Residual units in group rings and Novikov rings

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1 Introduction

1.1 Definitions

Let G be a residually finite group, and $n \in \mathbb{N}^*$. If R is a ring, $R[G]$ is the group ring with coefficients in R, and $\mathrm{M}_n(R)$ is the ring of (n, n)-matrices with coefficients in R.

- An element of $\mathrm{M}_n(\mathbb{Z}[G])$ is *a residual unit* if its image in $\mathrm{M}_n(\mathbb{Z}[G/H])$ is a unit for every subgroup $H \lhd_{f.i.} G$ (normal with of finite index).
- $\mathrm{M}_n(\mathbb{Z}[G])$ has *finitely detectable units* if every residual unit is a unit.

Recall that the ring $\mathbb{Z}[G]$ is *stably finite*, meaning that if $A \in \mathrm{M}_n(\mathbb{Z}[G])$, A is a unit $\Leftrightarrow$ A has a right inverse $\Leftrightarrow$ A has a left inverse.

The Novikov case. Let ξ be an element of $\mathcal{N}(G) = (\mathrm{Hom}(G, \mathbb{R}) \setminus \{0\})_{/\mathbb{R}_+^*}$. Recall that the Novikov ring $\mathbb{Z}[G]_\xi$ is the ring of series $\sum_{i=0}^{\infty} n_i g_i$ such that $n_i \in \mathbb{Z}$, $g_i \in G$ and $\xi(g_i) \to +\infty$. If H is a normal subgroup of G, we denote $\bar{\xi}_H \in \mathcal{N}(G/(H \cap \ker \xi))$ the image of ξ.

Recall that the Novikov ring is also stably finite. We adapt the previous definitions to the Novikov case:

- An element of $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ is a residual unit if its image in $\mathrm{M}_n(\mathbb{Z}[G/(H \cap \ker \xi)]_{\bar{\xi}_H})$ is a unit for every subgroup $H \lhd_{f.i.} G$.
- $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units if every residual unit is a unit.

1.2 Conjecture

Let G be a "nice" group. For instance:

- virtually (residually finite and right-orderable); by [Boyer-Rolfsen-Wiest] and Perelman, this is the case if $G = \pi_1(M)$ where M is a closed 3-dimensional manifold with $b_1(M) > 0$. By Agol, it suffices that if M be hyperbolic.
- (stronger) virtually (residuallly torsion-free nilpotent); by [Koberda 2013] this is the case if $G = \pi_1(M)$ where M is a closed 3-dimensional geometric manifold which is not Sol, in particular if M is hyperbolic. Other examples: free groups (Magnus), surface groups (Baumslag), right-angled Artin groups (Duchamp-Krob).

Then $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]$ and $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]_\xi)$ have finitely detectable units for every $n \in \mathbb{N}^*$.

Remarks. 1)One motivation for this conjecture in the Novikov case is that it implies that a class $\xi \in H^1(M, \mathbb{R}) \setminus \{0\})$ is represented by a nonsingular closed one-form if and only if that every twisted Alexander polynomial $\Delta_{M,u}^H$ is unitary. (cf [Sikorav, On Novikov homology]).

A related application is the following:

Proposition. *Let G be a group satisfying the conjecture in the Novikov case, and let ξ be a nonzero morphism from G to $\mathbb{Z}$. Then the following are equivalent:*

- $\ker \xi$ *is finitely generated*
- *for every subgroup $H <_{f.i.} G$, the Abelianization of* $\ker(\xi_{|H})$ *is finitely generated.*

2) If one replaces the quotients $\mathbb{Z}[G/H]$ with $H \triangleleft_{f.i.} G$ by the finite quotient rings $\mathbb{Z}[G]/I$, one obtains a weaker notion of residual unit. Indeed, assume that $\lambda \in \mathbb{Z}[G]$ (or a matrix ring, or the Novikov version) becomes a unit in every quotient $\mathbb{Z}[G/H]$, $H \triangleleft_{f.i.} G$. If $\mathbb{Z}[G]/I$ is a finite quotient, the normal subgroup

$$H := \{g \in G \mid \bar{g} = 1 \text{ in } \mathbb{Z}[G]/I\} = \ker(G \to \mathbb{Z}[G]/I)^*$$

has finite index in G and we have a factorization

$$\begin{tikzcd}
\mathbb{Z}[G] \arrow[r] \arrow[d] & \mathbb{Z}[G]/I \\
\mathbb{Z}[G/H]. \arrow[ur] &
\end{tikzcd}$$

Since λ becomes a unit in $\mathbb{Z}[G/H]$, it also becomes a unit in $\mathbb{Z}[G]/I$. (The converse would require to prove that if G is finite and $\lambda \in \mathbb{Z}[G]$ becomes a unit in $\mathbb{F}_p[G]$ for every prime p, then λ is a unit **mais justement c'est vrai : ça revient à dire que si le déterminant de la matrice $A \in \mathrm{M}_n(\mathbb{Z})$ associée est non divisible par p pour tout p premier, alors A est inversible** .)

1.3 Main result

Theorem. *Let G be a group which is residually torsion-free nilpotent. Then $\mathbb{Z}[G]$ has finitely detectable units.*

Outline. In Section 2, we prove the Theorem in the case G Abelian (also the Novikov version). In Section 3 we prove that the validity of the conjecture for a subgroup of finite index implies it for the whole group, also in the Novikov case. (Note that the validity of the conjecture for G easily implies it for any subgroup). In Section 4, we prove the Theorem for a torsion-free nilpotent group **la preuve est fausse**, and immediately deduce the general case in Section 5.

2 The case of Abelian groups

Let G be an Abelian group.

Proposition 1. *The ring $\mathrm{M}_n(\mathbb{Z}[G)$ has finitely detectable units for every $n \in \mathbb{N}^*$.*

Proof. Using the determinant, it suffices to treat the case of $\mathbb{Z}[G]$. Let $\lambda \in \mathbb{Z}[\mathbb{Z}^r]$ be a residual unit. By contradiction, assume that $\mathbb{Z}[G]/\langle\lambda\rangle$ is nonzero. Then it is a commutative unitary ring, thus it has a quotient F which is a field (without the axiom of choice since A is Noetherian). Also, F is finitely generated as an algebra over $\mathbb{Z}$, thus it is finite (cf mathoverflow 2011, *A finitely generated $\mathbb{Z}$-algebra that is a field has to be finite*).

Since λ becomes a unit in every quotient $\mathbb{Z}[G/H]$, $H <_{f.i.} G$, it becomes also a unit in every finite quotient $\mathbb{Z}[G]/I$ by the Remark 2 in 1.3. In particular, its image in F is a unit. Since F is a quotient of $\mathbb{Z}[G]/\langle\lambda\rangle$, this implies $F = 0$, contradiction.

Proposition 2. *The ring $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for every $n \in \mathbb{N}^*$.*

Proof. Using the determinant and the induction from a finite index subgroup (next section), it suffices to treat the case of $\mathbb{Z}[\mathbb{Z}^r]_\xi$. Let $\lambda \in \mathbb{Z}[\mathbb{Z}^r]_\xi$ be a residual unit. We can assume that its ξ-minimal part λ_0 belongs to $\mathbb{Z}[\ker\xi]$.

Since $\mathrm{supp}(\lambda_0)$ is finite, there exists a group homomorphism $f_0 : \ker \xi \to \mathbb{Z}$ which is injective on $\mathrm{supp}(\lambda_0)$. Thus we have a group morphism $f = f_0 \times \mathrm{Id} : \mathbb{Z}^r = \mathbb{Z}^s \times \mathbb{Z}^{r-s} \to \mathbb{Z} \times \mathbb{Z}^{r-s}$., whence a ring morphism

$$f = \mathbb{Z}[\mathbb{Z}^r]_\xi \to \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}^{r-s}]_{\overline{\xi}}.$$

The hypothesis implies that $f(\lambda)$ is again a residual unit. It suffices to prove that its $\overline{\xi}$-minimal part $f_0(\lambda_0)$ is a (trivial) unit in $\mathbb{Z}[\mathbb{Z}]$. This is not quite obvious since it might be a zero divisor.

Write $f(\lambda_0) = \sum_{i=0}^{d} a_i t^i = P(t)$. For every $n \in \mathbb{N}^*$, there is a natural ring morphism

$$\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}^{r-s}]_\xi \to \mathbb{Z}[\zeta_n][\mathbb{Z}^{r-s}]_{\overline{\xi}}.$$

The image of $f(\lambda)$ is a unit in $\mathbb{Z}[\zeta_n][\mathbb{Z}^{r-s}]_{\overline{\xi}}$. For n large enough, $P(\zeta_n) \neq 0$, thus it is the $\overline{\xi}$-minimal part of this image. Thus $P(\zeta_n)$ is a unit in $\mathbb{Z}[\zeta_n]$ for every n large enough. The proof of Proposition 2 will be finished thanks to the following Lemma, which is a little stronger than the fact that $\mathbb{Z}[\mathbb{Z}]$ has finitely detectable units

Lemma *Let $P(t) \in \mathbb{Z}[t, t^{-1}]$ be such that for every n large enough and not divisible by a given finite set F of prime numbers, $P(\zeta)$ is a unit in $\mathbb{Z}[\zeta]$ for every n-th root of unity (primitive or not). Then $P(t) = \pm t^m$.*

Proof of the Lemma. Multiplying by $\pm t^m$, we can assume that $\lambda = P(t) = a_0 + \cdots + a_d t^d \in \mathbb{Z}[t]$, with $a_0 \neq 0$ and $a_d > 0$. We want to show that $a_d = 1$ and $d = 0$.

Let $n \in \mathbb{N}^*$ large enough, not divisible by any element of F. Then if $\zeta^n = 1$, $P(\zeta)$ is a unit in $\mathbb{Z}[\zeta]$, thus an algebraic unit (an algebraic integer whose inverse is an algebraic integer). Thus $\prod_{\zeta^n=1} P(\zeta)$ is an algebraic unit belonging to $\mathbb{Q}$, ie ± 1. On the other hand, the computation in two ways of the resultant $\mathrm{Res}(P(t), t^n - 1)$ gives

$$\prod_{\zeta^n=1} P(\zeta) = \pm a_d^n \prod_{P(\alpha)=0} (\alpha^n - 1),$$

thus

$$\prod_{P(\alpha)=0} |\alpha^n - 1| = a_d^{-n}.$$

Denote $P_-(n)$, $P_+(n)$ and $P_0(n)$ the products of $|\alpha^n - 1|$ over the zeros of P satisfying $|\alpha| < 1$, $\alpha > 1$ and $\alpha = 1$. We thus have

$$P_-(n)P_+(n)P_0(n) = a_d^{-n}.$$

Now let us make $n \to +\infty$. Then $\lim P_-(n) = 1$, thus

$$\lim_{n \to +\infty} P_0(n)P_+(n) = \begin{cases} 1 \text{ if } a_d = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Moreover, $\lim P_+(n)$ always exists, and is $+\infty$ if P has a zero of modulus > 1, 1 if has no such zero. Thus $P_0(n)$ has a limit, equal to 0 or 1.

Let us show by contradiction that P has no zero of modulus 1. Let $g = (\alpha_1, \cdots, \alpha_r) \in (S^1)^r$, where $r > 0$ and $\alpha_1, \cdots, \alpha_r$ are all such zeros. We have $\alpha_i \neq 1$ since $P(1)$ is a unit in $\mathbb{Z}$.

By compactness of the group $(S^1)^r$, the ζ-limit set of $(g^{mp_1 \cdots p_k + 1})$ contains g, thus

$$\liminf_{n \to +\infty, n \wedge F = 1} \max_{1 \leq i \leq r} |\alpha_i^n - \alpha_i| = 0.$$

This implies

$$\limsup_{n \to +\infty, n \wedge F = 1} P_0(n) \geq \prod_{i=1}^{r} |\alpha_i - 1| > 0.$$

Since $\lim P_0(n) P_+(n)$ is finite, P has no zero of modulus > 1. Thus $\lim P_0(n) = 1$, and $a_d = 1$. Thus the product of the zeros of P is $\pm a_0$. Since their moduli are ≤ 1, every zero is of modulus 1. By [Kronecker 1857], $P(t)$ is a product of cyclotomic polynomials $\Phi_{n_j}(t)$. Since $P(1)$ is a unit in $\mathbb{Z}$, every $n_j = 1$ thus $P(t) = 1$, qed.

3 Induction from a subgroup of finite index

Proposition. *Let $\Gamma \subset G$ be a finite index subgroup.*

1) Assume that $\mathrm{M}_n(\mathbb{Z}[\Gamma])$ has finitely detectable units for every $n \in \mathbb{N}^*$. Then $\mathrm{M}_n(\mathbb{Z}[G])$ has finitely detectable units for every $n \in \mathbb{N}^*$.

2) Assume that $\mathrm{M}_n(\mathbb{Z}[\Gamma]_{\xi|\Gamma})$ has finitely detectable units for every $n \in \mathbb{N}^*$. Then $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for every $n \in \mathbb{N}^*$.

Proof. 1) Denote $\mathrm{card}(G/\Gamma) = m$. Using a section $\sigma : G/\Gamma \to G$, we have an isomorphism of $\mathbb{Z}[\Gamma]$-modules

$$\widetilde{\sigma} : (\mathbb{Z}[\Gamma])^m \approx \mathbb{Z}[G] \ , \ (a_1, \cdots, a_m) \mapsto \sum_{q \in G/\Gamma} a_q \sigma(q).$$

Thus $\mathbb{Z}[G]$ is isomorphic to a subring of $\mathrm{M}_m(\mathbb{Z}[\Gamma])$, and more generally, $\mathrm{M}_n(\mathbb{Z}[G])$ is isomorphic to a subring of $\mathrm{M}_{nm}(\mathbb{Z}[\Gamma])$.

Let $A \in \mathrm{M}_n(\mathbb{Z}[G])$ be a residual unit. The associated matrix $\widetilde{A} \in R_n$ is a residual unit in $\mathrm{M}_{nm}(\mathbb{Z}[\Gamma])$. By the hypothesis it has an inverse $B \in \mathrm{M}_{nm}(\mathbb{Z}[\Gamma])$, and the left multiplication by $\widetilde{A}$ on $\mathbb{Z}[\Gamma]^{nm}$ is bijective. Thus the left multiplication by A on $\mathbb{Z}[G]^n$ is bijective, thus A is a unit in $\mathrm{M}_n(\mathbb{Z}[G])$, qed.

2) The proof is essentially the same: $\mathrm{M}_n((\mathbb{Z}[G]_\xi)$ is isomorphic to a subring of $\mathrm{M}_{nm}(\mathbb{Z}[\Gamma]_{\xi|\Gamma})$, and if $A \in \mathrm{M}_n((\mathbb{Z}[G]_\xi)$ is a residual unit, then $\widetilde{A} \in \mathrm{M}_{nm}(\mathbb{Z}[\Gamma]_{\xi|\Gamma})$ is a unit, thus the left multiplication by $\widetilde{A}$ on $\mathbb{Z}[\Gamma]^{nm}_{\xi|\Gamma}$ is bijective, thus the left multiplication by A on $\mathbb{Z}[\Gamma]^{nm}_{\xi|/\Gamma}$ is bijective.

4 The case of torsion-free nilpotent groups (incomplet)

Proposition. *Let G be a torsion-free nilpotent group. Then $\mathbb{Z}[G]$ has finitely detectable units.*

Proof. We can assume that G is finitely generated. Let C be the center of G. By induction on the nilpotency class of G, we can assume that $\mathbb{Z}[G/C]$ has finitely detectable units. Fix a section $\sigma : G/C \to G$ and bi-invariant order on C and G/C. Since C is central, the lexicographic order on $G \approx C \times (G/C)$ is bivariant, and $\sigma : G/C \to G$ is increasing.

Let $\lambda \in \mathbb{Z}[G]$ be a residual unit, we want to prove that it is a unit. We can write

$$\lambda = \sum_{i=0}^{d} \lambda_i \sigma(\gamma_i) \ , \ \lambda_i \in \mathbb{Z}[C] \ , \ \lambda_0, \lambda_d \neq 0 \ , \ \gamma_i \in G/C \ , \ \ \gamma_0 < \gamma_1 \cdots < \gamma_k.$$

Let us prove that λ_0 is a unit in $\mathbb{Z}[C]$, ie $\lambda_0 \in \pm C$.

There exists a surjective morphism $u : C \to \mathbb{Z} = \langle t \rangle$ which is injective on $\operatorname{supp}(\lambda_0)$. It suffices to prove that $u(\lambda_0) = \pm t^k$.

We have an associated central extension

$$\langle t \rangle \to G/\ker u \to G/C$$

with the induced section $\overline{\sigma} : G/C \to G/\ker u$. Then $u(\lambda)$ is a residual unit in $\mathbb{Z}[G/\ker u]$, and we have a decomposition

$$u(\lambda) = \sum_{i=0}^{d} u(\lambda_i)\overline{\sigma}(\gamma_i).$$

Moreover, there exists a normal subgroup $H \lhd G$ with finite index such that $H \cap C \subset \ker u$. Thus $H/H \cap C$ is a subgroup of $G/\ker u$, of index $\leq [G : H] < \infty$. It is also a subgroup of G/C, thus by the hypothesis and the previous section **insuffisant car on n'a pas l'hypothèse matricielle**, $u(\lambda)$ is a unit in $\mathbb{Z}[G/\ker u]$.

For $n \in \mathbb{N}^*$, let π_n be the natural morphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. We have an associated central extension

$$\mathbb{Z}/n\mathbb{Z} \to G/\ker(\pi_n \circ u) \to G/C$$

with the induced section $(\overline{\sigma})_n : G/C \to G/\ker(\pi_n \circ u)$. Then $(\pi_n \circ u)(\lambda)$ is a unit in $\mathbb{Z}[G/\ker(\pi_n \circ u)]$, and we have a decomposition

$$(\pi_n \circ u)(\lambda) = \sum_{i=0}^{d} (\pi_n \circ u)(\lambda_i)(\overline{\sigma})_n(\gamma_i).$$

Since $(\pi_n \circ u)(\lambda)$ is a unit and $\gamma_0 < \cdots < \gamma_d$, $(\pi_n \circ u)(\lambda_0) = \pi_n(u(\lambda_0))$ is either a unit or a zero divisor in $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ **c'est ici que ça se casse la gueule pour les matrices**. The second possibility occurs only if n is divisible by some prime in a finite set. By the Proposition in section 2, $u(\lambda_0)$ is a unit in $\mathbb{Z}[\mathbb{Z}]$, thus λ_0 is a unit in $\mathbb{Z}[C]$, ie an element of $\pm C$.

The same argument for the reverse order proves that λ_d is a unit in $\mathbb{Z}[C]$. Thus the image of λ in $\mathbb{Z}[G/C]$ is $(\overline{\lambda})_1 = \sum_{0 \leq i \leq d} \overline{\lambda}_i \gamma_i$ with $\overline{\lambda}_i$ and $\overline{\lambda}_0, \overline{\lambda}_d \neq 0$. Since $\mathbb{Z}[G/C]$ has finitely detectable units, $(\overline{\lambda})_1$ a unit, thus a monomial, thus $d = 0$. Thus $\lambda = \lambda_0$ is a (trivial) unit in $\mathbb{Z}[G]$, qed.

Corollary. *If G is a group which is residually free nilpotent, $\mathbb{Z}[G]$ has finitely detectable units.*

Proof. By hypothesis, there exists a decreasing sequence (H_n) of normal subgroups such that G/H_n is torsion-free nilpotent and $\bigcap G/H_n = \{1\}$. Let $\lambda \in \mathbb{Z}[G]$ be a residual unit. Then each image $(\overline{\lambda})_n \in \mathbb{Z}[G/H_n]$ is a residual unit, thus a unit, ie an element of $\pm(G/H_n)$. This implies $\lambda \in \pm G$, qed.

5 Proof of the theorem

It is now immediate: if G is a residually torsion-free nilpotent group and $\lambda \in \mathbb{Z}[G]$ a residual unit, there exists a normal subgroup $H \lhd G$ such that G/H is torsion-free nilpotent and $\operatorname{supp}(\lambda)$ injects in G/H. The image $\overline{\lambda}$ of λ in $\mathbb{Z}[G/H]$ is a residual unit. By the previous section, $\overline{\lambda} \in \pm G/H$. SInce $\operatorname{supp}(\lambda)$ injects in G/H, $\lambda \in \pm G$, qed.

6 Finite detection of full left ideals

Definitions. Let G be a group. A left ideal $I \subset \mathbb{Z}[G]$ is *residually full* if its image in every quotient $\mathbb{Z}[G/H]$, $H \subset G$ normal subgroup of finite index, is equal to $\mathbb{Z}[G/H]$. Similarly, if ξ is a nonzero morphism from G to $\mathbb{R}$, a left ideal of $\mathbb{Z}[G]_\xi$ is residually full if its image is full in every quotient $\mathbb{Z}[G/H \cap \ker \xi]_{\overline{\xi}}$, $H \subset G$ normal subgroup of finite index.

Similarly, a left submodule $M \subset \mathbb{Z}[G]^n$ is residually full if its image in every quotient $\mathbb{Z}[G/H]^n$, $H \subset G$ normal subgroup of finite index, is equal to $\mathbb{Z}[G/H]^n$. An a left submodule $M \subset \mathbb{Z}[G]_\xi$ is residually full if its image is full in every quotient $\mathbb{Z}[G/H \cap \ker \xi]_{\overline{\xi}}^n$, $H \subset G$ normal subgroup of finite index.

The group ring $\mathbb{Z}[G]$ (resp. the Novikov ring $\mathbb{Z}[G]_\xi$) has *finitely detectable full left ideals* if every left ideal $I \subset \mathbb{Z}[G]$ (resp. $I \subset \mathbb{Z}[G]_\xi$) which is residually full is equal to $\mathbb{Z}[G]$. (resp. to $\mathbb{Z}[G]_\xi$).

Since an element of $\mathbb{Z}[G]$ or $\mathbb{Z}[G]_\xi$ is a unit if an only if it is left invertible, the finite detectability of invertibles is equivalent to the finite detectability of principal ideals.

Also, the finite detectability of full left ideals implies the finite detectability of full submodules of $\mathbb{Z}[G]^n$ and of the invertibility of matrices:

Proposition. *Assume that $\mathbb{Z}[G]$ (resp. $\mathbb{Z}[G]_\xi$) has finitely detectable full left ideals.*

(i) Let M be a residually full left submodule of $\mathbb{Z}[G]^n$ (resp. $\mathbb{Z}[G]_\xi^n$). Then $M = (\mathbb{Z}[G])^n$ (resp. $\mathbb{Z}[G]_\xi^n$).

(ii) The ring $\mathrm{M}_n(\mathbb{Z}[G])$ (resp. $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for every n.

Proof. It suffices to treat the case of $\mathbb{Z}[G]$, the case of $\mathbb{Z}[G]_\xi$ being completely analogous.

(i) By induction on n, the result being the hypothesis if $n = 1$. Assume that $n > 1$ and the result is true for $n - 1$. Consider the set I of $\lambda \in \mathbb{Z}[G]$ such that there exists $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{Z}[G]$ with $(\lambda_1, \cdots, \lambda_{n-1}, \lambda) \in M$. It is a left ideal, and its image is full in every quotient $\mathbb{Z}([G/H])$ with G/H finite. Thus $I = \mathbb{Z}[G]$, ie M contains an element $x = (\lambda_1, \cdots, \lambda_{n-1}, 1)$.

One has a direct sum decomposition $\mathbb{Z}[G]^n = \mathbb{Z}[G]^{n-1} \oplus \mathbb{Z}[G]x$. Substracting $\mu_n x$ from every element $(\mu_1, \cdots, \mu_n) \in M$, ones sees that $M = (M \cap (\mathbb{Z}[G]^{n-1}) \oplus \mathbb{Z}[G]x$. It suffices to prove that $M \cap \mathbb{Z}[G]^{n-1} = \mathbb{Z}[G]^{n-1}$. Clearly, $M \cap \mathbb{Z}[G])^{n-1}$ is a left submodule which has a full image in every quotient $(\mathbb{Z}[G/H])^{n-1}$, $H \subset G$ normal subgroup of finite index. By the induction hypothesis, $M \cap \mathbb{Z}[G]^{n-1} = \mathbb{Z}[G]^{n-1}$, qed.

(ii) Let $A \in \mathrm{M}_n(\mathbb{Z}[G])$ be a residual unit. The right multiplication by A is a left-linear map u from $\mathbb{Z}[G]^n$ to itself. By hypothesis, $\mathrm{im}(u)$ has full image in every quotient $(\mathbb{Z}[G/H])^n$, $H \subset G$ normal subgroup of finite index. By (i), $\mathrm{im}(u) = \mathbb{Z}[G]^n$, ie A is left invertible, thus invertible, qed.

7 Residually full left ideals and finite quotients

Proposition. *Let G be any group. Assume that $I \subset \mathbb{Z}[G]$ is a residually full left ideal.*

1) If $\mathbb{Z}[G]/I$ is finite, then $I = \mathbb{Z}[G]$.

2) The $\mathbb{Z}[G]$-module $\mathbb{Z}[G]/I$ has no nontrivial finite quotient.

Proof. 1) The subgroup

$$H := \{g \in G \mid \overline{g} = 1 \text{ in } \mathbb{Z}[G]/I\}$$

has finite index in G. Since I is residually full, its image in $\mathbb{Z}[G/H]$ is full. By definition of H, we have a factorization

$$\begin{tikzcd}
\mathbb{Z}[G] \arrow[r] \arrow[d] & \mathbb{Z}[G]/I \\
\mathbb{Z}[G/H]. \arrow[ur] &
\end{tikzcd}$$

Thus I surjects onto $\mathbb{Z}[G]/I$, which means that $I = \mathbb{Z}[G]$. $\square$

2) If M is a finite quotient of $\mathbb{Z}[G]$, $A = \mathbb{Z}[G]/I_1$ where $I_1 \supset I$. Then I_1 is residually full, thus $I_1 = \mathbb{Z}[G]$ by 1), $A = 0$.

Remark. Considering $(\mathbb{Z}[G]/I) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, one can replace "finite" by "finitely generated as a $\mathbb{Z}$-module".

8 From a finite index subgroup to the group

Proposition. *If $\Gamma < G$ has finite index and $\mathbb{Z}[\Gamma]$ has finitely detectable full left ideals, so has $\mathbb{Z}[G]$.*

We can assume that Γ is normal. Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal. As in section 4, $\mathbb{Z}[G]$ can be identified to $\mathbb{Z}[\Gamma]^m$, $m = [G : H]$, so that $\mathbb{Z}[G]$ embeds in $\mathrm{M}_m(\mathbb{Z}[\Gamma])$. Thus I is identified with a submodule $M \subset \mathbb{Z}[\Gamma]^m$.

Let $H < \Gamma$ be a normal subgroup of finite index. Then it contains H_1 which is normal in G and of finite index, and the isomorphism of $\mathbb{Z}[\Gamma]$-modules $\mathbb{Z}[G] \to \mathbb{Z}[\Gamma]^m$ induces an isomorphism of $\mathbb{Z}[\Gamma/H_1]$-modules $\mathbb{Z}[G/H_1] \to \mathbb{Z}[\Gamma/H_1]^m$. By hypothesis, the image of I in $\mathbb{Z}[G + H_1]$ is full, thus the image of M in $\mathbb{Z}[\Gamma/H_1]^m$ is full, and a fortiori the image in $\mathbb{Z}[\Gamma/H]^m$ is full. Since $\mathbb{Z}[\Gamma]$ has finitely detectable full left ideals, $M = \mathbb{Z}[\Gamma]^m$, thus $I = \mathbb{Z}[G]$, which proves the Proposition.

9 Finite detectability of full left ideals of $\mathbb{Z}[G]$ for G nilpotent

Proposition. *Let G be a nilpotent group. Then $\mathbb{Z}[G]$ has finitely detectable full left ideals. In particular, it has finitely detectable units.*

Proof. We can assume that G is finitely generated. Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal. By contradiction, assume that $M = \mathbb{Z}[G]/I$ is nonzero.

By Remark 2) in 1.3, M has no nontrivial finite quotient. Since $I \neq \mathbb{Z}[G]$, there exists a maximal left ideal $I_1 \supset I$ (without the axiom of choice, since $\mathbb{Z}[G]$ is Noetherian).

Thus $M_1 = \mathbb{Z}[G]/I_1$ is simple (nonzero and with no nontrivial submodule). By [Philip Hall 1959] (quoted in [Passman 1976, p.544]), M_1 is finite, a contradiction since M_1 is a quotient of M.

10 Finite detectability of full left ideals of $\mathbb{Z}[G]$ for G residually nilpotent

Proposition. *Let G be a residually nilpotent group. Then $\mathbb{Z}[G]$ has finitely detectable full left ideals.*

Proof. We can assume that G is finitely generated. Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal.

1) First case. Assume that G is residually torsion-free nilpotent and that I is principal, $I = \mathbb{Z}[G]\lambda$. Then λ is a residual unit, and there exists a normal subgroup $H \triangleleft G$ such that G/H is torsion-free nilpotent and $\mathrm{supp}(\lambda)$ injects in G/H.

The image $\overline{\lambda}$ of λ in $\mathbb{Z}[G/H]$ is a residual unit. By the previous section, $\overline{\lambda}$ is a unit. Since G/H is torsion-free nilpotent thus left orderable, $\overline{\lambda} \in \pm G/H$. Finally, since $\mathrm{supp}(\lambda)$ injects in G/H, $\lambda \in \pm G$, qed.

2) Second case. Assume that G is residually torsion-free nilpotent and that I is finitely generated, $I = \langle \lambda_1, \cdots, \lambda_n \rangle$. There exists a normal subgroup $H \lhd G$ such that G/H is torsion-free nilpotent and $\mathrm{supp}(\lambda_1) \cup \cdots \cup \mathrm{supp}(\lambda_n)$ injects in G/H.

There exists a bilateral order on G such that the projection $G \to G/H$ is non-decreasing. Let $a \in \mathbb{N}^*$ be the gcd of the minimal parts of the elements of I. We can assume that the minimal parts of $\lambda_1, \cdots, \lambda_n$ are all equal to a.

Question. Do we have $a = 1$? Let $I_1 = I + a\mathbb{Z}[G]$.

The image $\overline{I}$ of λ in $\mathbb{Z}[G/H]$ is residually full. By the previous section, $\overline{I} = \mathbb{Z}[G/H]$. Since G/H is torsion-free nilpotent thus left orderable, $\overline{\lambda} \in \pm G/H$. Finally, since $\mathrm{supp}(\lambda)$ injects in G/H, $\lambda \in \pm G$, qed.