Disordered systems and Hamilton-Jacobi equations (lecture 3)

Part 2. Rank-one matrix estimation.

Alternative methods: Lelarge Miolane; Barbier Macris.

I/ Introduction

Problem: Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N)$ be a vector of iid indep. r.v., law $P_N = P^{\otimes N}$. For some $t > 0$, we observe:

$$Y = \sqrt{\frac{2t}{N}} \underbrace{\bar{x}\,\bar{x}^T} + \underbrace{W},$$

where $W = (W_{ij})$ are indep. standard Gaussians.

For instance, we want to understand:

$$\mathrm{mmse}_N(t) = \frac{1}{N^2} \inf_{\hat{\theta}} \mathbb{E}\left[\left| \bar{x}\bar{x}^t - \hat{\theta}(Y) \right|^2 \right]$$

$$= \frac{1}{N^2} \mathbb{E}\left[\left| \bar{x}\bar{x}^t - \mathbb{E}\left[\bar{x}\bar{x}^t \mid Y\right] \right|^2 \right]$$

Informally:

$$\mathbb{P}\left[\bar{x} = x \text{ and } Y = y\right] \text{ "=" } dP_N(x)\, dy\, \exp\left(-\frac{1}{2}\left|Y - \sqrt{\frac{2t}{N}}\, x x^t\right|^2\right)$$

$$\mathbb{P}\left[\bar{x} = x \mid Y\right] = \frac{\exp\left(-\frac{1}{2}|Y - \dots|^2\right) dP_N(x)}{\int \exp\left(-\frac{1}{2}\underbrace{\left|Y - \sqrt{\frac{2t}{N}}\, x' x'^t\right|^2}\right) dP_N(x')}$$

Def: $$H_N^0(t,x) = \sqrt{\frac{2t}{N}}\, \underbrace{Y} \cdot x x^t - \frac{t}{N} |x|^4$$

$$H_N^0(t,x) = \underbrace{\sqrt{\frac{2t}{N}}\, x \cdot W x}_{\text{like spin glass}} + \underbrace{\frac{2t}{N}(x\cdot\bar{x})^2 - \frac{t}{N}|x|^4}_{\text{help us.}}$$

Denote

$$\langle f(x) \rangle := \frac{\int f(x) \exp(\overset{\text{still random}}{H_N^0}(t,x))\, dP_N(x)}{\int \exp(H_N^0(t,x))\, dP_N(x)}.$$

Exercise (credit): show that:

$$\langle f(x) \rangle = \mathbb{E}\left[f(\bar{x}) \mid Y \right].$$

Important ppty (Nishimori):

$$\mathbb{E} \langle f(x) \rangle = \mathbb{E}\left[f(\bar{x}) \right].$$

$$\mathbb{E} \langle f(x)\, g(x') \rangle = \mathbb{E}\left[\langle f(x)\rangle \langle g(x)\rangle \right]$$

indep. copy of x under $\langle \cdot \rangle$

$$= \mathbb{E}\left[\langle f(x)\rangle\, \mathbb{E}\left[g(\bar{x}) \mid Y\right] \right]$$

$$= \mathbb{E} \langle f(x)\, g(\bar{x}) \rangle.$$

$$\mathbb{E} \langle f(x, x') \rangle = \mathbb{E} \langle f(x, \bar{x}) \rangle.$$

We need to enrich the model somehow!

We also observe, for some $h \geq 0$:

$$\sqrt{2h}\, \bar{x} + z$$

$\uparrow$ standard Gaussian vector.

$$\begin{cases} H_N(t,h,x) = H_N^0(t,x) \\ \qquad + \underbrace{\sqrt{2h}\, x \cdot z} + 2h\, x\cdot\bar{x} - h|x|^2 \end{cases}$$

$$F_N(t,h) = \frac{1}{N} \log \int \exp\left(H_N(t,h,x)\right) dP_N(x).$$

$$\bar{F}_N(t,h) = \mathbb{E}\left[F_N(t,h)\right].$$

Thm: $\bar{F}_N \longrightarrow f$ solution of:

$$\begin{cases} \partial_t f - (\partial_h f)^2 = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \\ f(0,\cdot) = \bar{F}_1(0,\cdot). \end{cases}$$

II/ Main steps.

Prop: $$\partial_t \bar{F}_N - (\partial_h \bar{F}_N)^2 = \frac{1}{N^2} \mathbb{E} \langle (x\cdot\bar{x} - \mathbb{E}\langle x\cdot\bar{x}\rangle)^2 \rangle$$

and:

$$\frac{1}{N^2} \mathbb{E} \langle (x\cdot\bar{x} - \mathbb{E}\langle x\cdot\bar{x}\rangle)^2 \rangle$$
$$\leq \frac{1}{N} \partial_h^2 \bar{F}_N + \mathbb{E}\left[(\partial_h F_N - \partial_h \bar{F}_N)^2 \right]$$

Lemma 1: $$\partial_h \bar{F}_N = \mathbb{E} \left\langle \frac{x\cdot\bar{x}}{N} \right\rangle.$$

$$\partial_t \bar{F}_N = \mathbb{E} \left\langle \left(\frac{x\cdot\bar{x}}{N}\right)^2 \right\rangle.$$

Itô calculus without Itô.

$$\mathbb{E}\left[\exp\left(\sqrt{2t}\,X - t\right)\right] = 1$$

∂_t

So $$\mathbb{E}\left[\left(\frac{1}{\sqrt{2t}}X - 1\right)\exp(\dots)\right] = 0$$

$$\int x \underbrace{\exp\left(\sqrt{2t}\,x - t\right)} \exp\left(-\frac{x^2}{2}\right)dx$$

$$= \int \sqrt{2t}\,\exp(\dots)\;\exp\left(-\frac{x^2}{2}\right)dx.$$

Ex (credit) (1) prove Lemma 1 using GIP & Nishimori

(2) Relate mmse with $\partial_t \overline{F}_N$.

Second part of prop.

A a r.v. Notice that:

$$\mathbb{E}\left\langle \left(A - \mathbb{E}\langle A\rangle\right)^2\right\rangle$$

$$= \mathbb{E}\left\langle \left(A - \langle A\rangle\right)^2\right\rangle \quad \frac{1}{N}\partial_h^2 \overline{F}_N$$

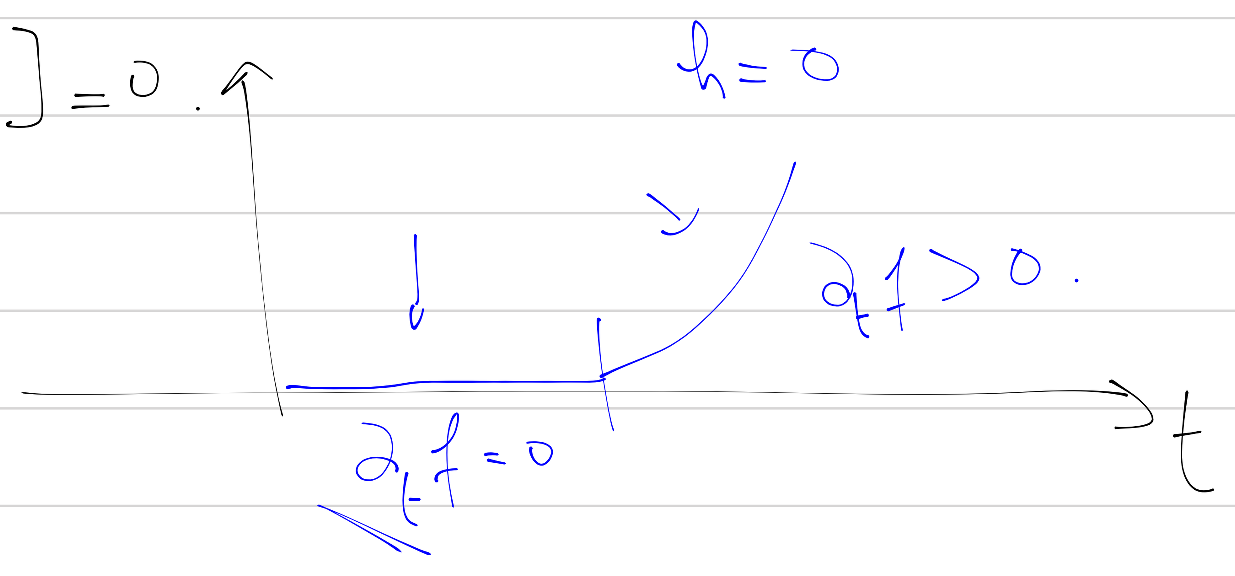
$$+ \mathbb{E}\Big[\underbrace{\left(\langle A\rangle - \mathbb{E}\langle A\rangle\right)^2}_{\partial_h F_N - \partial_h \overline{F}_N}\Big]$$

New ingredient:

Concentration estimate (Efron-Stein)

$$\mathbb{E}\left[\left|F_N - \overline{F}_N\right|^2\right] \text{ small.}$$

$\mathbb{E}[z_1] = 0$.



$$\partial_t f - (\partial_h f)^p. \qquad p\text{-spin.}$$

$$\left|\partial_t f - (\partial_h f)^3\right| \le \frac{C}{N}\left(\partial_h^2 f\right)^{1/2}$$