COLOR SYMMETRY BREAKING IN THE POTTS SPIN GLASS

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Abstract. The Potts spin glass is an analogue of the Sherrington-Kirkpatrick model in which each spin can take one of κ possible values, which we interpret as colors. It was suggested in [1] that the order parameter for this model is always invariant with respect to permutations of the colors. We show here that this is false whenever $\kappa \geqslant 58$.

Let $\kappa \geqslant 2$ and $N \geqslant 1$ be integers, and let $(g_{ij})_{i,j \geqslant 1}$ be independent centered Gaussians of variance 1. The energy function of the Potts spin glass is defined, for every $\sigma \in \{1, \ldots, \kappa\}^N$, by

$$H_N(\sigma) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \mathbf{1}_{\{\sigma_i = \sigma_j\}}. \tag{1}$$

The associated free energy at inverse temperature $\beta \geqslant 0$ is

$$F_N(\beta) := \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{1,\ldots,\kappa\}^N} \exp(\beta H_N(\sigma)). \tag{2}$$

For $\kappa = 2$, the Potts spin glass essentially coincides with the Sherrington-Kirkpatrick model [21]. Indeed, for $\kappa = 2$ we may as well consider that σ ranges in $\{-1,1\}^N$, and for every $\sigma \in \{-1,1\}^N$ and $i, j \in \{1, \ldots, N\}$, we can write

$$\mathbf{1}_{\{\sigma_i = \sigma_j\}} = \frac{1}{2}(\sigma_i \sigma_j + 1),$$

so that

$$\frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{1,2\}^N} \exp(\beta H_N(\sigma)) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{-1,1\}^N} \exp\left(\frac{\beta}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \sigma_i \sigma_j \right). \tag{3}$$

The right-hand side of the display above is the free energy of the Sherrington-Kirkpatrick model at inverse temperature $\beta/2$.

The asymptotic behavior of the free energy of the Potts spin glass has been obtained in [18]. The analysis proceeds by first identifying the asymptotics

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of the free energy restricted to configurations with a prescribed proportion of each color. To be precise, let

$$\mathscr{D} := \left\{ (d_1, \ldots, d_\kappa) \in [0,1]^\kappa \ \mid \ \sum_{k=1}^{\kappa} d_k = 1 \right\},$$

and for each $d \in \mathscr{D}$ and $\varepsilon > 0$, let

$$\Sigma_N(d,\varepsilon) := \left\{ \sigma \in \{1, \ldots, \kappa\}^N \ \mid \ \text{for every } k \in \{1, \ldots, \kappa\}, \ \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{\sigma_i = k\}} - d_k \right| \leqslant \varepsilon \right\},$$

$$F_N(\beta, d, \varepsilon) := \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \Sigma_N(d,\varepsilon)} \exp(\beta H_N(\sigma)).$$

We also define, for every $d \in \mathscr{D}$,

$$\Pi_d := \Big\{ \pi : [0,1] \to S_+^\kappa \ \mid \ \pi \text{ is left-continuous, non-decreasing, } \pi(0) = 0, \text{ and } \pi(1) = \operatorname{diag}(d_1, \ldots, d_\kappa) \Big\}, \tag{4}$$

where S_+^κ denotes the set of κ-by-κ positive semidefinite matrices. In (4), we say that the path π is non-decreasing to mean that for every $u \leqslant v \in [0,1]$, we have $\pi(v) - \pi(u) \in S_+^\kappa$.

Theorem 1 ([18])**.** *There exists an explicit functional $\mathscr{P}_\beta : \bigcup_{d \in \mathscr{D}} \Pi_d \to \mathbb{R}$ such that for every $d \in \mathscr{D}$,*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} F_N(\beta, d, \varepsilon) = \lim_{\varepsilon \to 0} \liminf_{N \to \infty} F_N(\beta, d, \varepsilon) = \inf_{\pi \in \Pi_d} \mathscr{P}_\beta(\pi).$$

As a consequence,

$$\lim_{N \to +\infty} F_N(\beta) = \sup_{d \in \mathscr{D}} \inf_{\pi \in \Pi_d} \mathscr{P}_\beta(\pi). \tag{5}$$

The reference [18] provides us with an explicit description of the functional $\mathscr{P}_\beta$ as an infimum over an additional parameter denoted by λ there. For the purposes of this note, we will only need an upper bound on $\mathscr{P}_\beta$ which is obtained by selecting $\lambda = 0$, and we will only need this upper bound on very simple (replica-symmetric) paths.

The definition of the Potts spin glass is clearly invariant under permutations of the κ different values that a spin can take. We interpret these different values as colors. We say that the Potts spin glass (at inverse temperature β) *preserves the color symmetry* if the supremum over $d \in \mathscr{D}$ in (5) is achieved at $d = (\frac{1}{\kappa}, \cdots, \frac{1}{\kappa})$, and if moreover, the infimum over $\pi \in \Pi_{(\frac{1}{\kappa}, \cdots, \frac{1}{\kappa})}$ in (5) is achieved at a path π that is color-symmetric (in other words, for each $u \in [0,1]$, the diagonal entries of $\pi(u)$ are all the same, and the non-diagonal entries of $\pi(u)$ are all the same). Otherwise, we say that the Potts spin glass *breaks the color symmetry.* It was suggested in [1] that the Potts spin glass always preserves the color symmetry. We postpone a more precise discussion of the literature and first show that this conjecture is false when κ is sufficiently large.

Theorem 2 (Color symmetry breaking)**.** *For every $N \geqslant 1$ and $\beta \geqslant 0$, we have*

$$F_N(\beta) \geqslant \left(\frac{N-1}{N}\right)^{3/2} \frac{2\beta}{3\sqrt{\pi}}, \tag{6}$$

and

$$\inf_{\pi \in \Pi_{(\frac{1}{\kappa},\dots,\frac{1}{\kappa})}} \mathscr{P}_\beta(\pi) \leqslant \log \kappa + \frac{\beta^2}{2\kappa}. \tag{7}$$

In particular, the claim that $F_N(\beta)$ converges to the left-hand side of (7) is false as soon as

$$\frac{\kappa}{\log \kappa} > \frac{9\pi}{2} \quad \text{and} \quad \left|\frac{3\sqrt{\pi}}{2\kappa}\beta - 1\right|^2 < 1 - \frac{9\pi \log \kappa}{2\kappa}. \tag{8}$$

Proof. We start with the proof of (6). By restricting the summation on $\sigma \in \{1,\dots,\kappa\}^N$ to a summation over $\sigma \in \{1,2\}^N$ in the definition of $F_N(\beta)$ in (2), we see that the term on the left side of (3) is a lower bound for $F_N(\beta)$. A simple lower bound for the term on the right side of (3) can be found in [8, Exercises 6.1 and 6.3 and solutions], and this yields (6).

As announced, for the proof of (7) we only need to consider a very special path, and we fix the additional parameter λ appearing in [18] to be zero. We choose the path π to be constant equal to $\mathrm{diag}(\frac{1}{\kappa},\dots,\frac{1}{\kappa})$ over $(0,1]$. In the notation of [18], this corresponds to the case of $r = 1$, $x_0 = 0$, $x_1 = 1$, $\gamma_0 = 0$, $\gamma_1 = \mathrm{diag}(\frac{1}{\kappa},\dots,\frac{1}{\kappa})$. Letting $(z_1,\dots,z_\kappa)$ be independent centered Gaussians of variance 1, we find that

$$\mathscr{P}_\beta(\pi) \leqslant \mathbb{E} \log \sum_{k=1}^{\kappa} \exp\left(\beta\sqrt{\frac{2}{\kappa}} z_k\right) - \frac{\beta^2}{2\kappa}.$$

We use Jensen's inequality to interchange the expectation and the logarithm in this last expression, and we obtain (7). The last part of the theorem follows by identifying the region in which the right-hand side of (6) exceeds the right-hand side of (7). $\square$

The condition in (8) is non-empty as soon as $\kappa \geqslant 58$. I made no attempt to obtain sharp bounds. In particular, the argument for (6) yields that the large-N limit of $F_N(\beta)$ is bounded from below by $\beta/2$ times the maximum of the Sherrington-Kirkpatrick energy function, which is expected to be about $\sqrt{2} \times 0.763\dots$ [6, 9, 20] (the extra $\sqrt{2}$ accounts for a difference in the choice of normalization). Using this bound instead, we find that color symmetry breaking occurs as soon as the number of colors κ is at least 21.

By analogy with the non-disordered version of the Potts model, one may expect that, at least for sufficiently large values of κ, the range of β's at which color symmetry is broken is unbounded. The bound (7) is however too crude to allow us to obtain this. By reasoning as in [8, Exercise 6.3 and solution], one can see that for every $N \geqslant 1$, $d \in \mathscr{D}$ and $\varepsilon > 0$,

$$F_N(\beta, d, \varepsilon) = \frac{\beta}{N} \mathbb{E} \max_{\sigma \in \Sigma_N(d,\varepsilon)} H_N(\sigma) + O(1) \qquad (\beta \to +\infty),$$

so the bound in (7) does not even capture the correct asymptotic behavior of this quantity as β tends to infinity, as it incorrectly scales like β^2 instead of scaling like β. There is clearly a lot of room to improve upon (7).

We now briefly review previous works on the topic. Most works in the physics literature also allow for the couplings (g_{ij}) to have a bias. In order to discuss this while keeping consistent notation, we thus define a more general version of the free energy by setting, for every $\beta \geqslant 0$ and $\gamma \in \mathbb{R}$,

$$(9) \qquad F_N(\beta, \gamma) \coloneqq \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{1,\dots,\kappa\}^N} \exp\left(\beta H_N(\sigma) + \frac{\beta\gamma}{N} \sum_{i,j=1}^N \mathbf{1}_{\{\sigma_i = \sigma_j\}} \right).$$

In the papers [10, 12], the authors restrict their analysis to the case $\gamma \leqslant \gamma_F(\kappa)$, where $\gamma_F(\kappa)$ is "the critical mean exchange for the highest-temperature transition to be to a ferromagnetic state". In [11], they announce a full resolution of the phase diagram in (β, γ) (also allowing for an external field); I cannot extract from there a precise condition on (β, γ) that would guarantee color symmetry, but they say that this requires to take $\gamma < 0$ for $\kappa > 4$, citing [17]. A concurrent work is [14], which focuses on the case $\gamma = 0$; the authors postulate color symmetry there, but they quickly correct this in [17] and propose a more sophisticated solution that is not color-symmetric for $\kappa > 4$. In [15], we read that "an appropriate nonzero value of γ must be chosen [10, 11, 12]" (notation and pointers adapted). In [7], the authors state that "ferromagnetic order is always preferred for $\kappa > 2$ for sufficiently low temperature", they cite [11], and they give a formula for the transition temperature which they denote by T_F. They specify that the transition is to a "colinear ferromagnet", which in the language of [11] is a phase in which only one color displays a non-zero overlap (in particular, this phase is not color-symmetric). They then say that "In the special case $\gamma = 0$ the ferromagnetic transition appears below $T = 1$ for $\kappa < 4$ and above that temperature for $\kappa > 4$. Our main interest in this paper is the study of the spin-glass transition. In order not to observe the ferromagnetic transition it will be necessary to add an antiferromagnetic coupling in the case $\kappa > 4$." (notation for γ and κ adapted). For their numerical simulations, they chose $\gamma = \frac{4-\kappa}{2\sqrt{2}}$ for $\kappa \geqslant 4$ and found it suitable to their stated needs. Similar statements can also be found in [2], although with a different formula for the transition temperature T_F, the existence of which is attributed to [15] there.

The recent paper [1] suggests that color symmetry is preserved for $\gamma = 0$ and arbitrary values of κ and β. As we have seen, this is invalid at least for $\kappa \geqslant 58$. The authors of [1] attribute this color-symmetry prediction to [10].

My own reading of the physics literature is different, as explained in the previous paragaph. The work [1] inspired [3], in which color symmetry is shown for all β with the choice of $\gamma = -\frac{\beta}{2}$. The results of [3] have been generalized in [16] to a broader class of models. One may also consult [4, 19] for results on the asymptotics of the free energy of more general spin-glass models, and [5, 13] for a thorough study of the non-disordered mean-field Potts model.

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