

FREE ENERGY OF NON-CONVEX MULTI-SPECIES SPHERICAL SPIN GLASSES

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ABSTRACT. We identify the limit free energy of all mean-field multi-species spin glasses defined on products of spheres. In particular, we do not make any convexity assumption on the covariance function of the Hamiltonian, and we allow for the presence of external fields. We also obtain a one-species reduction of the formula in the special case of balanced models.

1. INTRODUCTION

The aim of this paper is to identify the limit free energy of all multi-species spherical spin glasses, without imposing any convexity assumption on the covariance function of the Hamiltonian. We first describe the model. Let $\mathcal{S}$ be a finite set representing species labels. For each $N \in \mathbb{N}$, we let $(I_{N,s})_{s \in \mathcal{S}}$ be a partition of $[N] := \{1, \dots, N\}$, and we set

$$(1.1) \quad \lambda_N := (\lambda_{N,s})_{s \in \mathcal{S}}, \quad \text{with} \quad \lambda_{N,s} := \frac{|I_{N,s}|}{N}.$$

We assume that there exists $\lambda_\infty \in (0, 1]^{\mathcal{S}}$ such that

$$(1.2) \quad \lim_{N \rightarrow \infty} \lambda_N = \lambda_\infty.$$

For every $\sigma, \sigma' \in \mathbb{R}^N$, we define the species overlaps by

$$(1.3) \quad R_N(\sigma, \sigma') := (R_{N,s}(\sigma, \sigma'))_{s \in \mathcal{S}}, \quad \text{with} \quad R_{N,s}(\sigma, \sigma') := \frac{1}{N} \sum_{i \in I_{N,s}} \sigma_i \sigma'_i.$$

Let $\xi : \mathbb{R}^{\mathcal{S}} \rightarrow \mathbb{R}$ be a function such that, for some coefficients $(\beta_{s_1, \dots, s_k})$, we have the absolutely convergent expansion, for every $x \in \mathbb{R}^{\mathcal{S}}$,

$$(1.4) \quad \xi(x) = \sum_{k=1}^{+\infty} \sum_{s_1, \dots, s_k \in \mathcal{S}} \beta_{s_1, \dots, s_k}^2 \prod_{j=1}^k x_{s_j},$$

and let $(H_N(\sigma))_{\sigma \in \mathbb{R}^N}$ be a centered Gaussian field with covariance

$$(1.5) \quad \mathbb{E}[H_N(\sigma)H_N(\sigma')] = N\xi(R_N(\sigma, \sigma')).$$

As discussed in [21, Section 6], under the stated assumption on ξ , such a Gaussian field exists; and conversely, if such a Gaussian field exists and ξ admits an absolutely convergent power-series expansion, then this expansion must be of the form in (1.4). We stress that we do not assume any convexity property on ξ . For every $n \in \mathbb{N}$, let

$$(1.6) \quad \mathbb{S}_n := \{u \in \mathbb{R}^n : |u|^2 = n\}, \quad \text{and} \quad \mu_n := \text{the normalized surface measure on } \mathbb{S}_n.$$

We define the reference probability measure P_N on $\mathbb{R}^N$ as

$$(1.7) \quad dP_N(\sigma) := \bigotimes_{s \in \mathcal{S}} d\mu_{|I_{N,s}|}((\sigma_i)_{i \in I_{N,s}}).$$

We also fix a deterministic external field vector

$$(1.8) \quad h = (h^s)_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}},$$

so that every coordinate in species s feels the field h^s . The goal of the paper is to determine, for each $t \geq 0$, the limit as N tends to infinity of

$$(1.9) \quad \bar{F}_N(t, 0) := -\frac{1}{N} \mathbb{E} \log \int \exp \left(\sqrt{2t} H_N(\sigma) - Nt\xi(\lambda_N) + \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} \sigma_i \right) P_N(d\sigma).$$

The compensation term $-Nt\xi(\lambda_N)$ is harmless and is here to simplify the expression of the derivative in t of $\bar{F}_N$, and thus of the resulting partial differential equations.

Let $\mathcal{Q}$ be the set of right-continuous nondecreasing functions $q : [0, 1) \rightarrow \mathbb{R}_+$, with $\mathbb{R}_+ = [0, +\infty)$. For every $r \in [1, \infty]$, we set $\mathcal{Q}_r := \mathcal{Q} \cap L^r[0, 1)$, and for every $a \geq 0$, we write $\mathcal{Q}_{\infty, \leq a}$ to denote the set of functions in $\mathcal{Q}_\infty$ that remain bounded by a . We will in fact consider a larger family of quantities than those in (1.9), and the second argument of $\bar{F}_N$ can be any element of $\mathcal{Q}_1^\mathcal{S}$. This additional path variable encodes the ultrametric structure of a random magnetic field that we add to the model, see (2.11)–(2.15); the value displayed in (1.9) is that obtained for the choice of the null path for each of the species. As we set $t = 0$ in this extended free energy, the Hamiltonian H_N disappears, and we can determine, for each $q \in \mathcal{Q}_1^\mathcal{S}$, the limit

$$\psi(q) := \lim_{N \rightarrow +\infty} \bar{F}_N(0, q).$$

We point out that the function ψ depends on h , even though we keep it implicit in the notation. For every $t \geq 0$, $q, q' \in \mathcal{Q}_1^\mathcal{S}$, and $p \in \mathcal{Q}_\infty^\mathcal{S}$, we define

$$(1.10) \quad \mathcal{J}_{t,q}(q', p) := \psi(q') + \langle q - q', p \rangle_{L^2} + t \int_0^1 \xi(p(r)) dr,$$

where $\langle \kappa, p \rangle_{L^2} := \int_0^1 \kappa(u) \cdot p(u) du$ denotes the pairing of $\mathbb{R}^\mathcal{S}$ -valued paths. For every $\lambda \in \mathbb{R}_+^\mathcal{S}$, we write $\mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S} := \prod_{s \in \mathcal{S}} \mathcal{Q}_{\infty, \leq \lambda_s}$. Here is our main result.

Theorem 1.1. *For every $t \geq 0$ and $q \in \mathcal{Q}_2^\mathcal{S}$, we have*

$$(1.11) \quad \lim_{N \rightarrow \infty} \bar{F}_N(t, q) = \sup_{p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}} \inf_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}(q', p).$$

Moreover, denoting this limit by $f(t, q)$, the function f is the Lipschitz viscosity solution to

$$(1.12) \quad \begin{cases} \partial_t f - \int_0^1 \xi(\partial_q f) = 0, & \text{on } \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S}, \\ f(0, \cdot) = \psi, & \text{on } \mathcal{Q}_2^\mathcal{S}. \end{cases}$$

In particular, $f(t, 0)$ is the limit of the quantity in (1.9).

It will be shown in forthcoming work [8] that the variational formula on the right side of (1.11) is related to the Crisanti–Sommers formula [13] by a simple change of variables. When ξ is convex over $\mathbb{R}_+^\mathcal{S}$, the limit free energy (with $q = 0$) was already identified in [3, 4, 12, 28]. The approach pursued there is to first identify the limit free energy as a Parisi formula, and then to observe that this formula coincides with the Crisanti–Sommers formula. With our notation, the Parisi formula takes the form $\sup_{q'} \inf_p \mathcal{J}_{t,q}(q', p)$, and one can check that this expression becomes invalid for non-convex ξ , as was already observed in the context of ± 1 spins in [20, Section 6]. Here, we instead prove the formula (1.11) directly, as is required.

Prior to the present paper, the case when ξ is not convex was not very well understood. In the case when $|\mathcal{S}| = 2$, $h = 0$, and $\xi(x, y) = x^p y^q$ with $p, q \geq 96$, the limit of the ground-state energy was determined in [19]. Still with $h = 0$ but with ξ an arbitrary monomial, the limit free energy was determined in [26, 27] under the assumption that such a limit exists for a family of auxiliary models derived from ξ . Since Theorem 1.1 applies to every such model, the results of [26, 27] are now unconditional. The case of $|\mathcal{S}| = 2$, $h = 0$, and $\xi(x, y) = xy$ has been obtained unconditionally in [1, 2]. For $h = 0$ and special choices of the parameter λ_∞ , the ground-state energy of models with $\xi(x, y) = x^p y^q$ or with $\xi(x_1, \dots, x_d) = x_1 \cdots x_d$ has also been obtained unconditionally in [6, 14]. More precisely, the authors of [6] introduce a class of models called *balanced*, where particular symmetries suggest a possible reduction to an equivalent model with a single species, and they show the validity of this reduction for convex ξ , for the ground-state energy of the cases mentioned above, and at high temperature for all ξ ; see also [5, 18]. We prove that this reduction is indeed valid for all balanced spherical models in Section 7. All the non-convex cases in which these earlier works identified the limit free energy or ground-state energy display at most one step of replica symmetry breaking, whereas we handle models with a potentially infinite number of steps of replica symmetry breaking here.

The proof of Theorem 1.1 follows the strategy introduced in [9] for multi-species models with ± 1 spins and no external field. In contrast to the situation with ± 1 spins, for the spherical model the initial condition ψ is convex for all values of the external field, and this is the reason why we can cover all models in this case. Using this convexity property and the results of [11, 20, 21] based on Hamilton–Jacobi equations, we can readily obtain the bound

$$\liminf_{N \rightarrow +\infty} \bar{F}_N(t, q) \geq \sup_{p \in \mathcal{Q}_{\infty, \leq \lambda_{\infty}}^{\mathcal{J}}} \inf_{q' \in \mathcal{Q}_{\infty}^{\mathcal{J}}} \mathcal{J}_{t, q}(q', p),$$

as will be explained in more detail in Section 5. This is our replacement for Guerra’s interpolation bound [16] used for models with convex ξ . To obtain the converse bound, the cavity argument of [4, 12] guarantees the existence of a path $p \in \mathcal{Q}_{\infty}^{\mathcal{J}}$ such that

$$(1.13) \quad \limsup_{N \rightarrow +\infty} \bar{F}_N(t, q) \leq \mathcal{J}_{t, q}(q + t\nabla \xi(p), p).$$

This argument remains valid for non-convex ξ , but is no longer sufficient to conclude. In order to complete the argument, we will show that we can realize (1.13) for a path $p \in \mathcal{Q}_{\infty}^{\mathcal{J}}$ that satisfies the relation

$$(1.14) \quad p = \partial_q \psi(q + t\nabla \xi(p)).$$

Cavity arguments are more difficult to implement for spherical models than for models with ± 1 spins, since the uniform measure on a sphere is clearly not a product measure. For the purpose of proving (1.13) only, the inequality goes in the right direction for us to simply restrict integrals to “good sets” where we expect certain variables to concentrate. In order to establish (1.14), we also rely on certain cavity computations, however here we need to establish an equality, so the derivation of (1.14) will necessarily be more delicate. We give a detailed roadmap of our approach to these cavity calculations in the beginning of Section 3.

Organization of the paper. Section 2 defines the enriched free energy, identifies its initial condition ψ , and records their regularity properties. It ends with the precise statement of the claim (1.13)–(1.14), in Proposition 2.12, and with the short proof of Lemma 2.13 explaining why this would indeed provide the upper bound for Theorem 1.1. Section 3 contains the cavity calculations, whose outcome is the general cavity estimate of Theorem 3.24. Section 4 completes the proof of the claim (1.13)–(1.14), under a number of restrictions including that λ_{∞} must have rational coordinates. Section 5 defines viscosity solutions of (1.12), identifies the solution through the Hopf formula, and proves the lower bound for Theorem 1.1. Section 6 combines the two bounds into the identification of the limit free energy under the restrictions inherited from previous arguments, and then lifts these restrictions to complete the proof of Theorem 1.1. Section 7 discusses balanced models. Appendix A collects the identities for tilted discrete cascades used in Section 2, and Appendix B proves the concentration of the free energy for the Hamiltonians appearing in the cavity calculations.

Statement of AI use. OpenAI’s ChatGPT 5.5 and 5.6 and Anthropic’s Claude Fable 5 and 5.1 were used to help us in the development of the proofs of some technical parts and in the polishing of the paper.

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2. ENRICHED MODEL AND REGULARITY PROPERTIES

In this section, we introduce the enriched free energy $\bar{F}_N(t, q)$, in which the spins are coupled to an external Gaussian field organized along a Poisson–Dirichlet probability cascade. We show that $\bar{F}_N$ is Lipschitz continuous, differentiable, and semi-concave in q . These properties are collected in Subsection 2.1, together with the invariance of the overlap law, which is used throughout the cavity computations in the next section.

We also identify the initial condition $\psi = \lim_{N \rightarrow \infty} \bar{F}_N(0, \cdot)$. Following Talagrand [28], we replace the uniform measure on the sphere by a Gaussian measure of variance $1/b$, where b is

a Lagrange multiplier for the spherical constraint. The one-coordinate version of this model, the Gaussian block, is introduced in Subsection 2.2, and we initially define ψ in terms of this object. The regularity and convexity of ψ , as well as an explicit formula for its derivative, are proved in Subsection 2.3, and the convergence $\bar{F}_N(0, q) \rightarrow \psi(q)$ is proved in Subsection 2.4. The multi-species version of the Gaussian block reappears in Section 3 as the limit law of the cavity coordinates.

Finally, in Subsection 2.5, we give the precise statement of the claim (1.13)–(1.14), in Proposition 2.12, whose proof occupies Sections 3 and 4, and we explain how this proposition enters into the proof of Theorem 1.1.

2.1. Cascades and the enriched free energy. Recall the reference measure P_N from (1.7). For every $u \in \mathbb{R}^N$ and $s \in \mathcal{S}$, we write $u^{(s)} := (u_i)_{i \in I_{N,s}} \in \mathbb{R}^{I_{N,s}}$, and we denote by

$$(2.1) \quad \Sigma_N := \left\{ \sigma \in \mathbb{R}^N : \text{for every } s \in \mathcal{S}, \sigma^{(s)} \in \mathbb{S}_{|I_{N,s}|} \right\}$$

the support of P_N .

Continuous cascades. Let $\mathfrak{R}$ be a Poisson–Dirichlet probability cascade whose overlap is uniformly distributed over $[0, 1]$. This is a random probability measure on the unit sphere of an abstract separable Hilbert space, whose scalar product we denote by $\wedge$; we refer to [22, Chapter 2] or [10, Section 4] for its construction as a limit of discrete cascades, and to [22, Chapter 2] or [15, Chapter 5] for the properties of discrete cascades that we use below. We write $\mathfrak{U} := \text{supp } \mathfrak{R}$ for the (random) support of $\mathfrak{R}$, and $\alpha, \alpha', \alpha^1, \alpha^2, \dots$ for elements of $\mathfrak{U}$. For every $n \in \mathbb{N}$ and every integrable function g of $\alpha^1, \dots, \alpha^n$, we write

$$(2.2) \quad \langle g \rangle_{\mathfrak{R}} := \int g(\alpha^1, \dots, \alpha^n) \prod_{\ell=1}^n \mathfrak{R}(d\alpha^\ell).$$

The bracket $\langle \cdot \rangle_{\mathfrak{R}}$ is conditional on $\mathfrak{R}$, while $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$ also averages over the randomness of $\mathfrak{R}$. By construction, $\alpha \wedge \alpha'$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$, and $\mathfrak{U}$ is almost surely an ultrametric set.

Every path $q \in \mathcal{Q}_\infty$ is extended to $[0, 1]$ by setting

$$(2.3) \quad q(1) := \lim_{u \nearrow 1} q(u).$$

By [10, Proposition 4.1], for every $q \in \mathcal{Q}_\infty$ and almost every realization of $\mathfrak{R}$, there exists a centered Gaussian process $(w^q(\alpha))_{\alpha \in \mathfrak{U}}$, jointly measurable in α and in the underlying randomness, with covariance

$$(2.4) \quad \mathbb{E}[w^q(\alpha)w^q(\alpha')] = q(\alpha \wedge \alpha'), \quad \alpha, \alpha' \in \mathfrak{U}.$$

In particular, the variance of $w^q(\alpha)$ is $q(1)$. Whenever we take expectations of functionals involving both $\mathfrak{R}$ and such Gaussian processes, we first average over the Gaussian randomness, conditionally on $\mathfrak{R}$, and then over $\mathfrak{R}$; see [10, Lemma 4.5].

The cascade often appears tilted by a function of these Gaussian processes, and the property of the continuous cascade that we use most often is that such a tilt does not change the law of the overlaps. Let $n \in \mathbb{N}$, let $q_1, \dots, q_n \in \mathcal{Q}_\infty$, and, conditionally on $\mathfrak{R}$, let $w^{q_1}, \dots, w^{q_n}$ be independent Gaussian fields with the covariances (2.4); we write $\mathbf{w}(\alpha) := (w^{q_i}(\alpha))_{i \leq n} \in \mathbb{R}^n$. Given a Lipschitz function $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}$, let $\mathfrak{R}^{\mathbf{g}}$ be the random probability measure on $\mathfrak{U}$ with density proportional to $\exp(\mathbf{g}(\mathbf{w}(\alpha)))$ with respect to $\mathfrak{R}$, and let $\langle \cdot \rangle_{\mathfrak{R}^{\mathbf{g}}}$ be defined as in (2.2), with $\mathfrak{R}^{\mathbf{g}}$ in place of $\mathfrak{R}$. By [10, Proposition 4.8], applied to the diagonal matrix-valued path $\text{diag}(q_1, \dots, q_n)$, the law of the overlap array is invariant under this tilt: for every $n' \in \mathbb{N}$ and every bounded measurable $\mathbf{h} : [0, 1]^{n' \times n'} \rightarrow \mathbb{R}$,

$$(2.5) \quad \mathbb{E} \left\langle \mathbf{h} \left((\alpha^\ell \wedge \alpha^{\ell'})_{\ell, \ell' \leq n'} \right) \right\rangle_{\mathfrak{R}^{\mathbf{g}}} = \mathbb{E} \left\langle \mathbf{h} \left((\alpha^\ell \wedge \alpha^{\ell'})_{\ell, \ell' \leq n'} \right) \right\rangle_{\mathfrak{R}};$$

in particular, $\alpha \wedge \alpha'$ remains uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}^{\mathbf{g}}}$. In the enriched free energy introduced below, we take $n = N$ and $\mathbf{g}(\mathbf{w}(\alpha)) = \log \int \exp(H_N^{t,q}(\sigma, \alpha)) P_N(d\sigma)$, with $H_N^{t,q}$ as in (2.13); this function is Lipschitz in the fields, since $|\sigma|^2 = N$ on Σ_N , and since the disorder H_N is independent of the cascade and of the fields, we apply (2.5) conditionally on H_N .

Discrete cascades. Let $k \geq 0$ and $0 = m_0 < m_1 < \dots < m_k < m_{k+1} = 1$. A path $\rho \in \mathcal{Q}_\infty$ is called a finite-step path on the partition $(m_j)_{0 \leq j \leq k+1}$ if it can be written as

$$(2.6) \quad \rho = \sum_{j=0}^k \rho_j \mathbb{1}_{[m_j, m_{j+1})}, \quad 0 = \rho_{-1} \leq \rho_0 \leq \dots \leq \rho_k,$$

so that $\rho(1) = \rho_k$. We denote by $\mathcal{A} := \mathbb{N}^0 \cup \mathbb{N}^1 \cup \dots \cup \mathbb{N}^k$ the infinitary tree of depth k (with $\mathbb{N}^0 = \{\emptyset\}$), and for every $\alpha = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $\ell \in \{0, \dots, k\}$, we write $\alpha|_\ell = (n_1, \dots, n_\ell)$ for the ancestor of α at depth ℓ . The discrete Poisson–Dirichlet probability cascade with parameters $(m_1, \dots, m_k)$ is a random family of probability weights $(v_\alpha)_{\alpha \in \mathbb{N}^k}$ indexed by the leaves of $\mathcal{A}$. Its construction is explained in detail in [22, Section 2.3] or in [15, Section 5.6]; briefly, the probability weights are proportional to the products $w_\alpha := \prod_{j=1}^k u_{\alpha|_j}$, where, for every node $\beta \in \mathbb{N}^{j-1}$, the weights $(u_{\beta i})_{i \geq 1}$ attached to the children of β form a Poisson–Dirichlet point process with parameter m_j , independently over β and j . When $k = 0$, the tree is reduced to its root and $v_\emptyset = 1$. For two leaves α, α' , we write $\alpha \wedge \alpha' \in \{0, \dots, k\}$ for the depth of their most recent common ancestor; under two independent samples from $(v_\alpha)_{\alpha \in \mathbb{N}^k}$, averaged over the weights, this depth satisfies

$$(2.7) \quad \mathbb{P}(\alpha \wedge \alpha' = j) = m_{j+1} - m_j, \quad j \in \{0, \dots, k\},$$

by [15, Theorem 5.28]. Let $(z_\beta)_{\beta \in \mathcal{A}}$ be independent standard Gaussian random variables attached to the nodes of the tree, which we call marks. For a finite-step path ρ as in (2.6), the discrete field

$$(2.8) \quad \mathbf{w}^\rho(\alpha) := \sum_{j=0}^k (\rho_j - \rho_{j-1})^{1/2} z_{\alpha|_j}, \quad \alpha \in \mathbb{N}^k,$$

is a centered Gaussian field with covariance $\mathbb{E} \mathbf{w}^\rho(\alpha) \mathbf{w}^\rho(\alpha') = \rho_{\alpha \wedge \alpha'}$.

The relation between the continuous and the discrete constructions is the following. We denote by $\pi_m : [0, 1] \rightarrow \{m_0, \dots, m_k\}$ the coarse-graining map $\pi_m(u) := m_j$ for $u \in [m_j, m_{j+1})$, with $\pi_m(1) := m_k$, so that a finite-step path on the partition is a function of $\pi_m(u)$. Since $\alpha \wedge \alpha'$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$, the coarse-grained overlap $\pi_m(\alpha \wedge \alpha')$ of the continuous cascade takes the value m_j with probability $m_{j+1} - m_j$, which is also the law of $m_{\alpha \wedge \alpha'}$ under the discrete cascade, by (2.7). Both cascades satisfy the Ghirlanda–Guerra identities, which determine the law of the overlap array from that of a single overlap, and this is why the discrete cascade can replace the continuous one when the paths are finite-step paths on the partition; see the proof of [10, Proposition 4.6]. Precisely, let $q_1, \dots, q_n$ be finite-step paths on the partition $(m_j)_{0 \leq j \leq k+1}$, let $\mathbf{w}, \mathbf{g}$ and $\mathfrak{R}^{\mathbf{g}}$ be defined from these paths as before (2.5), and let $\mathbf{w}^{q_1}, \dots, \mathbf{w}^{q_n}$ be the discrete fields (2.8) built from independent families of marks $(z_{\beta,1})_{\beta \in \mathcal{A}}, \dots, (z_{\beta,n})_{\beta \in \mathcal{A}}$, with $\mathbf{w}(\alpha) := (\mathbf{w}^{q_i}(\alpha))_{i \leq n}$. Then, by [10, Proposition 4.6],

$$(2.9) \quad \mathbb{E} \log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp(\mathbf{g}(\mathbf{w}(\alpha))) = \mathbb{E} \log \int_{\mathfrak{U}} \exp(\mathbf{g}(\mathbf{w}(\alpha))) \mathfrak{R}(\mathrm{d}\alpha),$$

and, for every $n' \in \mathbb{N}$ and every bounded measurable $\mathbf{h}$,

$$(2.10) \quad \mathbb{E} \langle \mathbf{h}((m_{\alpha^\ell \wedge \alpha'^{\ell'}})_{\ell, \ell' \leq n'}) \rangle_{\mathbf{g}}^{\mathrm{d}} = \mathbb{E} \langle \mathbf{h}((\pi_m(\alpha^\ell \wedge \alpha'^{\ell'}))_{\ell, \ell' \leq n'}) \rangle_{\mathfrak{R}^{\mathbf{g}}},$$

where $\langle \cdot \rangle_{\mathbf{g}}^{\mathrm{d}}$ denotes the average with respect to the tilted discrete cascade $v_\alpha^{\mathbf{g}} \propto v_\alpha \exp(\mathbf{g}(\mathbf{w}(\alpha)))$, with independent replicas $\alpha^1, \alpha^2, \dots$. In the Gaussian block of Subsection 2.2, we use (2.9) with the quadratic function $\mathbf{g}(x) = (\sqrt{2}x_1 + a)/(2b)$, which is not Lipschitz; since it is nonnegative, (2.9) for this $\mathbf{g}$ follows from (2.9) for the bounded Lipschitz functions $\mathbf{g} \wedge L$ by monotone convergence as $L \rightarrow \infty$, both sides being then possibly equal to $+\infty$. Further identities for discrete cascades, concerning the law of the marks along the leaves sampled from a tilted cascade, are recalled in Appendix A.

The enriched Hamiltonian. Let $q = (q^s)_{s \in \mathcal{S}} \in \mathcal{Q}_\infty^{\mathcal{S}}$. For every $s \in \mathcal{S}$ and $i \in I_{N,s}$, let $(w_i^{q^s}(\alpha))_{\alpha \in \mathfrak{U}}$ be a centered Gaussian process with covariance (2.4) for the path q^s ; conditionally on $\mathfrak{R}$, these processes are independent as s and i vary, and independent of H_N . We set

$$(2.11) \quad W_N^q(\sigma, \alpha) := \sum_{s \in \mathcal{S}} \sum_{i \in I_{N,s}} w_i^{q^s}(\alpha) \sigma_i, \quad \sigma \in \mathbb{R}^N, \quad \alpha \in \mathfrak{U}.$$

By (1.3) and (2.4), this is a centered Gaussian field with covariance

$$(2.12) \quad \mathbb{E}[W_N^q(\sigma, \alpha)W_N^q(\sigma', \alpha')] = Nq(\alpha \wedge \alpha') \cdot R_N(\sigma, \sigma'),$$

where $a \cdot b := \sum_{s \in \mathcal{S}} a_s b_s$ for $a, b \in \mathbb{R}^{\mathcal{S}}$. For $t \geq 0$, we define the enriched Hamiltonian

$$(2.13) \quad \begin{aligned} H_N^{t,q}(\sigma, \alpha) &:= \sqrt{2t} H_N(\sigma) - Nt\xi(\lambda_N) + \sqrt{2} W_N^q(\sigma, \alpha) - Nq(1) \cdot \lambda_N \\ &\quad + \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} \sigma_i, \end{aligned}$$

where $q(1) := (q^s(1))_{s \in \mathcal{S}}$. Since $R_N(\sigma, \sigma) = \lambda_N$ for every $\sigma \in \Sigma_N$, the deterministic terms $Nt\xi(\lambda_N)$ and $Nq(1) \cdot \lambda_N$ are the variances of $\sqrt{t}H_N(\sigma)$ and of $W_N^q(\sigma, \alpha)$, so that the random part of (2.13) is a Gaussian field with its self-overlap correction, as in [9, 10]. The partition function and the free energy are

$$(2.14) \quad Z_N(t, q) := \iint \exp(H_N^{t,q}(\sigma, \alpha)) P_N(d\sigma) \mathfrak{R}(d\alpha),$$

$$(2.15) \quad \bar{F}_N(t, q) := -\frac{1}{N} \mathbb{E} \log Z_N(t, q).$$

Note the minus sign in (2.15). If $q = 0$, then (2.15) agrees with (1.9). For every $n \in \mathbb{N}$ and every bounded measurable function f of $(\sigma^\ell, \alpha^\ell)_{1 \leq \ell \leq n}$, we write

$$(2.16) \quad \langle f \rangle_{N,t,q} := \frac{1}{Z_N(t, q)^n} \int f((\sigma^\ell, \alpha^\ell)_{1 \leq \ell \leq n}) \prod_{\ell=1}^n e^{H_N^{t,q}(\sigma^\ell, \alpha^\ell)} P_N(d\sigma^\ell) \mathfrak{R}(d\alpha^\ell)$$

for the associated Gibbs bracket; it is conditional on all the randomness in (2.13), and $\mathbb{E} \langle \cdot \rangle_{N,t,q}$ also averages over this randomness. The pairs $(\sigma^\ell, \alpha^\ell)$ are called replicas, and we often write (σ, α) , (σ', α') for two of them.

Notation for paths. For $r \in [1, \infty]$, we equip $\mathcal{Q}_r^{\mathcal{S}} = \prod_{s \in \mathcal{S}} \mathcal{Q}_r$ with the norm $|q|_{L^r}$ of $L^r([0, 1]; \mathbb{R}^{\mathcal{S}})$, where $\mathbb{R}^{\mathcal{S}}$ carries the Euclidean norm $|\cdot|$, and we write $\langle \kappa, q \rangle_{L^2} := \int_0^1 \kappa(u) \cdot q(u) du$. For $a \geq 0$ and $\lambda \in \mathbb{R}_+^{\mathcal{S}}$, we recall that $\mathcal{Q}_{\infty, \leq a}$ denotes the set of paths in $\mathcal{Q}_\infty$ bounded by a , and that $\mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}} = \prod_{s \in \mathcal{S}} \mathcal{Q}_{\infty, \leq \lambda_s}$. Finite-step paths are dense in $\mathcal{Q}_r^{\mathcal{S}}$ for every $r \in [1, \infty]$.

We say that a function $g : \mathcal{Q}_2^{\mathcal{S}} \rightarrow \mathbb{R}$ is Gateaux differentiable at $q \in \mathcal{Q}_2^{\mathcal{S}}$ if there exists $y \in L^2([0, 1]; \mathbb{R}^{\mathcal{S}})$ such that, for every $\kappa \in L^2([0, 1]; \mathbb{R}^{\mathcal{S}})$ for which $q + \varepsilon \kappa \in \mathcal{Q}_2^{\mathcal{S}}$ for all sufficiently small $\varepsilon > 0$, we have

$$\lim_{\varepsilon \searrow 0} \frac{g(q + \varepsilon \kappa) - g(q)}{\varepsilon} = \langle \kappa, y \rangle_{L^2},$$

and if y is unique with this property; we then write $\partial_q g(q) := y$. Uniqueness holds automatically here, since the admissible directions include $\mathcal{Q}_\infty^{\mathcal{S}}$, and $\mathcal{Q}_\infty^{\mathcal{S}} - \mathcal{Q}_2^{\mathcal{S}}$ is dense in $L^2([0, 1]; \mathbb{R}^{\mathcal{S}})$.

Regularity of the enriched free energy.

Proposition 2.1 (Differentiability of $\bar{F}_N$). *For every $N \in \mathbb{N}$, $t, t' \geq 0$, and $q, q' \in \mathcal{Q}_\infty^{\mathcal{S}}$,*

$$(2.17) \quad |\bar{F}_N(t, q) - \bar{F}_N(t', q')| \leq |q - q'|_{L^1} + |t - t'| \sup_{a \in \prod_s [-\lambda_{N,s}, \lambda_{N,s}]} |\xi(a)|.$$

In particular, $\bar{F}_N$ extends uniquely by continuity to $\mathbb{R}_+ \times \mathcal{Q}_1^{\mathcal{S}}$, and (2.17) remains valid there. The restriction of $\bar{F}_N$ to $\mathbb{R}_+ \times \mathcal{Q}_2^{\mathcal{S}}$ is Gateaux differentiable in q , with

$$(2.18) \quad \partial_q \bar{F}_N(t, q) \in \mathcal{Q}_{\infty, \leq \lambda_N}^{\mathcal{S}}, \quad (t, q) \in \mathbb{R}_+ \times \mathcal{Q}_2^{\mathcal{S}},$$

and, for every $q \in \mathcal{Q}_\infty^{\mathcal{S}}$ and $\kappa \in L^2([0, 1]; \mathbb{R}^{\mathcal{S}})$,

$$(2.19) \quad \langle \kappa, \partial_q \bar{F}_N(t, q) \rangle_{L^2} = \mathbb{E} \langle \kappa(\alpha \wedge \alpha') \cdot R_N(\sigma, \sigma') \rangle_{N,t,q}.$$

Proof. This is [10, Proposition 5.1], in the diagonal multi-species form of [7, Proposition 4.1], and we only explain why the argument applies to the spherical reference measure. The only property of P_N that it uses is the bound $|R_{N,s}(\sigma, \sigma')| \leq \lambda_{N,s}$, which follows from the identity $R_N(\sigma, \sigma) = \lambda_N$ and the Cauchy-Schwarz inequality; in particular, $|R_N(\sigma, \sigma')| \leq |\lambda_N| \leq 1$ for the Euclidean norm, since the entries of λ_N are nonnegative and sum to 1. The membership of $\partial_q \bar{F}_N(t, q)$ in the cone

of nonnegative nondecreasing paths is obtained there by duality from a monotonicity property of $\bar{F}_N(t, \cdot)$, see [21, Proposition 3.8], whose proof only assumes that the reference measure is supported in the ball of radius $\sqrt{N}$, as is P_N . $\square$

Semi-concavity. For every $c \in (0, 1]$, we set

$$\mathcal{Q}_{\uparrow, c}^{\mathcal{S}} := \left\{ q \in \mathcal{Q}_1^{\mathcal{S}} : q^s(0) = 0 \text{ and } q^s(v) - q^s(u) \geq c(v - u) \text{ for every } s \in \mathcal{S} \text{ and } 0 \leq u \leq v < 1 \right\},$$

and

$$\mathcal{Q}_{\infty, \uparrow}^{\mathcal{S}} := \mathcal{Q}_{\infty}^{\mathcal{S}} \cap \bigcup_{c \in (0, 1]} \mathcal{Q}_{\uparrow, c}^{\mathcal{S}}.$$

For an increasing path q , we denote its distributional derivative by $\dot{q}$. The next result is a semi-concavity estimate for the free energy. It is used in Section 4 to compare the derivatives in q of two free energies that are close to each other.

Proposition 2.2 (Semi-concavity of the free energy). *There exists a constant $C < \infty$, depending only on ξ , such that, for every $N \in \mathbb{N}$, $c \in (0, 1]$, $t \geq 0$, $q, q' \in \mathcal{Q}_{\uparrow, c}^{\mathcal{S}}$ with $\dot{q} - \dot{q}' \in L^2([0, 1]; \mathbb{R}^{\mathcal{S}})$, and $r \in [0, 1]$,*

$$(2.20) \quad (1 - r)\bar{F}_N(t, q) + r\bar{F}_N(t, q') - \bar{F}_N(t, (1 - r)q + rq') \leq Cr(1 - r)c^{-2}|\dot{q} - \dot{q}'|_{L^2}^2.$$

The estimate is unchanged if one adds to the Hamiltonian a Gaussian field independent of the cascade field W_N^q , whose law does not depend on q , or a deterministic affine function of q .

Proof. This is [10, Propositions 3.7 and 3.8] at fixed t , in the diagonal multi-species form of [7, Proposition 4.5]; since t is fixed, the lower bound on t required there for the joint semi-concavity in (t, q) is not needed, and the ellipticity condition of the matrix-valued statement is void for scalar increments. The proofs use only the covariance (2.12) of the cascade field, the self-overlap correction in (2.13), and the bound $|R_{N,s}(\sigma, \sigma')| \leq \lambda_{N,s}$ verified in the proof of Proposition 2.1. For the last assertion, recall that the proof of [10, Proposition 3.7] consists of Hölder's inequality applied in the amplitudes $(q_j^s - q_{j-1}^s)^{1/2}$ of a finite-step path, for which the terms of the Hamiltonian that do not depend on the amplitudes play no role, and of a bound on the derivatives of the free energy in the amplitudes obtained by Gaussian integration by parts. A Gaussian field independent of the cascade field, with a law that does not depend on q , is such a term, and it does not affect the derivative bound; a deterministic affine function of q contributes zero to the left-hand side of (2.20). $\square$

2.2. The Gaussian block and the Crisanti–Sommers functional. We now introduce the function ψ appearing in Theorem 1.1. It is defined through a one-coordinate Gaussian model, the Gaussian block, in which a Lagrange multiplier b replaces the spherical constraint, and through an explicit functional of Crisanti–Sommers type in the variables (ρ, b) , which gives $\psi(\rho)$ after minimization over b . We then study the minimizing multiplier $b_a(\rho)$, introduce the path that will be identified with the derivative of ψ_a° in Subsection 2.3, and characterize $b_a(\rho)$ by a second-moment identity under the block. We treat one species at a time, and write $\rho \in \mathcal{Q}_{\infty}$ for the path and $a \in \mathbb{R}$ for the external field; the multi-species function ψ is a weighted sum of scalar functions. Throughout, $\gamma(d\tau) := (2\pi)^{-1/2}e^{-\tau^2/2}d\tau$ denotes the standard Gaussian measure on $\mathbb{R}$.

The Gaussian block. The idea behind the identification of $\lim_{N \rightarrow \infty} \bar{F}_N(0, q)$ is to replace the uniform measure on the sphere of radius $\sqrt{n}$ by the Gaussian measure $\propto e^{-b|x|^2/2}dx$ on $\mathbb{R}^n$, under which the coordinates decouple, and to choose $b > 0$ so that this Gaussian measure concentrates on the sphere; the parameter b is a Lagrange multiplier for the spherical constraint. This is the approach of [28, Section 3]. Since the enriched Hamiltonian at $t = 0$ is linear in the spins, the Gaussian model factorizes, conditionally on the cascade, into n copies of the following one-coordinate object. For $\rho \in \mathcal{Q}_{\infty}$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$, let

$$(2.21) \quad Z_{a,b}^\circ(\rho) := \int_{\mathbb{R}} \int_{\mathcal{U}} \exp\left((\sqrt{2}w^\rho(\alpha) + a)\tau - \frac{b-1}{2}\tau^2\right) \Re(d\alpha)\gamma(d\tau),$$

which may be infinite, and let $\langle \cdot \rangle_{\rho, b}^\circ$ denote the Gibbs bracket associated with the measure $\exp((\sqrt{2}w^\rho(\alpha) + a)\tau - \frac{b-1}{2}\tau^2) \Re(d\alpha)\gamma(d\tau)$ on $\mathbb{R} \times \mathcal{U}$, when $Z_{a,b}^\circ(\rho) < \infty$; replicas under this bracket

are denoted by $(\tau, \alpha), (\tau', \alpha')$, and the dependence on a is kept implicit. We call this object the Gaussian block with one coordinate. Since $\gamma(d\tau) \exp(-\frac{b-1}{2}\tau^2) = (2\pi)^{-1/2} e^{-b\tau^2/2} d\tau$, the Gaussian integral over τ can be computed, and, when $b > 0$,

$$(2.22) \quad Z_{a,b}^\circ(\rho) = b^{-1/2} \int_{\mathfrak{U}} \exp\left(\frac{(\sqrt{2}w^\rho(\alpha) + a)^2}{2b}\right) \mathfrak{R}(d\alpha).$$

Conditionally on α , the coordinate τ is Gaussian with mean $(\sqrt{2}w^\rho(\alpha) + a)/b$ and variance $1/b$ under $\langle \cdot \rangle_{\rho,b}^\circ$. The quadratic tilt in (2.22) makes the block finite only for b large enough, and the threshold is determined in terms of the quantity $\mathcal{K}$ we introduce next.

The Crisanti–Sommers functional. For $\rho \in \mathcal{Q}_\infty$, set

$$(2.23) \quad \mathcal{K}(\rho) := \rho(1) - \int_0^1 \rho(u) du, \quad \mathcal{K}_\rho(r) := \int_0^1 (\rho(1) - \rho(u) \vee r) du, \quad r \in [0, \rho(1)].$$

The function $\mathcal{K}_\rho$ is nonincreasing and 1-Lipschitz on $[0, \rho(1)]$, with $\mathcal{K}_\rho(0) = \mathcal{K}(\rho)$ and $\mathcal{K}_\rho(\rho(1)) = 0$. We say that $b \in \mathbb{R}$ is admissible for ρ if $b > 2\mathcal{K}(\rho)$. For $a \in \mathbb{R}$ and admissible b , we define

$$(2.24) \quad \mathcal{D}_a^\circ(\rho, b) := \frac{a^2}{2(b - 2\mathcal{K}(\rho))} + \int_0^{\rho(1)} \frac{dr}{b - 2\mathcal{K}_\rho(r)} + \frac{1}{2}(b - 1 - \log b) - \rho(1),$$

which is finite since $b - 2\mathcal{K}_\rho(r) \geq b - 2\mathcal{K}(\rho) > 0$, and we set

$$(2.25) \quad \psi_a^\circ(\rho) := - \inf_{b > 2\mathcal{K}(\rho)} \mathcal{D}_a^\circ(\rho, b).$$

With the fixed vector h from (1.8), we then define

$$(2.26) \quad \psi(q) := \sum_{s \in \mathcal{S}} \lambda_{\infty, s} \psi_{h^s}^\circ(q^s), \quad q \in \mathcal{Q}_\infty^\mathcal{S}.$$

We will show in Proposition 2.8 that ψ is Lipschitz continuous for the L^1 norm, and we extend it to $\mathcal{Q}_1^\mathcal{S}$ by continuity. The notation ψ always refers to this fixed h . More generally, for every probability vector λ with positive entries, we set

$$(2.27) \quad \psi_\lambda(q) := \sum_{s \in \mathcal{S}} \lambda_s \psi_{h^s}^\circ(q^s), \quad q \in \mathcal{Q}_\infty^\mathcal{S},$$

so that $\psi = \psi_{\lambda_\infty}$. This is the initial condition of the model with asymptotic species proportions λ , and it is used in Section 6, where λ_∞ is approximated by rational probability vectors. All the properties of ψ established below are proved species by species, and therefore hold for ψ_λ as well.

The relation between the block and the functional is that, for admissible b ,

$$(2.28) \quad \mathcal{D}_a^\circ(\rho, b) = \mathbb{E} \log Z_{a,b}^\circ(\rho) + \frac{1}{2}(b - 1) - \rho(1);$$

this is part of Lemma 2.7 below. The two corrections in (2.28) have the following origin, which will become apparent in the proof of Proposition 2.11. The term $-\rho(1)$ is the self-overlap correction $-Nq(1) \cdot \lambda_N$ of the enriched Hamiltonian (2.13), counted per coordinate. The term $\frac{1}{2}(b - 1)$ compensates the Gaussian weight $e^{-nb/2}$ carried by the sphere $|x|^2 = n$ in the Gaussian model, and the volume factor $\kappa_n \simeq e^{n/2}$ of the polar decomposition of $\mathbb{R}^n$. Up to the sign convention and the term $-\rho(1)$, the functional $\mathcal{D}_a^\circ(\rho, b)$ is the functional denoted by $W(x, b)$ in [28], and its multi-species version is the Parisi functional of [4] with the interaction term removed. The quantity $2\mathcal{K}(\rho)$ is denoted by d_1 in [12, 28]; we prefer the notation $\mathcal{K}(\rho)$, which stresses that it is a functional of the path.

Remark 2.3 (Finite-step paths). Let ρ be a finite-step path, written as in (2.6). An Abel summation gives $\mathcal{K}(\rho) = \sum_{i=1}^k m_i(\rho_i - \rho_{i-1})$, and for $r \in [\rho_{i-1}, \rho_i)$ with $i \geq 1$ we have $\mathcal{K}_\rho(r) = m_i(\rho_i - r) + \sum_{j>i} m_j(\rho_j - \rho_{j-1})$, while $\mathcal{K}_\rho(r) = \mathcal{K}(\rho)$ for $r \in [0, \rho_0)$. Writing

$$(2.29) \quad c_i := b - 2 \sum_{j=i}^k m_j(\rho_j - \rho_{j-1}), \quad 1 \leq i \leq k+1,$$

so that $c_1 = b - 2\mathcal{K}(\rho)$, $c_{k+1} = b$, and $c_1 \leq c_2 \leq \dots \leq c_{k+1}$, the integral in (2.24) can be computed explicitly:

$$(2.30) \quad \int_0^{\rho(1)} \frac{dr}{b - 2\mathcal{K}_\rho(r)} = \frac{\rho_0}{c_1} + \sum_{i=1}^k \frac{\log c_{i+1} - \log c_i}{2m_i}.$$

The multiplier equation. The endpoint $\rho(1)$ enters $\mathcal{D}_a^\circ$ through the admissibility threshold $2\mathcal{K}(\rho)$, through the domain of integration in (2.24), and through the term $-\rho(1)$. It is not controlled by the L^1 norm of the path, and neither is the minimizer $b_a(\rho)$. However, these occurrences cancel once b is measured from the boundary of the admissible set, in the variable $\ell := b - 2\mathcal{K}(\rho)$. For $\rho \in \mathcal{Q}_1$, $\ell > 0$, and $r \geq 0$, we set $\bar{\rho} := \int_0^1 \rho(u) du$,

$$(2.31) \quad d_{\rho,\ell}(r) := \ell + 2 \int_0^1 (r - \rho(v))_+ dv, \quad \text{and} \quad S_a(\rho, \ell) := \frac{a^2}{\ell^2} + 2 \int_0^\infty \frac{dr}{d_{\rho,\ell}(r)^2},$$

where $(\cdot)_+ := (\cdot) \vee 0$ denotes the positive part.

Lemma 2.4 (The multiplier equation). *Let $a \in \mathbb{R}$.*

(1) *For every $\rho \in \mathcal{Q}_1$, $\ell > 0$, and $r \geq 0$,*

$$(2.32) \quad \ell + 2(r - \bar{\rho})_+ \leq d_{\rho,\ell}(r) \leq \ell + 2r, \quad \text{hence} \quad \frac{1}{\ell} \leq S_a(\rho, \ell) \leq \frac{1}{\ell} + \frac{a^2 + 2\bar{\rho}}{\ell^2}.$$

The function $\ell \mapsto S_a(\rho, \ell)$ is continuous and strictly decreasing on $(0, \infty)$, and there is a unique $\ell_a(\rho) > 0$ such that $S_a(\rho, \ell_a(\rho)) = 1$. It satisfies

$$(2.33) \quad 1 \leq \ell_a(\rho) \leq \frac{1 + \sqrt{1 + 4a^2 + 8\bar{\rho}}}{2}.$$

(2) *Let $\rho \in \mathcal{Q}_\infty$ and $b > 2\mathcal{K}(\rho)$, and set $\ell := b - 2\mathcal{K}(\rho)$. Then $d_{\rho,\ell}(r) = b - 2\mathcal{K}_\rho(r)$ for $r \in [0, \rho(1)]$, $d_{\rho,\ell}(r) = b + 2(r - \rho(1))$ for $r \geq \rho(1)$, and*

$$(2.34) \quad \partial_b \mathcal{D}_a^\circ(\rho, b) = \frac{1}{2} (1 - S_a(\rho, b - 2\mathcal{K}(\rho))).$$

Consequently, $b \mapsto \mathcal{D}_a^\circ(\rho, b)$ is strictly convex on $(2\mathcal{K}(\rho), \infty)$, and the infimum in (2.25) is attained at the unique point

$$(2.35) \quad b_a(\rho) := \ell_a(\rho) + 2\mathcal{K}(\rho) \geq 1.$$

Proof. Part (1). The upper bound on $d_{\rho,\ell}$ follows from $(r - \rho(v))_+ \leq r$, and the lower bound from Jensen's inequality applied to the convex function $x \mapsto (r - x)_+$. Since $2 \int_0^\infty (\ell + 2r)^{-2} dr = \ell^{-1}$ and $2 \int_0^\infty (\ell + 2(r - \bar{\rho})_+)^{-2} dr = 2\bar{\rho}\ell^{-2} + \ell^{-1}$, the bounds on S_a follow. For fixed r , the function $\ell \mapsto d_{\rho,\ell}(r)^{-2}$ is continuous and strictly decreasing, and it is dominated by $(\ell_0 + 2(r - \bar{\rho})_+)^{-2}$, which is integrable on $[0, \infty)$, for $\ell \geq \ell_0 > 0$; by dominated convergence, $\ell \mapsto S_a(\rho, \ell)$ is continuous and strictly decreasing, and by (2.32), it tends to $+\infty$ as $\ell \searrow 0$ and to 0 as $\ell \rightarrow \infty$. Hence the equation $S_a(\rho, \ell) = 1$ has a unique solution $\ell_a(\rho)$. The lower bound $S_a(\rho, \ell) \geq \ell^{-1}$ gives $\ell_a(\rho) \geq 1$, and the upper bound gives $\ell_a(\rho)^2 - \ell_a(\rho) - (a^2 + 2\bar{\rho}) \leq 0$, which is the upper bound in (2.33).

Part (2). For $r \in [0, \rho(1)]$, the definition (2.23) gives $\mathcal{K}(\rho) - \mathcal{K}_\rho(r) = \int_0^1 (\rho(u) \vee r - \rho(u)) du = \int_0^1 (r - \rho(u))_+ du$, so that $b - 2\mathcal{K}_\rho(r) = d_{\rho,\ell}(r)$. For $r \geq \rho(1)$, we have $(r - \rho(v))_+ = r - \rho(v)$ for every v , so that $d_{\rho,\ell}(r) = \ell + 2r - 2\bar{\rho} = b + 2(r - \rho(1))$. Differentiating (2.24) in b under the integral sign, we get

$$\begin{aligned} \partial_b \mathcal{D}_a^\circ(\rho, b) &= -\frac{a^2}{2\ell^2} - \int_0^{\rho(1)} \frac{dr}{d_{\rho,\ell}(r)^2} + \frac{1}{2} \left(1 - \frac{1}{b}\right), \\ \frac{1}{b} &= 2 \int_{\rho(1)}^\infty \frac{dr}{(b + 2(r - \rho(1)))^2} = 2 \int_{\rho(1)}^\infty \frac{dr}{d_{\rho,\ell}(r)^2}, \end{aligned}$$

which is (2.34). Since $\ell \mapsto S_a(\rho, \ell)$ is strictly decreasing, $\partial_b \mathcal{D}_a^\circ(\rho, b)$ is strictly increasing in b , negative for $b < b_a(\rho)$ and positive for $b > b_a(\rho)$; this proves the strict convexity and identifies the minimizer. The bound $b_a(\rho) \geq 1$ follows from (2.33) and $\mathcal{K}(\rho) \geq 0$. $\square$

We next record the continuity properties of the multiplier, and introduce the path that will be identified with the derivative of ψ_a° .

Lemma 2.5 (Regularity of the multiplier and the derivative path). *Let $a \in \mathbb{R}$.*

(1) *For every $\rho, \rho' \in \mathcal{Q}_1$,*

$$(2.36) \quad |\ell_a(\rho) - \ell_a(\rho')| \leq 2|\rho - \rho'|_{L^1},$$

and, if $\rho, \rho' \in \mathcal{Q}_\infty$, then $|b_a(\rho) - b_a(\rho')| \leq 4|\rho - \rho'|_{L^1} + 2|\rho(1) - \rho'(1)|$.

(2) *For $\rho \in \mathcal{Q}_1$, define*

$$(2.37) \quad \zeta_{a,\rho}(u) := 1 - 2 \int_{\rho(u)}^\infty \frac{dr}{d_{\rho,\ell_a(\rho)}(r)^2}, \quad u \in [0, 1).$$

We have $\zeta_{a,\rho} \in \mathcal{Q}_{\infty, \leq 1}$, and $\zeta_{a,\rho} \geq a^2/\ell_a(\rho)^2$. If $\rho \in \mathcal{Q}_\infty$, then

$$(2.38) \quad \zeta_{a,\rho}(u) = \frac{a^2}{\ell_a(\rho)^2} + 2 \int_0^{\rho(u)} \frac{dr}{(b_a(\rho) - 2\mathcal{K}_\rho(r))^2} \quad \text{and} \quad \zeta_{a,\rho}(1) = 1 - \frac{1}{b_a(\rho)}.$$

Moreover, for every $\rho, \rho' \in \mathcal{Q}_1$ and $u \in [0, 1)$,

$$(2.39) \quad |\zeta_{a,\rho}(u) - \zeta_{a,\rho'}(u)| \leq 2|\rho(u) - \rho'(u)| + 8|\rho - \rho'|_{L^1},$$

and therefore, for every $r \in [1, \infty]$ and $\rho, \rho' \in \mathcal{Q}_r$,

$$(2.40) \quad |\zeta_{a,\rho} - \zeta_{a,\rho'}|_{L^r} \leq 10|\rho - \rho'|_{L^r}.$$

The path $\zeta_{a,\rho}$ will be identified in Proposition 2.8 with the derivative of ψ_a° at ρ . Note that $\ell_a(\rho)$ and $\zeta_{a,\rho}$ are defined for every integrable path, while $b_a(\rho)$ involves the endpoint; by (2.36), the map $\rho \mapsto \ell_a(\rho)$ is Lipschitz for the L^1 norm, and this is not the case for b_a .

Proof of Lemma 2.5. Part (1). Set $\delta := |\rho - \rho'|_{L^1}$. Since the positive part is 1-Lipschitz, $|d_{\rho,\ell}(r) - d_{\rho',\ell}(r)| \leq 2\delta$ for every r and ℓ , so that $d_{\rho',\ell+2\delta} \geq d_{\rho,\ell}$ and $S_a(\rho', \ell + 2\delta) \leq S_a(\rho, \ell)$. At $\ell = \ell_a(\rho)$, this gives $S_a(\rho', \ell_a(\rho) + 2\delta) \leq 1 = S_a(\rho', \ell_a(\rho'))$, hence $\ell_a(\rho') \leq \ell_a(\rho) + 2\delta$ by strict monotonicity. Exchanging the roles of ρ and ρ' proves (2.36), and the bound on b_a follows from $|\mathcal{K}(\rho) - \mathcal{K}(\rho')| \leq |\rho(1) - \rho'(1)| + \delta$.

Part (2). Write $\ell := \ell_a(\rho)$ and $d := d_{\rho,\ell}$. Since ρ is nondecreasing and right-continuous, and since d^{-2} is positive and continuous, the path $\zeta_{a,\rho}$ is nondecreasing and right-continuous, and it is at most 1. By the equation $S_a(\rho, \ell) = 1$, we have $\zeta_{a,\rho}(u) \geq 1 - 2 \int_0^\infty d^{-2} = a^2/\ell^2 \geq 0$, so that $\zeta_{a,\rho} \in \mathcal{Q}_{\infty, \leq 1}$. If ρ is bounded, then $1 - 2 \int_{\rho(u)}^\infty d^{-2} = a^2/\ell^2 + 2 \int_0^{\rho(u)} d^{-2}$, which is the first formula in (2.38) by Lemma 2.4, part (2); and letting $u \nearrow 1$ and using the same part again gives $\zeta_{a,\rho}(1) = 1 - 2 \int_{\rho(1)}^\infty d^{-2} = 1 - 1/b_a(\rho)$.

Let now $\rho' \in \mathcal{Q}_1$, $\delta := |\rho - \rho'|_{L^1}$, and $\tilde{d} := d_{\rho',\ell_a(\rho')}$. By (2.33), $d \geq 1$ and $\tilde{d} \geq 1$; by (2.36) and the 1-Lipschitz continuity of the positive part, $\|d - \tilde{d}\|_{L^\infty} \leq 4\delta$; and by the equation $S_a = 1$, we have $2 \int_0^\infty d^{-2} \leq 1$ and $2 \int_0^\infty \tilde{d}^{-2} \leq 1$. For $x, y \geq 1$,

$$|x^{-2} - y^{-2}| = |x - y| \frac{x + y}{x^2 y^2} \leq |x - y| (x^{-2} + y^{-2}),$$

so that $\int_0^\infty |d^{-2} - \tilde{d}^{-2}| dr \leq 4\delta$. Writing

$$\zeta_{a,\rho}(u) - \zeta_{a,\rho'}(u) = 2 \int_{\rho'(u)}^\infty (\tilde{d}(r)^{-2} - d(r)^{-2}) dr - 2 \int_{\rho(u)}^{\rho'(u)} d(r)^{-2} dr$$

and using $d^{-2} \leq 1$ gives (2.39). Taking the L^r norm and using $\delta \leq |\rho - \rho'|_{L^r}$ gives (2.40). $\square$

Linear tilts. The following identity is used here to show that the block is infinite outside the admissible range, and again in Section 3.

Lemma 2.6 (Linear tilts of the cascade). *Let $c \in \mathcal{Q}_\infty$, and let W be a centered Gaussian field on $\mathfrak{U}$ with covariance $\mathbb{E}W(\alpha)W(\alpha') = c(\alpha \wedge \alpha')$. We have*

$$(2.41) \quad \mathbb{E} \log \int \exp(W(\alpha)) \mathfrak{R}(d\alpha) = \frac{1}{2} \mathcal{K}(c) = \frac{1}{2} \left(c(1) - \int_0^1 c(r) dr \right).$$

Proof. For $v \in [0, 1]$, set $G(v) := \mathbb{E} \log \int \exp(\sqrt{v}W(\alpha)) \mathfrak{R}(d\alpha)$, which satisfies $0 \leq G(v) \leq vc(1)/2$ by Jensen's inequality. The field $\sqrt{v}W$ has covariance $vc(\alpha \wedge \alpha')$, which is affine in v . By Gaussian integration by parts in the form of [15, Theorem 4.6], and by the invariance (2.5) of the overlap law under the Lipschitz tilt $x \mapsto \sqrt{v}x$,

$$G'(v) = \frac{1}{2} \mathbb{E} \langle c(\alpha \wedge \alpha) - c(\alpha \wedge \alpha') \rangle_{\mathfrak{R}\sqrt{v}W} = \frac{1}{2} \left(c(1) - \int_0^1 c(r) dr \right),$$

which does not depend on v . Integrating over $[0, 1]$ and using $G(0) = 0$ gives (2.41). For a finite-step path c , the identity can also be read off from the recursion (A.1) of Appendix A, the tilt being affine in the marks at every level. $\square$

Properties of the block. For $\rho \in \mathcal{Q}_\infty$ and admissible b , we set

$$(2.42) \quad \mathcal{X}_a(\rho, b) := \mathcal{D}_a^\circ(\rho, b) - \frac{b-1}{2} + \rho(1) = \frac{a^2}{2(b-2\mathcal{K}(\rho))} + \int_0^{\rho(1)} \frac{dr}{b-2\mathcal{K}_\rho(r)} - \frac{1}{2} \log b,$$

the candidate value of $\mathbb{E} \log Z_{a,b}^\circ(\rho)$ in (2.28).

Lemma 2.7 (Gaussian block with one coordinate). *Let $\rho \in \mathcal{Q}_\infty$ and $a \in \mathbb{R}$.*

- (1) *If $b > 2\mathcal{K}(\rho)$, then $Z_{a,b}^\circ(\rho)$ is almost surely finite, $\log Z_{a,b}^\circ(\rho)$ is integrable, and (2.28) holds, that is, $\mathbb{E} \log Z_{a,b}^\circ(\rho) = \mathcal{X}_a(\rho, b)$.*
- (2) *If $b \leq 2\mathcal{K}(\rho)$, then $\mathbb{E} \log Z_{a,b}^\circ(\rho) = +\infty$.*
- (3) *The function $b \mapsto \mathbb{E} \log Z_{a,b}^\circ(\rho)$ is convex and differentiable on $(2\mathcal{K}(\rho), \infty)$, with $\mathbb{E} \langle \tau^2 \rangle_{\rho,b}^\circ < \infty$ and*

$$(2.43) \quad \partial_b \mathcal{D}_a^\circ(\rho, b) = \frac{1}{2} - \frac{1}{2} \mathbb{E} \langle \tau^2 \rangle_{\rho,b}^\circ.$$

Consequently, the minimizer $b_a(\rho)$ of Lemma 2.4 is the unique admissible b such that

$$(2.44) \quad \mathbb{E} \langle \tau^2 \rangle_{\rho,b}^\circ = 1.$$

Proof. Step 1: Finite-step paths.

Let ρ be as in (2.6), and let $b > 2\mathcal{K}(\rho)$. By (2.22) and (2.9), applied to the nonnegative function $g(x) := x^2/(2b)$ as explained after (2.10),

$$(2.45) \quad \mathbb{E} \log Z_{a,b}^\circ(\rho) = \mathbb{E} \log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp \left(g(\sqrt{2}w^\rho(\alpha) + a) \right) - \frac{1}{2} \log b,$$

as an identity in $(-\infty, +\infty]$. We compute the right-hand side with the recursive formula for averages along Poisson–Dirichlet cascades of [22, Theorem 2.9] or [15, Theorem 5.25], as in [28, Section 3]. Writing $\Delta_j := 2(\rho_j - \rho_{j-1})$ and letting z be a standard Gaussian random variable, define $Y_k := g$ and, for j from k down to 1,

$$(2.46) \quad Y_{j-1}(x) := \frac{1}{m_j} \log \mathbb{E} \exp \left(m_j Y_j \left(x + \Delta_j^{1/2} z \right) \right), \quad Y_{-1} := \mathbb{E} Y_0 \left(a + \Delta_0^{1/2} z \right).$$

The theorem is stated in [15] under the assumption that $\mathbb{E} \exp(m_k g(\sqrt{2}w^\rho(\alpha) + a)) < \infty$, which may fail for the quadratic function g even when b is admissible, and we apply it to the bounded functions $g \wedge L$ instead. As $L \rightarrow \infty$, the left-hand side of the resulting identity increases to $\mathbb{E} \log \sum_\alpha v_\alpha \exp(g(\sqrt{2}w^\rho(\alpha) + a))$, by monotone convergence, since $g \wedge L \geq 0$; and at each level, the function obtained from $g \wedge L$ through (2.46) increases to the function obtained from g , by monotone convergence in each expectation, the recursion being understood with values in $(-\infty, +\infty]$. Hence $\mathbb{E} \log \sum_\alpha v_\alpha \exp(g(\sqrt{2}w^\rho(\alpha) + a)) = Y_{-1}$ in $(-\infty, +\infty]$, and it remains to compute the recursion. If

$Y_j(x) = x^2/(2c_{j+1}) + d_{j+1}$ for some $c_{j+1} > m_j \Delta_j$ and $d_{j+1} \in \mathbb{R}$, then a Gaussian computation gives $Y_{j-1}(x) = x^2/(2c_j) + d_j$ with

$$c_j = c_{j+1} - m_j \Delta_j, \quad d_j = d_{j+1} + \frac{1}{2m_j} (\log c_{j+1} - \log c_j),$$

while $Y_{j-1} \equiv +\infty$ if $c_{j+1} \leq m_j \Delta_j$. Starting from $c_{k+1} = b$ and $d_{k+1} = 0$, the constants c_j are those of (2.29), and since they are nondecreasing in j , they are all positive if and only if $c_1 = b - 2\mathcal{K}(\rho) > 0$. Finally, $Y_{-1} = (a^2 + \Delta_0)/(2c_1) + d_1$, and comparing with (2.30) gives

$$(2.47) \quad Y_{-1} = \frac{a^2}{2(b - 2\mathcal{K}(\rho))} + \int_0^{\rho(1)} \frac{dr}{b - 2\mathcal{K}_\rho(r)}.$$

Together with (2.45), this proves $\mathbb{E} \log Z_{a,b}^\circ(\rho) = \mathcal{X}_a(\rho, b)$ for finite-step paths. In particular, $\mathbb{E} \log Z_{a,b}^\circ(\rho) < \infty$, so that $Z_{a,b}^\circ(\rho)$ is almost surely finite; and $\log Z_{a,b}^\circ(\rho)$ is bounded from below by a deterministic constant, as we now check for general paths, so that it is integrable.

Step 2: Cutoff and a deterministic lower bound.

Let $\rho \in \mathcal{Q}_\infty$ and $b \in \mathbb{R}$. For $L > 0$, let $Z_{a,b,L}^\circ(\rho)$ and $\langle \cdot \rangle_{\rho,b,L}^\circ$ be defined as $Z_{a,b}^\circ(\rho)$ and $\langle \cdot \rangle_{\rho,b}^\circ$ in (2.21), with the integration in τ restricted to $[-L, L]$, and let γ_L be the normalized restriction of γ to $[-L, L]$. By Jensen's inequality under $\gamma_L \otimes \mathfrak{R}$, and since $\int \tau \gamma_L(d\tau) = 0$,

$$(2.48) \quad \log Z_{a,b,L}^\circ(\rho) \geq \log \gamma([-L, L]) - \frac{|b-1|}{2} L^2 =: -C_{b,L}.$$

Since $Z_{a,b,L}^\circ(\rho)$ increases to $Z_{a,b}^\circ(\rho)$ as $L \rightarrow \infty$, and since $\log Z_{a,b,L}^\circ(\rho) \geq -C_{b,1}$ for $L \geq 1$, the expectation $\mathbb{E} \log Z_{a,b}^\circ(\rho)$ is well defined in $(-\infty, +\infty]$, and $\mathbb{E} \log Z_{a,b,L}^\circ(\rho) \rightarrow \mathbb{E} \log Z_{a,b}^\circ(\rho)$ as $L \rightarrow \infty$, by monotone convergence; this holds for every $b \in \mathbb{R}$.

Step 3: Admissible b for general paths.

Let $\rho \in \mathcal{Q}_\infty$ and $b > 2\mathcal{K}(\rho)$. For $\delta > 0$, we discretize ρ by setting

$$\rho^{(\delta)} := \min \{ \rho(1), \delta (\lfloor \rho/\delta \rfloor + 1) \}.$$

The path $\rho^{(\delta)}$ is a finite-step path, since ρ is bounded, nondecreasing, and right-continuous, and it satisfies $\rho \leq \rho^{(\delta)} \leq \rho + \delta$ and $\rho^{(\delta)}(1) = \rho(1)$. Since $\rho^{(\delta)} \geq \rho$ with the same endpoint, $\mathcal{K}(\rho^{(\delta)}) \leq \mathcal{K}(\rho)$, so that every $b' > 2\mathcal{K}(\rho)$ is admissible for $\rho^{(\delta)}$. We compare the block at (ρ, b) with the blocks at $(\rho^{(\delta)}, b \pm 2\delta)$, which Step 1 computes, along an interpolation in which the path and the multiplier vary simultaneously, the multiplier absorbing the effect of the change of path.

For $L > 0$ and $b' \in \mathbb{R}$, the function $g_{L,b'}(x) := \log \int_{-L}^L \exp(x\tau - \frac{b'}{2}\tau^2) \frac{d\tau}{\sqrt{2\pi}}$ is smooth, with $|g'_{L,b'}| \leq L$, and $Z_{a,b',L}^\circ(\rho) = \int \exp(g_{L,b'}(\sqrt{2}w^\rho(\alpha) + a)) \mathfrak{R}(d\alpha)$. Set $\kappa := \rho^{(\delta)} - \rho$, so that $0 \leq \kappa \leq \delta$ and $\kappa(1) = 0$, and, for $s \in [0, 1]$ and $\beta \in \mathbb{R}$, let $\rho_s := \rho + s\kappa$ and $b_s := b + s\beta$; we realize the fields jointly as $w^{\rho_s} = \sqrt{1-s}w^\rho + \sqrt{s}w^{\rho^{(\delta)}}$, with w^ρ and $w^{\rho^{(\delta)}}$ independent. Gaussian integration by parts and the invariance (2.5), as in the proof of [10, Proposition 5.1], together with the fact that the derivative of $\log Z_{a,b',L}^\circ(\rho)$ in b' is $-\frac{1}{2} \langle \tau^2 \rangle_{\rho,b',L}^\circ$, give

$$(2.49) \quad \frac{d}{ds} \mathbb{E} \log Z_{a,b_s,L}^\circ(\rho_s) = -\mathbb{E} \langle \kappa(\alpha \wedge \alpha') \tau \tau' \rangle_{\rho_s,b_s,L}^\circ - \frac{\beta}{2} \mathbb{E} \langle \tau^2 \rangle_{\rho_s,b_s,L}^\circ,$$

where the diagonal term of the integration by parts vanishes because $\kappa(1) = 0$, and where we used that $g'_{L,b_s}(\sqrt{2}w^{\rho_s}(\alpha) + a)$ is the conditional mean of τ given α under $\langle \cdot \rangle_{\rho_s,b_s,L}^\circ$. Since $0 \leq \kappa \leq \delta$ and $|\tau \tau'| \leq \frac{1}{2}(\tau^2 + \tau'^2)$, the first term on the right-hand side of (2.49) is at most $\delta \mathbb{E} \langle \tau^2 \rangle_{\rho_s,b_s,L}^\circ$ in absolute value. The derivative in (2.49) is therefore nonpositive when $\beta = 2\delta$, and nonnegative when $\beta = -2\delta$. Integrating over $s \in [0, 1]$ in both cases, we obtain

$$\mathbb{E} \log Z_{a,b+2\delta,L}^\circ(\rho^{(\delta)}) \leq \mathbb{E} \log Z_{a,b,L}^\circ(\rho) \leq \mathbb{E} \log Z_{a,b-2\delta,L}^\circ(\rho^{(\delta)}),$$

and letting $L \rightarrow \infty$, by the monotone convergence noted in Step 2, gives

$$(2.50) \quad \mathbb{E} \log Z_{a,b+2\delta}^\circ(\rho^{(\delta)}) \leq \mathbb{E} \log Z_{a,b}^\circ(\rho) \leq \mathbb{E} \log Z_{a,b-2\delta}^\circ(\rho^{(\delta)}).$$

For $4\delta < b - 2\mathcal{K}(\rho)$, the multipliers $b \pm 2\delta$ are admissible for $\rho^{(\delta)}$, and Step 1 identifies the outer terms in (2.50) with $\mathcal{X}_a(\rho^{(\delta)}, b \pm 2\delta)$. Since $|\mathcal{K}_{\rho^{(\delta)}}(r) - \mathcal{K}_\rho(r)| \leq \delta$ uniformly in $r \in [0, \rho(1)]$, the

denominators in (2.42) evaluated at $(\rho^{(\delta)}, b \pm 2\delta)$ are bounded from below by $\frac{1}{2}(b - 2\mathcal{K}(\rho))$ once $8\delta \leq b - 2\mathcal{K}(\rho)$, and dominated convergence gives $\mathcal{X}_a(\rho^{(\delta)}, b \pm 2\delta) \rightarrow \mathcal{X}_a(\rho, b)$ as $\delta \rightarrow 0$. Letting $\delta \rightarrow 0$ in (2.50) therefore yields $\mathbb{E} \log Z_{a,b}^\circ(\rho) = \mathcal{X}_a(\rho, b) < \infty$. In particular, $Z_{a,b}^\circ(\rho)$ is almost surely finite, and $\log Z_{a,b}^\circ(\rho) \geq -C_{b,1}$ is integrable. This completes the proof of part (1).

Step 4: Non-admissible b .

Let $b \leq 2\mathcal{K}(\rho)$ and $T > 0$, and let J be the interval $[T, 2T]$ if $a \geq 0$ and $[-2T, -T]$ otherwise, so that $a\tau \geq 0$ on J . The expectation $\mathbb{E} \log Z_{a,b}^\circ(\rho)$ is well defined in $(-\infty, +\infty]$, by Step 2. Restricting the integral in τ to J and applying Jensen's inequality with respect to the normalized Lebesgue measure on J , we obtain

$$\begin{aligned} \mathbb{E} \log Z_{a,b}^\circ(\rho) &\geq \log \frac{T}{\sqrt{2\pi}} + \frac{1}{T} \int_J \left(\mathbb{E} \log \int \exp(\sqrt{2}\tau w^\rho(\alpha)) \mathfrak{R}(d\alpha) + a\tau - \frac{b}{2}\tau^2 \right) d\tau \\ &= \log \frac{T}{\sqrt{2\pi}} + \frac{1}{T} \int_J \left(\left(\mathcal{K}(\rho) - \frac{b}{2} \right) \tau^2 + a\tau \right) d\tau \geq \log \frac{T}{\sqrt{2\pi}}, \end{aligned}$$

where we used Lemma 2.6 with the path $2\tau^2\rho$, for which $\mathcal{K}(2\tau^2\rho) = 2\tau^2\mathcal{K}(\rho)$. Letting $T \rightarrow \infty$ proves part (2).

Step 5: Dependence on b .

Since $Z_{a,b'}^\circ(\rho)$ is nonincreasing in b' , part (1), applied along a sequence of admissible values of b' decreasing to $2\mathcal{K}(\rho)$, shows that almost surely $Z_{a,b'}^\circ(\rho) < \infty$ for every $b' > 2\mathcal{K}(\rho)$. On this event, the function $b' \mapsto \log Z_{a,b'}^\circ(\rho)$ is the logarithm of the Laplace transform, in the variable $b'/2$, of the image of the positive measure $\exp((\sqrt{2}w^\rho(\alpha) + a)\tau + \frac{1}{2}\tau^2)\mathfrak{R}(d\alpha)\gamma(d\tau)$ under $(\tau, \alpha) \mapsto \tau^2$; it is therefore convex and differentiable on the open interval $(2\mathcal{K}(\rho), \infty)$, with derivative $-\frac{1}{2}\langle \tau^2 \rangle_{\rho, b'}^\circ$. Fix $b > 2\mathcal{K}(\rho)$. By convexity, for $\varepsilon > 0$, the difference quotients $\varepsilon^{-1}(\log Z_{a, b+\varepsilon}^\circ(\rho) - \log Z_{a, b}^\circ(\rho))$ decrease to $-\frac{1}{2}\langle \tau^2 \rangle_{\rho, b}^\circ$ as $\varepsilon \searrow 0$, and they are bounded above by the integrable random variable obtained for $\varepsilon = 1$. By monotone convergence, $-\frac{1}{2}\mathbb{E}\langle \tau^2 \rangle_{\rho, b}^\circ$ is the right derivative at b of the convex function $b' \mapsto \mathbb{E} \log Z_{a, b'}^\circ(\rho)$, which is finite on $(2\mathcal{K}(\rho), \infty)$ by part (1); in particular $\mathbb{E}\langle \tau^2 \rangle_{\rho, b}^\circ < \infty$. The same argument with $\varepsilon < 0$ identifies the left derivative with the same quantity. Hence $b' \mapsto \mathbb{E} \log Z_{a, b'}^\circ(\rho)$ is differentiable, with derivative $-\frac{1}{2}\mathbb{E}\langle \tau^2 \rangle_{\rho, b'}^\circ$, and (2.43) follows from (2.28). Finally, by Lemma 2.4, part (2), the derivative $\partial_b \mathcal{D}_a^\circ(\rho, b)$ vanishes exactly at $b = b_a(\rho)$, which is (2.44). $\square$

2.3. Regularity of the initial condition. We now show that ψ_a° is Lipschitz continuous for the L^1 norm, and that $\zeta_{a,\rho}$ is its derivative. Recall from Lemma 2.5 that $\zeta_{a,\rho}$ is defined for every $\rho \in \mathcal{Q}_1$.

Proposition 2.8 (Regularity of the initial condition). *Let $a \in \mathbb{R}$.*

(1) *For every $\rho, \rho' \in \mathcal{Q}_\infty$,*

$$(2.51) \quad |\psi_a^\circ(\rho) - \psi_a^\circ(\rho')| \leq |\rho - \rho'|_{L^1}.$$

Consequently, ψ_a° extends uniquely by continuity to $\mathcal{Q}_1$, and (2.51) holds on $\mathcal{Q}_1$.

(2) *For every $\rho, \rho' \in \mathcal{Q}_2$,*

$$(2.52) \quad \left| \psi_a^\circ(\rho') - \psi_a^\circ(\rho) - \int_0^1 (\rho' - \rho) \zeta_{a,\rho} \right| \leq 5|\rho' - \rho|_{L^2}^2.$$

In particular, ψ_a° is Fréchet differentiable on $\mathcal{Q}_2$, relative to $\mathcal{Q}_2$, with derivative $\zeta_{a,\rho}$ at ρ .

Proof. Step 1: The functional on the half-line.

For $\rho \in \mathcal{Q}_\infty$ and $\ell > 0$, set $\widehat{\mathcal{J}}_a(\rho, \ell) := \mathcal{D}_a^\circ(\rho, \ell + 2\mathcal{K}(\rho))$, so that, by Lemma 2.4,

$$(2.53) \quad \psi_a^\circ(\rho) = -\inf_{\ell > 0} \widehat{\mathcal{J}}_a(\rho, \ell) = -\widehat{\mathcal{J}}_a(\rho, \ell_a(\rho)).$$

Let $R > \rho(1)$, and write $b := \ell + 2\mathcal{K}(\rho)$. By part (2) of Lemma 2.4, $d_{\rho, \ell}(r) = b + 2(r - \rho(1))$ for $r \geq \rho(1)$, so that $\int_{\rho(1)}^R d_{\rho, \ell}(r)^{-1} dr = \frac{1}{2} \log d_{\rho, \ell}(R) - \frac{1}{2} \log b$; since moreover $\frac{1}{2}(b - 1) - \rho(1) =$

$\frac{1}{2}(\ell - 1) - \int_0^1 \rho$, the definition (2.24) and the same part of Lemma 2.4 give

$$(2.54) \quad \widehat{J}_a(\rho, \ell) = \frac{a^2}{2\ell} + \int_0^R \frac{dr}{d_{\rho, \ell}(r)} - \frac{1}{2} \log d_{\rho, \ell}(R) + \frac{\ell - 1}{2} - \int_0^1 \rho(u) du.$$

The right-hand side does not depend on the choice of $R > \rho(1)$, and the endpoint $\rho(1)$ no longer appears in it explicitly: the path enters only through $d_{\rho, \ell}$ and through its integral.

Step 2: Derivative along a segment, at fixed ℓ .

Let $\rho, \rho' \in \mathcal{Q}_\infty$, $\kappa := \rho' - \rho$, and $\rho_s := \rho + s\kappa$ for $s \in [0, 1]$, which belongs to $\mathcal{Q}_\infty$ with $\rho_s(1) = (1-s)\rho(1) + s\rho'(1)$. We fix $R > \max\{\rho(1), \rho'(1)\}$, which exceeds $\rho_s(1)$ for every s , and $\ell > 0$, and we show that $s \mapsto \widehat{J}_a(\rho_s, \ell)$ is differentiable on $[0, 1]$, with

$$(2.55) \quad \frac{d}{ds} \widehat{J}_a(\rho_s, \ell) = - \int_0^1 \kappa(u) \left(1 - 2 \int_{\rho_s(u)}^\infty \frac{dr}{d_{\rho_s, \ell}(r)^2} \right) du,$$

and that the right-hand side of (2.55) is a continuous function of $(s, \ell) \in [0, 1] \times (0, \infty)$. Since the positive part is 1-Lipschitz, we have $|d_{\rho_{s'}, \ell}(r) - d_{\rho_s, \ell}(r)| \leq 2|s' - s| |\kappa|_{L^1}$ for every $r \geq 0$ and $s, s' \in [0, 1]$, and $d_{\rho_s, \ell} \geq \ell$; the difference quotients of $s \mapsto d_{\rho_s, \ell}(r)^{-1}$ are therefore bounded by $2|\kappa|_{L^1} \ell^{-2}$, uniformly in r . Fix $s \in [0, 1]$. For every $r \geq 0$ and every u such that $\rho_s(u) \neq r$, the function $s' \mapsto (r - \rho_{s'}(u))_+$ is differentiable at $s' = s$, with derivative $-\kappa(u) \mathbb{1}_{\{r > \rho_s(u)\}}$. Since the sets $\{u : \rho_s(u) = r\}$, $r \geq 0$, are disjoint, at most countably many of them have positive measure, and for every other r , dominated convergence gives

$$\frac{d}{ds} d_{\rho_s, \ell}(r) = -2 \int_0^1 \kappa(u) \mathbb{1}_{\{r > \rho_s(u)\}} du.$$

By dominated convergence again, now in the variable $r \in [0, R]$, and by Fubini's theorem, using that $\rho_s(u) < R$ for every u ,

$$\frac{d}{ds} \int_0^R \frac{dr}{d_{\rho_s, \ell}(r)} = \int_0^R \frac{2 \int_0^1 \kappa(u) \mathbb{1}_{\{r > \rho_s(u)\}} du}{d_{\rho_s, \ell}(r)^2} dr = 2 \int_0^1 \kappa(u) \int_{\rho_s(u)}^R \frac{dr}{d_{\rho_s, \ell}(r)^2} du.$$

For the logarithmic term, since $R \geq \rho_s(u)$ for every u , we have $d_{\rho_s, \ell}(R) = \ell + 2R - 2 \int_0^1 \rho_s$, which is affine in s with derivative $-2 \int_0^1 \kappa$; and since $d_{\rho_s, \ell}(r) = d_{\rho_s, \ell}(R) + 2(r - R)$ for $r \geq R$, we have $d_{\rho_s, \ell}(R)^{-1} = 2 \int_R^\infty d_{\rho_s, \ell}(r)^{-2} dr$. Hence

$$\frac{d}{ds} \left(-\frac{1}{2} \log d_{\rho_s, \ell}(R) \right) = \frac{\int_0^1 \kappa}{d_{\rho_s, \ell}(R)} = 2 \int_0^1 \kappa(u) \int_R^\infty \frac{dr}{d_{\rho_s, \ell}(r)^2} du.$$

Adding these two contributions to the derivative $-\int_0^1 \kappa$ of the last term in (2.54) gives (2.55). For the joint continuity, let $(s', \ell') \rightarrow (s, \ell)$. Then $\rho_{s'}(u) \rightarrow \rho_s(u)$ for every u , and $d_{\rho_{s'}, \ell'} \rightarrow d_{\rho_s, \ell}$ uniformly on $[0, \infty)$, while, by (2.32), for $\ell' \geq \ell/2$ the functions $d_{\rho_{s'}, \ell'}^{-2}$ are bounded by $(\ell/2 + 2(r - \bar{\rho} \vee \bar{\rho}')_+)^{-2}$, which is integrable on $[0, \infty)$. By dominated convergence, the inner integrals in (2.55) converge for every u , and they are bounded uniformly in u ; since κ is bounded, the right-hand side of (2.55) converges as well.

Step 3: The envelope argument.

With the notation of Step 2, set $\ell_s := \ell_a(\rho_s)$ and $V(s) := \widehat{J}_a(\rho_s, \ell_s) = \inf_{\ell > 0} \widehat{J}_a(\rho_s, \ell)$, so that $\psi_a^\circ(\rho_s) = -V(s)$ by (2.53). By (2.36), $|\ell_{s'} - \ell_s| \leq 2|s' - s| |\kappa|_{L^1}$, so $s \mapsto \ell_s$ is continuous, and $\ell_s \geq 1$ by (2.33). For $s, s' \in [0, 1]$, the minimality of ℓ_s and of $\ell_{s'}$ gives

$$(2.56) \quad \widehat{J}_a(\rho_{s'}, \ell_{s'}) - \widehat{J}_a(\rho_s, \ell_{s'}) \leq V(s') - V(s) \leq \widehat{J}_a(\rho_{s'}, \ell_s) - \widehat{J}_a(\rho_s, \ell_s).$$

By Step 2 and the mean value theorem, the left-hand side of (2.56) is $(s' - s)$ times the right-hand side of (2.55) evaluated at $\ell = \ell_{s'}$ and at some point between s and s' , and similarly for the right-hand side of (2.56), with $\ell = \ell_s$. Dividing by $s' - s$ and letting $s' \rightarrow s$, the continuity of $s \mapsto \ell_s$ and the joint continuity established in Step 2 show that both bounds converge to the right-hand side of (2.55) at $\ell = \ell_s$, which is $-\int_0^1 \kappa \zeta_{a, \rho_s}$ by (2.37). Hence V is differentiable on $[0, 1]$, and

$$(2.57) \quad \frac{d}{ds} \psi_a^\circ(\rho_s) = \int_0^1 \kappa(u) \zeta_{a, \rho_s}(u) du.$$

This is the envelope theorem: the minimizer does not need to be differentiated. By (2.40) with $r = 1$, the right-hand side of (2.57) is a continuous function of s , so that (2.57) can be integrated over $[0, 1]$.

Step 4: Conclusion.

Since $0 \leq \zeta_{a,\rho_s} \leq 1$, integrating (2.57) gives $|\psi_a^\circ(\rho') - \psi_a^\circ(\rho)| \leq |\kappa|_{L^1}$, which is (2.51) on $\mathcal{Q}_\infty$; and by (2.40) with $r = 2$,

$$\begin{aligned} \left| \psi_a^\circ(\rho') - \psi_a^\circ(\rho) - \int_0^1 \kappa \zeta_{a,\rho} \right| &= \left| \int_0^1 \int_0^1 \kappa(u) (\zeta_{a,\rho_s}(u) - \zeta_{a,\rho}(u)) du ds \right| \\ &\leq \int_0^1 |\kappa|_{L^2} 10s |\kappa|_{L^2} ds = 5|\kappa|_{L^2}^2, \end{aligned}$$

which is (2.52) on $\mathcal{Q}_\infty$. Since $\mathcal{Q}_\infty$ is dense in $\mathcal{Q}_1$ for the L^1 norm, (2.51) shows that ψ_a° extends uniquely by continuity to $\mathcal{Q}_1$, and the extension satisfies the same estimate. For $\rho, \rho' \in \mathcal{Q}_2$, the truncations $\rho \wedge m$ and $\rho' \wedge m$ belong to $\mathcal{Q}_\infty$ and converge to ρ and ρ' in L^1 and in L^2 as $m \rightarrow \infty$, so that $\psi_a^\circ(\rho \wedge m) \rightarrow \psi_a^\circ(\rho)$, and $\zeta_{a,\rho \wedge m} \rightarrow \zeta_{a,\rho}$ in L^2 by (2.40); passing to the limit in (2.52) proves it on $\mathcal{Q}_2$. Finally, (2.52) says that the remainder is $O(|\rho' - \rho|_{L^2}^2)$ as $\rho' \rightarrow \rho$ in $\mathcal{Q}_2$, which implies the Fréchet differentiability relative to $\mathcal{Q}_2$. $\square$

Before turning to the species-weighted statements, we record the convexity of ψ_a° , which is the property that allows arbitrary species-dependent external fields, and which is the input of the Hopf formula of Section 5. Its proof is a short calculation on finite-step paths: at fixed $\ell = b - 2\mathcal{K}(\rho)$, the finite-step expression of $\mathcal{D}_a^\circ$ is concave in the increments of the path, because its logarithmic terms have nonnegative coefficients, and ψ_a° is a supremum of convex functions.

Lemma 2.9 (Convexity of the initial condition). *For every $q_0, q_1 \in \mathcal{Q}_1^\mathcal{S}$ and every $c \in [0, 1]$, we have $\psi((1-c)q_0 + cq_1) \leq (1-c)\psi(q_0) + c\psi(q_1)$. The same holds for ψ_λ , for every probability vector λ with positive entries.*

Proof. Since ψ_λ is a positive linear combination of the functions $\psi_{h^s}^\circ$, it suffices to prove that ψ_a° is convex on $\mathcal{Q}_1$, for every $a \in \mathbb{R}$. By the Lipschitz continuity (2.51) and the density of finite-step paths, it suffices to prove the convexity inequality for two finite-step paths on a common partition $0 = m_0 < m_1 < \dots < m_k < m_{k+1} = 1$. Along the segment between two such paths, the partition stays fixed and the values $\rho_0 \leq \dots \leq \rho_k$ move affinely; we parametrize them by the increments $t_i := \rho_i - \rho_{i-1} \geq 0$, $0 \leq i \leq k$, which depend affinely on the path. It is therefore enough to prove that $\psi_a^\circ(\rho)$ is a convex function of $t = (t_0, \dots, t_k) \in \mathbb{R}_+^{k+1}$.

We use the explicit form of $\mathcal{D}_a^\circ$ given in Remark 2.3, and the variable $\ell = b - 2\mathcal{K}(\rho) = c_1$ of Lemma 2.4, which ranges over $(0, \infty)$. In terms of ℓ and t , we have $c_{i+1} = \ell + 2 \sum_{j=1}^i m_j t_j$ for $0 \leq i \leq k$, $\rho_0 = t_0$, $\rho_k = \sum_{i=0}^k t_i$, and, by (2.30),

$$\mathcal{D}_a^\circ(\rho, b) = \frac{a^2}{2\ell} + \frac{t_0}{\ell} + \sum_{i=1}^k \frac{\log c_{i+1} - \log c_i}{2m_i} + \frac{1}{2} (c_{k+1} - 1 - \log c_{k+1}) - \sum_{i=0}^k t_i =: J_\ell(t).$$

When $k = 0$, that is, when ρ is constant, there is no logarithmic term and J_ℓ is an affine function of t_0 . For $k \geq 1$ and fixed ℓ , we collect the logarithmic terms: since $c_1 = \ell$ does not depend on t ,

$$J_\ell(t) = \sum_{j=2}^k \left(\frac{1}{2m_{j-1}} - \frac{1}{2m_j} \right) \log c_j + \left(\frac{1}{2m_k} - \frac{1}{2} \right) \log c_{k+1} + (\text{affine function of } t).$$

The coefficients in front of the logarithms are nonnegative since $m_1 < \dots < m_k < 1$, and each c_j is a positive affine function of t . Hence, in all cases, J_ℓ is concave in t , and $\psi_a^\circ(\rho) = \sup_{\ell > 0} (-J_\ell(t))$ is a supremum of convex functions of t , hence convex. $\square$

We now record the species-weighted statements. For a probability vector λ with positive entries and $\pi \in \mathcal{Q}_1^\mathcal{S}$, we define the path

$$(2.58) \quad \partial_q \psi_\lambda(\pi) := (\lambda_s \zeta_{h^s, \pi^s})_{s \in \mathcal{S}} \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S},$$

where the inclusion follows from $\zeta_{h^s, \pi^s} \in \mathcal{Q}_{\infty, \leq 1}$, and we set

$$(2.59) \quad \partial_q \psi(\pi) := \partial_q \psi_{\lambda_\infty}(\pi) \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^{\mathcal{S}}, \quad \pi \in \mathcal{Q}_1^{\mathcal{S}}.$$

Corollary 2.10 (Multi-species initial condition). *Let λ, λ' be probability vectors with positive entries. For every $\pi, \pi' \in \mathcal{Q}_1^{\mathcal{S}}$,*

$$(2.60) \quad |\psi_\lambda(\pi) - \psi_\lambda(\pi')| \leq |\pi - \pi'|_{L^1}$$

and

$$(2.61) \quad |\partial_q \psi_{\lambda'}(\pi') - \partial_q \psi_\lambda(\pi)|_{L^1} \leq \sum_{s \in \mathcal{S}} |\lambda'_s - \lambda_s| + 10|\pi' - \pi|_{L^1}.$$

For every $\pi, \pi' \in \mathcal{Q}_2^{\mathcal{S}}$,

$$(2.62) \quad |\psi_\lambda(\pi') - \psi_\lambda(\pi) - \langle \pi' - \pi, \partial_q \psi_\lambda(\pi) \rangle_{L^2}| \leq 5|\pi' - \pi|_{L^2}^2.$$

Proof. Since $\mathbb{R}^{\mathcal{S}}$ carries the Euclidean norm, we have $\sum_s \lambda_s |x_s| \leq |x|$ and $|x| \leq \sum_s |x_s|$ for $x \in \mathbb{R}^{\mathcal{S}}$. The bound (2.60) follows from (2.51), since $|\psi_\lambda(\pi) - \psi_\lambda(\pi')| \leq \sum_s \lambda_s |\pi^s - \pi'^s|_{L^1} \leq |\pi - \pi'|_{L^1}$. For (2.61), we write $\lambda'_s \zeta_{h^s, \pi'^s} - \lambda_s \zeta_{h^s, \pi^s} = (\lambda'_s - \lambda_s) \zeta_{h^s, \pi'^s} + \lambda_s (\zeta_{h^s, \pi'^s} - \zeta_{h^s, \pi^s})$, use $0 \leq \zeta \leq 1$ and (2.40), and sum over s . The remainder estimate (2.62) follows from (2.52) applied to each species, since $\sum_s \lambda_s |\pi'^s - \pi^s|_{L^2}^2 \leq |\pi' - \pi|_{L^2}^2$. $\square$

2.4. The spherical initial condition. We now prove that ψ is the limit of $\bar{F}_N(0, \cdot)$. For $n \in \mathbb{N}$ and $v \in \mathbb{R}^n$, we set

$$(2.63) \quad \Lambda_n(v) := \int_{\mathbb{S}_n} e^{v \cdot \sigma} \mu_n(d\sigma),$$

which depends only on $|v|$ by rotation invariance. By Jensen's inequality, we have $\Lambda_n \geq 1$. Moreover, for every v , the function $r \mapsto \log \Lambda_n(rv)$ is convex on $\mathbb{R}$, vanishes at $r = 0$, and is nonnegative; it is therefore nondecreasing on $[0, \infty)$. For one species with n coordinates, path $\rho \in \mathcal{Q}_\infty$, and external field $a \in \mathbb{R}$, the free energy (2.15) at $t = 0$ is

$$(2.64) \quad \bar{F}_{n,a}^\circ(\rho) := -\frac{1}{n} \mathbb{E} \log \int_{\mathbb{U}} \Lambda_n(\sqrt{2} w_{[n]}^\rho(\alpha) + a \vec{1}) \Re(d\alpha) + \rho(1),$$

where $w_{[n]}^\rho := (w_1^\rho, \dots, w_n^\rho)$ is a vector of independent copies of w^ρ and $\vec{1} := (1, \dots, 1)$. Proposition 2.1 gives $|\bar{F}_{n,a}^\circ(\rho) - \bar{F}_{n,a}^\circ(\rho')| \leq |\rho - \rho'|_{L^1}$; and since $|\sigma \cdot \vec{1}| \leq n$ on $\mathbb{S}_n$, we see that $|\partial_a \bar{F}_{n,a}^\circ(\rho)| \leq 1$.

Proposition 2.11 (Initial condition). *For every $q \in \mathcal{Q}_1^{\mathcal{S}}$, we have*

$$(2.65) \quad \lim_{N \rightarrow \infty} \bar{F}_N(0, q) = \psi(q).$$

For one species and a finite-step path, the identity (2.67) below is Talagrand's computation of the free energy at the initial point of Guerra's interpolation, [28, Proposition 3.1]; the multi-species statement, in the form of the convergence of the cavity functional as the number of cavity coordinates tends to infinity, is [4, Proposition 2.10]. For the reader's convenience, we include a proof that largely follows Talagrand's Gaussian comparison argument, except that for one bound, we use the tilted-cascade identities of Appendix A to replace his successive large-deviation estimates by a law of large numbers for the Gaussian radius at the optimizing multiplier.

Proof of Proposition 2.11. Since $\bar{F}_N(0, \cdot)$ and ψ are both 1-Lipschitz for the L^1 norm, by (2.17) and (2.60), and since finite-step paths are dense in $\mathcal{Q}_1^{\mathcal{S}}$, it suffices to prove (2.65) when every q^s is a finite-step path, which we assume from now on.

Step 1: Reduction to one species.

Since P_N is a product measure over the species, and since the field $\sqrt{2}W_N^q + \sum_s h^s \sum_{i \in I_{N,s}} \sigma_i$ is linear in σ , we have, for every α ,

$$\int \exp \left(\sqrt{2}W_N^q(\sigma, \alpha) + \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} \sigma_i \right) P_N(d\sigma) = \prod_{s \in \mathcal{S}} \Lambda_{N_s} \left(\sqrt{2}w_{I_{N,s}}^q(\alpha) + h^s \vec{1} \right),$$

where $N_s := |I_{N,s}|$ and $w_{I_{N,s}}^{q^s} := (w_i^{q^s})_{i \in I_{N,s}}$. The logarithm of the right-hand side is a Lipschitz function of the Gaussian fields, since $|\nabla \log \Lambda_n| \leq \sqrt{n}$. By (2.9), applied with a common partition for the paths q^s , the expectation of the logarithm of its integral against $\mathfrak{R}$ can be computed with the discrete cascade; the tilt is then a sum over the species of functions of the marks of the coordinates of that species, and by the additivity of the recursive formula over independent groups of marks, Lemma A.4, part (1), it equals the sum over s of the corresponding quantities for a single species, that is,

$$(2.66) \quad \bar{F}_N(0, q) = \sum_{s \in \mathcal{S}} \lambda_{N,s} \bar{F}_{N_s, h^s}^\circ(q^s).$$

Since $\lambda_{N,s} \rightarrow \lambda_{\infty,s}$, it suffices to prove that, for every finite-step $\rho \in \mathcal{Q}_\infty$ and $a \in \mathbb{R}$, we have

$$(2.67) \quad \lim_{n \rightarrow \infty} \bar{F}_{n,a}^\circ(\rho) = \psi_a^\circ(\rho).$$

Step 2: The Gaussian model.

Let ρ be a finite-step path, written as in (2.6), and let $b > 2\mathcal{K}(\rho)$. We use the discrete cascade with n marks per node, and we let $\mathbf{v}_i(\alpha) := \sqrt{2}w_i^\rho(\alpha) + a$, $1 \leq i \leq n$, where the w_i^ρ are the discrete fields (2.8) built from the marks of coordinate i . We write $\mathbf{v}(\alpha) := (\mathbf{v}_i(\alpha))_{i \leq n}$, so that, by (2.9), $\bar{F}_{n,a}^\circ(\rho) = -\frac{1}{n} \mathbb{E} \log A_n + \rho(1)$ with $A_n := \sum_\alpha v_\alpha \Lambda_n(\mathbf{v}(\alpha))$. We introduce the Gaussian model

$$(2.68) \quad Z_b := \sum_{\alpha \in \mathbb{N}^k} v_\alpha \int_{\mathbb{R}^n} \exp\left(\mathbf{v}(\alpha) \cdot x - \frac{b}{2}|x|^2\right) \frac{dx}{(2\pi)^{n/2}},$$

with Gibbs bracket $\langle \cdot \rangle_b$ on $\mathbb{R}^n \times \mathbb{N}^k$. Conditionally on α , the coordinates x_i are independent Gaussian random variables with mean $\mathbf{v}_i(\alpha)/b$ and variance $1/b$ under $\langle \cdot \rangle_b$, and integrating over x gives $Z_b = \sum_\alpha v_\alpha \prod_{i \leq n} b^{-1/2} e^{\mathbf{v}_i(\alpha)^2/(2b)}$. In the notation of Appendix A, this is a discrete cascade tilted by $X(\Omega_\alpha) := \sum_{i \leq n} g(\mathbf{v}_i(\alpha))$ with $g(x) = x^2/(2b)$, a sum of identical functions of the marks of the n coordinates. We collect the three facts about this tilted cascade that the proof uses.

First, by the additivity of the recursion, Lemma A.4, part (1), and by Step 1 of the proof of Lemma 2.7, in which the one-coordinate recursion was computed, we have

$$(2.69) \quad \frac{1}{n} \mathbb{E} \log Z_b = \mathbb{E} \log Z_{a,b}^\circ(\rho) = \mathcal{X}_a(\rho, b) = \mathcal{D}_a(\rho, b) - \frac{b-1}{2} + \rho(1).$$

Second, by Lemma A.4, part (2), the marks $(\Omega_{\alpha,i})_{i \leq n}$ of the n coordinates along the sampled leaf are independent and identically distributed under $\mathbb{E} \langle \cdot \rangle_b$, with a law that does not depend on n ; since, conditionally on α and on the marks, the coordinate x_i is Gaussian with a law that only depends on $\mathbf{v}_i(\alpha)$, the pairs $(\mathbf{v}_i(\alpha), x_i)_{i \leq n}$ are independent and identically distributed under $\mathbb{E} \langle \cdot \rangle_b$ as well, with a law ν that does not depend on n . Under ν , all moments are finite, since the tilted path law in (A.2) is Gaussian, the tilts X_j of the recursion being quadratic. Moreover, $b \mapsto \log Z_b$ is almost surely convex, with derivative $-\frac{1}{2} \langle |x|^2 \rangle_b$, so that the monotone convergence argument of Step 5 of the proof of Lemma 2.7 shows that $b \mapsto \frac{1}{n} \mathbb{E} \log Z_b$ is differentiable with derivative $-\frac{1}{2n} \mathbb{E} \langle |x|^2 \rangle_b = -\frac{1}{2} \mathbb{E} \langle x_1^2 \rangle_b$, by exchangeability. Comparing with (2.69) and (2.43), we get $\mathbb{E} \langle x_1^2 \rangle_b = \mathbb{E} \langle \tau^2 \rangle_{\rho,b}^\circ$, which equals 1 when $b = b_a(\rho)$, by (2.44). For this value of b , the independence gives the law of large numbers

$$(2.70) \quad \mathbb{E} \left\langle \left(\frac{1}{n} |x|^2 - 1 \right)^2 \right\rangle_{b_a(\rho)} = \frac{1}{n} \text{Var}_\nu(x_1^2) =: \frac{C_0}{n},$$

where $C_0 < \infty$ does not depend on n .

Third, by the variance bound of Lemma A.4, part (3), applied to the same tilt, we have $\text{Var}(\log Z_b) \leq Cn$; together with $|\mathbb{E} \log Z_b| = n|\mathcal{X}_a(\rho, b)|$, this gives

$$(2.71) \quad \mathbb{E}[(\log Z_b)^2] \leq Cn^2,$$

for a constant C depending only on ρ, a, b .

We now write $x \in \mathbb{R}^n$ in polar coordinates, $x = r\sigma$ with $r := |x|/\sqrt{n} \in [0, \infty)$ and $\sigma := \sqrt{n}x/|x| \in \mathbb{S}_n$. The Lebesgue measure decomposes as $(2\pi)^{-n/2}dx = \kappa_n r^{n-1}dr \mu_n(d\sigma)$, with

$$(2.72) \quad \kappa_n := \frac{2(n/2)^{n/2}}{\Gamma(n/2)}, \quad \text{so that} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \kappa_n = \frac{1}{2}$$

by Stirling's formula. Since $v(\alpha) \cdot x = r v(\alpha) \cdot \sigma$ and $|x|^2 = nr^2$, we obtain

$$(2.73) \quad Z_b = \int_0^\infty B(r) e^{-nbr^2/2} \kappa_n r^{n-1} dr, \quad B(r) := \sum_\alpha v_\alpha \Lambda_n(rv(\alpha)).$$

The function B is nondecreasing, by the monotonicity of $r \mapsto \Lambda_n(rv)$ noted above, and $B(1) = A_n$. Under $\langle \cdot \rangle_b$, the radial variable r has density $B(r)e^{-nbr^2/2} \kappa_n r^{n-1} / Z_b$. The identity (2.73) expresses the Gaussian model as a mixture of spherical models with radius $r\sqrt{n}$; the multiplier $b_a(\rho)$ is the one for which this mixture concentrates near $r = 1$.

Step 3: Lower bound on $\bar{F}_{n,a}^\circ(\rho)$.

Let $\delta > 0$. Restricting the integral in (2.73) to $[1, 1+\delta]$, and using $B(r) \geq B(1) = A_n$, $r^{n-1} \geq 1$, and $b > 0$ there, we obtain

$$Z_b \geq A_n \delta \kappa_n \exp\left(-\frac{nb(1+\delta)^2}{2}\right).$$

Taking logarithms and expectations, dividing by n , and using (2.69) and (2.72), we get

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{n} \mathbb{E} \log A_n - \rho(1) \right) \leq \mathcal{X}_a(\rho, b) - \rho(1) + \frac{b(1+\delta)^2}{2} - \frac{1}{2} = \mathcal{D}_a^\circ(\rho, b) + \frac{b}{2}((1+\delta)^2 - 1).$$

Letting $\delta \rightarrow 0$ and choosing $b = b_a(\rho)$, for which $\mathcal{D}_a^\circ(\rho, b) = -\psi_a^\circ(\rho)$, we obtain

$$(2.74) \quad \liminf_{n \rightarrow \infty} \bar{F}_{n,a}^\circ(\rho) \geq \psi_a^\circ(\rho).$$

Step 4: Upper bound on $\bar{F}_{n,a}^\circ(\rho)$.

We now fix $b := b_a(\rho)$ and $\delta \in (0, \frac{1}{2})$. Since $|r-1| \leq |r^2-1|$ for $r \geq 0$, Chebyshev's inequality gives $\langle \mathbb{1}_{\{|r-1|>\delta\}} \rangle_b \leq \delta^{-2} \langle (r^2-1)^2 \rangle_b$. Let G be the event $\{\langle (r^2-1)^2 \rangle_b \leq \delta^2/2\}$, whose complement has probability at most $2C_0/(n\delta^2)$ by (2.70) and Markov's inequality. On G , the radial variable lies in $[1-\delta, 1+\delta]$ with $\langle \cdot \rangle_b$ -probability at least $\frac{1}{2}$, so that, by (2.73) and the monotonicity of B ,

$$\frac{Z_b}{2} \leq \int_{1-\delta}^{1+\delta} B(r) e^{-nbr^2/2} \kappa_n r^{n-1} dr \leq 2\delta B(1+\delta) \kappa_n (1+\delta)^n e^{-nb(1-\delta)^2/2}.$$

Hence, on G ,

$$(2.75) \quad \log B(1+\delta) \geq \log Z_b + D_n, \quad D_n := \frac{nb(1-\delta)^2}{2} - \log \kappa_n - n \log(1+\delta) - \log(4\delta),$$

where D_n is deterministic with $|D_n| \leq C_\delta n$. Outside G , we only use $\log B(1+\delta) \geq 0$, which holds since $\Lambda_n \geq 1$. Multiplying (2.75) by $\mathbb{1}_G$ and taking expectations, we get

$$\mathbb{E} \log B(1+\delta) \geq \mathbb{E} \log Z_b - \mathbb{E}[\log Z_b \mathbb{1}_{G^c}] - |D_n| \mathbb{P}(G^c) + D_n,$$

and by the Cauchy-Schwarz inequality and (2.71),

$$\mathbb{E}[\log Z_b \mathbb{1}_{G^c}] \leq (\mathbb{E}[(\log Z_b)^2])^{1/2} \mathbb{P}(G^c)^{1/2} \leq C_\delta \sqrt{n}.$$

Therefore,

$$(2.76) \quad \mathbb{E} \log B(1+\delta) \geq \mathbb{E} \log Z_b + \frac{nb(1-\delta)^2}{2} - \log \kappa_n - n \log(1+\delta) - C_\delta(\sqrt{n}+1),$$

where C_δ depends on δ and on ρ, a , but not on n . Finally, we relate $B(1+\delta)$ to the free energy. Since $(1+\delta)v(\alpha) = \sqrt{2}(1+\delta)w^\rho(\alpha) + (1+\delta)a$ and $(1+\delta)w^\rho$ is the discrete field (2.8) associated with the path $\rho_\delta := (1+\delta)^2\rho$, we have $\mathbb{E} \log B(1+\delta) = n\rho_\delta(1) - n\bar{F}_{n,a_\delta}^\circ(\rho_\delta)$ with $a_\delta := (1+\delta)a$, by (2.9), and the Lipschitz bounds recorded below (2.64) give

$$|\bar{F}_{n,a_\delta}^\circ(\rho_\delta) - \bar{F}_{n,a}^\circ(\rho)| \leq (2\delta + \delta^2)|\rho|_{L^1} + \delta|a|.$$

Dividing (2.76) by n , inserting (2.69) and (2.72), and letting $n \rightarrow \infty$, we arrive at

$$\liminf_{n \rightarrow \infty} \left(-\bar{F}_{n,a}^\circ(\rho) \right) \geq \mathcal{D}_a^\circ(\rho, b) - \frac{b}{2} (1 - (1 - \delta)^2) - \log(1 + \delta) - ((1 + \delta)^2 - 1) \rho(1) - (2\delta + \delta^2) |\rho|_{L^1} - \delta |a|.$$

The right-hand side is $\mathcal{D}_a^\circ(\rho, b) + O(\delta)$ as $\delta \rightarrow 0$, and $\mathcal{D}_a^\circ(\rho, b_a(\rho)) = -\psi_a^\circ(\rho)$. Hence, we obtain that $\limsup_n \bar{F}_{n,a}^\circ(\rho) \leq \psi_a^\circ(\rho)$, which together with (2.74) proves (2.67), and completes the proof. $\square$

2.5. Critical paths and the inner infimum. Let $p \in \mathcal{Q}_\infty^\mathcal{S}$. Since the partial derivatives of ξ are nonnegative and nondecreasing in each coordinate on $\mathbb{R}_+^\mathcal{S}$, by the nonnegativity of the coefficients in (1.4), the path $\nabla \xi(p) := (\partial_s \xi(p(\cdot)))_{s \in \mathcal{S}}$ belongs to $\mathcal{Q}_\infty^\mathcal{S}$, with endpoint $\nabla \xi(p(1))$. Consequently, for every $t \geq 0$ and $r \in [1, \infty]$,

$$(2.77) \quad q \in \mathcal{Q}_r^\mathcal{S} \implies q + t \nabla \xi(p) \in \mathcal{Q}_r^\mathcal{S}.$$

For every $a \in \mathbb{R}^\mathcal{S}$, we set

$$(2.78) \quad \theta(a) := a \cdot \nabla \xi(a) - \xi(a),$$

and we define the Parisi functional

$$(2.79) \quad \mathcal{P}_{t,q}(p) := \psi(q + t \nabla \xi(p)) - t \int_0^1 \theta(p(r)) dr, \quad t \geq 0, \quad q \in \mathcal{Q}_1^\mathcal{S}, \quad p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}.$$

Comparing with (1.10), we see that $\mathcal{P}_{t,q}(p) = \mathcal{J}_{t,q}(q + t \nabla \xi(p), p)$.

The derivative of $\mathcal{J}_{t,q}(q', p)$ in p is $q - q' + t \nabla \xi(p)$, and its derivative in q' is $\partial_q \psi(q') - p$. Formally, the pair (q', p) is thus a critical point of $\mathcal{J}_{t,q}$ if and only if $q' = q + t \nabla \xi(p)$ and $p = \partial_q \psi(q + t \nabla \xi(p))$. We call the latter identity the *critical relation*, and a path p satisfying it a *critical path*. We now state the precise upper bound on the free energy proved in the next two sections.

Proposition 2.12 (Upper bound at a critical path). *Assume that $\lambda_\infty \in \mathbb{Q}^\mathcal{S}$, let $M \in \mathbb{N}$ be such that $M_s := M \lambda_{\infty, s} \in \mathbb{N}$ for every $s \in \mathcal{S}$, and consider the sequence of sizes*

$$(2.80) \quad N_n := nM, \quad |I_{N_n, s}| = nM_s, \quad n \in \mathbb{N},$$

along which $\lambda_{N_n} = \lambda_\infty$. For every $t > 0$ and $q \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$, there exists $p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$ such that

$$(2.81) \quad p = \partial_q \psi(q + t \nabla \xi(p)),$$

$$(2.82) \quad \limsup_{n \rightarrow \infty} \bar{F}_{N_n}(t, q) \leq \mathcal{P}_{t,q}(p).$$

The integer M is the size of the cavity block of Section 3, which adds M_s coordinates to species s , so that the passage from N_n to N_{n+1} in (2.80) adds one block. The class $\mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$ of regular paths, which are bounded, vanish at the origin, and have slope bounded from below, was introduced in Subsection 2.1; the regularity is used through the semi-concavity estimate of Proposition 2.2 when matching derivatives. The restrictions to rational proportions and to regular paths are removed in Section 6. The next lemma explains why Proposition 2.12 is enough to obtain the desired upper bound on the free energy.

Lemma 2.13 (The inner infimum at a critical path). *Let $t \geq 0$, $q \in \mathcal{Q}_\infty^\mathcal{S}$, and $p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$, and set $\pi := q + t \nabla \xi(p) \in \mathcal{Q}_\infty^\mathcal{S}$. If $p = \partial_q \psi(\pi)$, then*

$$(2.83) \quad \inf_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}(q', p) = \mathcal{J}_{t,q}(\pi, p) = \mathcal{P}_{t,q}(p).$$

Proof. Let $q' \in \mathcal{Q}_\infty^\mathcal{S}$, and set $\kappa := q' - \pi$, so that $\pi + \varepsilon \kappa = (1 - \varepsilon)\pi + \varepsilon q' \in \mathcal{Q}_\infty^\mathcal{S}$ for $\varepsilon \in [0, 1]$. By the convexity of ψ (Lemma 2.9),

$$(2.84) \quad \psi(q') - \psi(\pi) \geq \lim_{\varepsilon \searrow 0} \frac{\psi(\pi + \varepsilon \kappa) - \psi(\pi)}{\varepsilon} = \langle q' - \pi, \partial_q \psi(\pi) \rangle_{L^2}.$$

With $p = \partial_q \psi(\pi)$, it reads $\psi(q') - \langle q', p \rangle_{L^2} \geq \psi(\pi) - \langle \pi, p \rangle_{L^2}$, and adding $\langle q, p \rangle_{L^2} + t \int_0^1 \xi(p(r)) dr$ to both sides yields $\mathcal{J}_{t,q}(q', p) \geq \mathcal{J}_{t,q}(\pi, p)$. Since $\pi \in \mathcal{Q}_\infty^\mathcal{S}$ is itself an admissible competitor, the infimum in (2.83) equals $\mathcal{J}_{t,q}(\pi, p)$, and the second equality in (2.83) is the identity noted after (2.79). $\square$

Let $f(t, q)$ denote the right-hand side of (1.11), as in Theorem 1.1. Theorem 5.4 identifies f with the Lipschitz viscosity solution of (1.12), and Proposition 5.5 shows that $f(t, q) \leq \liminf_{N \rightarrow \infty} \bar{F}_N(t, q)$, for arbitrary species proportions, along the full sequence of sizes, and for every $q \in \mathcal{Q}_2^\mathcal{J}$. In the setting of Proposition 2.12, with p the critical path it provides, we therefore have

$$(2.85) \quad f(t, q) \leq \liminf_{n \rightarrow \infty} \bar{F}_{N_n}(t, q) \leq \limsup_{n \rightarrow \infty} \bar{F}_{N_n}(t, q) \leq \mathcal{P}_{t,q}(p) = \inf_{q' \in \mathcal{Q}_\infty^\mathcal{J}} \mathcal{J}_{t,q}(q', p) \leq f(t, q).$$

The third inequality is (2.82); the equality is Lemma 2.13, which applies since p satisfies (2.81) and q is bounded; and the last inequality holds because $p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{J}$ is a competitor in the supremum defining f . Once the Hamilton–Jacobi lower bound and the upper bound at a critical path are proved, in Sections 5 and 4 respectively, the chain (2.85) identifies the limit free energy for rational proportions and regular paths; this is Corollary 6.1, from which Theorem 1.1 follows by density and continuity arguments, as will be explained in Section 6.

3. CAVITY COMPUTATION AND SYNCHRONIZATION

The purpose of this section is to compare the systems with N and $N + M$ spins, with M kept fixed, and to provide the cavity input for Proposition 2.12. That proposition requires two conclusions for the same path p : the upper bound (2.82) and the critical relation (2.81). In [4, 12], the corresponding one-sided bound at $q = 0$, combined with Guerra’s bound under the convexity hypothesis, suffices to identify the free energy. Here we also need the critical relation to complete the identification (2.85).

The general strategy follows [10, Section 6], see also [9, Section 5]. A small perturbation, parametrized by x , enforces the Ghirlanda–Guerra identities, so that the bulk overlaps synchronize: the overlap $R_N(\sigma, \sigma')$ of two replicas becomes asymptotically a function $p(\alpha \wedge \alpha')$ of the overlap of their cascade variables, for a single path p . To establish the critical relation, Section 4 also compares the derivatives in q of the N - and $(N + M)$ -spin free energies. A small perturbation of the path q , parametrized by z , ensures that these derivatives are close to one another. The first of these two derivatives is an average of the bulk overlaps, and its limit defines p . By the exchangeability of the coordinates within each species, the second one is an average of the overlaps of the cavity coordinates, and in order to identify its limit as the right-hand side of (2.81), we will determine the limit law of these overlaps in terms of the limit behavior of the bulk.

This limit law is that described by the Gaussian block of Subsection 2.2, which is also the object through which ψ is defined. We recall its main features. In the version with one coordinate, a real variable τ and a cascade variable α are sampled with density proportional to

$$\exp\left(\left(\sqrt{2}w^\rho(\alpha) + a\right)\tau - \frac{b}{2}\tau^2\right)$$

with respect to $d\tau \mathfrak{R}(d\alpha)$, where $\mathfrak{R}$ is the Poisson–Dirichlet cascade of Subsection 2.1, w^ρ is the centered Gaussian field with covariance $\rho(\alpha \wedge \alpha')$, and a is the external field. The coefficient b is a Lagrange multiplier for the spherical constraint. The cascade field favors large values of τ , and the free energy of the block is finite if and only if $b > 2\mathcal{K}(\rho)$, with $\mathcal{K}$ as in (2.23); we then say that b is admissible for ρ . The free energy of the block is a convex function of b , and up to an affine term and the weighting of the species, the function $-\psi$ is defined by minimizing it over admissible b , see (2.25) and (2.28). The minimizer is the unique admissible b for which $\mathbb{E}\langle \tau^2 \rangle = 1$, as can be expected from a coordinate of a point on the sphere, see (2.44). In the present section, the block has M coordinates, M_s of them in each species s , with a path $\pi = (\pi^s)_{s \in \mathcal{S}}$ and a multiplier $b = (b^s)_{s \in \mathcal{S}}$; we call it the canonical Gaussian block, and we write b_π for the minimizing multiplier, see Subsection 3.9.

In order to perform the cavity calculation, and since P_{N+M} is not a product measure, we write a configuration in Σ_{N+M} as $(S_{r_N(\tau)}\sigma, \tau)$, where $\sigma \in \Sigma_N$, $\tau \in \mathbb{R}^M$, and where $S_{r_N(\tau)}$ rescales the coordinates of σ in species s by the factor $r_N^s(\tau)^{1/2}$, with $r_N^s(\tau) = 1 + (M_s - |\tau^{(s)}|^2)/N_s$; see (3.31). We refer to this parametrization of Σ_{N+M} as the chart. The rescaling factor deviates from 1 by $O(N^{-1})$, but it multiplies a radial derivative of the Hamiltonian whose Gibbs average can be of order N , so its contribution to the cavity increment is of order one and must be retained.

Explicitly, in the cavity computation of [10], the Hamiltonian of the $(N + M)$ -spin system is, up to negligible terms, the sum of the Hamiltonian of the N -spin system and of a term linear in τ . Here, an additional term of the form

$$(3.1) \quad \frac{1}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) \mathcal{E}_{N,s}(\sigma, \alpha)$$

appears, where $\mathcal{E}_{N,s}$ is the derivative of the Hamiltonian of the N -spin system with respect to the radius of species s , suitably normalized to be of order 1. In [4, 12], the cavity coordinates are confined to a thin annulus around the product of spheres of radii $\sqrt{M_s}$, on which this term is small and can be discarded at a cost that vanishes in the limit. The confinement operation is sufficient for the purpose of proving (2.82), but here we also aim to establish (2.81) and thus need to be more precise.

To control the term in (3.1), we introduce a small perturbation in the directions of $\mathcal{E}_{N,s}$, with a small coefficient parametrized by $y = (y^s)_{s \in \mathcal{S}}$. The logarithm of the partition function is convex in y , and a classical argument shows that $\mathcal{E}_{N,s}$ concentrates around its Gibbs average e_N^s for most values of y . Replacing $\mathcal{E}_{N,s}$ by e_N^s turns (3.1) into the additive constant $\frac{1}{2} \sum_s M_s e_N^s$ and the quadratic term $-\frac{1}{2} \sum_s e_N^s |\tau^{(s)}|^2$, so that in the limit, the cavity coordinates are those of a canonical Gaussian block with path $\pi := q + t \nabla \xi(p)$, whose multiplier b is an explicit function of $e := \lim e_N$. However, e_N is a quantity on which we have no direct information, so at this stage b is an unknown parameter of the limit. It is identified through the spherical constraint: by the permutation symmetry between the coordinates of each species in the $(N + M)$ -spin system, we have

$$\mathbb{E} \left(|\tau^{(s)}|^2 \right) = M_s.$$

For the limiting block, this identity is the normalization recalled above, and we can therefore conclude that $b = b_\pi$ (Lemma 3.17). This allows us to connect the Gaussian block with ψ , and thereby to complete the argument.

The unbounded nature of the cavity variables creates several difficulties, such as in justifying the replacement of $\mathcal{E}_{N,s}$ by e_N^s , in using that the density of the cavity coordinates is close to Gaussian, or in discussing Gaussian blocks for values of b that we do not know in advance to be admissible. We therefore insert a cutoff $\chi_L(\tau)$, under which all objects are defined for every b , and first let $N \rightarrow \infty$ at fixed L . Since the cutoff can only decrease the partition function of the $(N + M)$ -spin system, this gives a lower bound on the cavity increment $\mathbb{E} \log Z_{N+M} - \mathbb{E} \log Z_N$ by the free energy of the block with cutoff, for every L . The increment is bounded (Lemma 3.6), while the free energy of the block with cutoff diverges as $L \rightarrow \infty$ when b is not admissible (Lemma 3.20); hence b is admissible. We then let $L \rightarrow \infty$ (Lemma 3.23), and we show that the moments of the cavity coordinates are bounded uniformly in N , using the rotation invariance of the reference measure (Lemma 3.22). This extends the convergence of the Gibbs averages from bounded observables to observables of polynomial growth, such as $|\tau^{(s)}|^2$ in the identity above.

These steps are combined in Theorem 3.24, stated in Subsection 3.11. This theorem applies along sequences of perturbation parameters (x, y) for which two error terms vanish: the Ghirlanda–Guerra error Δ_N , which measures the failure of the Ghirlanda–Guerra identities in the bulk, and the radial error $\text{Rad}_{N,L}$, which measures the fluctuations of $\mathcal{E}_{N,s}$ at cutoff L , for every fixed L . The selection of such parameters is carried out in Section 4. Along a subsequence, the theorem provides a path p with $0 \leq p \leq \lambda$ describing the bulk overlaps; the convergence of the overlaps of the cavity coordinates under the $(N + M)$ -spin Gibbs measure, jointly with the cascade overlaps and including polynomial moments, to those of the canonical Gaussian block with path $\pi = q + t \nabla \xi(p)$ and multiplier $b = b_\pi$; and the asymptotic lower bound $-M \mathcal{P}_{t,q}(p)$ on the cavity increment, where $\mathcal{P}_{t,q}$ is the Parisi functional (2.79). In view of the sign convention in (2.15), a lower bound on the increment is what is needed for the upper bound (2.82).

The section is organized as follows. Subsections 3.1 and 3.2 set up the notation and introduce the perturbed Hamiltonians. The proof of Theorem 3.24 then goes through four stages.

- (1) *Geometry and scale.* Subsection 3.3 describes the chart and the chart density; Subsection 3.4 collects two Gaussian comparison lemmas; and Subsection 3.5 shows that the cavity increment is uniformly bounded.

- (2) *Comparison at fixed cutoff.* Subsections 3.6 to 3.8 replace the $(N + M)$ -spin system in the chart by an explicit Gaussian model for the cavity coordinates, at a fixed cutoff L , up to the radial error $\text{Rad}_{N,L}$ and a vanishing deterministic error. The main comparison result is Proposition 3.15.
- (3) *The canonical Gaussian block.* Subsection 3.9 introduces the block and relates it to ψ and to $\mathcal{P}_{t,q}$. It contains the characterization of b_π by the normalization, the formula for the derivative of ψ , and the divergence of the block with cutoff when b is not admissible.
- (4) *Passage to the limit and removal of the cutoff.* Subsection 3.10 contains the finite-overlap approximation, used to pass to the limit $N \rightarrow \infty$ at fixed cutoff, the uniform integrability estimate for the cavity coordinates, and the removal of the cutoff. Subsection 3.11 states and proves the theorem.

3.1. Setting and notation.

Standing assumptions. Throughout Sections 3 and 4, we assume that $\lambda_\infty \in \mathbb{Q}^\mathcal{S}$, and we write $\lambda := \lambda_\infty$. We fix positive integers $(M_s)_{s \in \mathcal{S}}$ and a partition $[M] = \sqcup_{s \in \mathcal{S}} M_s$ such that

$$(3.2) \quad M := \sum_{s \in \mathcal{S}} M_s, \quad \frac{M_s}{M} = \lambda_s \quad \text{for every } s \in \mathcal{S}, \quad |M_s| = M_s.$$

The set M_s is the set of labels of the cavity coordinates of species s . We restrict our attention to system sizes N that are multiples of M , and we assume that the species decomposition is compatible with the cavity block, in the sense that

$$(3.3) \quad N_s := |I_{N,s}| = \lambda_s N, \quad I_{N+M,s} = I_{N,s} \sqcup \{N + j : j \in M_s\}, \quad \text{for every } s \in \mathcal{S}.$$

In particular, the last M coordinates of $[N + M]$ are identified with $[M]$, and the cavity coordinates of species s are those with labels in $N + M_s$. Under (3.3), the proportions in (1.1) do not depend on N :

$$(3.4) \quad \lambda_N = \lambda_{N+M} = \lambda.$$

Bulk and cavity coordinates. For $\rho \in \mathbb{R}^{N+M}$, we write

$$(3.5) \quad \sigma := (\rho_1, \dots, \rho_N) \in \mathbb{R}^N \quad \text{and} \quad \tau := (\rho_{N+1}, \dots, \rho_{N+M}) \in \mathbb{R}^M,$$

and we call σ the bulk coordinates and τ the cavity coordinates of ρ . For $\tau \in \mathbb{R}^M$ and $s \in \mathcal{S}$, we write $\tau^{(s)} := (\tau_j)_{j \in M_s}$. For $r = (r^s)_{s \in \mathcal{S}} \in (0, \infty)^\mathcal{S}$ and $u \in \mathbb{R}^N$, we define the specieswise dilation $S_r u \in \mathbb{R}^N$ by

$$(3.6) \quad (S_r u)_i := \sqrt{r^s} u_i \quad \text{for every } s \in \mathcal{S} \text{ and } i \in I_{N,s}.$$

Thus S_r multiplies the squared norm of the species- s block by r^s , and

$$(3.7) \quad R_{N,s}(S_r \sigma, S_{r'} \sigma') = \sqrt{r^s r'^s} R_{N,s}(\sigma, \sigma'), \quad \sigma, \sigma' \in \mathbb{R}^N.$$

We set $\delta_0 := 1/2$ and

$$(3.8) \quad U_0 := [1 - \delta_0, 1 + \delta_0]^\mathcal{S}.$$

The field H_N is defined on all of $\mathbb{R}^N$, see (1.5), and the formula (2.11) defines $W_N^q(\sigma, \alpha)$ for every $\sigma \in \mathbb{R}^N$, with the covariance (2.12); in particular, both fields are defined at the dilated configurations $S_r \sigma$. By (3.7), the covariance of $H_N(S_r \sigma)$ and $H_N(S_{r'} \sigma')$ is $N \xi((\sqrt{r^s r'^s} R_{N,s}(\sigma, \sigma'))_{s \in \mathcal{S}})$, a smooth function of $(r, r') \in U_0^2$, and similarly for W_N^q . The fields are therefore differentiable in r in the L^2 sense, and their radial derivatives are jointly Gaussian with the fields themselves. All the radial derivatives appearing below are understood in this sense.

Fixed parameters. We fix $t > 0$ for the whole section. The dependence on t is kept in the notation $H_N^{t,q}$ from (2.13), but is suppressed from the notation of the objects that are specific to the cavity computation. We also fix the sequence

$$(3.9) \quad \eta_N := N^{-\frac{1}{16}}, \quad N \in \mathbb{N},$$

which sets the size of the perturbations of the Hamiltonian introduced in the next subsection. The exponent $1/16$ is chosen to match [10, Proposition 6.8], whose proof we invoke in Lemma 4.5 below, and any exponent in $(0, 1/16]$ would do.

Parameters and uniformity. The perturbations depend on parameters

$$x = (x_h)_{h \in \mathbb{N}^4} \in [0, 3]^{\mathbb{N}^4} \quad \text{and} \quad y = (y^s)_{s \in \mathcal{S}} \in [0, 3]^{\mathcal{S}},$$

which will be chosen in Section 4. We also fix a subset $Q \subseteq \mathcal{Q}^{\mathcal{S}}$ that is bounded in L^∞ , and a bounded subset $B \subseteq \mathbb{R}^{\mathcal{S}}$. Unless stated otherwise, when we say that an estimate holds uniformly over x , y , q , or b , we mean that it holds uniformly over

$$(3.10) \quad x \in [0, 3]^{\mathbb{N}^4}, \quad y \in [0, 3]^{\mathcal{S}}, \quad q \in Q, \quad b \in B,$$

and every supremum indexed by x , y , q , or b is taken over these sets. Constants denoted by C may depend on ξ , h , t , M , Q , and B , and may change from one occurrence to the next; additional dependencies are indicated by subscripts, as in C_L . In Section 4, the parameters x and y are restricted to the smaller sets $[1, 2]^{\mathbb{N}^4}$ and $[1, 2]^{\mathcal{S}}$; the estimates of the present section apply there verbatim.

Recall that $\mathfrak{U} = \text{supp } \mathfrak{R}$ denotes the support of the cascade, and that $\langle \cdot \rangle_{\mathfrak{R}}$ in (2.2) denotes the average with respect to independent copies of α distributed according to $\mathfrak{R}$, conditionally on $\mathfrak{R}$.

3.2. Perturbed Hamiltonians. We add two perturbations to the Hamiltonian $H_N^{t,q}$ in (2.13). The first one is the standard perturbation that yields the Ghirlanda–Guerra identities; the second one is specific to the spherical setting, and controls the radial derivative of the Hamiltonian.

3.2.1. Perturbation enforcing the Ghirlanda–Guerra identities. We fix enumerations $(\iota_m)_{m \geq 1}$ and $(a_m)_{m \geq 1}$ such that

$$\{\iota_m : m \geq 1\} = [0, 1] \cap \mathbb{Q}, \quad \{a_m : m \geq 1\} = ((0, \infty) \cap \mathbb{Q})^{\mathcal{S}}.$$

For $\mathbf{h} = (h_1, h_2, h_3, h_4) \in \mathbb{N}^4$, we define the kernel

$$(3.11) \quad C_{\mathbf{h}}(v, u) := (a_{h_1} \cdot v^{\odot h_2} + \iota_{h_3} u)^{h_4}, \quad v \in \mathbb{R}^{\mathcal{S}}, \quad u \in [0, 1],$$

where $\odot$ denotes componentwise multiplication, so that $v^{\odot h_2} = (v_s^{h_2})_{s \in \mathcal{S}}$. This is a polynomial in (v, u) with nonnegative coefficients. For each $\mathbf{h} \in \mathbb{N}^4$, let $(H_N^{\mathbf{h}}(\sigma, \alpha))_{\sigma \in \mathbb{R}^N, \alpha \in \mathfrak{U}}$ be a centered Gaussian process with covariance

$$(3.12) \quad \mathbb{E} H_N^{\mathbf{h}}(\sigma, \alpha) H_N^{\mathbf{h}}(\sigma', \alpha') = N C_{\mathbf{h}}(R_N(\sigma, \sigma'), \alpha \wedge \alpha'),$$

these processes being independent over $\mathbf{h}$ and independent of the other sources of randomness.

We briefly explain why these processes exist, in a form that yields versions which are polynomial in σ , and which is also used in Subsection 3.6 to couple the processes at sizes N and $N + M$. Expanding the power in (3.11) by the binomial formula,

$$(3.13) \quad C_{\mathbf{h}}(v, u) = \sum_{j=0}^{h_4} \binom{h_4}{j} \iota_{h_3}^j u^j (a_{h_1} \cdot v^{\odot h_2})^{h_4-j}.$$

For $\sigma \in \mathbb{R}^N$, let $\varphi_{N,s}(\sigma) := N^{-1/2} \sigma^{(s)} \in \mathbb{R}^{I_{N,s}}$, so that $\langle \varphi_{N,s}(\sigma), \varphi_{N,s}(\sigma') \rangle = R_{N,s}(\sigma, \sigma')$ by (1.3), and, for $0 \leq j \leq h_4$, consider the finite-dimensional space $E_{N,j} := (\oplus_{s \in \mathcal{S}} (\mathbb{R}^{I_{N,s}})^{\otimes h_2})^{\otimes (h_4-j)}$ and the polynomial map

$$(3.14) \quad \Phi_N^{\mathbf{h},j}(\sigma) := \left(\bigoplus_{s \in \mathcal{S}} \sqrt{a_{h_1}^s} \varphi_{N,s}(\sigma)^{\otimes h_2} \right)^{\otimes (h_4-j)} \in E_{N,j}.$$

The inner product of two direct sums is the sum of the inner products, and the inner product of two tensor powers is the power of the inner product, so

$$(3.15) \quad \langle \Phi_N^{h,j}(\sigma), \Phi_N^{h,j}(\sigma') \rangle = \left(\sum_{s \in \mathcal{S}} a_{h_1}^s R_{N,s}(\sigma, \sigma')^{h_2} \right)^{h_4-j}.$$

The path $u \mapsto \iota_{h_3}^j u^j$ belongs to $\mathcal{Q}_\infty$; it is constant when $j = 0$. By [10, Proposition 4.1], we can therefore attach to every element e of an orthonormal basis $\mathcal{E}_{N,j}$ of $E_{N,j}$ a cascade field $w_e^{h,j}$ on $\mathfrak{U}$ with covariance $\mathbb{E} w_e^{h,j}(\alpha) w_e^{h,j}(\alpha') = \iota_{h_3}^j(\alpha \wedge \alpha')^j$, as in (2.4), these fields being independent over (h, j, e) conditionally on $\mathfrak{R}$, jointly measurable in α and in the underlying randomness, and independent of the other sources of randomness. We set

$$(3.16) \quad H_N^h(\sigma, \alpha) := \sqrt{N} \sum_{j=0}^{h_4} \binom{h_4}{j}^{1/2} \sum_{e \in \mathcal{E}_{N,j}} w_e^{h,j}(\alpha) \langle \Phi_N^{h,j}(\sigma), e \rangle.$$

Since $\sum_{e \in \mathcal{E}_{N,j}} \langle \Phi(\sigma), e \rangle \langle \Phi(\sigma'), e \rangle = \langle \Phi(\sigma), \Phi(\sigma') \rangle$, the covariance of this field is given by (3.13) and (3.15), that is, by (3.12). By construction, $H_N^h(\sigma, \alpha)$ is a polynomial in σ whose coefficients are finitely many Gaussian random variables, measurable in α . In particular, it is defined for every $\sigma \in \mathbb{R}^N$, including the dilated configurations $S_r \sigma$ with $r \in U_0$, and its radial derivatives are obtained by differentiating the polynomial; they are linear combinations of the same fields $w_e^{h,j}$.

The kernel C_h and its derivatives are bounded on the relevant range of overlaps. Setting

$$(3.17) \quad \Lambda_h := 1 + \sum_{a=0}^4 \sup_{v \in [-2,2]^{\mathcal{S}}, u \in [0,1]} \|\partial_v^a C_h(v, u)\|,$$

we choose constants $c_h > 0$ such that

$$(3.18) \quad \sum_{h \in \mathbb{N}^4} c_h^2 \Lambda_h \leq 1.$$

Derivatives of order up to four are included in (3.17) because the radial perturbation introduced below involves derivatives of the covariance kernels, and its own covariance therefore involves up to four derivatives. For $x \in [0, 3]^{\mathbb{N}^4}$, we define

$$(3.19) \quad H_N^x(\sigma, \alpha) := \sum_{h \in \mathbb{N}^4} x_h c_h H_N^h(\sigma, \alpha) \quad \text{and} \quad H_N^{x,q}(\sigma, \alpha) := H_N^{t,q}(\sigma, \alpha) + \eta_N H_N^x(\sigma, \alpha).$$

By (3.12) and (3.18), the series defining $H_N^x(\sigma, \alpha)$ converges in L^2 , with variance at most $9N$, for every $\sigma \in \mathbb{R}^N$ with $R_N(\sigma, \sigma) \in [0, 2]^{\mathcal{S}}$; this covers Σ_N and the dilated configurations $S_r \sigma$ with $r \in U_0$, which are the only configurations at which H_N^x is evaluated. The coefficient η_N makes the contribution of this perturbation to the free energy per spin of order η_N^2 , hence negligible, while keeping it large enough to enforce the Ghirlanda–Guerra identities; see Lemma 4.5.

3.2.2. Radial evaluation and the radial perturbation. We now define the evaluation of the Hamiltonian along dilated configurations. For $r \in U_0$, set

$$(3.20) \quad \begin{aligned} H_N^{x,q}(r; \sigma, \alpha) &:= \sqrt{2t} H_N(S_r \sigma) - N t \xi(R_N(S_r \sigma, S_r \sigma)) + \sqrt{2} W_N^q(S_r \sigma, \alpha) - \sum_{s \in \mathcal{S}} q^s(1) |(S_r \sigma)^{(s)}|^2 \\ &+ \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} (S_r \sigma)_i + \eta_N H_N^x(S_r \sigma, \alpha). \end{aligned}$$

This is the expression of $H_N^{x,q}$ in (3.19), with σ replaced by $S_r \sigma$ everywhere, including in the deterministic self-overlap corrections. In particular, $H_N^{x,q}(\bar{\mathbf{1}}; \sigma, \alpha) = H_N^{x,q}(\sigma, \alpha)$, where $\bar{\mathbf{1}} = (1, \dots, 1)$.

As explained at the beginning of the section, writing the $(N+M)$ -spin system in the chart leads us to evaluate the Hamiltonian of the bulk at a dilation $S_r \sigma$ with $r - \bar{\mathbf{1}}$ of order N^{-1} . The radial derivatives of the Hamiltonian have means and covariances of order N , see (3.58) below, so that this small displacement produces a variation of order one, which is measured by the normalized radial derivatives

$$(3.21) \quad \mathcal{E}_{N,s}^{x,q}(\sigma, \alpha) := \frac{2}{N_s} \frac{\partial}{\partial r^s} H_N^{x,q}(r; \sigma, \alpha) \Big|_{r=\bar{\mathbf{1}}}, \quad s \in \mathcal{S}.$$

The factor $2/N_s$ is chosen so that, for a function f of σ , we have

$$\frac{2}{N_s} \partial_{r^s} f(S_r \sigma) \Big|_{r=\mathbf{1}} = \frac{1}{N_s} \sum_{i \in I_{N,s}} \sigma_i \partial_{\sigma_i} f(\sigma);$$

that is, $\mathcal{E}_{N,s}^{x,q}$ is the average over the species- s coordinates of $\sigma_i \partial_{\sigma_i} H_N^{x,q}$. We will need these quantities to be asymptotically deterministic under the Gibbs measure. For this purpose, we perturb the Hamiltonian in their direction: for $y \in [0, 3]^{\mathcal{S}}$, we set

$$(3.22) \quad V_N^{x,y,q}(\sigma, \alpha) := \eta_N \sum_{s \in \mathcal{S}} y^s N_s \mathcal{E}_{N,s}^{x,q}(\sigma, \alpha) \quad \text{and} \quad H_N^{x,y,q}(\sigma, \alpha) := H_N^{x,q}(\sigma, \alpha) + V_N^{x,y,q}(\sigma, \alpha).$$

The perturbed partition function, free energy, and Gibbs bracket are

$$(3.23) \quad \begin{aligned} Z_N^{x,y,q} &:= \iint \exp(H_N^{x,y,q}(\sigma, \alpha)) P_N(d\sigma) \mathfrak{R}(d\alpha), & \bar{F}_N^{x,y}(t, q) &:= -\frac{1}{N} \mathbb{E} \log Z_N^{x,y,q}, \\ \langle f \rangle_N^{x,y,q} &:= \frac{1}{(Z_N^{x,y,q})^n} \int f((\sigma^\ell, \alpha^\ell)_{\ell \leq n}) \prod_{\ell=1}^n e^{H_N^{x,y,q}(\sigma^\ell, \alpha^\ell)} P_N(d\sigma^\ell) \mathfrak{R}(d\alpha^\ell), \end{aligned}$$

where f is a bounded measurable function of n replicas. When $x = 0$ and $y = 0$, we recover $\bar{F}_N^{0,0}(t, q) = \bar{F}_N(t, q)$. The mechanism through which $V_N^{x,y,q}$ produces the concentration of $\mathcal{E}_{N,s}^{x,q}$ is explained in Subsection 3.8: the logarithm of the partition function is convex in y , with derivative proportional to $\langle \mathcal{E}_{N,s}^{x,q} \rangle$, and a convex function whose fluctuations are small has, for most values of its argument, a derivative whose fluctuations are small as well.

Remark 3.1 (Linear form of the perturbed Hamiltonians). The Hamiltonian $H_N^{x,y,q}$ depends on the cascade variable α only through the cascade fields $w_i^q(\alpha)$ and the fields $w_e^{h,j}(\alpha)$ of the construction (3.16), and it is linear in these fields, with coefficients that are polynomials in σ ; this remains true of its radial derivatives, hence of $V_N^{x,y,q}$. It is therefore of the form (B.1) considered in Appendix B, with the finite family of the cascade fields and the countable family of the perturbation fields. The tail condition (B.3) holds by (3.18), since the contribution to the variance of the indices h outside a finite set F is at most $CN\eta_N^2 \sum_{h \notin F} c_h^2 \Lambda_h$, for some constant $C < \infty$. The same applies to the bulk Hamiltonian $\tilde{H}_N^{x,y,q}$ of Subsection 3.6 and to the Hamiltonian of the $(N + M)$ -spin system. The invariance (2.5) is stated for finitely many fields; we apply it to the Hamiltonian in which the sum over h is restricted to a finite set F , conditionally on the fields that do not depend on the cascade, and we then let F increase to $\mathbb{N}^4$, using Lemma 3.4 below: by the Cauchy–Schwarz inequality, the covariances of the restricted and unrestricted Hamiltonians differ by at most the bound on the variance just given. Consequently, for every $n \in \mathbb{N}$, the overlap array $(\alpha^\ell \wedge \alpha^{\ell'})_{\ell, \ell' \leq n}$ of n replicas has the same law under every Gibbs measure built from these Hamiltonians as under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$; in particular, the overlap $\alpha \wedge \alpha'$ of two replicas is uniformly distributed on $[0, 1]$. Moreover, the corresponding free energies concentrate around their expectations at the rate given by Lemma B.1. These two facts are used repeatedly below without further comment.

3.2.3. Covariance of radially perturbed fields. All the Gaussian fields we consider have a covariance of the form $f(R_N(\sigma, \sigma'), \alpha \wedge \alpha')$, and the following lemma computes the covariance of their radial perturbation. For $y \in \mathbb{R}^{\mathcal{S}}$, we introduce the first-order differential operator

$$(3.24) \quad D_y := \sum_{s \in \mathcal{S}} y^s v_s \partial_{v_s},$$

acting on functions of $v \in \mathbb{R}^{\mathcal{S}}$, and we write $D_y^2 := D_y \circ D_y$.

Lemma 3.2 (Covariance of radial perturbations). *Let $f = f(v, u)$ be a function which is smooth in v on a neighborhood of $\{(\sqrt{r^s r'^s} v_s)_{s \in \mathcal{S}} : r, r' \in U_0, |v_s| \leq \lambda_s\}$, and let W be a centered Gaussian field on $\{S_r \sigma : r \in U_0, \sigma \in \Sigma_N\} \times \mathfrak{U}$ with covariance $\mathbb{E} W(\sigma, \alpha) W(\sigma', \alpha') = f(R_N(\sigma, \sigma'), \alpha \wedge \alpha')$. For $\eta > 0$ and $y \in \mathbb{R}^{\mathcal{S}}$, set*

$$W^y(\sigma, \alpha) := W(\sigma, \alpha) + 2\eta \sum_{s \in \mathcal{S}} y^s \partial_{r^s} W(S_r \sigma, \alpha) \Big|_{r=\mathbf{1}}, \quad \sigma \in \Sigma_N.$$

Then $\mathbb{E}W^y(\sigma, \alpha)W^y(\sigma', \alpha') = f_{\eta, y}(R_N(\sigma, \sigma'), \alpha \wedge \alpha')$, where

$$(3.25) \quad f_{\eta, y}(v, u) := f(v, u) + 2\eta D_y f(v, u) + \eta^2 D_y^2 f(v, u).$$

Proof. Set $v := R_N(\sigma, \sigma')$ and $u := \alpha \wedge \alpha'$. By (3.7), the covariance of $W(S_r \sigma, \alpha)$ and $W(S_{r'} \sigma', \alpha')$ is $f((\sqrt{r^s r'^s} v_s)_{s \in \mathcal{S}}, u)$. Since radial derivatives are L^2 derivatives, they commute with expectations, and

$$\mathbb{E}W^y(\sigma, \alpha)W^y(\sigma', \alpha') = \left(1 + 2\eta \sum_{s \in \mathcal{S}} y^s \partial_{r^s}\right) \left(1 + 2\eta \sum_{s \in \mathcal{S}} y^s \partial_{r'^s}\right) f((\sqrt{r^s r'^s} v_s)_{s \in \mathcal{S}}, u) \Big|_{r=r'=1}.$$

By the chain rule, $2\partial_{r^s} f((\sqrt{r^s r'^s} v_s)_{s \in \mathcal{S}}, u)|_{r=1} = v_s \partial_{v_s} f((\sqrt{r'^s} v_s)_{s \in \mathcal{S}}, u)$, and similarly for r' . Expanding the product of the two operators gives (3.25). $\square$

In particular, the covariance of $H_N^{x, y, q}$ is obtained from that of $H_N^{x, q}$ by applying the transformation in (3.25) to each covariance kernel, with $\eta = \eta_N$. Explicitly, since the fields H_N , W_N^q , and $(H_N^h)_h$ are independent, we obtain, for $v := R_N(\sigma, \sigma')$ and $u := \alpha \wedge \alpha'$,

$$(3.26) \quad \begin{aligned} \text{Cov}(H_N^{x, y, q}(\sigma, \alpha), H_N^{x, y, q}(\sigma', \alpha')) &= 2tN \xi_{\eta_N, y}(v) + 2N \sum_{s \in \mathcal{S}} (1 + \eta_N y^s)^2 q^s(u) v_s \\ &\quad + N \eta_N^2 \sum_{h \in \mathbb{N}^4} x_h^2 c_h^2(C_h)_{\eta_N, y}(v, u), \end{aligned}$$

where we used that $D_y(q^s(u) v_s) = y^s q^s(u) v_s$ for the middle term. The mean of $H_N^{x, y, q}(\sigma, \alpha)$ is

$$(3.27) \quad \begin{aligned} \mathbb{E}H_N^{x, y, q}(\sigma, \alpha) &= -Nt(\xi(\lambda) + 2\eta_N D_y \xi(\lambda)) - N \sum_{s \in \mathcal{S}} q^s(1) \lambda_s (1 + 2\eta_N y^s) \\ &\quad + \sum_{s \in \mathcal{S}} h^s (1 + \eta_N y^s) \sum_{i \in I_{N, s}} \sigma_i, \end{aligned}$$

since $R_N(S_r \sigma, S_r \sigma) = (r^s \lambda_s)_{s \in \mathcal{S}}$, $|(S_r \sigma)^{(s)}|^2 = r^s N_s$, and $\partial_{r^s} \sqrt{r^s}|_{r^s=1} = 1/2$. By the absolute convergence in (1.4) and by (3.18), the functions $\xi_{\eta_N, y}$ and $(C_h)_{\eta_N, y}$, as well as their first two derivatives in v , are bounded uniformly over $y \in [0, 3]^{\mathcal{S}}$ and N on the range of overlaps of interest.

3.3. The spherical chart. We now describe the chart on Σ_{N+M} that we use to separate the bulk and cavity coordinates. For $m, n \in \mathbb{N}$, set

$$(3.28) \quad \begin{aligned} \mathcal{B}_{m, n} &:= \{\tau \in \mathbb{R}^m : |\tau|^2 < n + m\}, \quad \phi_{m, n}(\sigma, \tau) := \left(\sqrt{\frac{n + m - |\tau|^2}{n}} \sigma, \tau \right), \\ p_{m, n}(\tau) &:= \begin{cases} \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2}) (\pi(n+m))^{m/2}} \left(1 - \frac{|\tau|^2}{n+m}\right)^{\frac{n}{2}-1}, & \tau \in \mathcal{B}_{m, n}, \\ 0, & \tau \in \mathbb{R}^m \setminus \mathcal{B}_{m, n}, \end{cases} \end{aligned}$$

where Γ is Euler's Gamma function. The map $\phi_{m, n}$ sends $\mathbb{S}_n \times \mathcal{B}_{m, n}$ onto $\mathbb{S}_{n+m}$ minus a null set. For $m \in \mathbb{N}$, we write

$$(3.29) \quad \gamma_m(du) := (2\pi)^{-m/2} \exp(-|u|^2/2) du$$

for the standard Gaussian measure on $\mathbb{R}^m$. The following lemma is a block version of the disintegration used in [4, 12], which peels off one coordinate at a time.

Lemma 3.3 (Disintegration of the surface measure). *For every measurable function $f : \mathbb{S}_{n+m} \rightarrow [0, \infty)$, we have*

$$(3.30) \quad \int_{\mathbb{S}_{n+m}} f(\rho) \mu_{n+m}(d\rho) = \iint_{\mathbb{S}_n \times \mathcal{B}_{m, n}} f(\phi_{m, n}(\sigma, \tau)) p_{m, n}(\tau) d\tau \mu_n(d\sigma).$$

Moreover, for each fixed m , the density $p_{m, n}$ converges to the density of γ_m locally uniformly on $\mathbb{R}^m$ as $n \rightarrow \infty$.

Proof. Set $R := \sqrt{n+m}$, and parametrize $\mathbb{S}_{n+m}$ minus a null set by

$$\mathcal{B}_{m,n} \times \{\omega \in \mathbb{R}^n : |\omega| = 1\} \ni (\tau, \omega) \mapsto (\sqrt{R^2 - |\tau|^2} \omega, \tau).$$

The partial derivative of this map with respect to τ_k is $(-\tau_k(R^2 - |\tau|^2)^{-1/2} \omega, e_k)$, and its differential in a unit direction ζ tangent to the sphere at ω is $((R^2 - |\tau|^2)^{1/2} \zeta, 0)$. Each vector of the second kind is orthogonal to each vector of the first kind, since $\zeta \perp \omega$. The Gram determinant of the m vectors of the first kind is $\det(I_m + \tau \tau^\top / (R^2 - |\tau|^2)) = R^2 / (R^2 - |\tau|^2)$, and that of the $n-1$ vectors of the second kind, with ζ ranging over an orthonormal basis of the tangent space, is $(R^2 - |\tau|^2)^{n-1}$. Denoting by ω_{n-1} the surface measure on the unit sphere of $\mathbb{R}^n$, and taking the square root of the product of these two determinants, we find that the surface measure on $\mathbb{S}_{n+m}$ is

$$R(R^2 - |\tau|^2)^{\frac{n-2}{2}} d\tau \omega_{n-1}(d\omega)$$

in these coordinates. Normalizing this measure, we see that under μ_{n+m} , the marginal law of τ has a density proportional to $(1 - |\tau|^2 / (n+m))^{n/2-1}$ on $\mathcal{B}_{m,n}$ and that, conditionally on τ , the vector ω is uniform on the unit sphere of $\mathbb{R}^n$; equivalently, $\sigma := \sqrt{n} \omega$ is distributed according to μ_n , and the first n coordinates of ρ are $\sqrt{(n+m - |\tau|^2)/n} \sigma$. This is the content of (3.30), up to the identification of the normalizing constant, which follows from the classical formula

$$\int_{\{|v| < 1\}} (1 - |v|^2)^{a-1} dv = \frac{\pi^{m/2} \Gamma(a)}{\Gamma(a + \frac{m}{2})}, \quad a > 0,$$

applied with $a = n/2$ after the change of variables $\tau = Rv$. For the last assertion, Stirling's formula gives $\Gamma(\frac{n+m}{2}) / \Gamma(\frac{n}{2}) = (\frac{n}{2})^{m/2} (1 + o(1))$ as $n \rightarrow \infty$, so that the constant in (3.28) converges to $(2\pi)^{-m/2}$, while $(1 - |\tau|^2 / (n+m))^{n/2-1}$ converges to $e^{-|\tau|^2/2}$ uniformly on compact sets. $\square$

We apply this disintegration to each species. We set

$$(3.31) \quad \mathcal{B}_N := \prod_{s \in \mathcal{S}} \mathcal{B}_{M_s, N_s}, \quad p_N^{\text{ch}}(\tau) := \prod_{s \in \mathcal{S}} p_{M_s, N_s}(\tau^{(s)}), \quad r_N^s(\tau) := 1 + \frac{M_s - |\tau^{(s)}|^2}{N_s}.$$

Comparing with (3.28), we see that $\sqrt{r_N^s(\tau)}$ is the factor by which ϕ_{M_s, N_s} rescales the species- s block of σ . For $(\sigma, \tau) \in \Sigma_N \times \mathcal{B}_N$, the point $(S_{r_N(\tau)} \sigma, \tau)$ belongs to Σ_{N+M} , with the identification in (3.3). We call this parametrization of Σ_{N+M} , minus a null set, by $\Sigma_N \times \mathcal{B}_N$ the *chart*, and we call p_N^{ch} the *chart density*; the superscript “ch” stands for “chart”. Since P_{N+M} is a product of surface measures over the species, Lemma 3.3 gives, for every nonnegative measurable f on Σ_{N+M} ,

$$(3.32) \quad \int_{\Sigma_{N+M}} f(\rho) P_{N+M}(d\rho) = \iint_{\Sigma_N \times \mathcal{B}_N} f(S_{r_N(\tau)} \sigma, \tau) p_N^{\text{ch}}(\tau) d\tau P_N(d\sigma).$$

We record two consequences. First, the second moment of the cavity coordinates under the reference measure is

$$(3.33) \quad \int_{\mathcal{B}_N} |\tau^{(s)}|^2 p_N^{\text{ch}}(\tau) d\tau = M_s, \quad s \in \mathcal{S},$$

since each coordinate of a point of $\mathbb{S}_{N_s+M_s}$ has second moment 1 under the uniform measure. Second, for every $L < \infty$, if $|\tau| \leq L+1$ then

$$(3.34) \quad |r_N^s(\tau) - 1| \leq \frac{M_s + (L+1)^2}{N_s}, \quad s \in \mathcal{S},$$

so that $r_N(\tau) \in U_0$ for all sufficiently large N , where U_0 is defined in (3.8). In the remainder of the section, we tacitly take N large enough, depending on L , for this to hold.

3.4. Two Gaussian comparison lemmas. The cavity computation consists of a chain of replacements of one Gaussian field by another, each of which is justified by the closeness of their means and covariances. We introduce a notation for such comparisons, and record the two interpolation lemmas that we use.

Asymptotic notation. Let (δ_N) be a sequence of positive numbers, not necessarily tending to zero as N tends to infinity. For two sequences (s_N) and (s'_N) of real numbers, we write

$$s_N \overset{\delta_N}{\approx} s'_N \quad \text{whenever} \quad |s_N - s'_N| = O(\delta_N) \quad \text{as } N \rightarrow \infty.$$

For two sequences (f_N) and (f'_N) of functions defined on sets $\mathbb{X}_N$, we write

$$f_N(x) \overset{\delta_N}{\approx} f'_N(x) \quad \text{uniformly over } x \in \mathbb{X}_N \quad \text{whenever} \quad \sup_{x \in \mathbb{X}_N} |f_N(x) - f'_N(x)| = O(\delta_N).$$

Let (δ'_N) be another sequence of positive numbers. For two sequences (W_N) and (W'_N) of Gaussian fields indexed by $\mathbb{X}_N$, we write

$$(3.35) \quad W_N(x) \overset{\delta_N, \delta'_N}{\approx} W'_N(x) \quad \text{uniformly over } x \in \mathbb{X}_N$$

whenever

$$\mathbb{E}W_N(x) \overset{\delta_N}{\approx} \mathbb{E}W'_N(x) \quad \text{and} \quad \text{Cov}(W_N(x), W_N(x')) \overset{\delta'_N}{\approx} \text{Cov}(W'_N(x), W'_N(x')),$$

uniformly over $x \in \mathbb{X}_N$ and over $(x, x') \in \mathbb{X}_N^2$ respectively. When δ_N and δ'_N tend to zero as N tends to infinity and we do not wish to keep track of the rates, we simply write $\approx$. The implicit constants in these relations are allowed to depend on the fixed parameters of the problem, including L when a cutoff is present, but not on N ; when the relation is claimed to hold uniformly over x, y, q , or b , the constants do not depend on these parameters either. Note that if $s_N \overset{\delta_N}{\approx} s'_N$ and $s'_N \overset{\delta'_N}{\approx} s''_N$, then $s_N \overset{\delta_N + \delta'_N}{\approx} s''_N$, and similarly for the other relations. Two remarks on the meaning of (3.35) are in order. First, the relation only compares the laws of the two fields, through their means and covariances; it does not assert a pathwise approximation, and in the estimates below a field may be replaced by any other field with the same law. Second, when the index set $\mathbb{X}_N$ contains a factor $\mathfrak{U}$, the fields are Gaussian conditionally on the cascade $\mathfrak{R}$, and the relation (3.35) is understood conditionally on $\mathfrak{R}$, with implicit constants that do not depend on its realization.

The first comparison lemma is the standard Gaussian interpolation, in a form that keeps track of the errors.

Lemma 3.4 (Gaussian comparison in covariance norm). *Let $\mathbb{P}$ be a probability measure on a measurable space $\mathbb{X}$, and let W^0 and W^1 be Gaussian fields indexed by $\mathbb{X}$, with bounded means m^0, m^1 and bounded covariances C^0, C^1 . Set $Z^i := \int_{\mathbb{X}} e^{W^i(x)} \mathbb{P}(dx)$, and let $\langle \cdot \rangle_i$ be the associated Gibbs bracket, for $i \in \{0, 1\}$. Then*

$$|\mathbb{E} \log Z^1 - \mathbb{E} \log Z^0| \leq \sup_{\mathbb{X}} |m^1 - m^0| + \sup_{\mathbb{X} \times \mathbb{X}} |C^1 - C^0|,$$

and, for every $n \in \mathbb{N}$, there is a constant $C_n < \infty$, depending only on n , such that, for every bounded measurable function f of n replicas,

$$|\mathbb{E} \langle f \rangle_1 - \mathbb{E} \langle f \rangle_0| \leq C_n \|f\|_{\infty} \left(\sup_{\mathbb{X}} |m^1 - m^0| + \sup_{\mathbb{X} \times \mathbb{X}} |C^1 - C^0| \right).$$

In particular, with the notation (3.35), if $W_N^0 \overset{\delta_N, \delta'_N}{\approx} W_N^1$ uniformly over $\mathbb{X}_N$, then

$$\mathbb{E} \log Z_N^0 \overset{\delta_N + \delta'_N}{\approx} \mathbb{E} \log Z_N^1 \quad \text{and} \quad \mathbb{E} \langle f \rangle_{0,N} \overset{\delta_N + \delta'_N}{\approx} \mathbb{E} \langle f \rangle_{1,N}.$$

Proof. Let $G^i := W^i - m^i$, and realize G^0 and G^1 as independent fields; this does not change the law of Z^0 , of Z^1 , nor of the individual brackets. For $u \in [0, 1]$, set $W^u := (1 - u)m^0 + um^1 + \sqrt{1 - u}G^0 + \sqrt{u}G^1$, and let Z^u and $\langle \cdot \rangle_u$ be the associated partition function and bracket. Gaussian integration by parts gives

$$\frac{d}{du} \mathbb{E} \log Z^u = \mathbb{E} \langle m^1(x) - m^0(x) \rangle_u + \frac{1}{2} \mathbb{E} \langle (C^1 - C^0)(x, x) \rangle_u - \frac{1}{2} \mathbb{E} \langle (C^1 - C^0)(x, x') \rangle_u,$$

where x, x' denote two replicas, and the first estimate follows by integration over $u \in [0, 1]$. Similarly, $\frac{d}{du} \mathbb{E} \langle f \rangle_u$ is a finite sum of Gibbs averages of products of f with $(m^1 - m^0)(x^\ell)$ or

$(C^1 - C^0)(x^\ell, x^{\ell'})$, for $\ell, \ell' \leq n + 2$, with coefficients depending only on n ; this gives the second estimate. The last assertion is a restatement of the first two. $\square$

The second lemma allows the difference between the two fields to grow quadratically in the cavity coordinates. It is tailored to the comparison of the $(N + M)$ -spin system in the chart with the N -spin system; see Lemmas 3.6 and 6.2.

Lemma 3.5 (Gaussian comparison with quadratic confinement). *For every $C_0 < \infty$ and $M \in \mathbb{N}$, the following holds. Let μ^0 be a probability measure on a measurable space $\mathbb{X}^0$, let $B \subseteq \mathbb{R}^M$ be a bounded measurable set, let ν be a probability measure on B with $\int |\tau|^2 \nu(d\tau) = M$, and set $\mathbb{X} := \mathbb{X}^0 \times B$ and $\mu := \mu^0 \otimes \nu$. Let W^0 and W^1 be Gaussian fields indexed by $z = (z_0, \tau) \in \mathbb{X}$, with means m^0, m^1 and covariances C^0, C^1 , such that $W^0(z_0, \tau)$ does not depend on τ , m^0 and C^0 are bounded, and*

$$(3.36) \quad |m^1(z) - m^0(z)| + |C^1(z, z') - C^0(z, z')| \leq C_0 (1 + |\tau|^2 + |\tau'|^2), \quad z, z' \in \mathbb{X}.$$

Let $Z^i := \int_{\mathbb{X}} e^{W^i(z)} \mu(dz)$ for $i \in \{0, 1\}$, and let $\langle \cdot \rangle_1$ be the Gibbs bracket associated with Z^1 . If

$$(3.37) \quad \mathbb{E} \langle |\tau|^2 \rangle_1 = M,$$

then $|\mathbb{E} \log Z^1 - \mathbb{E} \log Z^0| \leq 3C_0(1 + M)$.

Proof. Since B, m^0 , and C^0 are bounded, the fields have bounded means and variances on $\mathbb{X}$, so that Z^0 and Z^1 are almost surely finite with integrable logarithms. As in the previous proof, we take the centered fields $G^i := W^i - m^i$ to be independent. For $u \in [0, 1]$ and $c \geq 0$, we consider the interpolation with a quadratic confinement,

$$\Phi(u, c) := \mathbb{E} \log \int_{\mathbb{X}} \exp \left((1 - u)m^0(z) + um^1(z) + \sqrt{1 - u}G^0(z) + \sqrt{u}G^1(z) - c|\tau|^2 \right) \mu(dz),$$

and we denote by $\langle \cdot \rangle_{u,c}$ the associated Gibbs bracket. Gaussian integration by parts, as in the previous proof, together with (3.36), gives

$$(3.38) \quad |\partial_u \Phi(u, c)| \leq 3C_0 (1 + \mathbb{E} \langle |\tau|^2 \rangle_{u,c}) \quad \text{and} \quad \partial_c \Phi(u, c) = -\mathbb{E} \langle |\tau|^2 \rangle_{u,c}.$$

Set $c_0 := 3C_0$. Along the path $u \mapsto (u, c_0(1 - u))$, we obtain from (3.38) that

$$\frac{d}{du} \Phi(u, c_0(1 - u)) = \partial_u \Phi + c_0 \mathbb{E} \langle |\tau|^2 \rangle_{u, c_0(1 - u)} \geq -3C_0.$$

Moreover, since W^0 does not depend on τ and μ is a product measure, Jensen's inequality and the assumption on ν give

$$\Phi(0, c_0) - \Phi(0, 0) = \log \int_B e^{-c_0|\tau|^2} \nu(d\tau) \geq -c_0 M.$$

Combining the last two displays yields $\mathbb{E} \log Z^1 = \Phi(1, 0) \geq \Phi(0, c_0) - 3C_0 \geq \mathbb{E} \log Z^0 - c_0 M - 3C_0$. For the reverse inequality, we use the path $u \mapsto (u, c_0 u)$, along which (3.38) gives $\frac{d}{du} \Phi(u, c_0 u) \leq 3C_0$, so that $\Phi(1, c_0) \leq \Phi(0, 0) + 3C_0$. By Jensen's inequality and (3.37),

$$\Phi(1, 0) - \Phi(1, c_0) = \mathbb{E} \left[-\log \left(e^{-c_0|\tau|^2} \right)_1 \right] \leq c_0 \mathbb{E} \langle |\tau|^2 \rangle_1 = c_0 M.$$

Hence $\mathbb{E} \log Z^1 = \Phi(1, 0) \leq \mathbb{E} \log Z^0 + 3C_0 + c_0 M$, which completes the proof. $\square$

3.5. The cavity increment is bounded. We recall the partition function and the free energy from (3.23), and we define the cavity increment

$$(3.39) \quad A_N(x, y, q) := \mathbb{E} \log Z_{N+M}^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q} = -(N + M) \bar{F}_{N+M}^{x,y}(t, q) + N \bar{F}_N^{x,y}(t, q).$$

The main result of this section, Theorem 3.24, gives a lower bound on the $\liminf$ of A_N along suitable sequences. As a preliminary step, we show that A_N is bounded. The proof is a first instance of the use of the chart of Subsection 3.3, and it also introduces the estimates on overlaps that we use repeatedly below.

Lemma 3.6 (Boundedness of the cavity increment). *There exists a constant $C < \infty$ such that, for every sufficiently large N ,*

$$\sup_{x,y,q} |A_N(x,y,q)| \leq C.$$

Proof. The Hamiltonian $H_{N+M}^{x,y,q}$ of the $(N+M)$ -spin system is built with the coefficient η_{N+M} , whereas $H_N^{x,y,q}$ is built with η_N . We first compare $H_N^{x,y,q}$ with a version of the $(N+M)$ -spin Hamiltonian built with η_N , and then return to η_{N+M} .

Step 1: Overlaps in the chart.

Let $\rho = (S_{r_N(\tau)}\sigma, \tau)$ and $\rho' = (S_{r_N(\tau')}\sigma', \tau')$ with $(\sigma, \tau), (\sigma', \tau') \in \Sigma_N \times \mathcal{B}_N$. By (1.3), (3.7), and (3.31),

$$(3.40) \quad R_{N+M,s}(\rho, \rho') = \frac{N}{N+M} \sqrt{r_N^s(\tau)r_N^s(\tau')} R_{N,s}(\sigma, \sigma') + \frac{1}{N+M} \sum_{j \in \mathcal{M}_s} \tau_j \tau'_j.$$

Since $|\sqrt{a} - 1| \leq |a - 1|$ for $a \geq 0$ and $0 < r_N^s(\tau) \leq 2$, we have $|\sqrt{r_N^s(\tau)r_N^s(\tau')} - 1| \leq 2(|r_N^s(\tau) - 1| + |r_N^s(\tau') - 1|)$, and $N|r_N^s(\tau) - 1| \leq (M_s + |\tau^{(s)}|^2)/\lambda_s$ by (3.31) and (3.3). Using also $|R_{N,s}(\sigma, \sigma')| \leq \lambda_s$, we deduce from (3.40) that

$$(3.41) \quad (N+M) |R_{N+M}(\rho, \rho') - R_N(\sigma, \sigma')| \leq C(1 + |\tau|^2 + |\tau'|^2),$$

for a constant C depending only on M .

Step 2: The $(N+M)$ -spin system with coefficient η_N .

For $K \in \{N, N+M\}$, let $\widehat{H}_{K;N}^{x,y,q}$ be defined as $H_K^{x,y,q}$ in (3.19) and (3.22), except that η_K is replaced by η_N throughout, and let $\widehat{Z}_{K;N}^{x,y,q}$ be the associated partition function. Thus $\widehat{H}_{N;N}^{x,y,q} = H_N^{x,y,q}$. The mean and covariance of $\widehat{H}_{K;N}^{x,y,q}$ are given by (3.27) and (3.26), with N replaced by K in the prefactors and in the overlaps, and with η_N unchanged; we use here that $\lambda_K = \lambda$ by (3.4).

Let ρ, ρ' be as in Step 1. We claim that

$$(3.42) \quad \left| \mathbb{E} \widehat{H}_{N+M;N}^{x,y,q}(\rho, \alpha) - \mathbb{E} H_N^{x,y,q}(\sigma, \alpha) \right| \leq C(1 + |\tau|^2)$$

and

$$(3.43) \quad \left| \text{Cov}(\widehat{H}_{N+M;N}^{x,y,q}(\rho, \alpha), \widehat{H}_{N+M;N}^{x,y,q}(\rho', \alpha')) - \text{Cov}(H_N^{x,y,q}(\sigma, \alpha), H_N^{x,y,q}(\sigma', \alpha')) \right| \leq C(1 + |\tau|^2 + |\tau'|^2),$$

uniformly over $x, y, q, \alpha, \alpha' \in \mathcal{U}$, and the configurations. For (3.42), the first two terms in (3.27) are proportional to K with bounded coefficients, so the difference of their values at $K = N+M$ and $K = N$ is bounded by CM . For the external-field term, we note that $(S_{r_N(\tau)}\sigma)_i = \sqrt{r_N^s(\tau)}\sigma_i$ for $i \in I_{N,s}$, and that

$$\left| \left(\sqrt{r_N^s(\tau)} - 1 \right) \sum_{i \in I_{N,s}} \sigma_i \right| \leq N_s |r_N^s(\tau) - 1| \leq M_s + |\tau^{(s)}|^2, \quad \left| \sum_{j \in \mathcal{M}_s} \tau_j \right| \leq \sqrt{M_s} |\tau^{(s)}|,$$

by the Cauchy-Schwarz inequality and (3.31). This proves (3.42). For (3.43), each term in (3.26) is of the form $Kg(R_K, u)$, where g is one of the kernels $2t\xi_{\eta_N, y}$, $2\sum_s (1 + \eta_N y^s)^2 q^s(u) v_s$, and $\eta_N^2 \sum_h x_h^2 c_h^2(\mathcal{C}_h)_{\eta_N, y}$, and R_K stands for $R_{N+M}(\rho, \rho')$ or $R_N(\sigma, \sigma')$. These kernels and their gradients in v are bounded uniformly over x, y, q on the range of overlaps, by the remark following (3.27). Writing

$$(N+M)g(R_{N+M}, u) - Ng(R_N, u) = Mg(R_{N+M}, u) + N(g(R_{N+M}, u) - g(R_N, u))$$

and using (3.41) for the second term proves (3.43).

We now apply Lemma 3.5 with $\mathbb{X}^0 := \Sigma_N \times \mathcal{U}$, $\mu^0 := P_N \otimes \mathfrak{R}$, $B := \mathcal{B}_N$, $\nu := p_N^{\text{ch}}(\tau) d\tau$, and the fields

$$W^0(\sigma, \tau, \alpha) := H_N^{x,y,q}(\sigma, \alpha), \quad W^1(\sigma, \tau, \alpha) := \widehat{H}_{N+M;N}^{x,y,q}((S_{r_N(\tau)}\sigma, \tau), \alpha).$$

The set $\mathcal{B}_N$ is bounded, ν is a probability measure on $\mathcal{B}_N$ with second moment M by (3.33), and (3.36) holds by (3.42) and (3.43). By (3.32), we have $Z^0 = Z_N^{x,y,q}$ and $Z^1 = \widehat{Z}_{N+M;N}^{x,y,q}$, and the bracket $\langle \cdot \rangle_1$ is the Gibbs bracket of $\widehat{H}_{N+M;N}^{x,y,q}$ expressed in the chart. It remains to verify (3.37).

The mean and covariance of $\widehat{H}_{N+M;N}^{x,y,q}$ depend on the configurations only through their species overlaps and the sums $\sum_{i \in I_{N+M,s}} \rho_i$, which are invariant under permutations of the coordinates within each species. The law of the field is therefore invariant under such permutations, and so is $\mathbb{E} \langle \rho_i^2 \rangle_1$ as a function of $i \in I_{N+M,s}$. Since $\sum_{i \in I_{N+M,s}} \rho_i^2 = N_s + M_s$ on Σ_{N+M} , we get $\mathbb{E} \langle \rho_i^2 \rangle_1 = 1$ for every i , hence $\mathbb{E} \langle |\tau^{(s)}|^2 \rangle_1 = M_s$ and $\mathbb{E} \langle |\tau|^2 \rangle_1 = M$. Lemma 3.5 thus yields

$$(3.44) \quad \sup_{x,y,q} |\mathbb{E} \log \widehat{Z}_{N+M;N}^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q}| \leq C.$$

Step 3: Back to the coefficient η_{N+M} .

Set $K := N + M$. The fields $H_K^{x,y,q}$ and $\widehat{H}_{K;N}^{x,y,q}$ are built from the same Gaussian fields H_K , W_K^q , and $(H_K^h)_h$; they differ only in that η_K is replaced by η_N . By (3.27), their means differ by

$$(\eta_K - \eta_N) \left(-2KtD_y \xi(\lambda) - 2K \sum_{s \in \mathcal{S}} q^s(1) \lambda_s y^s + \sum_{s \in \mathcal{S}} h^s y^s \sum_{i \in I_{K,s}} \rho_i \right),$$

which is bounded by $CK|\eta_K - \eta_N|$ in absolute value. By (3.25), for every kernel f ,

$$f_{\eta_K, y} - f_{\eta_N, y} = (\eta_K - \eta_N) (2D_y f + (\eta_K + \eta_N) D_y^2 f),$$

and $|\eta_K^2 - \eta_N^2| \leq 2|\eta_K - \eta_N|$. Applying this to the three terms in (3.26), and using (3.18) to sum over h , we find that the covariances of $H_K^{x,y,q}$ and $\widehat{H}_{K;N}^{x,y,q}$ differ by at most $CK|\eta_K - \eta_N|$, uniformly over x, y, q and over $\Sigma_K \times \mathfrak{U}$. Lemma 3.4 therefore gives

$$(3.45) \quad \sup_{x,y,q} |\mathbb{E} \log Z_{N+M}^{x,y,q} - \mathbb{E} \log \widehat{Z}_{N+M;N}^{x,y,q}| \leq CN|\eta_{N+M} - \eta_N|,$$

and the right-hand side tends to zero, since $N|\eta_{N+M} - \eta_N| \leq MN^{-1/16}$ by (3.9). Combining (3.39), (3.44), and (3.45) proves the lemma. $\square$

3.6. The bulk system and the expansion of the Hamiltonian in the chart. We now begin the cavity computation proper. Throughout the rest of the section, we work in the chart of Subsection 3.3: a configuration $\rho \in \Sigma_{N+M}$ is written as $\rho = (S_{r_N(\tau)}\sigma, \tau)$ with $(\sigma, \tau) \in \Sigma_N \times \mathcal{B}_N$, and we often write $\tilde{\sigma} := S_{r_N(\tau)}\sigma$ for brevity.

3.6.1. Cutoff on the cavity coordinates. For each $L \geq 1$, we fix a continuous function $\chi_L : \mathbb{R}^M \rightarrow [0, 1]$ such that

$$(3.46) \quad \mathbb{1}_{\{|\tau| \leq L\}} \leq \chi_L(\tau) \leq \mathbb{1}_{\{|\tau| \leq L+1\}}, \quad \chi_L \leq \chi_{L'} \quad \text{if } L \leq L',$$

and we set $\chi_\infty := 1$. The partition function and Gibbs bracket of the $(N+M)$ -spin system with cutoff L are

$$(3.47) \quad \begin{aligned} Z_{N+M,L}^{x,y,q} &:= \iiint_{\Sigma_N \times \mathcal{B}_N \times \mathfrak{U}} \chi_L(\tau) \exp(H_{N+M}^{x,y,q}((S_{r_N(\tau)}\sigma, \tau), \alpha)) p_N^{\text{ch}}(\tau) d\tau P_N(d\sigma) \mathfrak{R}(d\alpha), \\ \langle f \rangle_{N+M,L}^{x,y,q} &:= \frac{1}{(Z_{N+M,L}^{x,y,q})^n} \int f((\rho^\ell, \alpha^\ell)_{\ell \leq n}) \prod_{\ell=1}^n \chi_L(\tau^\ell) e^{H_{N+M}^{x,y,q}(\rho^\ell, \alpha^\ell)} p_N^{\text{ch}}(\tau^\ell) d\tau^\ell P_N(d\sigma^\ell) \mathfrak{R}(d\alpha^\ell), \end{aligned}$$

where $\rho^\ell = (S_{r_N(\tau^\ell)}\sigma^\ell, \tau^\ell)$ and f is a bounded measurable function of n replicas. By (3.32), the bracket $\langle \cdot \rangle_{N+M,\infty}^{x,y,q}$ is the bracket $\langle \cdot \rangle_{N+M}^{x,y,q}$ in (3.23), and $\langle \cdot \rangle_{N+M,L}^{x,y,q}$ is the same Gibbs measure conditioned, in each replica, by the factor $\chi_L(\tau)$. Since $0 \leq \chi_L \leq 1$, we also have

$$(3.48) \quad Z_{N+M}^{x,y,q} = Z_{N+M,\infty}^{x,y,q} \geq Z_{N+M,L}^{x,y,q}.$$

This inequality is the source of the one-sided nature of our cavity estimate.

3.6.2. *The bulk system.* Let $\rho = (u, \tau)$ and $\rho' = (u', \tau')$ with $u, u' \in \mathbb{R}^N$ and $\tau, \tau' \in \mathbb{R}^M$. By the definition (1.3) of the overlaps and the identification (3.3),

$$(3.49) \quad R_{N+M}(\rho, \rho') = \mathbf{a}_N(u, u') + \mathbf{b}_N(\tau, \tau'), \quad \text{where} \\ \mathbf{a}_N(u, u') := \frac{N}{N+M} R_N(u, u') \quad \text{and} \quad \mathbf{b}_N(\tau, \tau') := \frac{1}{N+M} \left(\sum_{j \in \mathbf{M}_s} \tau_j \tau'_j \right)_{s \in \mathcal{S}}.$$

The covariance of H_{N+M} is therefore $(N+M)\xi(\mathbf{a}_N + \mathbf{b}_N)$. When τ and τ' are bounded, $\mathbf{b}_N$ is of order N^{-1} , and the leading term $(N+M)\xi(\mathbf{a}_N)$ only involves the bulk coordinates. This leads us to define the bulk covariance function

$$(3.50) \quad \tilde{\xi}_N(v) := \frac{N+M}{N} \xi\left(\frac{N}{N+M}v\right), \quad v \in \mathbb{R}^{\mathcal{S}},$$

and to let $\tilde{H}_N$ be a centered Gaussian field on $\mathbb{R}^N$ with covariance

$$(3.51) \quad \mathbb{E} \tilde{H}_N(u) \tilde{H}_N(u') = N \tilde{\xi}_N(R_N(u, u')) = (N+M)\xi(\mathbf{a}_N(u, u')), \quad u, u' \in \mathbb{R}^N.$$

The function $\tilde{\xi}_N$ is obtained from ξ by multiplying the coefficient of each monomial of total degree d in (1.4) by $(N/(N+M))^{d-1} \in (0, 1]$. Hence $\tilde{\xi}_N$ is again a covariance function of the form (1.4), so that the field $\tilde{H}_N$ exists on $\mathbb{R}^N$, as was the case for H_N , and $\tilde{\xi}_N$ together with its derivatives of order at most four converge to those of ξ , uniformly on $\prod_{s \in \mathcal{S}} [-(1+\delta_0)\lambda_s, (1+\delta_0)\lambda_s]$, as $N \rightarrow \infty$. In other words, the bulk system is the N -spin system with ξ replaced by $\tilde{\xi}_N$, and every estimate that we have obtained so far for the N -spin system applies to the bulk system, uniformly in N .

The fields H_N and $\tilde{H}_N$ differ by a field whose covariance is of order one. Explicitly, set

$$(3.52) \quad \theta_N(v) := \frac{N}{M} (\xi(v) - \tilde{\xi}_N(v)) = \frac{1}{M} \left[N\xi(v) - (N+M)\xi\left(\frac{N}{N+M}v\right) \right].$$

The coefficient of a monomial of degree d in θ_N is the corresponding coefficient in ξ multiplied by $\frac{N}{M} [1 - (\frac{N}{N+M})^{d-1}] \geq 0$, which converges to $d-1$ as $N \rightarrow \infty$. Since $\theta(v) = v \cdot \nabla \xi(v) - \xi(v)$ is obtained from ξ by multiplying the coefficient of each monomial of degree d by $d-1$, the absolute convergence in (1.4) gives

$$(3.53) \quad \lim_{N \rightarrow \infty} \sup_{v \in \prod_{s \in \mathcal{S}} [-\lambda_s, \lambda_s]} |\theta_N(v) - \theta(v)| = 0,$$

and θ_N is a valid covariance function, again of the form (1.4). We let $\tilde{Y}_N$ be a centered Gaussian field on $\mathbb{R}^N$, independent of $\tilde{H}_N$, with covariance $\mathbb{E} \tilde{Y}_N(u) \tilde{Y}_N(u') = \theta_N(R_N(u, u'))$, and we henceforth realize H_N as

$$(3.54) \quad H_N(u) = \tilde{H}_N(u) + \sqrt{M} \tilde{Y}_N(u), \quad u \in \mathbb{R}^N,$$

which is legitimate since the right-hand side has the covariance (1.5). This provides a coupling between the N -spin system and the bulk system, which includes the radial derivatives of the two fields. The field $\tilde{Y}_N$ will produce the “reaction term” $\mathcal{I}_{M,t}(p)$ in (3.116) below.

The Hamiltonian of the bulk system is defined as in (3.20)–(3.22), with H_N replaced by $\tilde{H}_N$. That is, for $r \in U_0$, we set

$$(3.55) \quad \tilde{H}_N^{x,q}(r; \sigma, \alpha) := \sqrt{2t} \tilde{H}_N(S_r \sigma) - Nt \tilde{\xi}_N(R_N(S_r \sigma, S_r \sigma)) + \sqrt{2} W_N^q(S_r \sigma, \alpha) - \sum_{s \in \mathcal{S}} q^s(1) |(S_r \sigma)^{(s)}|^2 \\ + \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} (S_r \sigma)_i + \eta_N H_N^x(S_r \sigma, \alpha), \quad \tilde{H}_N^{x,q}(\sigma, \alpha) := \tilde{H}_N^{x,q}(\bar{\mathbf{1}}; \sigma, \alpha),$$

and

$$(3.56) \quad \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha) := \frac{2}{N_s} \partial_{r,s} \tilde{H}_N^{x,q}(r; \sigma, \alpha) \Big|_{r=\bar{\mathbf{1}}}, \quad \tilde{V}_N^{x,y,q}(\sigma, \alpha) := \eta_N \sum_{s \in \mathcal{S}} y^s N_s \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha), \\ \tilde{H}_N^{x,y,q}(\sigma, \alpha) := \tilde{H}_N^{x,q}(\sigma, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha).$$

The partition function, free energy, and Gibbs bracket of the bulk system are

$$(3.57) \quad \begin{aligned} Z_N^{x,y,q} &:= \iint e^{\tilde{H}_N^{x,y,q}(\sigma,\alpha)} P_N(d\sigma) \mathfrak{R}(d\alpha), \quad \tilde{F}_N^{x,y}(t,q) := -\frac{1}{N} \mathbb{E} \log Z_N^{x,y,q}, \\ \langle f \rangle_N^{x,y,q} &:= \frac{1}{(Z_N^{x,y,q})^n} \int f((\sigma^\ell, \alpha^\ell)_{\ell \leq n}) \prod_{\ell=1}^n e^{\tilde{H}_N^{x,y,q}(\sigma^\ell, \alpha^\ell)} P_N(d\sigma^\ell) \mathfrak{R}(d\alpha^\ell). \end{aligned}$$

The mean and covariance of $\tilde{H}_N^{x,y,q}$ are given by (3.27) and (3.26) with ξ replaced by $\tilde{\xi}_N$. In particular, by Lemma 3.2 and the uniform bounds on the kernels and their derivatives, we have, for every radial multi-index ν with $1 \leq |\nu| \leq 2$,

$$(3.58) \quad \partial_r^\nu \tilde{H}_N^{x,q}(r; \sigma, \alpha) \stackrel{N,N}{\approx} 0 \quad \text{uniformly over } x, q, r \in U_0, \sigma \in \Sigma_N, \text{ and } \alpha \in \mathfrak{U},$$

that is, these derivatives have means and covariances of order N . In view of the normalizations in (3.56), this implies

$$(3.59) \quad \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha) \stackrel{1,N^{-1}}{\approx} 0 \quad \text{and} \quad \tilde{V}_N^{x,y,q}(\sigma, \alpha) \stackrel{N\eta_N, N\eta_N^2}{\approx} 0,$$

uniformly over $x, y, q, \sigma \in \Sigma_N$, and $\alpha \in \mathfrak{U}$. The same estimates hold for $\mathcal{E}_{N,s}^{x,q}$ and $V_N^{x,y,q}$ in (3.21)–(3.22):

$$(3.60) \quad \mathcal{E}_{N,s}^{x,q}(\sigma, \alpha) \stackrel{1,N^{-1}}{\approx} 0 \quad \text{and} \quad V_N^{x,y,q}(\sigma, \alpha) \stackrel{N\eta_N, N\eta_N^2}{\approx} 0.$$

Applying Lemma 3.4 to the fields $H_N^{x,y,q}$ and $H_N^{x,0,q} = H_N^{x,q}$, we deduce from (3.60) that the radial perturbation has a negligible effect on the free energy:

$$(3.61) \quad \sup_{x,y,q} \left| \overline{F}_N^{x,y}(t,q) - \overline{F}_N^{x,0}(t,q) \right| \leq C\eta_N.$$

Similarly, since the variance of $\eta_N H_N^x$ is at most $9N\eta_N^2$, Lemma 3.4 gives

$$(3.62) \quad \sup_{x,q} \left| \overline{F}_N^{x,0}(t,q) - \overline{F}_N(t,q) \right| \leq C\eta_N^2.$$

3.6.3. Decomposition of the $(N+M)$ -spin Hamiltonian. We now decompose each of the Gaussian fields entering $H_{N+M}^{x,q}$ into a bulk part, a part linear in the cavity coordinates, and a remainder. We start with the interaction field. In view of Taylor's formula applied to the covariance $(N+M)\xi(\mathbf{a}_N + \mathbf{b}_N)$ in (3.49), we let $(\tilde{Z}_{N,j})_{j \in [M]}$ and $\tilde{D}_N$ be centered Gaussian fields, indexed by $u \in \mathbb{R}^N$ and by $(u, \tau) \in \mathbb{R}^N \times \mathbb{R}^M$ respectively, independent of one another and of $\tilde{H}_N$, with covariances

$$(3.63) \quad \mathbb{E} \tilde{Z}_{N,j}(u) \tilde{Z}_{N,j'}(u') = \mathbb{1}_{\{j=j'\}} \partial_s \xi(\mathbf{a}_N(u, u')), \quad \text{for } j \in \mathbf{M}_s,$$

$$(3.64) \quad \mathbb{E} \tilde{D}_N(u, \tau) \tilde{D}_N(u', \tau') = (N+M) (\xi(\mathbf{a}_N + \mathbf{b}_N) - \xi(\mathbf{a}_N) - \mathbf{b}_N \cdot \nabla \xi(\mathbf{a}_N)),$$

where $\mathbf{a}_N = \mathbf{a}_N(u, u')$ and $\mathbf{b}_N = \mathbf{b}_N(\tau, \tau')$. The right-hand side of (3.64) is the sum of the monomials of $(N+M)\xi(\mathbf{a}_N + \mathbf{b}_N)$ that contain at least two factors from $\mathbf{b}_N$; since $\mathbf{a}_{N,s}$ and $\mathbf{b}_{N,s}$ are nonnegative definite kernels and the coefficients of ξ are nonnegative, it is a nonnegative definite kernel by the Schur product theorem, and the field $\tilde{D}_N$ exists. Comparing covariances, we see that the field

$$(3.65) \quad \tilde{H}_N(u) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \tilde{Z}_{N,j}(u) + \tilde{D}_N(u, \tau)$$

has the same law as $H_{N+M}((u, \tau))$. We henceforth realize H_{N+M} through (3.65); together with (3.54), this couples the three fields H_{N+M} , $\tilde{H}_N$, and H_N .

The cascade field and the external field are linear in ρ , and thus decompose exactly. By (2.11) and (3.3),

$$(3.66) \quad W_{N+M}^q((u, \tau), \alpha) = W_N^q(u, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j w_j^{q^s}(\alpha),$$

where $(w_j^{q^s})_{j \in \mathbf{M}_s}$ are the cascade fields attached to the cavity coordinates, which are independent of W_N^q ; and $\sum_s h^s \sum_{i \in I_{N+M,s}} \rho_i = \sum_s h^s \sum_{i \in I_{N,s}} u_i + \sum_s h^s \sum_{j \in \mathbf{M}_s} \tau_j$.

Finally, we couple the perturbation fields H_{N+M}^h and H_N^h through the construction (3.16). Regarding $\mathbb{R}^{I_{N,s}}$ as the subspace of $\mathbb{R}^{I_{N+M,s}}$ of vectors vanishing on the cavity coordinates $\mathbf{M}_s$, we have $E_{N,j} \subseteq E_{N+M,j}$ isometrically for every j ; we choose the orthonormal basis $\mathcal{E}_{N+M,j}$ so that it contains $\mathcal{E}_{N,j}$, and we build both fields from the same cascade fields $(w_e^{h,j})_{e \in \mathcal{E}_{N+M,j}}$, for every h and j . For $\rho = (u, \tau)$ with $u \in \mathbb{R}^N$ and $\tau \in \mathbb{R}^M$, and for $u' \in \mathbb{R}^N$, only the bulk coordinates contribute to the inner product

$$\langle \varphi_{N+M,s}(\rho), \varphi_{N,s}(u') \rangle = \frac{1}{\sqrt{N(N+M)}} \sum_{i \in I_{N,s}} u_i u'_i = \sqrt{\frac{N}{N+M}} R_{N,s}(u, u'),$$

so that $\langle \Phi_{N+M}^{h,j}(\rho), \Phi_N^{h,j}(u') \rangle = (a_{h_1} \cdot (\sqrt{N/(N+M)} R_N(u, u'))^{\odot h_2})^{h_4-j}$, while the fields $w_e^{h,j}$ contribute the factors $\iota_{h_3}^j(\alpha \wedge \alpha')^j$ as before. The binomial formula (3.13) therefore gives

$$(3.67) \quad \mathbb{E} \left[H_{N+M}^h((u, \tau), \alpha) H_N^h(u', \alpha') \right] = (N(N+M))^{1/2} C_h \left(\sqrt{\frac{N}{N+M}} R_N(u, u'), \alpha \wedge \alpha' \right),$$

for every $u, u' \in \mathbb{R}^N$, $\tau \in \mathbb{R}^M$, and $\alpha, \alpha' \in \mathfrak{U}$. Note that $(N(N+M))^{1/2} = N + O(1)$ and that $\sqrt{N/(N+M)} = 1 + O(N^{-1})$. This cross-covariance therefore differs by $O(\Lambda_h)$ from the covariance $N C_h(R_N(u, u'), \alpha \wedge \alpha')$ of H_N^h , and the same is true of the covariance $(N+M) C_h(R_{N+M}(\rho, \rho'), \alpha \wedge \alpha')$ of H_{N+M}^h when the cavity coordinates are bounded, by (3.41). Since the perturbation carries the prefactor η_N , these differences are negligible.

3.6.4. The cavity field and the expansion in the chart. The linear parts in (3.65) and (3.66) combine, together with the external field, into the field that drives the cavity coordinates. Its bulk part $\tilde{Z}_{N,j}$ still depends on N through $\mathbf{a}_N$, and we replace it by an N -independent version. Let $(Z_{N,j})_{j \in [M]}$ be centered Gaussian fields on Σ_N , independent over j and independent of all the other fields, such that

$$(3.68) \quad \mathbb{E} Z_{N,j}(\sigma) Z_{N,j}(\sigma') = \partial_s \xi(R_N(\sigma, \sigma')), \quad j \in \mathbf{M}_s,$$

and set

$$(3.69) \quad X_{N,j}^q(\sigma, \alpha) := h^s + \sqrt{2t} Z_{N,j}(\sigma) + \sqrt{2} w_j^q(\alpha), \quad s \in \mathcal{S}, \quad j \in \mathbf{M}_s.$$

The fields $(X_{N,j}^q)_{j \in [M]}$ are independent over j and independent of the fields entering $\tilde{H}_N^{x,y,q}$. We also introduce the coefficients

$$(3.70) \quad \vartheta^s := q^s(1) + t \partial_s \xi(\lambda), \quad s \in \mathcal{S},$$

which collect the quadratic terms in τ produced by the self-overlap corrections in the Hamiltonian: the term $-Nq(1) \cdot \lambda_N$ in (2.13) at size $N+M$ exceeds its value at size N by $-\sum_s q^s(1) |\tau^{(s)}|^2$ once the bulk is evaluated at the radius $r_N(\tau)$, and similarly for $-Nt\xi(\lambda_N)$, up to negligible terms. Note that ϑ does not depend on N , thanks to (3.4).

Lemma 3.7 (Expansion of the Hamiltonian in the chart). *Fix $L \geq 1$. Under the couplings described above, we have*

$$(3.71) \quad \begin{aligned} & H_{N+M}^{x,q}((S_{r_N(\tau)}\sigma, \tau), \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha) \\ & \approx \tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j X_{N,j}^q(\sigma, \alpha) - \sum_{s \in \mathcal{S}} \vartheta^s |\tau^{(s)}|^2, \end{aligned}$$

uniformly over x, y, q and $(\sigma, \tau, \alpha) \in \Sigma_N \times \{|\tau| \leq L+1\} \times \mathfrak{U}$.

Proof. Write $\tilde{\sigma} := S_{r_N(\tau)}\sigma$ and $\rho := (\tilde{\sigma}, \tau)$, and recall from (3.34) that $|r_N(\tau) - \bar{\mathbf{1}}| \leq CN^{-1}$ on the range of τ considered. We go through the terms of $H_{N+M}^{x,q}(\rho, \alpha)$ in (3.19) and (2.13), at size $N+M$. The term $\tilde{V}_N^{x,y,q}$ is the same on both sides of (3.71); since it is not independent of the other fields, we must also keep track of its cross-covariances with the terms that we modify.

Interaction field. By (3.65),

$$\sqrt{2t} H_{N+M}(\rho) = \sqrt{2t} \tilde{H}_N(\tilde{\sigma}) + \sqrt{2t} \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \tilde{Z}_{N,j}(\tilde{\sigma}) + \sqrt{2t} \tilde{D}_N(\tilde{\sigma}, \tau).$$

The first term is the interaction part of $\tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha)$ in (3.55). The last two terms are independent of all the other fields. By (3.64) and (3.49), the variance of $\tilde{D}_N(\tilde{\sigma}, \tau)$ is bounded by $(N+M) \sup |D^2 \xi| |\mathbf{b}_N(\tau, \tau)|^2 \leq C_L N^{-1}$, so that $\tilde{D}_N(\tilde{\sigma}, \tau) \stackrel{0, N^{-1}}{\approx} 0$. By (3.63), (3.7), and the Lipschitz continuity of $\nabla \xi$, the covariance of $\tilde{Z}_{N,j}(\tilde{\sigma})$ and $\tilde{Z}_{N,j}(\tilde{\sigma}')$ is

$$\partial_s \xi \left(\left(\frac{N}{N+M} \sqrt{r_N^u(\tau) r_N^u(\tau')} R_{N,u}(\sigma, \sigma') \right)_{u \in \mathcal{S}} \right) = \partial_s \xi(R_N(\sigma, \sigma')) + O(N^{-1}),$$

uniformly over the range considered, and the same holds after multiplication by $\tau_j \tau'_j$. Comparing with (3.68), we obtain

$$\sqrt{2t} \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \tilde{Z}_{N,j}(\tilde{\sigma}) + \sqrt{2t} \tilde{D}_N(\tilde{\sigma}, \tau) \stackrel{0, N^{-1}}{\approx} \sqrt{2t} \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j Z_{N,j}(\sigma),$$

and both sides are independent of all the other fields.

Self-overlap corrections. Since $\rho \in \Sigma_{N+M}$, the interaction correction in $H_{N+M}^{x,q}$ is $-(N+M)t\xi(\lambda)$. On the other hand, by (3.7) and (3.31), $R_{N,s}(\tilde{\sigma}, \tilde{\sigma}) = r_N^s(\tau)\lambda_s$, so that $\mathbf{a}_N(\tilde{\sigma}, \tilde{\sigma}) = \lambda - (|\tau^{(s)}|^2/(N+M))_{s \in \mathcal{S}}$. The interaction correction in $\tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha)$ is therefore

$$\begin{aligned} -Nt\tilde{\xi}_N(R_N(\tilde{\sigma}, \tilde{\sigma})) &= -(N+M)t\xi \left(\lambda - \left(\frac{|\tau^{(s)}|^2}{N+M} \right)_{s \in \mathcal{S}} \right) \\ &= -(N+M)t\xi(\lambda) + t \sum_{s \in \mathcal{S}} \partial_s \xi(\lambda) |\tau^{(s)}|^2 + O_L(N^{-1}), \end{aligned}$$

by Taylor's formula. Similarly, the cascade correction at size $N+M$ is $-\sum_s q^s(1)(N_s + M_s)$, while that in $\tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha)$ is $-\sum_s q^s(1)|\tilde{\sigma}^{(s)}|^2 = -\sum_s q^s(1)(N_s + M_s - |\tau^{(s)}|^2)$. Taking the difference of the two sides and recalling (3.70), the deterministic corrections account for the term $-\sum_s q^s |\tau^{(s)}|^2$ in (3.71), up to an error $O_L(N^{-1})$.

Cascade and external fields. By (3.66), $\sqrt{2}W_{N+M}^q(\rho, \alpha) = \sqrt{2}W_N^q(\tilde{\sigma}, \alpha) + \sqrt{2}\sum_s \sum_{j \in \mathbf{M}_s} \tau_j w_j^s(\alpha)$, where the first term is the cascade part of $\tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha)$ and the second is the cascade part of $\sum_j \tau_j \mathbf{X}_{N,j}^q$. The external field decomposes in the same way, producing the terms $h^s \tau_j$ in $\mathbf{X}_{N,j}^q$. These decompositions are exact.

Perturbation field. It remains to compare $\eta_{N+M} H_{N+M}^x(\rho, \alpha)$ with $\eta_N H_N^x(\tilde{\sigma}, \alpha)$, which is the perturbation part of $\tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha)$. Both are independent of all the fields considered so far, but not of $\tilde{V}_N^{x,y,q}$, which contains the term $2\eta_N^2 \sum_s y^s \partial_{r^s} H_N^x(S_r \sigma, \alpha)|_{r=\tilde{\mathbf{1}}}$. By (3.12), (3.67), and Lemma 3.2, all the relevant covariances are obtained by evaluating and differentiating the kernels C_h at overlaps that differ from $R_N(\sigma, \sigma')$ by $O_L(N^{-1})$, by (3.41) and (3.7), with prefactors $(N+M)\eta_{N+M}^2$, $(N(N+M))^{1/2}\eta_{N+M}\eta_N$, or $N\eta_N^2$, which differ from one another by $O(N|\eta_{N+M} - \eta_N| + \eta_N^2)$. Using (3.18) to sum over h , we find that replacing $\eta_{N+M} H_{N+M}^x(\rho, \alpha)$ by $\eta_N H_N^x(\tilde{\sigma}, \alpha)$ changes the variance by $O(N|\eta_{N+M} - \eta_N| + \eta_N^2)$, and the cross-covariance with $\tilde{V}_N^{x,y,q}$ by $O(\eta_N(N|\eta_{N+M} - \eta_N| + \eta_N^2))$. Both quantities vanish as $N \rightarrow \infty$, by (3.9). The means are zero.

Collecting the four steps, the means of the two sides of (3.71) differ by $O_L(N^{-1})$ and their covariances by $o(1)$, uniformly over the stated range. This is (3.71). $\square$

3.7. Radial linearization and replacement of the radial perturbation. The right-hand side of (3.71) still evaluates the bulk Hamiltonian at the random radius $r_N(\tau)$. Since $r_N(\tau) - \tilde{\mathbf{1}}$ is of order N^{-1} , while the radial derivatives of the Hamiltonian have means and covariances of order N by (3.58), the first-order term of the Taylor expansion in the radius is of order one and the second-order remainder is negligible; the first-order term is expressed through the normalized radial derivatives $\tilde{\mathcal{E}}_{N,s}^{x,q}$. We then show that the radial perturbation of the $(N+M)$ -spin system can be replaced by that of the bulk system.

Lemma 3.8 (First-order radial expansion). *Fix $L \geq 1$. We have*

$$(3.72) \quad \tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha) \stackrel{N^{-1}, N^{-1}}{\approx} \tilde{H}_N^{x,y,q}(\sigma, \alpha) + \frac{1}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha)$$

uniformly over x, y, q and $(\sigma, \tau, \alpha) \in \Sigma_N \times \{|\tau| \leq L + 1\} \times \mathfrak{U}$. The relation remains valid, with the same rates, if a Gaussian field independent of $\tilde{H}_N$, W_N^q , and H_N^x is added to both sides.

Proof. Set $\delta_s(\tau) := r_N^s(\tau) - 1 = (M_s - |\tau^{(s)}|^2)/N_s$, so that $|\delta_s(\tau)| \leq C_L N^{-1}$ by (3.34). Taylor's formula in the radial variable gives

$$\begin{aligned} \tilde{H}_N^{x,q}(r_N(\tau); \sigma, \alpha) &= \tilde{H}_N^{x,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \delta_s(\tau) \partial_{r^s} \tilde{H}_N^{x,q}(r; \sigma, \alpha) \Big|_{r=\bar{\mathbf{1}}} + \varepsilon_N(\sigma, \tau, \alpha), \\ \varepsilon_N(\sigma, \tau, \alpha) &:= \sum_{s, u \in \mathcal{S}} \delta_s(\tau) \delta_u(\tau) \int_0^1 (1-v) \partial_{r^s} \partial_{r^u} \tilde{H}_N^{x,q}(\bar{\mathbf{1}} + v\delta(\tau); \sigma, \alpha) dv. \end{aligned}$$

By the definition (3.56) of $\tilde{\mathcal{E}}_{N,s}^{x,q}$, the first-order term is $\frac{1}{2} \sum_s (M_s - |\tau^{(s)}|^2) \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha)$. By (3.58) and the bound on $\delta(\tau)$, the remainder satisfies $\varepsilon_N \stackrel{N^{-1}, N^{-3}}{\approx} 0$. The field on the right-hand side of (3.72) has variance at most CN , by (3.59) and the covariance formula of $\tilde{H}_N^{x,q}$; the Cauchy-Schwarz inequality therefore bounds its covariance with ε_N by $C_L N^{-1}$. This proves (3.72). A field independent of $\tilde{H}_N$, W_N^q , and H_N^x is independent of ε_N , which gives the last assertion. $\square$

We next compare the radial perturbations $V_{N+M}^{x,y,q}$ and $V_N^{x,y,q}$ in (3.22) with the bulk radial perturbation $\tilde{V}_N^{x,y,q}$ in (3.56). Recall that these three fields are, respectively, $2\eta_{N+M}$, $2\eta_N$, and $2\eta_N$ times a y -weighted sum of radial derivatives of $H_{N+M}^{x,q}$, $H_N^{x,q}$, and $\tilde{H}_N^{x,q}$. The radial derivative in $V_{N+M}^{x,y,q}$ acts on all the coordinates of species s , including the cavity ones. The fields H_{N+M} , H_N , and $\tilde{H}_N$, together with their radial derivatives, are coupled through (3.65) and (3.54); the cascade and perturbation fields are coupled through (3.66) and (3.67).

Lemma 3.9 (Replacement of the radial perturbation). *Fix $L \geq 1$, and write $\rho := (S_{r_N(\tau)}\sigma, \tau)$. Under the couplings above, we have*

$$(3.73) \quad H_{N+M}^{x,y,q}(\rho, \alpha) = H_{N+M}^{x,q}(\rho, \alpha) + V_{N+M}^{x,y,q}(\rho, \alpha) \approx H_{N+M}^{x,q}(\rho, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha)$$

uniformly over x, y, q and $(\sigma, \tau, \alpha) \in \Sigma_N \times \{|\tau| \leq L + 1\} \times \mathfrak{U}$, and

$$(3.74) \quad H_N^{x,y,q}(\sigma, \alpha) = H_N^{x,q}(\sigma, \alpha) + V_N^{x,y,q}(\sigma, \alpha) \approx H_N^{x,q}(\sigma, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha)$$

uniformly over x, y, q and $(\sigma, \alpha) \in \Sigma_N \times \mathfrak{U}$.

Proof. We first replace η_{N+M} by η_N in $V_{N+M}^{x,y,q}$. By (3.60) at size $N+M$, the field $V_{N+M}^{x,y,q}/\eta_{N+M}$ has mean and covariance of order N , and its covariance with $H_{N+M}^{x,q}$ is of order N as well. Multiplying it by η_N instead of η_{N+M} therefore changes the mean, the variance, and the covariance with $H_{N+M}^{x,q}$ of the left-hand side of (3.73) by at most $CN|\eta_{N+M} - \eta_N|$, which vanishes by (3.9).

Both sides of (3.73) and (3.74) are now of the form $H + 2\eta_N \sum_s y^s D_s$, where H is a Hamiltonian and D_s is a radial derivative. It therefore suffices to show that, in each of the two comparisons, the means of the two radial derivatives, their covariances with the Hamiltonian H and with the corresponding Hamiltonian evaluated at another point, and their mutual covariances, differ by $O_L(1)$; multiplication by $2\eta_N$ (once for the cross terms, twice for the covariances of the radial derivatives) then gives errors of order η_N and η_N^2 .

Consider (3.73). The means of the radial derivatives are obtained by differentiating the deterministic terms in (2.13), at sizes $N+M$ and N , as in (3.27). For species s , the derivative $2\partial_{r^s}$ of the interaction correction is $-2(N+M)t\lambda_s \partial_s \xi(\lambda)$ for the $(N+M)$ -spin system and $-2Nt\lambda_s \partial_s \tilde{\xi}_N(\lambda) = -2Nt\lambda_s \partial_s \xi(\frac{N}{N+M}\lambda)$ for the bulk system; these differ by $O(1)$. The derivatives of the cascade corrections are $-2q^s(1)(N_s + M_s)$ and $-2q^s(1)N_s$; and the derivatives of the external-field terms are $h^s \sum_{i \in I_{N+M,s}} \rho_i$ and $h^s \sum_{i \in I_{N,s}} \sigma_i$, whose difference is $h^s((\sqrt{r_N^s(\tau)} - 1) \sum_{i \in I_{N,s}} \sigma_i + \sum_{j \in M_s} \tau_j) = O_L(1)$, as in the proof of Lemma 3.6. The covariances are obtained by differentiating covariance kernels. For instance, by (1.5) and (3.7), and with $\rho' = (S_{r_N(\tau')}\sigma', \tau')$,

$$\text{Cov}(2\partial_{r^s} H_{N+M}(S_r \rho) \Big|_{r=\bar{\mathbf{1}}}, H_{N+M}(\rho')) = (N+M) R_{N+M,s}(\rho, \rho') \partial_s \xi(R_{N+M}(\rho, \rho')),$$

while, by (3.65), (3.51), and the independence of $\tilde{Z}_{N,j}$ and $\tilde{D}_N$ from $\tilde{H}_N$,

$$\text{Cov}(2\partial_{r^s} \tilde{H}_N(S_r \sigma) \Big|_{r=\bar{\mathbf{1}}}, H_{N+M}(\rho')) = N \sqrt{r_N^s(\tau')} R_{N,s}(\sigma, \sigma') \partial_s \xi\left(\frac{N}{N+M} R_N(\sigma, S_{r_N(\tau')}\sigma')\right).$$

By (3.41) and (3.34), the overlaps and radii appearing in these two expressions differ by $O_L(N^{-1})$, and the prefactors differ by M ; since ξ is smooth, the two expressions differ by $O_L(1)$. The same argument applies to the covariance of the radial derivatives with $H_{N+M}(\rho)$ itself, to the cascade field through (3.66) and (2.12), to the perturbation field through (3.67) and (3.18), in which case the kernels C_h are differentiated at most twice and the bound is $O_L(\Lambda_h)$ before summation, and to the covariances between two radial derivatives, which involve one more derivative of the kernels. This proves (3.73).

For (3.74), the cascade, external-field, and perturbation parts of $V_N^{x,y,q}$ and $\tilde{V}_N^{x,y,q}$ coincide, and the only difference lies in the interaction part, where H_N is replaced by $\tilde{H}_N$ and ξ by $\tilde{\xi}_N$. Under the coupling (3.54), we have $\text{Cov}(\partial_{r^s} \tilde{H}_N(S_r \sigma), H_N(\sigma')) = \text{Cov}(\partial_{r^s} \tilde{H}_N(S_r \sigma), \tilde{H}_N(\sigma'))$, and the kernels ξ and $\tilde{\xi}_N$, together with their derivatives of order at most three, differ by $O(N^{-1})$ uniformly on the overlap domain $[-1, 1]^{\mathcal{S}}$. The differences of means and covariances are therefore $O(1)$ again, and the conclusion follows as above. $\square$

We now introduce the partition functions in which the radial perturbation has been replaced by the bulk one. Set

$$(3.75) \quad Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}} := \iiint_{\Sigma_N \times \mathcal{B}_N \times \mathcal{U}} \chi_L(\tau) \exp(H_{N+M}^{x,q}((S_{r_N}(\tau)\sigma, \tau), \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha)) \times p_N^{\text{ch}}(\tau) d\tau P_N(d\sigma) \mathfrak{R}(d\alpha),$$

and let $\langle \cdot \rangle_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}}$ be the associated Gibbs bracket, defined as in (3.47). Similarly, set

$$(3.76) \quad Z_{N,\tilde{V}}^{x,y,q} := \iint \exp(H_N^{x,q}(\sigma, \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha)) P_N(d\sigma) \mathfrak{R}(d\alpha).$$

Recall the cavity increment $A_N(x, y, q)$ from (3.39), and define its counterpart with cutoff and bulk radial perturbation,

$$(3.77) \quad A_{N,L}^{\text{rad}}(x, y, q) := \mathbb{E} \log Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}} - \mathbb{E} \log Z_{N,\tilde{V}}^{x,y,q}.$$

Proposition 3.10 (Replacement of the radial perturbation at fixed cutoff). *Let $L \geq 1$. Uniformly over x, y, q , we have*

$$(3.78) \quad \mathbb{E} \log Z_{N+M,L}^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q} \approx A_{N,L}^{\text{rad}}(x, y, q),$$

$$(3.79) \quad \mathbb{E} \langle f \rangle_{N+M,L}^{x,y,q} \approx \mathbb{E} \langle f \rangle_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}},$$

for every bounded measurable function f of finitely many replicas of (τ, α) . Consequently, the increment with cutoff is asymptotically a lower bound for the true increment:

$$(3.80) \quad \lim_{N \rightarrow \infty} \sup_{x,y,q} (A_{N,L}^{\text{rad}}(x, y, q) - A_N(x, y, q))_+ = 0.$$

Proof. We apply Lemma 3.4 on the space $\Sigma_N \times \{|\tau| \leq L+1\} \times \mathcal{U}$, with the finite measure $\chi_L(\tau) p_N^{\text{ch}}(\tau) d\tau P_N(d\sigma) \mathfrak{R}(d\alpha)$ normalized to a probability measure (the normalization cancels in the differences of logarithms and does not affect the brackets), to the two fields in (3.73). The partition functions are $Z_{N+M,L}^{x,y,q}$ and $Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}}$, by (3.47) and (3.75), so that $\mathbb{E} \log Z_{N+M,L}^{x,y,q} \approx \mathbb{E} \log Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}}$ and (3.79) holds. Applying the same lemma on $\Sigma_N \times \mathcal{U}$ to the two fields in (3.74) gives $\mathbb{E} \log Z_N^{x,y,q} \approx \mathbb{E} \log Z_{N,\tilde{V}}^{x,y,q}$. Combining the two estimates on partition functions proves (3.78). Finally, by (3.48), we have $A_N(x, y, q) \geq \mathbb{E} \log Z_{N+M,L}^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q}$, so that $(A_{N,L}^{\text{rad}} - A_N)_+$ is bounded by the absolute value of the difference between the two sides of (3.78), which tends to zero uniformly over x, y, q . This proves (3.80). $\square$

We close this subsection with a comparison between the free energy of the bulk system and that of the $(N+M)$ -spin system, which is used in Section 4 to match their derivatives in q .

Lemma 3.11 (Bulk and true free energies). *There exists a constant $C < \infty$ such that, for every sufficiently large N ,*

$$(3.81) \quad \varepsilon_N(Q) := \sup_{x,y,q} \left| \tilde{F}_N^{x,y}(t, q) - \bar{F}_{N+M}^{x,y}(t, q) \right| \leq \frac{C}{N}.$$

Proof. We first compare the bulk system with the N -spin system. Under the coupling (3.54), the fields $H_N^{x,q} + \tilde{V}_N^{x,y,q}$ and $\tilde{H}_N^{x,q} + \tilde{V}_N^{x,y,q}$ differ by $\sqrt{2tM} \tilde{Y}_N(\sigma)$, whose covariance $2tM\theta_N(R_N(\sigma, \sigma'))$ is bounded by (3.53), and by the deterministic term $-Nt(\xi(\lambda) - \tilde{\xi}_N(\lambda)) = -tM\theta_N(\lambda)$, which is bounded as well. Lemma 3.4 therefore gives

$$\sup_{x,y,q} \left| \mathbb{E} \log Z_N^{\sim,x,y,q} - \mathbb{E} \log Z_{N,\tilde{V}}^{x,y,q} \right| \leq C.$$

By (3.74) and Lemma 3.4, we also have $\sup_{x,y,q} |\mathbb{E} \log Z_{N,\tilde{V}}^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q}| \leq C$ for all sufficiently large N . Dividing by N and recalling (3.23) and (3.57), we obtain $\sup_{x,y,q} |\tilde{F}_N^{x,y}(t, q) - \bar{F}_N^{x,y}(t, q)| \leq C/N$. Next, by (3.39),

$$\tilde{F}_N^{x,y}(t, q) - \frac{N+M}{N} \bar{F}_{N+M}^{x,y}(t, q) = \tilde{F}_N^{x,y}(t, q) - \bar{F}_N^{x,y}(t, q) + \frac{1}{N} A_N(x, y, q),$$

so that Lemma 3.6 gives $\sup_{x,y,q} |\tilde{F}_N^{x,y}(t, q) - \frac{N+M}{N} \bar{F}_{N+M}^{x,y}(t, q)| \leq C/N$. It remains to observe that $\bar{F}_{N+M}^{x,y}(t, q)$ is bounded uniformly over x, y, q and N , so that replacing $\frac{N+M}{N} \bar{F}_{N+M}^{x,y}$ by $\bar{F}_{N+M}^{x,y}$ costs at most CM/N . Indeed, Jensen's inequality and the self-overlap correction in (2.13) give

$$-t\xi(\lambda) \leq \frac{1}{N} \mathbb{E} \log Z_N(t, 0) \leq \frac{1}{N} \log \int \exp \left(\sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} \sigma_i \right) P_N(d\sigma) \leq \max_{s \in \mathcal{S}} |h^s|,$$

the Lipschitz estimate (2.17) extends this bound to $\bar{F}_N(t, q)$ for q in the bounded set Q , and (3.61)–(3.62) extend it to $\bar{F}_N^{x,y}(t, q)$. $\square$

3.8. Radial self-averaging and Gaussian reduction. Combining Lemmas 3.7 and 3.8, the Hamiltonian of the $(N+M)$ -spin system in the chart is, up to negligible terms,

$$(3.82) \quad \tilde{H}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in M_s} \tau_j \mathbf{X}_{N,j}^q(\sigma, \alpha) - \sum_{s \in \mathcal{S}} \vartheta^s |\tau^{(s)}|^2 + \frac{1}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha).$$

The first three terms are of the same form as in the cavity computation of [10]: a bulk Hamiltonian, a field linear in τ and independent of the bulk, and a deterministic quadratic term. The last term is specific to the spherical setting. In this subsection, we show that it can be replaced by its Gibbs average, at a cost controlled by the self-averaging of $\tilde{\mathcal{E}}_{N,s}^{x,q}$, and that this self-averaging holds for most values of the parameter y .

3.8.1. The Gaussian model and the interpolation. Recall the bulk bracket $\langle \cdot \rangle_N^{\sim,x,y,q}$ from (3.57), and set

$$(3.83) \quad e_N^s := \mathbb{E} \left\langle \tilde{\mathcal{E}}_{N,s}^{x,q} \right\rangle_N^{\sim,x,y,q}, \quad s \in \mathcal{S}.$$

This quantity depends on x, y, q , which we omit from the notation; it is bounded uniformly in N, x, y, q by Lemma 3.13 below. Replacing $\tilde{\mathcal{E}}_{N,s}^{x,q}$ by e_N^s in (3.82) turns the last term into $\frac{1}{2} \sum_s M_s e_N^s - \frac{1}{2} \sum_s e_N^s |\tau^{(s)}|^2$, which is a constant plus a quadratic term. Since the cavity coordinates will be integrated against the Gaussian measure γ_M , whose density already contains the factor $e^{-|\tau|^2/2}$, it is convenient to set

$$(3.84) \quad b_N^s := 1 + e_N^s + 2\vartheta^s, \quad s \in \mathcal{S},$$

so that, once $\tilde{\mathcal{E}}_{N,s}^{x,q}$ is replaced by e_N^s , the total quadratic term in (3.82) is $-\frac{1}{2} \sum_s (b_N^s - 1) |\tau^{(s)}|^2$. For a general vector $b = (b^s)_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$, we define the Gaussian model

$$(3.85) \quad Z_{N,L,\mathbf{X},b}^{x,y,q,\gamma} := \iiint \chi_L(\tau) \exp \left(\tilde{H}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in M_s} \tau_j \mathbf{X}_{N,j}^q(\sigma, \alpha) - \frac{1}{2} \sum_{s \in \mathcal{S}} (b^s - 1) |\tau^{(s)}|^2 \right) \gamma_M(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha),$$

and we let $\langle \cdot \rangle_{N,L,\mathbf{X},b}^{x,y,q,\gamma}$ be the associated Gibbs bracket on $\Sigma_N \times \mathbb{R}^M \times \mathfrak{U}$. The superscript γ indicates that the reference measure for the cavity coordinates is the Gaussian measure rather than the chart density p_N^{ch} .

The passage from (3.82) to the exponent in (3.85) with $b = b_N$ is achieved by interpolation. For $t \in [0, 1]$, set

$$(3.86) \quad \begin{aligned} \mathcal{H}_{N,t}(\sigma, \tau, \alpha) &:= \tilde{H}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \mathbf{X}_{N,j}^q(\sigma, \alpha) - \frac{1}{2} \sum_{s \in \mathcal{S}} (e_N^s + 2\vartheta^s) |\tau^{(s)}|^2 \\ &+ \frac{t}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) (\tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha) - e_N^s), \end{aligned}$$

and let $\langle \cdot \rangle_{N,t,L}$ be the Gibbs bracket associated with the measure

$$(3.87) \quad \chi_L(\tau) \exp(\mathcal{H}_{N,t}(\sigma, \tau, \alpha)) \gamma_M(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha)$$

on $\Sigma_N \times \mathbb{R}^M \times \mathfrak{U}$. A direct computation shows that the two endpoints of the interpolation are

$$(3.88) \quad \begin{aligned} \mathcal{H}_{N,1}(\sigma, \tau, \alpha) &+ \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e_N^s = \tilde{H}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \mathbf{X}_{N,j}^q(\sigma, \alpha) - \sum_{s \in \mathcal{S}} \vartheta^s |\tau^{(s)}|^2 \\ &+ \frac{1}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) \tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha), \\ \mathcal{H}_{N,0}(\sigma, \tau, \alpha) &= \tilde{H}_N^{x,y,q}(\sigma, \alpha) + \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} \tau_j \mathbf{X}_{N,j}^q(\sigma, \alpha) - \frac{1}{2} \sum_{s \in \mathcal{S}} (b_N^s - 1) |\tau^{(s)}|^2, \end{aligned}$$

so that $\mathcal{H}_{N,1}$ is (3.82) up to an additive constant, while $\mathcal{H}_{N,0}$ is the exponent in (3.85) with $b = b_N$. The cost of the interpolation is measured by

$$(3.89) \quad \text{Rad}_{N,L}(x, y, q) := \int_0^1 \sum_{s \in \mathcal{S}} \mathbb{E} \langle |\tilde{\mathcal{E}}_{N,s}^{x,q} - e_N^s| \rangle_{N,t,L} dt.$$

The main result of this subsection, Proposition 3.15, asserts that the chart partition function and Gibbs averages are, up to errors bounded by a constant times $\text{Rad}_{N,L}$ and by vanishing terms, those of the Gaussian model with $b = b_N$. Theorem 3.24 is then stated under the assumption that $\text{Rad}_{N,L}$ vanishes along the sequence under consideration, and Lemma 3.14 shows that this can be arranged by an appropriate choice of y .

3.8.2. Radial self-averaging. The mechanism through which the perturbation $\tilde{V}_N^{x,y,q}$ forces $\tilde{\mathcal{E}}_{N,s}^{x,q}$ to concentrate is the following elementary fact about random convex functions, which is a variant of [22, Lemma 3.2].

Lemma 3.12 (Derivatives of random convex functions). *Let $0 < \delta < 1/2$. Let $\mathcal{F}$ be a random differentiable convex function on $[0, 3]$, and set $f(u) := \mathbb{E}\mathcal{F}(u)$. Assume that f is finite and differentiable on $[0, 3]$ and that, for some $K < \infty$,*

$$(3.90) \quad \sup_{u \in [0,3]} |f'(u)| \leq K.$$

Then

$$(3.91) \quad \int_1^2 \mathbb{E} |\mathcal{F}'(u) - f'(u)| du \leq \frac{3}{\delta} \int_0^3 \mathbb{E} |\mathcal{F}(u) - f(u)| du + 2K\delta.$$

Proof. Set $A(u) := \mathcal{F}(u) - f(u)$. For every differentiable convex function $g : [0, 3] \rightarrow \mathbb{R}$ and every $u \in [1, 2]$, we have

$$\frac{g(u) - g(u - \delta)}{\delta} \leq g'(u) \leq \frac{g(u + \delta) - g(u)}{\delta}.$$

Using the upper difference quotient for $\mathcal{F}$ and the lower one for f , and then the other way around, we obtain

$$\begin{aligned} \mathcal{F}'(u) - f'(u) &\leq \frac{f(u + \delta) - 2f(u) + f(u - \delta) + A(u + \delta) - A(u)}{\delta}, \\ \mathcal{F}'(u) - f'(u) &\geq \frac{-f(u + \delta) + 2f(u) - f(u - \delta) + A(u) - A(u - \delta)}{\delta}. \end{aligned}$$

Since f is convex, $f(u + \delta) - 2f(u) + f(u - \delta) \geq 0$, and therefore

$$|\mathcal{F}'(u) - f'(u)| \leq \frac{f(u + \delta) - 2f(u) + f(u - \delta)}{\delta} + \frac{|A(u - \delta)| + |A(u)| + |A(u + \delta)|}{\delta}.$$

We integrate over $u \in [1, 2]$ and take expectations. The contribution of the second term is at most $\frac{3}{\delta} \int_0^3 \mathbb{E}|A(u)|du$. For the first term, we write

$$\int_1^2 (f(u+\delta) - 2f(u) + f(u-\delta)) du = \int_0^\delta \left(\int_{2-\delta+v}^{2+v} f'(w) dw - \int_{1-\delta+v}^{1+v} f'(w) dw \right) dv \leq 2K\delta^2,$$

by the fundamental theorem of calculus and (3.90). This proves (3.91). $\square$

We will apply this lemma to the free energy of the bulk system, as a function of y^s ; this yields the concentration of $\tilde{\mathcal{E}}_{N,s}^{x,q}$ under the bulk measure for most values of y . We then transfer this concentration to the interpolating measures (3.87) by a change of density. The point of this two-step procedure is that the centering e_N , which depends on y , never needs to be differentiated.

For $L \geq 1$, let $\langle \cdot \rangle_{N,L}^\otimes$ denote the Gibbs bracket associated with the measure

$$(3.92) \quad \chi_L(\tau) \exp(\tilde{H}_N^{x,y,q}(\sigma, \alpha)) \gamma_M(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha)$$

on $\Sigma_N \times \mathbb{R}^M \times \mathfrak{U}$. This is the product of the bulk Gibbs measure $\langle \cdot \rangle_N^{\sim, x, y, q}$ and of the probability measure proportional to $\chi_L \gamma_M$ on the cavity coordinates; as for the other brackets of this subsection, the dependence on x, y, q is kept implicit. Functions of (σ, α) alone have the same averages under $\langle \cdot \rangle_{N,L}^\otimes$ and under $\langle \cdot \rangle_N^{\sim, x, y, q}$. For $\mathfrak{t} \in [0, 1]$, set

$$(3.93) \quad \begin{aligned} W_{\mathfrak{t}}(\sigma, \tau, \alpha) &:= \mathcal{H}_{N,\mathfrak{t}}(\sigma, \tau, \alpha) - \tilde{H}_N^{x,y,q}(\sigma, \alpha) \\ &= \sum_{s \in \mathcal{S}} \sum_{j \in \mathbb{M}_s} \tau_j X_{N,j}^q(\sigma, \alpha) - \frac{1}{2} \sum_{s \in \mathcal{S}} (e_N^s + 2\vartheta^s) |\tau^{(s)}|^2 \\ &\quad + \frac{\mathfrak{t}}{2} \sum_{s \in \mathcal{S}} (M_s - |\tau^{(s)}|^2) (\tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma, \alpha) - e_N^s). \end{aligned}$$

Comparing (3.87) with (3.92), we see that the interpolating measure $\langle \cdot \rangle_{N,\mathfrak{t},L}$ has density

$$(3.94) \quad R_{\mathfrak{t}} := \frac{e^{W_{\mathfrak{t}}}}{\langle e^{W_{\mathfrak{t}}} \rangle_{N,L}^\otimes}$$

with respect to $\langle \cdot \rangle_{N,L}^\otimes$: for every nonnegative measurable function g on $\Sigma_N \times \mathbb{R}^M \times \mathfrak{U}$,

$$(3.95) \quad \langle g \rangle_{N,\mathfrak{t},L} = \langle g R_{\mathfrak{t}} \rangle_{N,L}^\otimes.$$

On the support of χ_L , the field $W_{\mathfrak{t}}$ is a linear combination of the fields $(\tilde{\mathcal{E}}_{N,u}^{x,q})_{u \in \mathcal{S}}$ and $(X_{N,j}^q)_{j \in [M]}$ with bounded coefficients, plus a bounded deterministic term. This is what makes the second moment of $R_{\mathfrak{t}}$ bounded at fixed L , as we now show.

Lemma 3.13 (Moment bounds under the cavity tilts). *Fix $L \geq 1$. There exists $C_L < \infty$ such that, for every N , $x, y \in [0, 3]^\mathcal{S}$, $q, \mathfrak{t} \in [0, 1]$, and $s \in \mathcal{S}$,*

$$(3.96) \quad \mathbb{E} \langle |\tilde{\mathcal{E}}_{N,s}^{x,q}|^4 \rangle_N^{\sim, x, y, q} + \mathbb{E} \langle |\tilde{\mathcal{E}}_{N,s}^{x,q}|^4 \rangle_{N,\mathfrak{t},L} + \mathbb{E} \langle R_{\mathfrak{t}}^2 \rangle_{N,L}^\otimes \leq C_L.$$

In particular, $\sup_N \sup_{x,y,q} \max_{s \in \mathcal{S}} |e_N^s| < \infty$ and $\sup_N \sup_{x,y,q} \text{Rad}_{N,L}(x, y, q) \leq C_L$.

Proof. We first prove a general estimate. Let μ be a finite measure on a space $\mathbb{X}$, let $\mathbf{G}$ be a centered Gaussian field on $\mathbb{X}$ and m a bounded function, and let $\langle \cdot \rangle$ be the Gibbs bracket associated with the measure $e^{\mathbf{G}(z) + m(z)} \mu(dz)$. Let X be a Gaussian field on $\mathbb{X}$, jointly Gaussian with $\mathbf{G}$, and let $A < \infty$ be such that

$$\sup_{z \in \mathbb{X}} (|\mathbb{E} X(z)| + \text{Var } X(z)) \leq A \quad \text{and} \quad \sup_{z, z' \in \mathbb{X}} |\text{Cov}(X(z), \mathbf{G}(z'))| \leq A.$$

We claim that, for every $\lambda \in \mathbb{R}$,

$$(3.97) \quad \mathbb{E} \langle e^{\lambda X} \rangle \leq \exp(3A|\lambda| + A\lambda^2/2).$$

To see this, fix $z \in \mathbb{X}$, write $X(z) = \mathbb{E} X(z) + X^\circ(z)$, and set $C_z(z') := \text{Cov}(X(z), \mathbf{G}(z'))$. The Cameron–Martin formula gives, for every nonnegative measurable functional Φ of the field $\mathbf{G}$,

$$\mathbb{E} \left[e^{\lambda X^\circ(z)} \Phi(\mathbf{G}) \right] = e^{\lambda^2 \text{Var}(X(z))/2} \mathbb{E} [\Phi(\mathbf{G} + \lambda C_z)].$$

We apply this identity to the normalized Gibbs density at $\mathbf{z}$, namely $\Phi(\mathbf{G}) := e^{\mathbf{G}(\mathbf{z})+m(\mathbf{z})} / \int e^{\mathbf{G}+m} d\mu$. Since $|C_{\mathbf{z}}| \leq A$, replacing $\mathbf{G}$ by $\mathbf{G} + \lambda C_{\mathbf{z}}$ multiplies the numerator and the denominator of Φ by factors in $[e^{-A|\lambda|}, e^{A|\lambda|}]$, so that $\Phi(\mathbf{G} + \lambda C_{\mathbf{z}}) \leq e^{2A|\lambda|} \Phi(\mathbf{G})$. Hence

$$\mathbb{E} \langle e^{\lambda X} \rangle = \int_{\mathbb{X}} \mathbb{E} [e^{\lambda X(\mathbf{z})} \Phi(\mathbf{G})] \mu(d\mathbf{z}) \leq e^{A|\lambda|+A\lambda^2/2+2A|\lambda|} \int_{\mathbb{X}} \mathbb{E} [\Phi(\mathbf{G})] \mu(d\mathbf{z}),$$

and the last integral equals 1. This proves (3.97). Since $|u|^4 \leq 24(e^u + e^{-u})$ and $|u|^8 \leq 8!(e^u + e^{-u})$, we deduce that

$$(3.98) \quad \mathbb{E}(|X|^4) \leq 48 \exp(3A + A/2) \quad \text{and} \quad \mathbb{E}(|X|^8) \leq 2 \cdot 8! \exp(3A + A/2).$$

We apply this estimate first to the bulk bracket $\langle \cdot \rangle_N^{\sim, x, y, q}$, with $\mathbb{X} := \Sigma_N \times \mathfrak{U}$, $\mu := P_N \otimes \mathfrak{R}$, and $\mathbf{G}$ the centered part of $\tilde{H}_N^{x, y, q}$, and with $X := \tilde{\mathcal{E}}_{N, s}^{x, q}$. The mean and the variance of X are bounded by (3.59). For the cross-covariances, we note that the covariance of $\tilde{\mathcal{E}}_{N, s}^{x, q}(\sigma, \alpha)$ with $\tilde{H}_N^{x, q}(\sigma', \alpha')$ is $2/N_s$ times a first radial derivative of a covariance kernel of order N , hence is bounded by (3.58), and that its covariance with $\tilde{V}_N^{x, y, q}(\sigma', \alpha') = \eta_N \sum_u y^u N_u \tilde{\mathcal{E}}_{N, u}^{x, q}(\sigma', \alpha')$ is $O(\eta_N)$ by (3.59). The hypotheses therefore hold with a constant $A = C$ independent of L , and (3.98) gives the bound on the first term in (3.96), together with

$$(3.99) \quad \mathbb{E}(|\tilde{\mathcal{E}}_{N, s}^{x, q}|^8)_N^{\sim, x, y, q} \leq C.$$

Since $e_N^s = \mathbb{E} \langle \tilde{\mathcal{E}}_{N, s}^{x, q} \rangle_N^{\sim, x, y, q}$ by (3.83), Jensen's inequality gives $|e_N^s| \leq C$, which is the first assertion after (3.96).

We next apply the estimate to the product bracket $\langle \cdot \rangle_{N, L}^{\otimes}$, with

$$\mathbb{X} := \Sigma_N \times \{|\tau| \leq L + 1\} \times \mathfrak{U}, \quad \mu := \chi_L(\tau) \gamma_M(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha),$$

the same field $\mathbf{G}$, and $X := W_t$. By (3.93), on the set $\mathbb{X}$, the field W_t is a linear combination of $(\tilde{\mathcal{E}}_{N, u}^{x, q})_{u \in \mathcal{S}}$ and $(X_{N, j}^q)_{j \in [M]}$ with coefficients bounded by C_L , plus a deterministic term bounded by C_L ; here we use $|\tau| \leq L + 1$, the bound $|e_N^s| \leq C$ just proved, and the boundedness of the coefficients ϑ^s in (3.70) for q in the bounded set Q of (3.10). By (3.69), the fields $X_{N, j}^q$ have mean h^s and variance bounded by C , and they are independent of $\tilde{H}_N^{x, y, q}$ and of $(\tilde{\mathcal{E}}_{N, u}^{x, q})_u$. Together with the bounds of the previous paragraph, this shows that the hypotheses hold with $A = C_L$, so that (3.97) gives

$$(3.100) \quad \mathbb{E} \langle e^{\lambda W_t} \rangle_{N, L}^{\otimes} \leq C_L \quad \text{for every } \lambda \in [-4, 4].$$

We now bound the second moment of the density (3.94). By the Cauchy-Schwarz inequality, $1 = (\langle e^{W_t/2} e^{-W_t/2} \rangle_{N, L}^{\otimes})^2 \leq \langle e^{W_t} \rangle_{N, L}^{\otimes} \langle e^{-W_t} \rangle_{N, L}^{\otimes}$, and by Jensen's inequality, $(\langle e^{-W_t} \rangle_{N, L}^{\otimes})^2 \leq \langle e^{-2W_t} \rangle_{N, L}^{\otimes}$. Hence

$$\begin{aligned} \mathbb{E} \langle R_t^2 \rangle_{N, L}^{\otimes} &= \mathbb{E} \left[\frac{\langle e^{2W_t} \rangle_{N, L}^{\otimes}}{(\langle e^{W_t} \rangle_{N, L}^{\otimes})^2} \right] \leq \mathbb{E} \left[\langle e^{2W_t} \rangle_{N, L}^{\otimes} \langle e^{-2W_t} \rangle_{N, L}^{\otimes} \right] \\ &\leq \left(\mathbb{E} \langle e^{4W_t} \rangle_{N, L}^{\otimes} \right)^{1/2} \left(\mathbb{E} \langle e^{-4W_t} \rangle_{N, L}^{\otimes} \right)^{1/2} \leq C_L, \end{aligned}$$

where we used the Cauchy-Schwarz inequality for $\mathbb{E}$, Jensen's inequality for the brackets, and the bound (3.100). This is the bound on the third term in (3.96).

For the second term, since $\tilde{\mathcal{E}}_{N, s}^{x, q}$ does not depend on τ , the change of density (3.95), the Cauchy-Schwarz inequality, and (3.99) give

$$\mathbb{E} \langle |\tilde{\mathcal{E}}_{N, s}^{x, q}|^4 \rangle_{N, t, L} = \mathbb{E} \langle |\tilde{\mathcal{E}}_{N, s}^{x, q}|^4 R_t \rangle_{N, L}^{\otimes} \leq \left(\mathbb{E} \langle |\tilde{\mathcal{E}}_{N, s}^{x, q}|^8 \rangle_N^{\sim, x, y, q} \right)^{1/2} \left(\mathbb{E} \langle R_t^2 \rangle_{N, L}^{\otimes} \right)^{1/2} \leq C_L.$$

Finally, by Jensen's inequality and $|e_N^s| \leq C$, we have $\mathbb{E} \langle |\tilde{\mathcal{E}}_{N, s}^{x, q} - e_N^s| \rangle_{N, t, L} \leq C_L$, and the bound on $\text{Rad}_{N, L}$ follows from its definition (3.89). $\square$

Lemma 3.14 (Radial self-averaging). *For every $L \geq 1$, we have*

$$\lim_{N \rightarrow \infty} \sup_{x, q} \int_{[1, 2]^{\mathcal{S}}} \text{Rad}_{N, L}(x, y, q) dy = 0.$$

Proof. Fix $L \geq 1$, N , x , and q , and write $E_s := \tilde{\mathcal{E}}_{N,s}^{x,q}$ and $\langle \cdot \rangle_0 := \langle \cdot \rangle_N^{x,y,q}$ for brevity. The constants C below may depend on L but not on $N, x, y, q, \mathbf{t}$. Recall from (3.83) that $e_N^s = \mathbb{E} \langle E_s \rangle_0$ depends on y . The proof has two steps: we first show that E_s concentrates around e_N^s under the bulk measure for most values of y , and we then transfer this concentration to the interpolating measures through the density $R_{\mathbf{t}}$.

Step 1: Concentration under the bulk measure.

Define

$$\mathcal{F}(y) := \frac{1}{N} \log Z_N^{x,y,q}, \quad f(y) := \mathbb{E} \mathcal{F}(y),$$

where $Z_N^{x,y,q}$ is the bulk partition function in (3.57). By (3.56), the dependence of $\tilde{H}_N^{x,y,q}$ on y is through the affine term $\eta_N \sum_s y^s N_s E_s$. Hence, using $N_s = \lambda_s N$,

$$\begin{aligned} \partial_{y^s} \mathcal{F}(y) &= \eta_N \lambda_s \langle E_s \rangle_0, & \partial_{y^s} f(y) &= \eta_N \lambda_s \mathbb{E} \langle E_s \rangle_0 = \eta_N \lambda_s e_N^s, \\ \partial_{y^s}^2 \mathcal{F}(y) &= \eta_N^2 \lambda_s^2 N \mathbb{E} \left((E_s - \langle E_s \rangle_0)^2 \right)_0. \end{aligned}$$

In particular, $\mathcal{F}$ and f are convex in each coordinate of y . Since $|e_N^s| \leq C$ by Lemma 3.13, we have $\sup_{y \in [0,3]^{\mathcal{S}}} |\partial_{y^s} \mathcal{F}(y)| \leq C \eta_N$. Moreover, the Hamiltonian $\tilde{H}_N^{x,y,q}$ is of the form (B.1), see Remark 3.1, and its variance is at most CN uniformly over x, y, q , by (3.26) and (3.59). Lemma B.1, which controls the fluctuations coming from the Gaussian fields and from the cascade, gives

$$\sup_{y \in [0,3]^{\mathcal{S}}} \mathbb{E} |\mathcal{F}(y) - f(y)| \leq CN^{-1/2}.$$

Set $\delta_N := (N^{-1/2}/\eta_N)^{1/2}$, which is smaller than $1/2$ for all sufficiently large N and tends to zero, by (3.9). Fix $s \in \mathcal{S}$ and the coordinates y^u , $u \neq s$. Applying Lemma 3.12 to $\mathcal{F}$ and f as functions of $u = y^s$, with $K = C\eta_N$ and $\delta = \delta_N$, gives

$$\begin{aligned} \eta_N \lambda_s \int_1^2 \mathbb{E} |\langle E_s \rangle_0 - e_N^s| dy^s &\leq \frac{3}{\delta_N} \int_0^3 \mathbb{E} |\mathcal{F} - f| dy^s + 2C\eta_N \delta_N \\ &\leq C \left(\frac{N^{-1/2}}{\delta_N} + \eta_N \delta_N \right) = 2C\eta_N \left(\frac{N^{-1/2}}{\eta_N} \right)^{1/2}. \end{aligned}$$

Dividing by $\eta_N \lambda_s$, we obtain

$$\int_1^2 \mathbb{E} |\langle E_s \rangle_0 - e_N^s| dy^s \leq C \left(\frac{N^{-1/2}}{\eta_N} \right)^{1/2}.$$

Next, integrating the formula for $\partial_{y^s}^2 \mathcal{F}$ over $y^s \in [1, 2]$ and using the bound on $\partial_{y^s} \mathcal{F}$ gives

$$\int_1^2 \mathbb{E} \left((E_s - \langle E_s \rangle_0)^2 \right)_0 dy^s \leq \frac{2C\eta_N}{\eta_N^2 \lambda_s^2 N} = \frac{C}{N\eta_N}.$$

Combining the last two displays with the triangle inequality and the Cauchy–Schwarz inequality, in the form

$$\mathbb{E} |\langle E_s - e_N^s \rangle_0| \leq \left(\mathbb{E} \left((E_s - \langle E_s \rangle_0)^2 \right)_0 \right)^{1/2} + \mathbb{E} |\langle E_s \rangle_0 - e_N^s|,$$

and using Jensen's inequality for the concave function $u \mapsto u^{1/2}$ after an integration over $y^s \in [1, 2]$, we get

$$\int_1^2 \mathbb{E} |\langle E_s - e_N^s \rangle_0| dy^s \leq \varepsilon_N := C \left((N\eta_N)^{-1/2} + (N^{-1/2}/\eta_N)^{1/2} \right),$$

and $\varepsilon_N \rightarrow 0$ by (3.9). Integrating over the remaining coordinates of $y \in [1, 2]^{\mathcal{S}}$ and summing over s , we conclude that

$$(3.101) \quad \sup_{x,q} \int_{[1,2]^{\mathcal{S}}} a_N(y) dy \leq |\mathcal{S}| \varepsilon_N, \quad \text{where} \quad a_N(y) := \sum_{s \in \mathcal{S}} \mathbb{E} |\langle E_s - e_N^s \rangle_0|.$$

Step 2: Transfer to the interpolating measures.

Fix $t \in [0, 1]$. Since E_s does not depend on τ , its averages under $\langle \cdot \rangle_{N,L}^\otimes$ coincide with those under $\langle \cdot \rangle_0$. By the change of density (3.95), the Cauchy–Schwarz inequality, and the bound on $\mathbb{E} \langle R_t^2 \rangle_{N,L}^\otimes$ in Lemma 3.13,

$$\begin{aligned} \mathbb{E} \langle |E_s - e_N^s| \rangle_{N,t,L} &= \mathbb{E} \langle |E_s - e_N^s| R_t \rangle_{N,L}^\otimes \leq \left(\mathbb{E} \langle |E_s - e_N^s|^2 \rangle_0 \right)^{1/2} \left(\mathbb{E} \langle R_t^2 \rangle_{N,L}^\otimes \right)^{1/2} \\ &\leq C \left(\mathbb{E} \langle |E_s - e_N^s|^2 \rangle_0 \right)^{1/2}. \end{aligned}$$

By Hölder's inequality, $\mathbb{E} \langle X^2 \rangle_0 \leq (\mathbb{E} \langle |X| \rangle_0)^{2/3} (\mathbb{E} \langle |X|^4 \rangle_0)^{1/3}$ for every random variable X , and for $X := E_s - e_N^s$ the fourth moment is bounded by Lemma 3.13 and $|e_N^s| \leq C$. Hence

$$\mathbb{E} \langle |E_s - e_N^s| \rangle_{N,t,L} \leq C (\mathbb{E} \langle |E_s - e_N^s| \rangle_0)^{1/3} \leq C a_N(y)^{1/3}.$$

Summing over s and integrating over $t \in [0, 1]$, we obtain from the definition (3.89) that

$$\text{Rad}_{N,L}(x, y, q) \leq C a_N(y)^{1/3}.$$

Integrating over $y \in [1, 2]^\mathcal{S}$, using Jensen's inequality for the concave function $u \mapsto u^{1/3}$, and then (3.101), we conclude that

$$\sup_{x,q} \int_{[1,2]^\mathcal{S}} \text{Rad}_{N,L}(x, y, q) dy \leq C (|\mathcal{S}| \varepsilon_N)^{1/3} \xrightarrow{N \rightarrow \infty} 0.$$

This completes the proof. $\square$

3.8.3. Gaussian reduction of the cavity block. We can now state the main result of this subsection: at fixed cutoff, the $(N + M)$ -spin system in the chart is described by the Gaussian model (3.85) with $b = b_N$, up to the radial error $\text{Rad}_{N,L}$.

Proposition 3.15 (Fixed-cutoff chart-to-Gaussian comparison). *Fix $L \geq 1$ and a bounded continuous function f of finitely many replicas of (τ, α) . There exist constants $C_L, C_{f,L} < \infty$ and sequences $(\varepsilon_{N,L})_N$ and $(\varepsilon_{N,f,L})_N$ of nonnegative numbers converging to zero, all independent of x, y, q , such that*

$$(3.102) \quad \left| \mathbb{E} \log Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}} - \mathbb{E} \log Z_{N,L,X,b_N}^{x,y,q,\gamma} - \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e_N^s \right| \leq C_L \text{Rad}_{N,L}(x, y, q) + \varepsilon_{N,L},$$

$$(3.103) \quad \left| \mathbb{E} \langle f \rangle_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}} - \mathbb{E} \langle f \rangle_{N,L,X,b_N}^{x,y,q,\gamma} \right| \leq C_{f,L} \text{Rad}_{N,L}(x, y, q) + \varepsilon_{N,f,L}.$$

Proof. We take N large enough that $\{|\tau| \leq L + 1\} \subseteq \mathcal{B}_N$, and we suppose that f depends on n replicas.

Step 1: Replacement of $p_N^{\text{ch}}(\tau) d\tau$ by $\gamma_M(d\tau)$.

For a finite measure ν on $\mathbb{R}^M$, set

$$Z_N[\nu] := \iiint \chi_L(\tau) e^{H_{N+M}^{x,q}((S_{r_N}(\tau)\sigma, \tau), \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha)} \nu(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha),$$

and let $\langle \cdot \rangle_{N,\nu}$ be the associated Gibbs bracket. With $\nu = p_N^{\text{ch}}(\tau) d\tau$, we recover $Z_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}}$ and $\langle \cdot \rangle_{N+M,L,\tilde{V}}^{x,y,q,\text{ch}}$ from (3.75). Set

$$\delta_{N,L} := \sup_{|\tau| \leq L+1} \left| \log \frac{p_N^{\text{ch}}(\tau)}{(2\pi)^{-M/2} e^{-|\tau|^2/2}} \right|,$$

which tends to zero as $N \rightarrow \infty$ by Lemma 3.3 and (3.31). On the support of χ_L , the two densities are within a factor $e^{\pm \delta_{N,L}}$ of each other, so that $Z_N[p_N^{\text{ch}} d\tau]$ and $Z_N[\gamma_M]$ are within a factor $e^{\pm \delta_{N,L}}$ of each other, and the Radon–Nikodym derivative of the one-replica Gibbs measure $\langle \cdot \rangle_{N,p_N^{\text{ch}} d\tau}$ with respect to $\langle \cdot \rangle_{N,\gamma_M}$ takes values in $[e^{-2\delta_{N,L}}, e^{2\delta_{N,L}}]$. Taking the n -fold product, we obtain

$$(3.104) \quad \begin{aligned} & \left| \mathbb{E} \log Z_N[p_N^{\text{ch}} d\tau] - \mathbb{E} \log Z_N[\gamma_M] \right| \leq \delta_{N,L} \quad \text{and} \\ & \left| \mathbb{E} \langle f \rangle_{N,p_N^{\text{ch}} d\tau} - \mathbb{E} \langle f \rangle_{N,\gamma_M} \right| \leq \|f\|_\infty (e^{2n\delta_{N,L}} - 1). \end{aligned}$$

Step 2: Expansion of the Hamiltonian.

By Lemma 3.7, then by Lemma 3.8 applied with the cavity field $\sum_j \tau_j \mathbf{X}_{N,j}^q$ added to both sides, which is allowed since this field is independent of $\tilde{H}_N$, W_N^q , and H_N^x , and finally by the first identity in (3.88), we have

$$H_{N+M}^{x,q}((S_{\tau_N}(\tau)\sigma, \tau), \alpha) + \tilde{V}_N^{x,y,q}(\sigma, \alpha) \approx \mathcal{H}_{N,1}(\sigma, \tau, \alpha) + \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e_N^s,$$

uniformly over x, y, q and $(\sigma, \tau, \alpha) \in \Sigma_N \times \{|\tau| \leq L+1\} \times \mathfrak{U}$. For $\mathbf{t} \in [0, 1]$, let

$$\mathcal{Z}_N(\mathbf{t}) := \iiint \chi_L(\tau) e^{\mathcal{H}_{N,\mathbf{t}}(\sigma, \tau, \alpha)} \gamma_M(d\tau) P_N(d\sigma) \mathfrak{R}(d\alpha),$$

whose Gibbs bracket is $\langle \cdot \rangle_{N,\mathbf{t},L}$. Since the constant $\frac{1}{2} \sum_s M_s e_N^s$ is deterministic and bounded, Lemma 3.4 gives sequences $\varepsilon'_{N,L}, \varepsilon'_{N,f,L} \rightarrow 0$, independent of x, y, q , such that

$$|\mathbb{E} \log \mathcal{Z}_N[\gamma_M] - \mathbb{E} \log \mathcal{Z}_N(1) - \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e_N^s| \leq \varepsilon'_{N,L} \quad \text{and} \quad |\mathbb{E} \langle f \rangle_{N,\gamma_M} - \mathbb{E} \langle f \rangle_{N,1,L}| \leq \varepsilon'_{N,f,L}. \quad (3.105)$$

Step 3: Interpolation in $\mathbf{t}$.

Differentiating in $\mathbf{t}$ and using (3.86), we find

$$\frac{d}{d\mathbf{t}} \mathbb{E} \log \mathcal{Z}_N(\mathbf{t}) = \frac{1}{2} \sum_{s \in \mathcal{S}} \mathbb{E} \langle (M_s - |\tau^{(s)}|^2) (\tilde{\mathcal{E}}_{N,s}^{x,q} - e_N^s) \rangle_{N,\mathbf{t},L}.$$

Since $|M_s - |\tau^{(s)}|^2| \leq M_s + (L+1)^2$ on the support of χ_L , we obtain from (3.89) that

$$|\mathbb{E} \log \mathcal{Z}_N(1) - \mathbb{E} \log \mathcal{Z}_N(0)| \leq C_L \text{Rad}_{N,L}(x, y, q). \quad (3.106)$$

Similarly, writing $z^\ell = (\sigma^\ell, \tau^\ell, \alpha^\ell)$ for the replicas,

$$\begin{aligned} \frac{d}{d\mathbf{t}} \mathbb{E} \langle f \rangle_{N,\mathbf{t},L} &= \frac{1}{2} \sum_{s \in \mathcal{S}} \mathbb{E} \left\langle f \left(\sum_{\ell=1}^n (M_s - |\tau^{\ell,(s)}|^2) (\tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma^\ell, \alpha^\ell) - e_N^s) \right. \right. \\ &\quad \left. \left. - n (M_s - |\tau^{n+1,(s)}|^2) (\tilde{\mathcal{E}}_{N,s}^{x,q}(\sigma^{n+1}, \alpha^{n+1}) - e_N^s) \right) \right\rangle_{N,\mathbf{t},L}, \end{aligned}$$

and since the replicas are exchangeable,

$$|\mathbb{E} \langle f \rangle_{N,1,L} - \mathbb{E} \langle f \rangle_{N,0,L}| \leq C_{f,L} \text{Rad}_{N,L}(x, y, q). \quad (3.107)$$

Step 4: Conclusion.

By the second identity in (3.88), we have $\mathcal{Z}_N(0) = Z_{N,L,\mathbf{X},b_N}^{x,y,q,\gamma}$ and $\langle \cdot \rangle_{N,0,L} = \langle \cdot \rangle_{N,L,\mathbf{X},b_N}^{x,y,q,\gamma}$. Combining (3.104), (3.105), (3.106), and (3.107) proves (3.102) and (3.103), with $\varepsilon_{N,L} := \delta_{N,L} + \varepsilon'_{N,L}$ and $\varepsilon_{N,f,L} := \|f\|_\infty (e^{2n\delta_{N,L}} - 1) + \varepsilon'_{N,f,L}$. $\square$

3.9. The canonical Gaussian block and the cavity functional. The cavity coordinates of the Gaussian model (3.85) interact with the bulk only through the fields $\mathbf{X}_{N,j}^q$ in (3.69), whose covariance is

$$\text{Cov}(\mathbf{X}_{N,j}^q(\sigma, \alpha), \mathbf{X}_{N,j}^q(\sigma', \alpha')) = 2t \partial_s \xi(R_N(\sigma, \sigma')) + 2q^s(\alpha \wedge \alpha'), \quad j \in \mathbf{M}_s.$$

Theorem 3.24 will show that, along suitable sequences, the overlaps of the bulk system synchronize, in the sense that $R_N(\sigma, \sigma')$ is asymptotically a function $p(\alpha \wedge \alpha')$ of the cascade overlap. The covariance above then becomes $2\pi^s(\alpha \wedge \alpha')$ with $\pi := q + t \nabla \xi(p)$, and the field $\mathbf{X}_{N,j}^q$ is replaced by $h^s + \sqrt{2} w_j^{\pi^s}(\alpha)$, where $w_j^{\pi^s}$ is a cascade field with covariance $\pi^s(\alpha \wedge \alpha')$. This subsection introduces the resulting limiting model, which involves only the cavity coordinates and the cascade, and relates it to the spherical initial condition ψ and to the functional $\mathcal{P}_{t,q}$ in (2.79).

3.9.1. *Definitions.* Recall $\mathcal{K}$ from (2.23). Let $\pi = (\pi^s)_{s \in \mathcal{S}} \in \mathcal{Q}_\infty^\mathcal{S}$ and $b = (b^s)_{s \in \mathcal{S}} \in \mathbb{R}^\mathcal{S}$. We say that b is admissible for π if

$$(3.108) \quad b^s > 2\mathcal{K}(\pi^s) \quad \text{for every } s \in \mathcal{S}.$$

For $s \in \mathcal{S}$ and $j \in \mathbb{M}_s$, let $w_j^{\pi^s}$ be independent centered Gaussian fields on $\mathfrak{U}$, with covariance $\mathbb{E}w_j^{\pi^s}(\alpha)w_j^{\pi^s}(\alpha') = \pi^s(\alpha \wedge \alpha')$ as in (2.4). Set

$$(3.109) \quad U_{\pi,b}(\tau, \alpha) := \sum_{s \in \mathcal{S}} \sum_{j \in \mathbb{M}_s} \left((\sqrt{2}w_j^{\pi^s}(\alpha) + h^s)\tau_j - \frac{b^s - 1}{2}\tau_j^2 \right), \quad \tau \in \mathbb{R}^M, \quad \alpha \in \mathfrak{U},$$

and, for $L \in [1, \infty]$,

$$(3.110) \quad \begin{aligned} Z_{M,L,b}^{\pi,\mathfrak{R}} &:= \iint \chi_L(\tau) e^{U_{\pi,b}(\tau, \alpha)} \gamma_M(d\tau) \mathfrak{R}(d\alpha), \\ \langle f \rangle_{M,L,b}^{\pi,\mathfrak{R}} &:= \frac{1}{(Z_{M,L,b}^{\pi,\mathfrak{R}})^n} \int f((\tau^\ell, \alpha^\ell)_{\ell \leq n}) \prod_{\ell=1}^n \chi_L(\tau^\ell) e^{U_{\pi,b}(\tau^\ell, \alpha^\ell)} \gamma_M(d\tau^\ell) \mathfrak{R}(d\alpha^\ell), \end{aligned}$$

where f is a function of n replicas. We write $Z_{M,b}^{\pi,\mathfrak{R}} := Z_{M,\infty,b}^{\pi,\mathfrak{R}}$ and $\langle \cdot \rangle_{M,b}^{\pi,\mathfrak{R}} := \langle \cdot \rangle_{M,\infty,b}^{\pi,\mathfrak{R}}$, and we call $\langle \cdot \rangle_{M,b}^{\pi,\mathfrak{R}}$ the canonical Gaussian block. When $L = \infty$, the integral defining $Z_{M,b}^{\pi,\mathfrak{R}}$ is finite with integrable logarithm precisely when b is admissible for π ; this is the content of Lemmas 3.20 and 3.23 below. We define the functional

$$(3.111) \quad \mathcal{D}_{M,L}(\pi, b) := \mathbb{E} \log Z_{M,L,b}^{\pi,\mathfrak{R}} + \frac{1}{2} \sum_{s \in \mathcal{S}} M_s(b^s - 1) - \sum_{s \in \mathcal{S}} M_s \pi^s(1), \quad \mathcal{D}_M(\pi, b) := \mathcal{D}_{M,\infty}(\pi, b).$$

The additive constant in (3.111) is chosen so that, in the proof of Theorem 3.24, $\mathcal{D}_{M,L}(\pi, b)$ is the limit of the chart partition function in (3.102), including the term $\frac{1}{2} \sum_s M_s e_N^s$. The cavity overlaps are

$$(3.112) \quad \tilde{R}_{M,s}(u, u') := \frac{1}{M} \sum_{j \in \mathbb{M}_s} u_j u'_j, \quad \tilde{R}_M := (\tilde{R}_{M,s})_{s \in \mathcal{S}}, \quad u, u' \in \mathbb{R}^M.$$

The functional $\mathcal{D}_M$ is a multi-species version of the functional $\mathcal{D}_a^\circ$ in Lemma 2.7. Indeed, the fields $(w_j^{\pi^s})_{s,j}$ are independent, and $U_{\pi,b}$ is a sum over (s, j) of terms each involving only τ_j and $w_j^{\pi^s}$. Since the recursive formula for cascade averages is additive over independent groups of fields, see [15, Corollary 5.26] and Step 2 of the proof of Proposition 2.11, the representation (2.28) gives

$$(3.113) \quad \mathcal{D}_M(\pi, b) = \sum_{s \in \mathcal{S}} M_s \mathcal{D}_{h^s}^\circ(\pi^s, b^s),$$

where the left-hand side is finite if and only if b is admissible for π , by Lemma 2.7. More precisely, the additivity of the recursion gives (3.113) for finite-step paths, and the general case follows by the argument of Step 3 of the proof of Lemma 2.7, which applies verbatim to the block with M coordinates: one discretizes each path π^s as there, and compares with the multipliers $(b^s \pm 2\delta)_{s \in \mathcal{S}}$; along the interpolation, the term produced by the change of paths is at most $\delta \sum_{s \in \mathcal{S}} \mathbb{E} \langle |\tau^{(s)}|^2 \rangle$ in absolute value, since $|\tau_j \tau'_j| \leq \frac{1}{2}(\tau_j^2 + \tau'^2_j)$, and it is therefore dominated by the term produced by the change of multipliers. Consequently, by (2.25) and (2.26),

$$(3.114) \quad \inf_b \mathcal{D}_M(\pi, b) = - \sum_{s \in \mathcal{S}} M_s \psi_{h^s}^\circ(\pi^s) = -M\psi(\pi),$$

where the infimum is over admissible b , and it is attained at the unique point

$$(3.115) \quad b_\pi := (b_{h^s}(\pi^s))_{s \in \mathcal{S}},$$

with $b_{h^s}(\pi^s)$ the minimizer of Lemma 2.4, characterized by the normalization (2.44) of Lemma 2.7.

Finally, we introduce the counterpart of the reaction field $\tilde{Y}_N$ in (3.54). For $p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$, let y^p be a centered Gaussian field on $\mathfrak{U}$ with covariance $\mathbb{E}y^p(\alpha)y^p(\alpha') = \theta(p(\alpha \wedge \alpha'))$; this is a valid covariance since θ has nonnegative coefficients and p is nondecreasing. Lemma 2.6, applied with $c = 2tM\theta(p)$, gives

$$(3.116) \quad \mathcal{I}_{M,t}(p) := \mathbb{E} \log \int \exp \left(\sqrt{2tM} y^p(\alpha) - tM\theta(p(1)) \right) \mathfrak{R}(d\alpha) = -tM \int_0^1 \theta(p(r)) dr.$$

We then define the cavity functional

(3.117)

$$\mathcal{P}_{M,t,q}(p, b) := \mathcal{D}_M(q + t\nabla\xi(p), b) - \mathcal{I}_{M,t}(p), \quad p \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}, \quad b \text{ admissible for } q + t\nabla\xi(p).$$

Theorem 3.24 bounds the cavity increment from below with this functional, and the next lemma identifies the functional with $-M\mathcal{P}_{t,q}(p)$ once b is determined by the spherical constraint.

3.9.2. Diagonal compensation. In the proof of Theorem 3.24, the limiting overlap array of the bulk system has a deterministic diagonal, $R_N(\sigma, \sigma) = \lambda$, while its off-diagonal entries are $p(\alpha \wedge \alpha')$; the endpoint $p(1)$ need not equal λ . Correspondingly, the cavity fields $X_{N,j}^q$ in (3.69) have the deterministic variance $2\vartheta^s = 2(q^s(1) + t\partial_s\xi(\lambda))$, whereas the cascade field of the canonical block with path $\pi = q + t\nabla\xi(p)$ has variance $2\pi^s(1) = 2(q^s(1) + t\partial_s\xi(p(1)))$, which may be smaller. The block is nevertheless insensitive to such a change of the diagonal variance, provided the quadratic coefficient is shifted accordingly. To formulate this, we parametrize the coefficient by the radial part

$$(3.118) \quad b(\pi, e) := (1 + e^s + 2\pi^s(1))_{s \in \mathcal{S}}, \quad \pi \in \mathcal{Q}_{\infty}^{\mathcal{S}}, \quad e \in \mathbb{R}^{\mathcal{S}},$$

so that, relative to the standard Gaussian measure γ_M , the exponent (3.109) of the block with coefficient $b(\pi, e)$ reads $\sum_s \sum_{j \in M_s} (\sqrt{2}w_j^{\pi^s}(\alpha) + h^s)\tau_j - \sum_s (\pi^s(1) + \frac{e^s}{2})|\tau^{(s)}|^2$. In this parametrization, the additive constant in (3.111) is $\frac{1}{2} \sum_s M_s e^s$, and does not depend on the path.

Lemma 3.16 (Diagonal compensation at fixed cutoff). *Let $L \geq 1$, $e \in \mathbb{R}^{\mathcal{S}}$, and $\pi, \pi' \in \mathcal{Q}_{\infty}^{\mathcal{S}}$. Then*

$$(3.119) \quad |\mathcal{D}_{M,L}(\pi, b(\pi, e)) - \mathcal{D}_{M,L}(\pi', b(\pi', e))| \leq M(L+1)^2 \sum_{s \in \mathcal{S}} |\pi^s - \pi'^s|_{L^1},$$

and, for every $n \in \mathbb{N}$ and every bounded measurable function f of n replicas of (τ, α) , there is a constant $C_n < \infty$, depending only on n , such that

$$(3.120) \quad \left| \mathbb{E} \langle f \rangle_{M,L,b(\pi,e)}^{\pi, \Re} - \mathbb{E} \langle f \rangle_{M,L,b(\pi',e)}^{\pi', \Re} \right| \leq C_n \|f\|_{\infty} M(L+1)^2 \sum_{s \in \mathcal{S}} |\pi^s - \pi'^s|_{L^1}.$$

Proof. For $t \in [0, 1]$, set $u_t := (1-t)\pi + t\pi' \in \mathcal{Q}_{\infty}^{\mathcal{S}}$ and $b_t := b(u_t, e)$, so that $(u_0, b_0) = (\pi, b(\pi, e))$ and $(u_1, b_1) = (\pi', b(\pi', e))$, and realize the cascade fields jointly as $w_j^{u_t} := \sqrt{1-t}w_j^{\pi^s} + \sqrt{t}w_j^{\pi'^s}$, with independent fields $w_j^{\pi^s}$ and $w_j^{\pi'^s}$. The covariance of the field $\sqrt{2}w_j^{u_t}(\alpha)\tau_j$ is $2u_t^s(\alpha \wedge \alpha')\tau_j\tau'_j$, whose derivative in t is $2(\pi'^s - \pi^s)(\alpha \wedge \alpha')\tau_j\tau'_j$. Gaussian integration by parts, as in the proof of Lemma 3.4, together with the derivative $\frac{d}{dt}b_t^s = 2(\pi'^s(1) - \pi^s(1))$ of the quadratic coefficient, gives

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \log Z_{M,L,b_t}^{u_t, \Re} &= \sum_{s \in \mathcal{S}} \sum_{j \in M_s} (\pi'^s(1) - \pi^s(1)) \mathbb{E} \langle \tau_j^2 \rangle_{M,L,b_t}^{u_t, \Re} - \frac{1}{2} \sum_{s \in \mathcal{S}} \sum_{j \in M_s} \frac{db_t^s}{dt} \mathbb{E} \langle \tau_j^2 \rangle_{M,L,b_t}^{u_t, \Re} \\ &\quad - \sum_{s \in \mathcal{S}} \sum_{j \in M_s} \mathbb{E} \langle \tau_j \tau'_j \rangle_{M,L,b_t}^{u_t, \Re} (\pi'^s - \pi^s)(\alpha \wedge \alpha'). \end{aligned}$$

The first term is the diagonal term of the integration by parts, the second one comes from the derivative of the quadratic coefficient, and the two cancel exactly, since $\frac{d}{dt}b_t^s = 2(\pi'^s(1) - \pi^s(1))$. This cancellation is the point of the parametrization (3.118). On the support of χ_L , we have $|\tau_j \tau'_j| \leq (L+1)^2$. Moreover, integrating out the cavity coordinates, the bracket $\langle \cdot \rangle_{M,L,b_t}^{u_t, \Re}$ tilts the cascade by the factor $\exp(\mathbf{g}(w^{u_t}(\alpha)))$, where $w^{u_t} = (w_j^{u_t})_{j \in [M]}$ and

$$\mathbf{g}(w) := \log \int \chi_L(\tau) \exp \left(\sum_{s \in \mathcal{S}} \sum_{j \in M_s} \left((\sqrt{2}w_j + h^s)\tau_j - \frac{1}{2}(b_t^s - 1)\tau_j^2 \right) \right) \gamma_M(d\tau)$$

is a Lipschitz function on $\mathbb{R}^M$, since its gradient is $\sqrt{2}$ times a Gibbs average of τ restricted to $\{|\tau| \leq L+1\}$. By the invariance (2.5), the overlap $\alpha \wedge \alpha'$ is therefore uniformly distributed on $[0, 1]$ under this bracket, so that the remaining sum is bounded in absolute value by $(L+1)^2 \sum_s M_s |\pi'^s - \pi^s|_{L^1} \leq M(L+1)^2 \sum_s |\pi^s - \pi'^s|_{L^1}$. Integrating over $t \in [0, 1]$, and recalling that the additive constant in (3.111) is $\frac{1}{2} \sum_s M_s e^s$ at both endpoints, proves (3.119). The same interpolation applied to $\mathbb{E} \langle f \rangle_{M,L,b_t}^{u_t, \Re}$ produces, as in the proof of Lemma 3.4, a finite sum of Gibbs

averages of f multiplied by the terms $(\pi'^s - \pi^s)(\alpha^\ell \wedge \alpha^{\ell'})\tau_j^\ell \tau_j^{\ell'}$ for $\ell, \ell' \leq n+2$, again with an exact cancellation of the diagonal terms; the overlaps $\alpha^\ell \wedge \alpha^{\ell'}$ with $\ell \neq \ell'$ are uniformly distributed by the same invariance, and (3.120) follows. $\square$

The lemma is a fixed-cutoff statement: nothing is claimed about the admissibility of $b(\pi, e)$, which is a separate matter, treated in Subsection 3.10 and in the proof of Theorem 3.24. Its role is to allow the passage to the limit along paths that converge in L^1 but whose endpoint values differ.

3.9.3. Stationarity in b and the link with the initial condition.

Lemma 3.17 (Stationarity in b). *Let $\pi \in \mathcal{Q}_\infty^\mathcal{S}$ and let b be admissible for π . The function $b \mapsto \mathcal{D}_M(\pi, b)$ is convex on the set of admissible b , with*

$$(3.121) \quad \partial_{b^s} \mathcal{D}_M(\pi, b) = \frac{M_s}{2} - \frac{1}{2} \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M,b}^{\pi, \mathfrak{R}}, \quad s \in \mathcal{S},$$

and $\mathcal{D}_M(\pi, b) \geq -M\psi(\pi)$. Moreover, if

$$(3.122) \quad \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M,b}^{\pi, \mathfrak{R}} = M_s \quad \text{for every } s \in \mathcal{S},$$

then $b = b_\pi$ and $\mathcal{D}_M(\pi, b) = -M\psi(\pi)$; if in addition $\pi = q + t\nabla\xi(p)$ for some $p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$, then

$$(3.123) \quad \mathcal{P}_{M,t,q}(p, b) = -M\mathcal{P}_{t,q}(p).$$

Proof. By (3.113) and part (2) of Lemma 2.4, the function $b \mapsto \mathcal{D}_M(\pi, b)$ is a sum over $s \in \mathcal{S}$ of strictly convex functions of b^s , hence it is convex. For each $s \in \mathcal{S}$, the function $b^s \mapsto \log Z_{M,b}^{\pi, \mathfrak{R}}$ is almost surely convex on the admissible set, with derivative $-\frac{1}{2} \langle |\tau^{(s)}|^2 \rangle_{M,b}^{\pi, \mathfrak{R}}$, and the monotone convergence argument of Step 5 of the proof of Lemma 2.7, which applies verbatim to the block with M coordinates, shows that $b^s \mapsto \mathbb{E} \log Z_{M,b}^{\pi, \mathfrak{R}}$ is differentiable with derivative $-\frac{1}{2} \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M,b}^{\pi, \mathfrak{R}}$; together with (3.111), this gives (3.121). The lower bound is (3.114). Under (3.122), the derivatives in (3.121) vanish, so that b is a critical point of the convex function $b \mapsto \mathcal{D}_M(\pi, b)$, hence a minimizer. By (3.113) and the uniqueness of the minimizer in Lemma 2.7, we get $b = b_\pi$ and $\mathcal{D}_M(\pi, b) = -M\psi(\pi)$. Finally, if $\pi = q + t\nabla\xi(p)$, then (3.117), (3.116), and (2.79) give

$$\mathcal{P}_{M,t,q}(p, b) = -M\psi(q + t\nabla\xi(p)) + tM \int_0^1 \theta(p(r)) dr = -M\mathcal{P}_{t,q}(p). \quad \square$$

The condition (3.122) is the species-wise form of the identity satisfied by the multiplier of a minimizer of the multi-species Parisi functional in [3, Theorem 2.12, (2.13a)], see also (4.3) there; in Theorem 3.24, it is derived from the spherical constraint of the finite system.

The derivative of the initial condition ψ can also be expressed through the canonical block. This is used in Section 4 to identify the critical relation satisfied by the limiting path p .

Lemma 3.18 (Derivative of the initial condition). *Let $\pi \in \mathcal{Q}_\infty^\mathcal{S}$, and let b_π be as in (3.115). For every $\kappa \in \mathcal{Q}_\infty^\mathcal{S}$, we have*

$$(3.124) \quad \langle \kappa, \partial_q \psi(\pi) \rangle_{L^2} = \sum_{s \in \mathcal{S}} \mathbb{E} \langle \kappa^s(\alpha \wedge \alpha') \widetilde{R}_{M,s}(\tau, \tau') \rangle_{M, b_\pi}^{\pi, \mathfrak{R}}.$$

Proof. Write $b := b_\pi$, $\bar{\kappa} := \max_{s \in \mathcal{S}} \kappa^s(1)$, and $\pi_\varepsilon := \pi + \varepsilon \kappa \in \mathcal{Q}_\infty^\mathcal{S}$ for $\varepsilon \geq 0$, and fix $\varepsilon_0 \in (0, 1]$ such that $4\varepsilon_0 \mathcal{K}(\kappa^s) \leq 1$ for every $s \in \mathcal{S}$. Since $\mathcal{K}$ is linear, and since $b^s - 2\mathcal{K}(\pi^s) = \ell_{h^s}(\pi^s) \geq 1$ by (2.35) and (2.33), we have $b^s - 2\mathcal{K}(\pi_\varepsilon^s) = \ell_{h^s}(\pi^s) - 2\varepsilon \mathcal{K}(\kappa^s) \geq \frac{1}{2}$ for every $\varepsilon \in [0, \varepsilon_0]$ and $s \in \mathcal{S}$; in particular, b is admissible for π_ε . We compute the derivative of $\varepsilon \mapsto \mathcal{D}_M(\pi_\varepsilon, b)$ at $\varepsilon = 0$ in two ways: from the explicit formula, and by Gaussian integration by parts. Since the covariance of the block without cutoff is unbounded, the integration by parts is performed with a cutoff, which is then removed.

Step 1: The explicit formula.

Recall the function $\widehat{J}_a$ from Step 1 of the proof of Proposition 2.8. By (3.113) and (2.35),

$$\mathcal{D}_M(\pi_\varepsilon, b) = \sum_{s \in \mathcal{S}} M_s \widehat{J}_{h^s}(\pi_\varepsilon^s, \ell_{h^s}(\pi_\varepsilon^s) - 2\varepsilon \mathcal{K}(\kappa^s)).$$

By Step 2 of that proof, applied to the segment from π^s to $\pi^s + \kappa^s$, the function $(\varepsilon, \ell) \mapsto \widehat{J}_{h^s}(\pi_\varepsilon^s, \ell)$ is differentiable in ε , and its derivative, given by (2.55), is jointly continuous on $[0, 1] \times (0, \infty)$. It is also differentiable in ℓ , with $\partial_\ell \widehat{J}_{h^s}(\pi_\varepsilon^s, \ell) = \frac{1}{2}(1 - S_{h^s}(\pi_\varepsilon^s, \ell))$ by (2.34), and this derivative is jointly continuous as well, by the dominated convergence argument given at the end of that step. We can therefore apply the chain rule. Since $S_{h^s}(\pi^s, \ell_{h^s}(\pi^s)) = 1$, the derivative in ℓ does not contribute at $\varepsilon = 0$, and by (2.55), (2.37), (2.58), and $M_s = \lambda_s M$, we obtain

$$(3.125) \quad \frac{d}{d\varepsilon} \mathcal{D}_M(\pi_\varepsilon, b) \Big|_{\varepsilon=0} = - \sum_{s \in \mathcal{S}} M_s \int_0^1 \kappa^s(u) \zeta_{h^s, \pi^s}(u) du = -M \langle \kappa, \partial_q \psi(\pi) \rangle_{L^2}.$$

Moreover, by (3.121), (3.113), and (2.34),

$$(3.126) \quad \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M, b}^{\pi_\varepsilon, \mathfrak{R}} = M_s S_{h^s}(\pi_\varepsilon^s, b^s - 2\mathcal{K}(\pi_\varepsilon^s)), \quad s \in \mathcal{S},$$

which is a continuous function of $\varepsilon \in [0, \varepsilon_0]$, by the joint continuity just mentioned, and is therefore bounded on this interval.

Step 2: Integration by parts with a cutoff.

Conditionally on $\mathfrak{R}$, let $(w_j^{\kappa^s})_{s \in \mathcal{S}, j \in \mathbf{M}_s}$ be independent Gaussian fields with covariances $\kappa^s(\alpha \wedge \alpha')$, independent of the fields $(w_j^{\pi^s})_{s \in \mathcal{S}, j \in \mathbf{M}_s}$, and realize the fields of the block with path π_ε as $w_j^{\pi_\varepsilon^s} := w_j^{\pi^s} + \sqrt{\varepsilon} w_j^{\kappa^s}$. The exponent in (3.109) then reads

$$U_\varepsilon := U_{\pi_\varepsilon, b} = U_{\pi, b} + \sqrt{\varepsilon} \mathbf{v}, \quad \mathbf{v}(\tau, \alpha) := \sqrt{2} \sum_{s \in \mathcal{S}} \sum_{j \in \mathbf{M}_s} w_j^{\kappa^s}(\alpha) \tau_j.$$

For $L \in [1, \infty]$ and $\varepsilon \in [0, \varepsilon_0]$, let $Z_{L, \varepsilon}$ and $\langle \cdot \rangle_{L, \varepsilon}$ be defined as in (3.110), with the exponent U_ε and with χ_L replaced by the indicator function of $\{|\tau| \leq L\}$, so that $Z_{\infty, \varepsilon} = Z_{M, b}^{\pi_\varepsilon, \mathfrak{R}}$ and $\langle \cdot \rangle_{\infty, \varepsilon} = \langle \cdot \rangle_{M, b}^{\pi_\varepsilon, \mathfrak{R}}$. We set

$$X_{L, \varepsilon} := \langle |\tau|^2 \rangle_{L, \varepsilon}, \quad D_{L, \varepsilon} := \sum_{s \in \mathcal{S}} \kappa^s(1) \langle |\tau^{(s)}|^2 \rangle_{L, \varepsilon}, \quad Y_{L, \varepsilon} := \sum_{s \in \mathcal{S}} \left\langle \kappa^s(\alpha \wedge \alpha') \sum_{j \in \mathbf{M}_s} \tau_j \tau_j' \right\rangle_{L, \varepsilon},$$

and $\Phi_L(\varepsilon) := \mathbb{E} D_{L, \varepsilon} - \mathbb{E} Y_{L, \varepsilon}$. Since $0 \leq \kappa^s \leq \bar{\kappa}$, the Cauchy–Schwarz inequality, the independence of the replicas under the bracket, and Jensen’s inequality give

$$(3.127) \quad 0 \leq D_{L, \varepsilon} \leq \bar{\kappa} X_{L, \varepsilon} \quad \text{and} \quad |Y_{L, \varepsilon}| \leq \bar{\kappa} \langle |\tau| |\tau'| \rangle_{L, \varepsilon} \leq \bar{\kappa} X_{L, \varepsilon}.$$

By (3.126), the random variable $X_{\infty, \varepsilon}$ is integrable, so that $\Phi_\infty(\varepsilon)$ is well defined. Let $L < \infty$. The covariance $2\pi_\varepsilon^s(\alpha \wedge \alpha') \tau_j \tau_j'$ of the field $\sqrt{2} w_j^{\pi_\varepsilon^s}(\alpha) \tau_j$ is bounded on $\{|\tau| \leq L\}$, and its derivative in ε is $2\kappa^s(\alpha \wedge \alpha') \tau_j \tau_j'$. The function $\varepsilon \mapsto \mathbb{E} \log Z_{L, \varepsilon}$ is continuous on $[0, \varepsilon_0]$, and Gaussian integration by parts, as in the proofs of Lemmas 3.4 and 3.16, shows that it is differentiable on $(0, \varepsilon_0]$ with derivative $\Phi_L(\varepsilon)$, the term $\mathbb{E} D_{L, \varepsilon}$ coming from the diagonal of the covariance. Since $|\Phi_L| \leq 2\bar{\kappa} L^2$, we obtain

$$(3.128) \quad \mathbb{E} \log Z_{L, \varepsilon} - \mathbb{E} \log Z_{L, 0} = \int_0^\varepsilon \Phi_L(\varepsilon') d\varepsilon', \quad \varepsilon \in [0, \varepsilon_0].$$

Step 3: Removal of the cutoff.

Fix $\varepsilon \in [0, \varepsilon_0]$. Almost surely, $Z_{\infty, \varepsilon}$ and $X_{\infty, \varepsilon}$ are finite, by Lemma 3.23 below and (3.126). For $L < \infty$, the average $X_{\infty, \varepsilon}$ is a convex combination of $X_{L, \varepsilon} \leq L^2$ and of an average of $|\tau|^2$ over $\{|\tau| > L\}$, which is at least L^2 ; hence $X_{L, \varepsilon} \leq X_{\infty, \varepsilon}$. By dominated convergence with respect to the measure $e^{U_\varepsilon} \gamma_M \otimes \mathfrak{R}$ and its tensor square, we have $D_{L, \varepsilon} \rightarrow D_{\infty, \varepsilon}$ and $Y_{L, \varepsilon} \rightarrow Y_{\infty, \varepsilon}$ almost surely as $L \rightarrow \infty$, and by (3.127), these random variables are bounded by $\bar{\kappa} X_{\infty, \varepsilon}$, which is integrable. Hence $\Phi_L(\varepsilon) \rightarrow \Phi_\infty(\varepsilon)$ as $L \rightarrow \infty$, with $|\Phi_L(\varepsilon)| \leq 2\bar{\kappa} \mathbb{E} X_{\infty, \varepsilon}$, and the right-hand side is bounded uniformly over $\varepsilon \in [0, \varepsilon_0]$ by Step 1. Moreover, $\mathbb{E} \log Z_{L, \varepsilon} \rightarrow \mathbb{E} \log Z_{\infty, \varepsilon}$ by monotone convergence,

as in the proof of Lemma 3.23, whose lower bound on the partition function uses only the unit ball. Letting $L \rightarrow \infty$ in (3.128), we obtain, by dominated convergence,

$$(3.129) \quad \mathbb{E} \log Z_{M,b}^{\pi_\varepsilon, \mathfrak{R}} - \mathbb{E} \log Z_{M,b}^{\pi, \mathfrak{R}} = \int_0^\varepsilon \Phi_\infty(\varepsilon') d\varepsilon', \quad \varepsilon \in [0, \varepsilon_0].$$

Step 4: Continuity at $\varepsilon = 0$.

We show that $\Phi_\infty(\varepsilon) \rightarrow \Phi_\infty(0)$ as $\varepsilon \rightarrow 0$. The term $\mathbb{E} D_{\infty, \varepsilon}$ is a continuous function of ε , by (3.126). For the other term, we use that $e^{\sqrt{\varepsilon}v} \leq 1 + e^{\sqrt{\varepsilon_0}v}$, so that $e^{U_\varepsilon} \leq e^{U_0} + e^{U_{\varepsilon_0}}$ for every $\varepsilon \in [0, \varepsilon_0]$. Almost surely, the function $(1 + |\tau|^2)(e^{U_0} + e^{U_{\varepsilon_0}})$ is integrable with respect to $\gamma_M \otimes \mathfrak{R}$, since $Z_{\infty, \varepsilon}$ and $X_{\infty, \varepsilon}$ are finite for $\varepsilon \in \{0, \varepsilon_0\}$. By dominated convergence, applied to the numerators and to the denominators of the brackets, we have $X_{\infty, \varepsilon} \rightarrow X_{\infty, 0}$ and $Y_{\infty, \varepsilon} \rightarrow Y_{\infty, 0}$ almost surely as $\varepsilon \rightarrow 0$. Moreover, $\mathbb{E} X_{\infty, \varepsilon} \rightarrow \mathbb{E} X_{\infty, 0}$ by (3.126). Since $\bar{\kappa} X_{\infty, \varepsilon} \pm Y_{\infty, \varepsilon} \geq 0$ by (3.127), Fatou's lemma gives

$$\bar{\kappa} \mathbb{E} X_{\infty, 0} \pm \mathbb{E} Y_{\infty, 0} \leq \liminf_{\varepsilon \rightarrow 0} (\bar{\kappa} \mathbb{E} X_{\infty, \varepsilon} \pm \mathbb{E} Y_{\infty, \varepsilon}) = \bar{\kappa} \mathbb{E} X_{\infty, 0} + \liminf_{\varepsilon \rightarrow 0} (\pm \mathbb{E} Y_{\infty, \varepsilon}),$$

that is, $\mathbb{E} Y_{\infty, \varepsilon} \rightarrow \mathbb{E} Y_{\infty, 0}$.

Step 5: Conclusion.

By (3.129) and Step 4, the function $\varepsilon \mapsto \mathbb{E} \log Z_{M,b}^{\pi_\varepsilon, \mathfrak{R}}$ is differentiable at $\varepsilon = 0$, with derivative $\Phi_\infty(0)$. By (3.111), we thus have

$$\frac{d}{d\varepsilon} \mathcal{D}_M(\pi_\varepsilon, b_\pi) \Big|_{\varepsilon=0} = \sum_{s \in \mathcal{S}} \kappa^s(1) \left(\mathbb{E} \left(|\tau^{(s)}|^2 \right)_{M, b_\pi}^{\pi, \mathfrak{R}} - M_s \right) - \sum_{s \in \mathcal{S}} \mathbb{E} \left(\kappa^s(\alpha \wedge \alpha') \sum_{j \in M_s} \tau_j \tau'_j \right)_{M, b_\pi}^{\pi, \mathfrak{R}},$$

where the terms $\kappa^s(1)$ come from the diagonal of the covariance and from the last term in (3.111). Since b_π minimizes $b \mapsto \mathcal{D}_M(\pi, b)$, the first sum vanishes by (3.121). Comparing with (3.125), dividing by $-M$, and using (3.112) gives (3.124). $\square$

3.9.4. Admissibility of b . The next two lemmas show that the truncated functional $\mathcal{D}_{M,L}(\pi, b)$ diverges as $L \rightarrow \infty$ when b is not admissible. This will be used in the proof of Theorem 3.24 to show that the limiting coefficient b is admissible, using only the boundedness of the cavity increment.

Lemma 3.19 (Quadratic gain from the cascade). *Let $\pi \in \mathcal{Q}_\infty^\mathcal{S}$, and for $\tau \in \mathbb{R}^M$, set $W_M^\pi(\tau, \alpha) := \sum_{s \in \mathcal{S}} \sum_{j \in M_s} \tau_j w_j^{\pi^s}(\alpha)$. We have*

$$\mathbb{E} \log \int \exp \left(\sqrt{2} W_M^\pi(\tau, \alpha) \right) \mathfrak{R}(d\alpha) = \sum_{s \in \mathcal{S}} \mathcal{K}(\pi^s) |\tau^{(s)}|^2.$$

Proof. For fixed τ , the field $\sqrt{2} W_M^\pi(\tau, \cdot)$ is a centered Gaussian field on $\mathfrak{U}$ with covariance $c(\alpha \wedge \alpha')$, where $c(r) := 2 \sum_{s \in \mathcal{S}} |\tau^{(s)}|^2 \pi^s(r)$ belongs to $\mathcal{Q}_\infty$. The result follows from (2.41) and the definition of $\mathcal{K}$ in (2.23). $\square$

Lemma 3.20 (Divergence for non-admissible b). *Let $\pi \in \mathcal{Q}_\infty^\mathcal{S}$ and $b \in \mathbb{R}^\mathcal{S}$. If b is not admissible for π , then*

$$\lim_{L \rightarrow \infty} \mathbb{E} \log Z_{M,L,b}^{\pi, \mathfrak{R}} = +\infty, \quad \text{and equivalently} \quad \lim_{L \rightarrow \infty} \mathcal{D}_{M,L}(\pi, b) = +\infty.$$

Proof. Set $\Delta_s := 2\mathcal{K}(\pi^s) - b^s$. Since b is not admissible, there is $s_* \in \mathcal{S}$ with $\Delta_{s_*} \geq 0$. For $\tau \in \mathbb{R}^M$, let

$$v(\tau) := \log \int \exp \left(\sqrt{2} W_M^\pi(\tau, \alpha) \right) \mathfrak{R}(d\alpha) \quad \text{and} \quad \Phi(\tau) := \frac{1}{2} \sum_{s \in \mathcal{S}} \Delta_s |\tau^{(s)}|^2 + \sum_{s \in \mathcal{S}} h^s \sum_{j \in M_s} \tau_j,$$

where W_M^π is as in Lemma 3.19. The random function v is convex in τ , and by Lemma 3.19, $\mathbb{E} v(\tau) = \sum_s \mathcal{K}(\pi^s) |\tau^{(s)}|^2$. By (3.109) and $\gamma_M(d\tau) = (2\pi)^{-M/2} e^{-|\tau|^2/2} d\tau$, we can therefore write

$$Z_{M,L,b}^{\pi, \mathfrak{R}} = (2\pi)^{-M/2} \int \chi_L(\tau) e^{\Phi(\tau)} e^{v(\tau) - \mathbb{E} v(\tau)} d\tau.$$

Let ν_L be the probability measure on $\mathbb{R}^M$ with density proportional to $\chi_L e^\Phi$, which is well defined since χ_L has compact support. Jensen's inequality with respect to ν_L gives

$$\log Z_{M,L,b}^{\pi,\mathfrak{R}} \geq -\frac{M}{2} \log(2\pi) + \log \int \chi_L(\tau) e^{\Phi(\tau)} d\tau + \int (v - \mathbb{E}v) d\nu_L.$$

The last term has zero expectation by Fubini's theorem; the required integrability follows from $\mathbb{E}|v(\tau)| \leq C(1 + |\tau|^2)$, which is a consequence of the lower bound $v(\tau) \geq \sqrt{2} \int W_M^\pi(\tau, \alpha) \mathfrak{R}(d\alpha)$ (Jensen's inequality) and of the upper bound $\mathbb{E}e^{v(\tau)} = \exp(\sum_s \pi^s(1) |\tau^{(s)}|^2)$. Hence

$$(3.130) \quad \mathbb{E} \log Z_{M,L,b}^{\pi,\mathfrak{R}} \geq -\frac{M}{2} \log(2\pi) + \log \int \chi_L(\tau) \exp \left(\frac{1}{2} \sum_{s \in \mathcal{S}} \Delta_s |\tau^{(s)}|^2 + \sum_{s \in \mathcal{S}} h^s \sum_{j \in \mathbb{M}_s} \tau_j \right) d\tau.$$

It remains to check that this deterministic integral tends to infinity with L . Let A_L be the set of $\tau \in \mathbb{R}^M$ such that $|\tau^{(s_*)}| \leq L/2$, $h^{s_*} \sum_{j \in \mathbb{M}_{s_*}} \tau_j \geq 0$, and $|\tau^{(u)}| \leq 1$ for every $u \neq s_*$. For $L \geq 2|\mathcal{S}|$, every $\tau \in A_L$ satisfies $|\tau|^2 \leq L^2/4 + |\mathcal{S}| \leq L^2$, so $\chi_L = 1$ on A_L by (3.46). On A_L , the term $\frac{1}{2} \Delta_{s_*} |\tau^{(s_*)}|^2$ and the external-field term of the species s_* are nonnegative, while the contributions of the other species are bounded from below by a constant $-C$ independent of L , since their coordinates lie in a fixed ball. By symmetry, the half-ball $\{\tau^{(s_*)} : |\tau^{(s_*)}| \leq L/2, h^{s_*} \sum_{j \in \mathbb{M}_{s_*}} \tau_j \geq 0\}$ has volume at least $cL^{M_{s_*}}$ for some $c > 0$ (if $h^{s_*} = 0$, it is the whole ball). Restricting the integral in (3.130) to A_L , we obtain

$$\mathbb{E} \log Z_{M,L,b}^{\pi,\mathfrak{R}} \geq -\frac{M}{2} \log(2\pi) - C + \log \left(cL^{M_{s_*}} \prod_{u \neq s_*} |\mathcal{B}_1(\mathbb{R}^{M_u})| \right) \xrightarrow{L \rightarrow \infty} +\infty,$$

where $|\mathcal{B}_1(\mathbb{R}^{M_u})|$ is the volume of the unit ball. The statement for $\mathcal{D}_{M,L}(\pi, b)$ follows from (3.111), since the remaining terms do not depend on L . $\square$

3.10. Finite-overlap approximation and cutoff removal. This subsection collects the tools needed to pass to the limit in the Gaussian model (3.85), and then to remove the cutoff L .

3.10.1. Approximation by finite overlap arrays. For replicas $(\sigma^\ell, \alpha^\ell)_{\ell \geq 1}$, $n \in \mathbb{N}$, and $p, q \in \mathcal{Q}_\infty^\mathcal{S}$, set

$$(3.131) \quad \begin{aligned} R_N^{\ell, \ell'} &:= R_N(\sigma^\ell, \sigma^{\ell'}), & R_\alpha^{\ell, \ell'} &:= \alpha^\ell \wedge \alpha^{\ell'}, & R_N^{\leq n} &:= (R_N^{\ell, \ell'})_{\ell, \ell' \leq n}, & R_\alpha^{\leq n} &:= (R_\alpha^{\ell, \ell'})_{\ell, \ell' \leq n}, \\ q(R_\alpha^{\leq n}) &:= \left(q(R_\alpha^{\ell, \ell'}) \right)_{\ell, \ell' \leq n}, & p(R_\alpha^{\leq n}) &:= \left(p(R_\alpha^{\ell, \ell'}) \right)_{\ell, \ell' \leq n}. \end{aligned}$$

For $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we write

$$(3.132) \quad a \asymp_\varepsilon b \quad \text{whenever} \quad |a - b| \leq \varepsilon.$$

This relation is not transitive; a chain of k such approximations yields an error of at most $k\varepsilon$.

In the Gaussian model (3.85), the cavity coordinates and the cascade fields attached to them can be integrated out conditionally on the bulk. The result is a functional of the Gaussian fields $X_{N,j}^q$ restricted to the bulk replicas, whose joint law depends only on the overlap array $(R_N^{\leq n}, R_\alpha^{\leq n}, q(R_\alpha^{\leq n}))$. The following lemma, which is the overlap-approximation argument of [10, Lemmas 6.6 and 6.7], makes this precise. Its main point is that the same approximating functions serve for the Gaussian model at finite N and for the canonical block, so that convergence in law of the overlap array transfers to convergence of the corresponding quantities. Recall the bulk bracket $\langle \cdot \rangle_N^{\sim, x, y, q}$ from (3.57) and the cascade bracket $\langle \cdot \rangle_\mathfrak{R}$ from (2.2).

Lemma 3.21 (Finite-overlap approximation). *For every $L \geq 1$ and $\varepsilon > 0$, we have the following.*

- (1) *There exist $n \in \mathbb{N}$ and a bounded continuous function f_ε such that, uniformly over x, y, q, b ,*

$$\mathbb{E} \log \frac{Z_{N,L,X,b}^{x,y,q,\gamma}}{Z_N^{\sim, x,y,q}} \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon \left(R_N^{\leq n}, q(R_\alpha^{\leq n}), b \right) \right\rangle_N^{\sim, x,y,q},$$

and, uniformly over $q \in Q$, $b \in B$, and $p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$,

$$\mathbb{E} \log Z_{M,L,b}^{q+t\nabla\xi(p), \mathfrak{R}} \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon \left(p(R_\alpha^{\leq n}), q(R_\alpha^{\leq n}), b \right) \right\rangle_\mathfrak{R}.$$

- (2) Let $m \geq 1$ and let $G : (\mathbb{R}^{\mathcal{S}} \times [0, 1])^{m \times m} \rightarrow \mathbb{R}$ be bounded and continuous. There exist $n \in \mathbb{N}$ and a bounded continuous function f_ε^G such that, uniformly over x, y, q, b ,

$$\mathbb{E} \left\langle G \left(\left(\tilde{R}_M(\tau^\ell, \tau^{\ell'}), R_\alpha^{\ell, \ell'} \right)_{\ell, \ell' \leq m} \right) \right\rangle_{N, L, X, b}^{x, y, q, \gamma} \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon^G \left(R_N^{\leq n}, R_\alpha^{\leq n}, q(R_\alpha^{\leq n}), b \right) \right\rangle_N^{\sim, x, y, q},$$

and, uniformly over $q \in Q$, $b \in B$, and $p \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}$,

$$\mathbb{E} \left\langle G \left(\left(\tilde{R}_M(\tau^\ell, \tau^{\ell'}), R_\alpha^{\ell, \ell'} \right)_{\ell, \ell' \leq m} \right) \right\rangle_{M, L, b}^{q + t \nabla \xi(p), \Re} \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon^G \left(p(R_\alpha^{\leq n}), R_\alpha^{\leq n}, q(R_\alpha^{\leq n}), b \right) \right\rangle_{\Re}.$$

- (3) For every $N \in \mathbb{N}$, let y_N be a centered Gaussian field on Σ_N , independent of the other Gaussian fields, with covariance $\mathbb{E} y_N(\sigma) y_N(\sigma') = \theta(R_N(\sigma, \sigma'))$. There exist $n \in \mathbb{N}$ and a bounded continuous function $f_\varepsilon^{\mathcal{I}}$ such that, uniformly over x, y, q ,

$$\mathbb{E} \log \left\langle \exp \left(\sqrt{2tM} y_N(\sigma) - tM \theta(\lambda) \right) \right\rangle_N^{\sim, x, y, q} \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon^{\mathcal{I}}(R_N^{\leq n}) \right\rangle_N^{\sim, x, y, q},$$

and, uniformly over $p \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}$,

$$\mathcal{I}_{M, t}(p) \asymp_\varepsilon \mathbb{E} \left\langle f_\varepsilon^{\mathcal{I}}(p(R_\alpha^{\leq n})) \right\rangle_{\Re}.$$

Proof. The quantities on the left-hand sides are of the form considered in [10, Lemmas 6.6 and 6.7], and the statements follow from [10, Proposition 4.4] as explained there. We only indicate how the arguments of the approximating functions arise. In the Gaussian model, the covariance of the cavity fields between two bulk replicas is $2t\partial_s \xi(R_N^{\ell, \ell'}) + 2q^s(R_\alpha^{\ell, \ell'})$, while in the canonical block with path $q + t\nabla \xi(p)$ it is $2t\partial_s \xi(p(R_\alpha^{\ell, \ell'})) + 2q^s(R_\alpha^{\ell, \ell'})$: these are the same continuous function of $(R_N^{\ell, \ell'}, q(R_\alpha^{\ell, \ell'}))$, respectively of $(p(R_\alpha^{\ell, \ell'}), q(R_\alpha^{\ell, \ell'}))$. Since $\nabla \xi$ is continuous, the dependence on the overlaps can be absorbed into the approximating functions, but the path q need not be continuous, so $q(R_\alpha^{\leq n})$ is kept as an argument. In part (2), the function G depends explicitly on $R_\alpha^{\ell, \ell'}$, which therefore remains a separate argument; in general, it cannot be recovered from $q(R_\alpha^{\leq n})$ when q is constant on an interval. Part (3) is the same argument applied to the scalar covariance $\theta(R_N^{\ell, \ell'})$, respectively $\theta(p(R_\alpha^{\ell, \ell'}))$. Since L is fixed, the cavity coordinates range over a compact set, and the boundedness of Q , B , and of the overlaps make the approximations uniform in the stated parameters. $\square$

3.10.2. Uniform integrability of the cavity coordinates. The extension of the cavity limit from bounded observables to observables of polynomial growth rests on the following moment bounds, which are a consequence of the rotation invariance of the reference measure within each species.

Lemma 3.22 (Uniform moments of the cavity coordinates). *For every integer $k \geq 1$, there exists a constant $C_k < \infty$, depending only on k , such that, for every $s \in \mathcal{S}$,*

$$\sup_N \sup_{x, y, q} \mathbb{E} \left\langle |\tau^{(s)}|^{2k} \right\rangle_{N+M}^{x, y, q} \leq C_k M_s^k,$$

$$\text{and consequently} \quad \lim_{a \rightarrow \infty} \sup_N \sup_{x, y, q} \mathbb{E} \left\langle \frac{|\tau^{(s)}|^2}{M_s} \mathbb{1}_{\{|\tau^{(s)}|^2 > a M_s\}} \right\rangle_{N+M}^{x, y, q} = 0.$$

Proof. Let $(e_i)_{i \leq N+M}$ be the standard basis of $\mathbb{R}^{N+M}$. For each $s' \in \mathcal{S}$, let $\mathcal{O}_{N, s'}$ be the compact group of orthogonal transformations of $\mathbb{R}^{I_{N+M, s'}}$ which fix the vector $\sum_{i \in I_{N+M, s'}} e_i$, and let $\mathcal{O}_N := \prod_{s' \in \mathcal{S}} \mathcal{O}_{N, s'}$ act on $\mathbb{R}^{N+M}$ blockwise. The reference measure P_{N+M} is invariant under $\mathcal{O}_N$. Every $U \in \mathcal{O}_N$ preserves the species overlaps and the sums $\sum_{i \in I_{N+M, s'}} \rho_i$, and commutes with the dilations S_r . By (3.27) and (3.26), the law of the field $(H_{N+M}^{x, y, q}(U\rho, \alpha))_{\rho, \alpha}$ is therefore the same as that of $(H_{N+M}^{x, y, q}(\rho, \alpha))_{\rho, \alpha}$. It follows that, for every bounded Borel function F on Σ_{N+M} ,

$$(3.133) \quad \mathbb{E} \langle F(\rho) \rangle_{N+M}^{x, y, q} = \mathbb{E} \left\langle \int_{\mathcal{O}_N} F(U\rho) dU \right\rangle_{N+M}^{x, y, q},$$

where dU denotes the Haar probability measure. We are thus reduced to a deterministic estimate on the average over $\mathcal{O}_{N, s}$.

Set $n_s := |I_{N+M,s}| = N_s + M_s$ and $\vec{1}_s := \sum_{i \in I_{N+M,s}} e_i$. Let $u \in \mathbb{R}^{I_{N+M,s}}$ with $|u|^2 = n_s$, and write

$$u = m\vec{1}_s + v, \quad m := \frac{1}{n_s} \sum_{i \in I_{N+M,s}} u_i, \quad v \perp \vec{1}_s,$$

so that $n_s m^2 + |v|^2 = n_s$; in particular, $|m| \leq 1$ and $|v|^2 = n_s(1 - m^2)$. Every $U \in \mathcal{O}_{N,s}$ fixes $\vec{1}_s$, so that $(Uu)_i = m + (Uv)_i$ for every $i \in I_{N+M,s}$. If $n_s = 1$, then $\mathcal{O}_{N,s}$ is trivial, $v = 0$, and $|(Uu)_i| = 1$. Assume now that $n_s \geq 2$, and fix $i \in I_{N+M,s}$. Under the Haar measure on $\mathcal{O}_{N,s}$, the vector Uv is uniform on the sphere of radius $|v|$ in the hyperplane $\vec{1}_s^\perp$, whose dimension is $d := n_s - 1$, and $(Uv)_i$ is the scalar product of Uv with the orthogonal projection of e_i onto $\vec{1}_s^\perp$, which has squared norm $1 - n_s^{-1} = d/n_s$. If Z is uniform on the unit sphere of $\mathbb{R}^d$, then $\mathbb{E}Z_1^{2k} = (2k-1)!! \prod_{j=0}^{k-1} (d+2j)^{-1}$: writing $Z = g/|g|$ with g a standard Gaussian vector in $\mathbb{R}^d$, and using the independence of $|g|$ and $g/|g|$, we get $\mathbb{E}g_1^{2k} = \mathbb{E}|g|^{2k} \mathbb{E}Z_1^{2k}$, with $\mathbb{E}g_1^{2k} = (2k-1)!!$ and $\mathbb{E}|g|^{2k} = \prod_{j=0}^{k-1} (d+2j)$. Hence

$$\int_{\mathcal{O}_{N,s}} (Uv)_i^{2k} dU = (2k-1)!! \left(\frac{d}{n_s}\right)^k |v|^{2k} \prod_{j=0}^{k-1} \frac{1}{d+2j} = (2k-1)!! (1-m^2)^k \prod_{j=0}^{k-1} \frac{d}{d+2j} \leq (2k-1)!!,$$

and, since $|m+w|^{2k} \leq 2^{2k-1}(m^{2k} + w^{2k})$ and $|m| \leq 1$,

$$(3.134) \quad \sup_N \sup_{u \in \mathbb{R}^{I_{N+M,s}} : |u|^2 = n_s} \max_{i \in I_{N+M,s}} \int_{\mathcal{O}_{N,s}} |(Uu)_i|^{2k} dU \leq 2^{2k-1} (1 + (2k-1)!!) =: C_k,$$

which also covers the case $n_s = 1$. Fix $j \in M_s$, so that $N+j \in I_{N+M,s}$ by (3.3), and let $F(\rho) := \rho_{N+j}^{2k} = \tau_j^{2k}$, which is bounded on Σ_{N+M} . For $\rho \in \Sigma_{N+M}$, we have $|\rho^{(s)}|^2 = n_s$, and $F(U\rho)$ depends only on the component of U in $\mathcal{O}_{N,s}$, so that (3.134), applied with $u := \rho^{(s)}$ and $i := N+j$, gives $\int_{\mathcal{O}_N} F(U\rho) dU \leq C_k$. By (3.133), we obtain $\mathbb{E} \langle \tau_j^{2k} \rangle_{N+M}^{x,y,q} \leq C_k$ for every $j \in M_s$. By Jensen's inequality, $|\tau^{(s)}|^{2k} = (\sum_{j \in M_s} \tau_j^2)^k \leq M_s^{k-1} \sum_{j \in M_s} \tau_j^{2k}$, and summing the previous bound over $j \in M_s$ gives the moment bound. The second assertion follows from the case $k = 2$: since $u \mathbf{1}_{\{u>a\}} \leq u^2/a$ for $u \geq 0$, the quantity under the limit is at most C_2/a . $\square$

3.10.3. Removal of the cutoff. For the $(N+M)$ -spin system, the exchangeability of the coordinates within each species gives an exact formula for the second moment of the cavity coordinates: for every $i \in I_{N+M,s}$, $\mathbb{E} \langle \rho_i^2 \rangle_{N+M}^{x,y,q}$ does not depend on i , and $\sum_{i \in I_{N+M,s}} \rho_i^2 = N_s + M_s$ on Σ_{N+M} , so that

$$(3.135) \quad \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{N+M}^{x,y,q} = M_s, \quad s \in \mathcal{S},$$

as already observed in the proof of Lemma 3.6. This identity is the spherical constraint in the form that we use.

Lemma 3.23 (Removal of the cutoff). *For every $n \in \mathbb{N}$, there exists $C_n < \infty$ such that the following holds.*

- (1) *For every bounded measurable function f of n replicas of (τ, α) , and every $L \geq 1$,*

$$\sup_N \sup_{x,y,q} \left| \mathbb{E} \langle f \rangle_{N+M,L}^{x,y,q} - \mathbb{E} \langle f \rangle_{N+M}^{x,y,q} \right| \leq C_n \|f\|_\infty \frac{M}{L^2}.$$

- (2) *Let $\pi \in \mathcal{Q}_\infty^\mathcal{S}$ and let b be admissible for π . The quantity $Z_{M,b}^{\pi,\Re}$ is almost surely finite with integrable logarithm, and for every bounded measurable function f of n replicas of (τ, α) ,*

$$\lim_{L \rightarrow \infty} \left| \mathbb{E} \langle f \rangle_{M,L,b}^{\pi,\Re} - \mathbb{E} \langle f \rangle_{M,b}^{\pi,\Re} \right| = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \mathcal{D}_{M,L}(\pi, b) = \mathcal{D}_M(\pi, b).$$

Proof. We first record an estimate valid for both brackets. Let $\langle \cdot \rangle$ be a Gibbs bracket without cutoff, let μ be the corresponding one-replica Gibbs measure, and let $\langle \cdot \rangle_L$ be the bracket obtained by inserting the factor $\chi_L(\tau^\ell)$ for each replica $\ell \leq n$. Setting $c_L := \langle \chi_L(\tau) \rangle \in (0, 1]$ and $u_L := 1 - c_L$, the bracket $\langle \cdot \rangle_L$ is the n -fold tensor power of the probability measure μ_L with density χ_L/c_L with respect to μ . Since $0 \leq \chi_L \leq 1$, the measure $\mu - c_L \mu_L$ is nonnegative with total mass u_L , so that $\mu = c_L \mu_L + u_L \nu_L$ for some probability measure ν_L (when $u_L = 0$, we have $\mu = \mu_L$). Expanding the

n -fold tensor power of this decomposition, we find $\mu^{\otimes n} = c_L^n \mu_L^{\otimes n} + (1 - c_L^n) \nu'$ for some probability measure ν' . Hence, for every bounded measurable f of n replicas,

$$|\langle f \rangle - \langle f \rangle_L| = (1 - c_L^n) |\nu'(f) - \langle f \rangle_L| \leq 2 \|f\|_\infty (1 - c_L^n) \leq 2n \|f\|_\infty u_L,$$

where we used that $1 - c^n \leq n(1 - c)$ for $c \in [0, 1]$. Taking expectations, we obtain, with $C_n := 2n$,

$$(3.136) \quad |\mathbb{E} \langle f \rangle_L - \mathbb{E} \langle f \rangle| \leq C_n \|f\|_\infty \mathbb{E} u_L.$$

For part (1), we apply this to $\langle \cdot \rangle_{N+M}^{x,y,q}$. By (3.46), Chebyshev's inequality, and (3.135),

$$\mathbb{E} u_L \leq \mathbb{E} \langle \mathbb{1}_{\{|\tau| > L\}} \rangle_{N+M}^{x,y,q} \leq \frac{1}{L^2} \sum_{s \in \mathcal{S}} \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{N+M}^{x,y,q} = \frac{M}{L^2}.$$

For part (2), we first verify the integrability statements. By (3.113) and Lemma 2.7, the quantity $\mathbb{E} \log Z_{M,b}^{\pi, \mathfrak{R}}$ is finite when b is admissible; in particular $Z_{M,b}^{\pi, \mathfrak{R}} < \infty$ almost surely. We also need a lower bound on $\log Z_{M,1,b}^{\pi, \mathfrak{R}}$. Let ν be the probability measure obtained by normalizing the restriction of γ_M to the unit ball $\mathcal{B}_1$ of $\mathbb{R}^M$. Since $\chi_1 = 1$ on $\mathcal{B}_1$, Jensen's inequality gives

$$\begin{aligned} \log Z_{M,1,b}^{\pi, \mathfrak{R}} &\geq \log \gamma_M(\mathcal{B}_1) + \iint_{\mathcal{B}_1 \times \mathfrak{U}} U_{\pi,b}(\tau, \alpha) \nu(d\tau) \mathfrak{R}(d\alpha) \\ &= \log \gamma_M(\mathcal{B}_1) - \frac{1}{2} \sum_{s \in \mathcal{S}} \sum_{j \in \mathbb{M}_s} (b^s - 1) \int_{\mathcal{B}_1} \tau_j^2 \nu(d\tau), \end{aligned}$$

where the terms linear in τ vanish because ν is symmetric. Since $Z_{M,1,b}^{\pi, \mathfrak{R}} \leq Z_{M,L,b}^{\pi, \mathfrak{R}} \leq Z_{M,b}^{\pi, \mathfrak{R}}$, both $\log Z_{M,1,b}^{\pi, \mathfrak{R}}$ and $\log Z_{M,b}^{\pi, \mathfrak{R}}$ are integrable, and by the monotonicity in (3.46), $\log Z_{M,L,b}^{\pi, \mathfrak{R}}$ increases to $\log Z_{M,b}^{\pi, \mathfrak{R}}$ as $L \rightarrow \infty$. Monotone convergence gives $\lim_{L \rightarrow \infty} \mathbb{E} \log Z_{M,L,b}^{\pi, \mathfrak{R}} = \mathbb{E} \log Z_{M,b}^{\pi, \mathfrak{R}}$, hence the convergence of $\mathcal{D}_{M,L}(\pi, b)$ by (3.111). Finally, since $Z_{M,b}^{\pi, \mathfrak{R}}$ is almost surely finite, dominated convergence gives $u_L = 1 - \langle \chi_L(\tau) \rangle_{M,b}^{\pi, \mathfrak{R}} \rightarrow 0$ almost surely, hence $\mathbb{E} u_L \rightarrow 0$, and (3.136) concludes the proof. $\square$

3.11. Ghirlanda–Guerra identities and the cavity limit. We finally combine the results of this section. We first quantify the extent to which the bulk system satisfies the Ghirlanda–Guerra identities. Recall the overlaps in (3.131) and the kernels $\mathbf{C}_h$ in (3.11), and set

$$R_{N,h}^{\ell, \ell'} := \mathbf{C}_h \left(R_N^{\ell, \ell'}, R_\alpha^{\ell, \ell'} \right) = \left(a_{h_1} \cdot \left(R_N^{\ell, \ell'} \right)^{\odot h_2} + \iota_{h_3} R_\alpha^{\ell, \ell'} \right)^{h_4}.$$

For $N \in \mathbb{N}$, an integer $n \geq 2$, $h \in \mathbb{N}^4$, and a bounded measurable function $\mathbf{f} : (\mathbb{R}^{\mathcal{S}} \times \mathbb{R})^{n \times n} \rightarrow \mathbb{R}$, we define, with $\langle \cdot \rangle = \langle \cdot \rangle_N^{x,y,q}$,

$$(3.137) \quad \begin{aligned} \Delta_N^{x,y,q}(\mathbf{f}, n, h) &:= \left| \mathbb{E} \left\langle \mathbf{f} \left(R_N^{\leq n}, R_\alpha^{\leq n} \right) R_{N,h}^{1,n+1} \right\rangle - \frac{1}{n} \mathbb{E} \left\langle \mathbf{f} \left(R_N^{\leq n}, R_\alpha^{\leq n} \right) \right\rangle \mathbb{E} \left\langle R_{N,h}^{1,2} \right\rangle \right. \\ &\quad \left. - \frac{1}{n} \sum_{\ell=2}^n \mathbb{E} \left\langle \mathbf{f} \left(R_N^{\leq n}, R_\alpha^{\leq n} \right) R_{N,h}^{1,\ell} \right\rangle \right|. \end{aligned}$$

Here and throughout this subsection, $\mathbb{E}$ integrates the Gaussian randomness in the Hamiltonian and the randomness of $\mathfrak{R}$, but not the parameters x, y . We enumerate as $((\mathbf{f}_j, n_j, h_j))_{j \in \mathbb{N}}$ all the triples $(\mathbf{f}, n, h)$ in which $n \geq 2$, $h \in \mathbb{N}^4$, and $\mathbf{f}$ is a monomial with coefficient 1 in the entries of the array. Since overlaps are bounded, we may modify each $\mathbf{f}_j$ outside a bounded set so that it becomes bounded, and then rescale it so that

$$(3.138) \quad \Delta_N^{x,y,q}(\mathbf{f}_j, n_j, h_j) \leq 1, \quad \text{for every } j \in \mathbb{N}, N \in \mathbb{N}, x \in [0, 3]^{\mathbb{N}^4}, y \in [0, 3]^{\mathcal{S}}, q \in \mathcal{Q}_\infty^{\mathcal{S}}.$$

We then set

$$(3.139) \quad \Delta_N(x, y, q) := \sum_{j=1}^{\infty} 2^{-j} \Delta_N^{x,y,q}(\mathbf{f}_j, n_j, h_j).$$

Thus $\Delta_N(x, y, q)$ vanishes asymptotically if and only if the overlap array of the bulk system asymptotically satisfies the Ghirlanda–Guerra identities. Lemma 4.5 below shows that this holds on average over (x, y) .

We can now state the main result of this section. Recall $\text{Rad}_{N,L}$ from (3.89), A_N from (3.39), $\mathcal{P}_{M,t,q}$ from (3.117), $\mathcal{P}_{t,q}$ from (2.79), the canonical block $\langle \cdot \rangle_{M,b}^{\pi, \mathfrak{R}}$ from (3.110), the cavity overlaps $\tilde{R}_M$ from (3.112), and b_π from (3.115). The four conclusions of the theorem are stated in their logical order: the synchronization of the bulk overlaps; the convergence of the cavity coordinates to the canonical block, with an admissible coefficient; the spherical normalization, which identifies the coefficient; and the lower bound on the increment, whose value is identified by the normalization.

Theorem 3.24 (Spherical cavity limit). *Fix $t > 0$ and $q \in \mathcal{Q}_\infty^\mathcal{S}$. Let $(N_k, x_k, y_k, q_k)_{k \geq 1}$ be a sequence such that $N_k \in M\mathbb{N}$ tends to infinity, $x_k \in [0, 3]^{\mathbb{N}^4}$, $y_k \in [0, 3]^\mathcal{S}$, and $q_k \in \mathcal{Q}_\infty^\mathcal{S}$ satisfies*

$$(3.140) \quad \sup_k |q_k|_{L^\infty} < \infty, \quad \lim_{k \rightarrow \infty} q_k(r) = q(r) \quad \text{for a.e. } r \in [0, 1], \quad \lim_{k \rightarrow \infty} q_k(1) = q(1).$$

Assume that

$$(3.141) \quad \lim_{k \rightarrow \infty} \Delta_{N_k}(x_k, y_k, q_k) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \text{Rad}_{N_k, L}(x_k, y_k, q_k) = 0 \quad \text{for every } L \in \mathbb{N}.$$

Then there exist a subsequence, which we still denote by $(N_k, x_k, y_k, q_k)_{k \geq 1}$, a path $p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$, and a vector $b \in \mathbb{R}^\mathcal{S}$ admissible for $\pi := q + t\nabla \xi(p)$, such that the following holds.

- (1) Synchronization of the bulk overlaps. *For every $n \in \mathbb{N}$, the array $(R_{N_k}^{\leq n}, R_\alpha^{\leq n})$ under $\mathbb{E} \langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k}$ converges in law, as $k \rightarrow \infty$, to*

$$(3.142) \quad \left(\left(p(R_\alpha^{\ell, \ell'}), R_\alpha^{\ell, \ell'} \right) \mathbb{1}_{\{\ell \neq \ell'\}} + (\lambda, 1) \mathbb{1}_{\{\ell = \ell'\}} \right)_{1 \leq \ell, \ell' \leq n} \quad \text{under } \mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}.$$

- (2) Convergence of the cavity coordinates. *For every $n \in \mathbb{N}$ and every continuous function $g : (\mathbb{R}^\mathcal{S} \times [0, 1])^{n \times n} \rightarrow \mathbb{R}$ such that $|g((a^{\ell, \ell'}, r^{\ell, \ell'})_{\ell, \ell' \leq n})| \leq C(1 + \sum_{\ell, \ell' \leq n} |a^{\ell, \ell'}|^d)$ for some $C < \infty$ and $d \in \mathbb{N}$, we have, with τ as in (3.5),*

$$(3.143) \quad \lim_{k \rightarrow \infty} \mathbb{E} \left\langle g \left(\left(\tilde{R}_M(\tau^\ell, \tau^{\ell'}), \alpha^\ell \wedge \alpha^{\ell'} \right)_{\ell, \ell' \leq n} \right) \right\rangle_{N_k + M}^{x_k, y_k, q_k} = \mathbb{E} \left\langle g \left(\left(\tilde{R}_M(\tau^\ell, \tau^{\ell'}), \alpha^\ell \wedge \alpha^{\ell'} \right)_{\ell, \ell' \leq n} \right) \right\rangle_{M, b}^{\pi, \mathfrak{R}}.$$

The arrays include their diagonal entries, so that (3.143) covers in particular the one-replica second moments $|\tau^{(s)}|^2 = M \tilde{R}_{M, s}(\tau, \tau)$.

- (3) Spherical normalization. *The canonical block satisfies*

$$(3.144) \quad \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M, b}^{\pi, \mathfrak{R}} = M_s \quad \text{for every } s \in \mathcal{S},$$

and consequently $b = b_\pi$.

- (4) The cavity increment. *We have*

$$(3.145) \quad \liminf_{k \rightarrow \infty} A_{N_k}(x_k, y_k, q_k) \geq \mathcal{P}_{M, t, q}(p, b) = -M \mathcal{P}_{t, q}(p).$$

The bounded set Q in (3.10) is understood to contain q and all the q_k , and the bounded set B to contain all the vectors b_{N_k} in (3.84), which are bounded by Lemma 3.13, as well as the vectors $\bar{b}$ and b constructed in the proof. The proof follows the order of limits described at the beginning of the section. Part 1 uses the Ghirlanda–Guerra identities and the synchronization theorem of [23] to identify the limiting overlap array, and defines the coefficient b by diagonal compensation. Part 2 computes the limits, at fixed cutoff, of the cavity increment and of the Gibbs averages of bounded functions of the cavity coordinates, using Proposition 3.15 and Lemmas 3.21 and 3.16. Part 3 shows that b is admissible, from the boundedness of the increment and Lemma 3.20. Part 4 removes the cutoff and proves assertions (2) and (3), using Lemmas 3.23 and 3.22. Part 5 completes the proof of assertion (4).

Proof of Theorem 3.24. Throughout the proof, we write $\vartheta_k^s := q_k^s(1) + t\partial_s \xi(\lambda)$ for the coefficient in (3.70) associated with q_k , and $\vartheta^s := q^s(1) + t\partial_s \xi(\lambda)$; by (3.140), $\vartheta_k \rightarrow \vartheta$.

Part 1: Convergence of the overlap array and choice of b .

The arrays $(R_{N_k}^{\ell, \ell'}, R_{\alpha}^{\ell, \ell'})_{\ell, \ell' \geq 1}$ are uniformly bounded. After passing to a subsequence, they converge in finite-dimensional distributions under $\mathbb{E} \langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k}$ to an array $(R_{\infty}^{\ell, \ell'}, R_{\alpha, \infty}^{\ell, \ell'})_{\ell, \ell' \geq 1}$; we denote by $\mathbf{E}$ the expectation under the law of the limiting array. The diagonal is deterministic, since $R_{N_k}^{\ell, \ell} = \lambda$ by (3.4) and $R_{\alpha}^{\ell, \ell} = 1$.

We next identify the off-diagonal entries. Since every term in (3.139) is nonnegative, the first limit in (3.141) gives $\lim_{k \rightarrow \infty} \Delta_{N_k}^{x_k, y_k, q_k}(\mathbf{f}_j, n_j, \mathbf{h}_j) = 0$ for every $j \in \mathbb{N}$. The linear span of the constants and of the monomials $\mathbf{f}_j$ is dense among the continuous functions of finitely many entries of the bounded overlap array, and, by polarization, the linear span of the constants and of the functions $(v, u) \mapsto \mathbf{C}_h(v, u)$, $\mathbf{h} \in \mathbb{N}^4$, is dense among the continuous functions on the overlap domain. Passing to the limit in (3.137), we obtain

$$\mathbf{E}[\mathbf{f}(\dots)\varphi(R_{\infty}^{1, n+1}, R_{\alpha, \infty}^{1, n+1})] = \frac{1}{n} \mathbf{E}[\mathbf{f}(\dots)] \mathbf{E}[\varphi(R_{\infty}^{1, 2}, R_{\alpha, \infty}^{1, 2})] + \frac{1}{n} \sum_{\ell=2}^n \mathbf{E}[\mathbf{f}(\dots)\varphi(R_{\infty}^{1, \ell}, R_{\alpha, \infty}^{1, \ell})]$$

for every $n \geq 2$, every bounded continuous function $\mathbf{f}$ of $((R_{\infty}^{\ell, \ell'}, R_{\alpha, \infty}^{\ell, \ell'}))_{\ell, \ell' \leq n}$, abbreviated as $\mathbf{f}(\dots)$, and every bounded continuous function φ . In other words, the limiting array satisfies the Ghirlanda–Guerra identities. To identify it, set

$$T^{\ell, \ell'} := \sum_{s \in \mathcal{S}} R_{\infty, s}^{\ell, \ell'} + R_{\alpha, \infty}^{\ell, \ell'},$$

where $R_{\infty, s}^{\ell, \ell'}$ is the s -component of $R_{\infty}^{\ell, \ell'}$. The entries of R_{∞} are nonnegative, by Talagrand's positivity principle applied to each species array, see [22, Theorem 2.16] and [23, Section 4], so that T takes values in $[0, 2]$. The synchronization theorem [23, Theorem 4] gives a Lipschitz, coordinatewise nondecreasing function $\Psi = (\Psi_{\mathcal{S}}, \Psi_{\alpha}) : [0, 2] \rightarrow \mathbb{R}_{+}^{\mathcal{S}} \times [0, 1]$ such that

$$(R_{\infty}^{\ell, \ell'}, R_{\alpha, \infty}^{\ell, \ell'}) = \Psi(T^{\ell, \ell'}) \quad \text{almost surely, for every } \ell, \ell'.$$

The scalar array $(T^{\ell, \ell'})_{\ell, \ell' \geq 1}$ has diagonal 2, because $\sum_s \lambda_s = 1$. We identify the limiting array by inverting Ψ_{α} . For $r \in [0, 1]$, set

$$\kappa(r) := \inf \{v \in [0, 2] : \Psi_{\alpha}(v) > r\}.$$

Fix $\ell \neq \ell'$, and let $U := R_{\alpha, \infty}^{\ell, \ell'} = \Psi_{\alpha}(T^{\ell, \ell'})$. By the invariance (2.5), the overlap $\alpha^{\ell} \wedge \alpha^{\ell'}$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k}$ for every k , hence so is U . In particular, for every $r \in [0, 1]$, the event $\{\Psi_{\alpha}(T^{\ell, \ell'}) > r\}$ has probability $1 - r > 0$, so that the set in the definition of $\kappa(r)$ is nonempty, and κ is a nondecreasing function from $[0, 1]$ to $[0, 2]$. It is right-continuous, since the sets $\{v \in [0, 2] : \Psi_{\alpha}(v) > r'\}$ increase to $\{v \in [0, 2] : \Psi_{\alpha}(v) > r\}$ as $r' \downarrow r$. Since Ψ_{α} is nondecreasing, each level set $\Psi_{\alpha}^{-1}(\{r\})$ is an interval, and the level sets containing more than one point have pairwise disjoint nonempty interiors; hence the set E of those $r \in [0, 1]$ for which $\Psi_{\alpha}^{-1}(\{r\})$ contains more than one point is countable. Let $r \in [0, 1] \setminus E$ and $v \in [0, 2]$ be such that $\Psi_{\alpha}(v) = r$. For $v' < v$, we have $\Psi_{\alpha}(v') \leq r$, while for $v' > v$, we have $\Psi_{\alpha}(v') \geq r$ and $\Psi_{\alpha}(v') \neq r$, since the level set of r is reduced to $\{v\}$. The set in the definition of $\kappa(r)$ is therefore $(v, 2]$, which is nonempty, so that $v < 2$ and $\kappa(r) = v$. Since U is uniformly distributed and $E \cup \{1\}$ is countable, we have $U \notin E \cup \{1\}$ almost surely, and applying the previous observation with $v = T^{\ell, \ell'}$ and $r = U$ gives

$$T^{\ell, \ell'} = \kappa(R_{\alpha, \infty}^{\ell, \ell'}) \quad \text{almost surely, for every } \ell \neq \ell'.$$

Set $p := \Psi_{\mathcal{S}} \circ \kappa$ on $[0, 1]$. Since $\Psi_{\mathcal{S}}$ is continuous and coordinatewise nondecreasing, with nonnegative values, and since κ is nondecreasing and right-continuous, the path p is nondecreasing, right-continuous, bounded, and nonnegative, so that $p \in \mathcal{Q}_{\infty}^{\mathcal{S}}$, with the endpoint convention (2.3). Applying $\Psi_{\mathcal{S}}$ to the previous display, we obtain

$$(3.146) \quad R_{\infty}^{\ell, \ell'} = p(R_{\alpha, \infty}^{\ell, \ell'}), \quad \ell \neq \ell'.$$

For every $s \in \mathcal{S}$, the Cauchy–Schwarz inequality gives $R_{N_k, s}^{\ell, \ell'} \leq \lambda_s$, so that $p^s \leq \lambda_s$ almost everywhere, and then everywhere on $[0, 1]$ by right-continuity and the endpoint convention. Hence $p \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}$. Finally, the invariance (2.5) identifies the joint law of the cascade overlaps, and not only the law of one overlap: for every $n \in \mathbb{N}$ and every k , the array $R_{\alpha}^{\leq n}$ has the same law

under $\mathbb{E}\langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k}$ as under $\mathbb{E}\langle \cdot \rangle_{\mathfrak{R}}$, hence so does the limiting array $(R_{\alpha, \infty}^{\ell, \ell'})_{\ell, \ell' \leq n}$. By (3.146) and the deterministic diagonal, the limiting array $(R_{\infty}^{\ell, \ell'}, R_{\alpha, \infty}^{\ell, \ell'})_{\ell, \ell' \leq n}$ is almost surely the image of $(R_{\alpha, \infty}^{\ell, \ell'})_{\ell, \ell' \leq n}$ under the map which applies p entrywise off the diagonal and puts $(\lambda, 1)$ on the diagonal, so its law is that of the array in (3.142). This proves part (1).

We now prepare the passage to the limit in the Gaussian model, which requires two additional ingredients. The first is a modification of p near $r = 1$, which allows us to represent the diagonal value λ of the limiting array by a cascade path. For $m \geq 2$, define

$$p_m(r) := \begin{cases} p(r), & 0 \leq r < 1 - m^{-1}, \\ \lambda, & 1 - m^{-1} \leq r \leq 1. \end{cases}$$

Since $p \leq \lambda$, we have $p_m \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}$, $p_m(1) = \lambda$, and $|p_m - p|_{L^1} \leq m^{-1}|\lambda - p|_{L^\infty} \rightarrow 0$. Since the off-diagonal overlaps $\alpha^\ell \wedge \alpha^{\ell'}$ are uniformly distributed on $[0, 1]$ under $\mathbb{E}\langle \cdot \rangle_{\mathfrak{R}}$ while the diagonal ones equal 1, we have, for every $n \in \mathbb{N}$,

$$(3.147) \quad \left(p_m(R_{\alpha}^{\ell, \ell'}), R_{\alpha}^{\ell, \ell'} \right)_{\ell, \ell' \leq n} \xrightarrow[m \rightarrow \infty]{\text{law}} \left(\left(p(R_{\alpha}^{\ell, \ell'}), R_{\alpha}^{\ell, \ell'} \right) \mathbb{1}_{\{\ell \neq \ell'\}} + (\lambda, 1) \mathbb{1}_{\{\ell = \ell'\}} \right)_{\ell, \ell' \leq n} \quad \text{under } \mathbb{E}\langle \cdot \rangle_{\mathfrak{R}}.$$

The second ingredient is the inclusion of the paths q_k in the convergence of part (1). For $\ell \neq \ell'$, the overlap $R_{\alpha}^{\ell, \ell'}$ is uniformly distributed on $[0, 1]$ both under $\mathbb{E}\langle \cdot \rangle_{\mathfrak{R}}$ and, by the invariance (2.5), under $\mathbb{E}\langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k}$. The uniform bound on the q_k and the almost everywhere convergence in (3.140) therefore give $\lim_k \mathbb{E} \left| q_k(R_{\alpha}^{\ell, \ell'}) - q(R_{\alpha}^{\ell, \ell'}) \right| = 0$ under both measures, while for $\ell = \ell'$ we use $q_k(1) \rightarrow q(1)$. Since q has at most countably many discontinuities and the off-diagonal overlaps are uniform, the continuous mapping theorem applied to the convergence of part (1) shows that, with

$$R_p^{\ell, \ell'} := p(R_{\alpha}^{\ell, \ell'}) \mathbb{1}_{\{\ell \neq \ell'\}} + \lambda \mathbb{1}_{\{\ell = \ell'\}},$$

we have, for every $n \in \mathbb{N}$,

$$(3.148) \quad \left(R_{N_k}^{\leq n}, R_{\alpha}^{\leq n}, q_k(R_{\alpha}^{\leq n}) \right) \text{ under } \mathbb{E}\langle \cdot \rangle_{N_k}^{\sim, x_k, y_k, q_k} \xrightarrow[k \rightarrow \infty]{\text{law}} \left(R_p^{\leq n}, R_{\alpha}^{\leq n}, q(R_{\alpha}^{\leq n}) \right) \text{ under } \mathbb{E}\langle \cdot \rangle_{\mathfrak{R}}.$$

Choice of b .

By Lemma 3.13, the vectors e_{N_k} in (3.83), computed with the parameters (x_k, y_k, q_k) , are bounded. Passing to a further subsequence, we may assume that $e_{N_k} \rightarrow e \in \mathbb{R}^{\mathcal{S}}$. By (3.84), the vectors $b_{N_k} = \bar{\mathbf{1}} + e_{N_k} + 2\vartheta_k$ then converge:

$$(3.149) \quad \lim_{k \rightarrow \infty} b_{N_k} = \bar{b} := \bar{\mathbf{1}} + e + 2\vartheta.$$

The vector $\bar{b}$ is the natural limit of the quadratic coefficients in the Gaussian model, whose cavity fields have the deterministic variance $2\vartheta^s$. In the canonical block with path $\pi = q + t\nabla\xi(p)$, however, the cascade fields have diagonal variance $\pi^s(1) = q^s(1) + t\partial_s\xi(p(1))$, which may be smaller than ϑ^s since $p(1) \leq \lambda$. We compensate for this difference in the quadratic coefficient, as explained in Subsection 3.9.2, and set

$$(3.150) \quad \pi := q + t\nabla\xi(p), \quad \pi_m := q + t\nabla\xi(p_m), \quad b := b(\pi, e) = (1 + e^s + 2\pi^s(1))_{s \in \mathcal{S}},$$

with the notation (3.118). By the nonnegativity of the coefficients in (1.4), $\pi, \pi_m \in \mathcal{Q}_{\infty}^{\mathcal{S}}$, and by the Lipschitz continuity of $\nabla\xi$ on the overlap domain, $|\pi_m - \pi|_{L^1} \rightarrow 0$ as $m \rightarrow \infty$. Moreover, $\pi_m(1) = q(1) + t\nabla\xi(\lambda) = \vartheta$, so that $\bar{b} = b(\pi_m, e)$ for every m . Lemma 3.16 therefore gives, for every $L \geq 1$ and every bounded continuous function f of finitely many replicas of (τ, α) ,

$$(3.151) \quad \lim_{m \rightarrow \infty} \mathcal{D}_{M, L}(\pi_m, \bar{b}) = \mathcal{D}_{M, L}(\pi, b), \quad \lim_{m \rightarrow \infty} \mathbb{E}\langle f \rangle_{M, L, \bar{b}}^{\pi_m, \mathfrak{R}} = \mathbb{E}\langle f \rangle_{M, L, b}^{\pi, \mathfrak{R}}.$$

This is the mechanism through which the diagonal value λ of the limiting overlap array, which differs from $p(1)$, is transferred to the quadratic coefficient of the block. We also record that, by (3.150),

$$(3.152) \quad \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e^s = \frac{1}{2} \sum_{s \in \mathcal{S}} M_s (b^s - 1) - \sum_{s \in \mathcal{S}} M_s \pi^s(1),$$

which is the additive constant in the definition (3.111) of $\mathcal{D}_{M,L}(\pi, b)$.

Part 2: The limits at fixed cutoff.

Fix $L \in \mathbb{N}$. We first consider the partition functions. By (3.102) and the second limit in (3.141), and after subtracting $\mathbb{E} \log Z_{N_k}^{\sim, x_k, y_k, q_k}$ from both partition functions,

$$(3.153) \quad \lim_{k \rightarrow \infty} \left(\mathbb{E} \log \frac{Z_{N_k+M, L, \tilde{V}}^{x_k, y_k, q_k, \text{ch}}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} - \mathbb{E} \log \frac{Z_{N_k, L, X, b_{N_k}}^{x_k, y_k, q_k, \gamma}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} - \frac{1}{2} \sum_{s \in \mathcal{S}} M_s e_{N_k}^s \right) = 0.$$

We compute the limit of the second term. Fix $\varepsilon > 0$, and let n and f_ε be given by part (1) of Lemma 3.21. For all sufficiently large k and m , we have

$$\begin{aligned} \mathbb{E} \log \frac{Z_{N_k, L, X, b_{N_k}}^{x_k, y_k, q_k, \gamma}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} &\stackrel{\text{L.3.21(1)}}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon(R_{N_k}^{\leq n}, q_k(R_\alpha^{\leq n}), b_{N_k}) \rangle_{N_k}^{\sim, x_k, y_k, q_k} \\ &\stackrel{(3.148), (3.149)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon(R_p^{\leq n}, q(R_\alpha^{\leq n}), \bar{b}) \rangle_{\mathfrak{R}} \\ &\stackrel{(3.147)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon(p_m(R_\alpha^{\leq n}), q(R_\alpha^{\leq n}), \bar{b}) \rangle_{\mathfrak{R}} \stackrel{\text{L.3.21(1)}}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \log Z_{M, L, \bar{b}}^{\pi_m, \mathfrak{R}}, \end{aligned}$$

where the second and third approximations use the continuity and boundedness of f_ε together with the convergences in law. Letting $k \rightarrow \infty$, then $m \rightarrow \infty$ using the first limit in (3.151) and the fact that the additive constant in (3.111) is the same for $(\pi_m, \bar{b})$ and (π, b) , and finally $\varepsilon \downarrow 0$, we obtain

$$\lim_{k \rightarrow \infty} \mathbb{E} \log \frac{Z_{N_k, L, X, b_{N_k}}^{x_k, y_k, q_k, \gamma}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} = \mathbb{E} \log Z_{M, L, b}^{\pi, \mathfrak{R}}.$$

Moreover, $\lim_k e_{N_k} = e$, so that, by (3.152) and the definition (3.111), we conclude from (3.153) that

$$(3.154) \quad \lim_{k \rightarrow \infty} \mathbb{E} \log \frac{Z_{N_k+M, L, \tilde{V}}^{x_k, y_k, q_k, \text{ch}}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} = \mathcal{D}_{M, L}(\pi, b).$$

We next identify the reaction term. Under the coupling (3.54), and by (3.52), the Hamiltonians in (3.76) and (3.57) satisfy $H_N^{x, q} + \tilde{V}_N^{x, y, q} = \tilde{H}_N^{x, y, q} + \sqrt{2tM} \tilde{Y}_N - tM\theta_N(\lambda)$, so that

$$\mathbb{E} \log \frac{Z_{N, \tilde{V}}^{x, y, q}}{Z_N^{\sim, x, y, q}} = \mathbb{E} \log \left\langle \exp \left(\sqrt{2tM} \tilde{Y}_N(\sigma) - tM\theta_N(\lambda) \right) \right\rangle_N^{\sim, x, y, q}.$$

Let y_N be the field in part (3) of Lemma 3.21. Conditionally on the fields entering $\tilde{H}_N^{x, y, q}$, the bracket $\langle \cdot \rangle_N^{\sim, x, y, q}$ is a probability measure, and the fields $\tilde{Y}_N$ and y_N are independent of it, with covariances $\theta_N(R_N(\sigma, \sigma'))$ and $\theta(R_N(\sigma, \sigma'))$. By (3.53) and Lemma 3.4,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sup_{x, y, q} &\left| \mathbb{E} \log \left\langle \exp \left(\sqrt{2tM} \tilde{Y}_N(\sigma) - tM\theta_N(\lambda) \right) \right\rangle_N^{\sim, x, y, q} \right. \\ &\quad \left. - \mathbb{E} \log \left\langle \exp \left(\sqrt{2tM} y_N(\sigma) - tM\theta(\lambda) \right) \right\rangle_N^{\sim, x, y, q} \right| = 0. \end{aligned}$$

It therefore suffices to compute the limit of

$$B_k := \mathbb{E} \log \left\langle \exp \left(\sqrt{2tM} y_{N_k}(\sigma) - tM\theta(\lambda) \right) \right\rangle_{N_k}^{\sim, x_k, y_k, q_k}.$$

Fix $\varepsilon > 0$, and let n and $f_\varepsilon^{\mathcal{I}}$ be given by part (3) of Lemma 3.21. For all sufficiently large k and m ,

$$\begin{aligned} B_k &\stackrel{\text{L.3.21(3)}}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon^{\mathcal{I}}(R_{N_k}^{\leq n}) \rangle_{N_k}^{\sim, x_k, y_k, q_k} \stackrel{(3.148)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon^{\mathcal{I}}(R_p^{\leq n}) \rangle_{\mathfrak{R}} \\ &\stackrel{(3.147)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle f_\varepsilon^{\mathcal{I}}(p_m(R_\alpha^{\leq n})) \rangle_{\mathfrak{R}} \stackrel{\text{L.3.21(3)}}{\underset{\sim_\varepsilon}{\approx}} \mathcal{I}_{M, t}(p_m). \end{aligned}$$

By (3.116), the continuity of θ , and the L^1 convergence of p_m to p , we have $\lim_{m \rightarrow \infty} \mathcal{I}_{M,t}(p_m) = \mathcal{I}_{M,t}(p)$. Letting $k \rightarrow \infty$, then $m \rightarrow \infty$, and finally $\varepsilon \downarrow 0$, we obtain

$$(3.155) \quad \lim_{k \rightarrow \infty} \mathbb{E} \log \frac{Z_{N_k, \tilde{V}}^{x_k, y_k, q_k}}{Z_{N_k}^{\sim, x_k, y_k, q_k}} = \mathcal{I}_{M,t}(p).$$

We finally consider the Gibbs averages, still at fixed cutoff L . Let $n \in \mathbb{N}$, let $g : (\mathbb{R}^{\mathcal{S}} \times [0, 1])^{n \times n} \rightarrow \mathbb{R}$ be bounded and continuous, and write $G := g((\tilde{R}_M(\tau^\ell, \tau^{\ell'}), \alpha^\ell \wedge \alpha^{\ell'})_{\ell, \ell' \leq n})$, a bounded continuous function of n replicas. By (3.79), (3.103), and the second limit in (3.141),

$$(3.156) \quad \lim_{k \rightarrow \infty} \left(\mathbb{E} \langle G \rangle_{N_k+M, L}^{x_k, y_k, q_k} - \mathbb{E} \langle G \rangle_{N_k, L, X, b_{N_k}}^{x_k, y_k, q_k, \gamma} \right) = 0.$$

Fix $\varepsilon > 0$, and let n' and f_ε^G be given by part (2) of Lemma 3.21, applied with $m = n$ replicas of the cavity coordinates and n' replicas of the bulk. For all sufficiently large k and m ,

$$\begin{aligned} \mathbb{E} \langle G \rangle_{N_k, L, X, b_{N_k}}^{x_k, y_k, q_k, \gamma} &\stackrel{\text{L.3.21(2)}}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \left\langle f_\varepsilon^G \left(R_{N_k}^{\leq n'}, R_\alpha^{\leq n'}, q_k(R_\alpha^{\leq n'}), b_{N_k} \right) \right\rangle_{N_k}^{\sim, x_k, y_k, q_k} \\ &\stackrel{(3.148), (3.149)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \left\langle f_\varepsilon^G \left(R_p^{\leq n'}, R_\alpha^{\leq n'}, q(R_\alpha^{\leq n'}), \bar{b} \right) \right\rangle_{\mathfrak{R}} \\ &\stackrel{(3.147)}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \left\langle f_\varepsilon^G \left(p_m(R_\alpha^{\leq n'}), R_\alpha^{\leq n'}, q(R_\alpha^{\leq n'}), \bar{b} \right) \right\rangle_{\mathfrak{R}} \\ &\stackrel{\text{L.3.21(2)}}{\underset{\sim_\varepsilon}{\approx}} \mathbb{E} \langle G \rangle_{M, L, \bar{b}}^{\pi_m, \mathfrak{R}}. \end{aligned}$$

Letting $k \rightarrow \infty$, then $m \rightarrow \infty$ using the second limit in (3.151), and finally $\varepsilon \downarrow 0$, and combining with (3.156), we get

$$(3.157) \quad \lim_{k \rightarrow \infty} \mathbb{E} \langle G \rangle_{N_k+M, L}^{x_k, y_k, q_k} = \mathbb{E} \langle G \rangle_{M, L, b}^{\pi, \mathfrak{R}}.$$

Part 3: Admissibility of b .

By (3.77), the increment $A_{N, L}^{\text{rad}}(x, y, q)$ is the difference of the two quantities whose limits are identified in (3.154) and (3.155). The one-sided comparison (3.80) of Proposition 3.10 therefore yields

$$(3.158) \quad \liminf_{k \rightarrow \infty} A_{N_k}(x_k, y_k, q_k) \geq \mathcal{D}_{M, L}(\pi, b) - \mathcal{I}_{M, t}(p) \quad \text{for every } L \in \mathbb{N}.$$

By Lemma 3.6, the left-hand side is finite, so that $\sup_{L \geq 1} \mathcal{D}_{M, L}(\pi, b) < +\infty$. Lemma 3.20 then shows that b is admissible for π . This is what makes the untruncated block $\langle \cdot \rangle_{M, b}^{\pi, \mathfrak{R}}$ well defined, by Lemma 3.23.

Part 4: Removal of the cutoff.

Since b is admissible for π , Lemma 3.23 allows us to let $L \rightarrow \infty$ on both sides of (3.157), uniformly in k on the left. This proves (3.143) for bounded continuous g .

We now remove the boundedness assumption on g . By hypothesis, there are $C < \infty$ and $d \in \mathbb{N}$ such that $|g((a^{\ell, \ell'}, r^{\ell, \ell'})_{\ell, \ell' \leq n})| \leq C(1 + \sum_{\ell, \ell' \leq n} |a^{\ell, \ell'}|^d)$. By the Cauchy–Schwarz inequality, $|\tilde{R}_{M, s}(\tau^\ell, \tau^{\ell'})| \leq M^{-1} |\tau^{\ell, (s)}| |\tau^{\ell', (s)}|$, and the Gibbs average over the replicas of a product of functions of τ^ℓ and $\tau^{\ell'}$ is bounded by the average of the square of either function, by the Cauchy–Schwarz inequality and the exchangeability of the replicas. Together with Jensen’s inequality, the moment bounds of Lemma 3.22, applied with an integer $k \geq d(1 + \delta)$, therefore imply that, for every $\delta > 0$,

$$(3.159) \quad \sup_k \mathbb{E} \langle |G|^{1+\delta} \rangle_{N_k+M}^{x_k, y_k, q_k} < +\infty.$$

For $C' > 0$, the function $|g|^{1+\delta} \wedge C'$ is bounded and continuous, so the bounded case and (3.159) give

$$\mathbb{E} \langle |G|^{1+\delta} \wedge C' \rangle_{M, b}^{\pi, \mathfrak{R}} = \lim_{k \rightarrow \infty} \mathbb{E} \langle |G|^{1+\delta} \wedge C' \rangle_{N_k+M}^{x_k, y_k, q_k} \leq \sup_k \mathbb{E} \langle |G|^{1+\delta} \rangle_{N_k+M}^{x_k, y_k, q_k},$$

and monotone convergence as $C' \rightarrow \infty$ shows that $\mathbb{E} \langle |G|^{1+\delta} \rangle_{M,b}^{\pi, \Re}$ is finite. Let $T_{C'}(u) := (-C') \vee (u \wedge C')$. The bounded case gives (3.143) with g replaced by $T_{C'} \circ g$, and since $|u - T_{C'}(u)| \leq |u| \mathbb{1}_{\{|u| > C'\}} \leq C'^{-\delta} |u|^{1+\delta}$, we have

$$\begin{aligned} \sup_k \mathbb{E} \langle |G - T_{C'}(G)| \rangle_{N_k+M}^{x_k, y_k, q_k} &\leq C'^{-\delta} \sup_k \mathbb{E} \langle |G|^{1+\delta} \rangle_{N_k+M}^{x_k, y_k, q_k}, \\ \mathbb{E} \langle |G - T_{C'}(G)| \rangle_{M,b}^{\pi, \Re} &\leq C'^{-\delta} \mathbb{E} \langle |G|^{1+\delta} \rangle_{M,b}^{\pi, \Re}. \end{aligned}$$

Both right-hand sides tend to zero as $C' \rightarrow \infty$, which proves (3.143) for every g in the statement, and completes the proof of assertion (2).

We next prove the normalization (3.144). Fix $s \in \mathcal{S}$, and let $g((a^{11}, r^{11})) := M a_s^{11}$, which is a continuous function of the array of one replica with growth of degree $d = 1$, and satisfies $g((\tilde{R}_M(\tau, \tau), \alpha \wedge \alpha)) = M \tilde{R}_{M,s}(\tau, \tau) = |\tau^{(s)}|^2$. Applying (3.143) with $n = 1$ and this function g , and using (3.135), we get

$$\mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{M,b}^{\pi, \Re} = \lim_{k \rightarrow \infty} \mathbb{E} \langle |\tau^{(s)}|^2 \rangle_{N_k+M}^{x_k, y_k, q_k} = M_s,$$

which is (3.144). The uniform moment bounds of Lemma 3.22 enter here through (3.159): the identity (3.135) alone would not allow the second moment to pass to the limit. Lemma 3.17 now gives $b = b_\pi$, which completes the proof of assertion (3).

Part 5: The cavity increment.

Since b is admissible for π , Lemma 3.23 gives $\lim_{L \rightarrow \infty} \mathcal{D}_{M,L}(\pi, b) = \mathcal{D}_M(\pi, b)$. Letting $L \rightarrow \infty$ in (3.158) and recalling (3.117), we obtain

$$(3.160) \quad \liminf_{k \rightarrow \infty} A_{N_k}(x_k, y_k, q_k) \geq \mathcal{D}_M(\pi, b) - \mathcal{I}_{M,t}(p) = \mathcal{P}_{M,t,q}(p, b).$$

Finally, since $b = b_\pi$ by Part 4, Lemma 3.17 gives $\mathcal{P}_{M,t,q}(p, b) = -M \mathcal{P}_{t,q}(p)$, which proves (3.145) and completes the proof. $\square$

4. UPPER BOUND AT A CRITICAL PATH

We now prove Proposition 2.12. We work under the standing assumptions of Section 3: $\lambda = \lambda_\infty \in \mathbb{Q}^{\mathcal{S}}$, the cavity block $(M_s, \mathbf{M}_s)_{s \in \mathcal{S}}$ satisfies (3.2) with the integers $M_s = M \lambda_s$ of the proposition, and the system sizes are those of the sequence (2.80), with the identification (3.3), so that the passage from N_n to $N_n + M = N_{n+1}$ adds M_s coordinates of species s . Since the free energy depends on the partition $(I_{N_n, s})_{s \in \mathcal{S}}$ only through the cardinalities $|I_{N_n, s}|$, by permutation invariance within each species, the identification (3.3) is a labeling convention and entails no loss of generality. We fix $t > 0$ throughout the section.

The bound (2.82) alone, for a path p describing the limiting overlaps and for $q = 0$, is essentially what the cavity argument of [4, 12] provides, in the limit of a large number of cavity coordinates; combined with Guerra's bound, it gives the Parisi formula in the convex case. The critical relation (2.81) requires the finer information on the cavity coordinates provided by Theorem 3.24.

The argument follows [9, Section 6], see also [10, Section 7], with the spherical cavity computation of Section 3 in place of the Ising cavity computation. Let us describe its structure. By the definition (3.39) of the cavity increment, the free energy at size N_n is, up to a negligible error, the Cesàro average $-\frac{1}{nM} \sum_{j < n} A_{N_j}$ of the cavity increments, so that an upper bound on $\bar{F}_{N_n}$ follows from a lower bound on $\liminf_j A_{N_j}$. Theorem 3.24 provides such a lower bound along sequences of perturbation parameters (x, y) for which the Ghirlanda–Guerra identities hold and the normalized radial derivatives self-average. Both properties hold on average over (x, y) , by Lemmas 4.5 and 3.14, and the averaging argument of [22, Lemma 3.3] allows us to select, for each n , parameters (x_n, y_n) for which they hold and for which $A_{N_n}(x_n, y_n)$ is not larger than its average. This gives (2.82), with p the path describing the limiting overlaps.

To prove the critical relation (2.81), we also randomize the path q , replacing it by $q + q_N(z)$ where $q_N(z)$ is a small random perturbation. The free energies of the bulk system and of the $(N + M)$ -spin system are within $O(N^{-1})$ of each other by Lemma 3.11, and semi-concave in q by Proposition 2.2; a one-dimensional argument then shows that their derivatives in q are close for

most values of z . Along the selected sequence, the derivative of the bulk free energy converges to p by part (1) of Theorem 3.24, while the derivative of the free energy of the $(N + M)$ -spin system converges to $\partial_q \psi(q + t \nabla \xi(p))$ by part (2) of the same theorem and Lemma 3.18, which applies since part (3) gives $b = b_\pi$. Compared with the Ising case, the new ingredients are the additional radial parameter y , which is averaged along with x ; the endpoint term that the radial perturbation produces in the dependence of the perturbed free energies on the path, because of which we work with directional derivatives, see Lemma 4.1; and the identification of the quadratic coefficient b through the spherical normalization.

Throughout the section, the parameters (x, y) range over the set

$$\mathfrak{X} := [1, 2]^{\mathbb{N}^4} \times [1, 2]^{\mathcal{S}},$$

and we denote by $\mathbb{E}_{x,y}$ the expectation with respect to the product uniform measure on $\mathfrak{X}$. The parameter z ranges over $[0, 3]^{\mathbb{N}}$ in the statements and over $[1, 2]^{\mathbb{N}}$ in the averaging arguments; $\mathbb{E}_z$ denotes the expectation with respect to the product uniform measure on $[1, 2]^{\mathbb{N}}$, and $\mathbb{E}_{x,y,z}$ the product of these expectations, with associated probability measure $\mathbb{P}_{x,y,z}$. Recall $\bar{F}_N^{x,y}$ and $\langle \cdot \rangle_N^{x,y,q}$ from (3.23), and $\tilde{F}_N^{x,y}$ and $\langle \cdot \rangle_N^{x,y,q}$ from (3.57).

4.1. Perturbation of the path and derivatives of the free energies.

A separating family of paths. Let $(e_s)_{s \in \mathcal{S}}$ denote the standard basis of $\mathbb{R}^{\mathcal{S}}$. Let $\varphi : \mathbb{R} \rightarrow [0, \infty)$ be smooth, with $\int_{\mathbb{R}} \varphi = 1$ and $\text{supp } \varphi \subseteq (0, 1)$, and set $\varphi_\varepsilon(v) := \varepsilon^{-1} \varphi(v/\varepsilon)$. Let $(\kappa_i)_{i \geq 1}$ enumerate the paths

$$\{(\mathbb{1}_{[a,\infty)} * \varphi_\varepsilon) e_s : a \in \mathbb{Q} \cap [0, 1), s \in \mathcal{S}, \varepsilon \in \mathbb{Q} \cap (0, 1)\}$$

restricted to $[0, 1]$, where $(\mathbb{1}_{[a,\infty)} * \varphi_\varepsilon)(u) := \int_{\mathbb{R}} \mathbb{1}_{[a,\infty)}(u-v) \varphi_\varepsilon(v) dv$. Each κ_i is a smooth path in $\mathcal{Q}_\infty^{\mathcal{S}}$ with values in $[0, 1]^{\mathcal{S}}$, and for every $p, p' \in \mathcal{Q}_2^{\mathcal{S}}$,

$$(4.1) \quad p = p' \iff \langle \kappa_i, p \rangle_{L^2} = \langle \kappa_i, p' \rangle_{L^2} \quad \text{for every } i \geq 1.$$

Indeed, as $\varepsilon \downarrow 0$, the paths $(\mathbb{1}_{[a,\infty)} * \varphi_\varepsilon) e_s$ converge in L^2 to $\mathbb{1}_{[a,1)} e_s$, so that equality of the pairings with every κ_i implies $\int_a^1 p^s = \int_a^1 p'^s$ for every $a \in \mathbb{Q} \cap [0, 1)$ and $s \in \mathcal{S}$, hence $p = p'$ almost everywhere; the converse is immediate. This is the family used in [9, Section 6]. Set

$$(4.2) \quad a_i := 3^{-1} 2^{-i} \max\{1, |\dot{\kappa}_i|_{L^\infty}\}^{-1}, \quad i \geq 1,$$

and, for $z = (z_i)_{i \geq 1} \in [0, 3]^{\mathbb{N}}$ and $N \in \mathbb{N}$, define the perturbed path

$$(4.3) \quad q_N(z) := \eta_N^{1/2} \sum_{i=1}^{\infty} a_i z_i \kappa_i,$$

with $\eta_N = N^{-1/16}$ as in (3.9). By (4.2) and since $0 \leq \kappa_i \leq 1$, the series in (4.3) and the series of the derivatives $a_i z_i \dot{\kappa}_i$ converge uniformly on $[0, 1]$, so that $q_N(z)$ is a nondecreasing, continuously differentiable path, with

$$(4.4) \quad \sup_{z \in [0, 3]^{\mathbb{N}}} (|q_N(z)|_{L^\infty} + |\dot{q}_N(z)|_{L^\infty}) \leq 2\eta_N^{1/2}.$$

For $i \geq 1$, $z \in [0, 3]^{\mathbb{N}}$, and $r \in [0, 3]$, we write $z^{i,r}$ for the element of $[0, 3]^{\mathbb{N}}$ obtained from z by replacing its i -th coordinate by r .

Dependence of the perturbed free energies on the path. The perturbed Hamiltonians depend on the path q both through the cascade field W^q and through the radial perturbation, which contains radial derivatives of W^q . The following lemma records that this dependence is again that of a cascade field, with a rescaled path, up to a deterministic term that involves the endpoint value $q(1)$. Because of this endpoint term, the perturbed free energies are not differentiable in q with respect to the L^2 norm. We therefore work with directional derivatives. Given $q \in \mathcal{Q}_\infty^{\mathcal{S}}$, we say that a direction $\kappa \in L^\infty([0, 1]; \mathbb{R}^{\mathcal{S}})$ is admissible at q if $q + \varepsilon \kappa \in \mathcal{Q}_\infty^{\mathcal{S}}$ for every ε in a neighborhood of 0. An admissible direction is a multiple of the difference of two elements of $\mathcal{Q}_\infty^{\mathcal{S}}$, so that the limit

$\kappa(1) := \lim_{u \nearrow 1} \kappa(u)$ exists, in agreement with the endpoint convention (2.3), and $|\kappa^s(1)| \leq |\kappa|_{L^\infty}$ for every $s \in \mathcal{S}$. For a function F of (t, q) , we set

$$(4.5) \quad D_\kappa F(t, q) := \left. \frac{d}{d\varepsilon} F(t, q + \varepsilon \kappa) \right|_{\varepsilon=0}$$

whenever this derivative exists. When F is Gateaux differentiable with an L^2 gradient, as is the case for $\bar{F}_N$ by Proposition 2.1, we simply have $D_\kappa F(t, q) = \langle \kappa, \partial_q F(t, q) \rangle_{L^2}$. We use the notation (4.5) only for the perturbed free energies.

Lemma 4.1 (Dependence on the path). *Let $K \in \{N, N+M\}$, $(x, y) \in \mathfrak{X}$, and $q \in \mathcal{Q}_\infty^\mathcal{S}$, and define the rescaled path*

$$(4.6) \quad T_{K,y}q := \left((1 + \eta_K y^s)^2 q^s \right)_{s \in \mathcal{S}} \in \mathcal{Q}_\infty^\mathcal{S}.$$

Then $H_K^{x,y,q}(\rho, \alpha)$ is the sum of three terms: a term that does not depend on q ; the cascade field $\sqrt{2}W_K^{T_{K,y}q}(\rho, \alpha)$, together with its self-overlap correction $-K(T_{K,y}q)(1) \cdot \lambda$; and the deterministic term

$$(4.7) \quad K \eta_K^2 \sum_{s \in \mathcal{S}} (y^s)^2 \lambda_s q^s(1),$$

which is affine in q . The same holds for $\tilde{H}_N^{x,y,q}$, with $K = N$. Consequently, for every direction κ admissible at q , the functions $\varepsilon \mapsto \tilde{F}_N^{x,y}(t, q + \varepsilon \kappa)$ and $\varepsilon \mapsto \bar{F}_K^{x,y}(t, q + \varepsilon \kappa)$ are twice continuously differentiable in a neighborhood of 0, and the directional derivatives (4.5) are given by

$$(4.8) \quad D_\kappa \tilde{F}_N^{x,y}(t, q) = \sum_{s \in \mathcal{S}} (1 + \eta_N y^s)^2 \mathbb{E} \langle \kappa^s(\alpha \wedge \alpha') R_{N,s}(\sigma, \sigma') \rangle_N^{\sim, x, y, q} - \eta_N^2 \sum_{s \in \mathcal{S}} (y^s)^2 \lambda_s \kappa^s(1),$$

$$(4.9) \quad D_\kappa \bar{F}_K^{x,y}(t, q) = \sum_{s \in \mathcal{S}} (1 + \eta_K y^s)^2 \mathbb{E} \langle \kappa^s(\alpha \wedge \alpha') R_{K,s}(\rho, \rho') \rangle_K^{x, y, q} - \eta_K^2 \sum_{s \in \mathcal{S}} (y^s)^2 \lambda_s \kappa^s(1).$$

In particular, there exists a constant $C < \infty$, independent of N, x, y, q , and κ , such that

$$(4.10) \quad \left| D_\kappa \tilde{F}_N^{x,y}(t, q) - \mathbb{E} \langle \kappa(\alpha \wedge \alpha') \cdot R_N(\sigma, \sigma') \rangle_N^{\sim, x, y, q} \right| \leq C \eta_N |\kappa|_{L^\infty},$$

$$(4.11) \quad \left| D_\kappa \bar{F}_{N+M}^{x,y}(t, q) - \mathbb{E} \langle \kappa(\alpha \wedge \alpha') \cdot R_{N+M}(\rho, \rho') \rangle_{N+M}^{x, y, q} \right| \leq C \eta_N |\kappa|_{L^\infty},$$

and such that, for every $q, q' \in \mathcal{Q}_\infty^\mathcal{S}$ and $F \in \{\tilde{F}_N^{x,y}, \bar{F}_K^{x,y}\}$,

$$(4.12) \quad |F(t, q) - F(t, q')| \leq C |q - q'|_{L^1} + C \eta_K^2 \sum_{s \in \mathcal{S}} |q^s(1) - q'^s(1)|.$$

Moreover, the semi-concavity estimate (2.20) of Proposition 2.2 holds for $\tilde{F}_N^{x,y}(t, \cdot)$ and $\bar{F}_{N+M}^{x,y}(t, \cdot)$ in place of $\bar{F}_N(t, \cdot)$, with a constant C independent of N, x, y .

The estimates (4.10)–(4.11) say that, up to an error of order η_N , the directional derivatives of the perturbed free energies are the overlap pairings appearing in (2.19); the endpoint term only contributes an error of order η_N^2 . The Lipschitz estimate (4.12) is the analogue of (2.17), with the endpoint term kept explicit; it is used in the telescoping argument of Lemma 4.6.

Proof of Lemma 4.1. By (3.20)–(3.22) and (2.11), the q -dependent part of $H_K^{x,y,q}(\rho, \alpha)$ is

$$\sqrt{2}W_K^q(\rho, \alpha) - \sum_{s \in \mathcal{S}} q^s(1) |\rho^{(s)}|^2 + 2\eta_K \sum_{s \in \mathcal{S}} y^s \partial_{r^s} \left(\sqrt{2}W_K^q(S_r \rho, \alpha) - \sum_{s' \in \mathcal{S}} q^{s'}(1) |(S_r \rho)^{(s')}|^2 \right) \Big|_{r=\bar{\mathbf{1}}}.$$

Since $W_K^q(S_r \rho, \alpha) = \sum_{s'} \sqrt{r^{s'}} \sum_{i \in I_{K,s'}} w_i^{q^{s'}}(\alpha) \rho_i$ and $|(S_r \rho)^{(s')}|^2 = r^{s'} |\rho^{(s')}|^2$, the radial derivative in r^s at $r = \bar{\mathbf{1}}$ equals $\frac{1}{2} \sqrt{2} \sum_{i \in I_{K,s}} w_i^{q^s}(\alpha) \rho_i - q^s(1) |\rho^{(s)}|^2$. Collecting terms and using $|\rho^{(s)}|^2 = |I_{K,s}| = K \lambda_s$ on Σ_K , the q -dependent part is

$$\sqrt{2} \sum_{s \in \mathcal{S}} (1 + \eta_K y^s) \sum_{i \in I_{K,s}} w_i^{q^s}(\alpha) \rho_i - K \sum_{s \in \mathcal{S}} (1 + 2\eta_K y^s) q^s(1) \lambda_s.$$

The first sum is a cascade field with path $T_{K,y}q$, since $(1 + \eta_K y^s) w_i^{q^s}$ has covariance $(1 + \eta_K y^s)^2 q^s(\alpha \wedge \alpha')$, and since $(1 + 2\eta_K y^s) = (1 + \eta_K y^s)^2 - \eta_K^2 (y^s)^2$, the second sum is the self-overlap correction for this field plus the term (4.7). The argument for $\tilde{H}_N^{x,y,q}$ is identical.

We now compute the directional derivatives. Let κ be admissible at q , and let $\varepsilon_0 > 0$ be such that $q_{\pm} := q \pm \varepsilon_0 \kappa \in \mathcal{Q}_{\infty}^{\mathcal{J}}$. For $|\varepsilon| < \varepsilon_0$, the path $q + \varepsilon \kappa$ is the convex combination $\frac{1}{2}(1 - \varepsilon/\varepsilon_0)q_- + \frac{1}{2}(1 + \varepsilon/\varepsilon_0)q_+$, and the same is true after applying the linear map $T_{K,y}$. As in the proof of [10, Proposition 5.1], we realize the cascade fields with paths $T_{K,y}(q + \varepsilon \kappa)$ jointly for all $|\varepsilon| < \varepsilon_0$ as $\sqrt{\frac{1}{2}(1 - \varepsilon/\varepsilon_0)} w^{T_{K,y}q_-} + \sqrt{\frac{1}{2}(1 + \varepsilon/\varepsilon_0)} w^{T_{K,y}q_+}$, with $w^{T_{K,y}q_-}$ and $w^{T_{K,y}q_+}$ independent. The Gaussian integration by parts leading to (2.19) applies without change to a Hamiltonian which is the sum of a cascade field, of its self-overlap correction, and of Gaussian and deterministic terms whose laws do not depend on the path of the cascade field: the derivative in ε of the free energy, at the path $T_{K,y}(q + \varepsilon \kappa)$ and in the direction $T_{K,y}\kappa$, is $\mathbb{E} \langle (T_{K,y}\kappa)(\alpha \wedge \alpha') \cdot R_K(\rho, \rho') \rangle_K^{x,y,q+\varepsilon\kappa}$. The deterministic term (4.7) enters the free energy $-\frac{1}{K} \mathbb{E} \log Z$ with a minus sign, and its derivative in ε is the constant $K \eta_K^2 \sum_s (y^s)^2 \lambda_s \kappa^s(1)$. Since $(T_{K,y}\kappa)^s = (1 + \eta_K y^s)^2 \kappa^s$, we obtain (4.9) at $\varepsilon = 0$; the same argument gives (4.8). Moreover, a second Gaussian integration by parts, as in the proof of [10, Proposition 5.1], shows that the derivative in ε computed above is itself continuously differentiable on $(-\varepsilon_0, \varepsilon_0)$, so that the free energies are twice continuously differentiable in ε there.

For (4.10) and (4.11), we use $0 \leq y^s \leq 3$, $\eta_{N+M} \leq \eta_N \leq 1$, and $|R_{K,s}| \leq 1$: the factors $(1 + \eta_K y^s)^2$ differ from 1 by at most $15\eta_N$, and the endpoint term is bounded by $9\eta_N^2 |\kappa|_{L^\infty}$.

For (4.12), let $q, q' \in \mathcal{Q}_{\infty}^{\mathcal{J}}$ and $\kappa := q' - q$. The segment $\varepsilon \mapsto q + \varepsilon \kappa$, $\varepsilon \in [0, 1]$, stays in $\mathcal{Q}_{\infty}^{\mathcal{J}}$, and, realizing the cascade fields along it as in the proof of [10, Proposition 5.1], the free energy is a continuous function of $\varepsilon \in [0, 1]$, continuously differentiable on $(0, 1)$, with derivative given by the right-hand side of (4.9) (or (4.8)) evaluated at the path $q + \varepsilon \kappa$. By the invariance (2.5), the overlap $\alpha \wedge \alpha'$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_K^{x,y,q+\varepsilon\kappa}$, so that, using $|R_{K,s}| \leq \lambda_s$ and $\sum_s \lambda_s |a_s| \leq |a|$ for $a \in \mathbb{R}^{\mathcal{J}}$, the first term of (4.9) is bounded by $16 \sum_s \lambda_s \mathbb{E} \langle |\kappa^s(\alpha \wedge \alpha')| \rangle_K^{x,y,q+\varepsilon\kappa} \leq 16 |\kappa|_{L^1}$, while the endpoint term is bounded by $9\eta_K^2 \sum_s |\kappa^s(1)|$. Integrating over $\varepsilon \in [0, 1]$ gives (4.12).

Finally, the last assertion of Proposition 2.2 allows for the addition of Gaussian fields independent of the cascade field and of affine deterministic terms, such as (4.7); applying it with the path $T_{K,y}q$, whose derivative differs from that of q by bounded factors, gives the semi-concavity estimate. $\square$

4.2. Matching the derivatives. The starting point is that the free energies of the bulk and of the $(N + M)$ -spin systems are close: by Lemma 3.11, for every bounded subset $Q \subseteq \mathcal{Q}_{\infty}^{\mathcal{J}}$,

$$(4.13) \quad \varepsilon_N(Q) = \sup_{x,y,q} \left| \tilde{F}_N^{x,y}(t, q) - \bar{F}_{N+M}^{x,y}(t, q) \right| \leq \frac{C_Q}{N}, \quad \text{so that} \quad \lim_{N \rightarrow \infty} \eta_N^{-1/2} \varepsilon_N(Q) = 0.$$

Lemma 4.2 (One-dimensional bounds). *Fix $c \in (0, 1]$. There exists $C_c < \infty$ such that the following holds for every $i \geq 1$, $N \in \mathbb{N}$, $(x, y) \in \mathfrak{X}$, $q \in \mathcal{Q}_{\uparrow,c}^{\mathcal{J}} \cap \mathcal{Q}_{\infty}^{\mathcal{J}}$, and $z \in [0, 3]^{\mathbb{N}}$. Define, for $r \in [0, 3]$,*

$$\mathbf{f}(r) := \tilde{F}_N^{x,y}(t, q + q_N(z^{i,r})), \quad \mathbf{g}(r) := \bar{F}_{N+M}^{x,y}(t, q + q_N(z^{i,r})).$$

Then $\mathbf{f}$ and $\mathbf{g}$ are twice continuously differentiable on $[0, 3]$, with

$$(4.14) \quad \mathbf{f}'(r) = \eta_N^{1/2} a_i D_{\kappa_i} \tilde{F}_N^{x,y}(t, q + q_N(z^{i,r})) \quad \text{and} \quad \mathbf{g}'(r) = \eta_N^{1/2} a_i D_{\kappa_i} \bar{F}_{N+M}^{x,y}(t, q + q_N(z^{i,r})),$$

and

$$(4.15) \quad |\mathbf{f}'| + |\mathbf{g}'| \leq C_c \eta_N^{1/2}, \quad \mathbf{f}'' \leq C_c \eta_N, \quad \mathbf{g}'' \leq C_c \eta_N$$

on $[0, 3]$.

Proof. By (4.3), the map $r \mapsto q + q_N(z^{i,r})$ is affine, with derivative $\eta_N^{1/2} a_i \kappa_i$, and it takes values in $\mathcal{Q}_{\uparrow,c}^{\mathcal{J}}$ because κ_i is nonnegative and nondecreasing. Since q has slope at least $c > 0$ while κ_i vanishes at the origin and is nondecreasing and Lipschitz, the direction κ_i is admissible at $q + q_N(z^{i,r})$ for every $r \in [0, 3]$, in the sense of Lemma 4.1. For $r \in [0, 3]$, the function $\varepsilon \mapsto \tilde{F}_N^{x,y}(t, q + q_N(z^{i,r}) + \varepsilon \eta_N^{1/2} a_i \kappa_i)$ coincides with $\mathbf{f}(r + \varepsilon)$ when $r + \varepsilon \in [0, 3]$, and it is twice continuously differentiable in a neighborhood of 0 by Lemma 4.1. Hence $\mathbf{f}$ is twice continuously differentiable

on $[0, 3]$, and (4.14) follows from the definition (4.5) of the directional derivative; the same applies to $\mathbf{g}$. By (4.10), (4.11), and $|R_{N,s}| \leq 1$,

$$|\mathbf{f}'(r)| + |\mathbf{g}'(r)| \leq 2\eta_N^{1/2} a_i (1 + C\eta_N) |\kappa_i|_{L^\infty}.$$

The semi-concavity estimate in Lemma 4.1, applied to two points of the segment $r \mapsto q + q_N(z^{i,r})$, says that $r \mapsto \mathbf{f}(r) - Cc^{-2}\eta_N a_i^2 |\kappa_i|_{L^2}^2 r^2$ is concave on $[0, 3]$; since $\mathbf{f}$ is twice differentiable, $\mathbf{f}'' \leq 2Cc^{-2}\eta_N a_i^2 |\kappa_i|_{L^2}^2$, and the same holds for $\mathbf{g}''$. Since $a_i |\kappa_i|_{L^\infty} \leq 1$ and $a_i |\kappa_i|_{L^2} \leq a_i |\kappa_i|_{L^\infty} \leq 1$ by (4.2), the bounds (4.15) follow, with a constant that does not depend on i . $\square$

Lemma 4.3 (One-dimensional semi-concavity estimate). *There exists $C > 0$ with the following property. Let $\mathbf{f}, \mathbf{g} : [0, 3] \rightarrow \mathbb{R}$ be twice continuously differentiable, with $|\mathbf{f}'|, |\mathbf{g}'| \leq a$ and $\mathbf{f}'', \mathbf{g}'' \leq b$ on $[0, 3]$, for some $a, b > 0$. We have*

$$\int_1^2 |\mathbf{f}'(r) - \mathbf{g}'(r)|^2 dr \leq C(a + b) \|\mathbf{f} - \mathbf{g}\|_{L^\infty[0,3]}.$$

Proof. This is [9, Lemma 6.2]; we recall the argument. Choose a smooth function $\zeta : [0, 3] \rightarrow [0, 1]$ with compact support in $(0, 3)$ and such that $\zeta = 1$ on $[1, 2]$. Integration by parts gives

$$\int_1^2 |\mathbf{f}' - \mathbf{g}'|^2 \leq \int_0^3 \zeta (\mathbf{f}' - \mathbf{g}')^2 = - \int_0^3 \zeta' (\mathbf{f} - \mathbf{g}) (\mathbf{f}' - \mathbf{g}') - \int_0^3 \zeta (\mathbf{f} - \mathbf{g}) (\mathbf{f}'' - \mathbf{g}'').$$

The first term on the right-hand side is bounded by $6a|\zeta'|_{L^\infty} \|\mathbf{f} - \mathbf{g}\|_{L^\infty}$. Since $b - \mathbf{f}'' \geq 0$ and $b - \mathbf{g}'' \geq 0$,

$$\int_0^3 |\mathbf{f}'' - \mathbf{g}''| \leq \int_0^3 ((b - \mathbf{f}'') + (b - \mathbf{g}'')) = 6b - \mathbf{f}'(3) - \mathbf{g}'(3) + \mathbf{f}'(0) + \mathbf{g}'(0) \leq 6b + 4a,$$

which bounds the second term. $\square$

Given $q \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$, we set, for $i \geq 1$, $N \in \mathbb{N}$, $(x, y) \in \mathfrak{X}$, and $z \in [0, 3]^\mathbb{N}$,

$$(4.16) \quad \mathfrak{D}_{N,i}(x, y, z) := \left| D_{\kappa_i} \tilde{F}_N^{x,y}(t, q + q_N(z)) - D_{\kappa_i} \bar{F}_{N+M}^{x,y}(t, q + q_N(z)) \right|^2,$$

where the directional derivatives are well defined by the observation on admissibility made in the proof of Lemma 4.2.

Lemma 4.4 (Derivative matching). *For every $q \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$ and every $i \geq 1$,*

$$(4.17) \quad \lim_{N \rightarrow \infty} \mathbb{E}_{x,y,z} \mathfrak{D}_{N,i}(x, y, z) = 0.$$

Proof. Choose $c \in (0, 1]$ such that $q \in \mathcal{Q}_{\uparrow, c}^\mathcal{S}$, and set $Q := \{q + q_N(z) : N \in \mathbb{N}, z \in [0, 3]^\mathbb{N}\}$, which is a bounded subset of $\mathcal{Q}_\infty^\mathcal{S}$ by (4.4). Fix $i \geq 1$, $(x, y) \in \mathfrak{X}$, and $z \in [1, 2]^\mathbb{N}$, and let $\mathbf{f}$ and $\mathbf{g}$ be as in Lemma 4.2. By that lemma, the assumptions of Lemma 4.3 hold with $a = C_c \eta_N^{1/2}$ and $b = C_c \eta_N$, and $\|\mathbf{f} - \mathbf{g}\|_{L^\infty[0,3]} \leq \varepsilon_N(Q)$ by the definition of $\varepsilon_N(Q)$ in (4.13). Lemma 4.3 therefore gives

$$\int_1^2 |\mathbf{f}'(r) - \mathbf{g}'(r)|^2 dr \leq C_c \left(\eta_N^{1/2} + \eta_N \right) \varepsilon_N(Q).$$

By (4.14) and (4.16), we have $|\mathbf{f}'(r) - \mathbf{g}'(r)|^2 = \eta_N a_i^2 \mathfrak{D}_{N,i}(x, y, z^{i,r})$, so that the left-hand side equals $\eta_N a_i^2 \int_1^2 \mathfrak{D}_{N,i}(x, y, z^{i,r}) dr$. Integrating over the remaining coordinates $(z_j)_{j \neq i} \in [1, 2]^{\mathbb{N} \setminus \{i\}}$, we obtain

$$\mathbb{E}_z \mathfrak{D}_{N,i}(x, y, z) \leq C_c a_i^{-2} \eta_N^{-1/2} \left(1 + \eta_N^{1/2} \right) \varepsilon_N(Q),$$

uniformly over $(x, y) \in \mathfrak{X}$. The right-hand side tends to zero by (4.13), which proves (4.17). $\square$

4.3. Ghirlanda–Guerra identities and choice of the parameters. Recall from (3.139) that

$$\Delta_N(x, y, q) = \sum_{j=1}^{\infty} 2^{-j} \Delta_N^{x,y,q}(\mathbf{f}_j, n_j, \mathbf{h}_j),$$

where $\Delta_N^{x,y,q}(\mathbf{f}, n, \mathbf{h})$, defined in (3.137), measures the failure of the Ghirlanda–Guerra identity for the bulk system $\mathbb{E} \langle \cdot \rangle_N^{x,y,q}$, tested against a function $\mathbf{f}$ of the overlaps of n replicas and the kernel $\mathbf{C}_h$, and where the triples $(\mathbf{f}_j, n_j, \mathbf{h}_j)$ enumerate the monomial test functions, normalized so that (3.138) holds.

Lemma 4.5 (Ghirlanda–Guerra identities on average). *For every bounded subset $Q \subseteq \mathcal{Q}_{\infty}^{\mathcal{S}}$,*

$$\lim_{N \rightarrow \infty} \sup_{q \in Q} \mathbb{E}_{x,y} \Delta_N(x, y, q) = 0.$$

Proof. Fix $\mathbf{h} \in \mathbb{N}^4$, and set

$$H_N^{\mathbf{h},y}(\sigma, \alpha) := H_N^{\mathbf{h}}(\sigma, \alpha) + 2\eta_N \sum_{s \in \mathcal{S}} y^s \partial_{r^s} H_N^{\mathbf{h}}(S_r \sigma, \alpha) \Big|_{r=\mathbf{1}}.$$

By (3.19), (3.55), and (3.56), the dependence of $\tilde{H}_N^{x,y,q}$ on the coordinate $x_{\mathbf{h}}$ is

$$\tilde{H}_N^{x,y,q}(\sigma, \alpha) = \tilde{H}_{N,\mathbf{h}}^{x,y,q}(\sigma, \alpha) + \eta_N x_{\mathbf{h}} \mathbf{C}_h H_N^{\mathbf{h},y}(\sigma, \alpha),$$

where $\tilde{H}_{N,\mathbf{h}}^{x,y,q}$ does not depend on $x_{\mathbf{h}}$ and is independent of $H_N^{\mathbf{h},y}$. By Lemma 3.2,

$$\mathbb{E} H_N^{\mathbf{h},y}(\sigma, \alpha) H_N^{\mathbf{h},y}(\sigma', \alpha') = N(\mathbf{C}_h)_{\eta_N, y} (R_N(\sigma, \sigma'), \alpha \wedge \alpha'),$$

and by (3.25) and (3.17),

$$\sup_{y \in [1,2]^{\mathcal{S}}} \sup_{v \in [-2,2]^{\mathcal{S}}, u \in [0,1]} |(\mathbf{C}_h)_{\eta_N, y}(v, u) - \mathbf{C}_h(v, u)| \leq C \Lambda_{\mathbf{h}}(\eta_N + \eta_N^2).$$

The perturbation $\eta_N x_{\mathbf{h}} \mathbf{C}_h H_N^{\mathbf{h},y}$ is thus of exactly the form treated in the proof of [10, Proposition 6.8], with the same coefficient $\eta_N = N^{-1/16}$, except that the covariance kernel $\mathbf{C}_h$ is replaced by $(\mathbf{C}_h)_{\eta_N, y}$. That proof uses the differentiation of the free energy in $x_{\mathbf{h}}$ followed by Gaussian integration by parts, the convexity of $\log Z_N^{x,y,q}$ in $x_{\mathbf{h}}$, and the concentration of $\log Z_N^{x,y,q}$ around its expectation. The latter is provided by Lemma B.1: the Hamiltonian $\tilde{H}_N^{x,y,q}$ is of the form (B.1), see Remark 3.1, and by (3.26), (3.58), and (3.59), its variance is bounded by CN uniformly over $q \in Q$, x , and $y \in [1,2]^{\mathcal{S}}$. Hence $\sup_{x,y,q} \mathbb{E} |\log Z_N^{x,y,q} - \mathbb{E} \log Z_N^{x,y,q}| \leq CN^{1/2}$, which is the concentration bound on which the argument of [10] rests. That argument then applies verbatim, and the estimates it yields are uniform over (q, y) . The Gaussian integration by parts produces the Ghirlanda–Guerra identities for the array $(\mathbf{C}_h)_{\eta_N, y}(R_N^{\ell, \ell'}, R_{\alpha}^{\ell, \ell'})$, and replacing this kernel by $\mathbf{C}_h$ changes each term of (3.137) by at most $C \Lambda_{\mathbf{h}}(\eta_N + \eta_N^2) \|\mathbf{f}\|_{\infty}$. Therefore, for every $j \in \mathbb{N}$,

$$\lim_{N \rightarrow \infty} \sup_{q \in Q} \mathbb{E}_{x,y} \Delta_N^{x,y,q}(\mathbf{f}_j, n_j, \mathbf{h}_j) = 0.$$

Since $0 \leq \Delta_N^{x,y,q}(\mathbf{f}_j, n_j, \mathbf{h}_j) \leq 1$ by (3.138), the definition (3.139) and dominated convergence complete the proof. $\square$

We now combine Lemmas 4.5, 3.14, and 4.4 to select the parameters. Given $q \in \mathcal{Q}_{\infty, \uparrow}^{\mathcal{S}}$, we set, with $\mathfrak{D}_{N,i}$ as in (4.16),

$$\mathfrak{D}_N(x, y, z) := \sum_{i=1}^{\infty} 2^{-i} \mathfrak{D}_{N,i}(x, y, z),$$

and we abbreviate

$$\begin{aligned} A_N(x, y, z) &:= A_N(x, y, q + q_N(z)), & \Delta_N(x, y, z) &:= \Delta_N(x, y, q + q_N(z)), \\ \text{Rad}_{N,L}(x, y, z) &:= \text{Rad}_{N,L}(x, y, q + q_N(z)), \end{aligned}$$

where A_N and $\text{Rad}_{N,L}$ are defined in (3.39) and (3.89).

Lemma 4.6 (Choice of the parameters). *For every $q \in \mathcal{Q}_{\infty, \uparrow}^{\mathcal{S}}$, there exist $(x_n, y_n) \in \mathfrak{X}$ and $z_n \in [1, 2]^{\mathbb{N}}$ such that, with $q_n := q + q_{N_n}(z_n)$,*

$$(4.18) \quad \lim_{n \rightarrow \infty} \Delta_{N_n}(x_n, y_n, z_n) = 0,$$

$$(4.19) \quad \lim_{n \rightarrow \infty} \text{Rad}_{N_n, L}(x_n, y_n, z_n) = 0 \quad \text{for every } L \in \mathbb{N},$$

$$(4.20) \quad \lim_{n \rightarrow \infty} \left(D_{\kappa_i} \tilde{F}_{N_n}^{x_n, y_n}(t, q_n) - D_{\kappa_i} \bar{F}_{N_n+M}^{x_n, y_n}(t, q_n) \right) = 0 \quad \text{for every } i \geq 1,$$

$$(4.21) \quad \limsup_{n \rightarrow \infty} \bar{F}_{N_n}(t, q) \leq \limsup_{n \rightarrow \infty} \left(-\frac{1}{M} A_{N_n}(x_n, y_n, z_n) \right).$$

Proof. Set $Q := \{q + q_N(z) : N \in \mathbb{N}, z \in [0, 3]^{\mathbb{N}}\}$, a bounded subset of $\mathcal{Q}_{\infty}^{\mathcal{S}}$ by (4.4). All the constants and the sets in (3.10) are understood with this Q .

Step 1: Averaged bounds.

Lemma 4.5 gives $\lim_{N \rightarrow \infty} \mathbb{E}_{x, y, z} \Delta_N(x, y, z) = 0$. Lemma 3.14, applied with the set Q , gives $\lim_{N \rightarrow \infty} \mathbb{E}_{x, y, z} \text{Rad}_{N, L}(x, y, z) = 0$ for every $L \in \mathbb{N}$. Moreover, by Lemma 3.13 and Jensen's inequality, for every $L \in \mathbb{N}$ there is $K_L < \infty$ such that $\sup_{N, x, y, z} \text{Rad}_{N, L}(x, y, z) \leq K_L$. Since $0 \leq \frac{2^{-L}}{1+K_L} \text{Rad}_{N, L} \leq 2^{-L}$, dominated convergence gives

$$\lim_{N \rightarrow \infty} \mathbb{E}_{x, y, z} \sum_{L=1}^{\infty} \frac{2^{-L}}{1+K_L} \text{Rad}_{N, L}(x, y, z) = 0.$$

Finally, (4.10), (4.11), and $0 \leq \kappa_i \leq 1$ give $\mathfrak{D}_{N, i} \leq C$ uniformly in i, N, x, y, z ; Lemma 4.4 and dominated convergence therefore yield $\lim_{N \rightarrow \infty} \mathbb{E}_{x, y, z} \mathfrak{D}_N(x, y, z) = 0$.

Step 2: Selection of the parameters.

Set

$$\mathfrak{E}_N(x, y, z) := \Delta_N(x, y, z) + \mathfrak{D}_N(x, y, z) + \sum_{L=1}^{\infty} \frac{2^{-L}}{1+K_L} \text{Rad}_{N, L}(x, y, z),$$

a nonnegative quantity with $\lim_{N \rightarrow \infty} \mathbb{E}_{x, y, z} \mathfrak{E}_N(x, y, z) = 0$ by Step 1. By Lemma 3.6, there is $C_A < \infty$ such that $|A_N(x, y, z)| \leq C_A$ for every sufficiently large N and every $(x, y, z) \in \mathfrak{X} \times [1, 2]^{\mathbb{N}}$. For $\varepsilon \in (0, C_A]$, consider the event

$$\Omega_{\varepsilon, N} := \{(x, y, z) : A_N(x, y, z) \leq \mathbb{E}_{x, y, z} A_N(x, y, z) + \varepsilon\}.$$

Since $A_N - \mathbb{E}_{x, y, z} A_N$ is centered, bounded below by $-2C_A$, and larger than ε on the complement of $\Omega_{\varepsilon, N}$, we have $0 \geq \varepsilon \mathbb{P}_{x, y, z}(\Omega_{\varepsilon, N}^c) - 2C_A \mathbb{P}_{x, y, z}(\Omega_{\varepsilon, N})$, hence

$$\mathbb{P}_{x, y, z}(\Omega_{\varepsilon, N}) \geq \frac{\varepsilon}{2C_A + \varepsilon} \geq \frac{\varepsilon}{3C_A},$$

which is the averaging argument of [22, Lemma 3.3]. On the other hand, Markov's inequality gives $\mathbb{P}_{x, y, z}(\mathfrak{E}_N \leq \varepsilon) \geq 1 - \varepsilon^{-1} \mathbb{E}_{x, y, z} \mathfrak{E}_N$. Take $\varepsilon_N := 2 \left(C_A (\mathbb{E}_{x, y, z} \mathfrak{E}_N + N^{-1}) \right)^{1/2}$, so that $\varepsilon_N \rightarrow 0$ and $\varepsilon_N^2 > 3C_A \mathbb{E}_{x, y, z} \mathfrak{E}_N$. For every sufficiently large N , we have $\varepsilon_N \leq C_A$, and the sum of the two probabilities above exceeds one, so that $\Omega_{\varepsilon_N, N} \cap \{\mathfrak{E}_N \leq \varepsilon_N\} \neq \emptyset$. We choose (x_n, y_n, z_n) in this intersection for $N = N_n$. Then

$$\lim_{n \rightarrow \infty} \mathfrak{E}_{N_n}(x_n, y_n, z_n) = 0 \quad \text{and} \quad A_{N_n}(x_n, y_n, z_n) \leq \mathbb{E}_{x, y, z} A_{N_n}(x, y, z) + \varepsilon_{N_n}.$$

Since all the terms in $\mathfrak{E}_N$ are nonnegative with positive weights, the first limit gives (4.18), (4.19), and $\lim_n \mathfrak{D}_{N_n}(x_n, y_n, z_n) = 0$; the latter implies $\lim_n \mathfrak{D}_{N_n, i}(x_n, y_n, z_n) = 0$ for every i , which is (4.20).

Step 3: Free energy as an average of cavity increments.

By (3.39) and the definition of $A_N(x, y, z)$,

$$\begin{aligned} & -N_{j+1} \bar{F}_{N_{j+1}}^{x, y}(t, q + q_{N_{j+1}}(z)) + N_j \bar{F}_{N_j}^{x, y}(t, q + q_{N_j}(z)) \\ & = A_{N_j}(x, y, z) - N_{j+1} \left(\bar{F}_{N_{j+1}}^{x, y}(t, q + q_{N_{j+1}}(z)) - \bar{F}_{N_{j+1}}^{x, y}(t, q + q_{N_j}(z)) \right). \end{aligned}$$

By (4.3), we have $q_{N_{j+1}}(z) - q_{N_j}(z) = (\eta_{N_{j+1}}^{1/2} - \eta_{N_j}^{1/2}) \sum_i a_i z_i \kappa_i$, and the path $\sum_i a_i z_i \kappa_i$ is bounded by 1 in L^∞ by (4.2). Since $\eta_{N_j}^{1/2} = (jM)^{-1/32}$, both $|q_{N_{j+1}}(z) - q_{N_j}(z)|_{L^1}$ and $\sum_s |q_{N_{j+1}}^s(z)(1) - q_{N_j}^s(z)(1)|$ are bounded by $Cj^{-1-1/32}$. The Lipschitz estimate (4.12), which applies to the perturbed free energy and accounts for the endpoint term in Lemma 4.1, therefore bounds the last term by $CN_{j+1}j^{-1-1/32} \leq Cj^{-1/32}$. Summing over $1 \leq j < n$ and dividing by $N_n = nM$, we obtain, uniformly over $(x, y, z) \in \mathfrak{X} \times [1, 2]^{\mathbb{N}}$,

$$(4.22) \quad \overline{F}_{N_n}^{x,y}(t, q + q_{N_n}(z)) = -\frac{1}{nM} \sum_{j=1}^{n-1} A_{N_j}(x, y, z) + o(1),$$

where we also used that the term $N_1 \overline{F}_{N_1}^{x,y}(t, q + q_{N_1}(z))$ is bounded. Moreover, removing the perturbations at the path $q + q_{N_n}(z)$ costs at most $C\eta_{N_n}$ by (3.61) and (3.62), and then (2.17) and (4.4) give $|\overline{F}_{N_n}(t, q + q_{N_n}(z)) - \overline{F}_{N_n}(t, q)| \leq C\eta_{N_n}^{1/2}$; hence

$$\sup_{(x,y,z) \in \mathfrak{X} \times [1,2]^{\mathbb{N}}} \left| \overline{F}_{N_n}^{x,y}(t, q + q_{N_n}(z)) - \overline{F}_{N_n}(t, q) \right| = o(1).$$

Averaging (4.22) over (x, y, z) therefore gives

$$(4.23) \quad \overline{F}_{N_n}(t, q) = -\frac{1}{nM} \sum_{j=1}^{n-1} \mathbb{E}_{x,y,z} A_{N_j}(x, y, z) + o(1).$$

Since $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} v_j \leq \limsup_{j \rightarrow \infty} v_j$ for every bounded sequence (v_j) of real numbers, we deduce from (4.23) and the choice of (x_n, y_n, z_n) in Step 2 that

$$\limsup_{n \rightarrow \infty} \overline{F}_{N_n}(t, q) \leq \limsup_{n \rightarrow \infty} \left(-\frac{1}{M} \mathbb{E}_{x,y,z} A_{N_n}(x, y, z) \right) \leq \limsup_{n \rightarrow \infty} \left(-\frac{1}{M} A_{N_n}(x_n, y_n, z_n) \right),$$

which is (4.21). $\square$

4.4. Proof of Proposition 2.12.

Proof. Step 1: The upper bound.

Let (x_n, y_n, z_n, q_n) be given by Lemma 4.6. By (4.4), the paths q_n are uniformly bounded and converge to q uniformly on $[0, 1]$, so that (3.140) holds. Pass to a subsequence along which $-\frac{1}{M} A_{N_n}(x_n, y_n, z_n)$ converges to its limsup. By (4.18) and (4.19), Theorem 3.24 applies along this subsequence, and provides a further subsequence, a path $p \in \mathcal{Q}_{\infty, \leq \lambda}^{\mathcal{S}}$, and a vector b admissible for $\pi := q + t\nabla \xi(p)$ such that, by (3.145),

$$(4.24) \quad \limsup_{n \rightarrow \infty} \left(-\frac{1}{M} A_{N_n}(x_n, y_n, z_n) \right) \leq \mathcal{P}_{t,q}(p).$$

Combining (4.21) and (4.24) gives (2.82).

Step 2: The critical relation.

We keep the subsequence, the path p , and the vector b of Step 1, and we show that p satisfies (2.81). Fix $i \geq 1$. As observed in the proof of Lemma 4.2, the direction κ_i is admissible at q_n , and by (4.10) with $\kappa = \kappa_i$, the directional derivative $D_{\kappa_i} \tilde{F}_{N_n}^{x_n, y_n}(t, q_n)$ differs from $\mathbb{E} \langle \kappa_i(\alpha \wedge \alpha') \cdot R_{N_n}(\sigma, \sigma') \rangle_{N_n}^{\sim, x_n, y_n, q_n}$ by at most $C\eta_{N_n}$. By part (1) of Theorem 3.24, the continuity and boundedness of κ_i , and the fact that $\alpha \wedge \alpha'$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$, we deduce that

$$(4.25) \quad \lim_{n \rightarrow \infty} D_{\kappa_i} \tilde{F}_{N_n}^{x_n, y_n}(t, q_n) = \mathbb{E} \langle \kappa_i(\alpha \wedge \alpha') \cdot p(\alpha \wedge \alpha') \rangle_{\mathfrak{R}} = \langle \kappa_i, p \rangle_{L^2}.$$

For the $(N + M)$ -spin system, we first use the exchangeability of the coordinates within each species. By (3.27) and (3.26), the law of $H_{N+M}^{x,y,q}$ is invariant under permutations of the coordinates

in $I_{N+M,s}$; so is the reference measure P_{N+M} , and therefore so is $\mathbb{E}(\cdot)_{N+M}^{x,y,q}$. Hence, for $j_0 \in \mathbf{M}_s$, using $|I_{N_n+M,s}|/(N_n+M) = \lambda_s = M_s/M$ by (2.80) and (3.2),

$$\begin{aligned}
 \mathbb{E} \langle \kappa_i^s(\alpha \wedge \alpha') R_{N_n+M,s}(\rho, \rho') \rangle_{N_n+M}^{x_n, y_n, q_n} &= \frac{|I_{N_n+M,s}|}{N_n+M} \mathbb{E} \langle \kappa_i^s(\alpha \wedge \alpha') \rho_{N_n+j_0} \rho'_{N_n+j_0} \rangle_{N_n+M}^{x_n, y_n, q_n} \\
 &= \frac{M_s}{M} \mathbb{E} \langle \kappa_i^s(\alpha \wedge \alpha') \tau_{j_0} \tau'_{j_0} \rangle_{N_n+M}^{x_n, y_n, q_n} \\
 &= \mathbb{E} \langle \kappa_i^s(\alpha \wedge \alpha') \tilde{R}_{M,s}(\tau, \tau') \rangle_{N_n+M}^{x_n, y_n, q_n}.
 \end{aligned}
 \tag{4.26}$$

Now apply part (2) of Theorem 3.24 with two replicas and $g((a^{\ell, \ell'}, r^{\ell, \ell'})_{\ell, \ell' \leq 2}) := \sum_{s \in \mathcal{S}} \kappa_i^s(r^{1,2}) a_s^{1,2}$, which is continuous with linear growth. Together with (4.11), (4.26), the identity $b = b_\pi$ from part (3), and Lemma 3.18 applied with $\kappa = \kappa_i \in \mathcal{Q}_\infty^\mathcal{S}$, this gives

$$\lim_{n \rightarrow \infty} D_{\kappa_i} \bar{F}_{N_n+M}^{x_n, y_n}(t, q_n) = \sum_{s \in \mathcal{S}} \mathbb{E} \langle \kappa_i^s(\alpha \wedge \alpha') \tilde{R}_{M,s}(\tau, \tau') \rangle_{M, b_\pi}^{\pi, \mathfrak{R}} = \langle \kappa_i, \partial_q \psi(\pi) \rangle_{L^2}.
 \tag{4.27}$$

Combining (4.20), (4.25), and (4.27), we obtain

$$\langle \kappa_i, p \rangle_{L^2} = \langle \kappa_i, \partial_q \psi(q + t \nabla \xi(p)) \rangle_{L^2}, \quad i \geq 1.$$

Since both p and $\partial_q \psi(q + t \nabla \xi(p))$ belong to $\mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$, by (2.58), the separating property (4.1) yields $p = \partial_q \psi(q + t \nabla \xi(p))$, which is (2.81). $\square$

5. HAMILTON–JACOBI EQUATION AND ONE-SIDED BOUND

In this section, we show that the Hamilton–Jacobi equation (1.12) has a unique Lipschitz viscosity solution f with initial condition ψ , and that this solution is given by the Hopf formula as displayed on the right side of (1.11) (Theorem 5.4). This relies on the well-posedness theory of [11], and on the convexity and regularity of ψ established in Section 2. We also show that f is a lower bound for the limit free energy (Proposition 5.5). This is an application of the free energy bound of [21], in the form restated in [11]. The arguments in this section do not require the species proportions to be rational, in contrast with the upper bound of Section 4; the rational approximation of Section 6 is only needed for the latter.

5.1. The equation on the cone. Let $\mathcal{H} := L^2([0, 1]; \mathbb{R}^\mathcal{S})$, with inner product $\langle \cdot, \cdot \rangle_\mathcal{H} := \langle \cdot, \cdot \rangle_{L^2}$, and view $\mathcal{Q}_2^\mathcal{S}$ as a closed convex cone in $\mathcal{H}$. This cone has empty interior in $\mathcal{H}$, and this is the main source of difficulty in the definition of viscosity solutions of (1.12). Its dual cone is

$$(\mathcal{Q}_2^\mathcal{S})^* := \{ \kappa \in \mathcal{H} : \langle \kappa, q \rangle_\mathcal{H} \geq 0 \text{ for every } q \in \mathcal{Q}_2^\mathcal{S} \}.
 \tag{5.1}$$

It has a concrete description, as in [11, Lemma 3.4 (2)] and [10, Lemma 3.5]: for $\kappa \in \mathcal{H}$,

$$\kappa \in (\mathcal{Q}_2^\mathcal{S})^* \iff \int_r^1 \kappa^s(v) dv \geq 0 \quad \text{for every } s \in \mathcal{S} \text{ and } r \in [0, 1].
 \tag{5.2}$$

Indeed, the condition on the right side is obtained by testing κ against the paths $\mathbb{1}_{[r,1]} e_s \in \mathcal{Q}_2^\mathcal{S}$, where $(e_s)_{s \in \mathcal{S}}$ is the canonical basis of $\mathbb{R}^\mathcal{S}$; conversely, every finite-step path in $\mathcal{Q}_2^\mathcal{S}$ is a linear combination with nonnegative coefficients of such paths, and finite-step paths are dense in $\mathcal{Q}_2^\mathcal{S}$. In particular, $\mathcal{Q}_2^\mathcal{S} \subseteq (\mathcal{Q}_2^\mathcal{S})^*$. Let g be a real-valued function defined on a subset G of $\mathcal{H}$, respectively of $\mathbb{R}^\mathcal{S}$. We say that g is $(\mathcal{Q}_2^\mathcal{S})^*$ -increasing, respectively $\mathbb{R}_+^\mathcal{S}$ -increasing, if $g(a) \geq g(b)$ whenever $a, b \in G$ satisfy $a - b \in (\mathcal{Q}_2^\mathcal{S})^*$, respectively $a - b \in \mathbb{R}_+^\mathcal{S}$. As for the free energy in [21, Proposition 3.8], the initial condition has this monotonicity, because its derivative is a path in the cone.

Lemma 5.1 (Monotonicity for the dual-cone order). *For every probability vector λ with positive entries, the function ψ_λ is $(\mathcal{Q}_2^\mathcal{S})^*$ -increasing on $\mathcal{Q}_2^\mathcal{S}$; in particular, so is ψ .*

Proof. Let $q, q' \in \mathcal{Q}_2^\mathcal{S}$ be such that $q' - q \in (\mathcal{Q}_2^\mathcal{S})^*$. Since $\mathcal{Q}_2^\mathcal{S}$ is convex, the segment $q_u := q + u(q' - q)$, $u \in [0, 1]$, stays in $\mathcal{Q}_2^\mathcal{S}$. By (2.62), the function $u \mapsto \psi_\lambda(q_u)$ is differentiable on $[0, 1]$, with derivative $\langle q' - q, \partial_q \psi_\lambda(q_u) \rangle_{L^2}$. By (2.58), $\partial_q \psi_\lambda(q_u)$ is a path in $\mathcal{Q}_\infty^\mathcal{S} \subseteq \mathcal{Q}_2^\mathcal{S}$, so this derivative is nonnegative, by the definition (5.1) of the dual cone. Hence $u \mapsto \psi_\lambda(q_u)$ is nondecreasing, and $\psi_\lambda(q') \geq \psi_\lambda(q)$. $\square$

Two issues arise in giving a meaning to (1.12) in the viscosity sense. The nonlinearity $p \mapsto \int_0^1 \xi(p)$ is naturally defined on the cone, whereas the gradient of a test function at a point of the cone need not belong to the cone; and the function ξ is only locally Lipschitz, whereas the well-posedness theory requires a Lipschitz nonlinearity. The following definitions, taken from [11, Section 4], address these two issues; the regularization of ξ and a finite-dimensional version of the extension (5.3) already appear in [21].

Definition 5.2 (Regularization). A function $\bar{\xi} : \mathbb{R}_+^{\mathcal{S}} \rightarrow \mathbb{R}$ is called a regularization of ξ if the following conditions hold.

- (1) The function $\bar{\xi}$ agrees with ξ on $\{a \in \mathbb{R}_+^{\mathcal{S}} : |a| \leq 1\}$.
- (2) The function $\bar{\xi}$ is Lipschitz and proper, in the sense that $\bar{\xi}$ is $\mathbb{R}_+^{\mathcal{S}}$ -increasing and, for every $b \in \mathbb{R}_+^{\mathcal{S}}$, the map $\mathbb{R}_+^{\mathcal{S}} \ni a \mapsto \bar{\xi}(a+b) - \bar{\xi}(a)$ is also $\mathbb{R}_+^{\mathcal{S}}$ -increasing.
- (3) If ξ is convex on $\mathbb{R}_+^{\mathcal{S}}$, then $\bar{\xi}$ is convex.

This is [11, Definition 4.2] in the present setting. The last condition plays no role here, since we do not assume ξ to be convex; it is included to match the definition in [11]. The function ξ itself is proper on $\mathbb{R}_+^{\mathcal{S}}$: since the series (1.4) has nonnegative coefficients and converges on all of $\mathbb{R}_+^{\mathcal{S}}$, the partial derivatives of ξ are nonnegative and nondecreasing in each coordinate on $\mathbb{R}_+^{\mathcal{S}}$, so that ξ is $\mathbb{R}_+^{\mathcal{S}}$ -increasing, and so is $a \mapsto \xi(a+b) - \xi(a) = \int_0^1 b \cdot \nabla \xi(a+ub) du$ for every $b \in \mathbb{R}_+^{\mathcal{S}}$. The function ξ is locally Lipschitz, but not globally Lipschitz in general, and the existence of a regularization follows from [11, Lemma 4.4], which builds on [21, Proposition 6.8]. Given a regularization $\bar{\xi}$, we define $H : \mathcal{H} \rightarrow \mathbb{R}$ by

$$(5.3) \quad H(\kappa) := \inf \left\{ \int_0^1 \bar{\xi}(p(r)) dr : p \in \mathcal{Q}_2^{\mathcal{S}} \cap \left(\kappa + (\mathcal{Q}_2^{\mathcal{S}})^* \right) \right\}, \quad \kappa \in \mathcal{H}.$$

In words, $H(\kappa)$ is the smallest value of $\int_0^1 \bar{\xi}(p)$ over the paths p in the cone that dominate κ for the dual-cone order. By [11, Lemma 4.6], the function H is Lipschitz, bounded below, and $(\mathcal{Q}_2^{\mathcal{S}})^*$ -increasing, and

$$(5.4) \quad H(p) = \int_0^1 \bar{\xi}(p(r)) dr, \quad p \in \mathcal{Q}_2^{\mathcal{S}}.$$

Thus, replacing ξ by $\bar{\xi}$ makes the nonlinearity Lipschitz, while the infimum in (5.3) extends the functional $p \mapsto \int_0^1 \bar{\xi}(p)$ from the cone to all of $\mathcal{H}$, in a way that preserves its monotonicity for the dual-cone order; this monotonicity is what makes a comparison principle available for the equation on the cone.

A function $\phi : (0, \infty) \times \mathcal{Q}_2^{\mathcal{S}} \rightarrow \mathbb{R}$ is called smooth if the following two conditions hold; this is [11, Definition 1.2].

- (1) For every $(t, q) \in (0, \infty) \times \mathcal{Q}_2^{\mathcal{S}}$, there exists a unique element of $\mathbb{R} \times \mathcal{H}$, denoted by $(\partial_t \phi(t, q), \partial_q \phi(t, q))$, such that, as (s, y) tends to (t, q) in $\mathbb{R} \times \mathcal{H}$ with $y \in \mathcal{Q}_2^{\mathcal{S}}$,

$$\phi(s, y) - \phi(t, q) = \partial_t \phi(t, q)(s - t) + \langle \partial_q \phi(t, q), y - q \rangle_{\mathcal{H}} + O(|s - t|^2 + |y - q|_{\mathcal{H}}^2).$$

- (2) The map $(t, q) \mapsto (\partial_t \phi(t, q), \partial_q \phi(t, q))$ is continuous from $(0, \infty) \times \mathcal{Q}_2^{\mathcal{S}}$ to $\mathbb{R} \times \mathcal{H}$.

The uniqueness required in the first condition is automatic, since $\mathcal{Q}_2^{\mathcal{S}} - \mathcal{Q}_2^{\mathcal{S}}$ is dense in $\mathcal{H}$.

Definition 5.3 (Viscosity solutions). Let $\bar{\xi}$ be a regularization of ξ , and let H be defined by (5.3). Consider the equation

$$(5.5) \quad \partial_t f - H(\partial_q f) = 0 \quad \text{on } \mathbb{R}_+ \times \mathcal{Q}_2^{\mathcal{S}}.$$

- (1) A continuous function $f : \mathbb{R}_+ \times \mathcal{Q}_2^{\mathcal{S}} \rightarrow \mathbb{R}$ is a viscosity subsolution of (5.5) if, for every $(t, q) \in (0, \infty) \times \mathcal{Q}_2^{\mathcal{S}}$ and every smooth function ϕ such that $f - \phi$ has a local maximum at (t, q) , relative to $(0, \infty) \times \mathcal{Q}_2^{\mathcal{S}}$, we have

$$(\partial_t \phi - H(\partial_q \phi))(t, q) \leq 0.$$

- (2) A continuous function $f : \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S} \rightarrow \mathbb{R}$ is a viscosity supersolution of (5.5) if, for every $(t, q) \in (0, \infty) \times \mathcal{Q}_2^\mathcal{S}$ and every smooth function ϕ such that $f - \phi$ has a local minimum at (t, q) , relative to $(0, \infty) \times \mathcal{Q}_2^\mathcal{S}$, we have

$$(\partial_t \phi - H(\partial_q \phi))(t, q) \geq 0.$$

- (3) A continuous function $f : \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S} \rightarrow \mathbb{R}$ is a viscosity solution of (5.5) if it is both a viscosity subsolution and a viscosity supersolution.

Finally, a continuous function $f : \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S} \rightarrow \mathbb{R}$ is called a viscosity solution of the equation in (1.12) if it is a viscosity solution of (5.5) for some regularization $\bar{\xi}$ of ξ .

The first part of the definition is [11, Definition 1.4], and the last part is [11, Definition 4.3]. The results of [11] are stated for paths with values in the cone of positive semidefinite matrices, which is the natural setting for vector spin glasses; in the present multi-species setting, the paths are $\mathbb{R}_+^\mathcal{S}$ -valued, which corresponds to diagonal matrices, and the arguments of [11] apply verbatim to this cone, as in [7, 9]. In the proof of Proposition 5.5, we nevertheless embed the spherical model into the matrix setting and identify the corresponding solution with f through the Hopf formula, so that the lower bound is a direct application of the results of [11].

5.2. The solution and its Hopf representation. Recall the functional $\mathcal{J}_{t,q}$ from (1.10).

Theorem 5.4 (Viscosity solution and Hopf formula). *Let ψ be the spherical initial condition defined in (2.26), restricted to $\mathcal{Q}_2^\mathcal{S}$.*

- (1) *For every regularization $\bar{\xi}$ of ξ , there exists a viscosity solution f of (5.5) with $f(0, \cdot) = \psi$, unique in the class of Lipschitz functions on $\mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S}$, and f does not depend on the choice of $\bar{\xi}$. In particular, f is the unique Lipschitz viscosity solution of (1.12).*
- (2) *For every $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S}$,*

$$(5.6) \quad f(t, q) = \sup_{p \in \mathcal{Q}_\infty^\mathcal{S}} \inf_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}(q', p) = \sup_{p \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}} \inf_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}(q', p).$$

At the level of the variational formula, the restriction of the supremum to $\mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$ is a consequence of the bound $\partial_q \psi(\pi) \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$ on the derivative of the initial condition.

Proof of Theorem 5.4. We verify the hypotheses of [11, Theorem 4.7]. The function ξ is locally Lipschitz, and it is proper on $\mathbb{R}_+^\mathcal{S}$, as shown after Definition 5.2. By Lemma 5.1 and Corollary 2.10, the function ψ is $(\mathcal{Q}_2^\mathcal{S})^*$ -increasing on $\mathcal{Q}_2^\mathcal{S}$ and satisfies

$$|\psi(q) - \psi(q')| \leq |q - q'|_{L^1}, \quad q, q' \in \mathcal{Q}_2^\mathcal{S},$$

and ψ is convex by Lemma 2.9. Part (1) is therefore [11, Theorem 4.7], in the diagonal setting discussed after Definition 5.3, together with [11, Remark 4.10] for the independence of f from the choice of $\bar{\xi}$; the last sentence of part (1) is a rephrasing, since a viscosity solution of the equation in (1.12) is by definition a viscosity solution of (5.5) for some regularization. Since ψ is convex, [11, Theorem 4.7] also gives the Hopf formula, which is the first equality in (5.6): the supremum and the infimum there are over bounded paths in the cone, that is, over $\mathcal{Q}_\infty^\mathcal{S}$.

It remains to show that the supremum can be restricted to $\mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$, and for this it suffices to show that, for $p \in \mathcal{Q}_\infty^\mathcal{S} \setminus \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$, the infimum over q' in (5.6) equals $-\infty$. By the endpoint convention (2.3), such a path has a coordinate $s \in \mathcal{S}$ and a point $r_0 \in [0, 1]$ with $p^s(r_0) > \lambda_{\infty, s}$, so that, by monotonicity,

$$\int_{r_0}^1 (p^s(v) - \lambda_{\infty, s}) dv > 0.$$

For $a > 0$, let $q'_a \in \mathcal{Q}_\infty^\mathcal{S}$ have s -component $a\mathbb{1}_{[r_0, 1]}$ and all other components equal to zero. Since ψ is convex, with derivative $\partial_q \psi(q'_a)$ at q'_a by (2.62), we have $\psi(0) \geq \psi(q'_a) - \langle q'_a, \partial_q \psi(q'_a) \rangle_{L^2}$, and the bound $\partial_q \psi(q'_a) \in \mathcal{Q}_{\infty, \leq \lambda_\infty}^\mathcal{S}$ in (2.59) gives $\langle q'_a, \partial_q \psi(q'_a) \rangle_{L^2} \leq a\lambda_{\infty, s}(1 - r_0)$. Therefore

$$\mathcal{J}_{t,q}(q'_a, p) \leq \psi(0) - a \int_{r_0}^1 (p^s(v) - \lambda_{\infty, s}) dv + \langle q, p \rangle_{L^2} + t \int_0^1 \xi(p(r)) dr \xrightarrow{a \rightarrow \infty} -\infty.$$

Hence such paths p do not contribute to the supremum, and the second equality in (5.6) follows. $\square$

Two comments are in order. First, the properties of ψ used in the proof are stated in Lemma 2.9, Corollary 2.10, Lemma 5.1, and (2.58) for the initial condition ψ_λ in (2.27) of a model with limiting proportions λ , for every probability vector λ with positive entries. The theorem thus applies as well with (ψ_λ, λ) in place of (ψ, λ_∞) , and we denote by f_λ the corresponding solution, which is given by the Hopf formula

$$(5.7) \quad f_\lambda(t, q) = \sup_{p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}} \inf_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}^\lambda(q', p), \quad (t, q) \in \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S},$$

where $\mathcal{J}_{t,q}^\lambda$ is defined as in (1.10) with ψ_λ in place of ψ ; the supremum may equivalently be taken over $\mathcal{Q}_\infty^\mathcal{S}$, and $f = f_{\lambda_\infty}$. This is used in Section 6. Second, since every $p \in \mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$ satisfies $|p(u)| \leq |\lambda| \leq 1$ for every u , the only term of $\mathcal{J}_{t,q}^\lambda(q', p)$ that depends on q , namely $\langle q, p \rangle_{L^2}$, is 1-Lipschitz in q for the L^1 norm, uniformly over p and q' . Taking the infimum over q' and then the supremum over p , we obtain

$$(5.8) \quad |f_\lambda(t, q) - f_\lambda(t, \tilde{q})| \leq |q - \tilde{q}|_{L^1}, \quad t \geq 0, \quad q, \tilde{q} \in \mathcal{Q}_2^\mathcal{S}.$$

This is the form of the Lipschitz continuity of f that is used in Section 6, where the identification of the limit is first obtained for regular bounded paths and then extended by density.

5.3. The lower bound.

Proposition 5.5 (Hamilton–Jacobi lower bound). *Let f be the viscosity solution of Theorem 5.4. Then, for every $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}_2^\mathcal{S}$,*

$$(5.9) \quad f(t, q) \leq \liminf_{N \rightarrow \infty} \bar{F}_N(t, q).$$

Proof. The lower bound is an application of [11, Theorem 4.14], which is a restatement of [21, Theorem 3.4]. This result concerns vector spin glasses in which the reference measure is an arbitrary probability measure supported in the closed ball of radius $\sqrt{N}$ of $(\mathbb{R}^N)^D$, for some $D \in \mathbb{N}$; in particular, no product structure of the reference measure is required, so that the spherical constraint is allowed. We first embed the spherical model into this setting, then verify the hypotheses of the theorem, and finally identify the solution of the matrix-valued equation appearing in its conclusion with f along diagonal paths.

Step 1: Embedding.

Let $D := |\mathcal{S}|$, and, as in [21, Section 6.2], associate with $\sigma \in \Sigma_N$ the vector-spin configuration $\widehat{\sigma} = (\widehat{\sigma}^s)_{s \in \mathcal{S}} \in (\mathbb{R}^N)^\mathcal{S}$ defined by

$$\widehat{\sigma}_i^s := \sigma_i \mathbb{1}_{\{i \in I_{N,s}\}}, \quad i \in [N], \quad s \in \mathcal{S}.$$

Then, for $\sigma, \sigma' \in \Sigma_N$,

$$\frac{1}{N} \widehat{\sigma}^s \cdot \widehat{\sigma}'^{s'} = R_{N,s}(\sigma, \sigma') \mathbb{1}_{\{s=s'\}}, \quad \text{and} \quad \sum_{s \in \mathcal{S}} |\widehat{\sigma}^s|^2 = N.$$

In other words, the overlap matrix $(\frac{1}{N} \widehat{\sigma}^s \cdot \widehat{\sigma}'^{s'})_{s, s' \in \mathcal{S}}$ is the diagonal matrix with diagonal $R_N(\sigma, \sigma')$, and the embedded configurations lie on the sphere of radius $\sqrt{N}$ of $(\mathbb{R}^N)^\mathcal{S}$. We absorb the external field into the reference measure: let P_N^h be the probability measure on Σ_N defined by

$$dP_N^h(\sigma) := \exp \left(N \bar{F}_N(0, 0) + \sum_{s \in \mathcal{S}} h^s \sum_{i \in I_{N,s}} \sigma_i \right) dP_N(\sigma),$$

which is indeed a probability measure by (1.9) at $t = 0$. The reference measure of the vector-spin model is the image of P_N^h under the embedding.

Step 2: Covariance function and free energy.

For a matrix $A \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}}$, we set $\Xi(A) := \xi((A_{ss})_{s \in \mathcal{S}})$, which is well defined since the series (1.4) converges on all of $\mathbb{R}^\mathcal{S}$. The function Ξ is smooth, hence locally Lipschitz, and it is proper on the cone of positive semidefinite matrices in the sense of [11, Definition 4.2]: the diagonal entries of a

positive semidefinite matrix are nonnegative, and if $B - A$ is positive semidefinite, then $A_{ss} \leq B_{ss}$ for every $s \in \mathcal{S}$, so that the properness of ξ on $\mathbb{R}_+^{\mathcal{S}}$, shown after Definition 5.2, transfers to Ξ . The setting of [11] requires a centered Gaussian field indexed by all of $(\mathbb{R}^N)^{\mathcal{S}}$, with covariance $N\Xi$ evaluated at the overlap matrix, that is, $N\xi((\frac{1}{N}\bar{\sigma}^s \cdot \bar{\sigma}'^s)_{s \in \mathcal{S}})$. Such a field exists: each kernel $(\bar{\sigma}, \bar{\sigma}') \mapsto \bar{\sigma}^s \cdot \bar{\sigma}'^s$ is nonnegative definite, hence so is every monomial $\prod_s (\bar{\sigma}^s \cdot \bar{\sigma}'^s)^{k_s}$ by the Schur product theorem, and the covariance is a convergent sum of such monomials with the nonnegative coefficients of (1.4). On the embedded configurations, this field has the same law as H_N , by (1.5). Likewise, the external field of [11] associated with a bounded matrix-valued path $\mathbf{q}$, nondecreasing for the positive semidefinite order, has covariance $\sum_{s, s' \in \mathcal{S}} \mathbf{q}_{ss'}(\alpha \wedge \alpha') \bar{\sigma}^s \cdot \bar{\sigma}'^{s'}$; on the embedded configurations, only the diagonal $\text{diag } \mathbf{q} := (\mathbf{q}_{ss})_{s \in \mathcal{S}}$ contributes, and the field has the law of $W_N^{\text{diag } \mathbf{q}}$ in (2.12). Note that $\text{diag } \mathbf{q} \in \mathcal{Q}_\infty^{\mathcal{S}}$, since the diagonal entries of a nondecreasing path of positive semidefinite matrices are nonnegative and nondecreasing. Conversely, for an $\mathbb{R}^{\mathcal{S}}$ -valued path p , we write $\text{diag}(p)$ for the path of diagonal matrices with diagonal p . Finally, the deterministic centerings of [11] are $Nt\Xi$ evaluated at the self-overlap matrix, which is $Nt\xi(\lambda_N)$ since $R_N(\sigma, \sigma) = \lambda_N$, and $\sum_{s, s'} \mathbf{q}_{ss'}(1) \bar{\sigma}^s \cdot \bar{\sigma}'^{s'} = N \text{diag } \mathbf{q}(1) \cdot \lambda_N$. Comparing with (2.13), we conclude that the free energy of [11] for the embedded model, with reference measure P_N^h , at time t and path $\mathbf{q}$, is

$$(5.10) \quad \bar{F}_N(t, \text{diag } \mathbf{q}) - \bar{F}_N(0, 0).$$

This identity extends from bounded to square-integrable matrix paths, since both sides are continuous for the L^1 norm; see (2.17) for the right-hand side.

Step 3: Initial condition and matrix-valued solution.

By (5.10) and Proposition 2.11, the initial condition of the embedded model converges: for every square-integrable matrix path $\mathbf{q}$ as above,

$$\lim_{N \rightarrow \infty} (\bar{F}_N(0, \text{diag } \mathbf{q}) - \bar{F}_N(0, 0)) = \psi(\mathbf{q}) := \psi(\text{diag } \mathbf{q}) - \psi(0).$$

The function ψ is convex, since ψ is convex and $\mathbf{q} \mapsto \text{diag } \mathbf{q}$ is linear. It also satisfies the hypotheses of [11, Theorem 4.7]: it is 1-Lipschitz for the L^1 norm, by (2.60) and since $|\text{diag } A| \leq |A|$ for the Frobenius norm; and it is increasing for the dual of the matrix cone, since if $\mathbf{q} - \mathbf{q}'$ belongs to this dual cone, then testing against the diagonal paths $\text{diag}(p)$ with $p \in \mathcal{Q}_2^{\mathcal{S}}$, which belong to the matrix cone, shows that $\text{diag } \mathbf{q} - \text{diag } \mathbf{q}' \in (\mathcal{Q}_2^{\mathcal{S}})^*$, and Lemma 5.1 gives $\psi(\text{diag } \mathbf{q}) \geq \psi(\text{diag } \mathbf{q}')$. Let $\mathbf{f}$ be the unique Lipschitz viscosity solution of the matrix-valued Hamilton–Jacobi equation with nonlinearity Ξ and initial condition ψ , given by [11, Theorem 4.7]. Since ψ is convex, the same theorem gives the Hopf formula

$$(5.11) \quad \mathbf{f}(t, \mathbf{q}) = \sup_{\mathbf{p}} \inf_{\mathbf{q}'} \left\{ \psi(\mathbf{q}') + \langle \mathbf{q} - \mathbf{q}', \mathbf{p} \rangle + t \int_0^1 \Xi(\mathbf{p}(r)) dr \right\},$$

where $\mathbf{p}$ and $\mathbf{q}'$ range over bounded paths that are nondecreasing for the positive semidefinite order, and $\langle \cdot, \cdot \rangle$ is the L^2 pairing associated with the Frobenius inner product.

Step 4: Identification along diagonal paths.

We claim that, for every $q \in \mathcal{Q}_\infty^{\mathcal{S}}$,

$$(5.12) \quad \mathbf{f}(t, \text{diag}(q)) = f(t, q) - \psi(0).$$

The restriction of a viscosity solution to a subcone is not, in general, a viscosity solution of the restricted equation, so we argue through the two Hopf formulas. Let $\mathbf{q} := \text{diag}(q)$. For every admissible $\mathbf{p}$, the path $p := \text{diag } \mathbf{p}$ belongs to $\mathcal{Q}_\infty^{\mathcal{S}}$, and restricting the infimum in (5.11) to the diagonal paths $\mathbf{q}' = \text{diag}(q')$ with $q' \in \mathcal{Q}_\infty^{\mathcal{S}}$, for which $\langle \mathbf{q} - \mathbf{q}', \mathbf{p} \rangle = \langle q - q', p \rangle_{L^2}$ and $\Xi(\mathbf{p}) = \xi(p)$, gives

$$\inf_{\mathbf{q}'} \left\{ \psi(\mathbf{q}') + \langle \mathbf{q} - \mathbf{q}', \mathbf{p} \rangle + t \int_0^1 \Xi(\mathbf{p}(r)) dr \right\} \leq \inf_{q' \in \mathcal{Q}_\infty^{\mathcal{S}}} \mathcal{J}_{t, q}(q', p) - \psi(0) \leq f(t, q) - \psi(0),$$

by (5.6). Taking the supremum over $\mathbf{p}$ proves that $\mathbf{f}(t, \text{diag}(q)) \leq f(t, q) - \psi(0)$. Conversely, we restrict the supremum in (5.11) to the diagonal paths $\mathbf{p} = \text{diag}(p)$ with $p \in \mathcal{Q}_\infty^{\mathcal{S}}$. For such paths, $\langle \mathbf{q} - \mathbf{q}', \mathbf{p} \rangle = \langle q - \text{diag } q', p \rangle_{L^2}$ and $\Xi(\mathbf{p}) = \xi(p)$, so that every term in the braces depends on q'

only through $\text{diag } \mathbf{q}'$, which ranges over all of $\mathcal{Q}_\infty^\mathcal{S}$ as $\mathbf{q}'$ varies. The infimum over $\mathbf{q}'$ is therefore $\inf_{\mathbf{q}' \in \mathcal{Q}_\infty^\mathcal{S}} \mathcal{J}_{t,q}(\mathbf{q}', p) - \psi(0)$, and taking the supremum over p gives $\mathbf{f}(t, \text{diag}(q)) \geq f(t, q) - \psi(0)$, again by (5.6). This proves (5.12).

Step 5: Conclusion.

Let $q \in \mathcal{Q}_\infty^\mathcal{S}$. By Steps 1 to 3, the embedded model satisfies the hypotheses of [11, Theorem 4.14], and the conclusion of this theorem at the path $\text{diag}(q)$ reads, by (5.10) and (5.12),

$$f(t, q) - \psi(0) = \mathbf{f}(t, \text{diag}(q)) \leq \liminf_{N \rightarrow \infty} (\bar{F}_N(t, q) - \bar{F}_N(0, 0)) = \liminf_{N \rightarrow \infty} \bar{F}_N(t, q) - \psi(0),$$

where we used (2.65) at $q = 0$ in the last step. This proves (5.9) for bounded q . For $q \in \mathcal{Q}_2^\mathcal{S}$, set $q_m^s := q^s \wedge m$ for every $s \in \mathcal{S}$ and $m \in \mathbb{N}$. Then $q_m \in \mathcal{Q}_\infty^\mathcal{S}$ and $|q_m - q|_{L^1} \rightarrow 0$ as $m \rightarrow \infty$. The estimates (5.8) and (2.17) give

$$f(t, q) \leq f(t, q_m) + |q_m - q|_{L^1} \leq \liminf_{N \rightarrow \infty} \bar{F}_N(t, q) + 2|q_m - q|_{L^1},$$

and letting $m \rightarrow \infty$ proves (5.9). $\square$

6. PROOF OF THE MAIN RESULT

Let us summarize what has been achieved so far. Proposition 2.12, proved in Section 4, bounds the limit free energy from above by the value $\mathcal{P}_{t,q}(p)$ of the Parisi functional at a critical path p , but only when the limiting proportions are rational, along the sequence of sizes $N \in MN$ compatible with the cavity block, and for regular bounded paths q . Section 5 identifies the candidate limit: the function f given by the Hopf formula (5.6) is the Lipschitz viscosity solution of (1.12), and Proposition 5.5 shows that it is a lower bound for the limit free energy, for arbitrary species proportions, along the full sequence of sizes, and for every $q \in \mathcal{Q}_2^\mathcal{S}$. In this section, we first combine these two results into the identification of the limit free energy for rational proportions, Corollary 6.1, and then remove the restrictions on the sizes and on the proportions, which completes the proof of Theorem 1.1.

6.1. Identification of the limit for rational proportions.

Corollary 6.1 (Identification of the limit for rational proportions). *Assume that $\lambda_\infty \in \mathbb{Q}^\mathcal{S}$, and let M and the sequence of sizes (2.80) be as in Proposition 2.12. Let f be the viscosity solution of Theorem 5.4. For every $t > 0$ and $q \in \mathcal{Q}_\infty^\mathcal{S}$, we have*

$$(6.1) \quad \lim_{n \rightarrow \infty} \bar{F}_{N_n}(t, q) = f(t, q).$$

Proof. Step 1: Regular paths.

Let $q \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$. Propositions 5.5 and 2.12 being now proved, the chain (2.85) holds, with $f(t, q)$ equal to the right-hand side of (1.11) by the Hopf formula (5.6). This proves (6.1) for $q \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$.

Step 2: Bounded paths.

Let $q \in \mathcal{Q}_\infty^\mathcal{S}$, and, for every integer $m \geq 1$, define q_m by

$$q_m^s(u) := q^s(u) \wedge (mu) + \frac{u}{m}, \quad u \in [0, 1), \quad s \in \mathcal{S}.$$

Each component of q_m is right-continuous, bounded by $|q|_{L^\infty} + 1$, vanishes at 0, and satisfies $q_m^s(v) - q_m^s(u) \geq (v - u)/m$ for $0 \leq u \leq v < 1$, so $q_m \in \mathcal{Q}_{\infty, \uparrow}^\mathcal{S}$. Moreover, $q^s(u) \wedge (mu)$ and $q^s(u)$ both belong to $[0, |q|_{L^\infty}]$, and they coincide for $u \geq |q|_{L^\infty}/m$, so that

$$|q_m - q|_{L^1} \leq \frac{|\mathcal{S}|}{m} (|q|_{L^\infty}^2 + 1) \xrightarrow{m \rightarrow \infty} 0.$$

Both $\bar{F}_{N_n}(t, \cdot)$ and $f(t, \cdot)$ are 1-Lipschitz for the L^1 norm, by (2.17) and (5.8). Hence, by Step 1,

$$\limsup_{n \rightarrow \infty} |\bar{F}_{N_n}(t, q) - f(t, q)| \leq 2|q_m - q|_{L^1} + \limsup_{n \rightarrow \infty} |\bar{F}_{N_n}(t, q_m) - f(t, q_m)| = 2|q_m - q|_{L^1},$$

and letting $m \rightarrow \infty$ proves (6.1). $\square$

It remains to extend (6.1) to arbitrary sizes and proportions. Lemma 6.3 shows that the free energy is not sensitive to the addition of a bounded number of coordinates, nor to small changes of the species proportions; this allows us to compare the actual system with a system with rational proportions λ close to λ_∞ , whose limit free energy is f_λ by Corollary 6.1. Lemma 6.5 then shows that $f_\lambda(t, q)$ is continuous in λ , which lets us pass from rational proportions to λ_∞ .

6.2. Finite-size comparison. The argument of Lemma 3.6 in Section 3 adapts to the comparison of systems of different sizes and species proportions: a single coordinate is added to one species, the Hamiltonian is unperturbed, and the dependence of the constant on t and q is made explicit. For an integer vector $\mathbf{n} = (n_s)_{s \in \mathcal{S}}$ with positive entries, we write $|\mathbf{n}|_1 := \sum_{s \in \mathcal{S}} n_s$, and we let $Z_{\mathbf{n}}(t, q)$ and $\bar{F}_{\mathbf{n}}(t, q)$ denote the partition function (2.14) and the free energy (2.15) of the system with n_s coordinates in species s , for every $s \in \mathcal{S}$; by permutation invariance within each species, these quantities do not depend on the choice of the partition $(I_{N,s})_{s \in \mathcal{S}}$ with $|I_{N,s}| = n_s$. We recall that $(e_s)_{s \in \mathcal{S}}$ denotes the canonical basis of $\mathbb{R}^{\mathcal{S}}$.

Lemma 6.2 (Adding one coordinate). *There exists a constant $C < \infty$, depending only on ξ and h , such that, for every integer vector $\mathbf{n}$ with positive entries, $s \in \mathcal{S}$, $t \geq 0$, and $q \in \mathcal{Q}_\infty^{\mathcal{S}}$,*

$$(6.2) \quad |\mathbb{E} \log Z_{\mathbf{n}}(t, q)| \leq C |\mathbf{n}|_1 (1 + t + |q|_{L^\infty}),$$

$$(6.3) \quad |\mathbb{E} \log Z_{\mathbf{n}+e_s}(t, q) - \mathbb{E} \log Z_{\mathbf{n}}(t, q)| \leq C (1 + t + |q|_{L^\infty}).$$

Proof. Write $K := |\mathbf{n}|_1$ and $\lambda := \mathbf{n}/K$, and let $(I_{K,u})_{u \in \mathcal{S}}$ denote the species partition of the system with n_u coordinates in species u . We first prove (6.2). By (2.17), we have $|\bar{F}_{\mathbf{n}}(t, q) - \bar{F}_{\mathbf{n}}(0, 0)| \leq |q|_{L^1} + t \sup_{[-1, 1]^{\mathcal{S}}} |\xi|$. Moreover, $|\bar{F}_{\mathbf{n}}(0, 0)| \leq \max_u |h^u|$, since $|\sum_u h^u \sum_{i \in I_{K,u}} \sigma_i| \leq K \max_u |h^u|$ on Σ_K , by the Cauchy–Schwarz inequality. Since $\mathbb{E} \log Z_{\mathbf{n}} = -K \bar{F}_{\mathbf{n}}$ and $|q|_{L^1} \leq |q|_{L^\infty}$, this proves (6.2).

We now turn to (6.3). The larger system has size $K+1$ and proportions $\lambda' := (\mathbf{n} + e_s)/(K+1)$. We apply the disintegration (3.30) with $m = 1$ and $n = n_s$ to the species- s sphere of the larger system, leaving the spheres of the other species unchanged: writing $\tau \in \mathcal{B}_{1, n_s}$ for the added coordinate, $r^s(\tau) := 1 + (1 - \tau^2)/n_s$, and $r^u(\tau) := 1$ for $u \neq s$, a configuration of the larger system is $(S_{r(\tau)}\sigma, \tau)$ with $\sigma \in \Sigma_K$, and its reference measure is $p_{1, n_s}(\tau) d\tau P_K(d\sigma)$. This is the chart of Subsection 3.3 for a cavity block consisting of one coordinate of species s , and we follow Steps 1 and 2 of the proof of Lemma 3.6, with $x = y = 0$. By (3.7), the analogue of (3.40) holds, and since $n_s |r^s(\tau) - 1| \leq 1 + \tau^2$ and $K |R_{K,s}(\sigma, \sigma')| \leq n_s$, the analogue of (3.41) reads

$$(K+1) |R_{K+1}((S_{r(\tau)}\sigma, \tau), (S_{r(\tau')}\sigma', \tau')) - R_K(\sigma, \sigma')| \leq C (1 + \tau^2 + \tau'^2),$$

for a constant C depending only on $|\mathcal{S}|$. We compare the enriched Hamiltonians (2.13) of the two systems, viewed as Gaussian fields indexed by (σ, α, τ) ; recall that their means and covariances depend on the configurations only through the overlaps and the sums $\sum_{i \in I_{K,u}} \sigma_i$. Both overlap vectors lie in the overlap domain $[-1, 1]^{\mathcal{S}}$, on which ξ is Lipschitz, so the covariance $(K+1)\xi(R_{K+1})$ of the interaction field of the larger system differs from the covariance $K\xi(R_K)$ of the smaller one by at most $C(1 + \tau^2 + \tau'^2)$; and the deterministic centerings $-(K+1)t\xi(\lambda')$ and $-Kt\xi(\lambda)$ differ by at most Ct , since $|\lambda' - \lambda| \leq 2/K$ and ξ is Lipschitz on $[0, 1]^{\mathcal{S}}$. For the cascade field, the covariance of $\sqrt{2}W_{K+1}^q$ in (2.12) is $2 \sum_{u \in \mathcal{S}} q^u (\alpha \wedge \alpha') (K+1) R_{K+1,u}$, which differs from the covariance of $\sqrt{2}W_K^q$ by at most $2|q|_{L^\infty} \sum_u |(K+1)R_{K+1,u} - K R_{K,u}| \leq C|q|_{L^\infty} (1 + \tau^2 + \tau'^2)$, while the self-overlap corrections $-(K+1)q(1) \cdot \lambda'$ and $-Kq(1) \cdot \lambda$ differ by the constant $q^s(1)$. The external-field terms differ by at most $C(1 + \tau^2)$, as in (3.42). Altogether, the means and covariances of the two Hamiltonians differ by at most $C(1 + t + |q|_{L^\infty})(1 + \tau^2 + \tau'^2)$.

We now apply Lemma 3.5 with $M = 1$, $\mathbb{X}^0 := \Sigma_K \times \mathfrak{U}$, $\mu^0 := P_K \otimes \mathfrak{A}$, $B := \mathcal{B}_{1, n_s}$, $\nu := p_{1, n_s}(\tau) d\tau$, and $C_0 := C(1 + t + |q|_{L^\infty})$. The set $\mathcal{B}_{1, n_s}$ is bounded, ν is a probability measure on it, and $\int \tau^2 p_{1, n_s}(\tau) d\tau = 1$ by (3.30), since each coordinate of a point of $\mathbb{S}_{n_s+1}$ has second moment 1 under the uniform measure. The condition (3.37) holds for the same reason as in the proof of Lemma 3.6: the law of the Hamiltonian of the larger system is invariant under permutations of the coordinates within species s , and $\sum_{i \in I_{K+1,s}} \rho_i^2 = n_s + 1$ on its configuration space, so that

$\mathbb{E}\langle \tau^2 \rangle_{\mathbf{n}+e_s} = 1$, where $\langle \cdot \rangle_{\mathbf{n}+e_s}$ denotes the Gibbs bracket associated with $Z_{\mathbf{n}+e_s}(t, q)$. The lemma yields (6.3). $\square$

We now translate this estimate into a comparison of free energies. If λ is a probability vector and $N \in \mathbb{N}$ is such that $N\lambda_s \in \mathbb{N}$ for every $s \in \mathcal{S}$, we write

$$(6.4) \quad \bar{F}_{N,\lambda}(t, q) := \bar{F}_{\mathbf{n}}(t, q), \quad \text{where } \mathbf{n} := (N\lambda_s)_{s \in \mathcal{S}},$$

for the free energy of a system of size N with species proportions λ . In particular, $\bar{F}_N = \bar{F}_{N,\lambda_N}$.

Lemma 6.3 (Finite-size comparison). *There exists a constant $C < \infty$, depending only on ξ and h , such that the following holds. Let $N, N' \in \mathbb{N}$ satisfy $N \leq N' \leq 2N$, and let λ, λ' be probability vectors such that $N\lambda_s, N'\lambda'_s \in \mathbb{N}$ for every $s \in \mathcal{S}$. For every $t \geq 0$ and $q \in \mathcal{Q}_\infty^\mathcal{S}$, we have*

$$(6.5) \quad |\bar{F}_{N,\lambda}(t, q) - \bar{F}_{N',\lambda'}(t, q)| \leq C(1+t+|q|_{L^\infty}) \left(|\lambda - \lambda'| + \frac{N' - N}{N} \right).$$

The constant involves $|q|_{L^\infty}$ rather than $|q|_{L^1}$, which is what the Gaussian comparison in Lemma 6.2 directly gives; this suffices, since the lemma is only applied at fixed bounded paths.

Proof of Lemma 6.3. We first translate the comparison of $\mathbb{E} \log Z_{\mathbf{n}}$ in Lemma 6.2 into a comparison of free energies. Let $\mathbf{n}$ be an integer vector with positive entries, and let $s \in \mathcal{S}$. Since $\bar{F}_{\mathbf{n}}(t, q) = -|\mathbf{n}|_1^{-1} \mathbb{E} \log Z_{\mathbf{n}}(t, q)$, we can write

$$\bar{F}_{\mathbf{n}+e_s}(t, q) - \bar{F}_{\mathbf{n}}(t, q) = \left(\frac{1}{|\mathbf{n}|_1} - \frac{1}{|\mathbf{n}|_1 + 1} \right) \mathbb{E} \log Z_{\mathbf{n}+e_s}(t, q) - \frac{1}{|\mathbf{n}|_1} (\mathbb{E} \log Z_{\mathbf{n}+e_s}(t, q) - \mathbb{E} \log Z_{\mathbf{n}}(t, q)).$$

By the two estimates of Lemma 6.2, the first term is bounded by $C(1+t+|q|_{L^\infty})/|\mathbf{n}|_1$ in absolute value, and so is the second. Hence

$$(6.6) \quad |\bar{F}_{\mathbf{n}+e_s}(t, q) - \bar{F}_{\mathbf{n}}(t, q)| \leq \frac{C}{|\mathbf{n}|_1} (1+t+|q|_{L^\infty}), \quad t \geq 0, \quad q \in \mathcal{Q}_\infty^\mathcal{S},$$

for a constant C depending only on ξ and h .

We now prove (6.5). Put $\mathbf{n} := (N\lambda_s)_{s \in \mathcal{S}}$ and $\mathbf{n}' := (N'\lambda'_s)_{s \in \mathcal{S}}$, and choose integer vectors $\mathbf{m}_0, \dots, \mathbf{m}_d$ with $\mathbf{m}_0 = \mathbf{n}$, $\mathbf{m}_d = \mathbf{n}'$, and $\mathbf{m}_{j+1} - \mathbf{m}_j \in \{e_s, -e_s : s \in \mathcal{S}\}$ for every $j < d$, such that each coordinate moves monotonically from n_s to n'_s along the path, and such that all the increments $+e_s$ come before all the decrements $-e_s$. Then

$$d = \sum_{s \in \mathcal{S}} |N'\lambda'_s - N\lambda_s| \leq N' \sum_{s \in \mathcal{S}} |\lambda'_s - \lambda_s| + (N' - N) \leq CN \left(|\lambda - \lambda'| + \frac{N' - N}{N} \right).$$

Moreover, for every j and s , the coordinate $(\mathbf{m}_j)_s$ lies between n_s and n'_s , and is therefore positive; and since $|\mathbf{m}_j|_1$ first increases from N and then decreases to $N' \geq N$, we have $|\mathbf{m}_j|_1 \geq N$ for every j . The estimate (6.6) applies to each pair $(\mathbf{m}_j, \mathbf{m}_{j+1})$, possibly after exchanging the roles of $\mathbf{n}$ and $\mathbf{n} + e_s$ there. Summing over $j \in \{0, \dots, d-1\}$ gives

$$|\bar{F}_{\mathbf{n}'}(t, q) - \bar{F}_{\mathbf{n}}(t, q)| \leq \frac{Cd}{N} (1+t+|q|_{L^\infty}),$$

and the bound on d proves (6.5). $\square$

6.3. Continuity of the limit free energy in the species proportions. Recall from (5.7) the function f_λ associated with a probability vector λ with positive entries. To make its dependence on λ explicit, we write $\psi_s := \psi_{h_s}^\circ$ for $s \in \mathcal{S}$, and we let ψ_s^* be the convex conjugate of the restriction of ψ_s to $\mathcal{Q}_\infty$,

$$(6.7) \quad \psi_s^*(a) := \sup_{\rho \in \mathcal{Q}_\infty} \{ \langle \rho, a \rangle_{L^2} - \psi_s(\rho) \} \in [-\psi_s(0), +\infty], \quad a \in \mathcal{Q}_\infty, \leq 1.$$

The lower bound is obtained by taking $\rho = 0$. Moreover, $\psi_s^*(0) = -\psi_s(0)$: for $\rho \in \mathcal{Q}_\infty$, the path ρe_s belongs to $\mathcal{Q}_2^\mathcal{S} \subseteq (\mathcal{Q}_2^\mathcal{S})^*$, so that Lemma 5.1 gives $\psi_\lambda(\rho e_s) \geq \psi_\lambda(0)$, that is, $\lambda_s \psi_s(\rho) \geq \lambda_s \psi_s(0)$ by (2.27). We set

$$(6.8) \quad c_s(a) := \psi_s^*(a) - \psi_s^*(0) = \psi_s^*(a) + \psi_s(0) \in [0, +\infty], \quad \text{so that} \quad c_s(0) = 0.$$

Lemma 6.4 (Normalized form of the Hopf formula). *For every probability vector λ with positive entries, $t \geq 0$, and $q \in \mathcal{Q}_2^\mathcal{S}$, we have*

$$(6.9) \quad f_\lambda(t, q) = \sum_{s \in \mathcal{S}} \lambda_s \psi_s(0) + \sup_{a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}} \left\{ A_\lambda(a) - \sum_{s \in \mathcal{S}} \lambda_s c_s(a_s) \right\},$$

where $A_\lambda(a) := \sum_{s \in \mathcal{S}} \lambda_s \langle q^s, a_s \rangle_{L^2} + t \int_0^1 \xi((\lambda_s a_s(r))_{s \in \mathcal{S}}) dr.$

Moreover, for every $a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}$,

$$(6.10) \quad 0 \leq A_\lambda(a) \leq C_{t,q} := \sum_{s \in \mathcal{S}} |q^s|_{L^1} + t \sup_{v \in [0,1]^\mathcal{S}} \xi(v),$$

and thus the supremum in (6.9) belongs to $[0, C_{t,q}]$.

Proof. The map $p \mapsto a := (p^s/\lambda_s)_{s \in \mathcal{S}}$ is a bijection from $\mathcal{Q}_{\infty, \leq \lambda}^\mathcal{S}$ onto $(\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}$, and $\langle q', p \rangle_{L^2} = \sum_s \lambda_s \langle q'^s, a_s \rangle_{L^2}$ for every $q' \in \mathcal{Q}_\infty^\mathcal{S}$, while $\psi_\lambda(q') = \sum_s \lambda_s \psi_s(q'^s)$ by (2.27). Since the components q'^s of $q' \in \mathcal{Q}_\infty^\mathcal{S}$ vary independently in $\mathcal{Q}_\infty$, we have

$$\sup_{q' \in \mathcal{Q}_\infty^\mathcal{S}} \{ \langle q', p \rangle_{L^2} - \psi_\lambda(q') \} = \sum_{s \in \mathcal{S}} \lambda_s \psi_s^*(a_s),$$

and therefore $\inf_{q'} \mathcal{J}_{t,q}^\lambda(q', p) = A_\lambda(a) - \sum_s \lambda_s \psi_s^*(a_s)$, so that, by (5.7),

$$(6.11) \quad f_\lambda(t, q) = \sup_{a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}} \left\{ A_\lambda(a) - \sum_{s \in \mathcal{S}} \lambda_s \psi_s^*(a_s) \right\};$$

the conjugates are bounded below and $A_\lambda(a)$ is finite, so that no indeterminate expression appears. Inserting $\psi_s^* = c_s - \psi_s(0)$ gives (6.9). The bounds (6.10) follow from $0 \leq a_s \leq 1$, $q^s \geq 0$, $0 \leq \lambda_s \leq 1$, and the nonnegativity of ξ on $[0, 1]^\mathcal{S}$, which holds since its coefficients are nonnegative. Finally, since $c_s \geq 0$, the expression in braces is at most $C_{t,q}$; and it vanishes at $a = 0$, since $c_s(0) = 0$ and $A_\lambda(0) = t\xi(0) = 0$. $\square$

The formula (6.9) isolates the dependence on λ : the proportions enter through explicit linear weights and through the argument of ξ , while the cost functions c_s do not depend on λ . The next lemma exploits this. It only uses near-maximizers of (6.9), for which the costs $c_s(a_s)$ are bounded; nothing is assumed on ψ_s^* away from these. For $\eta \in (0, 1]$, we set

$$(6.12) \quad \Lambda_\eta := \left\{ \lambda \in [\eta, 1]^\mathcal{S} : \sum_{s \in \mathcal{S}} \lambda_s = 1 \right\}.$$

Lemma 6.5 (Continuity in the species proportions). *Fix $\eta \in (0, 1]$. There exists $C < \infty$, depending only on ξ , h , and η , such that, for every $\lambda, \lambda' \in \Lambda_\eta$, $t \geq 0$, and $q \in \mathcal{Q}_2^\mathcal{S}$,*

$$(6.13) \quad |f_\lambda(t, q) - f_{\lambda'}(t, q)| \leq C(1 + t + |q|_{L^1}) |\lambda - \lambda'|.$$

Proof. Fix t and q , and let V_λ denote the supremum in (6.9). Let $\varepsilon \in (0, 1)$, and let $a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}$ satisfy $A_\lambda(a) - \sum_s \lambda_s c_s(a_s) \geq V_\lambda - \varepsilon$. Since $V_\lambda \geq 0$ and $A_\lambda(a) \leq C_{t,q}$, we get $\sum_s \lambda_s c_s(a_s) \leq C_{t,q} + 1$, hence $c_s(a_s) \leq (C_{t,q} + 1)/\eta$ for every $s \in \mathcal{S}$, as $\lambda_s \geq \eta$. We now evaluate the same a at λ' . Since ξ is Lipschitz on $[0, 1]^\mathcal{S}$, with a constant L_ξ depending only on ξ , and since $|(\lambda_s a_s(r))_s - (\lambda'_s a_s(r))_s| \leq |\lambda - \lambda'|$ for every r , we have

$$|A_\lambda(a) - A_{\lambda'}(a)| \leq \sum_{s \in \mathcal{S}} |\lambda_s - \lambda'_s| |q^s|_{L^1} + t L_\xi |\lambda - \lambda'| \quad \text{and} \quad \left| \sum_{s \in \mathcal{S}} (\lambda_s - \lambda'_s) c_s(a_s) \right| \leq \frac{C_{t,q} + 1}{\eta} \sum_{s \in \mathcal{S}} |\lambda_s - \lambda'_s|.$$

Using $\sum_s |x_s| \leq |\mathcal{S}|^{1/2} |x|$ for $x \in \mathbb{R}^\mathcal{S}$, and $\sum_s |q^s|_{L^1} \leq |\mathcal{S}|^{1/2} |q|_{L^1}$, we deduce that

$$V_{\lambda'} \geq A_{\lambda'}(a) - \sum_{s \in \mathcal{S}} \lambda'_s c_s(a_s) \geq V_\lambda - \varepsilon - C(1 + t + |q|_{L^1}) |\lambda - \lambda'|,$$

for a constant C depending only on ξ and η . Exchanging the roles of λ and λ' and letting $\varepsilon \downarrow 0$ bounds $|V_\lambda - V_{\lambda'}|$ by the last term. The first term in (6.9) is Lipschitz in λ with constant $\max_s |\psi_s(0)|$, and $\psi_s(0) = \psi_{h^s}^\circ(0)$ depends only on h^s . This proves (6.13). $\square$

6.4. Conclusion.

Proof of Theorem 1.1. Let $f = f_{\lambda_\infty}$ be the Lipschitz viscosity solution of (1.12) given by Theorem 5.4. By (5.6), $f(t, q)$ is the right-hand side of (1.11), so it remains to prove that $\bar{F}_N(t, q)$ converges to $f(t, q)$ for every $t \geq 0$ and $q \in \mathcal{Q}_2^\mathcal{S}$. At $t = 0$, this is the convergence of the initial condition in Proposition 2.11: $\bar{F}_N(0, q) \rightarrow \psi(q) = f(0, q)$ for every $q \in \mathcal{Q}_2^\mathcal{S}$. We fix $t > 0$ from now on.

Step 1: Bounded paths.

Let $q \in \mathcal{Q}_2^\mathcal{S}$. Set $\eta := \frac{1}{2} \min_{s \in \mathcal{S}} \lambda_{\infty, s} > 0$, and choose rational probability vectors $\lambda^{(m)} \in \Lambda_\eta \cap \mathbb{Q}^\mathcal{S}$, $m \in \mathbb{N}$, such that $\lambda^{(m)} \rightarrow \lambda_\infty$ as $m \rightarrow \infty$, together with integers $M_m \in \mathbb{N}$ such that $M_m \lambda_s^{(m)} \in \mathbb{N}$ for every $s \in \mathcal{S}$. Fix m . The systems of sizes $N \in M_m \mathbb{N}$ with proportions $\lambda^{(m)}$ form a model of the type considered in Section 1, with the same ξ and h , whose limiting proportions $\lambda^{(m)}$ are rational, and whose initial condition is $\psi_{\lambda^{(m)}}$; the sizes $N \notin M_m \mathbb{N}$ play no role, and can be assigned arbitrary proportions converging to $\lambda^{(m)}$. Corollary 6.1 applies to this model and gives

$$(6.14) \quad \lim_{k \rightarrow \infty} \bar{F}_{kM_m, \lambda^{(m)}}(t, q) = f_{\lambda^{(m)}}(t, q).$$

For $N \geq M_m$, let $K_N := M_m \lfloor N/M_m \rfloor$, so that $N - M_m < K_N \leq N \leq 2K_N$ and $K_N \lambda_s^{(m)} \in \mathbb{N}$ for every s . Lemma 6.3, applied with the sizes $K_N \leq N$ and the proportions $\lambda^{(m)}$ and λ_N , gives

$$|\bar{F}_N(t, q) - \bar{F}_{K_N, \lambda^{(m)}}(t, q)| \leq C(1 + t + |q|_{L^\infty}) \left(|\lambda_N - \lambda^{(m)}| + \frac{M_m}{K_N} \right),$$

with C depending only on ξ and h . Letting $N \rightarrow \infty$ at fixed m , using (6.14) and $\lambda_N \rightarrow \lambda_\infty$, and then applying Lemma 6.5 to $\lambda^{(m)}, \lambda_\infty \in \Lambda_\eta$, we obtain, since $|q|_{L^1} \leq |q|_{L^\infty}$,

$$\limsup_{N \rightarrow \infty} |\bar{F}_N(t, q) - f(t, q)| \leq C(1 + t + |q|_{L^\infty}) |\lambda^{(m)} - \lambda_\infty|,$$

with C depending only on ξ, h , and η . Letting $m \rightarrow \infty$ proves that $\bar{F}_N(t, q) \rightarrow f(t, q)$.

Step 2: General paths.

Let $q \in \mathcal{Q}_2^\mathcal{S}$, and set $q_j^s := q^s \wedge j$ for $j \in \mathbb{N}$ and $s \in \mathcal{S}$, so that $q_j \in \mathcal{Q}_2^\mathcal{S}$ and $|q_j - q|_{L^1} \rightarrow 0$ as $j \rightarrow \infty$, by monotone convergence. By (2.17), (5.8), and Step 1,

$$\limsup_{N \rightarrow \infty} |\bar{F}_N(t, q) - f(t, q)| \leq 2|q_j - q|_{L^1} + \limsup_{N \rightarrow \infty} |\bar{F}_N(t, q_j) - f(t, q_j)| = 2|q_j - q|_{L^1}.$$

Letting $j \rightarrow \infty$ completes the proof. $\square$

7. BALANCED MODELS

We show that, for balanced models, the formula of Theorem 1.1 is bounded from below and from above by formulas for a one-species spherical model, and that the two bounds coincide when all the components of q are equal, in particular at $q = 0$. Balanced models were introduced in [6], see also [18]; we follow the argument given for Ising spins in [9, Section 9], see also [17]. In this section, we write $\lambda = (\lambda_s)_{s \in \mathcal{S}}$ for λ_∞ , and $r\lambda := (r\lambda_s)_{s \in \mathcal{S}}$ for $r \in \mathbb{R}$.

Throughout this section, we assume that the external field does not depend on the species: there is $h_\star \in \mathbb{R}$ such that $h^s = h_\star$ for every $s \in \mathcal{S}$, and we write $\psi_\star^\circ := \psi_{h_\star}^\circ$, so that $\psi(q) = \sum_{s \in \mathcal{S}} \lambda_s \psi_\star^\circ(q^s)$. We define the effective one-species covariance function

$$(7.1) \quad \xi_\star(r) := \xi(r\lambda), \quad r \in \mathbb{R}.$$

Grouping the terms of (1.4) according to their total degree, we can write

$$(7.2) \quad \xi_\star(r) = \sum_{n \geq 1} \left(\sum_{s_1, \dots, s_n \in \mathcal{S}} \beta_{s_1, \dots, s_n}^2 \prod_{j=1}^n \lambda_{s_j} \right) r^n, \quad r \in \mathbb{R},$$

an absolutely convergent series with nonnegative coefficients. Thus, $\xi_\star$ satisfies the assumptions of Section 1 for a model with a single species.

Definition 7.1 (Balanced models). We say that ξ is balanced with respect to λ if

$$(7.3) \quad \xi((\lambda_s x_s)_{s \in \mathcal{S}}) \leq \sum_{s \in \mathcal{S}} \lambda_s \xi_*(x_s), \quad x = (x_s)_{s \in \mathcal{S}} \in [0, 1]^{\mathcal{S}}.$$

In (7.3), the variable x_s stands for the overlap of species s normalized by the number of coordinates in this species, so that $\lambda_s x_s$ is this overlap normalized by the total number of coordinates, as in (1.3). The inequality holds with equality when all the normalized overlaps x_s agree, by (7.1). The simplest example is the bipartite model with two species of equal sizes,

$$\mathcal{S} = \{1, 2\}, \quad \lambda = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \xi(a_1, a_2) = a_1 a_2, \quad \xi_*(r) = \frac{r^2}{4},$$

for which (7.3) reads $x_1 x_2 \leq \frac{1}{2}(x_1^2 + x_2^2)$. More generally, a pure monomial $\xi(a) = \beta^2 \prod_{s \in \mathcal{S}} a_s^{k_s}$ with $k_s \geq 1$ is balanced with respect to $\lambda_s := k_s/n$, where $n := |k|_1$: in this case, $\xi_*(r) = cr^n$ with $c := \beta^2 \prod_s \lambda_s^{k_s}$, and the inequality (7.3) becomes $\prod_s (x_s^n)^{\lambda_s} \leq \sum_s \lambda_s x_s^n$, which is the weighted arithmetic-geometric mean inequality. A general criterion on the coefficients in (1.4) ensuring (7.3) is given in [9, Lemma 9.2], which covers the balanced models of [6].

For $(t, \rho) \in \mathbb{R}_+ \times \mathcal{Q}_2$, $\rho' \in \mathcal{Q}_\infty$, and $a \in \mathcal{Q}_{\infty, \leq 1}$, we define the one-species analogue of the functional $\mathcal{J}_{t, q}$ in (1.10)

$$(7.4) \quad \mathcal{J}_{t, \rho}^*(\rho', a) := \psi_*^\circ(\rho') + \langle \rho - \rho', a \rangle_{L^2} + t \int_0^1 \xi_*(a(v)) dv,$$

and we set

$$(7.5) \quad u(t, \rho) := \sup_{a \in \mathcal{Q}_{\infty, \leq 1}} \inf_{\rho' \in \mathcal{Q}_\infty} \mathcal{J}_{t, \rho}^*(\rho', a).$$

This is the right-hand side of (1.11) for the one-species spherical model with covariance function ξ_* and external field h_* : for this model, the set of species is a singleton, $\lambda_\infty = 1$, and the initial condition (2.26) is ψ_*° . By Theorem 1.1 applied to this model, the function u is thus the limit of its enriched free energy, and it is the Lipschitz viscosity solution of (1.12) with ξ_* and ψ_*° in place of ξ and ψ .

For $q = (q^s)_{s \in \mathcal{S}} \in \mathcal{Q}_2^\mathcal{S}$, we write $\bar{q} := \sum_{s \in \mathcal{S}} \lambda_s q^s \in \mathcal{Q}_2$.

Proposition 7.2 (Balanced reduction). Assume that $h^s = h_*$ for every $s \in \mathcal{S}$ and that ξ is balanced with respect to λ . Let f be the limit free energy in Theorem 1.1. For every $t \geq 0$ and $q \in \mathcal{Q}_2^\mathcal{S}$, we have

$$(7.6) \quad u(t, \bar{q}) \leq f(t, q) \leq \sum_{s \in \mathcal{S}} \lambda_s u(t, q^s).$$

In particular, if $q^s = \rho$ for every $s \in \mathcal{S}$, for some $\rho \in \mathcal{Q}_2$, then $f(t, q) = u(t, \rho)$. Moreover, the first inequality in (7.6) holds even if ξ is not balanced.

Proof. Since $h^s = h_*$ for every $s \in \mathcal{S}$, the conjugate ψ_s^* in (6.7) does not depend on s : it is the conjugate of ψ_*° , and we denote it by ψ_*^* . By Theorem 1.1 and (6.11) with $\lambda = \lambda_\infty$, and by the definition (7.5) of u , we have, for every $\rho \in \mathcal{Q}_2$,

$$(7.7) \quad f(t, q) = \sup_{a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}} \left\{ \sum_{s \in \mathcal{S}} \lambda_s \langle q^s, a_s \rangle_{L^2} - \sum_{s \in \mathcal{S}} \lambda_s \psi_*^*(a_s) + t \int_0^1 \xi((\lambda_s a_s(v))_{s \in \mathcal{S}}) dv \right\},$$

$$(7.8) \quad u(t, \rho) = \sup_{a \in \mathcal{Q}_{\infty, \leq 1}} \left\{ \langle \rho, a \rangle_{L^2} - \psi_*^*(a) + t \int_0^1 \xi_*(a(v)) dv \right\}.$$

For the lower bound, we restrict the supremum in (7.7) to the families with $a_s = a$ for every $s \in \mathcal{S}$, for some $a \in \mathcal{Q}_{\infty, \leq 1}$. For such a family, the expression in braces in (7.7) is that in (7.8) with $\rho = \bar{q}$, since $\sum_s \lambda_s = 1$ and $\xi(a(v)\lambda) = \xi_*(a(v))$ by (7.1). We thus have $f(t, q) \geq u(t, \bar{q})$.

For the upper bound, let $a \in (\mathcal{Q}_{\infty, \leq 1})^\mathcal{S}$. For every $v \in [0, 1]$, the vector $(a_s(v))_{s \in \mathcal{S}}$ belongs to $[0, 1]^\mathcal{S}$, so that (7.3) gives $\xi((\lambda_s a_s(v))_{s \in \mathcal{S}}) \leq \sum_s \lambda_s \xi_*(a_s(v))$. The expression in braces in (7.7) is therefore at most

$$\sum_{s \in \mathcal{S}} \lambda_s \left\{ \langle q^s, a_s \rangle_{L^2} - \psi_*^*(a_s) + t \int_0^1 \xi_*(a_s(v)) dv \right\} \leq \sum_{s \in \mathcal{S}} \lambda_s u(t, q^s),$$

by (7.8). Taking the supremum over a proves the upper bound in (7.6).

Finally, if $q^s = \rho$ for every $s \in \mathcal{S}$, then $\bar{q} = \rho$ and $\sum_s \lambda_s u(t, q^s) = u(t, \rho)$, so that both sides of (7.6) are equal to $u(t, \rho)$. $\square$

Corollary 7.3 (One-species formula for balanced models). *Assume that $h^s = h_\star$ for every $s \in \mathcal{S}$ and that ξ is balanced with respect to λ . Let $(H_N^\star(\sigma))_{\sigma \in \mathbb{R}^N}$ be a centered Gaussian field with covariance*

$$\mathbb{E}[H_N^\star(\sigma)H_N^\star(\sigma')] = N\xi_\star \left(\frac{1}{N} \sum_{i=1}^N \sigma_i \sigma'_i \right), \quad \sigma, \sigma' \in \mathbb{R}^N,$$

and let

$$\bar{F}_N^\star(t) := -\frac{1}{N} \mathbb{E} \log \int_{\mathbb{S}_N} \exp \left(\sqrt{2t} H_N^\star(\sigma) - Nt\xi_\star(1) + h_\star \sum_{i=1}^N \sigma_i \right) \mu_N(d\sigma)$$

be the free energy (1.9) of the one-species spherical model with covariance function $\xi_\star$ and external field $h_\star$. For every $t \geq 0$, we have

$$(7.9) \quad \lim_{N \rightarrow \infty} \bar{F}_N(t, 0) = \lim_{N \rightarrow \infty} \bar{F}_N^\star(t) = \sup_{a \in \mathcal{Q}_{\infty, \leq 1}} \inf_{\rho' \in \mathcal{Q}_{\infty}} \left\{ \psi_\star^\circ(\rho') - \langle \rho', a \rangle_{L^2} + t \int_0^1 \xi_\star(a(v)) dv \right\}.$$

Proof. The right-hand side of (7.9) is $u(t, 0)$, which is equal to $f(t, 0) = \lim_N \bar{F}_N(t, 0)$ by Proposition 7.2 applied with $q = 0$ and Theorem 1.1. As observed below (7.5), Theorem 1.1 applied to the one-species model shows that $\lim_N \bar{F}_N^\star(t)$ is also equal to $u(t, 0)$. $\square$

APPENDIX A. TILTED DISCRETE CASCADES

This appendix collects the properties of discrete Poisson–Dirichlet cascades under a tilt that are used in Section 2 and in Appendix B: the recursive formula for the free energy and the variance bound, used in the concentration Lemma B.1, and the law of the marks along a sampled leaf, used for the Gaussian model with n coordinates in Proposition 2.11. They all follow from an invariance property of tilted cascades due to Panchenko and Talagrand, [24, Lemma 3], see also [25, Lemma 3.1], which we recall in the form we use: tilting the weights of the cascade by the exponential of a function of the marks has the same effect, in law, as changing the law of the marks along the tree, level by level, while leaving the weights untouched. The recursive formula of [22, Theorem 2.9] or [15, Theorem 5.25] is the identity in expectation which corresponds to this invariance.

Setting. We use the discrete cascade of Subsection 2.1: the tree $\mathcal{A}$ of depth $k \geq 0$, the parameters $0 = m_0 < m_1 < \dots < m_k < m_{k+1} = 1$, the unnormalized weights $w_\alpha = \prod_{j=1}^k u_{\alpha|_j}$, and the normalized weights v_α . We attach to every node $\beta \in \mathcal{A}$ a standard Gaussian vector $z_\beta := (z_{\beta,i})_{i \leq n} \in \mathbb{R}^n$, independently over the nodes and of the weights. For a node β of depth j , for a leaf α , and for $i \leq n$, we write

$$\Omega_\beta := (z_{\beta|_0}, \dots, z_{\beta|_j}) \in (\mathbb{R}^n)^{j+1} \quad \text{and} \quad \Omega_{\alpha,i} := (z_{\alpha|_0,i}, \dots, z_{\alpha|_k,i}) \in \mathbb{R}^{k+1}$$

for the marks along the path from the root to β , and for the marks of coordinate i along the path from the root to α . Given a measurable $X : (\mathbb{R}^n)^{k+1} \rightarrow \mathbb{R}$, we set $X_k := X$ and define recursively, for j from k down to 1,

$$(A.1) \quad X_{j-1}(x_0, \dots, x_{j-1}) := \frac{1}{m_j} \log \mathbb{E} \exp(m_j X_j(x_0, \dots, x_{j-1}, z_j)), \quad X_{-1} := \mathbb{E} X_0(z_0),$$

where z_j denotes a standard Gaussian vector in $\mathbb{R}^n$. We say that X is admissible if all the expectations in (A.1) are finite, that is, if $\mathbb{E} \exp(m_j X_j(x_0, \dots, x_{j-1}, z_j)) < \infty$ for every $j \in \{1, \dots, k\}$ and $(x_0, \dots, x_{j-1}) \in (\mathbb{R}^n)^j$, and if $X_0(z_0)$ is integrable. Every Lipschitz function is admissible. The recursive formula of [22, Theorem 2.9] or [15, Theorem 5.25] is stated under the stronger assumption $\mathbb{E} \exp(m_k X(z_0, \dots, z_k)) < \infty$, in which all the marks are integrated at once; the conditional formulation is needed for the quadratic tilts of the Gaussian block, for which this stronger assumption can fail even when the multiplier b is admissible for the path, that is, when $b > 2\mathcal{K}(\rho)$, while the recursion (A.1) is finite at every level exactly under this condition, see Step 1 of the

proof of Lemma 2.7. For admissible X , the tilted path law μ_X is the probability measure on $(\mathbb{R}^n)^{k+1}$ given by

$$(A.2) \quad \mu_X(dx) := \gamma^{\otimes n}(dx_0) \prod_{j=1}^k \exp(m_j X_j(x_0, \dots, x_j) - m_j X_{j-1}(x_0, \dots, x_{j-1})) \gamma^{\otimes n}(dx_j);$$

by (A.1), each factor integrates to one in x_j , so that μ_X is indeed a probability measure, under which the mark at the root is not tilted and the mark at level j is tilted by $\exp(m_j X_j)$ given the marks of the previous levels. Finally, $\langle \cdot \rangle_X$ denotes the average with respect to the tilted cascade $v_\alpha^X \propto v_\alpha \exp(X(\Omega_\alpha))$; that its normalization $\sum_\alpha v_\alpha \exp(X(\Omega_\alpha))$ is almost surely finite for admissible X is part of the next lemma.

Tilted marks. Let X be admissible. We define a second family of marks $(z'_\beta)_{\beta \in \mathcal{A}}$, generated along the tree: $z'_\emptyset := z_\emptyset$, and, recursively over the depth $j \in \{1, \dots, k\}$, conditionally on the marks of depth at most $j-1$, the marks $(z'_\beta)_{\beta \in \mathbb{N}^j}$ are independent, the mark of the node $\beta = \beta' i$ with parent $\beta' \in \mathbb{N}^{j-1}$ having the law

$$(A.3) \quad \exp(m_j X_j(\Omega'_{\beta'}, z) - m_j X_{j-1}(\Omega'_{\beta'})) \gamma^{\otimes n}(dz),$$

where $\Omega'_\beta := (z'_{\beta_0}, \dots, z'_{\beta_j})$ denotes the new marks along the path from the root to a node β of depth j . Comparing (A.3) with (A.2), we see that Ω'_α has the law μ_X , for every leaf α . We also set, for every node β of depth $j \geq 1$,

$$(A.4) \quad e_\beta := \exp(X_j(\Omega_\beta) - X_{j-1}(\Omega_{\beta_{j-1}})), \quad \text{so that} \quad \prod_{j=1}^k e_{\alpha|_j} = \exp(X(\Omega_\alpha) - X_0(z_\emptyset))$$

for every leaf α , by telescoping.

Lemma A.1 (Invariance of tilted cascades). *Let X be admissible. Conditionally on $z_\emptyset$, the point processes*

$$(A.5) \quad (u_{\alpha|_1} e_{\alpha|_1}, \dots, u_{\alpha|_k} e_{\alpha|_k}, \Omega_\alpha)_{\alpha \in \mathbb{N}^k} \quad \text{and} \quad (u_{\alpha|_1}, \dots, u_{\alpha|_k}, \Omega'_\alpha)_{\alpha \in \mathbb{N}^k}$$

on $(0, \infty)^k \times (\mathbb{R}^n)^{k+1}$ have the same law, the marks $(z'_\beta)_{\beta \in \mathcal{A}}$ in the second process being generated by (A.3) independently of the weights. In particular, by (A.4), conditionally on $z_\emptyset$,

$$(A.6) \quad (w_\alpha \exp(X(\Omega_\alpha) - X_0(z_\emptyset)), \Omega_\alpha)_{\alpha \in \mathbb{N}^k} \stackrel{d}{=} (w_\alpha, \Omega'_\alpha)_{\alpha \in \mathbb{N}^k}$$

as point processes on $(0, \infty) \times (\mathbb{R}^n)^{k+1}$, and $\sum_{\alpha \in \mathbb{N}^k} w_\alpha \exp(X(\Omega_\alpha))$ is almost surely finite.

Proof. This is [24, Lemma 3], for independent standard Gaussian marks and with the root mark $z_\emptyset$ frozen: the recursion (A.1), the laws (A.3), and the factors (A.4) are respectively (3.4), (3.6), and (3.7) there. The result is stated in [24] under the assumption $\mathbb{E} \exp X(\Omega_\alpha) < \infty$, but its proof is an induction on the depth of the tree, in which the invariance of Poisson–Dirichlet point processes with i.i.d. marks, [24, Lemma 2] or [15, Theorem 5.19], is applied to the children of each node β of depth $j-1$, conditionally on the marks of depth at most $j-1$, with the function $z \mapsto \exp(X_j(\Omega_\beta, z))$, whose moment of order m_j is $\exp(m_j X_{j-1}(\Omega_\beta))$. It therefore only uses the finiteness of the conditional expectations in (A.1), that is, the admissibility of X . The last assertion follows from (A.4) and from the almost sure finiteness of $\sum_\alpha w_\alpha$, [15, Lemma 5.23]. $\square$

We now derive the three consequences that we need. The first one is the recursive formula of [22, Theorem 2.9] or [15, Theorem 5.25], under the conditional integrability assumption, and the second one is [24, Theorem 6].

Corollary A.2 (Free energy, marks along the sampled leaf, and variance). *Let X be admissible. We have*

$$(A.7) \quad \mathbb{E} \log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp(X(\Omega_\alpha)) = X_{-1},$$

and, for every bounded measurable Φ on $(\mathbb{R}^n)^{k+1}$,

$$(A.8) \quad \mathbb{E} \langle \Phi(\Omega_\alpha) \rangle_X = \int \Phi d\mu_X.$$

If moreover $k \geq 1$ and $X_0(z_\emptyset)$ is square integrable, and if S denotes the sum of the points of a Poisson–Dirichlet point process with parameter m_1 , then $\text{Var}(\log S) < \infty$ and

$$(A.9) \quad \text{Var} \left(\log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp(X(\Omega_\alpha)) \right) \leq 2 \text{Var}(X_0(z_\emptyset)) + 4 \text{Var}(\log S).$$

In words, under $\mathbb{E}(\cdot)_X$, the marks along the sampled leaf have law μ_X ; and, apart from the contribution of the untilted root mark, the fluctuations of the tilted free energy are those of the weights at the first level, whatever the tilt.

Proof. When $k = 0$, the tree is reduced to its root, $v_\emptyset = 1$, $\langle \Phi \rangle_X = \Phi(z_\emptyset)$, μ_X is the law of $z_\emptyset$, and $X_{-1} = \mathbb{E}X(z_\emptyset)$, so that (A.7) and (A.8) are immediate. Let $k \geq 1$, and let $T := \sum_{\alpha \in \mathbb{N}^k} w_\alpha$, which is independent of $z_\emptyset$. We first check that T has the law of cS for some constant $c \in (0, \infty)$, and that $\log S$ is square integrable. For $k = 1$, $T = S$. For $k \geq 2$, we write $T = \sum_{i \geq 1} u_i T_i$, where $T_i := \sum_{\alpha: \alpha|_1 = i} w_\alpha / u_i$ is the sum of the weights of the subtree rooted at the node i of depth one. The random variables $(T_i)_{i \geq 1}$ are i.i.d., independent of $(u_i)_{i \geq 1}$, and distributed as the sum of the weights of the cascade of depth $k - 1$ with parameters $(m_2, \dots, m_k)$, which by induction has the law of $c'S'$, with S' the sum of a Poisson–Dirichlet point process with parameter $m_2 > m_1$; hence $\mathbb{E}T_1^{m_1} < \infty$ by [15, Proposition 5.18], and [15, Theorem 5.19] gives $T \stackrel{d}{=} (\mathbb{E}T_1^{m_1})^{1/m_1} S$. Moreover, S is at least the largest point u_1 of the process, whose law is $\mathbb{P}(u_1 \leq x) = \exp(-x^{-m_1})$, so that S has finite negative moments, and S has finite positive moments of order $p < m_1$ by [15, Proposition 5.18]; hence $\log S$ is square integrable, and so is $\log T$, with $\text{Var}(\log T) = \text{Var}(\log S)$.

Set $A := \log \sum_{\alpha} w_\alpha \exp(X(\Omega_\alpha))$, so that $\log \sum_{\alpha} v_\alpha \exp(X(\Omega_\alpha)) = A - \log T$. By (A.6), conditionally on $z_\emptyset$, the random variable $\sum_{\alpha} w_\alpha \exp(X(\Omega_\alpha) - X_0(z_\emptyset))$ has the same law as T ; since T is independent of $z_\emptyset$, the pair $(z_\emptyset, A)$ therefore has the same law as $(z_\emptyset, X_0(z_\emptyset) + \log T')$, with $T' \stackrel{d}{=} T$ independent of $z_\emptyset$. Since $X_0(z_\emptyset)$ and $\log T$ are integrable, taking expectations gives $\mathbb{E}A = X_{-1} + \mathbb{E} \log T$, which is (A.7). If $X_0(z_\emptyset)$ is square integrable, then $\text{Var}(A) = \text{Var}(X_0(z_\emptyset)) + \text{Var}(\log T)$ by independence, and $\text{Var}(A - \log T) \leq 2 \text{Var}(A) + 2 \text{Var}(\log T)$ gives (A.9). Finally, for (A.8), conditionally on $z_\emptyset$, (A.6) gives

$$\langle \Phi(\Omega_\alpha) \rangle_X = \frac{\sum_{\alpha} w_\alpha \exp(X(\Omega_\alpha) - X_0(z_\emptyset)) \Phi(\Omega_\alpha)}{\sum_{\alpha} w_\alpha \exp(X(\Omega_\alpha) - X_0(z_\emptyset))} \stackrel{d}{=} \frac{\sum_{\alpha} w_\alpha \Phi(\Omega'_\alpha)}{\sum_{\alpha} w_\alpha} = \sum_{\alpha} v_\alpha \Phi(\Omega'_\alpha).$$

Conditionally on $z_\emptyset$ and on the weights, each Ω'_α has the conditional law of μ_X given that its first coordinate is $z_\emptyset$, and $\sum_{\alpha} v_\alpha = 1$; taking expectations, first over the marks, then over the weights, and finally over $z_\emptyset$, gives (A.8). $\square$

We also record a quantitative bound on the last term in (A.9), which is used in Lemma B.1 with a parameter m_1 tending to zero.

Lemma A.3 (Variance of the logarithm of a Poisson–Dirichlet sum). *There exists $C < \infty$ such that, for every $m \in (0, \frac{1}{2}]$, the sum S of the points of a Poisson–Dirichlet point process with parameter m satisfies $\text{Var}(\log S) \leq Cm^{-2}$.*

Proof. Let $u_1 > u_2 > \dots$ be the points of the process, ordered decreasingly, and write $\log S = \log u_1 + \log(S/u_1)$. The random variable $E := u_1^{-m}$ is exponentially distributed with parameter one, so $\text{Var}(\log u_1) = m^{-2} \text{Var}(\log E)$. Conditionally on u_1 , the remaining points form a Poisson point process on $(0, u_1)$ with intensity $mx^{-m-1}dx$, so that

$$\begin{aligned} \mathbb{E} \left[\frac{S}{u_1} - 1 \mid u_1 \right] &= \int_0^{u_1} \frac{x}{u_1} mx^{-m-1} dx = \frac{m}{1-m} E, \\ \text{Var} \left(\frac{S}{u_1} \mid u_1 \right) &= \int_0^{u_1} \frac{x^2}{u_1^2} mx^{-m-1} dx = \frac{m}{2-m} E. \end{aligned}$$

Since $0 \leq \log(S/u_1) \leq S/u_1 - 1$, we get $\mathbb{E}[\log(S/u_1)^2] \leq \mathbb{E}[(S/u_1 - 1)^2] \leq Cm$ for $m \leq \frac{1}{2}$, and $\text{Var}(\log S) \leq 2 \text{Var}(\log u_1) + 2\mathbb{E}[\log(S/u_1)^2] \leq Cm^{-2}$. $\square$

Lemma A.4 (Tilts with a product structure). *Let $\{1, \dots, n\} = \sqcup_{g \in \mathcal{G}} I_g$ be a partition of the coordinates into groups, and for a leaf α and $g \in \mathcal{G}$, let $\Omega_{\alpha, g} := (\Omega_{\alpha, i})_{i \in I_g}$ denote the marks of*

the coordinates of the group g . For every $g \in \mathcal{G}$, let $X^{(g)}$ be a measurable function of the marks of the group g which is admissible for the cascade with $|I_g|$ marks per node, and set $X(\Omega_\alpha) := \sum_{g \in \mathcal{G}} X^{(g)}(\Omega_{\alpha,g})$.

- (1) The function X is admissible, the recursion (A.1) is additive: $X_j = \sum_g X_j^{(g)}$ for every j , the tilted path law is the product $\mu_X = \bigotimes_{g \in \mathcal{G}} \mu_{X^{(g)}}$, and

$$(A.10) \quad \mathbb{E} \log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp(X(\Omega_\alpha)) = \sum_{g \in \mathcal{G}} X_{-1}^{(g)}.$$

- (2) For every bounded measurable Φ ,

$$\mathbb{E} \langle \Phi((\Omega_{\alpha,g})_{g \in \mathcal{G}}) \rangle_X = \int \Phi d \bigotimes_{g \in \mathcal{G}} \mu_{X^{(g)}}.$$

In words, under $\mathbb{E} \langle \cdot \rangle_X$, the marks of the groups along the sampled leaf are independent, with laws $\mu_{X^{(g)}}$.

- (3) Assume moreover that $X_0^{(g)}(z_0)$ is square integrable for every g , and let S be as in Corollary A.2 if $k \geq 1$, with the convention $\text{Var}(\log S) := 0$ if $k = 0$. Then

$$\text{Var} \left(\log \sum_{\alpha \in \mathbb{N}^k} v_\alpha \exp(X(\Omega_\alpha)) \right) \leq 2 \sum_{g \in \mathcal{G}} \text{Var} \left(X_0^{(g)}(z_0) \right) + 4 \text{Var}(\log S).$$

Proof. Since the marks of different groups are independent at every level, the expectation in (A.1) factorizes over the groups, and an induction from $j = k$ down to $j = 0$ gives $X_j = \sum_g X_j^{(g)}$, each term being finite by assumption; in particular X is admissible, $X_{-1} = \sum_g X_{-1}^{(g)}$, and the density in (A.2) is the product of the corresponding densities of the groups. Part (1) then follows from (A.7), and part (2) from (A.8). This is the additivity of [15, Corollary 5.26], under the conditional integrability assumption. For part (3), the case $k = 0$ is immediate, since then $\log \sum_\alpha v_\alpha \exp(X(\Omega_\alpha)) = X(z_\emptyset) = \sum_g X_0^{(g)}(z_{\emptyset,g})$, whose variance is $\sum_g \text{Var}(X_0^{(g)}(z_0))$ by independence. When $k \geq 1$, it follows from (A.9) and from $\text{Var}(X_0(z_\emptyset)) = \sum_g \text{Var}(X_0^{(g)}(z_0))$, again by independence. $\square$

APPENDIX B. CONCENTRATION FOR HAMILTONIANS LINEAR IN THE CASCADE FIELDS

The Hamiltonians used in the cavity computation of Section 3 involve, besides the field W_N^q , a countable family of additional cascade fields. Conditionally on the cascade, the free energy of such a Hamiltonian is a Lipschitz function of Gaussian fields, and the Gaussian concentration inequality of [22, Theorem 1.2] controls its fluctuations around its conditional expectation. The fluctuations of the cascade itself, which is not a Gaussian object, require a separate argument. This second source of fluctuations was overlooked in [10, Proposition 6.8], where the Gaussian rate is invoked for the full free energy. We consider Hamiltonians of the form

$$(B.1) \quad G(\sigma, \alpha) := G_0(\sigma) + \mathbf{m}(\sigma) + \sum_{j \in J} w_j(\alpha) \phi_j(\sigma), \quad \sigma \in \Sigma_N, \quad \alpha \in \mathfrak{U},$$

where J is a finite or countable set, $(G_0(\sigma))_{\sigma \in \Sigma_N}$ is a centered Gaussian field independent of the cascade, $\mathbf{m} : \Sigma_N \rightarrow \mathbb{R}$ is bounded and measurable, the functions $\phi_j : \Sigma_N \rightarrow \mathbb{R}$ are bounded and measurable, and, conditionally on $\mathfrak{R}$, the fields $(w_j)_{j \in J}$ are independent centered Gaussian fields on $\mathfrak{U}$ with covariances $\rho_j(\alpha \wedge \alpha')$, for paths $\rho_j \in \mathcal{Q}_\infty$, independent of G_0 , and jointly measurable as in the construction of w^q in Subsection 2.1. We assume that

$$(B.2) \quad \mathbf{V} := \sup_{\sigma \in \Sigma_N} \left(\text{Var} G_0(\sigma) + \sum_{j \in J} \rho_j(1) \phi_j(\sigma)^2 \right) < \infty,$$

which is the supremum of the variance of $G(\sigma, \alpha)$, and that there exists an increasing sequence of finite sets $J_n \uparrow J$ such that

$$(B.3) \quad \lim_{n \rightarrow \infty} \sup_{\sigma \in \Sigma_N} \sum_{j \in J \setminus J_n} \rho_j(1) \phi_j(\sigma)^2 = 0,$$

a condition which is void when J is finite. Under these assumptions, the series in (B.1) converges in L^2 , uniformly over (σ, α) , and we let

$$Z := \iint \exp(G(\sigma, \alpha)) P_N(d\sigma) \mathfrak{R}(d\alpha)$$

be the associated partition function, with Gibbs bracket $\langle \cdot \rangle_G$. The enriched Hamiltonian $H_N^{t,q}$ in (2.13) is of this form, with $J = [N]$, $\rho_i = q^s$ and $\phi_i(\sigma) = \sqrt{2}\sigma_i$ for $i \in I_{N,s}$, $G_0 = \sqrt{2}tH_N$, and $V \leq 2Nt\xi(\lambda_N) + 2N \max_s q^s(1)$; the perturbed Hamiltonians of Section 3 are of this form as well, see Remark 3.1 there.

Lemma B.1 (Concentration of the free energy). *There exists a universal constant $C < \infty$ such that, under the assumptions above,*

$$(B.4) \quad \mathbb{E} |\log Z - \mathbb{E} \log Z| \leq C(1 + V)^{1/2}.$$

The bound (B.4) is the one given by the Gaussian concentration inequality when the cascade is frozen; the content of the lemma is that the fluctuations of the cascade do not degrade it. For the discrete cascades of Appendix A, the variance bound (A.9) shows that, apart from the contribution of the root mark, the fluctuations of a tilted free energy are bounded by those of the weights of the first level, uniformly in the tilt. The continuous cascade has levels arbitrarily close to the root, for which this bound degenerates. We therefore first cut off the levels below a small threshold δ , by modifying the paths ρ_j on $[0, \delta)$, at a cost controlled by a Gaussian interpolation performed conditionally on the cascade, and then apply the discrete bound to the modified model, whose first level is at height δ .

Proof of Lemma B.1. Throughout the proof, C denotes a universal constant.

Step 1: Integrability, and reduction to a finite family.

By Jensen's inequality under $P_N \otimes \mathfrak{R}$, we have $\log Z \geq \iint G dP_N d\mathfrak{R}$, and, conditionally on $\mathfrak{R}$, the right-hand side is a Gaussian random variable with mean in $[-\|\mathbf{m}\|_\infty, \|\mathbf{m}\|_\infty]$ and variance at most V ; moreover, $\mathbb{E}[Z \mid \mathfrak{R}] = \iint \exp(\mathbf{m}(\sigma) + \frac{1}{2} \text{Var } G(\sigma, \alpha)) dP_N d\mathfrak{R} \leq \exp(\|\mathbf{m}\|_\infty + V/2)$. Hence $\log Z$ is integrable, and, almost surely,

$$(B.5) \quad -\|\mathbf{m}\|_\infty \leq \mathbb{E} [\log Z \mid \mathfrak{R}] \leq \|\mathbf{m}\|_\infty + \frac{V}{2},$$

with the same bounds for $\mathbb{E} \log Z$. When J is finite, conditionally on $\mathfrak{R}$, the random variable $\log Z$ is the free energy of the Gaussian field G on $\Sigma_N \times \mathfrak{U}$, whose variance is at most V by (B.2), and the Gaussian concentration inequality, in the form used for free energies, see [22, Theorem 1.2] and [15, Theorem 4.7], gives Gaussian tails for $\log Z - \mathbb{E}[\log Z \mid \mathfrak{R}]$, uniformly over the realization of $\mathfrak{R}$; in particular,

$$(B.6) \quad \mathbb{E} \left[(\log Z - \mathbb{E}[\log Z \mid \mathfrak{R}])^2 \mid \mathfrak{R} \right] \leq CV$$

almost surely, and, in view of the paragraph preceding (B.5), $\log Z$ has finite moments of all orders.

We now reduce to the case of a finite family. Let $Z^{(n)}$ be the partition function of the Hamiltonian $G^{(n)}$ obtained by restricting the sum in (B.1) to J_n , with Gibbs bracket $\langle \cdot \rangle_{(n)}$, and let $T_n := G - G^{(n)} = \sum_{j \notin J_n} w_j(\alpha) \phi_j(\sigma)$, so that

$$\log Z - \log Z^{(n)} = \log \langle \exp(T_n) \rangle_{(n)}.$$

Conditionally on $\mathfrak{R}$, the field T_n is centered Gaussian, independent of $G^{(n)}$, with variance at most $\varepsilon_n := \sup_\sigma \sum_{j \notin J_n} \rho_j(1) \phi_j(\sigma)^2$, which tends to zero by (B.3). By Jensen's inequality and the inequality $\log u \leq u - 1$, we have $\langle T_n \rangle_{(n)} \leq \log Z - \log Z^{(n)} \leq \langle \exp(T_n) \rangle_{(n)} - 1$. Averaging first over T_n , with the bracket $\langle \cdot \rangle_{(n)}$ frozen, we get $\mathbb{E} \langle T_n \rangle_{(n)}^2 \leq \mathbb{E} \langle T_n^2 \rangle_{(n)} \leq \varepsilon_n$ and

$$\mathbb{E} \left[\left(\langle \exp(T_n) \rangle_{(n)} - 1 \right)^2 \right] = \mathbb{E} \langle \exp(T_n(z) + T_n(z')) \rangle_{(n)} - 2 \mathbb{E} \langle \exp(T_n) \rangle_{(n)} + 1 \leq e^{2\varepsilon_n} - 1,$$

where $z = (\sigma, \alpha)$ and $z' = (\sigma', \alpha')$ denote two replicas. Hence $\mathbb{E}|\log Z - \log Z^{(n)}| \leq \varepsilon_n^{1/2} + (e^{2\varepsilon_n} - 1)^{1/2}$, which tends to zero as $n \rightarrow \infty$, and

$$\mathbb{E}|\log Z - \mathbb{E} \log Z| \leq 2 \mathbb{E}|\log Z - \log Z^{(n)}| + \mathbb{E}|\log Z^{(n)} - \mathbb{E} \log Z^{(n)}|.$$

Since $G^{(n)}$ is again of the form (B.1) and satisfies (B.2) with the same $\mathbf{V}$, it therefore suffices to prove the lemma when J is finite, which we assume from now on. We also fix $\delta \in (0, \frac{1}{2}]$, to be chosen at the end.

Step 2: Flattening of the paths on $[0, \delta]$.

For $j \in J$, let $\tilde{\rho}_j(u) := \rho_j(u \vee \delta)$, which belongs to $\mathcal{Q}_\infty$, coincides with ρ_j on $[\delta, 1]$, and satisfies $0 \leq \tilde{\rho}_j - \rho_j \leq \rho_j(1)\mathbb{1}_{[0, \delta]}$. Let $(\tilde{w}_j)_{j \in J}$ be independent cascade fields with covariances $\tilde{\rho}_j(\alpha \wedge \alpha')$, independent of $(w_j)_{j \in J}$ and of G_0 conditionally on $\mathfrak{R}$, and, for $a \in [0, 1]$, let G^a be defined by (B.1) with $w_j^a := \sqrt{1-a}w_j + \sqrt{a}\tilde{w}_j$ in place of w_j , with partition function Z_a and bracket $\langle \cdot \rangle_a$. The field w_j^a has covariance $((1-a)\rho_j + a\tilde{\rho}_j)(\alpha \wedge \alpha')$, so that G^a is again of the form (B.1), with the same $\mathbf{V}$, and $Z_0 = Z$, while $\tilde{Z} := Z_1$ is the partition function of the flattened model. The derivative in a of the covariance of G^a is

$$\Delta(\sigma, \alpha; \sigma', \alpha') := \sum_{j \in J} (\tilde{\rho}_j - \rho_j)(\alpha \wedge \alpha') \phi_j(\sigma) \phi_j(\sigma'),$$

which vanishes when $\alpha = \alpha'$, since $\tilde{\rho}_j(1) = \rho_j(1)$, and satisfies $|\Delta(\sigma, \alpha; \sigma', \alpha')| \leq \mathbf{V} \mathbb{1}_{\{\alpha \wedge \alpha' < \delta\}}$ by the Cauchy-Schwarz inequality and (B.2). We compare the conditional expectations

$$\mu := \mathbb{E}[\log Z | \mathfrak{R}] \quad \text{and} \quad \tilde{\mu} := \mathbb{E}[\log \tilde{Z} | \mathfrak{R}],$$

which integrate the Gaussian fields with the cascade frozen. Conditionally on $\mathfrak{R}$, the family $(G^a)_{a \in [0, 1]}$ is a Gaussian interpolation, and Gaussian integration by parts, as in the proof of Lemma 3.4, gives

$$(B.7) \quad \frac{d}{da} \mathbb{E}[\log Z_a | \mathfrak{R}] = -\frac{1}{2} \mathbb{E}[\langle \Delta(z, z') \rangle_a | \mathfrak{R}],$$

where $z = (\sigma, \alpha)$ and $z' = (\sigma', \alpha')$ denote two replicas, the diagonal term of the integration by parts vanishing since $\Delta(z, z) = 0$. By the invariance (2.5), applied conditionally on G_0 , the overlap $\alpha \wedge \alpha'$ is uniformly distributed on $[0, 1]$ under $\mathbb{E} \langle \cdot \rangle_a$, so that $\mathbb{E} \langle |\Delta(z, z')| \rangle_a \leq \mathbf{V} \delta$ for every $a \in [0, 1]$. Integrating (B.7) over $a \in [0, 1]$ and taking expectations, we obtain

$$(B.8) \quad \mathbb{E}|\mu - \tilde{\mu}| \leq \frac{1}{2} \mathbf{V} \delta.$$

Since $\mathbb{E}\mu = \mathbb{E} \log Z$ and $\mathbb{E}\tilde{\mu} = \mathbb{E} \log \tilde{Z}$, the triangle inequality gives

$$\mathbb{E}|\log Z - \mathbb{E} \log Z| \leq \mathbb{E}|\log Z - \mu| + 2 \mathbb{E}|\mu - \tilde{\mu}| + \mathbb{E}|\tilde{\mu} - \mathbb{E} \log \tilde{Z}|.$$

We bound the first term by (B.6), the second one by (B.8), and the last one by the conditional Jensen inequality $\mathbb{E}|\tilde{\mu} - \mathbb{E} \log \tilde{Z}| \leq \mathbb{E}|\log \tilde{Z} - \mathbb{E} \log \tilde{Z}| \leq \text{Var}(\log \tilde{Z})^{1/2}$, and we get

$$(B.9) \quad \mathbb{E}|\log Z - \mathbb{E} \log Z| \leq C \mathbf{V}^{1/2} + \mathbf{V} \delta + \text{Var}(\log \tilde{Z})^{1/2}.$$

It remains to bound the variance of the free energy of the flattened model.

Step 3: Discrete approximation of the flattened model.

The flattened Hamiltonian depends on the cascade only through $(\alpha \wedge \alpha') \vee \delta$, since $\tilde{\rho}_j(u) = \rho_j(u \vee \delta)$. For $K \in \mathbb{N}$, consider the discrete cascade of depth K with parameters $m_1 := \delta$ and $m_{l+1} := \delta + l(1-\delta)/K$ for $1 \leq l \leq K$, so that $m_{K+1} = 1$, and let $\hat{\rho}_j$ be such that $\hat{\rho}_j = \rho_j(\delta)$ on $[0, \delta)$ and $\hat{\rho}_j = \rho_j(m_l)$ on $[m_l, m_{l+1})$ for $l \geq 1$. Let $\tilde{Z}^{(K)}$ be the partition sum obtained by replacing, in the flattened model, the cascade $\mathfrak{R}$ by this discrete cascade and the fields $\tilde{w}_j$ by the discrete fields $w_j^{\hat{\rho}_j}$ of (2.8), built from independent families of marks. As $K \rightarrow \infty$, the array $(m_{\alpha^\ell \wedge \alpha^{\ell'}} \vee \delta)_{\ell, \ell' \leq n'}$ of the discrete cascade converges in law to $((\alpha^\ell \wedge \alpha^{\ell'}) \vee \delta)_{\ell, \ell' \leq n'}$ under $\mathbb{E} \langle \cdot \rangle_{\mathfrak{R}}$, for every n' , by [15, Corollary 5.32], and since the off-diagonal overlaps are uniformly distributed while the paths $\tilde{\rho}_j$ have countably many discontinuities, the arrays $(\hat{\rho}_j(m_{\alpha^\ell \wedge \alpha^{\ell'}}))_{j, \ell, \ell'}$ converge in law to $(\tilde{\rho}_j(\alpha^\ell \wedge \alpha^{\ell'}))_{j, \ell, \ell'}$. Let $A > \|\mathbf{m}\|_\infty + \mathbf{V}$, and let $\log_A := (-A) \vee (\log \wedge A)$. The argument of the

proof of [10, Proposition 4.8], based on [10, Propositions 4.4 and 4.6], shows that $\mathbb{E} \log_A \tilde{Z}$ and $\mathbb{E}(\log_A \tilde{Z})^2$ are, up to an error which can be made arbitrarily small uniformly over the cascades under consideration, expectations of bounded continuous functions of these arrays: one truncates the Gaussian fields, using their Gaussian tails, approximates the bounded continuous functions $\log_A$ and $(\log_A)^2$ by polynomials on the compact range of the truncated partition function, and observes that the expectation of a polynomial in the partition function is, after integrating the Gaussian fields, a continuous function of the overlap array of finitely many replicas. Consequently, $\text{Var}(\log_A \tilde{Z}) = \lim_{K \rightarrow \infty} \text{Var}(\log_A \tilde{Z}^{(K)})$. Since $\log_A$ is the projection of $\log$ onto $[-A, A]$, and since $\mathbb{E} \log \tilde{Z}^{(K)}$ lies in $[-\|\mathbf{m}\|_\infty, \|\mathbf{m}\|_\infty + \mathbf{V}/2] \subseteq [-A, A]$, by (B.5), whose proof applies verbatim to the discrete model, we have $\text{Var}(\log_A \tilde{Z}^{(K)}) \leq \mathbb{E}[(\log_A \tilde{Z}^{(K)} - \mathbb{E} \log \tilde{Z}^{(K)})^2] \leq \text{Var}(\log \tilde{Z}^{(K)})$. Letting $A \rightarrow \infty$, and using that $\log \tilde{Z}$ is square integrable, by Step 1 applied to the flattened model, we conclude that

$$(B.10) \quad \text{Var}(\log \tilde{Z}) \leq \liminf_{K \rightarrow \infty} \text{Var}(\log \tilde{Z}^{(K)}).$$

Step 4: The discrete bound.

We fix K . In the notation of Appendix A, with $n = |J|$ marks per node, we have

$$\log \tilde{Z}^{(K)} = \log \sum_{\alpha \in \mathbb{N}^K} v_\alpha \exp(X(\Omega_\alpha)),$$

where

$$X(\Omega_\alpha) := \log \int \exp\left(G_0(\sigma) + \mathbf{m}(\sigma) + \sum_{j \in J} \mathbf{w}_j^{\hat{\rho}_j}(\alpha) \phi_j(\sigma)\right) P_N(d\sigma),$$

$$\mathbf{w}_j^{\hat{\rho}_j}(\alpha) = \sum_{l=0}^K (\hat{\rho}_{j,l} - \hat{\rho}_{j,l-1})^{1/2} z_{\alpha|l,j},$$

The gradient of X with respect to the marks is a Gibbs average of $((\hat{\rho}_{j,l} - \hat{\rho}_{j,l-1})^{1/2} \phi_j(\sigma))_{j,l}$, whose norm is at most $(\sum_j \hat{\rho}_j(1) \phi_j(\sigma)^2)^{1/2} \leq \mathbf{V}^{1/2}$; hence X is Lipschitz with respect to these variables, and admissible. Conditionally on G_0 , the variance bound (A.9) and the recursive formula (A.7) give

$$\text{Var}(\log \tilde{Z}^{(K)} | G_0) \leq 2 \text{Var}(X_0(z_\emptyset) | G_0) + 4 \text{Var}(\log S)$$

and $\mathbb{E}[\log \tilde{Z}^{(K)} | G_0] = \mathbb{E}[X_0(z_\emptyset) | G_0],$

where S is the sum of the points of a Poisson–Dirichlet point process with parameter $m_1 = \delta$, so that $\text{Var}(\log S) \leq C\delta^{-2}$ by Lemma A.3. Combining the two identities through the decomposition of the variance into conditional variance and variance of the conditional expectation, we obtain

$$\text{Var}(\log \tilde{Z}^{(K)}) \leq 2 \text{Var}(X_0(z_\emptyset)) + C\delta^{-2},$$

where the variance of $X_0(z_\emptyset)$ is now taken over G_0 and $z_\emptyset$ jointly. The random variable $X_0(z_\emptyset)$ depends on G_0 and $z_\emptyset$ only through the centered Gaussian field $\mathbf{G}(\sigma) := G_0(\sigma) + \sum_{j \in J} \hat{\rho}_{j,0}^{1/2} z_{\emptyset,j} \phi_j(\sigma)$ on Σ_N , whose variance is at most $\mathbf{V}$ by (B.2), since $\hat{\rho}_{j,0} = \rho_j(\delta) \leq \rho_j(1)$. Moreover, X_K is of the form $\log \int \exp(\mathbf{G}(\sigma) + \dots) P_N(d\sigma)$, whose derivative with respect to a bounded perturbation h of the field is $\int h d\nu$ for a probability measure ν on Σ_N , and the recursion (A.1) averages such derivatives with nonnegative weights summing to one; the same is therefore true of X_0 . The Gaussian concentration inequality, in the form used for free energies, see [22, Theorem 1.2] and [15, Theorem 4.7], thus applies to X_0 as a functional of $\mathbf{G}$, and gives $\text{Var}(X_0(z_\emptyset)) \leq C\mathbf{V}$. Hence $\text{Var}(\log \tilde{Z}^{(K)}) \leq C\mathbf{V} + C\delta^{-2}$, uniformly in K .

Step 5: Conclusion.

By (B.9), (B.10), and Step 4,

$$\mathbb{E}|\log Z - \mathbb{E} \log Z| \leq C\mathbf{V}^{1/2} + \mathbf{V}\delta + C\delta^{-1}.$$

Choosing $\delta := \min\{\frac{1}{2}, (1 + \mathbf{V})^{-1/2}\}$ gives (B.4). $\square$

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