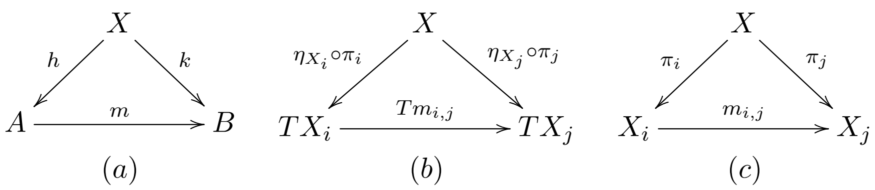


Figure 1: Commutative Triangles

We shall apply Thm. 5.5 to a functor

$$L \;:\; \mathbb{J} \;\longrightarrow\; (X \downarrow \mathsf{U})$$

As seen earlier, the category $(X \downarrow \mathsf{U})$ has objects of the form (A, α, h), where $(A, \alpha) \in \mathbf{FinSet}^T$ is a finite T-algebra and $h : X \to A$ is a continuous function, and morphisms $m : (A, \alpha, h) \to (B, \beta, k)$ are T-algebra morphisms from (A, α) and (B, β) such that Fig. 1 (a) commutes.
We consider the functor $L : \mathbb{J} \longrightarrow (X \downarrow \mathsf{U})$,

$$\begin{array}{rcl} i & \longmapsto & (TX_i,\, \mu_{X_i},\, \eta_{X_i} \circ \pi_i) \\ (i \to j) & \longmapsto & Tm_{i,j} \end{array}$$

As usual with monads, the naturality of μ implies that $Tm_{i,j}$ is indeed a T-algebra morphism from (TX_i, μ_{X_i}) to (TX_j, μ_{X_j}). Moreover, the commutation of Fig. 1 (b) follows from the naturality of η and the commutation of Fig. 1 (c)
Consider now the composite

$$\mathbb{J} \xrightarrow{L} (X \downarrow \mathsf{U}) \xrightarrow{\pi} \mathbf{FinSet}^T \xrightarrow{\mathsf{U}} \mathbf{Stone}$$

It takes $m_{i,j} : i \to j$ in $\mathbb{J}$ to $Tm_{i,j} : TX_i \to TX_j$ in $\mathbf{Stone}$, so that

$$\mathbb{J} \xrightarrow{L} (X \downarrow \mathsf{U}) \xrightarrow{\pi} \mathbf{FinSet}^T \xrightarrow{\mathsf{U}} \mathbf{Stone} \;\;=\;\; \mathbb{J} \xrightarrow{F} \mathbf{FinSet} \xrightarrow{T} \mathbf{FinSet}^T \xrightarrow{\mathsf{U}} \mathbf{Stone}$$

Hence $\mathrm{Lim}(\mathsf{U}\pi L) = \mathrm{Lim} UTF$. Since $\widehat{T}(X) = Lim(\mathsf{U}\pi)$, it remains to show that L is initial. This is deferred to App. D.

5.3. Profinite Powerset

We now look at the specific case of the finite powerset monad $\mathcal{P}$. Let X be a profinite Stone space, limit of $(X_i)_{i \in \mathbb{J}}$, with $\mathbb{J}$ a codirected category.

Following from the previous section, we define $\widehat{\mathcal{P}}$ as the profinite monad corresponding to $\mathcal{P}$. Then $\widehat{\mathcal{P}}(X)$ is the limit of $(\mathcal{P}(X_i))_{i \in \mathbb{J}}$, *i.e.* $\hat{\mathcal{P}}(X) = \{(S_i)_i \in \Pi X_i \mid \forall m : X_i \to X_j, m(S_i) = S_j\}$. One can show that monadic multiplication is computed pointwise by union, and that the monadic unit sends $x = (x_i)_{i \in \mathbb{J}} \in X$ to $(\{x_i\})_{i \in \mathbb{J}} \in \widehat{\mathcal{P}}(X)$, but we shall not use these structure morphisms in this report.

An important observation for $\widehat{\mathcal{P}}$ to be well defined as a monad through the pointwise presentation introduced in the previous section, is that the $\mathcal{P}$ operation on finite sets preserves surjectivity. Indeed, recall that we assumed for Prop. 5.3 that X was presented as the limit of a diagram with surjective morphisms. Since $\mathcal{P}$ preserves surjectivity, $\widehat{\mathcal{P}}(X)$ is once again the limit of a diagram with surjective morphisms, and as a consequence (see [CAMU16]) its projections are surjective as well. Thus, it is possible to see $\widehat{\mathcal{P}}\widehat{\mathcal{P}}(X)$ as the limit of $(\mathcal{P}(\mathcal{P}(X_i)))_{i\in\mathbb{J}}$ without problem. We do not use the monad structure in the rest of the report, but this ensures that it is compatible with the presentation of $\widehat{\mathcal{P}}$ that we use.

Notice also that $\widehat{\mathcal{P}}(X)$ is not equal to $\mathcal{P}(X)$, but we shall soon see that it represents the subset of $\mathcal{P}(X)$ comprised of the closed subsets of X.

Lemma 5.6. *Let* **closure** *and* **pref** *be the following operators:*

$$\begin{array}{rcrcl} \mathbf{closure} & : & \widehat{\mathcal{P}}(X) & \longrightarrow & \mathcal{P}(X) \\ & & (S_i)_i & \longmapsto & \{(x_i)_i \in X \mid \forall i, x_i \in S_i\} \end{array}$$

$$\begin{array}{rcrcl} \mathbf{pref} & : & \mathcal{P}(X) & \longrightarrow & \widehat{\mathcal{P}}(X) \\ & & T & \longmapsto & (\{x_i \mid x \in T\})_i \end{array}$$

Then **closure** *and* **pref** *form a Galois connection with the pointwise order on $\widehat{\mathcal{P}}(X)$. In other words, for $T \in \mathcal{P}(X)$ and $(S_i)_i \in \widehat{\mathcal{P}}(X)$:*

$$\forall i \in \mathbb{J}, \mathbf{pref}(T)_i \subseteq S_i \iff T \subseteq \mathbf{closure}((S_i)_i)$$

Proof. ($\Rightarrow$): Assume $T \subseteq \mathbf{closure}((S_i)_i)$. Let $i \in \mathbb{J}$ and $y \in \mathbf{pref}(T)_i$. Let $x \in T$ such that $x_i = y$. Then x is in $\mathbf{closure}((S_i)_i)$ and so $x_i = y$ is in S_i. So $\mathbf{pref}(T)_i \subseteq S_i$

($\Leftarrow$): Assume $\mathbf{pref}(T)_i \subseteq S_i$ for all $i \in \mathbb{J}$. Let $x \in T$. Then $x_i \in \mathbf{pref}(T)_i \subseteq S_i$ for all i. So $x \in \mathbf{closure}((S_i)_i)$, and so $T \subseteq \mathbf{closure}((S_i)_i)$. □

We now relate these two operators to the topology of X through the following lemma:

Proposition 5.7. *Let $T \in \mathcal{P}(X)$. The three following conditions are equivalent:*

(i) T is closed in X

(ii) T is in the image of **closure**

(iii) $T = \mathbf{closure}(\mathbf{pref}(T))$

Proof. Let us first recall what it means to be closed in X. In sight of 4.6, a subset $K \subseteq X$ is closed if, and only if, for each $x \notin T$ there exists $i \in \mathbb{J}$ and $K_i \subseteq X_i$, such that $T \cap \pi_i^{-1}(K_i) = \emptyset$ and $x_i \in K_i$.

- $(i) \Rightarrow (iii)$: Assume (i). The Galois connection between **pref** and **closure** ensures that $T \subseteq \mathbf{closure}(\mathbf{pref}(T))$. Let $x \notin T$. Let $i, (K_i)$ be as above. Then $x_i \in K_i$. Let us show that $x_i \notin \mathbf{pref}(T)_i$, thus proving that $x \notin \mathbf{closure}(\mathbf{pref}(T))$. Since $T \cap \pi_i^{-1}(K_i) = \emptyset$, any $y \in T$ is such that $y_i \notin K_i$, and in particular is such that $y_i \neq x_i$, so $x_i \notin K_i$.
- $(ii) \Rightarrow (i)$: Assume (ii). Let $S = (S_i)_{i\in\mathbb{J}} \in \widehat{\mathcal{P}}(X)$ such that $T = \mathbf{closure}(S)$. Let $x \notin T$. Then for some i, $x_i \notin S_i$. Since $\pi_i^{-1}(X_i \backslash S_i) \cap \mathbf{closure}(S) = \emptyset$ and $x_i \in X_i \backslash S_i$, we have proven that T is closed.
- $(iii) \Rightarrow (ii)$ is trivial. □

We now show an analogous statement for $\mathbf{pref} \circ \mathbf{closure}$. For $S \in \widehat{\mathcal{P}}(X)$, the Galois connection ensures that $\mathbf{pref}(\mathbf{closure}(S)) \leq S$, *i.e.* that $\mathbf{pref}(\mathbf{closure}(S))_i \subseteq S_i$ for all $i \in \mathbb{J}$.

Proposition 5.8. *We have* $S_i \subseteq \mathbf{pref}(\mathbf{closure}(S))_i$ *for all* $i \in \mathbb{J}$

Proof. Let $i \in \mathbb{J}$ and $\alpha \in S_i$. We want to show that α is in $\mathbf{pref}(\mathbf{closure}(S))_i$, *i.e.* that there is an $x \in \mathbf{closure}(S)$ such that $x_i = \alpha$. For F finite subset of the set of objects of $\mathbb{J}$, one may find an $x^F \in X$ such that $x^F_j \in S_j$ for $j \in F$ and $x^F_i = \alpha$. $(x^F)_{F \subseteq \mathbb{J}}$ is a net in X (see Def. B.2 in the appendix for recalls on nets), and by compactness has a converging subnet $(x^F)_{F \in I}$ where I is a subset of $\mathcal{P}_{fin}(\mathbb{J})$, with limit x. Let us show that x is in $\mathbf{closure}(S)$ and that $x_i = \alpha$.

For $j \in \mathbb{J}$, consider the open set $\pi^{-1}(\{x_j\})$ in X. It is a neighbourhood of x, and so there exists $F_j \in I$ such that for F containing F_j, $x^F \in \pi^{-1}(\{x_j\})$. In particular, for $F_j \cup \{j\}$, this means that $x_j \in S_j$. Thus, x is in $\mathbf{closure}(S)$, and so $\alpha \in \mathbf{pref}(\mathbf{closure}(S))_i$. □

5.4. Vietoris Space

What we have proven is that **pref** and **closure** realise a bijection between $\widehat{\mathcal{P}}(X)$ and the set of closed sets of X. In literature, like in [Kur67], the latter is seen as a topological space $\mathscr{V}(X)$, with a topology called the Vietoris topology or hypertopology. We refer to [GPR16] for more on the topic, and we simply give some recalls on this topology.

Notation 5.9. *For $I \subseteq X$, we introduce the sets $\Diamond I$ and $\Box I$ in $\mathcal{P}(\mathscr{V}(X))$ as follows:*

- $\Diamond I = \{F \in \mathscr{V}(X) \mid F \cap I \neq \emptyset\}$
- $\Box I = \{F \in \mathscr{V}(X) \mid F \subseteq I\}$

Intuitively, these sets correspond respectively to existential and universal quantification. Notice that $\Box I = \mathscr{V}(X) \setminus \Diamond(X \setminus I)$.

It is easy to see that $\Diamond$ preserves arbitrary unions, and that $\Box$ preserves arbitrary intersections. The Vietoris space over X has its topology induced by the following subbasis of opens:

$$\{\Diamond I \mid I \text{ open in } X\} \cup \{\Box I \mid I \text{ open in } X\}$$

Remark 5.10. *By definition of the topology, when I is open, so are $\Diamond I$ and $\Box I$, and it is easy to see that this extends to closed sets, and thus to clopen sets.*

Bearing this in mind, we recall the following:

Proposition 5.11. *The topology of $\mathscr{V}(X)$ has for a subbasis of clopens the following family: $\{\Diamond I \mid I \text{ clopen in } X\} \cup \{\Box I \mid I \text{ clopen in } X\}$*

Now, we want to prove that $\widehat{\mathcal{P}}(X)$ and $\mathscr{V}(X)$, which are in bijection via **pref** and **closure**, are also homeomorphic. Thus, we will prove that **pref** and **closure** are continuous. Since they are inverse of each other, it suffices to show that they are both open, *i.e.* that they each send basic or subbasic open sets to open sets.

Lemma 5.12. *The functions* **closure** *and* **pref** *are open.*

Proof. See Appendix D. □

We have now proven that the Vietoris space over X and $\widehat{\mathcal{P}}(X)$ are homeomorphic spaces. In the following sections, we will prefer using the Vietoris presentation of this space, however the homeomorphism with $\widehat{\mathcal{P}}(X)$ ensures, via results on profinite monads, that this space is a Stone space, and that it has some good properties. Using the bijection between the two spaces, we may explicit the semigroup operation on the Vietoris space by first applying **pref**, then taking the pointwise product in $\widehat{\mathcal{P}}(X)$, and applying **closure**.

6. Profinite Sum Fibration

In this section we combine the results of §3 and §5. We consider the fibred category Λ, where objects of $\Lambda(\Sigma)$ are tuples (Σ, S, I, f) with S a profinite semigroup, I a subset of S and $f : \Sigma \to S$ generating a dense subsemigroup of S (see App B.2 for details on this choice). The language recognized by (Σ, S, I, f) is $\mathcal{L} = f^{-1}I \subseteq \Sigma^+$.

On this fibration, we define existential and universal quantification, as follows:

$$\begin{array}{rcl} \exists_\Gamma(\Sigma \times \Gamma, S, I, f) & = & (\Sigma, \widehat{\mathcal{P}}(S), \Diamond I, \pi_\Gamma(f)) \\ \forall_\Gamma(\Sigma \times \Gamma, S, I, f) & = & (\Sigma, \widehat{\mathcal{P}}(S), \Box I, \pi_\Gamma(f)) \end{array}$$

To ensure that this is well-defined, we need to check that $\pi(f)$ has values in $\widehat{\mathcal{P}}(S)$. For a given $u \in \Sigma^+$ the set $\pi^{-1}(u)$ is finite, and therefore so is its image by f. Therefore, since singletons (and so finite sets) are closed in S, $\pi(f)(u)$ is closed. Both quantifications are functors, and respectively left and right adjoints to the substitution functor $\pi_\Sigma^\bullet$. The proofs are the same as in the finite case.

Applying the Sum construction on Λ yields the category $S(\Lambda)$ as follows:

- Objects are tuples (Σ, X, S, I, f) with Σ and X alphabets, S a profinite semigroup, I a subset of S, and $f : \Sigma \times X \to S$ a function generating a dense subset of S.
- Arrows from (Σ, X, S, I, f) to (Γ, Y, T, J, g) are triples (h, h_0, φ) with $h : \Sigma \to \Gamma$ a function, $h_0 : \Sigma \times X \to Y$ a function, and $\phi : (\Sigma \times X, S, I, f) \to (\Sigma \times X, T, J, g \circ \langle h \circ \pi, h_0 \rangle)$ an arrow in $\Lambda_{\Sigma \times X}$.

As in the finite case, the X set represents existential quantification, in the sense that the object (Σ, X, S, I, f) recognizes the language $\{\sigma \in \Sigma^+ \mid \exists w \in \pi_\Sigma^{-1}(\sigma), f(w) \in I\}$. On $S(\Lambda)(\Sigma)$, we once again define the powerset monad or determinization monad $\mathcal{P}$ as:

$$\mathcal{P}(\Sigma, X, S, I, f) \quad = \quad (\Sigma, 1, \widehat{\mathcal{P}}(S), \Diamond I, \pi_\Gamma(f))$$

It is easy to check that, like in the finite case, this operation is a monad and preserves languages.

As mentioned at the end of §4.3, we may want to study languages which are recognized by clopen, open, closed, or other classes of subsets I. Assume for instance that we want to study open languages. We consider the subcategory Λ^O of Λ of objects (Σ, S, I, f) with I open. Then notice that since the $\Diamond$ operator sends opens to opens, the $\exists_\Gamma$ functor restricts to $\Lambda^O_{\Sigma\times\Gamma} \to \Lambda^O_\Sigma$, and so quantification stabilizes open languages. This was a result that we expected, in fact we thought a priori that our notion of recognizer fibration would work only for open, closed and clopen sets. A surprising result is that the topology of $\widehat{\mathcal{P}}$ does not have any impact on the kind of subsets that we may consider: the fibration that we construct in this report has arbitrary subsets I.

7. Conclusion

We have described a profinite extension $\widehat{\mathcal{P}}$ to the powerset monad on semigroups, and used to define a fibred category Λ of profinite algebraic recognizers for finite words, which correspond to profinite-state automata. We have also applied the Sum construction on this fibration in order to obtain a more complex category where existential quantification comes with witnesses. What remains to be done is to see whether this setting solves the issues that occurred in the realm of finite-state automata.

Using profinite structures in the context of language theory has been done for some time, by Almeida [Alm05] and Pin [Pin09] for instance. However the idea of studying languages of profinite words is relatively new. Szymon Toruńczyk's thesis [Tor11] studies these languages and the corresponding notions of recognition and logic. During this internship, we also worked for some time on the subject of profinite word, as we intended to study recognition of such words as well. Appendix B.2 contains a more in-depth approach to profinite semigroups, and a discussion on profinite words and why we did not include them in the final version of this report.

As stated in the introduction, the motivation for exploring profinite-state automata comes from the difficulty of defining some operations on alternating automata. Appendix A gives some insight on the issue, using the notions and notations introduced in this report, along with a brief account of the role of Linear Logic in this setting.

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A. Linear Logic

In this section we elaborate on what is said at the end of section 3, and exhibit the link between some issues encountered and linear logic.

The Sum construction gives free existential quantification. There is an analogous construction, Prod, which gives free universal quantification. By applying Prod and then Sum, we obtain an operator $\mathsf{Dial} = \mathsf{Sum} \circ \mathsf{Prod}$. Consider a fibration p. $\mathsf{Dial}(p)$ is a category where formulas of the sort $\exists u, \forall x, P$ are freely possible: objects in this category are tuples $(A, X, U, \mathcal{A})$ with $\mathcal{A}$ an object in the fibre over $A \times X \times U$, and in the same way that X is a set which is existentially quantified upon, U is used for universal quantification. The object $(A, X, U, \mathcal{A})$ is thus interpreted as the formula $\exists x, \forall u, \alpha(a, x, u)$.

We saw that Sum constructs recognizers which work like non deterministic automata. Analogously, Prod constructs objects which act like universal automata, in which *every* run must be accepting for a word to be valid. In the same way, Dial is related to alternating automata, which may be seen as games where one player, ∃loise, is trying to find an accepting run on a word, and her opponent, ∀lexandre, is trying to prevent this from happening. The link between Sum and non deterministic automata is explored in [PR17], while [PR19] talks about Dial and alternating automata. For the sake of clarity, we will use Dial as an informal analogy for alternating automata, to explain the motivation behind our work, but keep in mind that the real situation is a bit more complex.

There is an operation called "simulation" which transforms an alternating automaton into a non-deterministic automaton, which translates into an operation from $\mathsf{Dial}(p)$ to $\mathsf{Sum}(p)$ in terms of category. This operation maps an object $\mathcal{A}$ over (Σ, X, U) into an object $!\mathcal{A}$ over $(\Sigma, !X, 1)$ by moving the universal part U into the existential part X.

A reader familiar with linear logic may have seen the ! symbol in that context. Let us recall some notions from this field of research. Linear logic was created from the following observation: intuitionistic logic differs from classical logic in that the weakening and contraction rules may only be applied to the left-hand side of sequents (the hypothesis) and not to the right-hand side. In linear logic, weakening and contraction may only be applied to specific formulas. Two operators, ! and ?, respectively called "of course" and "why not", quantify this idea: only formulas of the sort $!P$ may be contracted and weakened on the left-hand side, and only formulas of the sort $?P$ on the right-hand side.

In categorical models of linear logic (see [Mel09] for more), conjunction comes in the form of a tensor product $\otimes$, which acts as a parallelization: if we have two objects $\mathcal{A} = (\Sigma, X, U, ...)$ and $\mathcal{B} = (\Sigma, Y, V, ...)$, then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is $(\Sigma, X \times Y, U \times V, ...)$. Although we would like a categorical model of logic to be able to prove all sequents $P \vdash P \wedge P$, for an object $\mathcal{A}$ it is in general impossible to find a morphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Indeed, such a morphism would require a function $\Sigma \times X \times U \times U \to U$, in other words, a way to multiply elements of U. The ! operator, which removes the U from objects, has therefore another property: going from $!\mathcal{A}$ to $!\mathcal{A} \otimes !\mathcal{A}$ is possible. Dually, there is a disjunction operator $\oplus$, and ?, which removes the X part of objects, makes it possible to go from $?\mathcal{A} \oplus ?\mathcal{A}$ to $?\mathcal{A}$, which is not possible for general formulas.

When considering infinite automata, or infinite semigroups, ! and ? are functorial operators. However, when restricting to finite objects and infinite words, they cause a loss of information which makes it hard to ensure composition is preserved (in [Rib20] for instance). The question thus arises of whether or not it is possible to find a setting in which functoriality of these operators remained, while being somewhat restricted in order to represent faithfully questions of recognizability. We use profinite objects in order to find a balance between infinite objects, in which ! and ? are functorial, and finite objects, which are naturally studied in the area of recognizability.

B. On Stone Duality and Free pro-V Semigroups

This appendix gives some additional details on concepts introduced in §2 and §4, some of which were not used in the final paper, but still provide interesting insights and raise some questions. We begin with the proof of Lem. 4.8. Then we give some results on the structure of the Pro-**V** semigroup for a pseudo-variety **V**, and finally we discuss Stone duality.

B.1. Lemma 4.8

Lemma B.1. *Let* $\mathbf{V}$ *be an algebraic variety. Let* $F : \mathbb{J} \to \mathbf{V}$ *be a diagram, with* $\mathbb{J}$ *cofiltered. Then* $\mathrm{Lim}(F)$ *is a topological* $\mathbf{V}$*-algebra with a Stone topology if the* $F(i)$ *are finite.*

Proof. Consider the product $\prod_{i \in \mathbb{J}} F(i)$. For each fundamental operation f of arity n of $\mathbf{V}$ we have an n-ary function

$$\begin{array}{lcccl} f_\Pi & : & \left(\prod_{i \in \mathbb{J}} F(i)\right)^n & \longrightarrow & \prod_{i \in \mathbb{J}} F(i) \\ & & (x_1, \ldots, x_n) & \longmapsto & (f_i(x_1(i), \ldots, x_n(i)) \mid i \in \mathbb{J}) \end{array}$$

where f_i is the corresponding n-ary function on $F(i)$. It is easy to see that $\prod_{i \in \mathbb{J}} F(i)$ is a $\mathbf{V}$-algebra (since it satisfies any equation satisfied by all the $F(i)$'s).

Now, each f_Π restricts to a function

$$\mathrm{Lim}(F)^n \quad \longrightarrow \quad \mathrm{Lim}(F)$$

since for $h : i \to_{\mathbb{J}} j$ we have

$$\begin{array}{ccc} F(i)^n & \xrightarrow{f_i} & F(i) \\ {\scriptstyle F(h)^n}\big\downarrow & & \big\downarrow{\scriptstyle F(h)} \\ F(j)^n & \xrightarrow[f_j]{} & F(j) \end{array}$$

as the connecting maps $F(h) : F(i) \to F(j)$ are morphisms of $\mathbf{V}$-algebras. Moreover each such $f_\Pi : \mathrm{Lim}(F)^n \to \mathrm{Lim}(F)$ is continuous since

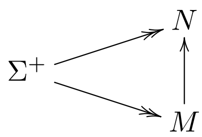
$$f_\Pi^{-1}(\pi_i^{-1}(S)) \quad = \quad (\langle \pi_i^{-1}, \ldots, \pi_i^{-1} \rangle \circ f_i^{-1})(S) \qquad \text{(in } \mathrm{Lim}(F)^n\text{)}$$

As a consequence, $\mathrm{Lim}(F)$ is a topological $\mathbf{V}$-algebra with a Stone topology if the $F(i)$ are finite. □

B.2. Free Pro-V Semigroup

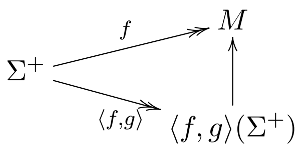
Let V be a pseudo-variety of finite semigroups. In general, it is impossible to find a free object in V, *i.e.* an object that generalizes every member of V at once, as such an object would be infinite. However, for a given alphabet Σ, it is possible to construct a profinite semigroup called the free pro-**V** semigroup over Σ (see e.g. [Alm05, §3.2] or [RS08, §3.2]), as a cofiltered limit $F : \mathbb{J} \to \mathbf{V}$, where $\mathbf{V}$ is seen as a category with semigroup maps as morphisms, as follows:

Consider first the category $\Sigma \Downarrow \mathbf{V}$ whose objects are pairs (M, f) where $M \in \mathbf{V}$ and $f : \Sigma \to M$ is a function which generates a surjective semigroup morphism $\Sigma^+ \twoheadrightarrow M$. The morphisms from $(\Sigma \to M)$ to $(\Sigma \to N)$ are semigroup morphisms $M \to N$ making the following triangle commute:

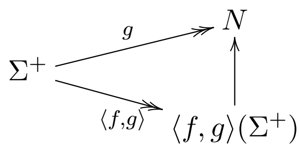


Note that by surjectivity of the semigroup morphisms $\Sigma^+ \twoheadrightarrow M$ and $\Sigma^+ \twoheadrightarrow N$, there is at most one morphism from $(\Sigma \to M)$ to $(\Sigma \to N)$, which is moreover necessarily surjective.

Then let $\mathbb{J}$ be a full subcategory of $\Sigma \Downarrow \mathbf{V}$ with one object for each isomorphism class of $\Sigma \Downarrow \mathbf{V}$. The category $\mathbb{J}$ is cofiltered since for any $(f : \Sigma \to M)$ and $(g : \Sigma \to N)$, we have



and



where $f : \Sigma^+ \twoheadrightarrow M$ is the (surjective) semigroup morphism induced by f (and similarly for g), where $\langle f, g\rangle(\Sigma^+) \subseteq M \times N$ is the image of pairing, and where $\langle f, g\rangle(\Sigma^+) \to M$ the restriction of the projection $M \times N \to M$ (and similarly for $\langle f, g\rangle(\Sigma^+) \to N$). Note that $M \times N \in \mathbf{V}$ since $M, N \in \mathbf{V}$, and thus $\langle f, g\rangle(\Sigma^+) \in \mathbf{V}$ since it is a subsemigroup of $M \times N$.

We then let $F : \mathbb{J} \to \mathbf{V}$ be the first projection and let $\overline{\Omega}_\Sigma(\mathbf{V})$ be the limit of F. It follows from results of §4.2 that $\overline{\Omega}_\Sigma(\mathbf{V})$ is a topological semigroup endowed with a Stone topology.

In particular, when $\mathscr{V}(\Sigma)$ is the class of regular languages on Σ, then the corresponding pseudo variety $\mathbf{V}$ is that of all finite Σ-generated semigroups, and the corresponding free pro-$\mathbf{V}$ semigroup over Σ is denoted **Pro-**Σ.

We want to study the structure of the free pro-$\mathbf{V}$ semigroup. To this end we recall some basic notions of general topology which we will use immediately. We begin by the definition of nets, which are a generalization of sequences in metric spaces (see [Wil70] for more).

Definition B.2. *Let $(D, \leq)$ be a preorder. We say that D is directed, or that $\leq$ is a direction, if, for all $d_1, d_2 \in D$, there exists $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$.*

Let S be a topological space, and $(x_d)_{d\in D} \in S^D$ be a family. If D is directed, Then we say that $(x_d)_{d\in D}$ is a net.

A subnet of$(x_d)_{d\in D}$ is a family $(x_{\varphi(d)})_{d\in D'}$ for some $\varphi : D' \to D$ both increasing ($d_1 \leq d_2 \Rightarrow \varphi(d_1) \leq \varphi(d_2)$) and cofinal (for $d \in D$, there exists $d' \in D'$ such that $d \leq \varphi(d')$).

We say that a net $(x_d)_{d\in D}$ converges to x if for every open U containing x, there exists $d_0 \in D$ such that $d_0 \leq d \Rightarrow x_d \in U$.

For example, a sequence in S is in particular a net, since $\mathbb{N}$, the index of sequences, is an order and thus a preorder. In a few lines, we will consider nets where the index is of the form $\mathcal{P}(X)$ for some set X, seen as a poset.

We may now recall some fundamental properties of nets:

Proposition B.3.

- *If S is Hausdorff, then each net has at most one limit.*
- *S is compact iff every net $(x_d)_{d\in D}$ has a subnet $(x_{\varphi(d)})_{d\in D'}$ that has a limit in S.*

- *Let $u : S \to T$ be a function between topological spaces. Then, u is continuous in $x \in S$ iff for every net $(x_d)_{d\in D}$ converging to x, We have $(u(x_d))_{d\in D}$ that converges to $u(x)$*

We now give two useful lemmas on the structure of free pro-$\mathbf{V}$ semigroups. We consider a pseudovariety $\mathbf{V}$, and its pro-$\mathbf{V}$ free semigroup over Σ, $\overline{\Omega}_\Sigma(\mathbf{V})$

Lemma B.4. *Let ι be the natural embedding of Σ^+ into $\overline{\Omega}_\Sigma(\mathbf{V})$. Explicitly, for every (M, f) in $(\Sigma \Downarrow \mathbf{V})$, the component of $\iota(w)$ on (M, f) is $f(w)$. Then $\{\iota(w) \mid w \in \Sigma^+\}$ is dense in $\overline{\Omega}_\Sigma(\mathbf{V})$*

Proof. Let U be an open subset of $\overline{\Omega}_\Sigma(\mathbf{V})$. Then U may be written as a union $U = \cup\pi_i^{-1}(K_i)$. Since for each (M, f) in $(\Sigma \Downarrow \mathbf{V})$, f is surjective, we may pick any $w \in \Sigma^+$ such that for some $i = (M, f)$, $f(w) \in K_i$, and this yields the announced result that $\iota(w) \in U$. □

Lemma B.5. *Let S be a pro-$\mathbf{V}$ semigroup, and $f : \Sigma \to S$ a function. We also denote f its extension to a morphism $\Sigma^+ \to S$. Let $G : \overline{\Omega}_\Sigma(\mathbf{V}) \to S$ be the unique continuous morphism such that $G \circ \iota = f$. The following conditions are equivalent:*

(a) The direct image of f is dense in S

(b) G is surjective.

Proof. Assume (b). Let U be an open subset of S. By continuity and surjectivity of G, $G^{-1}(U)$ is a nonempty open, and therefore meets $\mathsf{Im}(\iota)$. Let $w \in \Sigma^+$ such that $\iota(w) \in G^{-1}(U)$. Then $G \circ \iota(w) = f(w) \in U$. So f has a dense image in S.

Assume (a). We write S as a cofiltered limit of elements of $\mathbf{V}$: $S = \lim(S_i)_{i\in\mathbb{J}} \subseteq \prod_{i\in\mathbb{J}} S_i$. Let $s = (s_i)_i \in S$. We want to exhibit a $\beta \in \overline{\Omega}_\Sigma(\mathbf{V})$ such that $G(\beta) = s$. We exhibit a net whose limit verifies this.

For F finite subset of the set of objects of $\mathbb{J}$, which we denote J, we consider the open subset $U_F = \bigcap_{i\in F} \pi_i^{-1}(\{s_i\})$ of S. It is the set of elements of S whose components on objects of F are equal to those of s. It is nonempty since it contains s, and it is indeed open since F is finite. By density of f there exists a $u_F \in \Sigma^+$ such that $f(u_F) \in U_F$. The family $(\iota(u_F))_{F\subseteq J}$ is a net, such that $G(\iota(u_F))_i = s_i$ for $i \in F$.

Notice that since opens of S are unions of finite intersections of inverse projections, any open neighbourhood of s contains a U_F.

The net $(\iota(u_F))_{F\subseteq J}$ takes values in $\overline{\Omega}_\Sigma(\mathbf{V})$, which is a compact space. As such, this family has a converging subnet $(\iota(u_F))_{F\in I}$, where I is some (infinite) subset of $\mathcal{P}_{fin}(J)$. Let β be the limit of this subnet, which is unique in a Hausdorff space. By finality of subnets, for every subset F_0 of $\mathbb{J}$, there is a F in I containing F_0 (and in particular $U_{F_0} \subseteq U_F$.

By the two previous remark, for every open neighbourhood U of s, there exists a $F \subseteq J$ such that $U_F \subseteq U$, and we may in fact pick F in I. The subnet $(G(\iota(u_F)))_{F\in I}$ thus converges to s, and by continuity of G, $G(\beta) = s$ □

In particular, applying this proof to $S = \overline{\Omega}_\Sigma(\mathbf{V})$ and $f = id_S$ yields the interesting observation that elements of $\overline{\Omega}_\Sigma(\mathbf{V})$, which we call profinite words, may be constructed as limits of nets of finite words.

Profinite Recognition We studied profinite words during our work, and in particular we were hoping to be able to do recognition on profinite words directly, rather than finite words. However we encountered some difficulties in doing so, when dealing with quantification. Consider Σ, Γ two alphabets. The projection $\pi : \Sigma \times \Gamma \to \Sigma$ lifts to $\widehat{\pi} : \textbf{Pro-}(\Sigma \times \Gamma) \to \textbf{Pro-}\Sigma$ by embedding Σ into $\textbf{Pro-}\Sigma$ and taking the lift as defined in the previous section. Consider a recognizer $(\Sigma \times \Gamma, S, I, f)$. When studying finite words, we may define $\pi(f)(\sigma) = \{f(w) \mid w \in \pi^{-1}(\sigma)\}$ quite simply, however in the profinite case, things are not so easy, as we require the function of a recognizer to generate a dense image, or equivalently to lift to a surjective morphism, rather than to be surjective. As such, we must find an adequate candidate $\widehat{\pi(f)}$ which extends $\pi(f)$, but is also a semigroup morphism and continuous. Intuitively, we want to pick $\widehat{\pi(f)}(\sigma) = \{\widehat{f}(w) \mid w \in \widehat{\pi}^{-1}(\sigma)\}$. It is easy to see that this extends $\pi(f)$, but we were unsuccessful in proving that it is continuous and a morphism. The problem reduces to proving that $\widehat{\pi}$ is an open function, which we are for now unable to do.

B.3. Stone Duality

We follow the presentation of [GPR16, GPR17]. Given a set A we write $\beta(A)$ for the Stone-Čech compactification of A, seen as a discrete space. Explicitly, $\beta(A)$ consists of the ultrafilters on the Boolean algebra $\mathcal{P}(A)$ endowed with the Stone topology induced by the basis of clopens

$$\mathrm{ext}(S) \quad := \quad \{\mathcal{F} \in \mathfrak{Uf}(\mathcal{P}(A)) \mid S \in \mathcal{F}\} \qquad\qquad (\text{for } S \in \mathcal{P}(A))$$

It is well-known that β extends to a faithful functor. The inverse image of a function $f : A \to B$ is a morphism of Boolean algebras $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$, which, by Stone's Representation Theorem (see e.g. [Joh86, Cor. II.4.4]), induces a continuous function

$$\begin{array}{rcrcl} \beta(f) & : & \beta(A) & \longrightarrow & \beta(B) \\ & & \mathcal{F} & \longmapsto & \{S \in \mathcal{P}(B) \mid f^{-1}(S) \in \mathcal{F}\} \end{array}$$

Note that for $f, g : A \to B$, we have $f^{-1} = g^{-1}$ if and only if $f = g$, so that β is indeed faithful.

Moreover,

$$a \in A \quad \longmapsto \quad \hat{a} := \{S \in \mathcal{P}(A) \mid a \in S\} \in \mathfrak{Uf}(\mathcal{P}(A))$$

embeds the discrete space A as a dense subspace of $\beta(A)$ (since $\mathrm{ext}(\emptyset) = \emptyset$ while for any non-empty $S \in \mathcal{P}(A)$ we have $\hat{a} \in \mathrm{ext}(S)$ for every $a \in S$).

Note that the elements of $\beta(A)$ can be seen as maps of Boolean algebras

$$\mathcal{F} \quad : \quad \mathcal{P}(A) \quad \longrightarrow \quad \mathbf{2}$$

where $\mathbf{2}$ is the two-element Boolean algebra $\{0, 1\}$. Hence, each sub Boolean algebra $\mathcal{B}$ of $\mathcal{P}(A)$ induces a quotient X of $\beta(A)$. The elements of X can be described as composites

$$\mathcal{B} \hookrightarrow \mathcal{P}(A) \xrightarrow{\mathcal{F}} \mathbf{2}$$

For an alphabet Σ, following §B.3 (and [GPR16, GPR17]) we write $\beta(\Sigma^+)$ for the Stone dual of $\mathcal{P}(\Sigma^+)$, namely the space of ultrafilters over the Boolean algebra $\mathcal{P}(\Sigma^+)$.

The functoriality of β (see §B.3) implies that each function $f : \Sigma^+ \to \Gamma^+$ induces a continuous function

$$\begin{array}{rcrcl} \beta(f) & : & \beta(\Sigma^+) & \longrightarrow & \beta(\Gamma^+) \\ & & \mathcal{F} & \longmapsto & \{S \in \mathcal{P}(\Gamma^+) \mid f^{-1}(S) \in \mathcal{F}\} \end{array}$$

Note that the elements of $\beta(\Sigma^+)$ are maps of Boolean algebras

$$F \quad : \quad \mathcal{P}(\Sigma^+) \quad \longrightarrow \quad \mathbf{2}$$

Each sub Boolean algebra $\mathcal{B}$ of $\mathcal{P}(\Sigma^+)$ induces a quotient X of $\beta(\Sigma^+)$. The elements of X can be described as composites

$$\mathcal{B} \hookrightarrow \mathcal{P}(\Sigma^+) \xrightarrow{F} \mathbf{2}$$

Free Pro-V Semigroup. Consider a pseudo variety $\mathbf{V}$ of semigroups and let $\mathscr{V}$ be the corresponding variety of regular languages, with fibre over Σ denoted $\mathscr{V}(\Sigma)$. Then $\mathscr{V}(\Sigma)$ is a sub Boolean algebra of $\mathcal{P}(\Sigma^+)$:

$$\mathscr{V}(\Sigma) \hookrightarrow \mathcal{P}(\Sigma^+)$$

and thus induces a quotient $\beta(\Sigma^+)/\mathscr{V}$ of $\beta(\Sigma^+)$:

$$\beta(\Sigma^+) \twoheadrightarrow \beta(\Sigma^+)/\mathscr{V}$$

Proposition B.6. *The space $\beta(\Sigma^+)/\mathscr{V}$ is homeomorphic to the free pro-$\mathbf{V}$ semigroup over Σ.*

Proof of Proposition B.6. By Stone's Representation Theorem (see e.g. [Joh86, Cor. II.4.4]), it suffices to show that the Boolean algebra $\mathscr{V}(\Sigma)$ is isomorphic to the Boolean algebra of clopens of $\overline{\Omega}_\Sigma(\mathbf{V})$.

If K is a clopen of $\overline{\Omega}_\Sigma(\mathbf{V})$, then since $\mathbb{J}$ is cofiltered we can assume $K = \pi^{-1}_{(M,f)}(I)$ for some $(M, f) \in \mathbb{J}$ and some $I \subseteq M$. Then $f^{-1}(I) \in \mathscr{V}(\Sigma)$.

This defines a function from the clopens of $\overline{\Omega}_\Sigma(\mathbf{V})$ to $\mathscr{V}(\Sigma)$. Indeed, consider a clopen K which is the inverse image (under projection) of two distinct I, J with $I \subseteq M$ and $J \subseteq N$ for some $(M, f), (N, g) \in \mathbb{J}$. Then since $\mathbb{J}$ is cofiltered we can find $(L, h) \in \mathbb{J}$ with (surjective) morphisms to (M, f) and (N, g). Write $\tilde{I}$ (resp. $\tilde{J}$) for the inverse image of I (resp. J) in L. Note that the languages recognized by (M, f, I) and $(L, h, \tilde{I})$ (resp. (N, g, J) and $(L, h, \tilde{J})$) are the same. Moreover, K is the inverse image (under projection) of both $\tilde{I}$ and $\tilde{J}$. But since the connecting morphisms of $\mathbb{J}$ are surjective, it follows that $\tilde{I} = \tilde{J}$. Hence (M, f, I), $(L, h, \tilde{I})$, (N, g, J) and $(L, h, \tilde{J})$ all recognize the same language.

This function is injective, as if K, L are two distinct clopens of $\overline{\Omega}_\Sigma(\mathbf{V})$, then since $\mathbb{J}$ is cofiltered we can assume $K = \pi^{-1}_{(M,f)}(I)$ and $L = \pi^{-1}_{(M,f)}(J)$ for some $(M, f) \in \mathbb{J}$ and some distinct $I, J \subseteq M$. Since f induces a surjective morphism $\Sigma \twoheadrightarrow M$, the inverse images $f^{-1}(I)$ and $f^{-1}(J)$ are distinct.

This function is moreover surjective. Given $\mathcal{L} \in \mathscr{V}(\Sigma)$, by definition there are some $f : \Sigma \to M$ and $I \subseteq M$ such that $M \in \mathbf{V}$ and $\mathcal{L} = f^{-1}(I)$. Note that we can assume f to induce a surjective morphsim $\Sigma \twoheadrightarrow M$ since the image of Σ^+ is a subsemigroup of M and $\mathbf{V}$ is a pseudo-variety. Hence we can assume (M, f) to be an object of $\mathbb{J}$ and it follows from the results of §4.2 that $\pi^{-1}_{(M,f)}(I)$ is a clopen of $\overline{\Omega}_\Sigma(\mathbf{V})$.

We thus have a bijective correspondence between $\mathscr{V}(\Sigma)$ and the clopens of $\overline{\Omega}_\Sigma(\mathbf{V})$.

This bijection is an isomorphism of Boolean algebras. Indeed, consider $\mathcal{L}, \mathcal{L}' \in \mathscr{V}(\Sigma)$. Since $\mathbb{J}$ is cofiltered, we can assume $\mathcal{L}$ and $\mathcal{L}'$ to be recognized by (M, f, I) and (M, f, J) for some $(M, f) \in \mathbb{J}$ and some $I, J \subseteq M$. But then $\mathcal{L} \cup \mathcal{L}'$ (resp. $\mathcal{L} \cap \mathcal{L}'$) is the inverse image of $I \cup J$ (resp. $I \cap J$). □

The two spaces are homeomorphic as topological spaces, however $\beta(\Sigma^+)$ does not seem to have a natural product. An interesting observation is that as a Stone space, $\beta(\Sigma^+)$ is profinite, and therefore pro-$\mathbf{V}$ for $\mathbf{V}$ the pseudovariety of all finite semigroups. This result seems to indicate that $\beta(\Sigma^+)$ and $\mathbf{Pro}$-(Σ) may be isomorphic, or in other words that ultrafilters on languages are entirely determined by their image on regular languages, but we do not know that to be true.

C. Fibrations

In this appendix we give a more formal and detailed presentation of fibrations as introduced in §3, and give some details on the structure of the Sum fibration and its application to recognition. Refer to e.g. [Jac01, §1] for a more in-depth presentation of fibration theory.

C.1. Introduction to Fibrations

Definition C.1. *Let $p : \mathbb{E} \to \mathbb{B}$ be a functor.*

(a) Let $u : I \to J$ be a morphism in $\mathbb{B}$ and $u : X \to Y$ be a morphism in $\mathbb{E}$. We say that f is Cartesian over u if $pf = u$ and if for each $g : Z \to X$ such that $pg = wu$ for some $w : pZ \to Y$ there is a unique $h : Z \to X$ such that $ph = w$ and $g = fh$.

(b) We say that p is a fibration if for every Y in $\mathbb{E}$ and $u : I \to pY$ in $\mathbb{B}$ there is a morphism $f : X \to Y$ such that $pf = u$. When this f is universal among all possible choices, i.e. when any $f' : X' \to Y$ factors uniquely through f, we say that f is a Cartesian lifting of u.

When p is a fibration, we say that $\mathbb{E}$ is fibred over $\mathbb{B}$. We call $\mathbb{E}$ the total category, and $\mathbb{B}$ the base category.

The category $\mathbb{E}_I$ of objects of $\mathbb{E}$ whose image by p is I, and whose morphisms are above identities only (i.e. a morphism f in $\mathbb{E}_I$ is such that $p(f) = \mathrm{id}_I$), is called the fibre over I.

The fibre category $\mathbb{E}_I$ may be constructed as the following pullback:

$$\begin{array}{ccc} \mathbb{E}_I & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{1} & \xrightarrow[I]{} & \mathbb{B} \end{array}$$

where I is the constant functor which sends the only object of $\mathbf{1}$ to $I \in \mathbb{B}$.

Notice that there may be several choices for the Cartesian morphism f over u, however one may show that all Cartesian *liftings* of u are isomorphic. Assume that we now have made a choice, for each u, of a Cartesian lifting $\overline{u}(Y) : u^\bullet(Y) \to Y$. Then the operation $u^\bullet$ thus defined is a functor from $\mathbb{E}_J$ to $\mathbb{E}_I$ (contravariant to u), with $u^\bullet(f : X \to Y)$ defined as the unique morphism making the following diagram commute:

$$\begin{array}{ccc} u^\bullet(X) & \xrightarrow{\overline{u}(X)} & X \\ {\scriptstyle u^\bullet(f)}\,\vdots & & \downarrow{\scriptstyle f} \\ u^\bullet(Y) & \xrightarrow{\overline{u}(Y)} & Y \end{array}$$

The existence and uniqueness of $u^\bullet(f)$ come directly from the universal property of $\overline{u}(Y)$ as a Cartesian lifting of u with codomain Y.

Such a family of choices of liftings is called a *cleavage*, and we say that a fibration is *cloven* when it has a cleavage. When in addition we have $\mathrm{id}^\bullet = \mathrm{id}$ and $(v \circ u)^\bullet = u^\bullet v^\bullet$, we say that the fibration is *split*. Even when a fibration is cloven but not split, the two previous equalities are always natural isomorphisms.

Definition C.2 (Indexed Category)**.** *A* $\mathbb{B}$-indexed category *is a pseudo-functor* $F : \mathbb{B}^{op} \to \mathbf{Cat}$*, that is to say a mapping that associates to each* I *in* $\mathbb{B}$ *a category* $F(I)$*, and to each arrow* $u : I \to J$ *in* $\mathbb{B}$ *a functor* $u^\bullet = F(u) : F(J) \to F(I)$ *in the reverse direction, and which is equipped with natural isomorphisms*

$$\begin{array}{lcrcl} \eta_I & : & \mathrm{id} & \simeq & (id_I)^\bullet \\ \mu_{u,v} & : & u^\bullet v^\bullet & \simeq & (v \circ u)^\bullet \end{array}$$

such that the following two diagrams commute:

$$\begin{array}{ccccc} & & u^\bullet & & \\ & \swarrow^{\eta_I u^\bullet} & \| & \searrow^{u^\bullet \eta_J} & \\ (id_I)^\bullet u^\bullet & \xrightarrow[\mu_{id_I,u}]{} & u^\bullet & \xleftarrow[\mu_{u,id_J}]{} & u^\bullet (id_J)^\bullet \end{array} \qquad \begin{array}{ccc} u^\bullet v^\bullet w^\bullet & \xrightarrow{u^\bullet \mu_{v,w}} & u^\bullet (w \circ v)^\bullet \\ \downarrow{\scriptstyle \mu_{u,v} w^\bullet} & & \downarrow{\scriptstyle \mu_{u,w \circ v}} \\ (v \circ u)^\bullet w^\bullet & \xrightarrow[\mu_{v \circ u,w}]{} & (w \circ v \circ u)^\bullet \end{array}$$

In the particular case where the η*'s and* μ*'s are identities, we say that the indexed category is* strict*.*

Both indexed and fibered categories allow us to represent "global" objects that are indexed by a base category $\mathbb{B}$ such that the structure in $\mathbb{B}$ is reflected above. One may easily go from a cloven fibration to an indexed category by assigning to $I \in B$ the category $\mathbb{E}_I$ and to $u : I \to J$ the functor $u^\bullet$. If the fibration is moreover split, then the resulting indexed category is strict.

Reciprocally, it is also possible to construct a fibration from an indexed category, using the following construction.

Definition C.3 (Grothendieck Construction)**.** *Let* $\Lambda : \mathbb{B}^{op} \to \mathbf{Cat}$ *be an indexed category. The* Grothendieck construction $\int(\Lambda)$ *is the following category:*

- *Objects: couples* (I, X) *where* $I \in \mathbb{B}$ *and* $X \in \Lambda(I)$
- *Arrows* $(I, X) \to (J, Y)$ *are couples* (u, f) *with* $u : I \to J$ *in* $\mathbb{B}$ *and* $f : X \to u^\bullet(Y) = \Lambda(u)(Y)$ *in* $\Lambda(I)$*.*
- *Composition of*

  $$(I, X) \xrightarrow{(u,f)} (J, Y) \xrightarrow{(v,g)} (K, Z)$$

  is defined as:

  $$I \xrightarrow{u} J \xrightarrow{v} K$$

  $$X \xrightarrow{f} u^\bullet(Y) \xrightarrow{u^\bullet(g)} u^\bullet v^\bullet(Z) \xrightarrow[\cong]{\mu_{u,v}} (v \circ u)^\bullet(Z)$$

The Grothendieck construction does indeed yield a fibration, in the following way:

Proposition C.4. *The functor $p : \int \Lambda \to \mathbb{B}$ defined by:*

$$\begin{array}{rcl} p(I,X) & = & I \\ p((u,f):(I,X)\to(J,Y)) & = & u \end{array}$$

is a cloven fibration, with $u : I \to J$ lifting to $(u, \mathrm{id}) : (I, u^\bullet(Y)) \to (J,Y)$. If moreover Λ is a strict indexed category, then p is a split fibration.

C.2. Simple Fibration

In this paragraph we detail the composition and identities in the simple fibration.

Let $(A,X),(B,Y),(C,Z)$ be objects of $S(\mathbb{B})$ and $(g,g_0):(A,X)\to(B,Y)$, $(h,h_0):(B,Y)\to(C,Z)$.

- Identity on (A,X) is (id_A, π_X)
- Composition $(h,h_0)\circ(g,g_0)$ is (t,t_0), where $t = h\circ g$ and $t_0 = h_0 \circ \langle g\circ\pi_A, g_0\rangle : A\times X \to Z$

C.3. Sum Fibration

In this paragraph we do the same for Sum. To simplify notations, we write $\mathcal{A}(h(i), h_0(i,x))$ for $(\langle h\circ\pi, h_0\rangle)^\bullet(\mathcal{A})$, even though h and h_0 may not be functions. Similarly, for $\phi(b,y)$ a morphism in $\mathbb{E}_{B\times Y}$ we use the notation $\phi(h(a), h_0(x,a))$ for $(\langle h\circ\pi, h_0\rangle)^\bullet(\phi)$.

Let $(A,X,\mathcal{A})$, $(B,Y,\mathcal{B})$ and $(C,Z,\mathcal{C})$ be objects of $\mathsf{Sum}(\mathbb{E})$, and $(g,g_0,\psi):(A,X,\mathcal{A})\to(B,Y,\mathcal{B})$, $(h,h_0,\varphi):(B,Y,\mathcal{B})\to(C,Z,\mathcal{C})$ two arrows. The identity on $(A,X,\mathcal{A})$ is $(\mathrm{id}A, \pi_X, \mathrm{id}_{\mathcal{A}})$. The composition $(g,g_0,\psi)\circ(h,h_0,\varphi)$ is (t,t_0,θ) with:

- $t = h\circ g$
- $t_0(a,x) = h_0(h(a), g_0(a,x))$
- $\theta(a,x) = \varphi(g(a), g_0(a,x))\circ\psi(a,x) : \mathcal{A}(a,x)\to\mathcal{C}(h(g(a)), h_0(g(a), g_0(a,x)))$

Notice that the t and t_0 components behave like in the simple fibration.

C.4. Fibration Attemps

This appendix summarizes our first attempts at introducing a suitable fibration. It shows the different problems that arose each time, and thus gives some insight on the final version presented in the report. The main difference is the nature of the arrows.

C.4.1. Semigroup Morphisms

When studying language recognition, one may consider the following category.

- Objects: tuples $(\Sigma, S, I\subseteq S, f:\Sigma\to S)$ where S is an object of a subcategory $\mathbb{S}$ of **Sg**.
- An arrow from (Σ,S,I,f) to (Γ,T,J,g) is a tuple (u,h) with $u:\Sigma\to\Gamma$ and $h:S\to T$ a morphism such that $h\circ f = g\circ u$ and $h(f(w))\in J$ when $f(w)\in I$.
- The composition $(u',h')\circ(u,h)$ is $(u'\circ u, h'\circ h)$

- Identity morphisms are couples $(\mathrm{id}_{\Sigma+}, \mathrm{id}_S)$

This category is a fibration. This can be seen by applying the Grothendieck construction to the following. Let $\mathbb{S}$ be a category of semigroups, and consider the following indexed category $\Lambda(\mathbb{S}) : \mathbf{Alph} \to \mathbf{Cat}$:

- Objects of $\Lambda(\mathbb{S})(\Sigma)$: tuples $(S, I \subseteq S, f : \Sigma \to S)$
  Arrows from (S, I, f) to (T, J, g) : semigroup morphisms $h : S \to T$ such that $h \circ f = g$ and $h(f(w)) \in J$ when $f(w) \in I$.
- A function $u : \Sigma \to \Gamma$ gives a functor $u^\bullet : \Lambda(\mathbb{S})(\Gamma) \to \Lambda(\Sigma)$ by $u^\bullet(S, I, f) = (S, I, f \circ u)$ and $u^\bullet(h) = h$

Then the Grothendieck construction $\int \Lambda(\mathbb{S})$ is, following the definition, the following category:

- Objects: tuples $(\Sigma, S, I \subseteq S, f : \Sigma \to S)$
- An arrow from (Σ, S, I, f) to (Γ, T, J, g) is a tuple (u, h) with $u : \Sigma \to \Gamma$ and $h : S \to T$ a morphism such that $h \circ f = g \circ u$ and $h(f(w)) \in J$ when $f(w) \in I$.

Let us call $\mathbb{E}$ the previous fibration. Notice that in a given fibre $\mathbb{E}_\Sigma$, the u part of an arrow (u, h) is the identity function. When working inside a fibre, we will simply denote h for arrows (id, h).s Let us instantiate $\mathbb{S}$ by the category of finite semigroups. For Σ and Γ two alphabets, we have the first projection $\pi : \Sigma \times \Gamma \to \Sigma$. This gives the functor $\pi^\bullet$ between $\mathbb{E}_\Sigma$ and $\mathbb{E}_{\Sigma\times\Gamma}$ by:

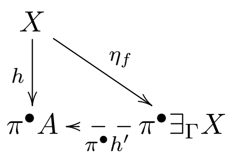
$$\pi^\bullet(\Sigma, S, I, f) = (\Sigma \times \Gamma, S, I, f \circ \pi)$$

$$\pi^\bullet(h : (\Sigma, S, I, f) \to (\Sigma, T, J, g)) = h : (\Sigma \times \Gamma, S, I, f \circ \pi) \to (\Sigma \times \Gamma, T, J, g \circ \pi)$$

Like in Lem. 2.6, we would like to show that the existential quantification is left adjoint to $\pi^\bullet$. Following [ML98, Th. IV.1.2], we need only to provide a map $\exists_\Gamma$ between objects of $\mathbb{E}_\Sigma$ and $\mathbb{E}_{\Sigma\times\Gamma}$, and for each (Σ, S, I, f) a universal arrow

$$\eta_f \quad : \quad (\Sigma \times \Gamma, S, I, f) \quad \longrightarrow \quad \pi^\bullet \exists_\Gamma(\Sigma \times \Gamma, S, I, f)$$

with the universal property being the following:



We pick as our adjoint candidate the various operations introduced in § 5, namely:

$$\exists_\Gamma(\Sigma \times \Gamma, S, I, f) = (\Sigma, \Diamond S, \Diamond I, \Diamond f)$$

with:

- $\Diamond S = \mathcal{P}_{fin}(S)$ with, for $A, B \subseteq S$, $A.B = \{a.b \mid a \in A, b \in B\}$
- $\Diamond I = \{A \in \Diamond S \mid I \cap A \neq \emptyset\}$
- $\Diamond f(w) = \{f(\langle w, v\rangle \mid v \in \Gamma^{|w|}\}$

What is an arrow $(\Sigma \times \Gamma, S, I, f) \to \pi^\bullet \exists_\Gamma(\Sigma \times \Gamma, S, I, f)$? It is an arrow $h : (\Sigma \times \Gamma, S, I, f) \to (\Sigma \times \Gamma, \Diamond S, \Diamond I, \Diamond f \circ \pi)$, *i.e.* a morphism $\eta_f : S \to \Diamond S$ such that:
$\eta_f \circ f = \Diamond f \circ \pi$ and $\eta_f(f(w)) \in \Diamond I$ when $f(w) \in I$.

i.e. such that

$$\eta_f(f(w)) = \{f(\langle w_\Sigma, v\rangle) \mid v \in \Gamma^{|w|}\}$$

Unfortunately this is not achievable, as the value of $f(w)$ does not provide enough information to recover all values $f(\langle w_\Sigma, v\rangle)$. The following is a counter-example:

- $\Sigma = \{a, b\}$
- $\Gamma = \{c, d\}$
- $S = \mathbb{F}_2 = \{0, 1\}$
- $f\left(\begin{smallmatrix} b \\ c \end{smallmatrix}\right) = 0$, $f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 1$ otherwise

Then, $\Diamond f$ is the function that sends w to $\{0, 1\}$ if w contains a b, and $\{|w| \mod 2\}$ otherwise. In this case, the existence of η_f implies that:

$$\{1\} = \eta_f(f\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)) = \eta_f(f\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)) = \{0, 1\}$$

which is absurd.

However, the $\exists_\Gamma$ operator is indeed a functor. To prove this without adjunction, we now need to define its effect on morphisms:

$\exists_\Gamma(h) : (\Sigma, \Diamond S, \Diamond I, \Diamond f) \to (\Sigma, \Diamond T, \Diamond J, \Diamond g)$ is the morphism between $\Diamond S$ and $\Diamond T$ defined by:

$$\exists_\Gamma(h)(A) = \{h(a) \mid a \in A\}$$

Lemma C.5. $\exists_\Gamma : \mathbb{E}_{\Sigma\times\Gamma} \to \mathbb{E}_\Sigma$ *is a functor.*

Proof. Let $P = (\Sigma \times \Gamma, S, I, f), Q = (\Sigma \times \Gamma, T, J, g), R = (\Sigma \times \Gamma, U, K, l)$ be three objects of $\mathbb{E}_{\Sigma\times\Gamma}$, and $h : P \to Q$, and $h' : Q \to R$. For $A \subseteq S$:

- $\exists_\Gamma(1_P)(A) = \{\mathrm{id}_S(a) \mid a \in S\} = A$
- $\exists_\Gamma(h' \circ h)(A) = \{h'(h(a)) \mid a \in A\} = \exists_\Gamma(h')(\{h(a) \mid a \in A\}) = \exists_\Gamma(h') \circ \exists_\Gamma(h)(A)$

So $\exists_\Gamma$ is indeed a functor. □

To make the adjunction between $\exists_\Gamma$ and $\pi^\bullet$ happen, we want to extend our fibration by making the morphisms inside a fibre more powerful. We introduce the revised fibration $\mathbb{E}' \to \mathbb{B}$:

- Objects: tuples $(\Sigma, S, I \subseteq S, f : \Sigma \to S)$ where S is an object of a subcategory $\mathbb{S}$ of **Sg**.
- An arrow from (Σ, S, I, f) to (Γ, T, J, g) is a tuple (u, h) with $u : \Sigma \to \Gamma$ and $h : S \times \Sigma^+ \to T$ a morphism such that $h \circ \langle f, \mathrm{id}_{\Sigma^+}\rangle = g \circ u$ and $h(f(w), w) \in J$ when $f(w) \in I$.
- the composition $(u', h') \circ (u, h)$ is $(u' \circ u, h'')$ with $h''(s, w) = h'(h(s, u(w)), w)$
- Identity morphism on (Σ, S, I, f) is (id_{Σ^+}, π_S)

Like previously, this fibration occurs as a Grothendieck construction.

A function $u : \Sigma \to \Gamma$ induces a functor $u^\bullet : \mathbb{E}'_\Gamma \to \mathbb{E}'_\Sigma$ by $u^\bullet(\Gamma, S, I, f) = (\Sigma, S, I, f \circ u)$ and $u^\bullet(h) = h \circ \langle id_S, u\rangle$

Lemma C.6. *We claim that this is indeed an extension of the first fibration, in the sense that for each morphism h in a fibre of $\mathbb{E}$ gives a morphism in the corresponding fibre of $\mathbb{E}'$.*

Proof. Let $h : (\Sigma, S, I, f) \to (\Sigma, T, J, g)$ in $\mathbb{E}_\Sigma$. Define $h' : S \times \Sigma^+ \to T$ by:

$$h'(s, w) = h(s)$$

Then h' is a semigroup morphism, and $h' \circ \langle f, id\rangle = h \circ f = g$, and $h'(f(w), w) \in J$ when $f(w) \in I$, so $h' : (\Sigma, S, I, f) \to (\Sigma, T, J, g)$ is a morphism in $\mathbb{E}'$

□

We now denote $\mathbb{E}$ for $\mathbb{E}'$.

Let $\pi : \Sigma \times \Gamma \to \Sigma$ be the first projection. Like previously, we introduce the map $\exists_\Gamma$ from objects of $\mathbb{E}_{\Sigma\times\Gamma}$ to objects of $\mathbb{E}_\Sigma$ defined by:

$$\exists_\Gamma(\Sigma \times \Gamma, S, I, f) = (\Sigma, \Diamond S, \Diamond I, \Diamond f)$$

with:

- $\Diamond S = \mathcal{P}_{fin}(S)$ with, for $A, B \subseteq S$, $A.B = \{a.b \mid a \in A, b \in B\}$
- $\Diamond I = \{A \in \Diamond S \mid I \cap A \neq \emptyset\}$
- $\Diamond f(w) = \{f(\langle w, v\rangle \mid v \in \Gamma^{|w|}\}$

To prove the adjunction with $\pi^\bullet$, remains to find a universal arrow

$$\eta_f : (\Sigma \times \Gamma, S, I, f) \to \pi^\bullet \exists_\Gamma(\Sigma \times \Gamma, S, I, f)$$

Such an arrow is a morphism $\eta_f : S \times (\Sigma \times \Gamma)^+ \to \Diamond S$ such that $\eta_f(f(w), w) = \Diamond f \circ \pi(w)$.

We may then take $\eta_f(s, w)$ to be precisely this, *i.e.* $\eta_f(s, w) = \Diamond f(\pi(w)) = \{f(\langle w_\Sigma, v\rangle) \mid v \in \Gamma^{|w|}\}$. This function is constant in s

Now to check the universal property of η_f. We take an arrow $h : (\Sigma \times \Gamma, S, I, f) \to (\Sigma \times \Gamma, T, J, g \circ \pi)$. We need to show that there exists a unique arrow $h' : (\Sigma, \Diamond S, \Diamond I, \Diamond f) \to (\Sigma, T, J, g)$ such that $h = \pi^\bullet(h') \circ \eta_f$.

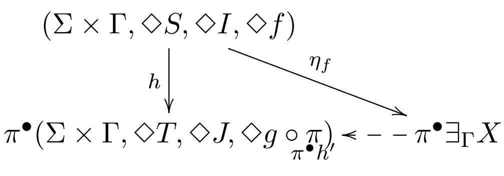
Such an arrow would need to satisfy the following:

$$h(s, w) = \pi^\bullet(h') \circ \eta_f(s, w)$$

i.e.:

$$h(s, w) = h'(\eta_f(s, w), w_\Sigma)$$

If we relax the universal property by only requiring it to work for arrows $h : X \to \pi^\bullet A$, then we want to satisfy the following diagram:



It seems that this does not add new information to the situation, since the hypotheses on h and η_f only affect inputs of the form $(f(w), w)$. In fact, more generally a morphism $h : (\Sigma, S, I, f) \to (\Gamma, T, J, g)$ only needs to satisfy $h(f(w), w) = g(w)$, which does not give info on the behaviour of h over S

C.4.2. Relational Morphisms

In a second attempt, we looked at another form of morphism for objects of our fibration: relational morphisms.

The study of language recognition motivates the study of a category $\Lambda(\Sigma)$, with Σ an alphabet, where objects are tuples (S, I, f) where:

- S is a semigroup
- I is a subset of S
- f is a function $\Sigma \to S$ such that the induced semigroup morphism $\Sigma^+ \to S$, also denoted f, spans all of S.

The language recognized by such an object is the subset $\mathcal{L} = f^{-1}I \subseteq \Sigma^+$.

For the morphisms in this category, we choose the following:

Morphisms $R : (S, I, f) \to (T, J, g)$ are semigroup relations on $S \times T$, *i.e.* subsemigroups of $S \times T$, satisfyfing the following property:

$$\mathsf{Im}\langle f, g\rangle \subseteq R \subseteq (I^* \Rightarrow J^*)$$

where:

- $\mathsf{Im}\langle f, g\rangle$ is the subsemigroup equal to $\{(f(w), g(w) \mid w \in \Sigma^+\}$
- $I^* = I \times T$, $J^* = S \times J$
- for $A, B \subseteq C$, $A \Rightarrow B = \{x \in C \mid (x \in A \Rightarrow x \in B)\}$

The motivation for this choice of arrows comes from the following lemma:

Lemma C.7. *Let (S, I, f) and (T, J, g) be two objects of $\Lambda(\Sigma)$. Let $\mathcal{L}$ (resp. $\mathcal{L}'$ be the language recognized by (S, I, f) (resp. (T,J,g)). Then Lang $\subseteq \mathcal{L}' \iff \mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*)$*

Proof.

$$\begin{aligned} \mathsf{Im}\langle f, g\rangle \subseteq (I^* \Rightarrow J^*) &\iff \text{for all } w \in \Sigma^+,\ f(w) \in I \Rightarrow g(w) \in J \\ &\iff \text{for all } w,\ w \in \mathcal{L} \Rightarrow w \in \mathcal{L}' \\ &\iff \mathcal{L} \subseteq \mathcal{L}' \end{aligned}$$

□

As an immediate corollary, $\mathcal{L} \subseteq \mathcal{L}'$ if and only if there is an arrow from (S, I, f) to (T, J, g) , since $\mathsf{Im}\langle f, g\rangle$ is a semigroup relation on $S \times T$. In other words, arrows in $\Lambda(\Sigma)$ exactly reflect language inclusions.

We now introduce the operations $\pi^\bullet$ and $\exists_\Gamma$ which will reflect those introduced in Lem. 2.6. To this end, we will exhibit the indexed structure behind the $\Lambda(\Sigma)$ categories, and use the Grothendieck construction to build an indexed category, where $\pi^\bullet$ and $\exists_\Gamma$ will form an adjunction.

Proposition C.8. *The map* Λ : **Alph** $\to$ **Cat** *is a strict indexed category, with the function* $u : \Sigma \to Gamma$ *yielding the following functor* $u^\bullet : \Lambda(\Gamma) \to \Lambda(\Sigma)$*:*

- $u^\bullet(S, I, f) = (S, I, f \circ u)$
- $u^\bullet(R : (S, I, f) \to (T, J, g)) = R : (S, I, f \circ u) \to (T, J, g \circ u)$

Proof. It is clear that $u^\bullet$ is functorial, and that the action of Λ is strictly functorial, as long as $u^\bullet(R)$ satisfies the required property of arrows, namely:

$$\mathsf{Im}\langle f \circ u, g \circ u\rangle \subseteq R \subseteq (I^* \Rightarrow J^*)$$

But since $\mathsf{Im}\langle f \circ u, g \circ u\rangle \subseteq \mathsf{Im}\langle f, g\rangle \subseteq R \subseteq (I^* \Rightarrow J^*)$ this is verified.

∎

Applying the Grothendieck construction on this indexed category gives the following fibration $\mathbb{E} \to$ **Alph**:

- objects of $\mathbb{E}$: tuples (Σ, S, I, f) with $\Sigma \in$ **Alph**, S a semigroup, $I \subseteq S$, and $f : \Sigma \to S$.
- arrows from (Σ, S, I, f) to (Γ, T, J, g) are couples (u, h) with $u : \Sigma \to Gamma$ in **Alph**, and $h : (S, I, f) \to u^\bullet(T, J, g) = (T, J, g \circ u)$ in $\Lambda(\Sigma)$

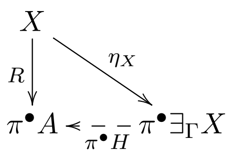
Let us look at the effect of the projection function $\pi : \Sigma \times \Gamma \to \Sigma$. This function yields a functor $\pi^\bullet : \Lambda(\Sigma) \to \Lambda(\Sigma \times \Gamma)$

Notice that if $\mathcal{L}$ is recognized by (Σ, S, I, f), then the language recognized by $\pi^\bullet(\Sigma, S, I, f)$ is precisely $\pi^\bullet(\mathcal{L})$ as introduced in Lem. 2.6. As is commonly done in categorical logic, we want to exhibit an adjunction between this projection functor and some existential quantifier functor.

Following [ML98, Th. IV.1.2], we need only to provide a map $\exists_\Gamma$ between objects of $\mathbb{E}_\Sigma$ and $\mathbb{E}_{\Sigma\times\Gamma}$, and for each $X = (\Sigma, S, I, f)$ a universal arrow

$$\eta_X : (\Sigma \times \Gamma, S, I, f) \to \pi^\bullet \exists_\Gamma(\Sigma \times \Gamma, S, I, f)$$

with the universal property being the following:



We pick as our adjoint candidate the various operations introduced in § 5, namely:

$$\exists_\Gamma(\Sigma \times \Gamma, S, I, f) = (\Sigma, \Diamond S, \Diamond I, \Diamond f)$$

with:

- $\Diamond S = \mathcal{P}_{fin}(S)$ with, for $A, B \subseteq S$, $A.B = \{a.b \mid a \in A, b \in B\}$
- $\Diamond I = \{A \in \Diamond S \mid I \cap A \neq \emptyset\}$
- $\Diamond f(w) = \{f(\langle w, v\rangle \mid v \in \Gamma^{|w|}\}$

Following sections 3.2 and 5, $\exists_\Gamma(\Sigma \times \Gamma, S, I, f)$ recognizes exactly the language $\exists_\Gamma(\mathcal{L})$

However, the universal property does not seem realizable. Assuming we take η_X to be $\mathsf{Im}\langle f, \Diamond f\rangle$, then for each arrow $R : (\Sigma \times \Gamma, S, I, f) \to (\Sigma \times \Gamma, T, J, g \circ \pi)$ we need a unique arrow $H : (\Sigma, \Diamond S, \Diamond I, \Diamond f) \to (\Sigma, T, J, g)$ such that $R = \pi^\bullet H \circ \eta_X$.

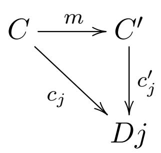
An analysis of the unfolding of definitions leads to the conclusion that $H \subseteq \Diamond S \times T$ needs to be defined as $EHt \iff \exists s$ s.t. sRt and $s\eta_X E$, however such an H is not necessarily a subset of $\Diamond I^\bullet \Rightarrow J^\bullet$.

D. Recalls and Proofs for §5 (Powersets of (Profinite) Semigroups)

In this appendix we give some categorical recalls useful for §5.1, as well as the proofs of §5 which were on the more technical and long side, and which were not included in the main report in order to streamline it as much as possible.

D.1. Categorical Limits

We recall the notion of limits in category theory. Let $D : \mathbb{J} \to \mathbb{C}$ be a functor. Intuitively, $\mathbb{J}$ is an "simple" category used to index $\mathbb{C}$ through D. We call such a functor a diagram of type $\mathbb{J}$ in $\mathbb{C}$. A cone to D is an object $C \in \mathbb{C}$ and a family of arrows $(c_j)_j \in \mathbb{J}$ in $\mathbb{C}$ such that for each arrow $\alpha : i \to j$ in $\mathbb{J}$, we have $c_j = D\alpha \circ c_i$. We consider the category $\mathrm{Cone}(D)$ of such cones. In this category, a morphism of cones $m : (C, (c_j)_j) \to (C', (c'_j)_j)$ is an arrow in $\mathbb{C}$ such that for each $j \in \mathbb{J}$, the following commutes:

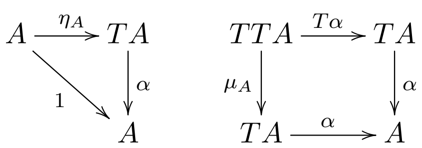


A limit of D is a terminal object in $\mathrm{Cone}(D)$, *i.e.* an object $C_0 \in C$ along with morphisms $(\pi_j : C \to Dj)_{j\in\mathbb{J}}$, called projections, such that for any other cone $(C, (c_j)_j)$, there is a unique arrow $\varphi : C \to C_0$ such that $c_j = \pi_j \circ u$ for each $j \in \mathbb{J}$.

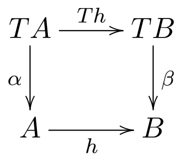
Limits are a key concept of category theory, and many basic notions, like pullbacks, products, equalizers, may be expressed as limits. We refer to [Awo06] for a detailed and clear presentation of the subject.

D.2. Eilenberg-Moore Algebras

We recall the well-known categorical concept of Eilenberg-Moore algebras (once again, refer to e.g. [Awo06] for a more detailed approach). Let (T, η, μ) be a monad on a category $\mathbb{C}$. Eilenberg-Moore algebras for T are couples (A, α) with $A \in \mathbb{C}$ and $\alpha : TA \to A$ such that $\alpha \circ \eta_A = 1_A$ and $\alpha \circ \mu_A = \alpha \circ T\alpha$ as in the following diagrams:



The category $\mathbb{C}^T$ of Eilenberg-Moore algebras for T has for morphisms arrows $h : (A, \alpha) \to (B, \beta)$ in $\mathbb{C}$ such that $h \circ \alpha = \beta \circ Th$, as in the following diagram:



For example, if T is the free monoid monad, that sends a set A to the set of finite words over A, A^*, then Eilenberg-Moore algebras for T are exactly monoids.

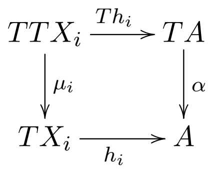
D.3. Lemma 5.3

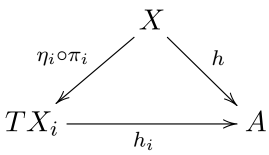
Let $(A, \alpha : TA \to A, h : X \to A) \in (X \downarrow U)$, that we will simply denote A. Recall that we consider the functor $L : \mathbb{J} \longrightarrow (X \downarrow \mathsf{U})$,

$$\begin{array}{rcl} i & \longmapsto & (TX_i,\, \mu_{X_i},\, \eta_{X_i} \circ \pi_i) \\ (i \to j) & \longmapsto & Tm_{i,j} \end{array}$$

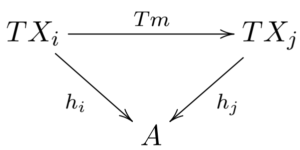
We need to prove that the category $(L \downarrow A)$ is nonempty and connected, in order to show that L is an initial functor.

Let us first give an explicit description of $(L \downarrow A)$. Objects are couples $(i, h_i : (TX_i, \eta_i \circ \pi_i, \mu_i) \to (A, h, \alpha))$ with h_i an arrow of $(X \downarrow U)$, and η_i, μ_i the components of η, μ on X_i. Unfolding definitions, h_i needs to satisfy the two following commutative diagrams:

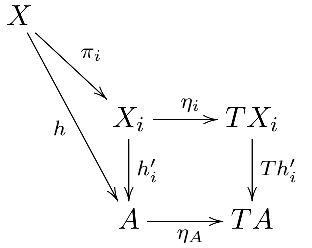




Similarly, a morphism $m : (i, h_i) \to (j, h_j)$ is a function $m : X_i \to X_j$ making the following commute:

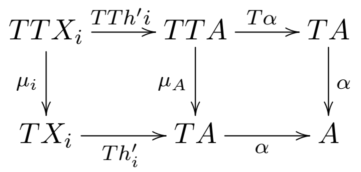


Non-emptiness follows from [CAMU16, Rem. 2.5]: any morphism $h : X \to A$ with A finite factors through some $\pi_i : X \to X_i$ as $h = h'_i \circ \pi_i$, which yields the following commutative diagram by naturality of η:



Post-composing by α, we get that $\alpha \circ Th'_i \circ \eta_i \circ \pi_i = \alpha \circ \eta_A \circ h'_i \circ \pi_i = h'_i \circ \pi_i = h$, which is the triangle identity needed for (L, A), with $h_i = \alpha \circ Th'_i$.

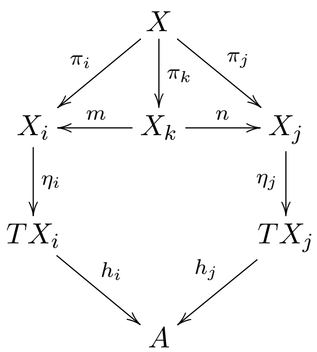
The square identity comes by checking that the outer rectangle of the following diagram commutes:



This follows from the naturality of μ and the T-algebra structure of (A, α), which make the two smaller squares commute.

Thus, (i, h_i) is in (L, A), which is non-empty. To prove connectedness, we take $(i, h_i), (j, h_j)$ in (L, A), and exhibit a (k, h_k) with arrows $m : (k, h_k) \to (i, h_i)$ and $n : (k, h_k) \to (j, h_j)$.

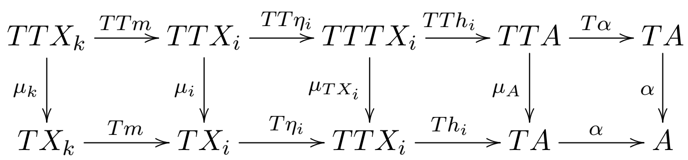
Since X is a directed limit, there exists X_k and $: X_k \to X_i, n : X_k \to X_j$ with $\pi_i = m \circ \pi_k$ and $\pi_j = n \circ \pi_k$. Consider the following diagram:



The paths from X to A are all equal to h, and so by surjectivity of π_k, the two arrows from X_k to A , namely $h_i \circ \eta_i \circ m$ and $h_j \circ \eta_j \circ n$, are equal. We call h'_k this arrow. Then applying T we get $Th'_k : TX_k \to TA$, and we set $h_k = \alpha \circ Th'_k$

The previous diagram shows that h_k satisfies the triangle identity of $(L \downarrow A)$.

To check the square identity, we consider the following diagram:



The three leftmost innersquares commute due to naturality of μ, and the last square commutes by T-algebra properties of (A, α). Thus the outer rectangle commutes, ending the proof that (k, h_k) is an object of $(L \downarrow A)$.

Finally, it remains to show that Tm, Tn are arrows of $(L \downarrow A)$, *i.e.* that $h_k = h_i \circ Tm = h_j \circ Tn$. We show the result for m, the reasoning is identical for n. By monadic properties, we have:

$$h_i = h_i \circ \mu_i \circ T\eta_i$$

Then, since h_i is a morphism of T-algebra,

$$h_i = \alpha \circ Th_i \circ T\eta_i$$

Post-composing by Tm, we get the announced result:

$$h_i \circ Tm = \alpha \circ Th_i \circ T\eta_i = h_k$$

Finally, $(L \downarrow A)$ is non-empty and connected, and so L is initial.

D.4. Lemma 5.12

Lemma D.1. *The function* **pref** *is open.*

Proof. We show that **pref** sends basic opens of $\mathscr{V}(X)$ to opens of $\widehat{\mathcal{P}}(X)$. Let I be an open of X. As we have seen in the preliminary section, we may write I as a union of inverse projections: $I = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(S_i)$ with $S_i \subseteq X_i$. We prove the following:

(a) $\mathbf{pref}(\Diamond I) = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Diamond S_i)$

(b) $\mathbf{pref}(\Box I) = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Box S_i)$

Let $U = (U_i)_{i \in \mathbb{J}} \in \widehat{\mathcal{P}}(X)$.

(a) Assume that $U \in \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Diamond S_i)$. Then there exists i such that $U_i \in \Diamond S_i$, *i.e.* such that $U_i \cap S_i \neq \emptyset$. We choose some $\alpha \in U_i \cap S_i$ and, using nets like in §B.2, we may construct $x \in X$ such that $x_j \in U_j$ for each j, and $x_i = \alpha$. Then x is in $\mathbf{closure}(U)$ and in $\pi_i^{-1}(S_i)$, and so $\mathbf{closure}(U) \in \Diamond(\cup_{i \in \mathbb{J}} \pi_i^{-1}(S_i)) = \Diamond I$. Applying **pref**, we find that $U \in \mathbf{pref}(\Diamond I)$.

Conversely, if $U \in \mathbf{pref}(\Diamond I)$, then $U = \mathbf{pref}(T)$ with $T \in \Diamond I$. Then $T \cap \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(S_i) \neq \emptyset$, and so there is an i such that $T \cap \pi_i^{-1}(S_i) \neq \emptyset$. Let $x \in T \cap \pi_i^{-1}(S_i)$. Since $\mathbf{pref}(T) = U$, $x_i \in U_i$, and $x_i \in S_i$, so $U_i \in \Diamond(S_i)$, so $U \in \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Diamond S_i)$.

Therefore, $\mathbf{pref}(\Diamond I) = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Diamond S_i)$.

(b) Assume that $U \in \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Box S_i)$. There exists i such that $U \in \pi_i^{-1}(\Box S_i)$. Thus, $U_i \subseteq S_i$. We show that $\mathbf{closure}(U) \in \Box I$, which, by applying **pref**, will proves that $U \in \mathbf{pref}(\Box I)$. Let $x \in \mathbf{closure}(U)$, we want to show that $x \in I = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(S_i)$. By definition of **closure**, we have $x_i \in U_i \subseteq \S_i$, and so $x \in I = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(S_i)$.

Conversely, if $U \in \mathbf{pref}(\Box I)$, then $U = \mathbf{pref}(T)$ with $T \in \Diamond I$. Then $T \subseteq \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(S_i)$. T is a closed subset of X, which is compact, so T is compact, and therefore there is a finite subset F of objects of $\mathbb{J}$ such that $T \subseteq \bigcup_{i \in F} \pi_i^{-1}(S_i)$. And since $\mathbb{J}$ is codirected, we may even restrict F to a singleton. Thus, there exists $i \in \mathbb{J}$ such that $T \subseteq \pi_i^{-1}(S_i)$. Now we show that $\mathbf{pref}(T) \subseteq \pi_i^{-1}(\Box S_i)$, *i.e.* that $\mathbf{pref}(T)_i \subseteq S_i$. If $\alpha \in \mathbf{pref}(T)_i$, then there is $x \in T$ such that $x_i = \alpha \in T_i$, and since $T \subseteq \pi_i^{-1}(S_i)$, $x_i = \alpha \in S_i$. So, $U = \mathbf{pref}(T) \in \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Box S_i)$.

Therefore, $\mathbf{pref}(\Box I) = \bigcup_{i \in \mathbb{J}} \pi_i^{-1}(\Box S_i)$.

$\square$

Lemma D.2. *The function* **closure** *is open.*

Proof. Similarly, we show that **closure** sends basic opens of $\widehat{\mathcal{P}}(X)$ to opens of $\mathscr{V}(X)$. Let $i \in \mathbb{J}$ and $K_i \subseteq \mathcal{P}(X_i)$. We may decompose K_i as function of $\Diamond$ and $\Box$ as follows:

$$K_i = \bigcup_{I \in K_i} (\bigcap_{a \in I} \Diamond\{a\} \cap \Box I)$$

We now show that the following holds:

$$\mathbf{closure}(\pi_i^{-1}(K_i)) = \bigcup_{I \in K_i} (\bigcap_{a \in I} \Diamond(\pi_i^{-1}\{a\}) \cap \Box(\pi_i^{-1} I))$$

which implies that **closure** send basic opens to opens.

Let $U \in \mathcal{P}(X)$.

Assume $U \in \mathbf{closure}(\pi_i^{-1}(K_i))$. Then $U = \mathbf{closure}(T)$ for some $T \in \pi_i^{-1}(K_i)$. Let us show that $U \in \bigcap_{a \in T_i} \Diamond(\pi_i^{-1}\{a\}) \cap \Box(\pi_i^{-1} T_i)$.

For $a \in T_i$, we must show that $U \cap \pi_i^{-1}(\{a\})$ is nonempty, *i.e.* that there exists $x \in U$ such that $x_i = a$. $U = \mathbf{closure}(T)$, and for such sets we have already shown how to construct a net converging to such an x. Now we must show that $U \subseteq \pi_i^{-1}(T_i)$. If $x \in U = \mathbf{closure}(T)$, then $x_i \in T_i$, and so $x \in \pi_i^{-1}(T_i)$.

Thus, $U \in \bigcup_{I \in K_i} (\bigcap_{a \in I} \Diamond(\pi_i^{-1}\{a\}) \cap \Box(\pi_i^{-1} I))$.

Now assume $U \in \bigcup_{I \in K_i} (\bigcap_{a \in I} \Diamond(\pi_i^{-1}\{a\}) \cap \Box(\pi_i^{-1} I))$. Then for some $I \in K_i$, $U \in \bigcap_{a \in I} \Diamond(\pi_i^{-1}\{a\}) \cap \Box(\pi_i^{-1} I)$. We will show that $\mathbf{pref}(U) \in \pi_i^{-1}(K_i)$, and applying **closure** to both sides will yield that $U \in \mathbf{closure}(\pi_i^{-1}(K_i)$.

In fact, we will show that $\mathbf{pref}(U) = I$, by proving that it is a subset of I and contains every element of I.

Let $\alpha \in \mathbf{pref}(U)_i$. There exists $x \in U$ such that $x_i = \alpha$. Since $U \in \Box(\pi_i^{-1}(I)$, we have $x \in \pi_i^{-1}(I)$ and so $x_i = \alpha \in I$.

Let $a \in I$. We must show that $a \in \mathbf{pref}(U)_i$. Since $U \in \Diamond(\pi_i^{-1}(\{a\})$, there exists $x \in U$ such that $x_i = a$. Therefore a is in $\mathbf{pref}(U)_i$.

Thus, $U \in \mathbf{closure}(\pi_i^{-1}(K_i))$. $\square$

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