PACKING AND COVERING INDUCED SUBDIVISIONS

O-JOUNG KWON AND JEAN-FLORENT RAYMOND

Abstract. A graph class $\mathcal{F}$ has the induced Erdős-Pósa property if there exists a function f such that for every graph G and every positive integer k, G contains either k pairwise vertex-disjoint induced subgraphs that belong to $\mathcal{F}$, or a vertex set of size at most $f(k)$ hitting all induced copies of graphs in $\mathcal{F}$. Kim and Kwon (SODA'18) showed that for a cycle C_ℓ of length ℓ, the set of C_ℓ-subdivisions has the induced Erdős-Pósa property if and only if $\ell \leq 4$. In this paper, we investigate whether subdivisions of H have the induced Erdős-Pósa property for other graphs H.

We completely settle the case where H is a forest or a complete bipartite graph. Regarding the general case, we identify necessary conditions on H for its subdivisions to have the induced Erdős-Pósa property. For this, we describe three basic constructions that can be used to prove that subdivisions of a graph do not have the induced Erdős-Pósa property. Among remaining graphs, we prove that the subdivisions of the diamond, the 1-pan, and the 2-pan have the induced Erdős-Pósa property.

1. Introduction

All graphs in this paper are finite and without loops or parallel edges. In this paper we are concerned with the induced version of the Erdős-Pósa property. This property expresses a duality between invariants of packing and covering related to a class of graphs. Its name originates from the following result.

Theorem 1.1 (Erdős-Pósa Theorem, [EP62]). *There is a function $f(k) = O(k \log k)$ such that, for every graph G and every $k \in \mathbb{N}$, either G has k vertex-disjoint cycles, or there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ and such that $G - X$ has no cycle.*

In general, we say that a class has the *Erdős-Pósa property* if a similar statement holds: either we can find in a graph many occurrences of members of the class, or we hit them all with a small number of vertices[1]. Since the proof of Theorem 1.1 by Paul Erdős and Lajós Pósa, the line of research of identifying new classes that have the Erdős-Pósa property has been very active (see surveys [Ree97, RT17]). These results are not only interesting because they express some duality between apparently unrelated parameters: they can also be used to design algorithms (see e.g. [Tho88, FLM$^+$16, CRST17]).

Several authors attempted to extend Theorem 1.1 in various directions. One of them is to consider long cycles, i.e. cycles of length at least ℓ for some fixed $\ell \in \mathbb{N}_{\geq 3}$.

Theorem 1.2 ([MNŠW17], see also [BBR07, FH14]). *There is a function $f(k, \ell) = O(k\ell + k \log k)$ such that, for every graph G and every $k \in \mathbb{N}$, either G has k vertex-disjoint cycles of length at least ℓ, or there is a set $X \subseteq V(G)$ with $|X| \leq f(k, \ell)$ and such that $G - X$ has no such cycle.*

Date: Draft of Tuesday 20$^{\text{th}}$ March, 2018, 19:06.

[1] A formal definition will be given shortly after.

The topic we address in this paper is the Erdős-Pósa property of induced objects. As every cycle contains an induced cycle, Theorem 1.1 also holds if one replaces *cycle* with *induced cycle* in its statement. This is not so clear with Theorem 1.2 since a long cycle in a graph does not always contain a long induced cycle. In [JP17], Jansen and Ma. Pilipczuk asked whether Theorem 1.2 holds for induced cycles of length at least ℓ for $\ell = 4$. This was recently proved to be true by Kim and the first author [KK17]:

Theorem 1.3 ([KK17])**.** *There is a function $f(k) = O(k^2 \log k)$ such that, for every graph G and every $k \in \mathbb{N}$, either G has k vertex-disjoint induced cycles of length at least 4, or there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G - X$ has no such cycle.*

They also showed that extensions to $\ell > 4$ (even with a different order of magnitude for f) were not possible. A *subdivision of* H (H*-subdivision* for short) is a graph obtained from H by subdividing some of its edges. A subgraph of a graph G is called an *induced subdivision of* H if it is an induced subgraph of G that is a subdivision of H. Theorem 1.3 can be reformulated in terms of Erdős-Pósa property of induced subdivisions of C_4. The authors of [KK17] noted that it is an interesting topic to investigate the Erdős-Pósa property of induced subdivisions of other graphs.

In this paper, we determine whether H-subdivisions have the induced version of the Erdős-Pósa property or not, for various graphs H. We note that the (classic, i.e. non-induced) Erdős-Pósa property of subdivisions has been investigated before [Tho88]. In order to present our results, let us formally define the induced Erdős-Pósa property[2]. For graphs H and G, we denote by $\nu_H(G)$ the maximum number of vertex-disjoint induced subgraphs of G that are subdivisions of H. We denote by $\tau_H(G)$ the minimum size of a subset $X \subseteq V(G)$ such that no induced subgraph of $G - X$ is a subdivision of H. We say that subdivisions of H have the *induced Erdős-Pósa property* if there is a *bounding function* $f\colon \mathbb{N} \to \mathbb{R}$ such that, for every graph G, the following holds:

$$\tau_H(G) \leq f(\nu_H(G)).$$

Our results. Towards a classification of graphs based on the induced Erdős-Pósa property of their subdivisions, we first consider several simple extensions of cycles, depicted in Figure 1 (see Section 2 for a formal definition).

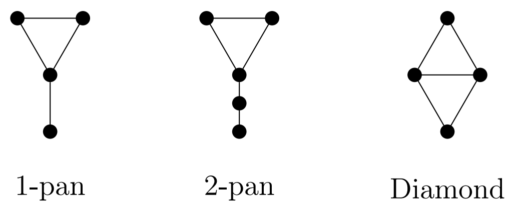

FIGURE 1. The graphs mentioned in the statement of Theorem 1.4.

Theorem 1.4. *If H is either a diamond, a 1-pan, or a 2-pan, then H-subdivisions have the induced Erdős-Pósa property with a polynomial bounding function.*

[2]We decide to use this terminology because the classic Erdős-Pósa property considered the subgraph relation as a containment relation and therefore the sentence "induced H-subdivisions have the Erdős-Pósa property" might be confusing.

For 1- and 2-pans, we furthermore give a polynomial-time algorithm that constructs a packing or a hitting set of bounded size. This will be explicitly mentioned in the statement of the theorems for each of the graphs.

We then give negative results for graphs satisfying certain general properties.

Theorem 1.5. *Let H be a graph which satisfies one of the following:*

(1) H is a forest and two vertices of degree at least 3 lie in the same connected component;
(2) H contains an induced cycle of length at least 5;
(3) H contains a cycle C and two adjacent vertices having no neighbors in C;
(4) H contains a cycle C and three vertices having no neighbors in C;
(5) $H = K_{2,n}$ with $n \geq 3$;
(6) H is not planar,

then H-subdivisions do not have the induced Erdős-Pósa property.

We remark that if a forest F has no two vertices of degree at least 3 in the same connected component, then F is a disjoint union of subdivisions of stars, and therefore every subdivision of F contains F as an induced subgraph. Thus, for such a forest F, F-subdivisions trivially have the induced Erdős-Pósa property (see Lemma 4.3). By item (1) of Theorem 1.5, other forests F will not satisfy this property. Therefore, we can focus on graphs containing a cycle.

The rest of Theorem 1.5 provides, for graphs that have cycles, necessary conditions for their subdivisions to not have the induced Erdős-Pósa property.

By combining together Theorems 1.3, 1.5, and 1.4, we obtain the following dichotomies:

Corollary 1.6.

(1) Let F be a forest. Then F-subdivisions have the induced Erdős-Pósa property if and only if no connected component of F has at least two vertices of degree at least 3.
(2) Let $n, m \in \mathbb{N}_{\geq 1}$ with $n \leq m$. Then $K_{n,m}$-subdivisions have the induced Erdős-Pósa property if and only if $n \leq 1$ or $m \leq 2$.
(3) Let $n \in \mathbb{N}_{\geq 1}$. Then subdivisions of the n-pan have the induced Erdős-Pósa property if and only if $n \leq 2$.

Our techniques. To obtain negative results, we describe three constructions. Their common point is that induced subdivisions of the considered graph H will have very constrained positions. This will ensure that two distinct induced H-subdivisions to meet. On the other hand, they contain several distinct induced H-subdivisions, so one needs many vertices to hit them all. Compared to their non-induced counterparts, induced subdivisions of a fixed graph do not appear in very dense subgraphs such as cliques, which leaves more freedom in the design of our constructions.

The proofs of the positive results for 1-pan and 2-pan start similarly. We first show that of S is smallest 1-pan- (resp. 2-pan-) subdivision in G, then either G contains k pairwise vertex-disjoint induced subdivisions of the 1-pan, or all the 1-pan-(resp. 2-pan-)subdivisions that intersect S can be hit with $O(k \log k)$ vertices. By applying inductively this result we can conclude that subdivisions of the 1-pan (resp. 2-pan) have the induced Erdős-Pósa property with gap $O(k^2 \log k)$. The proof for the diamond is much more involved and relies on tools such as the regular partition lemma of [CKOW17].

2. Preliminaries

Basics. We denote by $\mathbb{R}$ the set of reals and by $\mathbb{N}$ the set of non-negative integers. For an integer $p \geq 1$, we denote by $\mathbb{N}_{\geq p}$ the set of integers greater than or equal to p. Let G be a graph. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively, and, for every $X \subseteq V(G)$ (resp. $X \subseteq E(G)$), by $G - X$ the graph obtained by removing all vertices (resp. edges) of X from G. We use $|G|$ as a shorthand for $|V(G)|$.

For two graphs G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

For $v, w \in V(G)$, we denote by $\mathrm{dist}_G(v, w)$ the number of edges of a shortest path from v to w in G; that is the *distance* between v and w in G. This notation is extended to sets $V, W \subseteq V(G)$ as $\mathrm{dist}_G(V, W) = \min_{(v,w) \in V \times W} \mathrm{dist}_G(v, w)$. Let $r \in \mathbb{N}$. The *r-neighborhood* of a subset $S \subseteq V(G)$, that we denote by $N_G^r[S]$, is the set of all vertices w such that $\mathrm{dist}_G(w, S) \leq r$.

Subdivisions. We recall the definition of subdivision given in the introduction. Let G be a graph and $uv \in E(G)$. The operation of *subdividing* an edge e of G adds a new vertex of degree 2 adjacent to the endpoints of e and removes e. We say that a graph H' is a subdivision of a graph H if H' can be obtained from H by a sequence of edge subdivisions. We also say that G has an *induced subdivision* of H, or shortly an induced H-subdivision, if there is an induced subgraph of G that is (isomorphic to) an H-subdivision. The *dissolution* of a vertex v of degree 2 of a graph G consists in removing v and adding an edge between its neighbors.

Models. A *model* of H in G is an injective function φ with domain $V(H) \cup E(H)$ that maps vertices of H to vertices of G and edges of H to induced paths of G and such that:

- for every edge $uv \in E(H)$, $\varphi(u)$ and $\varphi(v)$ are the endpoints of $\varphi(uv)$;
- for every distinct edges $e, f \in E(H)$, there is no edge between $V(\varphi(e)) \setminus V(\varphi(f))$ and $V(\varphi(f)) \setminus V(\varphi(e))$;
- for every non-adjacent vertices $u, v \in V(H)$, $\varphi(u)\varphi(v) \notin E(G)$.

It is easy to see that a graph G contains an induced subdivision of a graph H if and only if there is a model of H in G. We will often used this fact implicitly. If H' is an induced subgraph of H, then we denote by $\varphi(H')$ the subgraph of G induced by $\bigcup_{v \in V(H')} V(\varphi(v)) \cup \bigcup_{uv \in E(H')} V(\varphi(uv))$.

Small graphs. We denote by K_n the complete graph on n vertices, and by C_n the cycle on n vertices. The *claw* is the complement of the disjoint union of K_3 and K_1. The *n-pan* is the graph obtained from the disjoint union of a path on n edges and K_3 by identifying one of end vertices of the path with a vertex in K_3. The *diamond* is the graph obtained by removing an edge in K_4.

General tools. We collect in this subsection general results that will be used in the paper. The first one is a lemma used by Simonovits [Sim67] to give a new proof of Theorem 1.1.

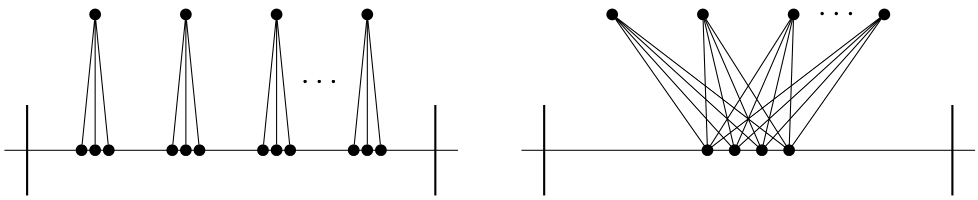

FIGURE 2. Given a path P and a set S of many vertices outside P each having at most k neighbors on P, we can use the regular partition lemma to find a large subset S' of S and a partition of P into at most k parts such that S' connects to each part like one of the depicted configurations.

Lemma 2.1 ([Sim67], see also [Die10, Lemma 2.3.1]). *Let $k \in \mathbb{N}_{\geq 1}$. If a cubic multigraph has at least $24k \log k$ vertices, then it contains at least k vertex-disjoint cycles. Furthermore, such k cycles can be found in polynomial time.*

The second one is related to an Erdős-Pósa type problem about paths intersecting a fixed set of vertices. Let G be a graph and let $A \subseteq V(G)$. An *A-path* in G is a path with both end vertices in A and all internal vertices in $V(G) \setminus A$. An A-path is *non-trivial* if it contains an internal vertex. An *A-cycle* is a cycle containing at least one vertex of A.

Theorem 2.2 ([Gal64]). *Let G be a graph, $A \subseteq V(G)$, and k be a positive integer. Then one can find in time $\mathcal{O}(kn^2)$ either*

(1) $k+1$ vertex-disjoint A-paths, or
(2) $X \subseteq V(G)$ with $|X| \leq 2k$ such that $G - X$ has no A-paths.

We use the regular partition lemma introduced in [CKOW17]. For a sequence $(A_1, \dots, A_\ell)$ of finite subsets of an interval $I \subseteq \mathbb{R}$, a partition $\{I_1, \dots, I_k\}$ of I into intervals is called a *regular partition* of I with respect to $(A_1, \dots, A_\ell)$ if for all $i \in \{1, \dots, k\}$, either

- $A_1 \cap I_i = A_2 \cap I_i = \cdots = A_\ell \cap I_i \neq \emptyset$, or
- $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \dots, \ell\}$ with $j < j'$, $\max(A_j \cap I_i) < \min(A_{j'} \cap I_i)$, or
- $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \dots, \ell\}$ with $j < j'$, $\max(A_{j'} \cap I_i) < \min(A_j \cap I_i)$.

The number of parts k is called the *order* of the regular partition. The following can be obtained using multiple applications of the Erdős-Szekeres Theorem [ES87].

Lemma 2.3 (Regular partition lemma [CKOW17]). *Let $I \subseteq \mathbb{R}$ be an interval. There exists a function N such that for all positive integers n, ℓ, the value $N = N(n, \ell)$ satisfies the following. For every sequence $(A_1, \dots, A_N)$ of n-element subsets of I, there exist a subsequence $(A_{j_1}, \dots, A_{j_\ell})$ of $(A_1, \dots, A_N)$ and a regular partition of I with respect to $(A_{j_1}, \dots, A_{j_\ell})$ that has order at most n.*

We will use this lemma with $n \in \{2, 3, 4\}$. We note that for fixed k, the function $N(n, \ell)$ is a polynomial function, but the order is big. For instance, $N(n, \ell) = \mathcal{O}(\ell^{968})$.

3. NON-PLANAR GRAPHS

We prove here item (6) of Theorem 1.5 (which is Lemma 3.2): subdivisions of a non-planar graph never have the induced Erdős-Pósa property.

We use the notion of *Euler genus* of a graph G. The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [MT01] for more details on graph embedding. The *Euler genus* $\gamma(G)$ of a graph G is the minimum Euler genus of a surface where G can be embedded.

We will adapt the proof of [RT17, Lemma 5.2] (similar to that of [RS86, Lemma 8.14] but dealing with subdivisions) to the setting of induced subdivisions.

Lemma 3.1 (from the proof of [RT17, Lemma 5.2])**.** *For every non-planar graph H, there is a family of graphs $(G_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,*

(i) $\gamma(G_n) = \gamma(H)$; and
(ii) $G_n - X$ contains a subdivision of H, for every $X \subseteq V(G_n)$ with $|X| < n$.

Lemma 3.2. *For every non-planar graph H, the subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. We construct a family of graphs $(\dot{G}_n)_{n\in\mathbb{N}}$ such that $\nu_H(\dot{G}_n) = 1$ and $\tau_H(\dot{G}_n) \geq n$, for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let G_n be the graph of Lemma 3.1 and let $\dot{G}_n$ be the graph obtained by subdividing once every edge of G_n. Notice that for every subdivision G' of a graph G we have $\gamma(G) = \gamma(G')$ (an embedding of one in a given surface can straightforwardly be deduced from an embedding of the other). Hence $\gamma(\dot{G}_n) = \gamma(G_n) = \gamma(H)$. As in the proof of [RT17, Lemma 5.2]), we observe that since H is not planar the disjoint union of two subdivisions of H has Euler genus larger than that of H (see [BHK62]). Therefore $\dot{G}_n$ does not contain two disjoint (induced) subdivisions of H: $\nu_H(\dot{G}_n) \leq 1$. For every subgraph J of G_n, the subgraph of $\dot{G}_n$ obtained by subdividing once every edge of J is induced. Hence property (ii) of $(G_n)_{n\in\mathbb{N}}$ implies that for every $X \subseteq V(\dot{G}_n)$ with $|X| < n$ the graph $\dot{G}_n - X$ contains an induced subdivision of H. In other words $\tau_H(\dot{G}_n) \geq n$. □

4. Negative results using the triangle-wall

We prove in this section items (1) and (5) of Theorem 1.5. The cornerstone of both proofs is the use of a triangle-wall, which we describe hereafter. These results are then used to show the dichotomies (1) and (2) of Corollary 1.6.

Let $n \in \mathbb{N}_{\geq 2}$. The *$n$-garland* is the graph obtained from the path $x_0y_0 \dots x_iy_i \dots x_{n-1}y_{n-1}$ by adding, for every $i \in \{0, \dots, n-1\}$, a new vertex z_i adjacent to x_i and y_i. Let $Q_0, \dots, Q_{n-1}$ be n copies of the $2n$-garland. For every $i \in \{0, \dots, n-1\}$, we respectively denote by x^i_j, y^i_j, z^i_j the copies of the vertices x_j, y_j, z_j in Q_i. The graph Γ_n (informally called *triangle-wall*) is constructed from the disjoint union of $Q_0, \dots, Q_{n-1}$ as follows:

- for every even $i \in \{0, \dots, n-1\}$ and every odd $j \in \{0, \dots, 2n-1\}$, we add an edge between the vertex z^i_j and the vertex z^{i+1}_j;
- for every odd $i \in \{0, \dots, n-1\}$ and every even $j \in \{0, \dots, 2n-1\}$, we add an edge between the vertex z^i_j and the vertex z^{i+1}_j.

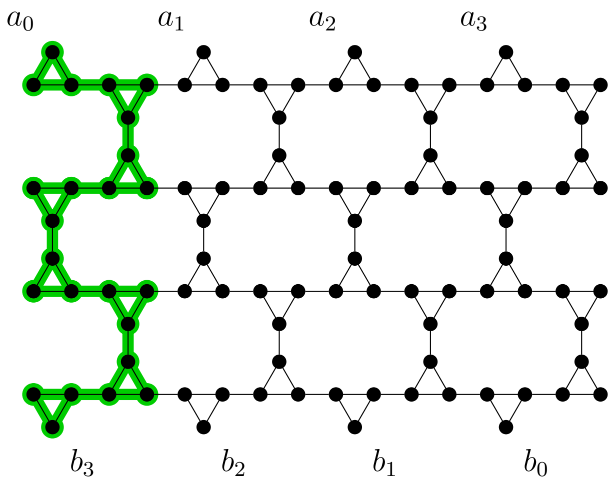

FIGURE 3. The graph Γ_4, with C_1 depicted in green.

For $i \in \{0, \dots, n-1\}$, we set

$$a_i = z^0_{2i}, \qquad b_i = \begin{cases} z^p_{2(n-i-1)} & \text{if } p \text{ is even,} \\ z^p_{2(n-i)-1} & \text{otherwise.} \end{cases},$$

$$\text{and } C_i = \bigcup_{j=1}^{n} \{x^j_{2i}, y^j_{2i}, z^j_{2i}, x^j_{2i+1}, y^j_{2i+1}, z^j_{2i+1}\}.$$

Intuitively, the Q_i's form the rows of Γ_n and the C_i's its columns. The graph Γ_4 is depicted in Figure 3. Observe that there is no induced claw in Γ_n.

4.1. **Complete bipartite patterns.** In this section, we complete the study of the induced Erdős-Pósa property of subdivisions of complete bipartite graphs by proving item (2) of Corollary 1.6.

Lemma 4.1 (item (5) of Theorem 1.5)**.** *For every integer $r \geq 3$, the subdivisions of $K_{2,r}$ do not have the induced Erdős-Pósa property.*

Proof. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq 1}}$ of graphs such that $\nu_{K_{2,r}}(G_n) = O(1)$ while $\tau_{K_{2,r}}(G_n) = \Omega(n)$.

For $n \in \mathbb{N}_{\geq 1}$, the graph G_n is obtained as follows from the graph Γ_{rn}:

- for every $i \in \{0, \dots, n-1\}$, add a new vertex u_i and make it adjacent to $a_{ir}, a_{ir+1}, \dots, a_{(i+1)r-1}$;
- for every $i \in \{0, \dots, n-1\}$, add a new vertex v_i and make it adjacent to $b_{ir}, b_{ir+1}, \dots, b_{(i+1)r-1}$;
- for every distinct $i, j \in \{0, \dots, n-1\}$, add all possible edges between $\{u_i, v_i\}$ and $\{u_j, v_j\}$ (thus, u_i and v_j are not adjacent iff $i = j$).

Let us show that $\nu_{K_{2,r}}(G_n) \leq 1$. For this we consider a model φ of $K_{2,r}$ in G_n. Notice that $\varphi(K_{2,r})$ has exactly two vertices of degree r, and that they are not adjacent. As each of them has $r \geq 3$ pairwise non-adjacent neighbors, we observe that none of these two vertices of degree r belongs to the subgraph Γ_{rn} of G_n. Furthermore, as they are not adjacent, one is u_i and the other one v_i, for some $i \in \{0, \dots, n-1\}$.

Let us now focus on the (u_i, v_i)-paths of $\varphi(K_{2,r})$. Observe there is no edge between the interior vertices of two distinct (u_i, v_i)-paths of $\varphi(K_{2,r})$. Therefore if two such paths contain a vertex of $\{u_j, v_j,\ j \in \{0, \dots, n-1\} \setminus \{i\}\}$ each, one of these vertices is u_j and the other one is v_j, for some $j \in \{0, \dots, n-1\}$. As $r \geq 3$, we deduce that at least one (u_i, v_i)-path of $\varphi(K_{2,r})$ does not contain any vertex of $\{u_j, v_j,\ j \in \{0, \dots, n-1\} \setminus \{i\}\}$. The interior of this path lies in Γ_{rn} and connects a neighbor of u_i to a neighbor of v_i, i.e. a vertex of $a_{ir}, a_{ir+1}, \dots, a_{(i+1)r-1}$ to a vertex of $b_{ir}, b_{ir+1}, \dots, b_{(i+1)r-1}$.

We just proved that every induced subdivision of $K_{2,r}$ in G_n contains a path of the subgraph Γ_{rn} connecting a vertex of $a_{ir}, a_{ir+1}, \dots, a_{(i+1)r-1}$ to a vertex of $b_{ir}, b_{ir+1}, \dots, b_{(i+1)r-1}$, for some $i \in \{1, \dots, n\}$. As every two such paths meet for different values of i, we deduce $\nu_{K_{2,r}}(G_n) \leq 1$.

We now show that when $n \geq 2r$, $\tau_{K_{2,r}}(G_n) \geq \frac{n}{2}$. For this we consider the sets defined for every $i \in \{0, \dots, n-1\}$ as follows:

$$C_i^+ = \{u_i, v_{n-i-1}\} \cup \bigcup_{j=(i+1)r-1}^{ir} C_j.$$

The set C_i^+ contains u_i, v_{n-i-1} and the vertices that are, intuitively, in the columns that are between them. Let X be a subset of $V(G_n)$ with $\lceil \frac{n}{2} \rceil - 1$ vertices. Observe that for distinct $i, j \in \{1, \dots, n\}$, the sets C_i^+ and C_j^+ are disjoint. Hence the $\lceil n/2 \rceil$ sets $\left\{C_i^+ \cup C_{n-i-1}^+,\ i \in \left\{0, \dots, \left\lceil \frac{n-1}{2} \right\rceil\right\}\right\}$ are disjoint. As $|X| = \lceil \frac{n}{2} \rceil - 1$, we have $X \cap (C_i^+ \cup C_{n-i-1}^+) = \emptyset$ for some $i \in \left\{0, \dots, \left\lceil \frac{n-1}{2} \right\rceil\right\}$.

Recall that Γ_n is composed of n disjoint garlands $Q_1, \dots Q_n$ connected together. As $|X| < \frac{n}{2}$ and $r \leq \frac{n}{2}$, some r of these garlands do not intersect X. Let $\pi \colon \{1, \dots, r\} \to \{1, \dots, n\}$ be a increasing function such that $Q_{\pi(i)} \cap X = \emptyset$, for every $i \in \{1, \dots, r\}$. Then there is a collection of disjoint paths $P_1, \dots, P_r$ of $G - X$ so that P_j connects a_{ir+j} to b_{ir+j}, for every $j \in \{1, \dots, r\}$. The path P_j can be obtained by following in $G_n[C_{ir+j}]$ (intuitively, the j-th column of C_i^+) a shortest path from a_{ir+j} to $Q_{\pi(j)}$, then chordlessly following $Q_{\pi(j)}$ up to a vertex of $C_{(n-i-1)r+j}$ and finally following a chordless path to $b_{(i-1)r+j}$ in $G_n[C_{(n-i+2)r+j}]$. We deduce that $G - X$ contains a model of $K_{2,r}$. As this argument holds for every $X \subseteq V(G)$ such that $|X| \leq \lceil \frac{n}{2} \rceil - 1$, we deduce that $\tau_{K_{2,r}}(G_n) \geq \frac{n}{2}$.

Therefore $\nu_{K_{2,r}}(G_n) = O(1)$ and $\tau_{K_{2,r}}(G_n) = \Omega(n)$. This concludes the proof. □

Corollary 4.2 (item (2) of Corollary 1.6)**.** *Let $r, r' \in \mathbb{N}_{\geq 1}$ with $r \leq r'$. Subdivisions of $K_{r,r'}$ have the induced Erdős-Pósa property if and only if either $r \leq 1$ or $r' \leq 2$.*

Proof. When $r = 1$, the result holds by Lemma 4.3 (to be proved in the next section). When $r = r' = 2$, $K_{2,r}$ is a cycle on four vertices and the result holds by Theorem 1.3. When $r = 2$ and $r' \geq 3$, the result follows from Lemma 4.1. In the case where $r \geq 3$, then $K_{r,r'}$ is not planar and the result follows from Lemma 3.2. □

4.2. **Acyclic patterns.** We show in this section that if a graph is acyclic, then its subdivisions either have the induced Erdős-Pósa property with a linear bounding function (Lemma 4.3), or they do not have the induced Erdős-Pósa property (Lemma 4.4). This proves item (1) of Corollary 1.6.

Lemma 4.3. *Let H be a graph whose connected components are paths or subdivided stars. Then subdivisions of H have the induced Erdős-Pósa property with a bounding function of order $O(k)$.*

Proof. We show that for every graph G we have $\tau_H(G) \leq \nu_H(G) \cdot |H|$. Towards a contradiction we assume that the above statement has a counterexample G, that we choose to have the minimum number of vertices. Clearly $\nu_H(G) \geq 1$. Let M be a subdivision of H with the minimum number of vertices. As the connected components of H are paths and subdivided stars, M is a copy of H. Hence $|M| = |H|$. By minimality of G, we have $\tau_H(G - V(M)) \leq f(\nu_H(G - V(M)))$. We deduce

$$\begin{aligned}\tau_H(G) &\leq \tau_H(G - V(M)) + |M| \\ &\leq f(\nu_H(G) - 1) + k \\ &= f(k).\end{aligned}$$

This contradicts the definition of G. □

Lemma 4.4 (item (1) of Theorem 1.5)**.** *Let H be a forest, one connected component of which contains at least two vertices of degree at least 3. Then subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. Our goal is to construct an infinite family of graphs $(G_n)_{n \in \mathbb{N}_{\geq 1}}$ such that $\nu_H(G_n) = O(1)$ while $\tau_H(G_n) = \Omega(n)$. Let J be a connected component of H which contains at least two vertices of degree at least 3 and let u and u' be two such vertices, that are at minimal distance of each other. Let $\{v, v'\}$ be a edge of the path connecting u to u', with the convention that u is closer to v than to v'. We call T the connected component of $H - \{vv'\}$ that contains v, T' that that contains v', and D the union of the remaining connected components. We also call F (resp. F') the graph obtained from T (resp. T') by removing the vertices of the path from u to v (resp. u' to v').

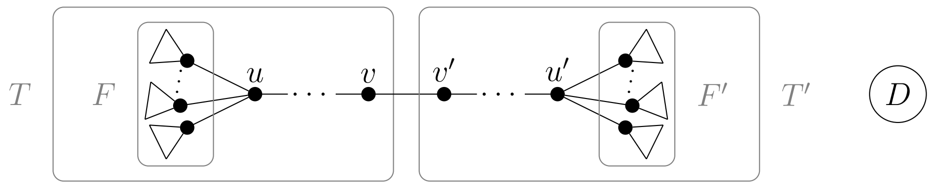

FIGURE 4. The forest H.

Recall that the triangle-wall Γ_n is defined in Section 2 and has special vertices called a_i and b_i. For every $n \in \mathbb{N}_{\geq 1}$, we construct the graph G_n from a copy of Γ_n as follows:

(1) for every $i \in \{0, \dots, n-1\}$, we add a new copy T_i of T and add an edge from the vertex v of this copy to a_i;
(2) for every $i \in \{0, \dots, n-1\}$, we add a new copy T'_i of T' and add an edge from the vertex v' of this copy to b_i;
(3) for every $i \in \{0, \dots, n-1\}$, we add a new copy D_i of D;
(4) for every distinct $i, j \in \{0, \dots, n-1\}$, we add all edges between $F_i \cup D_i$ and $F_j \cup D_j$, between $F'_i \cup D_i$ and $F'_j \cup D_j$ and between $F_i \cup D_i$ and $F'_j \cup D_j$.

Let $n \in \mathbb{N}$ and let us show that every subdivision of H in G_n has a very restricted position. For convenience we refer to the copy of Γ_n in G_n as Γ_n and if w is a vertex of H, we refer by w_i to its copy in T_i, T'_i, or D_i. We denote by P_{xy} the unique path of T (resp. T') with endpoints $x, y \in V(T)$ (resp. $x, y \in V(T')$) and similarly for $P_{x_iy_i}$ in T_i and T'_i.

Let φ be a model of H in G_n.

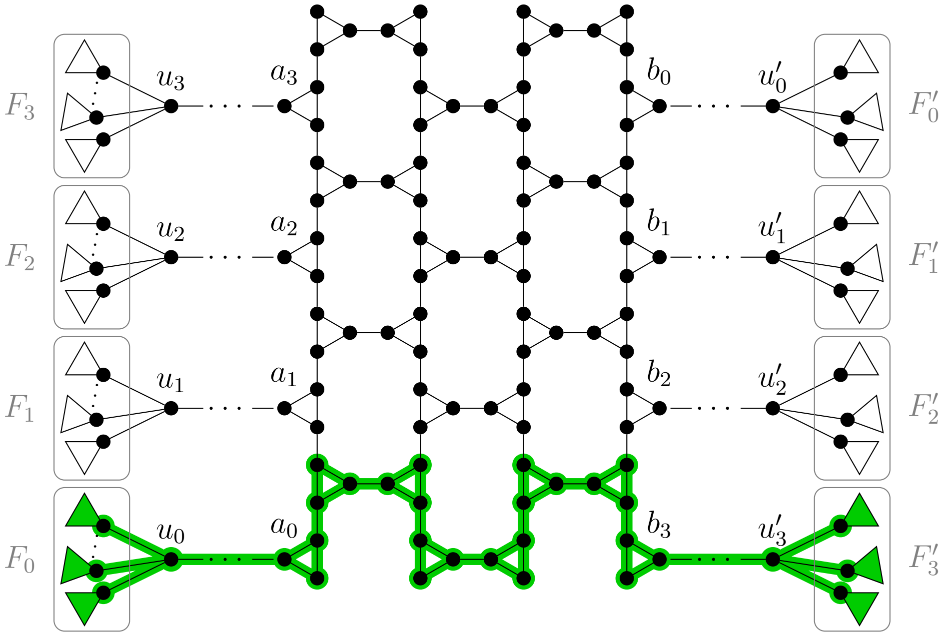

FIGURE 5. The graph G_4. The D_i's and the edges between the $V(F_i) \cup V(F'_i)$'s (framed) are not depicted.

Claim 4.5. *There is an integer $i \in \{0, \dots, n-1\}$ such that every vertex of degree at least 3 of $\varphi(H)$ belongs to $V(T_i) \cup V(T'_i) \cup V(D_i)$.*

Proof. Let x, y be two vertices of degree 3 in $\varphi(H)$. As noted mentioned above, these vertices do not belong to Γ_n. Let us assume towards a contradiction that for distinct $i, j \in \{0, \dots, n-1\}$ we have

$$x \in V(T_i) \cup V(T'_i) \cup V(D_i) \quad \text{and} \quad y \in V(T_j) \cup V(T'_j) \cup V(D_j).$$

The first case we consider is when $x \in \{u_i, u'_i\}$. Then at most one of its neighbors (in $\varphi(H)$) belongs to $P_{u_i v_i}$ or $P_{u'_i v'_i}$ so x has two neighbors x_1 and x_2 that belong to $V(F_i) \cup V(F'_i) \cup V(D_i)$. If additionally $y \in \{u_j, u'_j\}$, then we can similarly deduce that it has (in $\varphi(H)$) two neighbors that belong to $V(F_j) \cup V(F'_j)$. By construction they are both adjacent to x_1 and x_2 and form together with them an induced C_4, a contradiction. We deduce $y \notin \{u_j, u'_j\}$, i.e. $y \in V(F_j) \cup V(F'_j)$. But then y is adjacent to x_1 and x_2, which again forms an induced C_4 and hence is not possible. Notice that the case where $x \notin \{u_i, u'_i\}$ and $y \in \{u_j, u'_j\}$ is symmetric.

The only remaining case to consider is then when $x \notin \{u_i, u'_i\}$ and $y \notin \{u_j, u'_j\}$. Then $xy \in E(\varphi(H))$. At most one neighbor of x (in $\varphi(H)$) is u_i, so x has a neighbor $x_1 \neq y$ in $V(F_k) \cup V(F'_k) \cup V(D_k)$ for some $k \in \{1, \dots, n\}$. As this neighbor cannot be adjacent to y without creating a cycle, it belongs in fact to $V(F_j) \cup V(F'_j) \cup V(D_j)$. Symmetrically, y has a neighbors distinct from x in $V(F_i) \cup V(F'_i) \cup V(D_i)$. But then $G_n[\{x, x_1, y, y_1\}]$ contains a cycle, a contradiction. This concludes the proof. ◇

Let i be as in the statement of Claim 4.5. Observe that $\varphi(H)$ contains no vertex of $V(T_j) \cup V(T'_j) \cup V(D_j)$ for $j \in \{0, \dots, n-1\} \setminus \{i\}$, otherwise it would contain a cycle:

$$V(\varphi(H)) \subseteq V(T_i) \cup V(T'_i) \cup V(D_i) \cup V(\Gamma_n).$$

By construction $|E(T_i \cup T_i' \cup D_i)| = |E(H)| - 1$. We deduce that $\varphi(H)$ contains a path Q from a vertex x of T_i to a vertex y of T_i'. As Q does not intersect $V(T_j) \cup V(T_j') \cup V(D_j)$ for $j \in \{0, \dots, n-1\} \setminus \{i\}$, a subpath of it links a_i to b_i in Γ_n.

We proved that if there is an induced subdivision of H in G_n, it contains an path of Γ_n from a_i to b_i for some $i \in \{0, \dots, n-1\}$. Notice that two such paths meet for distinct values of i. Consequently, $\nu_H(G_n) \leq 1$. On the other hand, $G_p[V(T_i) \cup P \cup V(T_i') \cup V(D_i)]$ is an induced subdivision of H for every $i \in \{1, \dots, n\}$ and every chordless path P of Γ_n connecting a_i to b_i. Hence $\nu_H(G_n) = 1$. We now show that $\tau_H(G_n) = \Omega(n)$.

Claim 4.6. *For every $n \in \mathbb{N}$, $\tau_H(G_n) \geq \frac{n}{2}$.*

Proof. This proof is similar to the end of the proof of Lemma 4.1. Let $n \in \mathbb{N}$. We consider a set X of $\lceil \frac{n}{2} \rceil - 1$ vertices of G_n and show that $G_n - X$ contains a induced subdivision of H. For every $i \in \{0, \dots, n-1\}$, we set

$$C_i^+ = V(T_i) \cup V(D_i) \cup V(T'_{p-i-1}) \cup C_i.$$

Intuitively C_i contains the vertices of the copies T_i and T_{p-i+1}, a path linking them, and D_i; see Figure 5 for a depiction of C_1 in G_4 (in green). As in the proof of Lemma 4.1 we deduce the existence of an integer i such that none of C_i^+ and C_{n-i-1}^+ intersects X. As $|X| < n$, for some $j \in \{0, \dots, n-1\}$ the garland Q_j (which is a subgraph of Γ_n) does not contain a vertex of X. Besides, any of these garlands intersect each of $C_0, \dots, C_{n-1}$. We deduce that in $G - X$, C_i and C_{n-i-1} (which by definition of i do not intersect X) belong to the same connected component. An induced subdivision of H can then be found by connecting the vertex a_i of T_i to the vertex b_i of T_{n-i-1} using a chordless path. As this argument works for every $X \subseteq V(G)$ such that $|X| \leq \lceil \frac{n}{2} \rceil - 1$, we deduce that $\tau_H(G_n) \geq \frac{n}{2}$. ◇

We constructed a family $(G_n)_{n \in \mathbb{N}}$ of graphs such that $\nu_H(G_n) = O(1)$ and $\tau_H(G_n) = \Omega(n)$. This proves that subdivisions of H do not have the induced Erdős-Pósa property. □

5. Negative results using hypergraphs

We prove in this section negative results about the Erdős-Pósa property of the subdivisions of graphs containing long cycles (Theorem 5.2) and of graphs that have three vertices away from a cycle (Theorem 5.4). We start with an easy lemma.

Lemma 5.1. *Let H be a graph containing a cycle C, let H' be a subdivision of H, and let e be an edge of H' contained in the subdivision of C. Then $H' - \{e\}$ does not contain an induced subdivision of H.*

Proof. For a graph G, let $c(G)$ denote the number of distinct (and not disjoint) cycles in G. Observe that there is a bijection between the cycles of G and those of a subdivision G' of G, hence $G(G) = c(G')$. Besides, if G' is a subgraph of G, every cycle of G' is a cycle of G, hence $c(G') \leq c(G)$.

As e belongs to a cycle, $c(H' - \{e\}) < c(H') = c(H)$. If $H' - \{e\}$ contained a subdivision H'' of H we would have the following contradiction:

$$c(H) = c(H'') \leq c(H' - \{e\}) < c(H') = c(H).$$

□

Our proofs of Theorem 5.2 and Theorem 5.4 rely on a suitable modification of the construction given in [KK17] to show that for $\ell \geq 5$, C_ℓ-subdivisions have no induced Erdős-Pósa property.

A pair (X, E) of a set X and a family E of non-empty subsets of X is called a *hypergraph*. Each element in E is called a *hyperedge*, and for a hypergraph H, let $E(H)$ denote the set of hyperedges in H. A subset Y of X is called a *hitting set* if for every $F \in E$, $Y \cap F \neq \emptyset$. For positive integers a, b with $a \geq b$, let $U_{a,b}$ be the hypergraph (X, E) such that $X = \{1, \dots, a\}$ and E is the set of all subsets of X of size b. It is not hard to observe that for every positive integer k, every two hyperedges of $U_{2k-1,k}$ intersect and that the minimum size of a hitting set of $U_{2k-1,k}$ is precisely k.

The uniform bipartite graph UB_k with bipartition (A, B) is defined as follows:

- $A = \{x_1, x_2, \dots, x_{2k-1}\}$;
- for every hyperedge $F = \{a_1, a_2, \dots, a_k\}$ of $U_{2k-1,k}$ with $a_1 < a_2 < \dots < a_k$, we introduce a path $P_F = p_1^F q_1^F p_2^F q_2^F \cdots p_k^F q_k^F p_{k+1}^F$, and identify x_{a_i} with q_i^F for each $i \in \{1, \dots, k\}$.

We also set $\mathcal{P}(UB_k) = \{P_F : F \in E(U_{2k-1,k})\}$.

We are ready to prove one of the main results of this section.

Theorem 5.2 (item (2) of Theorem 1.5)**.** *If a graph H contains an induced cycle of length at least 5, then subdivisions of H have no induced Erdős-Pósa property.*

Proof. Let H be a graph containing an induced cycle C of length at least 5, and let $m := |V(H)|$. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq m}}$ of graphs such that $\nu_H(G_n) = 1$ while $\tau_H(G_n) \geq n$ for every $n \in \mathbb{N}_{\geq m}$.

Let $n \geq m$ be an integer and let us fix an edge uv of C. Let H' be the graph obtained from H by subdividing each edge into a path of length $2n$. In particular, we subdivide uv into $uw_1 w_2 \cdots w_{2n-1} v$. We construct a graph $G = G_n$ from UB_n as follows:

- for each path P of $\mathcal{P}(UB_n)$, we take a copy H_P of H' and identify the copy of $w_1 w_2 \cdots w_{2n-1}$ with the path P, in the same order
- then we add all possible edges between $V(H_P) \setminus V(P)$ and $V(H_{P'}) \setminus V(P')$ for two distinct paths P and P' in $\mathcal{P}(UB_n)$.

Note that A is still an independent set in G. Clearly, each H_P is an induced subdivision of H. We claim that every induced subdivision of H in G must contain P for some $P \in \mathcal{P}(UB_n)$.

Claim 5.3. *Every induced subdivision of H in G is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$, and contains the path P.*

Proof. Let φ be a model of H in G and $C' := \varphi(C)$.

We show that C' is contained in H_P for some $P \in \mathcal{P}(UB_n)$. Since A is an independent set, C' contains a vertex v of $H_P - A$ for some $P \in \mathcal{P}(UB_n)$. Suppose for contradiction that C' is not contained in H_P. Then C' also contains a vertex v' of $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n)$, $P' \neq P$. By construction v is adjacent to v'. Furthermore, as C' is an induced cycle of length at least 5, C' does not contain any vertex in $H_{P''}$ for $P'' \in \mathcal{P}(UB_n) \setminus \{P, P'\}$.

We analyze the order of $C'[A]$. Let us observe that if I is an independent set of a cycle M, we have $|N_M(I)| \geq |I|$ since we can define an injective mapping from I to $V(M) \setminus I$ by mapping $v \in V(T)$ to its left neighbor. Since $C'[A]$ is an independent set in G, the previous observation implies that $|C' - A| \geq |C'[A]|$.

Suppose $C' - (\{v, v'\} \cup A)$ has two vertices w and w'. Then either C' has a vertex with three neighbors in C' (when both w and w' belong to one of $H_P - A$ and $H_{P'} - A$), or it contains an induced C_4 (when $\{w, w'\}$ intersects both $V(H_P - A)$ and $V(H_{P'} - A)$). This is not possible as C' is an induced cycle of length at least 5, hence $C' - A$ has at most three vertices.

Observe that if $|C' - A| = |C'[A]| = 3$, then C' is a C_6 and $C' - A$ is an independent set. This is not possible as $vv' \in E(C' - A)$, hence $|C'| \leq 5$. Since $C'[A]$ is an independent set we deduce $|C'[A]| \leq 2$. Notice that in the case where $|C'[A]| = 2$, $C' - A$ has three vertices and only one edge. Again this is not possible, because $C' - A$ intersects both $H_P - A$ and $H_{P'} - A$. Consequently, C' has length 4, which is a contradiction.

We conclude that C' is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$. Note that every cycle in H_P contains more than m vertices of $H_P - V(P)$, by construction. Therefore, if $\varphi(H)$ contains a vertex of $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n)$, $P' \neq P$. then this vertex should have degree at least m in $\varphi(H)$, which cannot happen in an induced subdivision of H. Thus, all other vertices are also contained in $V(H_P) \cup A$, which proves the first part of the claim. Since P is a part of the subdivisions of the cycle of H in H_P, by Lemma 5.1, the subdivision of H should contain the path P. ⋄

Since every induced subdivision of H in G contains a path in $\mathcal{P}(UB_n)$, no two induced subdivisions of H in G are vertex-disjoint, hence $\nu_H(G) = 1$. For every vertex subset S of size at most $n - 1$, there exists $P \in \mathcal{P}(G)$ such that H_P does contain any vertex of S. Therefore $\tau_H(G) \geq n$, as required. □

Using the same construction, we can also prove the following.

Theorem 5.4 (item (3) of Theorem 1.5)**.** *Let H be a graph containing a cycle C and two adjacent vertices having no neighbors in C. Then subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. We use the same construction in Theorem 5.2. We also claim that every induced subdivision of H in G is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$.

Let φ be a model of H in G and let $C' := \varphi(C)$. Let vw be an edge of $\varphi(H)$ having no neighbors in C'. Such an edge exists by the assumption that H contains two adjacent vertices having no neighbors in C.

As A is independent, one of v and w is not contained in A. Without loss of generality, we assume $v \in V(H_P) \setminus A$ for some $P \in \mathcal{P}(UB_n)$. Since v has no neighbor in C', C' should be contained in $V(H_P) \cup A$. By construction, C' has at least $|V(H)|$ vertices in $H_P - A$. Thus, if $\varphi(H)$ contained a vertex in $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n) \setminus \{P\}$, then it would have degree at least $|V(H)|$, which is not possible. This implies that $V(F) \subseteq V(H_P) \cup A$, and again by Lemma 5.1, F contains P. The remaining part is the same as in Theorem 5.2. □

6. Negative results using the semi-grid

In this section, we show that if H contains a cycle and three vertices that have no neighbors in C, then H-subdivisions have no induced Erdős-Pósa property. For example, we may consider a graph F that is the disjoint union of C_3 and three isolated vertices. Clearly, triangle-walls contain F as an induced subgraph. Also, in the construction based on hypergraphs in Section 5, we have an independent set A, and we can choose some C_4 by picking two vertices in $V(H_P) \setminus A$ and $V(H_{P'}) \setminus A$ for some distinct $P, P' \in \mathcal{P}(UB_n)$, and then choose three vertices in A having no neighbors in C_4 (for this, we choose P, P' so that $A \setminus (P \cup P')$ has 3 vertices). Thus, we need a new construction.

For $p \in \mathbb{N}_{\geq 3}$, we define the *semi-grid* SG_n of order n as follows:

- $V(SG_n) = \{v_{i,j} : i, j \in \{1, \dots, n-1\}, i \geq j\}$;
- we add the edges of the paths $P_1 = v_{1,1}v_{1,2} \cdots v_{1,n-1}$ and $P_n = v_{n-1,1}v_{n-1,2} \cdots v_{n-1,n-1}$, and, for every $i \in \{2, \dots, n-1\}$,
$$P_i = v_{1,i-1}v_{2,i-1} \cdots v_{i-1,i-1}v_{i,i} \cdots v_{i,n-1};$$
- additionally, we add an edge between two vertices v and w if there is no $i \in \{1, \dots, n\}$ such that $v, w \in V(P_i)$.

Observe that SG_n satisfies the following, for every $i, j \in \{1, \dots, n\}$ with $i < j$:

- P_i is an induced path;
- $V(P_i) \cap V(P_j) = \{v_{i,j-1}\}$;
- every vertex of SG_n belongs to exactly two paths of $P_1, \dots, P_n$;
- $V(SG_n) \setminus V(P_i)$ has $n-3$ neighbors in P_i.

Theorem 6.1 (item (4) of Theorem 1.5)**.** *Let H be a graph that contains a cycle C and three vertices having no neighbors in C. Then subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. Let $m := |V(H)|$. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq m+3}}$ of graphs such that $\nu_H(G_n) = 1$ while $\tau_H(G_n) \geq \frac{n}{2}$ for every $n \in \mathbb{N}_{\geq m+3}$.

Let $n \geq m+3$ be an integer and let H' be the graph obtained from H by subdividing each edge into a path of length n. In particular, we subdivide uv into $uw_1w_2 \cdots w_{n-1}v$. Let us fix an edge uv of C. The graph $G = G_n$ is constructed from SG_n as follows:

- we take copies $H_1, H_2, \dots, H_n$ of H' and for each $j \in \{1, \dots, n\}$, we identify the copy of $w_1w_2 \cdots w_{n-1}$ in H_j with the path P_j;
- then we add all possible edges between $V(H_j) \setminus V(P_j)$ and $V(G) \setminus V(H_j)$.

Clearly, each H_j is an induced subdivision of H. We show that every induced subdivision of H in G is H_j for some $j \in \{1, \dots, n\}$.

Claim 6.2. *Every induced subdivision of H in G is contained in H_j for some $j \in \{1, \dots, n\}$ and contains the path P_j.*

Proof. Let φ be a model of H in G, and let $F := \varphi(H)$ and $C' := \varphi(C)$, and $Z = \{z_1, z_2, z_3\}$ be a set of three vertices in F having no neighbors in C'. Such a set Z exists because H contains three vertices having no neighbors in C.

We consider two cases depending on whether Z contains a vertex of $V(H_j) \setminus V(P_j)$ for some $j \in \{1, \dots, n\}$, or not.

First case: Z contains a vertex of $V(H_j) \setminus V(P_j)$ for some $j \in \{1, 2, \dots, n\}$. By construction, for all $a \in V(H_j) \setminus V(P_j)$ and $b \in V(G) \setminus V(H_j)$, a is adjacent to b. Thus, C' is contained in H_j.As P_j is a path, C' contains a vertex of $V(H_j) \setminus V(P_j)$. It implies that Z is also contained in H_j.

First assume C' contains P_j. In this case, every vertex in $V(G) \setminus V(H_j)$ has at least $n-3 \geq |V(H)|$ neighbors in P_j. As the maximum degree of H is at most $m-1 < n-3$, no vertex of F belongs to $V(G) \setminus V(H_j)$. Hence F are contained in H_j. By assumption, F contains P_j so we are done.

In the remaining case C' does not contain P_j. As $V(C') \subseteq H_j$ and P_j is a path, we deduce that C' belongs to $V(H_j) \setminus V(P_j)$. By construction every vertex in $V(G) \setminus V(H_j)$

dominates C'. Since C' contains at least m vertices while H has maximum degree less than m, all other vertices of F are contained in H_j. If F does not contain an edge in P_j, then F contains no induced subdivision of H by Lemma 5.1. Therefore, F contains P_j, as required.

Second case: $Z \subseteq V(SG_n)$. First assume z_1, z_2, z_3 are all contained in P_i for some $i \in \{1, \dots, n\}$. We can observe that every vertex of $V(G) \setminus V(P_i)$ has a neighbor in $\{z_1, z_2, z_3\}$. Thus, C' is contained in H_i. Since P_i is a path and z_1 is already in P_i, C' is contained in $V(H_i) \setminus V(P_i)$. Since C' contains at least m vertices and every vertex in $V(G) \setminus V(H_i)$ dominates C', all other vertices of F are contained in H_i. If F does not contain an edge in P_j, then F contains no induced subdivision of H by Lemma 5.1. Therefore, F contains P_j.

We can assume that two of z_1, z_2, z_3 are not contained in the same path P_i of SG_n. Without loss of generality, we assume $z_1 \in V(P_a) \cap V(P_{a'})$ and $z_2 \in V(P_b) \cap V(P_{b'})$ such that a, a', b, b' are all distinct. Therefore the four vertices $u_{a,b} \in V(P_a) \cap V(P_b)$, $u_{a,b'} \in V(P_a) \cap V(P_{b'})$, $u_{a',b} \in V(P_{a'}) \cap V(P_b)$, and $u_{a',b'} \in V(P_{a'}) \cap V(P_{b'})$ are the only vertices in G that are not adjacent to both z_1 and z_2. Thus, $V(C') \subseteq \{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\}$.

Now, if z_3 is not contained in one of $P_a, P_{a'}, P_b, P_{b'}$, then z_3 has a neighbor in C', a contradiction. Therefore, z_3 is contained in one of these four paths. Without loss of generality we assume that z_3 is contained in P_a.

First assume $z_3 \in \{u_{a,b}, u_{a,b'}\}$. Since C' is a cycle, $V(C') = \{u_{a,b}, u_{a,b'}, u_{a',b}, u_{a',b'}\} \setminus \{z_3\}$. But then, z_3 has a neighbor ($u_{a',b'}$ or $u_{a',b}$) in C', a contradiction. We may then assume $z_3 \notin \{u_{a,b}, u_{a,b'}\}$. Since C' contains a cycle, it has at least 3 vertices; thus, it contains a vertex in $\{u_{a',b}, u_{a',b'}\}$. Thus, z_3 has a neighbor in C', a contradiction.

This concludes the proof of the claim. ◇

By Claim 6.2, every induced H-subdivision in G contains P_j for some $j \in \{1, \dots, n\}$. Thus, two induced H-subdivisions always intersect, and hence $\nu_H(G) = 1$. Let S be a vertex subset of size less than $\frac{n}{2}$. Clearly each vertex of S hits at most two models in the graph. Thus, S hits less than $2(\frac{n}{2}) = n$ models, and $G - S$ contains an induced subdivision of H. This implies that $\tau_H(G) \geq \frac{n}{2}$, as required. □

7. Subdivisions of 1-pan or 2-pan have the induced Erdős-Pósa property

Recall that for $p \in \mathbb{N}$, the p-pan is the graph obtained by adding an edge between a triangle and a path on p vertices. We show in this section that for every $p \in \{1, 2\}$, the subdivisions of the p-pan have the induced Erdős-Pósa property with bounding function $O(k^2 \log k)$ (Theorem 7.2 and Theorem 7.7). If $p \geq 3$, the p-pan has an edge that has no neighbor in its triangle, and by Theorem 5.4, the subdivisions of the p-pan do not have the induced Erdős-Pósa property. This proves the dichotomy (3) of Corollary 1.6. Furthermore, the proofs of our positive results yield polynomial-time algorithms for finding a packing of induced subdivisions of the p-pan or a hitting set of bounded size.

The proofs for 1-pan and 2-pan start similarly and the end of the proof for the 2-pan is more technical. Let $p \in \{1, 2\}$, let G be a graph and let H be such that:

- H is an induced subdivision of the p-pan in G with the minimum number of vertices, and
- $G - V(H)$ has no induced subdivision of the p-pan.

With these assumptions, we will show that for every $k \in \mathbb{N}$, G contains either k pairwise vertex-disjoint induced subdivisions of the p-pan, or a vertex set S of size $O(k \log k)$

hitting all induced subdivisions of the p-pan. By applying inductively this result we can conclude that a graph G contains either k pairwise vertex-disjoint induced subdivision of the p-pan, or a vertex set of size $O(k^2 \log k)$ hitting all induced subdivisions of the p-pan. The following will be necessary in the proofs.

Lemma 7.1. *Let p be a fixed positive integer. Given a graph G, one can in time $\mathcal{O}(|V(G)|^{p+5})$ find an induced subdivision of the p-pan with the minimum number of vertices, if one exists.*

Proof. We first describe how, given vertices $v_1, v_2, \ldots, v_p$ and w_1, w_2, w_3, one can in $O(n^2)$ steps,

- either find an induced subdivision of the p-pan where $v_1 \ldots v_p w_2$ is the path, both w_1 and w_2 adjacent to w_1, and that, additionally, has minimum order;
- or correctly conclude that such a p-pan does not exist in G.

For this, we first check that $v_1 v_2 \cdots v_p w_2$ is an induced path, and that no edge exists between $\{v_1, v_2, \ldots, v_p\}$ and $\{w_1, w_2, w_3\}$ other than $v_p w_2$, $w_1 w_2$, and $w_2 w_3$. If one of them is not satisfied, we answer negatively.

In $G - \{v_1, v_2, \ldots, v_p, w_2\} - (N_G(\{v_1, v_2, \ldots, v_p, w_2\}) \setminus \{w_1, w_3\})$, we then compute a shortest path P from w_1 to w_3. If such a path does not exist, we answer negatively. Otherwise, $G[\{v_1, v_2, \ldots, v_p, w_2\} \cup V(P)]$ is the desired induced subdivision of the p-pan.

We apply the above procedure for every choice of $p+3$ vertices of G. In the end, we output an induced subdivision of the p-pan of minimum order among those returned, if any. Overall this takes $\mathcal{O}(|V(G)|^{p+5})$ steps. If S is an induced subdivision of the p-pan in G, then one choice of $v_1, v_2, \ldots, v_p$ corresponds to the path of length p of S and w_1 and w_3 to the two neighbors of w_2 on the cycle. A shortest path from w_1 to w_3 yields an induced subdivision of the p-pan of order at most $|S|$. This guarantees that the algorithm outputs a induced subdivision of minimum order. ☐

7.1. **On subdivisions of the 1-pan.** The aim of this section is to prove the following theorem.

Theorem 7.2. *There is a polynomial-time algorithm which, given a graph G and a positive integer k, finds either k vertex-disjoint induced subdivisions of the 1-pan in G or a vertex set of size at most $\mathcal{O}(k^2 \log k)$ hitting every induced subdivision of the 1-pan in G.*

Let us refer to an induced subdivision of the 1-pan as a pair (v, C), where v is the vertex of degree one and C is the cycle. We start with structural lemmas.

Lemma 7.3. *Let G be a graph and let (u, C) be an induced subdivision of the 1-pan in G with the minimum number of vertices. If $|V(C)| \geq 7$ then every vertex of $V(G) \setminus V(C)$ has at most one neighbor on C.*

Proof. Clearly, u has exactly one neighbor in C. Suppose a vertex $v \in V(G) \setminus (V(C) \cup \{u\})$ has at least two neighbors in C. We choose two neighbors w_1, w_2 of v that are at minimum distance on C. Let Q be a shortest path from w_1 to w_2 in C. Then $G[V(Q) \cup \{v\}]$ is an induced cycle, and it is strictly shorter than C, as $|V(Q)| + 1 \leq |V(C)|/2 + 1 < |V(C)|$.

For each $i \in \{1, 2\}$, let w_i' be the neighbor of w_i in C, which is not on the path Q. Since (u, C) is an induced subdivision with minimum number of vertices, $G[V(Q) \cup \{v, w_i'\}]$

is not an induced subdivision of the 1-pan, and thus v should be adjacent to w_1' and w_2'. Note that w_1' is not adjacent to w_2'; otherwise, $\mathrm{dist}_C(w_1, w_2) \leq \mathrm{dist}_C(w_1', w_2') = 1$ and we would have $|V(C)| \leq 4$, a contradiction. Therefore, $G[\{v, w_1, w_1', w_2'\}]$ is isomorphic to the 1-pan, contradicting the assumption that (u, C) is an induced subdivision with minimum number of vertices.

We conclude that every vertex in $V(G) \setminus V(C)$ has at most one neighbor in C. □

Lemma 7.4. *Let $C = v_1 v_2 \cdots v_m v_1$ be an induced cycle of length at least 4 in a graph G, and let $v \in V(G) \setminus V(C)$ such that v is adjacent to v_3, and non-adjacent to v_1, v_2, v_4. Then $G[V(C) \cup \{v\}]$ contains an induced subdivision of the 1-pan.*

Proof. If v has no neighbors in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$, then it is clear. We may assume v has a neighbor in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$. We choose a neighbor v_i of v in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$ with minimum i. Then $G[\{v, v_2, v_3, v_4, \dots, v_i\}]$ is an induced subdivision of the 1-pan in G. □

Suppose (u, C) is an induced subdivision of the 1-pan in a graph G. In the next lemma, we apply Lemma 2.1 to obtain, given many $V(C)$-paths, many induced subdivisions of the 1-pan.

Lemma 7.5. *Let $k \in \mathbb{N}_{\geq 1}$, let G be a graph, and (v, C) be an induced subdivision of the 1-pan in G with the minimum number of vertices. Given a set of at least $24k \log 2k$ vertex-disjoint $V(C)$-paths in $G - E(C)$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 1-pan.*

Proof. Let $\mathcal{P}$ be a given set of $\lceil 24k \log 2k \rceil$ vertex-disjoint $V(C)$-paths of $G - E(C)$.

We consider the subgraph H on the vertex set $V(C) \cup (\bigcup_{P \in \mathcal{P}} V(P))$ and edge set $E(C) \cup (\bigcup_{P \in \mathcal{P}} E(P))$. This graph has maximum degree 3 and has at least $48k \log 2k$ vertices of degree 3, as each path of $\mathcal{P}$ contributes for two vertices of degree 3.

According to Lemma 2.1, one can in polynomial time construct a set $\mathcal{Q}'$ of $2k$ vertex-disjoint $V(C)$-cycles in H. Observe that C intersects all other cycles, and thus, C is not contained in $\mathcal{Q}'$. Note that for each cycle U of $\mathcal{Q}'$, $G[V(U)]$ contains an induced cycle that has at least one edge of C. This is always possible: in the cycle U, every chord e divides the cycle into two paths, one of which, together with e, is again a cycle containing an edge of C and less chords. Let $\mathcal{Q}$ be the collection of $2k$ resulting induced $V(C)$-cycles of G obtained from $\mathcal{Q}'$.

If U is a cycle of G and $q \in V(G) \setminus V(U)$, we call (q, U) a *good pair* if $G[\{u\} \cup V(U)]$ contains an induced subdivision of the 1-pan. For each cycle $U \in \mathcal{Q}$, there are four consecutive vertices u, v, w, z in C such that $vw \in E(C) \cap E(U)$ and $u \notin E(C)$. Let q be the neighbor of v in C other than w. Note that q is adjacent to v but has no neighbors in $\{u, w, z\}$. Therefore, $G[\{q\} \cup V(U)]$ contains an induced subdivision of the 1-pan by Lemma 7.4: (q, U) is a good pair. Note that the vertex q is uniquely determined by the vertices u, v, w, z of U.

Based on this fact, to find k pairwise vertex-disjoint induced subdivisions of the 1-pan, it is sufficient to find a collection of k good pairs $(q_1, C_1), (q_2, C_2), \dots, (q_k, C_k)$ such that

- $C_1, \dots, C_k \in \mathcal{Q}$ are distinct;
- for $i, j \in \{1, \dots, k\}$, q_i is not contained in C_j.

A collection of good pairs satisfying the above conditions is called *valid*.

We consider the auxiliary graph F with a vertex for each cycle of $\mathcal{Q}$ and where two vertices are adjacent if and only if there is an edge between the corresponding cycles. For every $u \in V(F)$, we denote by C_u the corresponding cycle of $\mathcal{Q}$. For convenience, let $C = z_1 z_2 \cdots z_m z_1$ and let $z_0 := z_m$.

We first consider the set I of all isolated vertices of F. For each cycle $U \in \mathcal{Q}$ corresponding to a vertex of I, let j be the minimum integer such that $z_j \in V(U) \cap V(C)$, and $f(U) := z_{j-1}$. Then $\mathcal{U} := \{(f(C_u), C_u) : u \in I\}$ is a valid set of good pairs. We may assume $|I| < k$, otherwise, we already found the k required good pairs.

We will add good pairs to $\mathcal{U}$ using the remaining cycles. For this we construct a sequence $F_{|I|}, F_{|I|+1}, \ldots, F_{k-1}$ defined as follows.

Let $F_{|I|} := F - I$. Note that $F_{|I|}$ contains no isolated vertices. Since $|V(F)| \geq 2k$, $F_{|I|}$ contains at least $2k - |I|$ vertices. For each $i \in \{|I|, \ldots, k-1\}$, we do the following.

(1) Suppose F_i contains a vertex of degree 1. We choose a vertex a in F_i adjacent to maximum number of degree-1 vertices in F_i. Let $a_1, \ldots, a_t$ be the leaf neighbors of a in F_i. By construction of $\mathcal{Q}$, two cycles C_{a_p} and C_{a_q} never have the same neighbor in C_a. Thus, we can choose t vertices $x_1, \ldots, x_t \in V(C_a) \cap V(C)$ such that

$$\mathcal{U} \cup \{(x_p, C_{x_p}) : p \in \{1, \ldots, t\}\}$$

is a valid set of good pairs.

We add these t good pairs to $\mathcal{U}$, and let $F_m := F_i - \{a, a_1, a_2, \ldots, a_t\}$ where $m = \min(k, i + t)$.

(2) Suppose F_i contains no vertices of degree 1. We choose any edge ab of F_i and choose $u \in V(C_b) \cap V(C)$ that is a vertex having a neighbor in C_a. Note that $\mathcal{U} \cup \{(u, C_a)\}$ is a valid set of good pairs. Let $F_{i+1} := F_i - \{a, b\}$.

In the above procedure, we preserve the property that for $|I| \leq i < k$, $|V(F_i)| \geq 2(k - i) > 0$ and F_i contains no isolated vertices. Therefore, we can always apply (1) or (2), and finally obtain F_k at the end, which means that we obtain a required set of k good pairs. Using Lemma 7.4, one can output k vertex-disjoint induced subdivisions of the 1-pan in polynomial time. □

We are now ready to prove the main result of this section.

Proof of Theorem 7.2. We construct sequences $G_1, \ldots, G_{\ell+1}$ and $F_1, \ldots, F_\ell$ such that

- $G_1 = G$,
- for every $i \in \{1, \ldots, \ell\}$, F_i is an induced subdivision of the 1-pan in G_i with the minimum number of vertices,
- for every $i \in \{1, \ldots, \ell\}$, $G_{i+1} = G_i - V(F_i)$.

Such a sequence can be constructed in polynomial time repeatedly applying Lemma 7.1. If $\ell \geq k$, then we have found a packing of k induced subdivisions of the 1-pan. Hence, we may assume that $\ell \leq k - 1$.

Let $\mu_k := 48k \log k + 20k + 1$. The rest of the proof relies on the following claim.

Claim 7.6. *Let $j \in \{1, \ldots, \ell + 1\}$. One can find in polynomial time either k vertex-disjoint induced subdivisions of the 1-pan, or a vertex set X_j of G_j of size at most $(\ell + 1 - j)\mu_k$ such that $G_j - X_j$ has no induced subdivision of the 1-pan.*

Proof. We prove the claim by induction for $j = \ell + 1$ down to $j = 1$. The claim trivially holds for $j = \ell + 1$ with $X_{\ell+1} = \emptyset$ because $G_{\ell+1}$ has no induced subdivision of the 1-pan. Let us assume that for some $j \leq \ell$, we obtained a required vertex set X_{j+1} of

G_{j+1} of size at most $(\ell - j)\mu_k$. Then in $G_j - X_{j+1}$, F_j is an induced subdivision of the 1-pan with the minimum number of vertices. If F_j has at most 7 vertices, then we set $X_j := X_{j+1} \cup V(F_j)$. Clearly, $|X_j| \leq (\ell - j + 1)\mu_k$. We may thus assume F_j has at least 8 vertices. Let $F_j := (u, C)$.

We first apply Gallai's A-path Theorem (Theorem 2.2) with $A = V(C)$ for finding at least $24k \log 2k$ pairwise vertex-disjoint $V(C)$-paths in $(G_j - X_{j+1}) - E(C)$. Assume it outputs such $V(C)$-paths. Then, by applying Lemma 7.5 to $G_j - X_{j+1}$ and $V(C)$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 1-pan. Thus, we may assume that Theorem 2.2 outputs a vertex set S of size at most $48k \log 2k$ hitting all $V(C)$-paths in $(G_j - X_{j+1}) - E(C)$.

Now, we consider the graph $G'_j := G_j - (X_{j+1} \cup S \cup \{u\})$. Suppose G'_j contains an induced subdivision $Q = (d, D)$ of the 1-pan. Then $G'_j[V(C) \cap V(Q)]$ is connected; otherwise, G'_j contains a $V(C)$-path in $(G_j - X_{j+1}) - E(C)$, contradicting with that S hits all such $V(C)$-paths. Furthermore, $G'_j[V(C) \cap V(Q)]$ contains no edge of D; otherwise, we also have a $V(C)$-path in $(G_j - X_{j+1}) - E(C)$. Thus, we have that $|V(C) \cap V(Q)| \leq 2$.

We recursively construct sets $U \subseteq V(C)$ and $\mathcal{J}$ as follows. At the beginning set $U := \emptyset$ and $\mathcal{J} := \emptyset$. For every set of four consecutive vertices v_1, v_2, v_3, v_4 of C with $v_2, v_3 \notin U$, we test whether $G'_j[V(G'_j) \setminus V(C) \cup \{v_2, v_3\}]$ contains an induced subdivision H of the 1-pan, and if so, add vertices v_1, v_2, v_3, v_4 to U, and add H to $\mathcal{J}$ and increase the counter by 1. If the counter reaches k, then we stop. Note that graphs in $\mathcal{J}$ are pairwise vertex-disjoint; otherwise, we can find a $V(C)$-path, a contradiction. Thus, if the counter reaches k, we can output k pairwise vertex-disjoint induced subdivisions of the 1-pan.

Assume the counter does not reach k. Then we have $|U| \leq 4(k-1)$. We take the 1-neighborhood U' of U in C, and observe that $|U'| \leq 12(k-1)$. We claim that $G'_j - U'$ has no induced subdivision of the 1-pan. Suppose for contradiction that there is an induced subdivision F of the 1-pan in $G'_j - U_j$. The intersection of F on C is a set of at most two consecutive vertices, say T. In case when $|T| = 1$, neighbors of T in C are not contained in U, as U' is the 1-neighborhood of U in C. Thus, we can increase the counter by adding this induced subdivision to $\mathcal{J}$, a contradiction. Therefore, $G'_j - U'$ has no induced subdivision of the 1-pan, and $X_j := X_{j+1} \cup S \cup \{u\} \cup U'$ satisfies the claim. ◇

The result follows form the claim with $j = 1$. □

7.2. **On subdivisions of the 2-pan.** This section is devoted to the proof of the following result.

Theorem 7.7. *There is a polynomial-time algorithm which, given a graph G and a positive integer k, finds either k vertex-disjoint induced subdivisions of the 2-pan in G or a vertex set of size at most $\mathcal{O}(k^2 \log k)$ hitting every induced subdivision of the 2-pan in G.*

We refer to an induced subdivision of the 2-pan as a tuple (v_1, v_2, C), where v_1, v_2 are respectively the vertices of degree one and two that are not contained in the cycle, and C is the cycle.

As in the 1-pan case, we first obtain a structural property. For a subgraph H of a graph G and $v \in V(G) \setminus V(H)$, we say that v *dominates* H if v is adjacent to every vertex of H.

Lemma 7.8. *Let G be a graph and let (v_1, v_2, C) be an induced subdivision of the 2-pan in G with the minimum number of vertices such that $|V(C)| \geq 11$. For every vertex v in $V(G) \setminus V(H)$, either v has at most one neighbor in C or it dominates C.*

Proof. Observe that each of v_1 and v_2 has at most one neighbor in C. Let $v \in V(G) \setminus (V(C) \cup \{v_1, v_2\})$ and suppose it has at least two neighbors in C and has a non-neighbor z. We choose two neighbors w_1, w_2 of v at minimum distance on C and that, additionally, minimize the distance to z on C. Let Q be a shortest path of C from w_1 to w_2. Then $G[V(Q) \cup \{v\}]$ is an induced cycle, and it is strictly shorter than C, as $|V(Q)| + 1 \leq |V(C)|/2 + 1 < |V(C)|$.

For each $i \in \{1, 2\}$, let x_i be the neighbor of w_i in C, which is not on the path Q, and let y_i be the neighbor of x_i in C other than w_i. Note that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$ and furthermore, there are no edges between $\{x_1, y_1\}$ and $\{x_2, y_2\}$; otherwise, C would have length at most 10, a contradiction.

Observe that for each $i \in \{1, 2\}$, v has a neighbor in $\{x_i, y_i\}$; otherwise, $G[V(Q) \cup \{v, x_i, y_i\}]$ is an induced subdivision of the 2-pan, which has smaller number of vertices than (v_1, v_2, C).

We claim that v is complete to $\{x_1, y_1, x_2, y_2\}$. Suppose not. Without loss of generality, we may assume v has a non-neighbor in $\{x_1, y_1\}$ and a neighbor in $\{x_2, y_2\}$. Hence there is an induced cycle C' in $G[\{v, w_2, x_2, y_2\}]$ that contains v. Depending which of x_1, y_1 is not adjacent to v, one of (x_1, y_1, C') and (y_1, x_1, C') is an induced subdivision of the 2-pan with less vertices than (v_1, v_2, C). Consequently, v is complete to $\{x_1, y_1, x_2, y_2\}$. By the choice of w_1, w_2, this also implies that these vertices are adjacent. But then z is closer on C to one of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ than to $\{w_1, w_2\}$. As this contradicts the definition of w_1 and w_2, we conclude that v dominates C. □

The following is a 2-pan counterpart of Lemma 7.4.

Lemma 7.9. *Let $C = v_1 v_2 \cdots v_m v_1$ be an induced cycle of length at least 4 in a graph G and $w_1, w_2, w_3, w_4 \in V(G) \setminus V(C)$ such that*

(1) $v_2 w_1 w_2 w_3 w_4$ is an induced path,
(2) $v_2 w_1$ is the only edge between $\{v_1, v_2, v_3, v_4\}$ and $\{w_1, w_2, w_3, w_4\}$, and
(3) each vertex in C has at most one neighbor in $\{w_1, w_2, w_3, w_4\}$.

Then $G[V(C) \cup \{w_1, w_2, w_3, w_4\}]$ contains an induced subdivision of the 2-pan.

Proof. If $v_2 w_1$ is the only edge between $\{w_1, w_2\}$ and C, then (w_1, w_2, C) is an induced subdivision of the 2-pan. We may assume C has length at least 5, and w_1 or w_2 has a neighbor in $C - \{v_1, v_2, v_3, v_4\}$. We choose $i \in \{5, \ldots, m\}$ such that:

- v_i has a neighbor in $\{w_1, w_2\}$; and
- no internal vertex of the path $P = v_i v_{i+1} \ldots v_m v_1 v_2$ has a neighbor in $\{w_1, w_2\}$.

We may assume $i = 5$; otherwise, the induced cycle in $G[V(P) \cup \{w_1, w_2\}]$ that contains v_i together with v_3, v_4 forms an induced subdivision of the 2-pan. On the other hand, we can observe that P contains at most 2 internal vertices; otherwise the induced cycle in $G[\{w_1, w_2, v_2, v_3, v_4, v_5\}]$ that contains v_i together with two internal vertices in P forms an induced subdivision of the 2-pan. We distinguish cases depending the number of internal vertices of P.

First case: P has a unique internal vertex, i.e. $m = 5$. As v_i has a neighbor in $\{w_1, w_2\}$, it has none in $\{w_3, w_4\}$ by our assumption. Therefore, $G[V(P) \cup \{w_1, w_2, w_3, w_4\}]$ contains an induced subdivision of the 2-pan.

Second case: P has exactly 2 internal vertices, i.e. $m = 6$. If v_6 has a neighbor in $\{w_1, w_2, w_3, w_4\}$, then the induced cycle in $G[\{v_6, v_1, v_2, w_1, w_2, w_3, w_4\}]$ that contains v_2 together with v_3, v_4 induces a subdivision of the 2-pan. Hence we may assume v_6 has no neighbors in $\{w_1, w_2, w_3, w_4\}$. Since v_5 has no neighbor in $\{w_3, w_4\}$ (as above), $G[V(P) \cup \{w_1, w_2, w_3, w_4\}]$ contains an induced subdivision of the 2-pan.

This concludes the lemma. □

Lemma 7.10. *Let $k \in \mathbb{N}_{\geq 1}$, let G be a graph, (v_1, v_2, C) be an induced subdivision of the 2-pan in G with the minimum number of vertices, and D be the set of all vertices dominating C. Given a set of at least $408k \log k$ vertex-disjoint $V(C)$-paths in $G - E(C) - D$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 2-pan.*

Proof. When $k = 1$ we output (v_1, v_2, C). In the sequel we assume that $k \geq 2$. Let $C = z_1 z_2 \cdots z_m z_1$ and for each $i \in \{0, 1, 2, 3\}$, let $z_{-i} := z_{m-i}$. Let $\mathcal{P}$ be the given set of $\lceil 408k \log k \rceil$ vertex-disjoint $V(C)$-paths in $G - E(C) - D$. For each $P \in \mathcal{P}$, let $\mathrm{end}(P)$ be the set of end vertices of P. We construct a subset $\mathcal{P}'$ of $\mathcal{P}$ with the following property:

$$\forall P_1, P_2 \in \mathcal{P}',\ \mathrm{dist}_C(\mathrm{end}(P_1), \mathrm{end}(P_2)) \geq 5.$$

This can be done by repeatedly choosing a path P in $\mathcal{P}$ and discarding from $\mathcal{P}$ all paths Q that have an endpoint at distance at most 4 from one of P. For each path added to $\mathcal{P}'$, at most 16 are discarded, hence $|\mathcal{P}'| \geq |\mathcal{P}|/17$.

We consider the subgraph H on the vertex set $V(C) \cup (\bigcup_{P \in \mathcal{P}'} V(P))$ and edge set $E(C) \cup (\bigcup_{P \in \mathcal{P}'} E(P))$. This graph has maximum degree 3 and has at least $24k \log k$ vertices of degree 3, as each path of $\mathcal{P}'$ contributes for two vertices of degree 3. According to Lemma 2.1, one can in polynomial time construct a set $\mathcal{Q}$ of k vertex-disjoint cycles of C. Observe that C intersects all other cycles, and thus, C is not contained in $\mathcal{Q}$.

As in the proof of Lemma 7.5 we note that for each cycle U of $\mathcal{Q}$, $G[V(U)]$ has an induced cycle containing at least one edge of C. Let $\mathcal{Q}'$ be the collection of k resulting induced $V(C)$-cycles obtained from $\mathcal{Q}$.

Now, for each cycle U of $\mathcal{Q}$, we choose a vertex $z_{\ell_U} \in V(U) \cap V(C)$ such that

$$z_{\ell_U - 1} \notin V(U) \quad \text{and} \quad z_{\ell_U + 1} \in V(U).$$

It is easy to see that such a vertex always exists, as U contains an edge of C and H has maximum degree 3.

Let $U \in \mathcal{Q}'$ and let $x, y \in V(U)$ be such that $x z_{\ell_U} z_{\ell_U+1} y$ is a subpath of U. By Lemma 7.8 and since we work in $G - D$, every vertex of U has at most one neighbor in C. In particular, $z_{\ell_U - 1} z_{\ell_U}$ is the only edge between $\{x, z_{\ell_U}, z_{\ell_U+1}, y\}$ and $\{z_i,\ \ell_U - 4 \leq i \leq \ell_U - 1\}$. Therefore, one can find an induced subdivision of the 2-pan in $G[V(U) \cup \{z_i : \ell_U - 4 \leq i \leq \ell_U - 1\}]$ using Lemma 7.9.

From the definition of $\mathcal{P}'$, the vertices in $\{z_i,\ \ell_U - 4 \leq i \leq \ell_U - 1\}$ are not contained in any of cycles in $\mathcal{Q}'$. Hence by applying Lemma 7.9 as above for every $U \in \mathcal{Q}'$ we obtain in polynomial time a collection of k vertex-disjoint induced subdivisions of the 2-pan. This concludes the proof. $\square$

The last thing to show is that in fact, an induced subdivision of the 2-pan never contains a vertex dominating C.

Lemma 7.11. *Let $k \in \mathbb{N}_{\geq 1}$, let G be a graph, (v_1, v_2, C) be an induced subdivision of the 2-pan in G with the minimum number of vertices and $|V(C)| \geq 18$, and D be the set of all vertices dominating C. Every induced subdivision of the 2-pan in $G - \{v_1, v_2\}$ has no C-dominating vertices.*

Proof. Let H be an induced subdivision of the 2-pan in $G - \{v_1, v_2\}$ containing a C-dominating vertex v. We prove two claims.

Claim 7.12. *There is no induced path $p_1p_2p_3$ in $H - v$ such that $p_1 \in V(C)$ and $p_2, p_3 \in V(G) \setminus (V(C) \cup D)$.*

Proof. Let q_1, q_2 be the neighbors of p_1 in C. As v is C-dominating, for each $i \in \{1, 2\}$, vp_1q_iv is a triangle. By minimality of (v_1, v_2, C), we deduce that q_i does not belong to H. For the same reason, v has no neighbor in $\{p_2, p_3\}$.

Let $i \in \{1, 2\}$. Consider the subgraph induced on $\{v, q_i, p_1, p_2, p_3\}$. As above the graph $G[\{v, q_i, p_1, p_2, p_3\}]$ is not a 2-pan. So q_i is adjacent to either p_2 or p_3. Since $p_2 \notin D$, p_2 has exactly one neighbor in C (which is p_1) and thus it is not adjacent to q_i. Therefore, both q_1 and q_2 are adjacent to p_3. By Lemma 7.8, p_3 is C-dominating, and thus p_3 is adjacent to p_1. This contradicts the assumption that $p_1p_2p_3$ is an induced path. We conclude that there is no such an induced path. $\diamond$

A similar observation can be made for a path continuing from a C-dominating vertex.

Claim 7.13. *There is no induced path $wvp_1p_2p_3p_4p_5$ in H such that $w \in V(C)$ and $p_1, p_2, p_3, p_4 \in V(G) \setminus (V(C) \cup D)$.*

Proof. Let q_1, q_2 be the neighbors of w in C. As above we can deduce that none of q_1 and q_2 belongs to H. Also, for every $i \in \{1, 2\}$, $G[\{v, w, q_i, p_1, p_2\}]$ is not a 2-pan, so q_i is adjacent to either p_1 or p_2. If both q_1 and q_2 are adjacent to p_1 (or p_2), then by Lemma 7.8, p_1 (or p_2) is C-dominating, a contradiction. Therefore, we may assume, without loss of generality, q_1 is adjacent to p_1, but not to p_2, and q_2 is adjacent to p_2 but not to p_1. Then the cycle $p_1p_2q_2wq_1$ with two more vertices in C induces a 2-pan, which is smaller than (v_1, v_2, C), a contradiction. We conclude that there is no such an induced path. $\diamond$

Observe that $|V(H) \cap V(C)| \leq 3$, otherwise the C-dominating vertex in H has degree 4 in H. Also, since H contains a vertex of C, we have $|V(H) \cap D| \leq 3$. So $H - (V(C) \cup D)$ has at least 14 vertices divided into at most 7 connected components, one such component has an induced path Q on at least 2 vertices. Let q be an end vertex of this path with a neighbor in $V(H) \cap (V(C) \cup D)$, that we call x. If $x \in V(C)$, then by Claim 1, we obtain a contradiction. Suppose $x \in D$. Note that since H does not contain v_1, v_2, H contains a vertex of C, say y. Since y has no neighbors in Q, by Claim 2, we obtain a contradiction.

We conclude that there is no such an induced subdivision of the 2-pan. $\square$

We can now prove Theorem 7.7.

Proof of Theorem 7.7. As in the proof for 1-pans, we construct sequences $G_1, \dots, G_{\ell+1}$ and $F_1, \dots, F_\ell$ such that

- $G_1 = G$;
- for each $i \in \{1, \dots, \ell\}$, F_i is an induced subdivision of the 2-pan in G_i with the minimum number of vertices; and
- for each $i \in \{1, \dots, \ell\}$, $G_{i+1} = G_i - V(F_i)$.

Such a sequence can be constructed in polynomial time by repeatedly applying Lemma 7.1. If $\ell \geq k$, then we have found a packing of k induced subdivisions of the 2-pan. Hence, we assume in the sequel that $\ell \leq k-1$. Let $\mu_k := 48k \log k + 20k + 1$.

Claim 7.14. *Let $j \in \{\ell+1, \dots, 1\}$. One can find in polynomial time either k vertex-disjoint induced subdivisions of the 2-pan, or a vertex set X_j of G_j of size at most $(\ell+1-j)\mu_k$ such that $G_j - X_j$ has no induced subdivision of the 2-pan.*

Proof. We prove claim by induction for $j = \ell+1$ down to $j = 1$. The claim trivially holds for $j = \ell+1$ with $X_{\ell+1} = \emptyset$ because $G_{\ell+1}$ has no induced subdivision of the 2-pan. Let us assume that for some $j \leq \ell$, we obtained a required vertex set X_{j+1} of G_{j+1} of size at most $(\ell-j)\mu_k$. Then in $G_j - X_{j+1}$, F_j is an induced subdivision of the 2-pan with the minimum number of vertices. If F_j has less than 18 vertices, then we set $X_j := X_{j+1} \cup V(F_j)$. Clearly, $|X_j| \leq (\ell - j + 1)\mu_k$. We may assume F_j has at least 18 vertices. Let $F_j := (v_1, v_2, C)$, and let D be the set of vertices in $G_j - X_{j+1}$ that dominate C.

According to Lemma 7.11, there are no induced subdivisions of the 2-pan intersecting D. Therefore, we can ignore the vertex set D.

We apply Gallai's A-path Theorem (Theorem 2.2) with $A = V(C)$ for finding at least $24k \log k$ pairwise vertex-disjoint $V(C)$-paths in $(G_j - D - X_{j+1}) - E(C)$. Assume it outputs such $V(C)$-paths. Then, by applying Lemma 7.10 to $G_j - X_{j+1}$ and $V(C)$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 2-pan. Thus, we may assume that Theorem 2.2 outputs a vertex set S of size at most $48k \log k$ hitting all $V(C)$-paths in $(G_j - D - X_{j+1}) - E(C)$.

Now, we consider the graph $G'_j := G_j - (X_{j+1} \cup D \cup S \cup \{u\})$. Suppose G'_j contains an induced subdivision $Q = (w_1, w_2, W)$ of the 2-pan. Then $G'_j[V(C) \cap V(Q)]$ is connected; otherwise, G'_j contains a $V(C)$-path in $(G_j - D - X_{j+1}) - E(C)$, contradicting the fact that S hits all such $V(C)$-paths. Furthermore, $G'_j[V(C) \cap V(Q)]$ contains no edge of W; otherwise, we also have a $V(C)$-path in $(G_j - D - X_{j+1}) - E(C)$. Thus, we have that $|V(C) \cap V(Q)| \leq 3$.

We recursively construct sets $U \subsetneq V(C)$ and $\mathcal{J}$ as follows. We start with $U := \emptyset$ and $\mathcal{J} := \emptyset$. For every set of five consecutive vertices v_1, v_2, v_3, v_4, v_5 of C with $v_2, v_3, v_4 \notin U$, we test whether $G'_j[V(G'_j) \setminus V(C) \cup \{v_2, v_3, v_4\}]$ contains an induced subdivision H of the 2-pan, and if so, add vertices v_1, v_2, v_3, v_4, v_5 to U, and add H to $\mathcal{J}$ and increase the counter by 1. If the counter reaches k, then we stop. Note that graphs in $\mathcal{J}$ are pairwise vertex-disjoint; otherwise, we can find a $V(C)$-path, a contradiction. Thus, if the counter reaches k, we can output k pairwise vertex-disjoint induced subdivisions of the 2-pan.

Assume the counter does not reach k. Then we have $|U| \leq 5(k-1)$. We take the 2-neighborhood U' of U in C, and observe that $|U'| \leq 25(k-1)$. We claim that $G'_j - U'$ has no induced subdivision of the 2-pan. Suppose for contradiction that there is an induced subdivision F of the 2-pan in $G'_j - U_j$. The intersection of F on C is a set of at most three consecutive vertices, say T. In case when $|T| = 1$, the vertices of C that are at distance at most 2 from T in C are not included in U, as U' is the 2-neighborhood of U in C. Thus, we can increase the counter by adding this induced subdivision to $\mathcal{J}$, a contradiction. Therefore, $G'_j - U'$ has no induced subdivision of the 2-pan, and $X_j := X_{j+1} \cup S \cup \{u\} \cup U'$ satisfies the claim. ⋄

The result follows form the claim with $j = 1$. □

8. Subdivisions of the diamond have the induced Erdős-Pósa property

In this section, we prove that subdivisions of the diamond have the induced Erdős-Pósa property.

Theorem 8.1. *There exists a polynomial function $g : \mathbb{N} \to \mathbb{N}$ satisfying the following: For every graph G and positive integer k, either G contains k vertex-disjoint induced subdivisions of the diamond in G or it has a vertex set of size at most $g(k)$ hitting every induced subdivision of the diamond.*

Remark that while g is a polynomial, our upper-bound on its order is large: $g(k) = O(N(4, 3k)) = O(k^{968})$, where N is the function of Lemma 2.3.

We follow a similar line of proofs as in the previous section. We first deal with the case where the considered graph G has an induced diamond subdivison S such that $G - V(S)$ does not have any. Observe that a subdivision of the diamond consists of three disjoint paths from a vertex to another vertex. With this observation, we can reduce problem to the setting where, given a graph G and a vertex set A inducing a path, we aim at finding either many vertex-disjoint induced subdivisions of the diamond or a small vertex set hitting all the induced subdivisions of the diamond that meet A.

For a vertex subset S of a graph G and $v \in V(G) \setminus S$, a path is called a (v, S)-path, if it starts with v and ends at a vertex in S and contains no other vertices in S.

8.1. Structural lemmas.

We use the Erdős-Pósa property of A-ℓ-combs, recently developed by Bruhn, Heinlein, and Joos [BHJ17], to first exclude induced subdivisions of a special type. Since we only use it with $\ell = 3$, we name it A-claw for simplicity. Given a graph G and a vertex subset A, we say that a subgraph H of G is called an *A-claw* if

- H contains a vertex $v \in V(G) \setminus A$,
- H consists of three (v, A)-paths P_1, P_2, P_3 where for distinct $i, j \in \{1, 2, 3\}$, $(V(P_i) \setminus \{v\}) \cap (V(P_j) \setminus \{v\}) = \emptyset$.

For convenience, if H is a subgraph of G, then we refer to $V(H)$-claws as *H-claws*. The *leaves* of an A-claw are its vertices in A.

Lemma 8.2. *Let Q be an induced path and F be a Q-claw in a graph G. Then $G[V(F) \cup V(Q)]$ contains an induced subdivision of the diamond.*

Proof. We apply induction on $|V(F) \cup V(Q)|$. Let v be the vertex of $V(F) \setminus V(Q)$ that is connected via the paths P_1, P_2, P_3 to Q, forming the claw F. Without loss of generality, we assume that the end vertices of P_1, P_2, P_3 appear in Q in this order and that P_1 and P_3 meet Q on its endpoints. For each $i \in \{1, 2\}$, let Q_i be the subpath of Q from the end vertex of P_i to the end vertex of P_{i+1}. Let $H := G[V(F) \cup V(Q)]$.

If H contains no edges other than those in $P_1 \cup P_2 \cup P_3 \cup Q$, then clearly, it is an induced subdivision of the diamond. Therefore, we may assume that H contains some edge, which is not an edge of P_1, P_2, P_3, or Q. We may also assume that each of

P_1, P_2, P_3 is an induced path; otherwise we could shorten it and apply the induction hypothesis.

Suppose there is an edge wz such that w is an internal vertex of P_1 and z is an internal vertex of $P_2 \cup Q_1$. Then from z, there are three paths to P_1, where the end vertices of two paths are the end vertices of P_1. As P_1 is an induced path, by induction hypothesis, $G[V(P_1 \cup P_2 \cup Q_1)]$ contains an induced subdivision of the diamond. We may assume that there is no such an edge. We can apply the same argument for all pairs $(P_2, P_1 \cup Q_1), (P_2, P_3 \cup Q_2), (P_3, P_2 \cup Q_2)$.

Since Q is an induced path, one of w and z is contained in $F - V(Q)$. If $w \in V(P_2) \setminus V(Q)$, then z is contained in $P_1 \cup P_3 \cup Q$, which is not possible by the previous argument. Therefore, we may assume that w is an internal vertex of P_1 without loss of generality. Since $z \notin V(P_2 \cup Q_1)$, z is an internal vertex of $P_3 \cup Q_2$.

Let r be the neighbor of v in P_2, and let P_1' be the subpath of P_1 from v to w. Note that $G[V(P_1') \cup \{r\}]$ is an induced path, and there are three paths from z to P_1', namely, two paths along $P_2 \cup P_3 \cup Q_2$ and wz. Since the union of those paths do not contain the end vertex of P_1 in Q, we obtained a smaller induced subgraph graph satisfying the premisses of the lemma. By the induction hypothesis, it contains an induced subdivision of the diamond. This completes the proof. ☐

Bruhn, Heinlein, and Joos [BHJ17] showed that A-claws have the Erdős-Pósa property.

Theorem 8.3 (Bruhn, Heinlein, and Joos [BHJ17])**.** *There exists a constant α satisfying the following. For every graph G and every $A \subseteq V(G)$, G either contains k vertex-disjoint A-claws, or has a vertex set of size αk hitting all A-claws.*

For a set $A \subseteq V(G)$, a *Tutte bridge* of A in G is a subgraph of G consisting of one component C of $G - A$ and all edges joining C and A and all vertices of A incident with those edges.

Lemma 8.4. *Let Q be an induced path in a graph G and H be a Tutte bridge of $V(Q)$ in G such that $|V(H) \cap V(Q)| \geq 3$. Then H contains a Q-claw.*

Proof. Let a, b, c be three distinct vertices in $V(H) \cap V(Q)$, and let a', b', c' be their (not necessarily distinct) neighbors in $H - V(Q)$, respectively. Note that $H - V(Q)$ is connected by definition. Let P be a shortest path from a' to b' in $H - V(Q)$. If P contains c', then P together with aa', bb', cc' forms a Q-claw, which consists of three paths from c' to Q. We may assume that P does not contain c'. Let R be a shortest path from c' to P in $H - V(Q)$. Then we have a Q-claw starting from the end vertex of R other than c', to $V(Q)$, as required. ☐

After hitting all Q-claws using Theorem 8.3, we may assume that there is no Tutte bridge of $V(Q)$ in G such that $|V(H) \cap V(Q)| \geq 3$ by Lemma 8.4. The following lemma shows that if there is a Tutte bridge containing a cycle C and connected to Q via two disjoint paths (with some additional properties), then one can also find an induced subdivision of the diamond. Note that first two conditions do not always imply the existence of an induced subdivision of the diamond; if ab is an edge that is a component of $G - V(Q)$ and $N(a) \cap V(Q) = N(b) \cap V(Q) = \{p, q\}$ for some two consecutive vertices p, q in Q, then this Tutte bridge induces a K_4, that does not contain an induced subdivision of the diamond. To avoid this, we require that two vertices in $V(H) \cap V(Q)$ are not consecutive.

Lemma 8.5. *Let Q be an induced path in a graph G and H be a Tutte bridge of $V(Q)$ in G. If H contains a cycle connected to Q via two vertex-disjoint paths whose endpoints on Q are not adjacent, then $G[V(H) \cup V(Q')]$ contains an induced subdivision of the diamond, where Q' is a minimal subpath of Q containing $V(H) \cap V(Q)$.*

Proof. We consider a minimal counterexample. Let $V(H) \cap V(Q) = \{v_1, v_2\}$, let P_1 and P_2 be the two paths from $V(H) \cap V(Q)$ to C such that $v_1 \in V(P_1)$ and $v_2 \in V(P_2)$. By the given condition, v_1 is not adjacent to v_2. We may assume that $V(H) = V(C) \cup V(P_1) \cup V(P_2)$; if H contains more vertices, then we may consider the graph obtained by removing a vertex not contained in $V(C) \cup V(P_1) \cup V(P_2)$. Also, we may assume that P_1 and P_2 are induced, otherwise, we can take a shorter path.

Let w_1 and w_2 be the end vertices of P_1 and P_2 on C, respectively, and let Q_1 and Q_2 be the two subpaths from w_1 to w_2 in C.

We claim that C is induced. If w_1 is adjacent to w_2, then C is induced, otherwise, we can take an induced cycle in $G[V(C)]$ containing the edge w_1w_2, which is shorter than C. We may assume that $w_1w_2 \notin E(G)$. In this case, each of Q_1 and Q_2 is induced, otherwise, we can take a shorter cycle. For contradiction, suppose that there is an edge between an internal vertex z_1 of Q_1 and an internal vertex z_2 of Q_2. If $Q_1 = w_1z_1w_2$ and $Q_2 = w_1z_2w_2$, then $V(Q_1) \cup V(Q_2)$ induces the diamond, as w_1 is not adjacent to w_2. Therefore, we may assume that one of Q_1 and Q_2 has length at least 3. But then the deletion of a vertex of Q_1 or Q_2 distinct from w_1, w_2, z_1, z_2 yields a smaller bridge that still satisfies the assumptions of the lemma. This contradicts the definition of H, hence there are no edges between the interiors of the paths Q_1 and Q_2, which implies that C is an induced cycle.

Now, we claim that w_1 has no neighbors in $P_2 - w_2$ and similarly, w_2 has no neighbors in $P_1 - w_1$. Suppose that w_1 has a neighbor z in $P_2 - w_2$. Note that one of Q_1 and Q_2 has length at least 2, say Q_2. Then Q_1 and the subpath from w_2 to z in P_2 form a cycle, and there are two paths from v_1, v_2 to this cycle. As their union avoids internal vertices of Q_2, $G[V(H) \cup V(Q')]$ contains an induced subdivision of the diamond. The same argument holds for w_2 and $P_1 - w_1$. This proves the claim.

Observe that there are no edges between $H - V(Q)$ and $Q - \{v_1, v_2\}$, because of the condition that $|V(H) \cap V(Q)| = 2$. As C is induced and w_i has no neighbors in $P_{3-i} - w_{3-i}$ for each i, we may assume that there is an edge between $P_1 \cup P_2 - \{w_1, w_2\}$ and $C - \{w_1, w_2\}$; otherwise $G[V(H) \cup V(Q')]$ is an induced subdivision of the diamond. By symmetry, we may assume that there is a vertex $y \in P_1 - w_1$ having a neighbor in the interior of Q_1. We choose such a vertex y with $\text{dist}_{P_1}(y, w_1)$ is minimum. If y has more than one neighbor in the interior of Q_1, then by Lemma 8.2, $G[V(P_1) \cup V(Q_1)]$ contains an induced subdivision of the diamond. Thus, y has exactly one neighbor, say z. If $Q_2 = w_1w_2$, then the union of C and the subpath of P_1 form y to w_1 induces a subdivision of the diamond. Otherwise, the union of the subpath of P_1 form y to w_1 and the subpath of Q_1 from z to w_1 forms a cycle, and there are two disjoint paths from v_1, v_2 to this cycle avoiding internal vertices of Q_2. Thus, by the minimality, it contains an induced subdivision of the diamond, a contradiction.

We conclude that $G[V(H) \cup V(Q')]$ contains an induced subdivision of the diamond. □

We use the following lemma to deal with some claws not covered by Lemma 8.5.

Lemma 8.6. *Let Q be an induced path in a graph G and H_1, H_2 be two Tutte bridges of $V(Q)$ in G such that*

- $|V(H_i) \cap V(Q)| = 2$ *for each* i,
- Q_1 *and* Q_2 *share an edge where* Q_i *is a minimal subpath of* Q *containing* $V(H) \cap V(Q)$.

Then $G[V(H_1) \cup V(H_2) \cup V(Q_1) \cup V(Q_2)]$ *contains an induced subdivision of the diamond.*

Proof. Let $Q = v_1 v_2 \cdots v_m$, and let v_{a_i}, v_{b_i} be the end vertices of Q_i such that $a_i < b_i$. Without loss of generality, we may assume that $a_1 \le a_2$. As Q_1 and Q_2 share an edge, $b_1 \ge a_2 + 1$.

Let x_1 and y_1 be neighbors of v_{a_1} and v_{b_1} in $H_1 - V(Q)$, respectively, and let R_1 be a path from x_1 to y_1 in $H_1 - V(Q)$. Similarly, let x_2 and y_2 be neighbors of v_{a_2} and v_{b_2} in $H_2 - V(Q)$, respectively, and let R_2 be a shortest path from x_2 to y_2 in $H_2 - V(Q)$. Let R be the subpath of Q from v_{a_1} to v_{a_2}.

Observe that $G[V(R) \cup \{x_1, x_2\}]$ is an induced path from x_1 to x_2, because $V(H_i) \cap V(Q)$ is exactly $\{a_i, b_i\}$. Let $j = \min\{b_1, b_2\}$. It is easy to see that there are three paths from v_j to the induced path $G[V(R) \cup \{x_1, x_2\}]$, namely, a subpath of Q from v_{a_2} to v_j, and two paths along R_1 and R_2. Thus, by Lemma 8.2, $G[V(R_1) \cup V(R_2) \cup V(Q_1) \cup V(Q_2)]$ contains an induced subdivision of the diamond. □

8.2. **The main proof.** We can now describe the main proof of this section. The following proposition asserts that subdivisions of the diamond intersecting a given path have the induced Erdős-Pósa property.

Proposition 8.7. *There exists a polynomial function $g_2 : \mathbb{N} \to \mathbb{N}$ satisfying the following. For every graph G, for every induced path P of G, and every $k \in \mathbb{N}$, either G contains k vertex-disjoint induced subdivisions of the diamond, or it has a vertex set of size at most $g_2(k)$ hitting all the induced subdivision of the diamond that intersect P.*

Proposition 8.8. *There exists a polynomial function $g_1 : \mathbb{N} \to \mathbb{N}$ satisfying the following. For every G, H, k such that*

- *G is a graph;*
- *H is an induced subdivision H of the diamond in G such that $G - V(H)$ has no induced subdivision of the diamond;*
- $k \in \mathbb{N}$,

then either G contains k vertex-disjoint induced subdivisions of the diamond, or it has a vertex set of size at most $g_1(k)$ hitting every induced subdivision of the diamond.

The following is a restatement of the main result of this section (Theorem 8.1).

Theorem 8.9. *There exists a polynomial function $g : \mathbb{N} \to \mathbb{N}$ satisfying the following. For every graph G and $k \in \mathbb{N}$, either G contains k vertex-disjoint induced subdivisions of the diamond, or it has a vertex set of size at most $g(k)$ hitting every induced subdivision of the diamond.*

We first discuss how to prove Theorem 8.1 using Proposition 8.7. We assign $g(k) := k \cdot g_1(k)$, and $g_1(k) := 3g_2(k)$. We can prove Theorem 8.1 using Proposition 8.8, with exactly the same argument in the proof for 1-pan (Theorem 7.2). Suppose H is a given induced subdivision of the diamond in Proposition 8.8, and let P_1, P_2, P_3 be the three paths between degree 3 vertices of H. By applying Proposition 8.7 to P_1, P_2, P_3, we can find either k pairwise vertex-disjoint induced subdivisions of the diamond, or a vertex set of size at most $3g_2(k)$ hitting all induced subdivisions intersecting one of P_1, P_2, P_3. This implies Proposition 8.8.

Proof of Proposition 8.7. Let $P = v_1 v_2 \cdots v_m$, and $I = \{1, \ldots, m\}$.

We first apply the A-claw lemma to P. Let α be the constant described in Theorem 8.3. By Theorem 8.3. There are either $N(3, 3k)$ pairwise vertex-disjoint P-claws, or a vertex set of size $\alpha \cdot N(3, 3k)$ hitting all P-claws. We show below that in the former case, we can find k pairwise vertex-disjoint induced subdivisions of the diamond.

Claim 8.10. *If G has a set of $N(3, 3k)$ vertex-disjoint P-claws, then it has k vertex-disjoint induced subdivisions of the diamond.*

Proof. Let $T_1, T_2, \ldots, T_{N(3,3k)}$ be a given set of pairwise vertex-disjoint P-claws. For each $i \in \{1, \ldots, N(3, 3k)\}$, let $A_i := \{j \in \mathbb{N} : v_j \in V(T_i)\}$. By definition, $|A_i| = 3$.

We apply the regular partition lemma to A_i's with $n = 3$. Then there exist a subsequence $(A_{c_1}, \ldots, A_{c_{3k}})$ of $(A_1, \ldots, A_{N(3,3k)})$ and a regular partition of I with respect to $(A_{c_1}, \ldots, A_{c_{3k}})$ that has order at most 3. Since given P-claws are pairwise vertex-disjoint, in each part, restrictions of $A_{c_1}, \ldots, A_{c_{3k}}$ are not the same.

We first suppose the order of the partition is 1. One of the following holds

- for all $j, j' \in \{1, \ldots, 3k\}$ with $j < j'$, $\max(A_{c_j}) < \min(A_{c_{j'}})$, and
- for all $j, j' \in \{1, \ldots, 3k\}$ with $j < j'$, $\max(A_{c_{j'}}) < \min(A_{c_j})$.

For each $i \in \{1, \ldots, 3k\}$, let P_i be the minimal subpath of P containing the vertices in $\{v_j : j \in A_{c_i}\}$. By Lemma 8.2, each $G[V(T_{C_i}) \cup V(P_i)]$ contains an induced subdivision of the diamond, so we are done.

Suppose the order of the partition is 2 or 3. Let I_1 and I_2 be the first two parts in the partition, and we may assume that A_{c_i} has an element in each of the parts. For each $i \in \{1, \ldots, 3k-2\}$, let Q_i be the minimal subpath of P containing the vertices in

$$\{v_j : j \in (A_{c_i} \cup A_{c_{i+1}} \cup A_{c_{i+2}}) \cap I_1\},$$

and let R_i be the minimal subpath of P containing the vertices in

$$\{v_j : j \in (A_{c_i} \cup A_{c_{i+1}} \cup A_{c_{i+2}}) \cap I_2\}.$$

Then subgraphs in

$$\{T_{C_{3\ell-2}} \cup T_{C_{3\ell-1}} \cup T_{C_{3\ell}} \cup Q_{3\ell-2} \cup R_{3\ell-2} : 1 \le \ell \le k\}$$

are pairwise vertex-disjoint, and each subgraph contains an induced subdivision of the diamond, by Lemma 8.2. Therefore, we obtain k pairwise vertex-disjoint induced subdivisions of the diamond, as required. ◇

By Claim 8.10, we may assume that G contains a vertex subset X_1 of size at most $\alpha \cdot N(3, 3k)$ such that $G - X_1$ has no P-claws. Let $G_1 := G - X_1$. By Lemma 8.4, G_1 contains no Tutte bridge of $V(P)$ such that $|V(H) \cap V(P)| \ge 3$.

Now, we greedily construct a maximal set $\mathcal{U}$ of pairwise vertex-disjoint Tutte bridges H of $V(P)$ in G_1 such that

- $|V(H) \cap V(P)| = 2$,
- $V(H) \cap V(P)$ are not consecutive vertices of P, and
- H contains a cycle.

Suppose $|\mathcal{U}| \geq N(2,3k)$. In this case, we apply the regular partition lemma with $n = 2$. Then by the same line of proofs in Claim 8.10, we can find k pairwise vertex-disjoint induced subdivisions of the diamond. Otherwise, let $X_2 := \bigcup_{H \in \mathcal{U}}(V(H) \cap V(P))$. Then we have that $|X_2| \leq 2N(2,3k)$ and X_2 hits all Tutte bridges of $V(P)$ satisfying the three conditions above. Let $G_2 := G_1 - X_2$.

In the next step, we greedily build a maximal set $\mathcal{W}$ of pairwise vertex-disjoint tuples of Tutte bridges (H_1, H_2) of $V(P)$ in G_2 such that

- $|V(H_i) \cap V(P)| = 2$,
- Q_1 and Q_2 share an edge, where Q_i is a minimal subpath of P containing $V(H) \cap V(P)$.

Suppose $|\mathcal{W}| \geq 4N(4,3k)$.

Note that H_1 might intersect H_2, and therefore, there are four types of a pair (H_1, H_2). Let v_{a_i}, v_{b_i} be the vertices of $V(H_i) \cap V(P)$ such that $a_i < b_i$. There are types depending on whether $v_{a_1} = v_{a_2}$ and $v_{b_1} = v_{b_2}$. As $|\mathcal{W}| \geq 4N(4,3k)$, there is a subset $\mathcal{W}_1$ of $\mathcal{W}$ of size at least $N(4,3k)$ which consists of pairs of the same type.

In this case, we apply the regular partition lemma with n equal to the size of $(V(H_1) \cup V(H_2)) \cap V(P)$ for pairs in $\mathcal{W}_1$. Then by the same line of proofs in Claim 8.10, we get k pairwise vertex-disjoint induced subdivisions of the diamond. Otherwise, let $X_3 := \bigcup_{(H_1,H_2) \in \mathcal{W}}((V(H_1) \cup V(H_2)) \cap V(P))$. Then we have that $|X_3| \leq 16N(4,3k)$ and X_3 hits all pairs of Tutte-bridges satisfying the above conditions. Let $G_3 := G_2 - X_3$.

Observe that if G_3 contains an induced subdivision X of the diamond, then $V(X) \cap V(P)$ consists of at most two consecutive vertices of P. Therefore, we can find in polynomial time either k pairwise vertex-disjoint induced subdivisions of the diamond, or a vertex set X_4 of size at most $2k$ hitting all remaining induced subdivisions of the diamond. In the latter case, we obtain a hitting set of size at most

$$|X_1 \cup X_2 \cup X_3 \cup X_4| \leq \alpha N(3,3k) + N(2,3k) + 16N(4,3k) + 2k.$$

So, the function $g_1(k) = \alpha N(3,3k) + N(2,3k) + 16N(4,3k) + 2k$ satisfies the statement. □

9. A lower-bound on the bounding function

We provide in this section a lower-bound on the bounding function for the induced Erdős-Pósa property of subdivisions. We first need a few more definitions. For every graph G, the *girth* of G, denoted by $\mathbf{girth}(G)$ is the minimum order of a cycle in G. The treewidth of G, denoted by $\mathbf{tw}(G)$ is a graph invariant that can be defined using tree-decompositions. We avoid the technical definition here and only state the two well-known properties of treewidth that we use:

- deleting a vertex or an edge in a graph decreases its treewidth by at most one;
- for every planar graph H of maximum degree 3, there is a constant $c \in \mathbb{N}$ such that every graph of treewidth at least c contains a subdivision of H (Grid Minor Theorem, [RS86]).

We refer the reader to [Die10] for an introduction to treewidth.

Lemma 9.1. *Let H be a graph that has a cycle and no vertex of degree more than 3. There is no function $f(k) = o(k \log k)$ such that subdivisions of H have the induced Erdős-Pósa property with bounding function f.*

Proof. When H is not planar, the result follows from Lemma 3.2. We therefore assume for now that H is planar. By the Grid Minor Theorem, there is a constant c such that every graph G satisfying $\mathbf{tw}(G) \geq c$ contains a subdivision of H. We will construct sequences $(G_n)_{n\in\mathbb{N}}$ (graphs) and $(k_n)_{n\in\mathbb{N}}$ (integers) such that $\nu_H(G_n) = O(k_n)$ while $\tau_H(G_n) = \Omega(k_n \log k_n)$.

We start with an infinite family $(R_n)_{n\in\mathbb{N}}$ of 3-regular graphs (of increasing order), called Ramanujan graphs, whose existence is proved in [Mor94, Theorem 5.13]. These graphs have the following properties:

(1) $\forall n \in \mathbb{N}$, $\mathbf{girth}(R_n) \geq \frac{2}{3}\log|R_n|$ ([Mor94, Theorem 5.13]);
(2) $\exists n' \in \mathbb{N}$, $\exists \alpha \in \mathbb{R}_{>0}$, $\forall n \in \mathbb{N}$, $n \geq n' \Rightarrow \mathbf{tw}(R_n) \geq \alpha|R_n|$ (see [BEM+04, Corollary 1]).

Recall that $n \mapsto |R_n|$ is increasing. We denote by n'' the minimum integer that is larger than n' and such that $|R_n| \geq \frac{c}{\alpha}$ for every $n \geq n''$. For every integer $n \geq n''$, we define k_n as the maximum positive integer such that

$$\frac{1}{\alpha}\left(c + k_n \log k_n\right) \leq |R_n|$$

Such a value exists by definition of n''. Notice that $n \mapsto k_n$ is non-decreasing and is not upper-bounded by a constant. Observe that for every $n \geq n''$, $\mathbf{tw}(R_n) \geq c + k_n \log k_n$.

Let us define, for every integer $n \in \mathbb{N}$, G_n as the graph obtained from R_n by subdividing once every edge. We then have, for every $n \geq n''$:

$$\begin{aligned} |G_n| &= \frac{5}{2}\cdot|R_n| \\ &\geq \frac{5}{2\alpha}(c + k_n \log k_n) \qquad (1) \quad \text{and} \\ \mathbf{girth}(G_n) &= 2\cdot\mathbf{girth}(R_n) \\ &\geq \frac{4}{3}\cdot\log\left(\frac{2}{5}\cdot|G_n|\right). \qquad (2) \end{aligned}$$

Claim 9.2. *For every integer $n \geq n''$, $\tau_H(G_n) \geq k_n \log k_n$.*

Proof. Let $X \subseteq V(G_n)$ be such that $|X| < k_n \log k_n$. We show that $G_n - X$ contains an induced subdivision of H. Recall that each vertex of degree 2 of G_n was obtained by subdividing an edge of R_n. We define X^e as the set of edges of R_n corresponding to the vertices of degree 2 in X and set $X^v = X \cap V(R_n)$. As the deletion of an edge or a vertex in a graph decreases its treewidth by at most one, the graph obtained from R_n by deleting X^v and X^e has treewidth at least c. By definition of c, this graph contains a subdivision S of H. Notice that the corresponding subdivision of H in G_n (i.e. that obtained by subdividing once every edge of S) is induced and does not contain any vertex of X. As this holds for every subset of $V(G_n)$ with less than $k_n \log k_n$ vertices, we deduce that $\tau_H(G_n) \geq k_n \log k_n$. ⋄

Claim 9.3. *There is a $n''' \in \mathbb{N}$, such that for every $n \geq n'''$, $\nu_H(G_n) < k_n$.*

Proof. Let $n \geq n''$ and let us assume that G contains k_n disjoint induced subdivisions of H. As H has a cycle, the order of each of these subdivisions is at least the girth of G.

We deduce:

$$\begin{aligned}|G_n| &\geq k_n \cdot \mathbf{girth}(G_n) \\ &\geq k_n \cdot \frac{4}{3} \cdot \log\left(\frac{2}{5} \cdot |G_n|\right) && \text{by (2)} \\ &\geq k_n \cdot \frac{4}{3} \cdot \log\left(\frac{1}{\alpha}(c + k_n \log k_n)\right) && \text{by (1).}\end{aligned}$$

On the other hand, from the maximality of k_n we get:

$$|G_n| < \frac{5}{2\alpha}(c + k_n \log k_n).$$

Combining these bounds together we obtain:

$$k_n \cdot \frac{4}{3} \cdot \log\left(\frac{1}{\alpha}(c + k_n \log k_n)\right) < \frac{5}{2\alpha}(c + k_n \log k_n). \tag{3}$$

The left-hand side of (3) is a $\Omega(k_n \log(k_n \log k_n))$ while its right-hand side is a $O(k_n \log k_n)$. As $n \mapsto k_n$ is not upper-bounded by a constant, there is a positive integer $n''' \geq n''$ such that for every $n \geq n'''$, (3) does not hold. Therefore, when $n \geq n'''$ we have $\nu_H(G_n) < k_n$. ◇

The sequences $(G_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ have the property that $\tau_H(G_n) = \Omega(k_n \log k_n)$ (Claim 9.2) while $\nu_H(G_n) = O(k_n)$ (Claim 9.3), as required.

□

10. Conclusion and open problems

In this paper, we investigated the induced Erdős-Pósa property of subdivisions beyond know results about cycles and obtained both positive and negative results. We note that our positive results for pans come with polynomial-time algorithms that output either a large packing of induced subdivisions of the considered graph H, or a small hitting set. These can be directly used to design approximation algorithms for computing the maximum size of a packing of induced subdivisions of H and the minimum size of a hitting set (as in [CRST17] for instance). For 1-pans and 2-pans, this gives a polynomial-time $O(\mathsf{OPT} \log \mathsf{OPT})$-approximation. On the other hand, our negative results cover a vast class of graphs.

The most general open problem on the topic discussed in this paper is to characterize the graphs H whose subdivisions have the induced Erdős-Pósa property. Intermediate steps are the study of subdivisions of specific graphs.

We observe that the constructions we used for our counterexamples contain arbitrarly large complete subgraphs. Therefore the landscape of the induced Erdős-Pósa property of subdivisions might be much different if one restricts their attention to graphs excluding a dense subgraph. We note that in this direction, Weißauer recently proved that for every $s, \ell \in \mathbb{N}$, subdivisions of C_ℓ have the induced Erdős-Pósa property in $K_{s,s}$-subgraph-free graphs [Wei18].

Another line of research in the study of the Erdős-Pósa property of graph classes is to optimize the bounding function. We note that all our positive results hold with a polynomial bounding function. On the other hand, we obtained a lower bound of $\Omega(k \log k)$ for non-acyclic subcubic graphs. We do not expect our upper-bounds to be tight and it is an open question to find the correct order of magnitude of the bounding functions for the graphs we considered. In this direction it is also open to determine the correct order of magnitude of the bounding function in Theorem 1.3.

References

[BBR07] Etienne Birmelé, J. Adrian Bondy, and Bruce A. Reed. The Erdős–Pósa property for long circuits. *Combinatorica*, 27(2):135–145, Mar 2007.

[BEM+04] S. Bezrukov, R. Elsässer, B. Monien, R. Preis, and J.-P. Tillich. New spectral lower bounds on the bisection width of graphs. *Theoretical Computer Science*, 320(2-3):155–174, 2004.

[BHJ17] Henning Bruhn, Matthias Heinlein, and Felix Joos. Frames, A-paths and the Erdős-Pósa property. *Arxiv preprint*, 2017. arXiv:1707.02918.

[BHK62] Joseph Battle, Frank Harary, and Yukihiro Kodama. Additivity of the genus of a graph. *Bulletin of the American Mathematical Society*, 68(6):565–568, 11 1962.

[CKOW17] Hojin Choi, O-joung Kwon, Sang-il Oum, and Paul. Wollan. Chi-boundedness of graph classes excluding wheel vertex-minors. *Arxiv preprint*, 2017. arXiv:1702.07851.

[CRST17] Dimitris Chatzidimitriou, Jean-Florent Raymond, Ignasi Sau, and Dimitrios M. Thilikos. An $O(\log OPT)$-approximation for covering and packing minor models of θ_r. *Algorithmica*, Apr 2017. arXiv:1510.03945.

[Die10] Reinhard Diestel. Graph theory, volume 173 of. *Graduate texts in mathematics*, 2010. Fourth edition.

[EP62] Paul Erdős and Lajós Pósa. On the maximal number of disjoint circuits of a graph. *Publ. Math. Debrecen*, 9:3–12, 1962.

[ES87] P. Erdös and G. Szckeres. *A Combinatorial Problem in Geometry*, pages 49–56. Birkhäuser Boston, Boston, MA, 1987.

[FH14] Samuel Fiorini and Audrey Herinckx. A tighter Erdős-Pósa function for long cycles. *Journal of Graph Theory*, 77(2):111–116, 2014. arXiv:1205.0940.

[FLM+16] Fedor V Fomin, Daniel Lokshtanov, Neeldhara Misra, Geevarghese Philip, and Saket Saurabh. Hitting forbidden minors: Approximation and kernelization. *SIAM Journal on Discrete Mathematics*, 30(1):383–410, 2016. arXiv:1010.1365.

[Gal64] Tibor Gallai. Maximum-minimum sätze und verallgemeinerte faktoren von graphen. *Acta Mathematica Academiae Scientiarum Hungarica*, 12(1):131–173, Mar 1964.

[JP17] Bart MP Jansen and Marcin Pilipczuk. Approximation and kernelization for chordal vertex deletion. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1399–1418. SIAM, 2017. arXiv:1605.03001.

[KK17] Eun Jung Kim and O-joung. Kwon. Erdős-Pósa property of chordless cycles and its applications. *Arxiv preprint*, 2017. arXiv:1711.00667.

[MNŠW17] Frank Mousset, Andreas Noever, Nemanja Škorić, and Felix Weissenberger. A tight Erdős–Pósa function for long cycles. *Journal of Combinatorial Theory, Series B*, 125(Supplement C):21 – 32, 2017. arXiv:1603.07588.

[Mor94] Moshe Morgenstern. Existence and explicit constructions of $q+1$ regular ramanujan graphs for every prime power q. *Journal of Combinatorial Theory, Series B*, 62(1):44 – 62, 1994.

[MT01] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces.* Johns Hopkins University Press, Baltimore, MD, 2001.

[Ree97] Bruce A. Reed. *Tree width and tangles: A new connectivity measure and some applications*, pages 87–162. Cambridge University Press, 1997.

[RS86] Neil Robertson and Paul D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986.

[RT17] Jean-Florent Raymond and Dimitrios M Thilikos. Recent techniques and results on the Erdős–Pósa property. *Discrete Applied Mathematics*, 231(Supplement C):25 – 43, 2017. arXiv:1603.04615.

[Sim67] Miklós Simonovits. A new proof and generalizations of a theorem of Erdős and Pósa on graphs without $k+1$ independent circuits. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(1):191–206, 1967.

[Tho88] Carsten Thomassen. On the presence of disjoint subgraphs of a specified type. *Journal of Graph Theory*, 12(1):101–111, 1988.

[Wei18] D. Weißauer. In absence of long chordless cycles, large tree-width becomes a local phenomenon. *ArXiv e-prints*, March 2018.

(O. Kwon)
Department of Mathematics
Incheon National University
Incheon, South Korea
Email address: ojoungkwon@gmail.com

(J.-F. Raymond)
Logic and Semantics Research Group
Technische Universität Berlin
Berlin, Germany
Email address: raymond@tu-berlin.de