Induced minors and well-quasi-ordering*

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Monday 22nd January, 2018

Abstract

A graph H is an induced minor of a graph G if it can be obtained from an induced subgraph of G by contracting edges. Otherwise, G is said to be H-induced minor-free. Robin Thomas showed that K_4-induced minor-free graphs are well-quasi-ordered by induced minors [*Graphs without* K_4 *and well-quasi-ordering*, Journal of Combinatorial Theory, Series B, 38(3):240–247, 1985].

We provide a dichotomy theorem for H-induced minor-free graphs and show that the class of H-induced minor-free graphs is well-quasi-ordered by induced minors if and only if H is an induced minor of the Gem (the path on 4 vertices plus a dominating vertex) or of the graph obtained by adding a vertex of degree 2 to the complete graph on 4 vertices. To this end we prove two decomposition theorems which are of independent interest.

Similar dichotomy results were previously given for subgraphs by Guoli Ding in [*Subgraphs and well-quasi-ordering*, Journal of Graph Theory, 16(5):489–502, 1992] and for induced subgraphs by Peter Damaschke in [*Induced subgraphs and well-quasi-ordering*, Journal of Graph Theory, 14(4):427–435, 1990].

*This work was partially done while J. Błasiok was student at the Institute of Computer Science, University of Warsaw, Poland and while J.-F. Raymond was affiliated to LIRMM, Université de Montpellier, France and to the Institute of Computer Science, University of Warsaw, Poland. The research was supported by the Foundation for Polish Science (Jarosław Błasiok and Marcin Kamiński), the (Polish) National Science Centre grants SONATA UMO-2012/07/D/ST6/02432 (Marcin Kamiński and Jean-Florent Raymond) and PRELUDIUM 2013/11/N/ST6/02706 (Jean-Florent Raymond), the Warsaw Center of Mathematics and Computer Science (Jean-Florent Raymond and Théophile Trunck), and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, ERC consolidator grant DISTRUCT, agreement No 648527 (Jean-Florent Raymond). Emails: jblasiok@g.harvard.edu, mjk@mimuw.edu.pl, raymond@tu-berlin.de, and theophile.trunck@ens-lyon.org.

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1 Introduction

A *well-quasi-order* (*wqo* for short) is a quasi-order which contains no infinite decreasing sequence and no infinite collection of pairwise incomparable elements (called an *antichain*). One of the most important results in this field is arguably the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by the minor relation [23]. Other natural containment relations are not so generous; they usually do not wqo all graphs. In the last decades, much attention has been brought to the following question: given a partial order $(S, \preceq)$, what subclasses of S are well-quasi-ordered by $\preceq$? For instance, Fellows et al. proved in [8] that graphs with bounded feedback-vertex-set are well-quasi-ordered by topological minors. Another result is that of Oum [19] who proved that graphs of bounded rank-width are wqo by vertex-minors. Other papers considering this question include [1, 3–7, 10, 17, 20, 24].

One way to approach this problem is to consider graph classes defined by excluded substructures. In this direction, Damaschke proved in [4] that a class of graphs defined by one forbidden induced subgraph H is wqo by the induced subgraph relation if and only if H is the path on four vertices. Similarly, a bit later Ding proved in [5] an analogous result for the subgraph relation. Other authors also considered this problem (see for instance [2, 11–13]). In this paper, we provide an answer to the same question for the induced minor relation, which we denote $\leq_{\mathrm{im}}$. This generalizes a result of Thomas who proved that graphs with no K_4-minor are wqo by $\leq_{\mathrm{im}}$ [24]. Before stating our main result, let us introduce two graphs which play a major role in this paper (see Figure 1). The first one, $\widehat{K}_4$, is obtained by adding a vertex of degree two to K_4, and the second one, called the Gem, is constructed by adding a dominating vertex to P_4.

Figure 1: The graph $\widehat{K}_4$ (on the left) and the Gem (on the right).

2 Induced minors and well-quasi-ordering

Our main result is the following.

Theorem 1 (Dichotomy Theorem)**.** *Let H be a graph. The class of H-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$ iff H is an induced minor of $\widehat{K}_4$ or the* Gem.

Our proof naturally has two parts: for different graphs H, we need to show wqo of H-induced minor-free graphs or exhibit an H-induced minor-free antichain.

Classes that are wqo. The following two theorems describe the structure of graphs with H forbidden as an induced minor, when H is $\widehat{K}_4$ and the Gem, respectively.

Theorem 2 (Decomposition of $\widehat{K}_4$-induced minor-free graphs)**.** *Let G be a 2-connected graph such that $\widehat{K}_4 \not\leq_{\mathrm{im}} G$. Then one of the following holds:*

(i) $K_4 \not\leq_{\mathrm{im}} G$; or

(ii) G is a subdivision of a graph among K_4, $K_{3,3}$, and the prism; or

(iii) $V(G)$ has a partition (W, M) such that $G[W]$ is a wheel on at most 5 vertices and $G[M]$ is a complete multipartite graph; or

(iv) $V(G)$ has a partition (C, I) such that $G[C]$ is a cycle, I is an independent set and every vertex of I has the same neighborhood on C.

Theorem 3 (Decomposition of Gem-induced minor-free graph)**.** *Let G be a 2-connected graph such that* Gem $\not\leq_{\mathrm{im}} G$*. Then G has a subset $X \subseteq V(G)$ of at most six vertices such that every connected component of $G \setminus X$ is either a cograph or a path whose internal vertices are of degree two in G.*

Using the two above structural results, we are able to show the well-quasi-ordering of the two classes with respect to induced minors. For every graph H, a graph not containing H as induced minor is said to be *H-induced minor-free*.

Theorem 4. *The class of $\widehat{K}_4$-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$.*

Theorem 5. *The class of* Gem*-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$.*

Organization of the paper. After a preliminary section introducing notions and notation used in this paper, we present in Section 4 several infinite antichains for induced minors. Section 5 is devoted to the proof of Theorem 1, assuming Theorem 4 and Theorem 5, the proof of which are respectively given in Section 6 and Section 7. Finally, we give in Section 8 some directions for further research.

3 Preliminaries

The notation $[\![i, j]\!]$ stands for the interval of integers $\{i, \ldots, j\}$. We denote by $\mathcal{P}(S)$ the power set of a set S and by $\mathcal{P}^{<\omega}(S)$ the set of all its finite subsets.

3.1 Graphs and classes

The graphs in this paper are simple and loopless. Given a graph G, $V(G)$ denotes its vertex set and $E(G)$ its edge set. For every positive integer n, K_n is the complete graph on n vertices and P_n is the path on n vertices. For every integer $n \geq 3$, C_n is the cycle on n vertices. For H and G graphs, we write $H + G$ the disjoint union of H and G. The *complement* of a graph G, denoted by $\overline{G}$, is obtained by remplacing every edge by a non-edge, and vice-versa. Also, for every $k \in \mathbb{N}$, $k \cdot G$ is the disjoint union of k copies of G. For every pair u, v of vertices a path P there is exactly one subpath in P between u and v, that we denote by uPv. Two vertices $u, v \in V(G)$ are said to be *adjacent* if $\{u, v\} \in E(G)$. The *neighborhood* of $v \in V(G)$, denoted $N_G(v)$, is the set of vertices that are adjacent to v. If H is a subgraph of G, we write $N_H(v)$ for $N_G(v) \cap V(H)$. Given two sets X, Y of vertices of a graph, we say that there is an edge between X and Y (or that X and Y are adjacent) if there is $x \in X$ and $y \in Y$ such that $\{x, y\} \in E(G)$. The number of connected components of a graph G is denoted $\mathrm{cc}(G)$. We call *prism* the complement of C_6.

A *cograph* is a graph not containing the path on four vertices as induced subgraph. The following notion will be used when decomposing graphs not containing Gem as induced minor. An induced subgraph of a graph G is said to be *basic in* G if it is either a cograph, or an induced path whose internal vertices are of degree two in G. A *linear forest* is a disjoint union of paths. The *closure* of a class $\mathcal{G}$ by a given operation is the class obtained from graphs of $\mathcal{G}$ by a finite application of this operation.

Complete multipartite graphs. A graph G is said to be *complete multipartite* if its vertex set can be partitioned into sets $V_1, \ldots, V_k$ (for some positive integer k) in a way such that two vertices of G are adjacent iff they belong to different V_i's. The class of complete multipartite graphs is referred to as $\mathcal{K}_{\mathbb{N}^\star}$.

Wheels. For every positive integer k, a *k-wheel* is a graph obtained from $C + K_1$, where C is an induced cycle of order at least k, by connecting the isolated vertex to k distinct vertices of the cycle. C is said to be the *cycle* of the k-wheel, whereas the vertex corresponding to K_1 is its *center*.

Cutsets. In a graph G, a K_2-cutset (resp. $\overline{K_2}$-cutset) is a subset $S \subseteq V(G)$ such that $G - S$ is not connected and $G[S]$ is isomorphic to K_2 (resp. $\overline{K_2}$).

Labels and roots. Let $(\Sigma, \preceq)$ be a poset. A $(\Sigma, \preceq)$-labeled graph is a pair (G, λ) such that G is a graph, and $\lambda\colon V(G) \to \mathcal{P}^{<\omega}(\Sigma)$ is a function referred as the *labeling of the graph*. For the sake of simplicity, we will refer to the labeled graph of a pair (G, λ) by G and to λ by λ_G. If $\mathcal{G}$ is a class of (unlabeled) graphs, $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G})$ denotes the class of $(\Sigma, \preceq)$-labeled graphs of $\mathcal{G}$. Observe that any unlabeled graph can be seen as a $\emptyset$-labeled graph. A *rooted graph* is a graph with a distinguished edge called *root*.

Labels will allow us to focus on labeled 2-connected graphs, as stated in the following proposition.

Proposition 1 ([8]). *Let $\mathcal{G}$ be a class of graphs that is closed by taking induced subgraphs. If for any wqo $(S, \preceq)$ the class of $(S, \preceq)$-labeled 2-connected graphs of $\mathcal{G}$ is wqo by $\leq_{\mathrm{im}}$, then $\mathcal{G}$ is wqo by $\leq_{\mathrm{im}}$.*

3.2 Sequences, posets and well-quasi-orders

In this section, we introduce basic definitions and facts related to the theory of well-quasi-orders. In particular, we recall that being well-quasi-ordered is preserved by several operations.

Sequences. A *sequence* of elements of a set A is an ordered countable collection of elements of A. Unless otherwise stated, sequences are finite. The sequence of elements $s_1, \ldots, s_k \in A$ in this order is denoted by $\langle s_1, \ldots, s_k \rangle$. We use the notation $A^\star$ for the class of all finite sequences over A (including the empty sequence). The length of a finite sequence $s \in A^\star$ is denoted by $|s|$.

Posets ans wqos. A *partially ordered set* (*poset* for short) is a pair $(A, \preceq)$ where A is a set and $\preceq$ is a binary relation on A which is reflexive, antisymmetric and transitive. An *antichain* is a sequence of pairwise non-comparable elements. In a sequence $\langle x_i \rangle_{i \in I \subseteq \mathbb{N}}$ of a poset $(A, \preceq)$, a pair (x_i, x_j), $i, j \in I$ is a *good pair* if $x_i \preceq x_j$ and $i < j$. A poset $(A, \preceq)$ is a *well-quasi-order* (*wqo* for short)[1], and its elements are said to be *well-quasi-ordered* (*wqo* for short) by $\preceq$, if every infinite sequence has a good pair, or equivalently, if $(A, \preceq)$ has neither an infinite decreasing sequence, nor an infinite antichain. An infinite sequence containing no good pair is called an *bad sequence*.

Ordering sequences. For any partial order $(A, \preceq)$, we define the relation $\preceq^\star$ on $A^\star$ as follows: for every $r = \langle r_1, \ldots, r_p \rangle$ and $s = \langle s_1, \ldots, s_q \rangle$ of $A^\star$, we have $r \preceq^\star s$ if there is a increasing function $\varphi \colon [\![1, p]\!] \to [\![1, q]\!]$ such that for every $i \in [\![1, p]\!]$ we have $r_i \preceq s_{\varphi(i)}$. Observe that $=^\star$ is then the subsequence relation. This order relation is extended to the class $\mathcal{P}^{<\omega}(A)$ of finite subsets of A as follows, generalizing the subset relation: for every $B, C \in \mathcal{P}^{<\omega}(A)$, we write $B \preceq^{\mathcal{P}} C$ if there is an injection $\varphi \colon B \to C$ such that $\forall x \in B,\ x \preceq \varphi(x)$. Observe that $=^{\mathcal{P}}$ is the subset relation.

[1] Usually in literature the term well-quasi-order is defined for more general structures than posets, namely *quasi-orders*. Those relations are like posets, except they are not required to be antisymmetric. This is mere technical detail, as every poset is a quasi-order, and from a quasi-order one can make a poset by taking a quotient by the equivalence relation $a \preceq b \wedge b \preceq a$.

Monotonicity. In order to stress that the domain $(A, \preceq_A)$ and codomain $(B, \preceq_B)$ of a function φ are posets, we sometimes use the notation $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$. A function $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$ is said to be *monotone* if it satisfies the following property:

$$\forall x, y \in A,\ x \preceq_A y \Rightarrow f(x) \preceq_B f(y).$$

In order to prove that a function is monotone, one can focus on each argument separately, as noted in the following remark.

Remark 1. Let $(A, \preceq_A)$, $(B, \preceq_B)$, and $(C, \preceq_C)$ be posets and let $f\colon (A \times B, \preceq_A \times \preceq_B) \to (C, \preceq_C)$ be a function. If we have both

$$\forall a \in A,\ \forall b, b' \in B,\ b \preceq_B b' \Rightarrow f(a, b) \preceq_C f(a, b') \tag{1}$$

$$\text{and } \forall a, a' \in A,\ \forall b \in B,\ a \preceq_A a' \Rightarrow f(a, b) \preceq_C f(a', b) \tag{2}$$

then f is monotone. Indeed, let $(a, b), (a', b') \in A \times B$ be such that $(a, b) \preceq_A \times \preceq_B (a', b')$. By definition of the relation $\preceq_A \times \preceq_B$, we have both $a \preceq a'$ and $b \preceq b'$. From line (1) we get that $f(a, b) \preceq_C f(a, b')$ and from line (2) that $f(a, b') \preceq_C f(a', b')$, hence $f(a, b) \preceq_C f(a', b')$ by transitivity of $\preceq_C$. Thus f is monotone. Observe that this remark can be generalized to functions with more than two arguments.

Well-quasi-orders can be constructed from smaller ones by simple operations. The following proposition lists well-known properties of well-quasi-orders. For a reference, the reader can refer to [9] and [15].

Proposition 2. *Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be two wqos, then*

- *their union $(A \cup B, \preceq_A \cup \preceq_B)$, which is the poset defined as follows:*

  $$\forall x, y \in A \cup B,\ x \preceq_A \cup \preceq_B y \text{ if } (x, y \in A \text{ and } x \preceq_A y) \text{ or } (x, y \in B \text{ and } x \preceq_B y),$$

  is a wqo;

- *their Cartesian product $(A \times B, \preceq_A \times \preceq_B)$ which is the poset defined by:*

  $$\forall (a, b), (a', b') \in A \times B,\ (a, b) \preceq_A \times \preceq_B (a', b') \text{ if } a \preceq_A a' \text{ and } b \preceq_B b',$$

  is a wqo;

- *$(A^\star, \preceq^\star)$ is a wqo (Higman's Lemma);*

- *$(\mathcal{P}^{<\omega}(A), \preceq^{\mathcal{P}})$ is a wqo;*

- *if $(C, \preceq_C)$ is a poset included in the image of a monotone function with domain $(A, \preceq_A)$, then $(C, \preceq_C)$ is a wqo.*

3.3 Graph operations and containment relations

Most of the common order relations on graphs, sometimes called *containment relations*, can be defined in two equivalent ways: either in terms of *graph operations*, or by using *models*. Let us look closer at them.

Local operations. If $\{u, v\} \in E(G)$, the *edge contraction* of $\{u, v\}$ adds a new vertex w adjacent to the neighbors of u and v and then deletes u and v. In the case where G is labeled, we set $\lambda_G(w) = \lambda_G(u) \cup \lambda_G(v)$. On the other hand, an *edge subdivision* of $\{u, v\}$ adds a new vertex adjacent to u and v and deletes the edge $\{u, v\}$. The *identification* of two vertices u and v adds the edge $\{u, v\}$ if it was not already existing, and contracts it. If G is $(\Sigma, \preceq)$-labeled (for some poset $(\Sigma, \preceq)$), a *label contraction* is the operation of relabeling a vertex $v \in V(G)$ with a label l such that $l \preceq^{\mathcal{P}} \lambda_G(v)$. The motivation for this definition of label contraction is the following. Most of the time, labels will be used to encode connected graphs into 2-connected graphs. Given a connected graph which is not 2-connected, we can pick an arbitrary block (i.e. a maximal 2-connected component), delete the rest of the graph and label each vertex v by the subgraph it was attached to in the original graph if v was a cutvertex, and by $\emptyset$ otherwise. That way, contracting the label of a vertex v in the labeled 2-connected graph corresponds to reducing (for some containment relation) the subgraph which was dangling at vertex v in the original graph (see also Proposition 1).

Models. Let $(\Sigma, \preceq)$ be any poset. A *containment model* of a $(\Sigma, \preceq)$-labeled graph H in a $(\Sigma, \preceq)$-labeled graph G (*H-model* for short) is a function $\mu \colon V(H) \to \mathcal{P}^{<\omega}(V(G))$ satisfying the following conditions:

(i) for every two distinct $u, v \in V(H)$, the sets $\mu(u)$ and $\mu(v)$ are disjoint;

(ii) for every $u \in V(H)$, the subgraph of G induced by $\mu(u)$ is connected;

(iii) for every $u \in V(H)$, $\lambda_H(u) \preceq^\star \bigcup_{v \in \mu(u)} \lambda_G(v)$ (label conservation).

When in addition μ is such that for every two distinct $u, v \in V(H)$, the sets $\mu(u)$ and $\mu(v)$ are adjacent in G iff $\{u, v\} \in E(H)$, then μ is said to be an *induced minor model* of H in G.

If μ is an induced minor model of H in G satisfying the following condition:

$$\bigcup_{v \in V(H)} \mu(v) = V(G),$$

then μ is a *contraction model* of H in G.

If μ is an induced minor model of H in G satisfying the following condition:

$$\forall v \in V(H),\ |\mu(v)| = 1,$$

then μ is an *induced subgraph model* of H in G.

An H-model in a graph G witnesses the presence of H as substructure of G (which can be induced subgraph, induced minor, contraction, etc.), and the subsets of $V(G)$ given by the image of the model indicate which subgraphs to keep and to contract in G in order to obtain H.

When dealing with rooted graphs, the aforementioned models must in addition preserve the root, that is, if $\{u, v\}$ is the root of H then the root of G must have one endpoint in $\mu(u)$ and the other in $\mu(v)$.

Containment relations. Local operations and models can be used to express that a graph is *contained* in an other one, for various definitions of "contained". We say that a graph H is an *induced minor* (resp. a contraction, induced subgraph) of a graph G if there is an induced minor model (resp. a contraction model, an induced subgraph model) μ of H in G, what we note $H \leq_{\mathrm{im}} G$ (resp. $H \leq_{\mathrm{c}} G$, $H \leq_{\mathrm{isg}} G$).

Otherwise, G is said to be *H-induced minor-free* (resp. *H-contraction-free*, *H-induced subgraph-free*). The class of H-induced minor-free graphs will be referred to as $\mathrm{Excl}_{\mathrm{im}}(H)$.

Remark 2. In terms of local operation, these containment relations are defined as follows for every H, G graphs:

- $H \leq_{\mathrm{isg}} G$ iff there is a (possibly empty) sequence of vertex deletions and label contractions transforming G into H;
- $H \leq_{\mathrm{im}} G$ iff there is a (possibly empty) sequence of vertex deletions, edge contractions and label contractions transforming G into H;
- $H \leq_{\mathrm{c}} G$ iff there is a (possibly empty) sequence of edge contractions and label contractions transforming G into H.

Subdivisions. A subdivision of a graph H, or H-subdivision, is a graph obtained from H by edge subdivisions. The vertices added during this process are called *subdivision vertices.*

Containing K_4-subdivisions. A graph G contains K_4 as an induced minor if and only if G contains K_4-subdivision as a subgraph. This equivalence is highly specific to the graph K_4 and in general neither implication would be true. We will freely change between those two notions for containing K_4, depending on which one is more convenient in the given context.

A graph G will be said to contain a *proper K_4 subdivision*, if there is some vertex $v \in V(G)$, such that $G \setminus v$ contains a K_4-subdivision.

4 Antichains for induced minors

An infinite antichain is an obstruction for a quasi-order to be a wqo. As we will see in Section 5, the study of infinite antichains can provide helpful information when looking for graphs H such that $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo. In this section we present enumerate some of the known infinite antichains for induced minors.

In 1985, Thomas [24] presented an infinite sequence of planar graphs (also mentioned later in [22]), that is an antichain for induced minors. This shows that induced minors do not well-quasi-order planar graphs. The elements of this antichain, called *alternating double wheels*, are constructed from an even cycle by adding two nonadjacent vertices and connecting one to one color class of the cycle, and connecting the other vertex to the other color class (cf. Figure 2 for the three first such graphs). This infinite antichain shows that $(\mathrm{Excl}_{\mathrm{im}}(K_5), \leq_{\mathrm{im}})$ is not a wqo since no alternating double wheel contains K_5 as (induced) minor. As a consequence, $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is not a wqo as soon as H contains K_5 as induced minor.

Therefore, in the quest for all graphs H such that $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is wqo, we can focus the cases where H is K_5-induced minor-free.

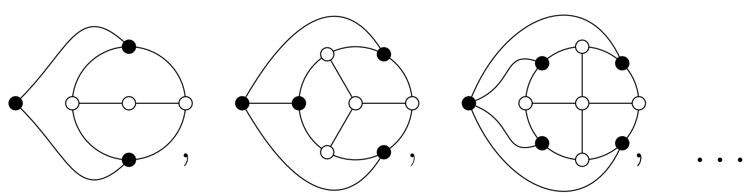

Figure 2: Thomas' alternating double wheels.

The infinite antichain $\mathcal{A}_M$ depicted in Figure 3 was introduced in [18], where it is also proved that none of its members contains K_5^- as induced minor. Similarly as the above remark, it follows that if $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo then $K_5^- \not\leq_{\mathrm{im}} H$. Notice that graphs in this antichain have bounded maximum degree.

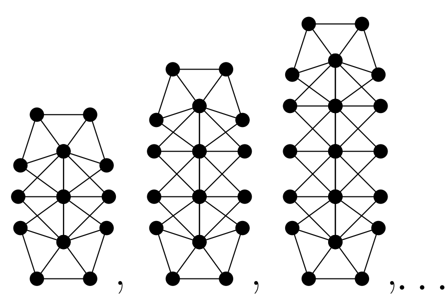

Figure 3: The infinite antichain $\mathcal{A}_M$ of Matoušek, Nešetřil, and Thomas.

An *interval graph* is the intersection graph of segments of $\mathbb{R}$. A well-known property of interval graphs that we will use later is that they do not contains C_4 as induced minor.

In order to show that interval graphs are not wqo by $\leq_{\mathrm{im}}$, Ding introduced in [6] an infinite sequence of graphs defined as follows. For every $n \in \mathbb{N}$, $n > 2$, let T_n be the set of closed intervals

- $[i, i]$ for i in $[\![-2n, -1]\!] \cup [\![1, 2n]\!]$;
- $[-2, 2]$, $[-4, 1]$, $[-2n+1, 2n]$, $[-2n+1, 2n-1]$;
- $[-2i+1, 2i+1]$ for i in $[\![1, n-2]\!]$; and
- $[-2i, 2i-2]$ for i in $[\![3, n]\!]$.

Figure 4 depicts the intervals of T_6: the real axis (solid line) is folded up and an interval $[a, b]$ is represented by a dashed line between a and b.

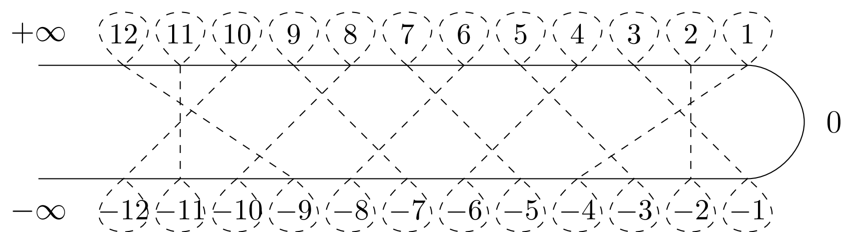

Figure 4: An illustration of the intervals in T_6.

For every $n \in \mathbb{N}$, $n > 2$, let A_n^D be the intersection graph of segments of T_n. Let $\mathcal{A}_D = \langle A_n^D \rangle_{n>2}$. Ding proved in [6] that $\mathcal{A}_D$ is an antichain for $\leq_{\mathrm{im}}$, thus showing that interval graphs are not wqo by induced minors.

Let us now present two infinite antichains that were, to our knowledge, not mentioned elsewhere earlier. Let $A_{\overline{C}} = \langle \overline{C_n} \rangle_{n \geq 3}$ be the sequence of *antiholes*, whose first elements are represented in Figure 5.

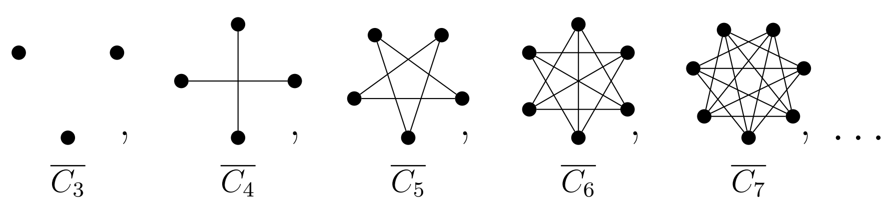

Figure 5: Antiholes antichain.

Lemma 1. *If $H \leq_{\mathrm{im}} \overline{C_n}$ and $|V(H)| < n$ for some integer $n \geq 3$, then $\overline{H}$ is a linear forest.*

Proof. Towards a contradiction, let us assume that $\overline{H}$ is not a linear forest.

First case: $\overline{H}$ has a vertex v of degree at least 3. Let x, y, z be three neighbors of v. In the graph $H[\{v, x, y, z\}]$, the vertex v is adjacent to none of x, y, z. In an antihole, every vertex has exactly two non-neighbors, so $H[\{v, x, y, z\}]$ is not an induced minor of any element of $\mathcal{A}_{\overline{C}}$. In particular, $H \not\leq_{\mathrm{im}} \overline{C_n}$, a contradiction.

Second case: $\overline{H}$ contains an induced cycle as an induced subgraph. Then for some integer $n' \geq 3$, we have $\overline{C_{n'}} \leq_{\mathrm{im}} H \leq_{\mathrm{im}} \overline{C_n}$. Let μ be an induced minor model of $\overline{C_{n'}}$ in $\overline{C_n}$. Let u, v, w be three vertices of $\overline{C_{n'}}$ that appear consecutively and in this order in its complement, $C_{n'}$. Notice that v is adjacent to none of u, w in $\overline{C_{n'}}$. Therefore, there is no edge from $\mu(v)$ to any of $\mu(u)$ and $\mu(w)$ in $\overline{C_n}$. Since a vertex of $\overline{C_n}$ has exactly two non-neighbors and distinct vertices have different sets of non-neighbors, we deduce that $\mu(u), \mu(v), \mu(w)$ are singletons and that they contain vertices that are consecutive (in this order) on the cycle of the complement of $\overline{C_n}$. Applying this argument for every triple such as u, v, w implies the existence of a cycle of order n' in C_n, a contradiction. $\square$

Corollary 1. *$\mathcal{A}_{\overline{C}}$ is an antichain.*

We will meet again the antichain $\mathcal{A}_{\overline{C}}$ in the proof of Theorem 1. Another infinite antichain which shares with $\mathcal{A}_M$ the properties of planarity and bounded maximum degree is depicted in Figure 6. We will not go more into detail about it here as this antichain will be of no use in the rest of the paper.

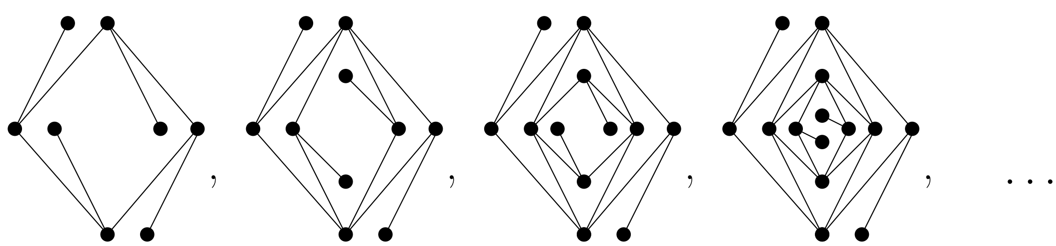

Figure 6: Nested lozenges.

5 The dichotomy theorem

The purpose of this section is to prove Theorem 1, that is, to characterize all graphs H such that $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo. To this end, we will assume Theorem 4 and Theorem 5, which we will prove later, in Section 6 and Section 7 respectively.

The main ingredients of the proof are the infinite antichains presented in Section 4, together with Theorem 4 and Theorem 5. Infinite antichains will be used to discard every graph H that is not induced minor of all but finitely many elements of some infinite antichain. On the other hand, knowing that $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo gives that $(\mathrm{Excl}_{\mathrm{im}}(H'), \leq_{\mathrm{im}})$ is a wqo for every $H' \leq_{\mathrm{im}} H$ in the virtue of the following remark.

Remark 3. For every H, H' such that $H' \leq_{\text{im}} H$, we have $\text{Excl}_{\text{im}}(H') \subseteq \text{Excl}_{\text{im}}(H)$.

As a consequence of Lemma 1, if $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$ is a wqo then $\overline{H}$ is a linear forest. Because this statement concerns the complement of H, we will be led below to work with this graph rather than with H. The following lemma presents step by step the properties that we can deduce on $\overline{H}$ by assuming that $\text{Excl}_{\text{im}}(H)$ is wqo by $\leq_{\text{im}}$.

Lemma 2. *If* $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$ *is a wqo, then we have*

(R1) $\overline{H}$ *has at most 4 connected components;*

(R2) at most one connected component of $\overline{H}$ *is not a single vertex;*

(R3) the largest connected component of $\overline{H}$ *has at most 4 vertices;*

(R4) if $n = |V(H)|$ *and* $c = \text{cc}(\overline{H})$ *then* $n \leq 7$ *and* $\overline{H} = (c-1) \cdot K_1 + P_{n-c+1}$*;*

(R5) if $\text{cc}(\overline{H}) = 3$ *then* $|V(H)| \leq 5$.

(R6) if $\text{cc}(\overline{H}) = 4$ *then* $|V(H)| \leq 4$.

Proof. Proof of item (R1). The infinite antichain $\mathcal{A}_M$ does not contain K_5 as (induced) minor, hence $K_5 \not\leq_{\text{im}} H$ and so $\overline{H}$ does not contain $5 \cdot K_1$ as induced minor. Therefore it has at most 4 connected components.

Proof of items (R2) and (R3). The infinite antichain $\mathcal{A}_D$ does not contain C_4 as induced minor (as it is an interval graph), hence neither does H. Therefore $\overline{H}$ does not contain $2 \cdot P_2$ as induced minor. This implies that $\overline{H}$ does not contain P_5 as induced minor and that given two connected components of $\overline{H}$ at least one must be of order one. As connected components of H are paths (by Lemma 1), the largest connected component of H has order at most 4.

Item (R4) follows from the above proofs and from the fact that $\overline{H}$ is a linear forest.

Proof of item (R5). Similarly as in the proof of item (R1), $\mathcal{A}_M$ does not contain K_5^- as induced minor so $\overline{K_5^-} = K_2 + 3 \cdot K_1$ is not an induced minor of $\overline{H}$. If we assume that $\text{cc}(\overline{H}) = 3$ and $|V(H)| \geq 6$ vertices, the largest component of $\overline{H}$ is a path on (a least) 4 vertices, so it contains $K_1 + K_2$ as induced subgraph. Together with the two other (single vertex) components, this gives an $K_2 + 3 \cdot K_1$ induced minor, a contradiction.

Proof of item (R6). Let us assume that $\text{cc}(\overline{H}) = 4$. If the largest connected component has more than one vertex, then $\overline{H}$ contains $K_2 + 3 \cdot K_1$ induced minor, which is not possible (as in the proof of item (R5)). Therefore $\overline{H} = 4 \cdot K_1$ and so $\left|V(\overline{H})\right| = 4$. $\square$

We are now able to describe more precisely graphs H for which $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$ could be a wqo. Let K_3^+ be the complement of $P_3 + K_1$ and let K_4^- be the complement of $K_2 + 2 \cdot K_1$, which is also the graph obtained from K_4 by deleting an edge (sometimes referred as *diamond graph*).

Lemma 3. *If* $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ *is a wqo, then* $H \leq_{\mathrm{im}} \widehat{K}_4$ *or* $H \leq_{\mathrm{im}} \mathrm{Gem}$*.*

Proof. Using the information on $\overline{H}$ given by Lemma 2, we can build a table of possible graphs $\overline{H}$ depending on $\mathrm{cc}(\overline{H})$ and $|V(\overline{H})|$. Table 1 is such a table: each column corresponds for a number of connected components (between one and four according to item (R1)) and each line corresponds to an order (at most seven, by item (R4)). A grey cell means either that there is no such graph (for instance a graph with one vertex and two connected components), or that for all graphs $\overline{H}$ matching the number of connected components and the order associated with this cell, the poset $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is not a wqo.

$\lvert V(H)\rvert \setminus \mathrm{cc}(\overline{H})$	1	2	3	4
1	K_1			
2	K_2	$2 \cdot K_1$		
3	P_3	$K_2 + K_1$	$3 \cdot K_1$	
4	P_4	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$
5	(R3)	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 1: If $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo, then $\overline{H}$ belongs to this table.

From Table 1 we can easily deduce Table 2 of corresponding graphs H.

$\lvert V(H)\rvert \setminus \mathrm{cc}(\overline{H})$	1	2	3	4
1	K_1			
2	$2 \cdot K_1$	K_2		
3	$K_2 + K_1$	P_3	K_3	
4	P_4	K_3^+	K_4^-	K_4
5	(R3)	Gem	$\widehat{K}_4$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 2: If $(\mathrm{Excl}_{\mathrm{im}}(H), \leq_{\mathrm{im}})$ is a wqo, then H belongs to this table.

Observe that we have

- $K_1 \leq_{\mathrm{im}} 2 \cdot K_1 \leq_{\mathrm{im}} K_2 + K_1 \leq_{\mathrm{im}} P_4 \leq_{\mathrm{im}} \mathrm{Gem}$;
- $K_2 \leq_{\mathrm{im}} P_3 \leq_{\mathrm{im}} K_3^+ \leq_{\mathrm{im}} \mathrm{Gem}$;
- $K_3 \leq_{\mathrm{im}} K_4^- \leq_{\mathrm{im}} \widehat{K}_4$; and

- $K_4 \leq_{\text{im}} \widehat{K}_4$.

This concludes the proof.

□

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. If $H \not\leq_{\text{im}}$ Gem and $H \not\leq_{\text{im}} \widehat{K}_4$, then by Lemma 3 $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$ is not a wqo. On the other hand, by Theorem 4 and Theorem 5 we know that both $\text{Excl}_{\text{im}}(\widehat{K}_4)$ and $\text{Excl}_{\text{im}}(\text{Gem})$ are wqo by $\leq_{\text{im}}$. Consequently, by Remark 3, $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$ is wqo as soon as $H \leq_{\text{im}}$ Gem or $H \leq_{\text{im}} \widehat{K}_4$. □

6 Graphs not containing $\widehat{K}_4$

The main goal of this section is to provide a proof to Theorem 4. To this purpose, we first prove in Section 6.1 that graphs of $\text{Excl}_{\text{im}}(\widehat{K}_4)$ admit a simple structural decomposition (Theorem 2). This structure is then used in Section 6.2 to show that graphs of $\text{Excl}_{\text{im}}(\widehat{K}_4)$ are well-quasi-ordered by the relation $\leq_{\text{im}}$.

6.1 A decomposition theorem for $\text{Excl}_{\text{im}}(\widehat{K}_4)$

This section is devoted to the proof of Theorem 2. This theorem states that every graph in the class $\text{Excl}_{\text{im}}(\widehat{K}_4)$, either does not contain K_4 as induced minor, or is a subdivision of some small graph, or can be partitioned into "simple" parts.

The proof is split into several parts. Recall that a *proper K_4-subdivision* in a graph G is a K_4-subdivision that does not uses all vertices. First we show (Section 6.1.1) that if G is neither a subdivision of a small graph, nor a wheel, nor a K_4-subdivision free graph, then G has a proper K_4-subdivision S. We then deduce properties about vertices of G that do not belong to S (Section 6.1.2). In Section 6.1.3, we handle the case where S is in fact a K_4-subgraph and show that in the other cases, we can focus on the case where S is a 3-wheel. The last part of the proof is addressed in Section 6.1.4.

6.1.1 Finding a proper K_4-subdivision

In this section we show that, unless G has a simple structure, it contains a proper K_4-subdivision. The proof of the following easy lemma is left to the reader.

Lemma 4. *If G can be obtained by adding an edge between two vertices of a $K_{3,3}$-subdivision (resp. a prism-subdivision), then G has a proper K_4-subdivision.*

Lemma 5. *If graph G contains a K_4-subdivision, then one of the following holds:*

(i) G has a proper K_4-subdivision, or

(ii) G is a wheel, or

(iii) G is a subdivision of K_4, $K_{3,3}$, or the prism.

Proof. Looking for a contradiction, let G be a counterexample with the minimum number of vertices and, subject to that, the minimum number of edges. Let S be a K_4-subdivision in G. As G has no proper K_4-subdivision, S is a spanning subgraph. Besides, G is not a K_4-subdivision so there is an edge $e \in E(G) \setminus E(S)$. Notice that since the minimum degree of K_4 is 3, contracting an edge incident with a vertex of degree 2 in G would yield a smaller counterexample. Therefore G has minimum degree at least 3. Let $G' = G \setminus \{e\}$. This graph clearly contains S. By minimality of G, the graph G' is either a wheel, or a subdivision of a graph among K_4, $K_{3,3}$, and the prism. Observe that G' cannot have a proper K_4-subdivision because it would also be a proper K_4-subdivision in G.

First case: G' is a wheel. Let C be the cycle of the wheel and let r be its center. In G the edge e does not have r as an endpoint, because otherwise G would also be a wheel. Therefore e is incident with two vertices of C. Let P and P' be the two subpaths of C whose endpoints are the endpoints of e. Observe that none of P and P' contains more than two neighbors of r. Indeed, if, say, P contained at least three neighbors of r, then the subgraph of G induced by the vertices of P, e, an r would contain a K_4-subdivision, hence contradicting the fact that G has no proper K_4-subdivision.

Therefore G is the cycle C with exactly one chord, e, and the vertex r which has at most 4 neighbors on C. Because G has minimum degree at least 3, it has at most 7 vertices. We can easily check that if r has three neighbors on C, then either one of P and P' contain exactly one of them, in which case G is a subdivision of the prism, or both contain two of them (one being contained in both P and P') and G is a wheel (with a center which is the neighbor of C lying on both P and P'). If r has four neighbors on C, the interior of P and P' must each contain two of them according to the above remarks. The deletion of any neighbor of r in this graph yields a K_4-subdivision of non-subdivision vertices r and the remaining neighbors. Observe that both cases contradict the assumptions made on G.

Second case: G' is a subdivision of K_4, or $K_{3,3}$, or the prism. In the two latter cases the result follows by Lemma 4. We therefore assume that G' is a subdivision of K_4. A *branch* of S is a maximal path, the internal vertices of which have degree two (in the subgraph S). Notice that every branch of S is chordless in, otherwise one could shortcut it and thus find a proper K_4-subdivision in G. In the case where the endpoints of e belong to the interior of two different branches, then it is easy to see that G is a prism-subdivision if these branches share a vertex and a subdivision of $K_{3,3}$ otherwise. Let $\{x, y, z, t\}$ be the non-subdivision vertices of the K_4-subdivision. We denote by $B_{s,t}$ (for $s, t \in \{x, y, z, t\}$) the branch ending at vertices s and t. Finally, let us assume that the one endpoint of e is a non-subdivision vertex, say x, and the other one, that we call u, is a subdivision vertex of a branch, say $B_{y,z}$. If X is the set of interior vertices of one of $B_{x,y}$, $B_{x,z}$, or $B_{x,t}$, then the graph $G \setminus X$ has a K_4 subdivision of non-subdivision vertices x, u, z, t, x, y, y, t or x, y, u, z respectively. In this case G has a proper K_4-subdivision. If none of $B_{x,y}$, $B_{x,z}$, and $B_{x,t}$ has internal vertices, then G is a wheel of center x.

In all the possible cases we reached the contradiction we were looking for. This concludes the proof. □

6.1.2 On the neighbors of proper K_4-subdivisions

The outcomes (ii) and (iii) of Lemma 5 match possible outcomes of Theorem 2, as noted in its proof in the end of Section 6.1.4, therefore we now focus on the case where G has a proper K_4-subdivision. The following lemma describes a structure that forces $\widehat{K}_4$-induced minors and will be used to deduce properties of $\widehat{K}_4$-induced minor-free graphs.

Lemma 6. *If G contains as induced minor any graph H consisting of:*

- *a K_4-subdivision S;*
- *an extra vertex x linked by exactly two paths L_1 and L_2 to two distinct vertices $s_1, s_2 \in V(S)$, where the only common vertex of L_1 and L_2 is x;*
- *and possibly extra edges between the vertices of S, or between L_1 and L_2, or between the interior of the paths and S,*

then $\widehat{K}_4 \leq_{\mathrm{im}} G$.

Proof. Let us call $V = \{v_1, v_2, v_3, v_4\}$ the non-subdivision vertices of S, i.e. vertices corresponding to vertices of K_4. We present here a sequence of edge contractions in H leading to $\widehat{K}_4$. As long as there is a path between two elements of $V \cup \{s_1, s_2, x\}$, internally disjoint with this set, we contract the whole path to a single edge.

Once we cannot apply this contraction any more, we end up with a graph that has two parts: the K_4-subdivision with at most 2 subdivisions (with vertex set $V \cup \{s_1, s_2\}$) and the vertex x, which is now only adjacent to s_1 and s_2.

First case: $s_1, s_2 \in V$. The graph H is isomorphic to $\widehat{K}_4$: it is K_4 plus a vertex of degree two.

Second case: $s_1 \in V$ and $s_2 \notin V$ (and the symmetric case). As vertices of V are the only vertices of H that have degree 3 in S, s_2 is of degree 2 in S (it is introduced by subdivision). The contraction of the edge between s_2 and one of its neighbors in S that is different from s_1 leads to the first case.

Third case: $s_1, s_2 \notin V$. As in second case, these two vertices have degree two in S. Since no two different edges of K_4 can have the same endpoints, the neighborhoods of s_1 and s_2 have at most one common vertex. Then for every $i \in \{1, 2\}$ there is a neighbor t_i of s_i that is not adjacent to s_{3-i}. Contracting the edges $\{s_1, t_1\}$ and $\{s_2, t_2\}$ leads to the first case. □

Corollary 2 (from Lemma 6)**.** *Let $G \in \mathrm{Excl}_{\mathrm{im}}(\widehat{K}_4)$ be a 2-connected graph containing a proper K_4-subdivision S. For every vertex $x \in V(G) \setminus V(S)$, $N_S(x) \geq 3$.*

Proof. As G is 2-connected, we can find when $N_S(x) \leq 2$ two paths from x to S that satisfy the conditions of Lemma 6. Therefore, $N_S(x) \geq 3$. □

6.1.3 Small K_4-subdivisions

We handle separately the case where G contains a subgraph isomorphic to K_4.

Lemma 7. *Let $G \in \mathrm{Excl}_{\mathrm{im}}(\widehat{K}_4)$ be a 2-connected graph. If G has a subgraph S that is isomorphic to K_4, then $G \setminus V(S)$ is complete multipartite.*

Proof. Let us first show that for every non-adjacent $u, v \in V(G) \setminus V(S)$, $N_S(u) = N_S(u)$. According to Corollary 2, each of u and v has at least 3 neighbors in S, hence $3 \leq |N_S(u)|, |N_S(v)| \leq 4$. If $|N_S(u)| = |N_S(v)| = 4$, then $N_S(u) = N_S(v)$ and we are done. Towards a contradiction, we assume $N_S(u) \neq N_S(v)$. In the case where $|N_S(u)| = |N_S(v)| = 3$ (resp. $|N_S(u)| = 3$ and $|N_S(u)| = 4$, or the other way around), we have $\widehat{K}_4 \leq_{\mathrm{im}} G$ as depicted on Figure 7.(a), a contradiction. Let us now show that

(a) (b)

Figure 7: Finding $\widehat{K}_4$ as induced minor in the proof of Lemma 7. Vertices of S are black and u, v, w are white. Dotted lines represent edges that may be present or not. The numbers indicate which vertices of $\widehat{K}_4$ (following the convention of Figure 1) correspond to the subsets of vertices depicted in blue.

$G \setminus V(S)$ is complete multipartite. A graph is complete multipartite iff it does not contain $K_1 + K_2$ as induced subgraph. Towards a contradiction, we therefore assume that there are three vertices $u, v, w \in V(G) \setminus V(S)$ such that $\{u, v\}$ is the only edge in $G[\{u, v, w\}]$. According to the paragraph above applied to u and w and then to w and v, we have $N_S(u) = N_S(v) = N_S(w)$. As noted above, each of u, v and w have at least three neighbors on S. In this case again we are able to find $\widehat{K}_4$ as an induced minor (in fact, as an induced subgraph), as depicted in Figure 7.(b). This is a contradiction, hence $G \setminus (V(S) \setminus s)$ is complete multipartite. □

Notice that the conclusion of Lemma 7 is an outcomes of Theorem 2. We now show that some minimum K_4-subdivision is a 3-wheel. This will allow us to focus on the case where S is a 3-wheel.

Lemma 8. *If $G \in \mathrm{Excl}_{\mathrm{im}}(\widehat{K}_4)$ is a 2-connected graph with a proper K_4-subdivision, then some K_4-subdivision of G with the minimum number of vertices is a 3-wheel.*

Proof. We show that for every proper K_4-subdivision S of G, we can find a 3-wheel with at most the same number of vertices. Let $x \in V(G) \setminus V(S)$ and let V be as in the proof of Lemma 6. We call two neighbors of x in S *equivalent* if they lie on the same path between two elements of V. Intuitively, equivalent vertices correspond to the same edge of K_4. By Corollary 2, we can assume $|N_S(x)| \geq 3$.

We start with the case where no cycle of S contains three neighbors of x. Notice that for every pair of edges of K_4, there is a cycle using these edges. This implies that x does neither have two equivalent neighbors, nor a neighbor in V. Besides, for every choice of four edges of K_4, there is a Hamiltonian cycle containing three of them. We deduce $|N_S(x)| = 3$. Let us consider the induced minor H of $G[V(S) \cup \{x\}]$ obtained by iteratively contracting all edges that are incident with at most one vertex of $V \cup N_S(x) \cup \{x\}$. By the above remark and as x does not have three neighbors on a cycle, there is a vertex of $V(H) \setminus \{x\}$ adjacent to the three neighbors of x. Contracting two of the edges incident with this vertex merges two neighbors of x and the graph we obtain is a K_4 subdivision (corresponding to S) together with a vertex of degree 2 (corresponding to x). By Lemma 6, we have $\widehat{K}_4 \leq_{\mathrm{im}} G$, a contradiction.

In the remaining case, S has a cycle containing three neighbors of x. Let C be such a cycle with the minimum number of vertices. Notice that $V(S) \setminus V(C)$ is then not empty, hence $C \cup \{x\}$ is the desired wheel and does not have more vertices than S. $\square$

6.1.4 Dealing with proper K_4-subdivisions

In the sequel, we deal with a graph G that is 2-connected and has a proper K_4-subdivision, but contains neither $\widehat{K}_4$ as induced minor, nor K_4 as subgraph. Let us consider a subgraph S of G that is a 3-wheel with the minimum number of vertices and, subject to this requirement, has a minimum number of chords (i.e. edges of $E(G) \setminus E(S)$ with both endpoints in $V(S)$). We denote by C the cycle of this 3-wheel, by r its center, and set $R = G \setminus V(C)$. These assumptions are implicit in the following lemmas.

As S is not necessarily an induced subgraph of G, the cycle C may have chords. We consider this case hereafter.

Lemma 9. *If C is not an induced cycle in G, then $G[V(S)]$ is the prism.*

Proof. Let $u, v, w \in N_C(r)$ be three distinct neighbors of r in C and let C_u be the path of C between v and w that does not contain the vertex u, and similarly for C_v and C_w. Let us assume that C has a chord $\{x, y\}$. Observe that x and y cannot both belong to C_l for some $l \in \{u, v, w\}$, as the deletion of any interior vertex of xC_ly (which exist as $\{x, y\}$ is a chord of C) would leave a K_4-subdivision in S, contradicting its minimality (see Figure 8.(a)). Therefore, x and y belong to different C_l's, say without loss of generality that $x \in V(C_w)$ and $y \in V(C_v)$.

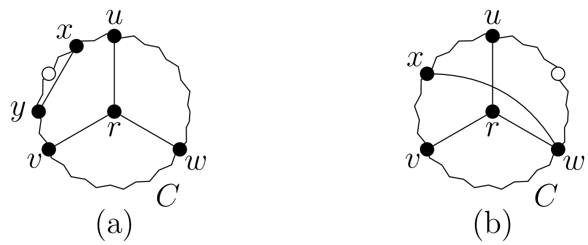

Figure 8: When C has a chord. In these examples, there is a K_4-subdivision that does not use the white vertex, which contradicts the minimality of S. Zigzag lines depict paths with at least one edge.

First case: $y = w$ and x belongs to the interior of C_w. Observe that if one of C_u or C_v has an internal vertex z, then $S \setminus z$ still has a K_4-subdivision (see Figure 8.(b)), which would contradict the minimality of S. Hence each of C_u and C_v is reduced to an edge. Now, notice that if r is adjacent to an internal vertex of C_w, then again one can find a smaller K_4-subdivision, for instance by deleting u. The path C_w together with r forms the cycle C' of a 4-wheel of center w. As noticed above, C_w is chordless and r has no neighbors on this path. Therefore C' is chordless: it has fewer chords than C. This contradicts the definition of S, hence this case is not possible.

Second case: x and y are interior vertices of C_w and C_v, respectively. We first show that $|C| = 5$. As in the previous case, it is easy to see that if one of uC_wx, xC_wv, uC_vy, yC_vw or C_u has an internal vertex z, then $S \setminus z$ is not K_4-subdivision free. As this contradicts the definition of S, we deduce that each of them is reduced to an edge, proving that $|C| = 5$. In the light of the previous case, no endpoints of a chord of C can belong to $\{u, v, w\}$. As C has 5 vertices, it has only one chord. This concludes the proof. □

Corollary 3. *If C has a chord, then G is the prism.*

Proof. From Lemma 9, we get that S is the prism. Observe that for every choice of three vertices of the prism, there is a cycle of length at most 4 containing them. Let $v \in V(G) \setminus V(S)$. The vertex v has at least 3 neighbors in S (Corollary 2), thus it is the center of a 3-wheel on size at most 5, using a cycle as mentioned above. This contradicts the minimality of S, the prism, which has six vertices. Hence we deduce that $V(G) \setminus V(S)$ is empty: G is the prism. □

The case where C has chords being fixed, we assume in the remaining of this section that C is a chordless cycle.

Lemma 10. *If for some $t \in V(G) \setminus V(S)$, $N_C(t) \leq 2$, then $|C| = 4$ and every $t' \in V(G) \setminus V(S)$ such that $N_C(t') \leq 2$ has the same neighborhood on S as t.*

Proof. Let $t \in V(G) \setminus V(S)$. As $N_S(t) \geq 3$ (Corollary 2) and $N_C(v) \leq 2$, we deduce that $r \in N_S(t)$ and $|N_S(t)| = 3$. Let x, y be the two neighbors of t on C. We define u, v, w, C_u, C_v, and C_w as in the proof of Lemma 9. We consider different cases according to the positions of x and y.

First case: both x and y belong to one of C_u, C_v, C_w, say C_u, without loss of generality, at least one of them being in the interior of the path. Then contracting the subpath of C_u that links x to y yield a graph that has a vertex of degree 2, t, with exactly two neighbors on a K_4-subdivision, contradicting Corollary 2. Hence this case is not possible.

Second case: $x, y \in \{u, v, w\}$, say $x = u$ and $y = v$, without loss of generality. Observe that if one of C_v or C_u has an interior vertex, then the graph induced by r, t and $V(C_w)$ has a K_4-subdivision that is smaller than S, a contradiction. We deduce that each of these paths is reduced to an edge. In order to show the same thing for C_w, we assume that there is an interior vertex z to C_w. Note that $G[u, v, w, r, t]$ has a K_4-subdivision. As C is induced, z has degree 2 in S. Then it has at most two neighbors in the aforementioned K_4-subdivision, a contradiction to Corollary 2. Therefore, C_w is an edge. We deduce that $|S| = 4$. That is, S is a K_4-subgraph, where we assumed the opposite. Consequently, this case is not possible either.

Third case: x and y belong to the interior of two different paths among C_u, C_v, and C_w, say without loss of generality C_u and C_v, respectively. Then by contracting the subpath of C_u linking x to w, we reach the first case.

Fourth case: for some $z \in \{u, v, w\}$, x belongs to the interior of C_z and $y = z$. Without loss of generality we assume that $z = w$. As in the previous lemmas, it is easy to check that if one of C_u, C_v, uC_wx, and xC_wv has an interior vertex, then deleting it and deleting v, u, u, or v, respectively, does not make $G[V(S) \cup \{t\}]$ K_4-subdivision free. This contradicts the minimality of S, hence each of these paths is reduced to an edge. We deduce $|C| = 4$. In the light of the previous remarks and as $|C| = 4$, the only possible neighbors for a vertex t' as in the statement of the lemma are x and y, which concludes the proof. $\square$

We can now focus on vertices of $V(G) \setminus V(S)$ that are adjacent to at least three vertices of C.

Lemma 11. *If some $s \in V(G) \setminus V(S)$ satisfies $|N_C(s)| \geq 3$, then $N_C(s) = N_C(r)$.*

Proof. Towards a contradiction, we assume that some $u \in V(C)$ is adjacent to r but not to s. As s and r play a symmetric role, this is the only case to consider.

Let v, w (both distinct from u) and u', v', w' be neighbors of r and s on C, respectively. We consider the graph H obtained from $G[V(C) \cup \{r, s\}]$ by iteratively contracting every edge of C that is not incident with two vertices of $\{u, v, w, u', v', w'\}$. This graph is an induced cycle (as C is induced) on at most 6 vertices, that we call C', plus the two vertices r and s that have at least three neighbors each on C. Observe that while two

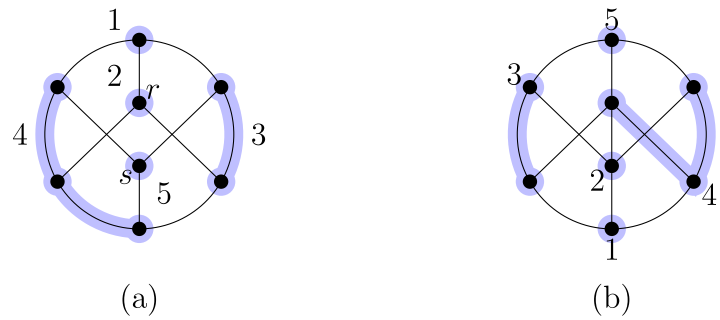

Figure 9: Models of $\widehat{K}_4$ in the proof of Lemma 11. The numbers indicate which vertices of $\widehat{K}_4$ (following the convention of Figure 1) correspond to the subsets of vertices depicted in blue.

neighbors of s are adjacent and are not both neighbors of r, we can contract the edge between them and decrease by one the degree of s, without changing degree of r. If the degree of s reaches two by such means, then by Lemma 6, $\widehat{K}_4 \leq_{\text{im}} H$, a contradiction. We can thus assume that every vertex of C' adjacent to a neighbor of s is a neighbor of r. This is also true when r and s are swapped since this argument can be applied to r too. This observation implies that $N_S(r) \cap N_S(s) = \emptyset$ (as u is adjacent to r but not to s, none of its neighbors on C can be adjacent to r, and so on along the cycle) and that the neighbors of r and s are alternating on C'. Without loss of generality, we suppose that $C' = uu'vv'ww'$. Figure 9.(a) and Figure 9.(b) shows how a model of $\widehat{K}_4$ can then be found in this case, depending whether $\{r, s\} \in E(G)$, respectively, a contradiction. □

Lemma 12. *$G \setminus V(S)$ is complete multipartite.*

Proof. Let us consider the graph obtained from G by contracting S to K_4. Observe that this does not impact the adjacencies in $G \setminus V(S)$. The result then follows from Lemma 7. □

Lemma 13. *Either $|C| \leq 4$ or $V(G) \setminus V(C)$ is an independent set.*

Proof. Assuming that $|C| \geq 5$, let us show that $V(G) \setminus V(C)$ is an independent set. By Lemma 10 and Lemma 11, the vertices of $V(G) \setminus V(C)$ all have the same neighborhood on C, which has size at least 3. Towards a contradiction, let us assume that there exist two adjacent vertices $x, y \in V(G) \setminus V(C)$. We define u, v, w, C_u, C_v, and C_w as in the proof of Lemma 9. Observe that the graph induced by x, y, and C_w contains a K_4-subdivision. We deduce that none of C_u and C_v contains an internal vertex, otherwise the deletion of this vertex and w would produce a graph violating the minimality of S. Symmetrically, C_w has no internal vertex. Hence $|C| = 3$, a contradiction. This proves that $V(G) \setminus V(C)$ is an independent set. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $G \in \mathrm{Excl}_{\mathrm{im}}(\widehat{K}_4)$ be a 2-connected graph. If G does not contain a K_4-subdivision, or if G is a subdivision of K_4, $K_{3,3}$, then we are done (outcomes (i) or (ii) of the theorem). If G contains a K_4-subdivision but not a proper one, from Lemma 5 we get that G is a subdivision of one of K_4, $K_{3,3}$, or the prism, in which case the theorem holds (outcome (ii)), or that G is a wheel, which has a trivial partition $(V(G) \setminus \{r\}, \{r\})$ (where r is the center of the wheel) satisfying item (iv) of the statement of the theorem.

Therefore we assume that G does not fall in one of the aforementioned cases. By Lemma 5, G then has a proper K_4-subdivision. If G has a K_4-subgraph S, then $(V(S), V(G) \setminus V(S))$ is a partition satisfying item (iii) of the desired statement, according to Lemma 7.

We now focus on the case where G has no K_4-subgraph and we consider a minimal 3-wheel as defined at the beginning of Section 6.1.4, using the same notation. The case where C is not induced is not possible as we assume that G is not a prism (cf. Corollary 3). If $V(G) \setminus V(S)$ has a vertex t such that $|N_C(t)| = 2$, then, by the virtue of Lemma 10 and Lemma 12, $|S| = 5$ and the partition $(V(S), V(G) \setminus V(S))$ suits the requirements of (iii). Otherwise, every vertex $t \in V(G) \setminus V(S)$ satisfies $|N_C(t)| \geq 3$. Then, by Lemma 11 and Lemma 13, $G \setminus V(S)$ is an independent set and its vertices have the same neighborhood on C. Therefore, the partition $(V(C), V(G) \setminus V(C))$ satisfies the conditions of outcome (iv). □

6.2 From a decomposition theorem to well-quasi-ordering

This section is devoted to the proof of Theorem 4. We define the two following classes of graphs:

- $\mathcal{WM}$ is the class of the graphs that admit a partition (W, M) of their vertex set such that W induces a wheel on at most 5 vertices and M a complete multipartite graph;
- $\mathcal{CI}$ is the class of the graphs that admit a partition (C, I) of their vertex set such that C induces a cycle, I is an independent set, and every vertex of I has the same neighborhood on C.

These classes respectively correspond to the outcomes (iii) and (iv) of Theorem 2. Our proof of Theorem 4 relies on the two following lemmas which are proved in the next sections.

Lemma 14. *For every (unlabeled) graph G and every wqo $(\Sigma, \preceq)$, the class of $(\Sigma, \preceq)$-labeled G-subdivisions is well-quasi-ordered by contractions.*

Lemma 15. *For every wqo $(\Sigma, \preceq)$, the classes $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{CI})$ and $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{CI})$ are wqo by induced minors.*

We also use the following result by Thomas.

Proposition 3 ([24])**.** *For every wqo* $(\Sigma, \preceq)$*, the class of* $(\Sigma, \preceq)$*-labeled* K_4*-induced minor-free graphs is wqo by induced minors.*

We now show that $\widehat{K}_4$-induced minor-free graphs are wqo by induced minors.

Proof of Theorem 4. According to Proposition 1, it is enough to show that for every wqo $(\Sigma, \preceq)$, the class of $(\Sigma, \preceq)$-labeled 2-connected graphs not containing $\widehat{K}_4$ as induced minor is wqo by induced minors. By Theorem 2, this class can be divided into three subclasses:

- K_4-induced minor-free graphs;
- subdivisions of a graph among K_4, $K_{3,3}$, and the prism;
- graphs of $\mathcal{WM} \cup \mathcal{CI}$.

Proposition 3, Lemma 14, and Lemma 15 respectively handle these three cases. Since it is a finite union of wqos, the class of $(\Sigma, \preceq)$-labeled $\widehat{K}_4$-induced minor free graphs is a wqo as well (wrt. induced minors). This concludes the proof. □

The following sections contain the proofs of Lemma 14 and Lemma 15. The technique that we repeatedly use in order to show that a poset $(A, \preceq_A)$ is a wqo is the following:

1. we define a function $f\colon A' \to A$. Intuitively, elements of A' can be seen as descriptions (or encodings) of objects of A and f is the function constructing the objects from the descriptions;
2. we show that A' is wqo by some relation $\preceq_{A'}$. Usually, A' is a product, union or sequence over known wqos so this can be done using Proposition 2;
3. we prove that $f\colon (A', \preceq_{A'}) \to (A, \preceq_A)$ is monotone and surjective (sometimes using Remark 1) and by Proposition 2 we conclude that $(A, \preceq_A)$ is a wqo.

6.3 Well-quasi-ordering subdivisions

Let $\mathcal{OP}$ denote the class of paths with at least two vertices and whose endpoints are distinguished, i.e. one end is said to be the beginning and the other one the end. In the sequel, $\mathrm{fst}(P)$ denotes the first vertex of the path P and $\mathrm{lst}(P)$ its last vertex. We extend the relation $\leq_{\mathrm{im}}$ to $\mathcal{OP}$ as follows: for every $G, H \in \mathcal{OP}$, $G \leq_{\mathrm{im}} H$ if there is an induced minor model μ of G in H such that $\mathrm{fst}(H) \in \mu(\mathrm{fst}(G))$ and $\mathrm{lst}(H) \in \mu(\mathrm{lst}(G))$, and similarly for $\leq_{\mathrm{c}}$.

We omit the proof of the following lemma, which follows from the natural correspondence between labeled paths with distinguished ends and sequences of these labels.

Lemma 16. *For every wqo* $(\Sigma, \preceq)$*, the poset* $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ *is a wqo.*

We can now show that labeled subdivisions of a fixed graph are well-quasi-ordered by the contraction relation, i.e. Lemma 14.

Proof of Lemma 14. Let G be a non labeled graph, let $(\Sigma, \preceq)$ be a wqo and let $\mathcal{G}$ be the class of all $(\Sigma, \preceq)$-labeled G-subdivisions. We set $m = |E(G)|$. Let us show that $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo. First, we arbitrarily choose an orientation to every edge of G and an enumeration $e_1, \dots, e_m$ of these edges. We now consider the function f that, given a tuple $(Q_1, \dots, Q_m)$ of m paths of $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})$, returns the graph constructed from G by, for every $i \in [\![1, m]\!]$, replacing the edge e_i by the path Q_i, while respecting the orientation, i.e. the first (resp. last) vertex of Q_i goes to the first (resp. last) vertex of e_i. As a Cartesian product of wqos and since $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ is a wqo (Lemma 16), the domain $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})^m$ of f is well-quasi-ordered by $\leq_{\mathrm{c}}^m$. Notice that f is surjective on $\mathcal{G}$. In order to show that $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo, it is enough to prove that

$$f\colon (\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}}^m) \to (\mathcal{G}, \leq_{\mathrm{c}})$$

is an monotone, as explained in Proposition 2, that is, to prove that for every $\mathcal{Q}, \mathcal{R} \in \mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})^m$ such that $\mathcal{Q} \leq_{\mathrm{c}}^m \mathcal{R}$, we have $f(\mathcal{Q}) \leq_{\mathrm{c}} f(\mathcal{R})$. According to Remark 1, we only need to care of the case where $\mathcal{Q}$ and $\mathcal{R}$ differ by only one coordinate. By symmetry we may assume that they only differ by the first one, i.e. $\mathcal{Q} = (Q, Q_2, \dots, Q_m)$ and $\mathcal{R} = (R, Q_2, \dots, Q_m)$ with $Q \leq_{\mathrm{c}} R$. Let $\mu\colon V(Q) \to \mathcal{P}^{<\omega}(V(R))$ be a contraction model of Q in R and let $\mu'\colon V(f(\mathcal{Q})) \to \mathcal{P}^{<\omega}(V(f(\mathcal{Q})))$ be the trivial contraction model of $f(\mathcal{Q}) \setminus V(Q)$ in itself defined by $\forall u \in V(f(\mathcal{Q})) \setminus V(Q),\ \mu'(u) = \{u\}$. We now consider the function $\nu\colon V(f(\mathcal{Q})) \to \mathcal{P}^{<\omega}(V(f(\mathcal{R})))$ defined as follows:

$$\nu\colon \begin{cases} V(f(\mathcal{Q})) & \to & \mathcal{P}^{<\omega}(V(f(\mathcal{R}))) \\ u & \mapsto & \mu(u) \quad \text{if } u \in V(Q) \\ u & \mapsto & \mu'(u) \quad \text{otherwise.} \end{cases}$$

It can be easily checked that ν is a contraction model of $f(\mathcal{Q})$ in $f(\mathcal{R})$. Hence $f(\mathcal{Q}) \leq_{\mathrm{c}} f(\mathcal{R})$, as required. This proves that $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo. $\square$

6.4 Well-quasi-ordering $\mathcal{WM}$ and $\mathcal{CI}$

In this section, we prove Lemma 15 by first dealing with $\mathcal{WM}$ (Lemma 17) and then with $\mathcal{CI}$ (Lemma 18). The following result is straightforward consequence of Higman's lemma (see Proposition 2).

Corollary 4. *If $(\Sigma, \preceq)$ is wqo then the class of $(\Sigma, \preceq)$-labeled independent sets (resp. cliques) is wqo induced subgraphs.*

Corollary 5. *If a class of $(\Sigma, \preceq)$-labeled graphs $(\mathcal{G}, \leq_{\mathrm{im}})$ is wqo, then so is its closure by finite disjoint union (resp. join).*

Proof. Let $\mathcal{U}$ be the closure of $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}, \leq_{\mathrm{im}})$ by disjoint union. Observe that every graph of $\mathcal{U}$ can be partitioned in a family of pairwise non-adjacent graphs of $\mathcal{G}$. Therefore we can define a function mapping every $\mathcal{G}$-labeled independent set to the graph of $\mathcal{U}$ obtained from G by replacing each vertex by its label (which is an $(\Sigma, \preceq)$-labeled graph). It is easy to check that this function of $(\mathcal{G}, \leq_{\mathrm{im}}) \to (\mathcal{U}, \leq_{\mathrm{im}})$ is monotone and surjective. Together with Proposition 2 and Corollary 4, this yields the desired result. □

Corollary 6. *If $(\Sigma, \preceq)$ is a wqo then the class of $(\sigma, \preceq)$-labeled complete multipartite graphs are wqo by induced subgraphs.*

Lemma 17. *If $(\Sigma, \preceq)$ is a wqo then the class of $(\Sigma, \preceq)$-labeled graphs of $\mathcal{WM}$ is well-quasi-ordered by induced subgraphs.*

Proof. For every graph G of $\mathcal{WM}$, let (W_G, M_G) be a partition of $V(G)$ as in the definition of $\mathcal{WM}$. Let S be a wheel on at most 5 vertices and let us consider the subclass $\mathcal{WM}(S)$ of all $(\Sigma, \preceq)$-labeled graphs $G \in \mathcal{WM}$ such that W_G is (isomorphic to) the wheel S. We set $s = |S|$ and choose an ordering $v_1, \ldots, v_s$ of the vertices of S. In this proof, for every $G, H \in \mathcal{WM}$, we write $H \leq_{\mathrm{isg}'} G$ if H is an induced subgraph of G that can be obtained without deleting vertices of W_G. Observe that well-quasi-ordering by $\leq_{\mathrm{isg}'}$ implies well-quasi-ordering by $\leq_{\mathrm{isg}}$. Let us show that $\mathcal{WM}(S)$ is well-quasi-ordered by $\leq_{\mathrm{isg}'}$.

For every $G \in \mathcal{WM}(S)$ and every $v \in V(G)$, let $\tau(v) \subseteq \{0,1\}^s$ be a tuple encoding the adjacencies of v on S. Formally, for a fixed ordering of $V(S)$, the i-th coordinate of $\tau(v)$ is equal to 1 if v is adjacent to the i-th vertex of S and to 0 otherwise, for every $i \in [\![1, s]\!]$. Recall that for every $G \in \mathcal{WM}(S)$, the label $\lambda_G(v)$ of a vertex $v \in V(G)$ is a finite subset of Σ. Let $\lambda'_G(v) = \{(l, \tau(v)),\ l \in \lambda(v)\}$. Informally, we add to the label of v information about its adjacency in S. The new label is a subset of $\Sigma \times \{0,1\}^s$. Let f be the function mapping every $G \in \mathcal{WM}(S)$ to $(\lambda_G(v_1), \ldots, \lambda_G(v_s), J_G)$, where J_G is the graph obtained from $G[M_G]$ by relabeling every vertex v with $\lambda'_G(v)$. We see J_G as a $(\Sigma \times \{0,1\}^s, \preceq \times =)$-labeled graph. Observe that f is injective and that for every $G, H \in \mathcal{WM}(S)$,

$$H \leq_{\mathrm{isg}'} G \iff f(H) \underbrace{\preceq^\star \times \cdots \times \preceq^\star}_{s \text{ times}} \times \leq_{\mathrm{isg}} f(G).$$

According to Proposition 2, $(\mathcal{WM}(S), \leq_{\mathrm{isg}'})$ (the domain of f) is a wqo iff the image of f is wqo by $\underbrace{\preceq^\star \times \cdots \times \preceq^\star}_{s \text{ times}} \times \leq_{\mathrm{isg}}$. For every G, J_G is a complete multipartite graph labeled with a wqo hence, according to Corollary 6, $\mathcal{J} = \{J_G,\ G \in \mathcal{WM}(S)\}$ is well-quasi-ordered by $\leq_{\mathrm{isg}}$. The image of f is a subset of the Cartesian product of several occurrences of $(\Sigma, \preceq)$ with $(\mathcal{J}, \leq_{\mathrm{isg}})$, thus it is a wqo. This implies that $(\mathcal{WM}(S), \leq_{\mathrm{isg}})$ is a wqo.

Since there is a finite number of non-isomorphic wheels on at most 5 vertices, the $(\Sigma, \preceq)$-labeled graphs of $\mathcal{WM}$ form a finite union of wqo, hence they are wqo by $\leq_{\mathrm{isg}}$. □

Lemma 18. *If $(\Sigma, \preceq)$ is a wqo then the class of $(\Sigma, \preceq)$-labeled graphs of $\mathcal{CI}$ is wqo by induced minors.*

Proof. We consider the function

$$f\colon (\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})^\star \times (\mathcal{P}^{<\omega}(\Sigma), \preceq^{\mathcal{P}})^\star, \leq_{\mathrm{c}}^\star \times \preceq^{\mathcal{P}\star}) \to (\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{CI}), \leq_{\mathrm{im}})$$

that, given a sequence $\langle R_0, \dots, R_{k-1}\rangle$ of $(\Sigma, \preceq)$-labeled paths of $\mathcal{OP}$ and a sequence A of subsets of Σ, returns the graph constructed as follows:

(i) for every element $a \in A$, create a new vertex and label it with a;

(ii) in the disjoint union of these vertices and the paths of $\{R_i\}_{i\in[\![0,k-1]\!]}$, add an edge between $\mathrm{lst}(R_i)$ and $\mathrm{fst}(R_{(i+1) \bmod k})$, for every $i \in [\![0, k-1]\!]$;

(iii) add all possible edges between $\mathrm{fst}(R_i)$ and the vertices created in the first step, for every $i \in [\![0, k-1]\!]$.

The domain of f is a wqo, as a Cartesian product of wqos (cf. Lemma 16). Observe that its image is $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{CI})$. To show that this set is wqo, it is enough to prove that f is monotone, according to Proposition 2. By Remark 1, this can be done by proving the two following implications:

$$\forall R \in \mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})^\star,\ \forall A, B \in \Sigma^{\mathcal{P}\star},\ A \preceq^\star B \Rightarrow f(R, A) \leq_{\mathrm{im}} f(R, B),\ \text{and}$$
$$\forall Q, R \in \mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{OP})^\star,\ \forall A \in \Sigma^{\mathcal{P}\star},\ Q \leq_{\mathrm{c}}^\star R \Rightarrow f(Q, A) \leq_{\mathrm{im}} f(R, J).$$

First implication. Let us call $a_1, \dots, a_{|A|}$ and $b_1, \dots, b_{|B|}$ the elements of A and B, respectively. As $A \preceq^\star B$, there is a function $\varphi\colon [\![1, |A|]\!] \to [\![1, |B|]\!]$ such that for every $i \in [\![1, |A|]\!]$, we have $a_i \preceq b_{\varphi(i)}$. Let us call v_i the vertex labeled b_i in step (i) of the construction of $f(R, B)$. Then $f(R, A)$ can be obtained from $f(R, B)$ by first deleting the vertices the form v_i with $i \in [\![1, |B|]\!] \setminus \varphi([\![1, |A|]\!])$ and then, for every $i \in [\![1, |A|]\!]$, contracting the label of the vertex $v_{\varphi(i)}$ (which is $b_{\varphi(i)}$) to a_i. Hence $f(R, A) \leq_{\mathrm{im}} f(R, B)$.

Second implication. Let $Q_0, \dots, Q_{k-1}$ and $R_0, \dots, R_{l-1}$ be the elements of Q and R, respectively. By definition of the relation $\leq_{\mathrm{c}}^\star$, there is an increasing function $\varphi\colon [\![0, k-1]\!] \to [\![0, l-1]\!]$ such that

$$\forall i \in [\![0, k-1]\!],\ Q_i \leq_{\mathrm{c}} R_{\varphi(i)}.$$

Let us call μ_i a contraction model of Q_i in R_i, for every $i \in [\![0, k-1]\!]$. Let $\mu\colon V(f(Q, A)) \to \mathcal{P}^{<\omega}(V(f(R, A)))$ be the function that maps a vertex v

- to $\{v\}$ if it has been created during step (i) of the construction of $f(Q, A)$;
- to $\mu_i(v)$ otherwise, where i is such that $v \in Q_i$.

It can be checked that μ is a contraction model of $f(Q, A)$ in $f(R, A)$. As a consequence, $f(Q, A) \leq_{\mathrm{im}} f(R, A)$.

We deduce that $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{CI})$ is wqo by induced minors, as desired. □

7 Graphs not containing Gem

The purpose of this section to give a proof to Theorem 5. This will be done by first proving a decomposition theorem for graphs of $\mathrm{Excl}_{\mathrm{im}}(\mathrm{Gem})$, and then using this theorem to prove that $(\mathrm{Excl}_{\mathrm{im}}(\mathrm{Gem}), \leq_{\mathrm{im}})$ is a wqo.

7.1 A Decomposition theorem for $\mathrm{Excl}_{\mathrm{im}}(\mathrm{Gem})$

This section is devoted to the proof of Theorem 3, which is split in several lemmas. In the sequel, G is a 2-connected graph of $\mathrm{Excl}_{\mathrm{im}}(\mathrm{Gem})$. When G is 3-connected, we will rely on the following result originally proved by Ponomarenko.

Proposition 4 ([21])**.** *Every 3-connected* Gem*-induced minor-free graph is either a cograph, or has an induced subgraph S isomorphic to P_4, such that every connected component of $G \setminus S$ is a cograph.*

Therefore we will here focus on the case where G is 2-connected but not 3-connected. A *rooted diamond* is a graph which can be constructed from a rooted C_4 by adding a chord incident with exactly one endpoint of the root (cf. Figure 10).

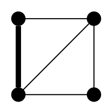

Figure 10: A rooted diamond, the root being the thick edge.

Lemma 19. *Let $S = \{v_1, v_2\} \in E(G)$ be a cutset in a graph G and let J be a component of $G \setminus S$. Let H be the graph $G[V(J) \cup \{v_1, v_2\}]$ rooted at $\{v_1, v_2\}$. If H has a rooted diamond as induced minor, then* Gem $\leq_{\mathrm{im}} G$*.*

Proof. Let J' be a component of $G \setminus S$ other than J and let G' be the graph obtained from G by:

1. applying the necessary operations (contractions and vertex deletions) to transform $G[V(J) \cup \{v_1, v_2\}]$ into a rooted diamond;

2. deleting every vertex not belonging to $V(J) \cup V(J') \cup \{v_1, v_2\}$;

3. contracting J' to a single vertex.

The graph G' is then a rooted diamond and a vertex adjacent to both endpoints of its root, that is, G' is isomorphic to Gem. $\square$

Let us now characterize these 2-connected graphs avoiding rooted diamonds.

Lemma 20. *Let G be a 2-connected graph rooted at $\{u, v\} \in E(G)$. If $\{u, v\}$ is not a cutset of G and G does not contain a rooted diamond as induced minor, then either G is an induced cycle, or both u and v are dominating in G.*

Proof. Assuming that u is not dominating and G is not an induced cycle, let us prove that G contains a rooted diamond as induced minor. Let $w \in V(G)$ be a vertex such that $\{u, w\} \notin E(G)$. Such a vertex always exists given that u is not dominating. Let C be a shortest cycle using the edge $\{u, v\}$ and the vertex w (which exists since G is 2-connected), let P_u be the subpath of C linking u to w without meeting v and similarly let P_v be the subpath of C linking v to w without meeting u. By the choice of C, both P_u and P_v are induced paths. Notice that if there is an edge other than $\{u, v\}$ connecting a vertex of $P_u \setminus \{w\}$ to vertex of $P_v \setminus \{w\}$, then G contains a rooted diamond as induced minor. Therefore we can now assume that C is an induced cycle.

Since we initially assumed that G is not an induced cycle, G contains a vertex not belonging to C. Let G' be the graph obtained from G by contracting to one vertex x some connected component of $G \setminus C$ and deleting all the other components. Obviously we have $G' \leq_{\text{im}} G$. Let us show that G' contains a rooted diamond as induced minor.

Observe that G' is a k-wheel of center x, for some $k \geq 2$. Furthermore, the neighborhood of x, which has size at least two (as G is 2-connected), is not equal to $\{u, v\}$, otherwise $\{u, v\}$ would be a cutset in G. It is then not hard to see that G' contains a rooted diamond as induced minor, a contradiction. □

Remark 4. In a Gem-induced minor-free graph G, every induced subgraph H dominated by a vertex $v \in V(G) \setminus V(H)$ is a cograph.

Indeed, assuming that H is not a cograph, let P be a path on four vertices which is induced subgraph of H. Then $G[V(P) \cup \{v\}]$ is isomorphic to Gem, a contradiction.

Recall that we say that an induced subgraph of G is *basic in G* if it is either a cograph, or an induced path whose internal vertices are of degree two in G.

Lemma 21. *If G has a K_2-cutset $S = \{v_1, v_2\}$, then every connected component of $G \setminus S$ is basic in G.*

Proof. By Lemma 19, for every connected component J of $G \setminus S$ we know that the graph $G[V(J) \cup S]$ rooted at $\{u, v\}$ contains no rooted diamond. By the virtue of Lemma 20, this graph either is an induced cycle, or has a dominating vertex among u and v. In the first case, J is a path whose all internal vertices are of degree two in G, hence H is basic. If one of u and v is dominating, then J is a cograph according to Remark 4. Therefore in both cases C is basic in G. □

Let us now focus on 2-connected graphs with a $\overline{K_2}$-cutset, which is the last case in our characterization theorem.

Corollary 7. *If G has a $\overline{K_2}$-cutset S such that $G \setminus S$ contains more than two connected components, then every connected component of $G \setminus S$ is basic in G.*

Proof. It follows directly from Lemma 21. Indeed, if the connected components of $G \setminus S$ are $J_1, J_2, \ldots J_k$, let us contract J_1 to an edge between the two vertices of S. The obtained graph fulfills the assumptions of Lemma 21: S is a K_2-cutset. Therefore each of the components $J_2, \ldots, J_k$ is basic in G. Applying the same argument with J_2 instead of J_1 yields that J_1 is basic in G as well. $\square$

Lemma 22. *Let $S = \{u, v\}$ be a $\overline{K_2}$-cutset, such that and $G \setminus S$ has only two connected components J_1 and J_2. Then G contains a cycle C as induced subgraph such that every connected component of $G \setminus C$ is basic in G.*

Proof. For every $i \in \{1, 2\}$, let Q_i be a shortest path linking u to v in $G[V(J_i) \cup \{u, v\}]$. Notice that the cycle $C = G[V(Q_1) \cup V(Q_2)]$ is then an induced cycle. For contradiction, let us assume that some connected component J of $G[V \setminus C]$ is not basic in G. By symmetry, we can assume that $J \subset J_1$.

Notice that since G is 2-connected, J has at least two distinct neighbors x, y on C. Let G' be the graph obtained from G by contracting Q_1 to an edge between u and v in a way such that x is not contracted to y (that is, x is contracted to one of u, v and y to the other one). In G', $\{u, v\}$ is a K_2-cutset, therefore by Lemma 21, every connected component of $G \setminus S$ is basic in G'. As this consequence holds for every choice of J and G' is an induced minor of G, we eventually get that every connected component of $G \setminus C$ is basic in G. $\square$

In the sequel, S, u, v, C, H_1, and H_2 follow the definitions of the statement of Lemma 22.

Remark 5. Every connected component J of $G \setminus C$ has at least two and at most three neighbors on C.

Indeed, it has at least two neighbors on C because G is 2-connected. Besides if J has at least four neighbors on C, then contracting in $G[V(C) \cup V(J)]$ the component J to a single vertex, deleting a vertex of C not belonging to $N(J)$ (which exists since J belongs to only one of the components of $G \setminus S$) and then contracting every edge incident with a vertex of degree two would yield Gem.

Lemma 23. *If C has at least one vertex of degree two, then for every distinct connected components J_1 and J_2 of $G \setminus C$ we have $N_C(J_1) \subseteq N_C(J_2)$ or $N_C(J_2) \subseteq N_C(J_1)$.*

Proof. Let us assume, for contradiction, that the claim is not true and let G be a minimal counterexample with respect to induced minors. In such a case both J_1 and J_2 are single vertices (say j_1 and j_2 respectively) and they are the only connected components of $G \setminus C$. We now argue that any such minimal counterexample must contain as induced minor one of graphs presented on Figure 11 (where thick edges represent the cycle C). This would conclude the proof as each of these graphs contains Gem as induced minor, as shown in Figure 11.

First of all, in such a minimal counterexample there is only one vertex in C of degree 2, let us call it c. We will consider all the ways that the vertices j_1 and j_2 can be connected to the neighbors of c, and show that in every such case we can contract our graph to one of the graphs on Figure 11. According to Remark 5, each of j_1 and j_2 will have either two or three neighbors on C.

First case: both j_1 and j_2 are connected with both neighbors of c. As $N(j_1) \not\subseteq N(j_2)$ and $N(j_2) \not\subseteq N(j_1)$, each of j_1, j_2 has a neighbor which is not adjacent to the other. But since j_1 and j_2 can have at most three neighbors, the neighborhood of j_1 and j_2 is now completely characterized. Figure 11.(a) presents the only possible graph for this case.

Second case: j_1 is connected with exactly one of neighbors of c and j_2 is connected with the other one. In this case, as each of j_1, j_2 has at least two neighbors on C, contracting all the edges of C whose both endpoints are at distance at least two from c gives the graph depicted in Figure 11.(b).

Third case: j_1 is connected with both neighbors of c, and j_2 is connected with at most one of them. In this case, as long as C has more than 4 edges, we can contract an edge of C to find a smaller counterexample. Precisely, if it has more than 4 edges, there are two edges e_1, e_2 in C within distance exactly one to c and those two do not share an endpoint. Moreover j_2 has a neighbor s in $C \setminus N(c)$, which is not a neighbor of j_1. Now one of the edges e_1, e_2 is not incident to s, and contracting this edge yields a smaller counterexample. Therefore, we only have to care about the case where C has exactly 4 edges, and this case is exactly the graph represented on Figure 11.(c).

In each of the induced minor-minimal counterexamples, a Gem can be found as induced minor, as depicted in Figure 11. This concludes the proof. □

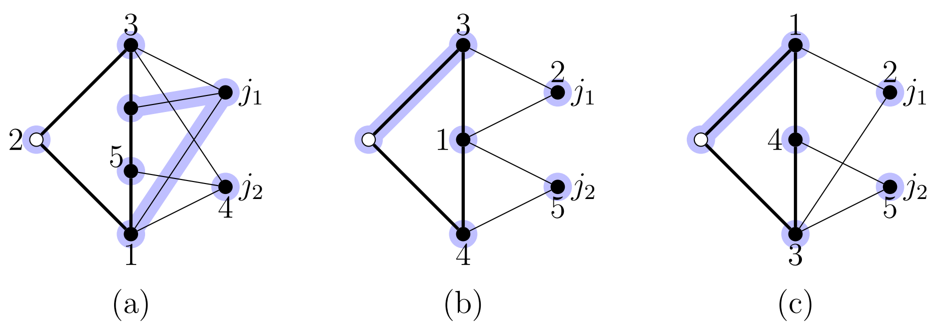

Figure 11: Induced minor-minimal counterexamples in the proof Lemma 23 contain the Gem as induced minor. The vertex c is depicted in white. The numbers indicate which vertices of the Gem (following the convention of Figure 1) correspond to the subsets of vertices depicted in blue.

Corollary 8. *If C has at least one vertex of degree two, then it has at most three vertices of degree greater than two.*

Proof. Notice that the set of vertices of C that have degree greater then two is exactly the union of $N_C(J)$ over all connected components J of $G \setminus C$. We just saw in Lemma 23 that for every two connected components of $G \setminus C$, the neighborhood on C of one is contained in the neighborhood on C of the other. Besides these neighborhoods have size at most three, otherwise we would be able to find a Gem as induced minor. Therefore their union have size at most three as well. □

Corollary 9. *Every connected component of $G \setminus C$ is basic and C has at most six vertices of degree greater than two.*

Proof. Notice that contracting J_1 to a single vertex h in G gives a graph G' and a cycle C' (contraction of C) such that every connected component of $G' \setminus C'$ is basic and C' has at least one vertex of degree 2, h. By Corollary 8, C' has at most three vertices of degree greater than two. Notice that these vertices belong to $G' \setminus h$ which is isomorphic to $G \setminus J_1$. Hence $G \setminus J_1$ has at most three vertices of degree greater than two. Applying the same argument with J_2 instead of J_1 we get the desired result. □

Now we are ready to prove the main decomposition theorem for Gem-induced minor-free graphs.

Proof of Theorem 3. Recall that we are looking for a subset X of $V(G)$ of size at most 6 such that each component of $G \setminus X$ is basic in G.

If G is 3-connected, by Proposition 4 it is either a cograph, or has a subset X of four vertices such that every connected component of $G \setminus X$ is a cograph. Let us now assume that G is not 3-connected.

In the case where G has a K_2-cutset S, or if G has a $\overline{K_2}$-cutset S such that $G \setminus S$ has more than two connected components, then according to Lemma 21 and Corollary 7 respectively, S satisfies the required properties. In the remaining case, by Corollary 9 G has a cycle C such that every connected component of $G \setminus C$ is basic in G and which has at most six vertices of degree more than two in G. Let X be the set containing those vertices of degree more than two. Observe that every connected component of $G \setminus X$ is either a connected component of $G \setminus C$ (hence it is basic) or a part of C, i.e. a path whose internal vertices are of degree two in G (which is basic as well). As $|X| \leq 6$, X satisfies the desired properties. □

7.2 Well-quasi-ordering Gem-induced minor-free graphs

In this section we give a proof of Theorem 5. We proved in the previous section that the structure of 2-connected Gem-induced minor-free graphs is essentially very simple, with building blocks being cographs and long induced paths. To conclude that labeled 2-connected Gem-induced minor-free graphs are wqo by induced minor relation, we need the fact that the building blocks, in particular labeled cographs, are themselves well-quasi-ordered by the induced minor relation. For this we rely on the following extension of the results of Damaschke [4] to labeled graphs due to Atminas and Lozin.

Proposition 5 (from [1]). *For any wqo $(\Sigma, \preceq)$, the class of $(\Sigma, \preceq)$-labeled cographs is wqo by induced subgraphs.*

Fellows et al. proved that if a (labelled) graph class $\mathcal{G}$ is wqo by subgraphs, then for every $k \in \mathbb{N}$, the class of graphs that have k vertices whose deletion results in a graph of $\mathcal{G}$ is also wqo by subgraphs [8, Theorem 4]. We here prove a counterpart of this result for labeled induced minors. For every graph class $\mathcal{G}$ ad every integer k, we denote by $\mathcal{G}^{(+k)}$ the class of graphs that have at most k vertices whose deletion results in a graph of $\mathcal{G}$.

Lemma 24. *Let $\mathcal{G}$ be a class of graphs such that for every wqo $(\Sigma, \preceq)$, the class $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G})$ is wqo by induced minors. Then for every $k \in \mathbb{N}$, and every wqo $(\Sigma', \preceq')$, the class $\mathrm{lab}_{(\Sigma',\preceq')}(\mathcal{G}^{(+k)})$ is wqo by induced minors.*

Proof. Graphs of $\mathcal{G}^{+k}$ can be partitioned into $k+1$ classes depending on the minimum number of vertices to delete in order to obtain a graph of $\mathcal{G}$. In each of these classes the partition can be refined depending on the subgraph induced by the vertices to remove (for an arbitrarly choice of these vertices). Since a finite union of wqos is a wqo, it is enough to focus on the class $\mathcal{H}$ of graphs that have a set X of exactly k vertices such that:

- $G \setminus X \in \mathcal{G}$;
- after forgeting the labels, $G[X]$ is (isomorphic to) the same unlabeled graph H.

We fix an ordering $h_1, \dots, h_k$ of the vertices of H. Let $\Sigma = \Sigma' \times \mathcal{P}^{<\omega}(\llbracket 1, k \rrbracket)$ and let $\preceq$ be the order $\preceq' \times =$ on Σ. For every $\lambda \in \mathcal{P}^{<\omega}(\Sigma)$, we define $\pi(\lambda)$ as the union of the sets $A \subseteq \llbracket 1, k \rrbracket$ such that $(s, A) \in \lambda$ for some $s \in \Sigma'$. We also set $\tau(\lambda) = \{s,\ \exists A \subseteq \llbracket 1, k \rrbracket,\ (s, A) \in \lambda\}$.

Informally, the label of a vertex will encode some adjacencies together with a label and the functions π and τ can be used to retrieve this information. Let f be the function that, given a k-uple $(\lambda_1, \dots, \lambda_k) \in \mathcal{P}^{<\omega}(\Sigma')^k$ and a graph $G \in \mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G})$, constructs the graph $f(G)$ from the disjoin union of H and G as follows:

- label h_i with λ_i, for every $i \in \llbracket 1, k \rrbracket$;
- make every vertex $v \in V(G)$ adjacent to the vertices of $\{h_1, \dots, h_k\}$ whose indices are given by $\pi(\lambda_G(v))$;
- relabel every vertex $v \in V(G)$ with the label given by $\tau(\lambda_G(v))$.

One can easily check that $\mathcal{H}$ is included in the image of f. Besides, the domain of f is a wqo, as it is a Cartesian product of wqos. In order to show that $\mathcal{H}$ is wqo by

$\leq_{\text{im}}$, we can prove that f is monotone, according to Proposition 2. As usual we focus on proving two implications:

$$\forall \lambda, \lambda', \lambda_2, \ldots, \lambda_k \in \mathcal{P}^{<\omega}(\Sigma'), \forall G \in \mathcal{P}^{<\omega}(\Sigma'),$$
$$\lambda \preceq'^{\star} \lambda' \Rightarrow f(\lambda, \lambda_2, \ldots, \lambda_k, G) \leq_{\text{im}} f(\lambda', \lambda_2, \ldots, \lambda_k, G), \text{ and}$$
$$\forall \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathcal{P}^{<\omega}(\Sigma'), \forall G, G' \in \mathcal{P}^{<\omega}(\Sigma'),$$
$$G \leq_{\text{im}} G' \Rightarrow f(\lambda_1, \lambda_2, \ldots, \lambda_k, G) \leq_{\text{im}} f(\lambda_1, \lambda_2, \ldots, \lambda_k, G').$$

We do not consider the cases where the other parameters of f are different since these situations are symmetric to that addressed with the first implication.

First implication. We can obtain $f(\lambda, \lambda_2, \ldots, \lambda_k, G)$ from $f(\lambda', \lambda_2, \ldots, \lambda_k, G)$ by contracting to λ the label λ' carried by h_1. Hence $f(\lambda, \lambda_2, \ldots, \lambda_k, G) \leq_{\text{im}} f(\lambda', \lambda_2, \ldots, \lambda_k, G)$.

Second implication. For the sake of clarity we set $R = f(\lambda_1, \lambda_2, \ldots, \lambda_k, G)$ and $R' = f(\lambda', \lambda_2, \ldots, \lambda_k, G)$. Let μ be an induced-minor model of G in G'. Let $\mu' \colon V(R) \to \mathcal{P}^{<\omega}(V(R'))$ be the function which maps a vertex $v \in V(R)$ to $\{v\}$ if $v \in V(H)$ and $\mu(v)$ otherwise.

Let us show that μ' is an induced minor model of R in R'. The fact that μ' is a containment model either follow from the properties of μ (for vertices of $V(G)$) or is trivial (for vertices of $V(H)$). To show that it is an induced minor model, we have to prove that for every pair of adjacent vertices x, y of R, the sets $\mu'(x)$ and $\mu'(y)$ are connected by an edge in R'. Again this is straightforward when $x, y \in V(H)$ or $x, y \in V(G)$, hence we assume that $x \in V(H)$ and $y \in V(G)$. Without loss of generality we may assume that $x = h_1$. By construction and since h_1 are y adjacent in R, the label $\lambda_G(y)$ of y in G contains a pair (s, A) for some $s \in \Sigma'$ and some $A \in \mathcal{P}^{<\omega}(\llbracket 1, k \rrbracket)$ that contains 1. Besides, by definition of μ, we have $\lambda_G(y) \preceq^{\star} \bigcup_{z \in \mu(y)} \lambda_{G'}(z)$. Hence there is a vertex $z \in \mu(y)$ sucht that $\lambda_{G'}(z)$ contains a pair (s', A') for some $s' \in \Sigma'$ and some $A' \in \mathcal{P}^{<\omega}(\llbracket 1, k \rrbracket)$ that contains 1. Therefore there is an edge between a vertex of $\mu'(y)$ and one of $\mu'(x)$, as desired. This proves that μ' is an induced minor model of R in R'. Hence $R \leq_{\text{im}} R'$, as required. $\square$

Proof of Theorem 5. According to Proposition 1, it is enough to prove that for every wqo $(S, \preceq)$, the class of $(S, \preceq)$-labeled 2-connected graphs which does not contain Gem as induced minor is well-quasi-ordered by induced minors. By Theorem 3, these graphs can be turned into a disjoint union of paths and cographs by the deletion of at most six vertices. As a consequence of Lemma 24 (for $k = 6$ and where $\mathcal{G}$ is the class of disjoint unions of cographs and paths), these graphs are well-quasi-ordered by induced minors and we are done. $\square$

8 Concluding remarks

In this paper we characterized all graphs H such that the class of H-induced minor-free graphs is a well-quasi-order with respect to the induced minor relation. This allowed us to identify the *boundary graphs* (Gem and $\widehat{K}_4$) and to give a dichotomy theorem for this problem. Our proof relies on two decomposition theorems and a study of infinite antichains of the induced minor relation. This work can be seen as the induced minor counterpart of previous dichotomy theorems by Damaschke [4] and Ding [5].

The question of characterizing ideals which are well-quasi-ordered can also be asked for ideals defined by forbidding several elements. To the knowledge of the authors, very little [16] is known on these classes for the induced minor relation, and thus their investigation could be the next target in the study of induced minors ideals. Partial results have been obtained when considering the induced subgraph relation [12, 14].

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