Well-quasi-ordering H-contraction-free graphs*

Marcin Kamiński[a], Jean-Florent Raymond[a,b], and Théophile Trunck[c]

[a] *Institute of Computer Science, University of Warsaw, Poland.*
[b] *LIRMM, University of Montpellier, France.*
[c] *LIP, ÉNS de Lyon, France.*

Abstract

A *well-quasi-order* is an order which contains no infinite decreasing sequence and no infinite collection of incomparable elements. In this paper, we consider graph classes defined by excluding one graph as contraction. More precisely, we give a complete characterization of graphs H such that the class of H-contraction-free graphs is well-quasi-ordered by the contraction relation. This result is the contraction analogue of the previous dichotomy theorems of Damsaschke [*Induced subgraphs and well-quasi-ordering*, Journal of Graph Theory, 14(4):427–435, 1990] on the induced subgraph relation, Ding [*Subgraphs and well-quasi-ordering*, Journal of Graph Theory, 16(5):489–502, 1992] on the subgraph relation, and Błasiok et al. [*Induced minors and well-quasi-ordering*, ArXiv e-prints, 1510.07135, 2015] on the induced minor relation.

1 Introduction

A *well-quasi-order* is a quasi-order where every decreasing sequence and every collection of incomparable elements (called an *antichain*) are finite. Well-quasi-orders enjoy nice combinatorial properties that can be used in several contexts, from algebra to computational complexity and algorithms. Since its introduction more than sixty years ago, the theory of well-quasi-orders led to major results in Graph Theory and Combinatorics. In particular, Kruskal showed in [9] that trees are well-quasi-ordered by homeomorphic embedding, and Robertson and

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Seymour proved that both the minor relation and the immersion relation are well-quasi-orders on the class of finite graphs [11, 12]. Most of the usual quasi-orders on graphs are not well-quasi-orders in general, though. Given one of these quasi-orders, a natural line of research is to identify the subclasses that are well-quasi-ordered. Our work is motivated by the following results.

Theorem 1 ([2])**.** *The class of H-induced subgraph-free graphs is well-quasi-ordered by induced subgraphs iff H is an induced subgraph of P_4.*[1]

Theorem 2 ([3])**.** *The class of H-subgraph-free graphs is well-quasi-ordered by subgraphs iff H is a subgraph of P_n, for some $n \in \mathbb{N}$.*[1]

Theorem 3 ([7])**.** *The class of H-topological minor-free multigraphs is well-quasi-ordered by topological minors iff H is a topological minor of R_n, for some $n \in \mathbb{N}$.*[2]

Theorem 4 ([1])**.** *The class of H-induced minor-free graphs is well-quasi-ordered by induced minors iff H is an induced minor of the* gem *or $\hat{K}_4$.*[3]

These results characterize the closed classes defined by one forbidden substructure that are well-quasi-orders. Like the four containment relations on graphs mentioned in the above results, the contraction relation is not a well-quasi-order in general. Let the diamond be the graph obtained from K_4 by deleting an edge. Our main contribution in this direction is the following result.

Theorem 5. *The class of connected H-contraction-free graphs is well-quasi-ordered by contractions iff H is a contraction of the diamond.*

The requirement of connectivity in Theorem 5 is necessary in the sense that for every graph H, the class of (not necessarily connected) H-contraction-free graphs contains the infinite antichain $\{\overline{K_i},\ i \in \mathbb{N}_{\geq h+1}\}$ (where $h = |V(H)|$) and therefore is not a well-quasi-order. Theorem 5 can be seen as contraction counterpart of the results mentioned above.

Another line of research when dealing with quasi-orders that are not well-quasi-orders in general is to look at canonical antichains. An antichain is *canonical* if for every closed subset F of the quasi-order, F is a wqo iff F has a finite intersection with this antichain. Intuitively, a canonical antichain represents all infinite antichains of a quasi-order. As shown by the results below, the question of the presence or absence of a canonical antichain has been studied for several containment relations and graph classes.

Theorem 6 ([4])**.** *Under the subgraph relation, the class of finite graphs has a canonical antichain.*

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[1] P_n is the path on n vertices, for every $n \in \mathbb{N}$.

[2] R_n is the multigraph obtained by doubling every edge of a path on n edges, for every $n \in \mathbb{N}$.

[3] The gem is the graph obtained by adding a dominating vertex to P_4 and $\hat{K}_4$ is the graph obtained by adding a vertex of degree 2 to K_4.

Theorem 7 ([4]). *Under the induced subgraph relation, the class of finite graphs does not have a canonical antichain.*

Theorem 8 ([4]). *Under the induced subgraph relation, both the class of interval graphs and the class of bipartite permutation graphs have a canonical antichain.*

Theorem 9 ([8]). *Under the multigraph contraction relation, the class of finite (loopless) multigraphs has a canonical antichain.*

We give an answer to this question for the containment relation with the following result.

Theorem 10. *Under the contraction relation, the class of finite graphs does not have a canonical antichain.*

The proof of Theorem 10 relies on the tools introduced in [4] that can be used to prove that a quasi-order does not have a canonical antichain.

Organization of the paper. The proof of Theorem 5 contains three parts. The first one, given in Section 3, is a study of infinite antichains of the contraction relation from which we can deduce that if the class of H-contraction-free graphs is well-quasi-ordered by contractions, then H is a contraction of the diamond. Section 4 contains the second part which is a decomposition theorem for diamond-contraction-free graphs. The last part uses this decomposition to show the well-quasi-ordering result and is presented in Section 5. The proof of Theorem 10 is given in Section 6. Definitions of the terms and notations used are introduced in Section 2.

2 Preliminaries

We use the notation $\mathbb{N}_{\geq k}$ for the set $\{i \in \mathbb{N},\ i \geq k\}$, for every $k \in \mathbb{N}$. For every set S, we denote by $\mathcal{P}(S)$ the collection of subsets of S.

Graphs. All graphs in this paper are finite, simple, and undirected. We denote by $V(G)$ the vertex set of a graph G and by $E(G)$ its edge set. If $X \subseteq V(G)$, the *subgraph of G induced by X*, which we write $G[X]$, is the graph with vertex set X and edge set $E(G) \cap X^2$. Let C be a (not necessarily induced) cycle in a graph G. A pair of vertices $\{u, v\} \subseteq V(C)$ that are not adjacent in C is a *chord of C in G* if $\{u, v\} \in E(G)$. Otherwise $\{u, v\}$ is a *non-chord of C in G*.

A vertex v of a graph G is a *cutvertex* if $G \setminus \{v\}$ has more connected components than G. A *block* is a maximal subgraph that has no cutvertex. A *clique-cactus graph* is a graph whose blocks are cycles and cliques (cf. Figure 1 for an example).

If G is a graph, then $\overline{G}$ is the graph obtained by replacing all non-edges by edges and vice versa. For every positive integer r we denote by D_r the graph

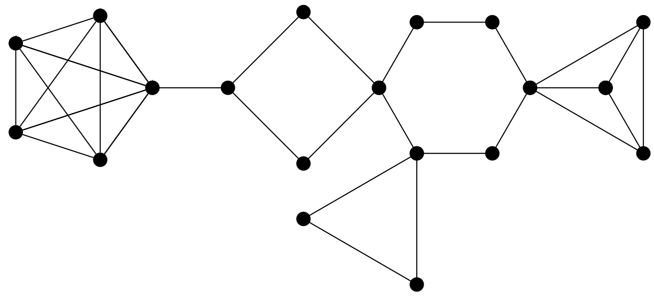

Figure 1: A clique-cactus graph.

$\overline{2 \cdot K_1 \cup \cdot K_r}$. In particular, D_2 is the diamond. We set $\mathcal{D} = \{D_r,\ r \in \mathbb{N}\}$ (cf. Figure 2) and $\mathcal{S} = \{K_{1,r},\ r \in \mathbb{N}\}$.

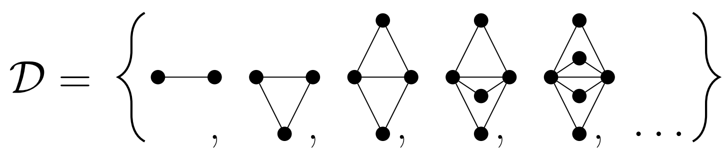

Figure 2: Graphs of $\mathcal{D}$.

Subsets of vertices. If G is a graph, the *degree* of a subset $S \subseteq V(G)$ is the number of vertices of $V(G) \setminus S$ that have a neighbor in S. The subset S is said to be *connected* if $G[S]$ is connected. We say that S is *adjacent* to some vertex v (respectively some subset $S' \subseteq V(G)$) if there is an edge from v to a vertex of S (respectively from a vertex of S to a vertex of S').

Contractions. In a graph G, a *contraction of the edge* $\{u, v\} \in E(G)$ is the operation which adds a new vertex adjacent to the neighbors of u and v and then deletes u and v. We say that a graph H is a *contraction* of a graph G whenever H can be obtained from G by a sequence of edge contractions, what we denote by $H \leq_{\mathrm{ctr}} G$.

A *contraction model* of a graph H in a graph G is function $\varphi\colon V(H) \to \mathcal{P}(V(G))$ such that:

(i) for every $v \in V(H)$, $\varphi(v)$ is connected;

(ii) $\{\varphi(v),\ v \in V(H)\}$ is a partition of $V(G)$;

(iii) for every $u, v \in V(H)$, the vertices u and v are adjacent in H iff the subsets $\varphi(u)$ and $\varphi(v)$ are adjacent in G.

This definition has several consequences. In particular, the degree of $v \in V(H)$ in H is at most the degree of $\varphi(v)$ in G. Also, there is no model of a graph with no dominating vertex in a graph with a dominating vertex.

It is easy to check that H is a contraction of G iff there is a contraction model of H in G. A graph G is said to *exclude a graph H as contraction*, or to be *H-contraction-free*, if H is not a contraction of G. We denote the class of connected H-contraction-free graphs by $\mathrm{Excl}(H)$.

We say that a graph H is an *induced minor* of a graph G if it can be obtained from G by deleting vertices and contracting edges.

Sequences and orders. We write $\langle s_1, \dots, s_n \rangle$ the sequence containing the elements $s_1, \dots, s_n$ in this order. For every set S, we denote by $S^\star$ the set of all finite sequences over S, including the empty sequence. For any partial order $(A, \preceq)$, we define the relation $\preceq^\star$ on $A^\star$ as follows: for every $r = \langle r_1, \dots, r_p \rangle$ and $s = \langle s_1, \dots, s_q \rangle$ of $A^\star$, we have $r \preceq^\star s$ if there is an increasing function $\varphi\colon \{1, \dots, p\} \to \{1, \dots, q\}$ such that for every $i \in \{1, \dots, p\}$ we have $r_i \preceq s_{\varphi(i)}$. This generalizes the subsequence relation.

Well-quasi-orders and antichains. Given an order $\preceq$ over S, a sequence over S is said to be an antichain of $(S, \preceq)$ if its elements are pairwise incomparable with respect to $\preceq$. A *well-quasi-order* (wqo for short) is a quasi-order where every decreasing sequence and every antichain is finite.

We will use the two classical results stated below.

Proposition 1 (Folklore)**.** *Let $(A, \preceq)$ be a quasi-order and $B, C \subseteq A$. If both $(B, \preceq)$ and $(C, \preceq)$ are wqo, then so is $(B \cup C, \preceq)$.*

Proposition 2 (Higman's Lemma [6])**.** *$(A, \preceq_A)$ is a wqo, then so is $(A^\star, \preceq_A^\star)$.*

If $(S, \preceq)$ is a quasi-order that is not a wqo, a *minimal antichain* [10] of $(S, \preceq)$ is an antichain $\langle a_i \rangle_{i \in \mathbb{N}}$ where for every $i \in \mathbb{N}$, a_i is a minimal element (with respect to $\preceq$) such that there is an infinite antichain of $(S, \preceq)$ starting with $\langle a_j \rangle_{j \in \{0, \dots, i\}}$. Observe that every quasi-order that is not a wqo and that has no infinite decreasing sequence has a minimal antichain. For every subset $A \subseteq S$, we define:

$$\mathrm{Incl}(A) = \{x \in S,\ \exists y \in A,\ x \preceq y \text{ and } x \neq y\}.$$

An antichain A is *fundamental* if $(\mathrm{Incl}(A), \preceq)$ is a wqo. A set $F \subseteq S$ is said to be $\preceq$-closed if it satisfies the following property: $\forall x \in F, \forall y \in S,\ y \preceq x \Rightarrow y \in F$.

An antichain A of a quasi-order $(S, \preceq)$ is *canonical* if for every $\preceq$-closed subset $F \subseteq S$, we have

$$F \cap A \text{ is finite} \iff (F, \preceq) \text{ is a wqo}.$$

Let us end this section by a simple observation.

Observation 1. Every sequence of graphs that is decreasing with respect to contraction is finite. In fact, in such a sequence the number of edges is also decreasing, as every edge contraction decreases the number of edges of a graph by at least one.

A consequence of the above observation is that infinite antichains are the only obstacles for a class of graphs to be well-quasi-ordered by the contraction relation. We deal with them in the next section.

3 Infinite antichains

A simple but crucial observation in the study of the well-quasi-orderability of classes that are defined by forbidden structures (of any kind) is the following. If none of the graphs of an infinite antichain $\mathcal{A}$ contains some graph H, then excluding H does not give a well-quasi-order. Indeed, the class obtained still contains the infinite antichain $\mathcal{A}$. Let us restate this observation in terms of contractions.

Observation 2. Let $\mathcal{A}$ be an infinite antichain of the contraction relation. If $(\mathrm{Excl}(H), \leq_{\mathrm{ctr}})$ is a wqo, then all but finitely many graphs of $\mathcal{A}$ contain H as contraction.

For this reason, we deal here with infinite antichains of the contraction relation. The simplest one is certainly the class of complete bipartite graphs with one part of size two: $\mathcal{A}_K = \{K_{2,r},\ r \in \mathbb{N}_{\geq 2}\}$.

Lemma 1. *For every $p, q, p', q' \in \mathbb{N}_{\geq 2}$ such that $p \leq p'$ and $q < q'$, there is no model of $K_{p,q}$ in $K_{p',q'}$.*

Proof. Let us assume for a contradiction that there is a model φ of $K_{p,q}$ in $K_{p',q'}$. As $K_{p',q'}$ has more vertices than $K_{p,q}$, there is a vertex v of $K_{p,q}$ such that $|\varphi(v)| \geq 2$. Observe that every subset of at least two vertices of $K_{p,q}$ that induced a connected subgraph is dominating. Indeed, such a subset must contain at least a vertex from each part of the bipartition. It follows from the definition of a model that $K_{p',q'}$ has a dominating vertex, a contradiction. Therefore, there is no model of $K_{p,q}$ in $K_{p',q'}$. □

Corollary 1. *$\{K_{2,p},\ p \in \mathbb{N}_{\geq 2}\}$ is an antichain of $\leq_{\mathrm{ctr}}$.*

Recall that $\mathcal{S}$ is the class of stars and that $\mathcal{D} = \{D_r, r \in \mathbb{N}\}$, where D_r is the graph that can be obtained by contracting an edge of $K_{2,r+1}$, for every $r \in \mathbb{N}$ (cf. Section 2). The following will be useful later.

Observation 3. For every $p \in \mathbb{N}_{\geq 1}$, contracting one edge in D_p gives either D_{p-1}, or $K_{1,p}$, depending on which edge is contracted.

As we want to identify graphs H such that $(\mathrm{Excl}(H), \leq_{\mathrm{ctr}})$ is a wqo, we must consider every graph H such that $\mathcal{A} \cap \mathrm{Excl}(H)$ is finite, for every antichain $\mathcal{A}$. A first step towards this goal is the following observation.

Lemma 2. *Let $p \in \mathbb{N}_{\geq 2}$. If $H \leq_{\mathrm{ctr}} K_{2,p}$ and $H \neq K_{2,p}$, then $H \in \mathcal{D} \cup \mathcal{S}$.*

Proof. Given that $H \leq_{\text{ctr}} K_{2,p}$, there is a sequence of contractions transforming $K_{2,p}$ into H. If this sequence contains only one contraction, then it is straightforward that $H = D_{p-1}$. Therefore in the other cases H is a contraction of D_{p-1}. We get the result from Observation 3 and the observation that every contraction of a graph of $\mathcal{S}$ (i.e. the class of stars) belongs to $\mathcal{S}$. $\square$

Observation 4. For every positive integers p, q such that $p < q$, we have $D_p \leq_{\text{ctr}} K_{2,q}$.

Indeed, if F is a collection of $q - p$ edges of $K_{2,q}$ that all incident with the same vertex of degree p, then it is easy to check that contracting F in $K_{2,q}$ yields D_p. An immediate consequence of Observation 4 is that $\mathcal{A}_K \cap \operatorname{Excl}(D_p)$ is finite for every positive integer p.

From the fact that every graph of $\mathcal{D} \cup \mathcal{S}$ is a contraction of D_p for some positive integer p, Observation 4 gives.

Observation 5. If $(\operatorname{Excl}(H), \leq_{\text{ctr}})$ is a wqo, then $H \leq_{\text{ctr}} D_p$ for some $p \in \mathbb{N}_{\geq 1}$

However, we will need another antichain in order to find more properties that H must satisfy. Let us consider the antichain of antiholes, which already appeared in [1] in the context of induced minors: $\mathcal{A}_{\overline{C}} = \{\overline{C}_i,\ i \in \mathbb{N}_{\geq 6}\}$ (cf. Figure 3). This connection with the induced minor relation (where edge contractions and vertex deletions are allowed) is not surprising: as every contraction is an induced minor, every antichain of the induced minor relation is also an antichain of the contraction relation.

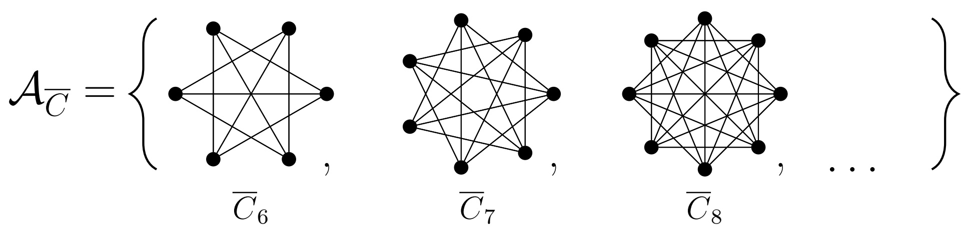

Figure 3: Antiholes antichain.

For completeness, we include the following proof.

Lemma 3 (See also [1, Lemma 1])**.** *$\mathcal{A}_{\overline{C}}$ is an antichain of the contraction relation.*

Proof. Towards a contradiction, let us assume that there is a contraction model φ of $\overline{C}_p$ in $\overline{C}_q$ for some integers $p, q \in \mathbb{N}_{\geq 3}$ such that $p < q$. Recall that the image of φ is a partition of $\overline{C}_q$. As $|\overline{C}_p| < |\overline{C}_q|$, there is a vertex v of $\overline{C}_p$ such that $|\varphi(v)| \geq 2$. Observe that for every choice of two vertices of $\overline{C}_p$ there is at most one vertex which is not adjacent to one of them. Therefore, there is at most one set in $\{\varphi(u),\ u \in V(\overline{C}_p) \setminus \{v\}\}$ that is not adjacent to $\varphi(v)$. This contradicts the fact that φ is a model of $\overline{C}_p$ in $\overline{C}_q$ as every vertex of $\overline{C}_q$ is adjacent to all but two vertices. Consequently there is no contraction model of $\overline{C}_p$ in $\overline{C}_q$, for every integers $p, q \in \mathbb{N}_{\geq 3}$, $p < q$. $\square$

Again, we look at graphs H such that $\mathrm{Excl}(H) \cap \mathcal{A}_{\overline{C}}$ is finite. As a wqo must contain none of $\mathcal{A}_K$ and $\mathcal{A}_{\overline{C}}$, it is enough to consider graphs such that $\mathrm{Excl}(H) \cap \mathcal{A}_{\overline{C}}$ is finite among those for which $\mathrm{Excl}(H) \cap \mathcal{A}_K$ is finite.

Lemma 4. *If $p \geq 3$ then $\mathrm{Excl}(D_p) \cap \mathcal{A}_{\overline{C}}$ is infinite.*

Proof. For every $p \geq 3$, then graph D_p has independence number at least 3. Let $q > p$. As contracting edges can only decrease the independence number, there is no sequence of contractions transforming $\overline{C}_q$ (which has independent number 2) to D_p, for every integer $q > p$. Therefore $\overline{C}_q \in \mathrm{Excl}(D_p)$, for every integer $q > p$. □

Corollary 2. *If $(\mathrm{Excl}(H), \leq_{\mathrm{ctr}})$ is a wqo, then $H \leq_{\mathrm{ctr}} D_2$.*

The next sections are devoted to graphs not containing D_2 as contraction. We will first prove a decomposition theorem for the graphs in this class, which we will use to show that $(\mathrm{Excl}(D_2), \leq_{\mathrm{ctr}})$ is a wqo.

4 On graphs with no diamond

In this section we show that graphs in $\mathrm{Excl}(D_2)$ have a simple structure. More precisely, we prove the following lemma. Recall that clique-cactus graphs are the graphs whose blocks are cycles or cliques.

Lemma 5. *Graphs of $\mathrm{Excl}(D_2)$ are exactly the connected clique-cactus graphs.*

The proof of Lemma 5 will be given after a few lemmas. If C is a cycle of a graph G and $\{u, v\}, \{u', v'\} \subseteq V(C)$, we say that $\{u, v\}$ and $\{u', v'\}$ are *crossing in C* if u, v, u', v' are distinct and are appearing in this order on the cycle.

Lemma 6. *Let G be a graph and let C be a cycle in G. If C has at least one chord and one non-chord in G, then it has one chord and one non-chord that are crossing in C.*

Proof. Let $\{x, x'\}$ be a non-chord of C in G and let P and Q be the connected components of $C \setminus \{x, x'\}$ which obviously are paths. Let us assume that every chord of C in G has both endpoints either in P or in Q (otherwise we are done) and let $\{y, y'\}$ be a chord of C in G, the endpoints of which belong, say, to P. Let z be a vertex of the subpath of P delimited by y and y' such that $z \notin \{y, y'\}$, and let z' be a vertex of Q. If $\{z, z'\}$ is a chord of C in G, then $\{x, x'\}$ and $\{z, z'\}$ are satisfying the required property. Otherwise, $\{z, z'\}$ is a non-chord and now $\{y, y'\}$ and $\{z, z'\}$ are crossing. □

Lemma 7. *Let $G \in \mathrm{Excl}(D_2)$. Every cycle of G is either an induced cycle, or it induces a clique in G.*

Proof. Let G be a graph of $\mathrm{Excl}(D_2)$ and let C be a cycle of G. Towards a contradiction, let us assume that C has at least one chord $\{u, u'\}$ and one non-chord $\{v, v'\}$. According to Lemma 6 we can assume without loss of generality that they are crossing in C. Let P and Q be the two connected components of $C \setminus \{v, v'\}$. Contracting P to a single vertex x and Q to y yields a graph G' such that:

- v, x, v', y lie on the cycle in this order;
- $\{v, v'\} \notin E(G')$; and
- $\{x, y\} \in E(G')$ (as $\{u, u'\}$ connects the subgraphs that are respectively contracted to x and y).

Notice that $G'[\{v, x, v', y\}]$ is isomorphic to D_2, however G' may also contains other vertices. Let us consider $G' \setminus \{v, x, v', y\}$. While $G' \setminus \{v, v', x, y\}$ contains a connected component adjacent to x or y, we contract it to this vertex (that we keep calling with the same name). Then, while it has a connected component adjacent to v but not v' (respectively v' but not v), we contract it to v (respectively v'), again keeping the same name for that vertex. Finally, the only remaining connected components (if any) are adjacent to exactly v and v': we contract each of them to a single vertex, adjacent to v and v'. Notice that none of these operations create an edge connecting v to v', thus the subset $\{v, v', x, y\}$ still induces a subgraph isomorphic to D_2. Let us call G'' the obtained graph. As observed above, G'' consists of the subgraph isomorphic to D_2 induced by $\{v, v', x, y\}$ plus k extra vertices of degree two, $z_1, \dots, z_k$, each of which is adjacent to v and v'. In the case where $k = 0$, G'' is isomorphic to D_2 and we reached the contradiction we were looking for. Otherwise, we contract $\{v', y\}$ (naming the resulting vertex v'), which produces a complete subgraph on vertices v, v', x, and then we contract $\{v, z_i\}$ for every $i \in \{2, \dots, k\}$ (naming the resulting vertex v). The obtained graph is a complete graph on v, v', x, two vertices of which (that are v, v') are adjacent to an extra vertex, v_1. This graph is isomorphic to D_2, as x is not adjacent to v_1, therefore we reached a contradiction. Therefore C has either no chords or no non-chords in G. It is clear that in the first case C is an induced cycle of G and that in the second case it induces a clique. $\square$

Lemma 8. *Let $G \in \mathrm{Excl}(D_2)$ be a 2-connected graph. Then G is either a cycle, or a clique.*

Proof. We assume that $|V(G)| > 1$, otherwise the result is trivial. Let C be a longest cycle of G. By Lemma 7 the cycle C is either an induced cycle, or it induces a clique in G. Let us treat these two cases separately. For contradiction we assume that $V(G) \setminus V(C)$ is not empty and we call $H_1, \dots, H_t$ the connected components of $G \setminus C$, for some $t \in \mathbb{N}_{\geq 1}$. Let us consider the graph G' where H_i, which is connected, has been contracted to a single vertex h_i, for every $i \in \{1, \dots, t\}$. Observe that G' is 2-connected, given that G is 2-connected. Also, $G' \in \mathrm{Excl}(D_2)$.

First case: C induces a clique in G'. Notice that C is then a maximal clique. Let $u = h_1$. As C is maximal, there is a vertex $v \in V(C)$ such that $\{u, v\} \notin E(G')$. Let x and y be two neighbors of u on C (they exist since G' is 2-connected). These vertices define two subpaths of C. Let R be the longest of these paths that contains v. Observe that in this case, R has at least three vertices. The union of $\{u, x\}$, $\{u, y\}$ and R is a cycle of G' that we call C'. According to Lemma 7, this cycle is either induced or it induces a clique. As $\{u, v\} \notin E(G')$, C' cannot induce a clique in G. On the other hand, C is not an induced cycle as every pair of vertices of R are adjacent (and $|V(R)| \geq 3$ as mentioned earlier). We reached the contradiction we were looking for.

Second case: C is an induced cycle and has at least 4 vertices. Let $i \in \{1, \dots, t\}$. As G' is 2-connected, h_i has at least two neighbors on C: let x and y be two of them.

Claim 1. x and y are not adjacent.

Proof. Let us assume that $\{x, y\} \in E(G')$. Let C' be the cycle obtained from C by replacing the edge $\{x, y\}$ by the path xh_iy. This cycle is not induced as x, y are not adjacent in C' whereas $\{x, y\} \in E(G)$. It does not induce a clique either since x is not adjacent with the other neighbor of y on C (which is not x as we assume that C has at least 4 vertices). This contradicts Lemma 7 and therefore proves that $\{x, y\} \notin E(G)$. □

Every pair of distinct vertices of the cycle C defines two subpaths of C meeting only at these vertices. Let u and v be two vertices of C such that h_i has at least one neighbor in the interior of each of the subpaths of C defined by u and v, that we will respectively call P and Q. Such vertices exist, as a consequence of Claim 1.

Let us consider the contraction H of G' obtained by contracting the interior path of P (respectively Q) to a single vertex w_P (respectively w_Q) and then by contracting the edge connecting h_i to w_P. This edge exists by definition of u and v. Then uw_Pvw_Q is a cycle of H where $\{w_P, w_Q\}$ is a chord (because we contracted to w_P the vertex h_1 which was adjacent to both w_P and w_Q) and $\{u, v\}$ is a non-chord (as they were non-adjacent vertices of the induced cycle C and that nothing has been contracted to them). According to Lemma 7, the graph H contains D_2 as contraction. As H is a contraction of G, then $D_2 \leq_{\text{ctr}} G$, a contradiction.

In both cases we reached a contradiction, therefore $V(G) \setminus V(C)$ is empty: G is a clique or an induced cycle. □

We are now ready to prove Lemma 5.

Proof of Lemma 5. The fact that a graph of $\text{Excl}(D_2)$ is clique-cactus is a straightforward corollary of Lemma 8. It is easy to see that a clique-cactus graph does not contain D_2 as contraction by noticing that D_2 is a contraction of a graph if and only if it is a contraction of one of its 2-connected components. As D_2 is neither a contraction of a cycle, nor of a clique, we get the desired result. □

5 Well-quasi-ordering clique-cactus graphs

We proved in the previous section that graphs of $\mathrm{Excl}(D_2)$ are exactly the connected clique-cactus graphs. This section contains the last part of the proof of Theorem 5, which is the following lemma. We conclude this section with the proof of Theorem 5.

Lemma 9. *Connected clique-cactus graphs are well-quasi-ordered by* $\leq_{\mathrm{ctr}}$.

In this section, we deal with rooted graphs. A *rooted graph* is a graph which has a distinguished vertex, called *root*. The contraction relation is extended to the setting of rooted graphs by requiring that a model of a rooted graph H in a rooted graph G maps the root of H to a connected subgraph of G containing the root of G.

Let us denote by $\mathcal{C}$ the class of rooted connected clique-cactus graphs. In this class, two isomorphic graphs with a different root are seen as different. It is clear that proving that $(\mathcal{C}, \leq_{\mathrm{ctr}})$ is a wqo implies Lemma 9. This is what we will do.

Building blocks. Let us define three graph constructors $\mathsf{stick}\colon \mathcal{C}^\star \to \mathcal{C}$, $\mathsf{cycle}\colon \mathcal{C}^\star \to \mathcal{C}$, and $\mathsf{clique}\colon \mathcal{C}^\star \to \mathcal{C}$. Given a sequence $\langle G_0, \dots, G_{p-1}\rangle \in \mathcal{C}^\star$ (for some $p \in \mathbb{N}$), if U denote the union of the graphs $G_1, \dots, G_{p-1}$, then we define;

- $\mathsf{stick}(G_0, \dots, G_{p-1})$ is the graph obtained from U by identifying the vertices $\mathrm{root}(G_0), \dots, \mathrm{root}(G_{p-1})$;
- $\mathsf{cycle}(G_0, \dots, G_{p-1})$ is the graph obtained from U by adding the edges $\{\mathrm{root}(G_i), \mathrm{root}(G_{(i+1) \mod p})\}$ for every $i \in \{0, \dots, p-1\}$; and
- $\mathsf{clique}(G_0, \dots, G_{p-1})$ is the graph obtained from U by adding the edges $\{\mathrm{root}(G_i), \mathrm{root}(G_j)\}$ for every distinct $i, j \in \{0, \dots, p-1\}$.

The root of $\mathsf{stick}(G_0, \dots, G_{p-1})$, $\mathsf{cycle}(G_0, \dots, G_{p-1})$ and $\mathsf{clique}(G_0, \dots, G_{p-1})$ is the vertex that is the root of G_0. These constructors will allow us to encode graphs of $\mathcal{C}$ into sequences.

We will now decompose graphs of $\mathcal{C}$ along blocks.

For every block B of a graph G, let $\mathsf{dec}_B(G)$ denote the collection of all the graphs H that can be constructed from some connected component C of $G \setminus V(B)$ by adding a new vertex v adjacent to the vertices of C that are adjacent to a vertex of B in G, and setting $\mathrm{root}(H) = v$.

Observe that as soon as $\mathrm{root}(G) \in V(B)$, every graph of $\mathsf{dec}_B(G)$ is a proper contraction of G. Let $\mathsf{dec}(G)$ denote the union of the sets $\mathsf{dec}_B(G)$ for every block B of G containing the root of G. The following observation is a consequence of Lemma 5.

Observation 6. For every graph $G \in \mathcal{C}$ there is a (not necessarily unique) sequence $\langle \mathcal{G}_0, \dots, \mathcal{G}_{p-1}\rangle \in \mathsf{dec}(G)^\star$ (for some $p \in \mathbb{N}$) such that $G = \mathsf{cycle}(\mathsf{stick}(\mathcal{G}_0), \dots, \mathsf{stick}(\mathcal{G}_{p-1}))$ or $G = \mathsf{clique}(\mathsf{stick}(\mathcal{G}_0), \dots, \mathsf{stick}(\mathcal{G}_{p-1}))$.

From encodings to well-quasi-ordering. The following lemma will allow us to work on sequences in order to show that two graphs are comparable.

Lemma 10. *Let $\mathcal{G}, \mathcal{H} \in \mathcal{C}^\star$. If $\mathcal{H} \leq_{\mathrm{ctr}}{}^\star \mathcal{G}$, then*

(i) $\mathsf{cycle}(\mathcal{H}) \leq_{\mathrm{ctr}} \mathsf{cycle}(\mathcal{G})$*;*

(ii) $\mathsf{clique}(\mathcal{H}) \leq_{\mathrm{ctr}} \mathsf{clique}(\mathcal{G})$*; and*

(iii) $\mathsf{stick}(\mathcal{H}) \leq_{\mathrm{ctr}} \mathsf{stick}(\mathcal{G})$*.*

Proof. Let $\mathcal{H} = \langle H_1, \dots, H_p \rangle$ and $\mathcal{G} = \langle G_1, \dots, G_q \rangle$ (for some positive integers p, q), and let $H = \mathsf{cycle}(\mathcal{H})$ and $G = \mathsf{cycle}(\mathcal{G})$. For the sake of readability we will refer to H_i's (respectively G_i's) either as elements of $\mathcal{H}$ (respectively $\mathcal{G}$) or as subgraphs of H (respectively G).

If $\mathcal{H} \leq_{\mathrm{ctr}}{}^\star \mathcal{G}$, then there is, by definition of $\leq_{\mathrm{ctr}}{}^\star$, an increasing function $\varphi\colon \{1, \dots, p\} \to \{1, \dots, q\}$ such that $\forall i \in \{1, \dots, p\}\,,\ H_i \leq_{\mathrm{ctr}} G_\varphi(i)$. Therefore there is a sequence of edge contractions transforming $G_{\varphi(i)}$ into H_i for every $i \in \{1, \dots, p\}$. Let us perform the following operations on G:

1. for every $j \in \{1, \dots, q\} \setminus \{\varphi(i),\ i \in \{1, \dots, p\}\}$ we contract the subgraph G_j to a single vertex v_j and we then contract some edge incident with v_j;
2. for every $i \in \{1, \dots, p\}$ we contract the subgraph G_i in order to obtain the subgraph $H_\varphi(i)$.

Observe that after step 1., we obtain the graph $\mathsf{cycle}(\mathcal{G}^-)$, where $\mathcal{G}^-$ can be obtained from $\mathcal{G}$ be deleting elements of indices in $\{1, \dots, q\} \setminus \{\varphi(i),\ i \in \{1, \dots, p\}\}$. Intuitively, we contracted the graphs that do not appear in H and removed their attachment point from the cycle. Then we replace in step 2. every graph of $\mathcal{G}^-$ by its corresponding contraction of $\mathcal{H}$. Therefore the graph obtained at the end is $\mathsf{cycle}(\mathcal{H})$, that is H, as required.

The cases (ii) and (iii) are very similar: H can be obtained from G by following the same operations as above. ☐

Proof of Lemma 9. Let us assume by contradiction that $(\mathcal{C}, \leq_{\mathrm{ctr}})$ is not a wqo. All decreasing sequences of this quasi-order are finite (as each contraction decreases the number of edges by at least one), therefore $(\mathcal{C}, \leq_{\mathrm{ctr}})$ contains an infinite antichains. Let us consider a minimal antichain $\{A_i\}_{i \in \mathbb{N}}$ of $(\mathcal{C}, \leq_{\mathrm{ctr}})$. Let $\mathcal{B} = \bigcup_{i \in \mathbb{N}} \mathsf{dec}(A_i)$, and let us show that $(\mathcal{B}, \leq_{\mathrm{ctr}})$ is a wqo. For contradiction, let us assume that it is not a wqo and let $\{B_i\}_{i \in \mathbb{N}}$ be a minimal antichain of this quasi-order.

By definition of $\mathcal{B}$, for every $H \in \mathcal{B}$ there is an integer $i \in \mathbb{N}$ such that $H \leq_{\mathrm{ctr}} A_i$ (for instance, an integer i such that $H \in \mathsf{dec}(A_i)$). Therefore for every $i \in \mathbb{N}$ there is an integer $\pi(i)$ such that $B_i \leq_{\mathrm{ctr}} A_{\pi(i)}$. Let $k \in \mathbb{N}$ be the integer such that $\pi(k)$ is minimum. Then the following sequence

$$\mathcal{A} = A_0, \dots, A_{\pi(k)-1}, B_k, B_{k+1}, \dots$$

is an infinite antichain of $(\mathcal{C}, \leq_{\mathrm{ctr}})$. Indeed, as both $\{A_i\}_{i \in \mathbb{N}}$ and $\{B_i\}_{i \in \mathbb{N}}$ are antichains, every pair of comparable graphs of $\mathcal{A}$ involves one graph of $\{A_i\}_{i \in \{1, \dots, \pi(k)-1\}}$

and one graph of $\{B_i\}_{i\in\mathbb{N}_{\geq k}}$. Let us assume that for some $i \in \{0,\dots,\pi(k)-1\}$ and $j \in \mathbb{N}_{\geq k}$ we have $A_i \leq B_j$. Then $A_i \leq B_j \leq A_{\pi(i)}$, a contradiction with the fact that $\{A_i\}_{i\in\mathbb{N}}$ is an antichain. The case $B_j \leq A_i$ is not possible by the choice of k. This proves that $(\mathcal{B}, \leq_{\text{ctr}})$ is a wqo. According to Proposition 2, $(\mathcal{B}^\star, {\leq_{\text{ctr}}}^\star)$ is also a wqo. Let $\mathcal{B}' = \{\mathsf{stick}(\mathcal{H}),\ \mathcal{H} \in \mathcal{B}^\star\}$. Item (iii) of Lemma 10 implies that any antichain in $(\mathcal{B}', \leq_{\text{ctr}})$ can be translated into an antichain of the same length in $(\mathcal{B}^\star, {\leq_{\text{ctr}}}^\star)$, hence $(\mathcal{B}', \leq_{\text{ctr}})$ is a wqo. By the same argument (now using items (i) and (ii) of Lemma 10), we deduce that the quasi-orders

$$(\{\mathsf{cycle}(\mathcal{H}),\ \mathcal{H} \in \mathcal{B}'^\star\}, \leq_{\text{ctr}}) \quad \text{and} \quad (\{\mathsf{clique}(\mathcal{H}),\ \mathcal{H} \in \mathcal{B}'^\star\}, \leq_{\text{ctr}})$$

are well-quasi-orders. Therefore $\mathcal{U} = \{\mathsf{cycle}(\mathcal{H}),\ \mathcal{H} \in \mathcal{B}'^\star\} \cup \{\mathsf{clique}(\mathcal{H}),\ \mathcal{H} \in \mathcal{B}'^\star\}$ is well-quasi-ordered by $\leq_{\text{ctr}}$, as a consequence of Proposition 1. According to Observation 6, we have $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$. This contradicts the fact that $\{A_i\}_{i\in\mathbb{N}}$ is an infinite antichain. Therefore $(\mathcal{C}, \leq_{\text{ctr}})$ is a wqo and we are done. □

We would like to point out that with a proof similar to Lemma 9, it is in fact possible prove that if a class $\mathcal{G}$ of 2-connected graphs is wqo by $\leq_{\text{ctr}}$, then so is the class of graphs whose blocks belong to $\mathcal{G}$. The interested reader may have a look at [8, Lemma 5] for a result of this flavour, see also [5, Theorem 5] and [1].

The proof of Theorem 5 is now immediate.

Proof of Theorem 5. Let H be a graph such that $\mathrm{Excl}(H)$ is a wqo. Then $H \leq_{\text{ctr}} D_2$, by Corollary 2. On the other hand, if $H \leq_{\text{ctr}} D_2$ then $\mathrm{Excl}(H) \subseteq \mathrm{Excl}(D_2)$. Observe that every antichain (respectively decreasing sequence) of $(\mathrm{Excl}(H), \leq_{\text{ctr}})$ is an antichain (respectively a decreasing sequence) of $(\mathrm{Excl}(D_2), \leq_{\text{ctr}})$. As a consequence of Lemma 9 we get that $(\mathrm{Excl}(H), \leq_{\text{ctr}})$ is a wqo and we are done. □

6 On canonical antichains

In this section, we will use the following result of Ding in order to prove Theorem 10. Figure 4 illustrates the requirements of the lemma.

Lemma 11 ([4, Theorem 1.1])**.** *Let $(S, \preceq)$ be a quasi-order, let $\langle a_i \rangle_{i\in\mathbb{N}}$ be a sequence of elements of S and let $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ be a sequence of sequences of elements of S. If we have*

(i) $\langle a_i \rangle_{i\in\mathbb{N}}$ is a fundamental infinite antichain; and

(ii) for every $i \in \mathbb{N}$, $\mathcal{W}_i$ is a fundamental infinite antichain; and

(iii) for every $i \in \mathbb{N}$ and every $H \in \mathcal{W}_i$, $a_i \preceq H$ and there are no other comparable pairs of elements in $\bigcup_{i\in\mathbb{N}} \mathcal{W}_i \cup \{a_i\}_{i\in\mathbb{N}}$

then $(S, \preceq)$ does not have a canonical antichain.

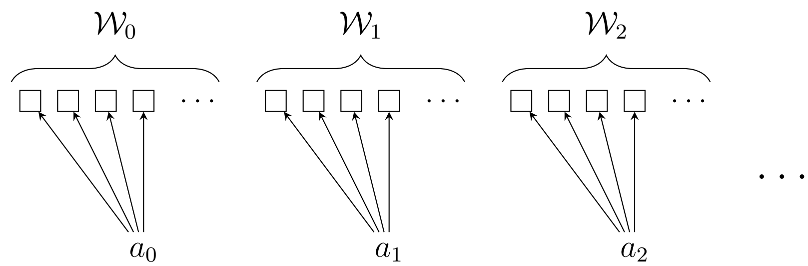

Figure 4: The situation described in Lemma 11. Arrows are directed towards larger elements.

Note that [4, Theorem 1.1] mentions other obstructions to the existence of a canonical antichain, however we will only use that described in Lemma 11. We will now define some sequences of graphs and show that they satisfy the properties of Lemma 11.

For every $p, q \in \mathbb{N}$, let $W_{p,q}$ be the graph obtained by adding two non-adjacent dominating vertices to the disjoint union of $\overline{K}_p$ and $K_{2,q}$ (see Figure 5). These two vertices are called *poles*, and the two vertices corresponding to the part of $K_{2,q}$ of size 2 are called *semipoles*. Observe that the other vertices either have degree two (in which case they are adjacent to the two poles, only), or have degree four (and they are adjacent to both poles and both semipoles).

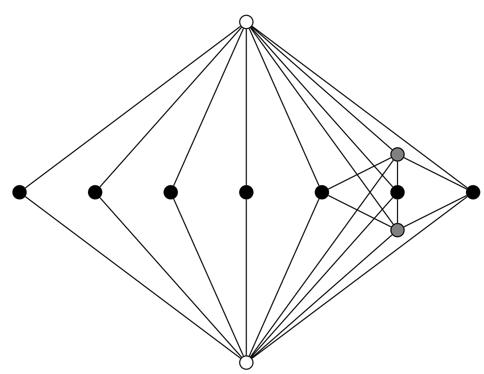

Figure 5: The graph $W_{4,3}$. Poles are drawn in white and semipoles in gray.

Lemma 12. *For every $p, p', q, q' \in \mathbb{N}_{\geq 3}$, there is a model of $W_{p,q}$ in $W_{p',q'}$ iff $(p, q) = (p', q')$.*

Proof. Let us assume that there is a model φ of $W_{p,q}$ in $W_{p',q'}$. Let v be a vertex of $W_{p,q}$ of degree two. By definition of a model, $\varphi(v)$ must be a connected subset of degree 2 of $V(W_{p',q'})$. Let us consider the possible choices for this subset.

First case: $\varphi(v)$ is of the form $V(W_{p',q'}) \setminus \{x, y\}$, for some $u, v \in V(W_{p',q'})$. Therefore, $V(W_{p',q'}) \setminus \varphi(v)$ has two vertices. As $W_{p,q}$ has more than 3 vertices (recall that $p, q \geq 3$) which are mapped by φ to disjoint subsets of $V(W_{p',q'})$, this case is not possible.

Second case: $\varphi(v)$ is the subset of vertices inducing the subgraph $K_{2,q'}$ used in the construction of $W_{p',q'}$. Observe that the poles and semipoles of $W_{p,q}$ (4 vertices in total, as they are distinct from v) all have degree at least 3. Moreover, $W_{p',q'} \setminus \varphi(v)$ is isomorphic to $K_{2,p'}$. As every connected subset of degree at least 3 of $K_{2,p'}$ must contain a pole of this graph, there are at most two such subsets that are disjoint. This contradicts the fact that the images by φ of the four poles and semipoles of $W_{p,q}$ are disjoint connected subsets of degree at least 3 of $W_{p',q'} \setminus \varphi(v)$. Therefore, this case is not possible neither.

Third case: $\varphi(v) = \{x\}$ for some vertex $x \in V(W_{p',q'})$ of degree 2. As we reach this case for every choice of a vertex of degree 2 of $W_{p,q}$, we deduce $p \leq p'$. The same argument applied to vertices of degree 4 yields $q \leq q'$. Let us now consider poles and semipoles.

Let u be a pole. Observe that according to the above remarks, $\varphi(u)$ must be adjacent to vertices of degree two, so it should contain a pole of $W_{p,q}$. If $\varphi(u)$ contains in addition a vertex of degree 2 or 4 of $W_{p,q}$, then $\varphi(u)$ is dominating. This is not possible since u is not dominating, therefore $\varphi(u) = \{v\}$ for some pole v of $W_{p',q'}$. Let us now assume that u is a semipole of $W_{p,q}$. As previously, the above remarks imply that $\varphi(u)$ is adjacent to vertices of degree 4 of $W_{p',q'}$. Hence $\varphi(u)$ contains a semipole of $W_{p',q'}$ (it cannot contain a pole as both belong to the image of poles of $W_{p,q}$). Therefore each semipole of $W_{p,q}$ is sent to a subset of $V(W_{p',q'})$ containing a semipole. Observe that $\varphi(u)$ cannot contain a vertex of degree two otherwise it would not be connected. Besides, it cannot contain a vertex of degree 4 otherwise it would be adjacent to the image by φ of the other semipole of $W_{p,q}$. Consequently $\varphi(u)$ contains a semipole of $W_{p',q'}$ and no other vertex. We proved that for every $u \in V(W_{p,q})$, the set $\varphi(u)$ is a singleton. Therefore $|V(W_{p,q})| = |V(W_{p',q'})|$. Given that $p \leq p'$ and $q \leq q'$ (as proved above), this is possible only if $p = p'$ and $q = q'$. This concludes the proof. □

Corollary 3. *$\{W_{p,q}\}_{p,q\geq 3}$ is an antichain for $\leq_{\mathrm{ctr}}$.*

For every $i \in \mathbb{N}_{\geq 3}$, let $\mathcal{W}_i = \{W_{i,q}\}_{q \in \mathbb{N}_{\geq 3}}$.

Lemma 13. *For every $p, q, r \in \mathbb{N}_{\geq 3}$, we have $K_{2,r} \leq_{\mathrm{ctr}} W_{p,q}$ iff $r = p + 1$.*

Proof. Let us consider a contraction model φ of $K_{2,r}$ in $W_{p,q}$. We call X the vertices of $W_{p,q}$ inducing the $K_{2,q}$ used in the construction of this graph. Let us consider a vertex u of degree 2 of $K_{2,r}$. Exactly as in the proof of Lemma 12, there are three possible choices for $\varphi(u)$. For the same reason as in this proof, the case where $\varphi(u) = V(W_{p,q}) \setminus \{x, y\}$ (for some $x, y \in V(W_{p,q})$) is not possible. Therefore, either $\varphi(u) = X$, or $\varphi(u) = \{x\}$ for some $x \in V(W_{p,q})$ of degree 2. Since this holds for every vertex of degree 2 of $K_{2,r}$, and as $W_{p,q}$ has exactly p vertices of degree 2, we deduce that $r \leq p + 1$.

Let v, w denote the poles of $K_{2,r}$. Because of the observation above and of the definition of a contraction model, each of $\varphi(v)$ and $\varphi(w)$ must be adjacent to some vertex of degree 2 of $W_{p,q}$ and these sets should not be adjacent. The only possible choice for them is to let $\varphi(v)$ be the singleton containing one pole of $W_{p,q}$ and $\varphi(w)$ be the singleton containing the other pole. Observe that, in the case where $p+1>r$, either one vertex of degree 2 of $W_{p,q}$ or a vertex of X does not belong to the image of φ. This contradicts the definition of a model, hence this case is not possible.

The only remaining case is thus $r=p+1$. Observe that X induces a connected subgraph. It is not hard to see that contracting X to a single vertex yields $K_{2,p+1}$. □

Observation 7. Let $p,q \in \mathbb{N}_{\geq 3}$. There is no induced path on four vertices in $W_{p,q}$, neither in $K_{2,p}$.

Then we successively deduce the following consequences.

Corollary 4. *For every $p,q \in \mathbb{N}_{\geq 3}$, none of the graphs $W_{p,q}$ and $K_{2,p}$ contains the gem as induced minor.*

Corollary 5. *No graph of* $\mathrm{Incl}(\mathcal{W}_i)$ *and of* $\mathrm{Incl}(\{K_{2,p}\}_{p\in\mathbb{N}_{\geq 3}})$ *contains the gem as induced minor, for every $i \in \mathbb{N}_{\geq 3}$.*

The following observation will allow us to use Lemma 14, which deals with induced minors.

Observation 8. Let H and G be two graphs. If both of them have a dominating vertex, then H is a contraction of G iff H is an induced minor of G.

Lemma 14 ([1])**.** *Graphs not containing the gem as induced minor are wqo by the induced minor relation.*

The following corollaries are direct consequences of Lemma 14, Observation 8 and Corollary 5.

Corollary 6. $\mathrm{Incl}(\{K_{2,p}\}_{p\in\mathbb{N}_{\geq 3}})$ *is wqo by* $\leq_{\mathrm{ctr}}$.

Corollary 7. *The graphs of* $\mathrm{Incl}(\mathcal{W}_i)$ *with a dominating vertex are wqo by* $\leq_{\mathrm{ctr}}$, *for every $i \in \mathbb{N}_{\geq 3}$.*

Lemma 15. *$\mathcal{W}_i$ is a fundamental antichain, for every $i \in \mathbb{N}_{\geq 3}$.*

Proof. Let $i \in \mathbb{N}_{\geq 3}$. We need to show that $(\mathrm{Incl}(\mathcal{W}_i), \leq_{\mathrm{ctr}})$ is a wqo. Let us call *inner edge* every edge of $W_{p,q}$ that is not incident with a pole, for every $p,q \in \mathbb{N}_{\geq 3}$. Observe that if a graph $H \in \mathrm{Incl}(\mathcal{W}_i)$ has been obtained by contracting at least one edge incident with a pole, then H has a dominating vertex. According to Corollary 7, these graphs are wqo by $\leq_{\mathrm{ctr}}$, therefore we will here consider graphs of $\mathrm{Incl}(\mathcal{W}_i)$ that have been obtained by only contracting inner edges. We call $\mathcal{I}$ this class.

We first show that $\mathcal{I}$ is the union of the two following classes:

- the class $\mathcal{I}_0$ of graphs that can be obtained by adding two non-adjacent dominating vertices to $\overline{K}_i + D_q$ for some $q \in \mathbb{N}_{\geq 0}$; and

- the class $\mathcal{I}_1$ of graphs that can be obtained by adding two non-adjacent dominating vertices to $\overline{K}_i + S_q$ for some $q \in \mathbb{N}_{\geq 0}$.

Again we use the notion of *poles* to denote the two dominating vertices added to construct graphs of $\mathcal{I}_0$ and $\mathcal{I}_1$. A *semipole* is either a dominating vertex of D_q (when dealing with graphs of $\mathcal{I}_0$), or the dominating vertex of S_q (when dealing with graphs of $\mathcal{I}_1$).

Contracting an inner edge in $W_{i,q}$ clearly yields a graph of $\mathcal{I}_0$. Now, observe that any further contraction of an edge connecting a vertex of degree 4 to a semipole gives a graph of $\mathcal{I}_0$ again. If, on the other hand, we contract the edge connecting the two semipoles, then we get a graph of $\mathcal{I}_1$. On a graph of $\mathcal{I}_1$, contracting an edge of the star (used in the construction of this graph) still gives a graph of $\mathcal{I}_1$. Therefore $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$.

Let us assume that $\mathcal{I}$ is not wqo by $\leq_{\mathrm{ctr}}$. Therefore it has an infinite antichain. As $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, one of $\mathcal{I}_0$ and $\mathcal{I}_1$ (at least) has an infinite antichain. Let $\mathcal{A}$ be such an infinite antichain.

We now look at vertices of graphs of $\mathcal{A}$ that are neither poles, nor semipoles, nor have degree 2. These vertices are the vertices of degree 2 of the copy of D_q or the vertices of degree one of the copy of S_q used in the construction of the graphs of $\mathcal{A}$ (depending whether $\mathcal{A} \subseteq \mathcal{I}_0$ or $\mathcal{A} \subseteq \mathcal{I}_1$). We call them *inner vertices.*

Let A and A' be two graphs of $\mathcal{A}$ such that A has less inner vertices than A'. These graphs exist since the elements of $\mathcal{A}$ are distinct. Let q be the number of inner vertices of A and q' the one of A'.

In both cases $\mathcal{A} \subseteq \mathcal{I}_0$ and $\mathcal{A} \subseteq \mathcal{I}_1$ we can obtain A from A' by contracting $q' - q$ inner vertices of A' to a semipole. This contradicts the fact that $\mathcal{A}$ is an antichain. Therefore $(\mathcal{I}, \leq_{\mathrm{ctr}})$ is a wqo. This implies that $\mathcal{W}_i$ is fundamental, as required. □

We are now ready to prove Theorem 10.

Proof of Theorem 10. Let $A_i = K_{2,i+1}$ for every $i \in \mathbb{N}_{\geq 3}$.

By the virtue of Corollary 6, $\{A_i\}_{i \in \mathbb{N}_{\geq 3}}$ is a fundamental antichain, as well as $\mathcal{W}_i$, for every $i \in \mathbb{N}_{\geq 3}$, according to Lemma 15. Also, for every $i \in \mathbb{N}_{\geq 3}$, we have $A_i \leq_{\mathrm{ctr}} H$ for every $H \in \mathcal{W}_i$ (Lemma 13) and there are no other comparable pairs of elements in $\bigcup_{i \in \mathbb{N}_{\geq 3}} \mathcal{W}_i \cup \{A_i\}_{i \in \mathbb{N}_{\geq 3}}$ (Lemma 12 and Lemma 13).

Hence these sequences of graphs satisfy the requirements of Lemma 11, which implies that there is no canonical antichain for the contraction relation. □

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