PROFILE AND NEIGHBOURHOOD COMPLEXITY OF GRAPHS WITH EXCLUDED MINORS AND TREE-STRUCTURED GRAPHS

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Abstract. The *r-neighbourhood complexity* of a graph G is the function counting, for a given integer k, the largest possible number, over all vertex-subsets A of size k, of subsets of A realized as the intersection between the r-neighbourhood of some vertex and A. A refinement of this notion is the *r-profile complexity*, that counts the maximum number of distinct distance-vectors from any vertex to the vertices of A, ignoring distances larger than r. Typically, in structured graph classes such as graphs of bounded VC-dimension or chordal graphs, these functions are bounded, leading to insights into their structural properties and efficient algorithms.

We improve existing bounds on the r-profile complexity (and thus on the r-neighbourhood complexity) for graphs in several structured graph classes. We show that the r-profile complexity of graphs excluding K_h as a minor is in $O_h(r^{3h-3}k)$. For graphs of treewidth at most t we give a bound in $O_t(r^{t+1}k)$, which is tight up to a function of t as a factor. These bounds improve results and answer a question of Joret and Rambaud [Combinatorica, 2024].

For outerplanar graphs, we can improve our treewidth bound by a factor of r and conjecture that a similar improvement holds for graphs with bounded simple treewidth. For graphs of treelength at most ℓ, we give the upper bound in $O(k(r^2(\ell+1)^k))$.

Our bounds also imply relations between the order, diameter and metric dimension of graphs in these classes, improving results from [Beaudou et al., SIDMA 2017].

1. Introduction

An important structural property of a graph or hypergraph is the way its neighbourhoods are structured. A prominent parameter measuring this aspect is the *Vapnik-Chervonenkis Dimension*, or VC-dimension, of a graph [KKR+97] or a hypergraph [VC71]. By the Perles-Sauer-Shelah Lemma [Sau72, She72], for a graph G of VC-dimension at most c, the number of distinct intersections within any set A of vertices and the neighbourhood of any vertex of G is in $O(|A|^c)$, instead of $2^{|A|}$. This has led to the definition of the neighbourhood complexity of a graph, and more generally, for any integer $r \geq 1$, the r-neighbourhood complexity of a graph. Informally speaking, this is the function assigning to an integer k the maximum number of distinct subsets of any vertex set A of size k that are realised as the r-neighbourhood (within A) of some vertex of G.

The VC-dimension was originally defined for hypergraphs in the context of machine learning [VC71], by taking the hyperedges instead of the neighbourhoods. When the dataset (seen as a hypergraph) has bounded VC-dimension, and thus polynomial neighbourhood complexity, there is, for example, an efficient algorithm for the PAC-learning problem [BEHW89]. For graphs, having bounded VC-dimension also has important algorithmic applications, see for example [BBB+21, BLL+15, MV17]. Graph classes whose members have bounded VC-dimension (and thus neighbourhood complexity polynomial in k) include for example sparse

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graphs (such as graphs of bounded degeneracy), geometric intersection graphs (e.g. interval graphs, line graphs, or disk graphs), graphs with no 4-cycles, and structured dense graphs (for instance graphs of bounded clique-width or twin-width). More generally, many of these types of graphs also have their r-neighbourhood complexity polynomial in k for any integer $r \geq 1$. This is the case for example for graphs from graph classes of bounded expansion [RVS19] or those that are nowhere dense [EGK+17].

Our goal is to improve known upper bounds on the r-neighbourhood complexity and the related r-profile complexity of graphs in structured graph classes. We next formally define these notions.

Neighbourhood and profile complexity. Let us formally define neighbourhood and profile complexity.

Definition 1 (neighborhood complexity, nc_r, N_r)**.** *The* r-neighbourhood complexity *is the function defined, for a graph G and a positive integer k, by:*

$$\mathrm{nc}_r(G,k) = \max_{A \in \binom{V(G)}{k}} \left|\{N_r(v) \cap A, v \in V(G)\}\right|,$$

where $N_r(v)$ denotes the r-neighbohood *of v, i.e., the set of vertices of G at distance at most r from v.*

It turns out that the r-neighbourhood complexity of sparse graphs is linear in k for every r. More precisely, Reidl, Sánchez Villaamil, and Stavropoulos proved in [RVS19] that a graph class $\mathcal{C}$ that is closed under taking subgraphs has *bounded expansion* if and only if there is a function f such that for every $G \in \mathcal{C}$ and $r, k \in \mathbb{N}$, $\mathrm{nc}_r(G,k) \leq f(r) \cdot k$. (Bounded expansion is a very general notion of sparsity that includes graph classes excluding a fixed graph as a minor or subdivision, and thus classes whose members have bounded treewidth, and classes whose members have bounded maximum degree, among others.) More generally, nowhere dense classes of graphs have near-linear neighbourhood complexity [EGK+17]. We refer to the book [NdM12] for more details on classes of bounded expansion and graph classes that are nowhere dense.

The neighbourhood complexity is not only an important structural graph measure, but it also has algorithmic applications. For example, in kernelization (a subarea of parameterized complexity) for problems related to distances such as r-Dominating Set or r-Independent Set, bounding the number of r-neighbourhoods may allow to discard vertices behaving the same way and keep only one representative for each neighbourhood. For algorithmic use of neighbourhood complexity, see for instance [EGK+17] or the discussion in the introduction of [RVS19]. Bounds for specific values of r (typically $r = 1$) have also been used to design algorithms, see e.g. [BBGR24, LPS+22]

A useful refinement of the intersection of the r-neighbourhood of a vertex with a given set is the vector of r-truncated distances to vertices of this set, called *profile* and formally defined as follows.

Definition 2 (profile, p_r, Cap_r)**.** *Given a graph G, a set A of its vertices, and an integer r, we define the* r-profile *of some vertex v of G with respect to A as the function*

$$p_r(v,A) : \begin{cases} A & \rightarrow \quad \mathbb{N} \\ a & \mapsto \quad \mathrm{Cap}_r(\mathrm{dist}(a,v)) \end{cases}$$

where $\mathrm{Cap}_r(\ell) = \ell$ if $\ell \leq r$, and $\mathrm{Cap}_r(\ell) = +\infty$ otherwise.

When studying the r-neighbourhood complexity, several authors (see [EGK+17, JR24]) have used this refinement and defined the notion of profile complexity.

Definition 3 (profile complexity, pc_r)**.** *The r-profile complexity of a graph G is the function defined by:*

$$\mathrm{pc}_r(G,k) = \max_{A \in \binom{V(G)}{k}} \mathrm{pc}_r(G,A),$$

where $\mathrm{pc}_r(G,A)$ counts the number of different r-profiles with respect to the subset A except the "all $+\infty$" profile[1]*, i.e.* $\mathrm{pc}_r(G,A) = \left|\{p_r(v,A), v \in V(G)\} \setminus \{a \mapsto +\infty\}\right|$.

For a graph G and two integers k and r, one can check that we always have $\mathrm{nc}_r(G,k) \leq \mathrm{pc}_r(G,k)+1$ [EGK+17, JR24]. Moreover, Joret and Rambaud proved the following lemma, further tightening the relation between these two functions for many natural classes of graphs.

Lemma 4 ([JR24, Lemma 8])**.** *Let $\mathcal{C}$ be a graph class stable by the operation of adding pendant vertices. If there exists a function $f_{\mathcal{C}} : \mathbb{N}^2 \to \mathbb{N}$ such that for any graph G in $\mathcal{C}$, and any integers r and k we have $\mathrm{nc}_r(G,k) \leq f_{\mathcal{C}}(r,k)$, then for any G, r, k as above, $\mathrm{pc}_r(G,k) \leq f_{\mathcal{C}}(r,(r+1)k)$.*

Lemma 4 implies that for any graph class $\mathcal{C}$ satisfying the mild condition of being closed under adding pendant vertices, any upper bound obtained on the neighbourhood complexity can be extended to an upper bound on the profile complexity. In the specific case when the dependency on k is linear (meaning $f : (r,k) \mapsto g(r) \cdot k$ for some function $g : \mathbb{N} \to \mathbb{N}$), the ratio between both bounds is $r+1$. Joret and Rambaud [JR24, Corollary 37] use this fact in the context of sparse graphs to extend lower bounds on the profile complexity to the neighbourhood complexity of such graphs.

The notions of neighbourhood complexity and profile complexity are closely related to *graph identification problems*, as we will see now.

Connection to Metric Dimension and other identification problems. In the area of identification problems, one wishes to distinguish the elements of a graph or discrete structure by the means of a small substructure.

A prominent example in this area is the concept of a *resolving set* of a graph G, which is a set S of vertices such that for any two distinct vertices u, v of G, there is a vertex in S with $\mathrm{dist}(u,s) \neq \mathrm{dist}(v,s)$. In other words, the $\mathrm{diam}(G)$-profiles of the vertices of G with respect to S, are all distinct, where $\mathrm{diam}(G)$ is the diameter of G.

The *metric dimension* $\mathrm{md}(G)$ of a graph G is the smallest size of a resolving set of G. The concept was introduced in the 1970s [HM76, Sla75] and extensively studied since then, with applications such as detection problems in networks, graph isomorphism, coin-weighing problems, or machine learning; see the surveys [KY21, TFL21]. As shown by Joret and Rambaud [JR24], the notion of r-profile complexity is closely connected to the study of the metric dimension of a graph with bounded diameter. Indeed, for a graph G and any integer k, if $\mathrm{pc}_r(G,k) \leq f(r,k)$ for some function f, then G has at most $f(\mathrm{diam}(G), \mathrm{md}(G))$ vertices, since every vertex in G needs a distinct $\mathrm{diam}(G)$-profile with respect to any optimal resolving set S. The question of finding the best possible upper bound on the number of vertices of a graph as a function of the metric dimension and diameter was studied for various graph classes in [BDF+18, FMN+17, HMP+10]. In fact, as attested by our work, the methods from these papers are at times applicable to the r-profile complexity as well.

[1]We exclude this profile because it is more convenient in the proofs.

As observed in [BFLP24], a similar connection for the r-neighbourhood complexity exists with another type of identification problem, namely the concept of an *r-locating-dominating set*, which is a set S of vertices of a graph G such that for every vertex in $V(G) \setminus S$, the intersection between its r-neighbourhood and S is non-empty and unique [Hon09]. This concept (and the close variant of r-identifying codes) is widely studied, with many applications. We refer to the book chapter [LHC20] and the extensive online bibliography maintained at [DL12] for more details. Clearly, if $\mathrm{nc}_r(G,k) \leq f(r,k)$ for some function f, then G has at most $f(r, \mathrm{ld}_r(G))$ vertices, where $\mathrm{ld}_r(G)$ is the smallest size of an r-locating-dominating set of G. The question of finding the best possible upper bound on the number of vertices of a graph G as a function of r and $\mathrm{ld}_r(G)$ was studied for various classes of graphs, especially for $r = 1$, see [CFPW24, FGN+13, FMN+17, RS84, Sla87].

Previous work. Improving in particular a previous bound of Sokolowski [Sok23] for planar graphs, Joret and Rambaud showed the following upper bounds on the profile complexity (and thus, neighbourhood complexity) of several classes of sparse graphs:

Theorem 5 ([JR24]). *For every graph G,*

(1) $\mathrm{pc}_r(G,k) \in O\left((t+1)\binom{r+t}{t} r^{2t} k\right)$ *if G has treewidth at most t;*
(2) $\mathrm{pc}_r(G,k) \in O_h(r^{h^2-1}k)$ *if G excludes K_h as a minor;*
(3) $\mathrm{pc}_r(G,k) \in O_g(r^5 k)$ *if G has Euler genus at most g;*
(4) $\mathrm{pc}_r(G,k) \in O(r^4 k)$ *if G is planar.*

Moreover, they described, for infinitely many integers t, a graph G of treewidth t such that $\mathrm{pc}_r(G,k) \in \Omega(r^{t+1}k/t^t)$; by Lemma 4, this also implies that $\mathrm{nc}_r(G,k) \in \Omega(r^t k/t^t)$.

The r-neighbourhood complexity of dense graph classes has also been studied, see for example [PP20] for graphs of bounded clique-width or [BFLP24] for graphs of bounded twin-width.

As described by Joret and Rambaud [JR24], their bounds improve results from [BDF+18] on the metric dimension. Let G be a graph with order n, metric dimension k and diameter d. As mentioned above, if $\mathrm{pc}_r(G,k) \leq f(r,k)$ for some function f, then G has at most $f(d,k)$ vertices, and the above bounds on pc_r can be directly applied. It was proved in [HMP+10] that for any graph G, we have $n \in O(k(2d/3)^k)$, and this is asymptotically tight. It is known that $n \in O(dk^2)$ if G is an interval graph or a permutation graph, and $n \in O(dk)$ if G is a cograph, a unit interval graph, or a bipartite permutation graph, and these bounds are also tight [FMN+17]. The bound $n \in O((dk)^{d \cdot 2^{O(w)}})$ holds if G has rank-width at most w [BDF+18]. It is known that $n \in O(kd^2)$ if G is outerplanar [BDF+18]. Moreover, $n \leq kd^2(2\ell+1)^{3w+1}$ if G has a tree-decomposition of width w and length ℓ [BDF+18], which implies that $n \in O(2^{2^{O(k)}} d^2)$ if G is chordal.

Our results. In this paper we optimally improve bounds from the literature on the r-profile complexity of several classes of sparse graphs and give new bounds for chordal and similar tree-structured graphs. As discussed above, these results also provide improved bounds on the r-neighbourhood complexity and metric dimension of the considered graph classes.

We first turn our attention to graphs of bounded treewidth and prove the following.

Theorem 6. *Let t, r be two positive integers and let G be a graph of treewidth at most t. Then,* $\mathrm{pc}_r(G,k) \in O(t^{O(t)} r^{t+1} k)$.

Note that Theorem 6 is asymptotically tight by a construction from [JR24, Theorem 36] providing (for every positive integer t and arbitrarily large values of r) graphs G of treewidth t with $\mathrm{pc}_r(G, k) \in \Omega_t(r^{t+1}k)$. Our proof relies on the notion of *guarding sets* introduced by Joret and Rambaud [JR24]. The main innovation of our argument is that we construct a guarding set based on a least common ancestor closure in the decomposition tree of the bags containing the vertices of A. This is inspired by an algorithm in [FLMS12] for a different problem (hitting planar minors). Compared to the generalised colouring numbers approach followed in [JR24] to obtain the bound of Theorem 5.(3), our proof results in a smaller guarding set (albeit with slightly larger members), and consequently in better bounds.

Item (4) of Theorem 5 by Joret and Rambaud gives an $O(r^4k)$ bound for the profile complexity of planar graphs. Meanwhile our Theorem 6 gives a $O(r^3k)$ for treewidth 2 graphs (which are planar). We improve this further for outerplanar graphs with the following bound, extending a result for metric dimension from [BDF+18].

Theorem 7. *Let G be an outerplanar graph, then* $\mathrm{pc}_r(G, k) \in O(r^2k)$.

We conjecture that this result can be extended to an $O(t^{O(t)}r^tk)$ bound for graphs with bounded simple treewidth (see Section 7). We then consider the general case of graphs excluding a fixed minor.

Theorem 8. *Let $h \geq 4$ and r be positive integers and let G be a graph with no K_h minor. Then,* $\mathrm{pc}_r(G, k) \in O(h^{O(h)}r^{3h-3}k)$.

Theorem 8 is as substantial improvement over the previous bound from Joret and Rambaud [JR24] (see item (2) of Theorem 5) and positively answers one of their open questions. Our approach builds on the guarding sets used by Joret and Rambaud, which are constructed in terms of the weak colouring numbers of the graph. We use a mix of the strong and weak colouring numbers and capitalise on the fact that graphs excluding a fixed minor have orderings which give good (and different) upper bounds on these numbers. The members of our guarding set are substantially smaller, giving us this improved result.

Our approach for Theorem 8 can be used in general for classes with bounded expansion. We discuss some possible applications in Section 5.1, and as an example consider intersection graphs of balls in $\mathbb{R}^d$.

We then turn our attention to tree-structured dense graphs, namely, chordal graphs. Actually our results in this direction hold for the more general setting of graphs of bounded treelength, i.e., graphs that admit a tree decomposition where every pair of vertices in the same bag are at distance at most some constant ℓ (for chordal graphs, $\ell = 1$).

Theorem 9. *Let G be a graph of treelength at most ℓ. Then* $\mathrm{pc}_r(G, k) \in O\left(k \cdot (r^2(\ell+1)^k)\right)$.

Our results have the following consequences about the metric dimension of the considered graph classes.

Corollary 10. *Let G be an n-vertex graph with diameter d and metric dimension k.*

(1) If G has treewidth at most t, then $n \in O_t(d^{t+1}k)$.
(2) If G excludes K_h as a minor for an integer $h \geq 4$, then $n \in O_h(d^{3(h-1)}k)$.
(3) If G is outerplanar, then $n \in O(d^2k)$.
(4) If G has a treelength at most ℓ, then $n \in O\left(k \cdot (d^2(\ell+1)^k)\right)$.

The two first bounds improve results from [JR24] and the two last bounds improve those from [BDF+18]. The bound for outerplanar graphs was already proved in [BDF+18] and

Theorem 7 allows us to easily recover it. We note that the bound for treewidth is asymptotically tight, indeed the construction provided in [JR24, Theorem 36] actually gives (for every positive integer t and arbitrarily large values of d) graphs of treewidth t, metric dimension k, diameter $O(d)$, and $\Omega_t(d^{t+1}k)$ vertices.

Outline. In Section 2 we introduce the necessary definitions. We give bounds for graphs of bounded treewidth and outerplanar graphs in Sections 3 and 4 respectively. In Section 5 we discuss the case of graph classes excluding a fixed minor. Section 6 is devoted to graphs of bounded treelength. Finally, we conclude in Section 7 with open questions.

2. Preliminaries

In this paper all graphs are simple, loopless, and undirected.

2.1. **Distances, profiles, and guarding sets.** The *distance* between two vertices u and v of a graph G is the minimum number of edges of a path starting at u and ending at v. The *distance* $\mathrm{dist}_G(X, Y)$ between two vertex sets X and Y of G is the minimum distance in G between a vertex of X and a vertex of Y. We drop the subscript when there is no ambiguity. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance between any two of its vertices.

Recall that the definitions of the neighbourhood complexity nc_r, r-profiles p_r, the profile complexity pc_r, and the truncating function Cap_r have been given above in Definitions 1, 2, and 3.

In the course of proving the different items of Theorem 5, Joret and Rambaud introduced the concept of a *guarding set*, that we define now. Let G be a graph, $A \subseteq V(G)$, and $r, p \in \mathbb{N}$. A family $\mathcal{S} \subseteq 2^{V(G)}$ is a (r,p)*-guarding set* for A if:

(1) $|S| \leq p$ for every $S \in \mathcal{S}$, and
(2) for every $v \in V(G)$, there exists $S \in \mathcal{S}$ such that S intersects every path of length at most r in G from v to a vertex of A (if any).

Guarding sets are interesting because of the following result.

Lemma 11 ([JR24, Lemma 12])**.** *Let r, p be nonnegative integers, G a graph, and $\mathcal{S}$ an (r,p)-guarding set for $A \subseteq V(G)$. Suppose that for some non-decreasing function f and every $A' \subseteq V(G)$,*

$$\mathrm{pc}_r(G, A') \leq f(r, |A'|).$$

Then

$$\mathrm{pc}_r(G, A) \leq f(r, p)|\mathcal{S}|.$$

2.2. **Tree representations and widths.** A *tree representation* of a graph $G = (V, E)$ is a pair $\mathcal{T} = (T, \{T_v\}_{v \in V(G)})$ such that T is a tree, for every $v \in V(G)$, T_v is a subtree of T, and for every $u, v \in V(G)$, if $uv \in E(G)$ then T_u and T_v have a common node in T. To avoid any possible confusion between the vertices of the graphs T and G that play different roles here, we will use the synonym *node* to refer to a vertex of the tree of a tree representation.

For a vertex t of T, we define $\beta_{\mathcal{T}}(t)$ as the set of all vertices $v \in V(G)$ such that $t \in V(T_v)$ and call this set the *bag* at t. We drop the subscript when there is no ambiguity. The *width* of $\mathcal{T}$ is defined as $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a representation of G.[2] Similarly we can define the *treelength* $\mathrm{tl}(G)$ of G, where the

[2]This is not the usual definition of treewidth but it can easily be checked that the two definitions are equivalent.

length of a representation $(T, \{T_v\}_{v\in V(G)})$ is defined as $\max_{t\in V(T)} \max_{u,v\in\beta(t)} \operatorname{dist}_G(u, v)$. The length of a tree-decomposition was defined in [DG07], see also [BS24] for a recent characterization. *Chordal graphs* can be defined as graphs that have treelength at most one.

2.3. **Least common ancestors.** For a rooted tree T and $M \subseteq V(T)$, the *least common ancestor closure (LCA-closure)* M' of M, is the output of the following process. First set $M' = M$, and then, as long as there are $u, v \in M'$ whose least common ancestor w in T is not in M', add w to M'. We will need the following folklore lemma (see [FLMS12, Lemma 1] for a proof).

Lemma 12 ([FLMS12, Lemma 1]). *Let T be a tree and, for $M \subseteq V(T)$, let M' be the LCA-closure of M. Then we have $|M'| \leq 2|M|$ and, for every component C of $T \setminus M'$, $|N_T(C)| \leq 2$.*

2.4. **Generalised colouring numbers.** The following parameters were introduced by Kierstead and Yang [KY03]. Consider $r \in \mathbb{N}$, a graph G, and a linear ordering L of $V(G)$. We say that a vertex $u \in V(G)$ is *weakly r-reachable* from $v \in V(G)$ if there exists an uv-path P of length at most r such that u is minimum with respect to L in $V(P)$. If we additionally have $v \leq_L w$ for all vertices $w \in P \setminus \{u\}$, we say that u is *strongly r-reachable* from v. Let $\operatorname{WReach}_r[G, L, v]$ and $\operatorname{SReach}_r[G, L, v]$ be the sets of vertices that are weakly r-reachable and strongly r-reachable from v, respectively. We set

$$\operatorname{wcol}_r(G, L) = \max_{v\in V(G)} |\operatorname{WReach}_r[G, L, v]|, \qquad \operatorname{scol}_r(G, L) = \max_{v\in V(G)} |\operatorname{SReach}_k[G, L, v]|,$$

and define the *weak r-colouring number*, $\operatorname{wcol}_r(G)$, and the *strong r-colouring number*, $\operatorname{scol}_r(G)$, of G, respectively, as follows:

$$\operatorname{wcol}_r(G) = \min_L \operatorname{wcol}_r(G, L), \qquad \operatorname{scol}_r(G) = \min_L \operatorname{scol}_r(G, L).$$

3. Graphs of bounded treewidth

In this section we prove Theorem 6, that we restate below for convenience.

Theorem 6. *Let t, r be two positive integers and let G be a graph of treewidth at most t. Then, $\operatorname{pc}_r(G, k) \in O(t^{O(t)} r^{t+1} k)$.*

The tightness of our bound is witnessed by Theorem 36 of [JR24]. Our bound relies on a carefully chosen guarding set, given by the following lemma.

Lemma 13. *Let G be a graph with treewidth at most t. For every $A \in V(G)$ there is an $(r, 2(t + 1))$-guarding set for A in G of size at most $4|A|$.*

Proof. We take a tree representation $(T, \{T_v\}_{v\in V(G)})$ of width at most t of G, and root T at an arbitrary node s. Note that we may assume that the tree representation is such that $|\beta(u) \cap \beta(v)| \leq t$ for every $uv \in E(T)$ as otherwise $\beta(u) = \beta(v)$ and we could identify these two nodes without changing the width of the reprensentation. For every vertex $x \in V(G)$, we let s_x be the (unique) node of T_x that is the closest to the root s.

We let $B = \{s_x \mid x \in A\}$ and B' be the LCA-closure of B in T with root s. For every node $b \in B'$, we let $p'(b)$ be the first vertex of $B' \setminus \{b\}$ met along the path from b to the root s, or $p'(b) = b$ if no such node exists. We let $\mathcal{S} = \{\beta(b) \mid b \in B'\} \cup \{\beta(b) \cup \beta(p'(b)) \mid b \in B'\}$. We clearly have $|\mathcal{S}| \leq 2|B'|$, and thus by Lemma 12 we have

$$|\mathcal{S}| \leq 4|B| \leq 4|A|. \tag{1}$$

Moreover, by extending the argument of the proof of Lemma 12 it is not hard to see that for every component C of $T \setminus B'$, the set $N_T(C)$ is of the form $\{b\}$ or $\{b, p'(b)\}$ for some $b \in B'$. The definition of $\mathcal{S}$ and the fact that the tree representation has width at most t imply the following:

$$\text{for every } S \in \mathcal{S},\ |S| \leq 2(t+1). \tag{2}$$

By Equations (1) and (2) all we have to show now is that for every $x \in V(G)$, there exists $S \in \mathcal{S}$ which intersects every (x, A)-path of length at most r in G. We assume that there is such an (x, A)-path, as otherwise there is nothing to prove. If there is a node $b \in B'$ such that $x \in \beta(b)$, then, trivially, $\beta(b) \in \mathcal{S}$ intersects every (x, A)-path. So, we assume there is no such node, and thus T_x is contained in some component C of $T \setminus B'$. As mentioned earlier, $N_T(C)$ is of the form $\{b\}$ or $\{b, p'(b)\}$ for some $b \in B'$.

Let a be some vertex in A and first assume that there is some node $c \in V(C)$ such that $c \in V(T_a)$. Since we have $s_a \in B$ and T_a being connected, then at least one of b and $p'(b)$ belongs to T_a. Thus $a \in \beta(b)$ or $a \in \beta(p'(b))$. In particular the set $\beta(b) \cup \beta(p'(b)) \in \mathcal{S}$ intersects every (x, a)-path, as claimed. Let us now deal with the remaining case where T_a does not have a node in common with C. In this case, we use the property of tree representations that bags are vertex separators and thus the set $\beta(b) \cup \beta(p'(b))$ separates in G every vertex x such that T_x is contained in C from any other vertex. In particular it separates x form A. We conclude that for every $a \in A$, the set $\beta(b) \cup \beta(p'(b)) \in \mathcal{S}$ intersects every (x, a)-path. So $\mathcal{S}$ is indeed the claimed guarding set. □

Bousquet and Thomassé [BT15], defined the *distance VC-dimension* of a graph G as follows. Let $\mathcal{H}$ be the hypergraph with vertex set $V(G)$ and having $\{N_r(v) \mid v \in G,\ r \geq 0\}$ as its edge set. The distance VC-dimension of G is the VC-dimension of $\mathcal{H}$. Bousquet and Thomassé showed that graphs excluding K_h as a minor have distance VC-dimension at most $h - 1$. Beaudou et al. [BDF+18] used this result, together with the Perles-Sauer-Shelah Lemma [Sau72, She72] to bound the metric dimension of graphs excluding a fixed complete minor. Following similar steps, Joret and Rambaud [JR24] obtained the following result, which will be useful for us.

Theorem 14 (Joret and Rambaud [JR24])**.** *Let $t \geq 3$ be an integer and G a graph with no K_t minor. Then for every nonempty set $A \subseteq V(G)$ and every $r \geq 0$, we have*

$$\mathrm{pc}_r(G, A) \leq (r+1)^{t-1}|A|^{t-1}.$$

We are now ready to prove Theorem 6.

Proof of Theorem 6. We will actually show the following more accurate bound: for every graph G with treewidth at most t, every subset $A \subseteq V(G)$ and every integer $r \geq 0$,

$$\mathrm{pc}_r(G, A) \leq 2^{t+3}(r+1)^{t+1}(t+1)^{t+1} \cdot |A|.$$

Let $\mathcal{S}$ be a $(r, 2(t+1))$-guarding set for A in G of size at most $4|A|$ as given by Lemma 13. Since graphs with treewidth at most t exclude K_{t+2} as a minor, by Theorem 14 we have for every $A' \subseteq V(G)$ the bound $\mathrm{pc}_r(G, A') \leq (r+1)^{t+1}|A'|^{t+1}$. Therefore, by Lemma 11 we obtain

$$\begin{aligned}\mathrm{pc}_r(G, A) &\leq (r+1)^{t+1}(2(t+1))^{t+1}|\mathcal{S}| \\ &\leq 2^{t+3}(r+1)^{t+1}(t+1)^{t+1}|A|,\end{aligned}$$

as claimed. □

4. Outerplanar graphs

By Theorem 6, if G has treewidth at most 2, then its profile complexity is in $O(r^3|A|)$, and this is tight (up to a constant factor) as noted in the introduction. In this section, we extend a result from [BDF+18] about the metric dimension, to show that the profile complexity of outerplanar graphs is in $O(r^2|A|)$. As mentioned in the introduction, we conjecture that a similar improvement is possible for all graphs with bounded simple treewidth (see Conjecture 26 in Section 7). Proposition 20 of [BDF+18] implies that the following is asymptotically tight.

Theorem 7. *Let G be an outerplanar graph, then $\mathrm{pc}_r(G,k) \in O(r^2k)$.*

Proof. Actually we show the following stronger statement: For every outerplanar graph G, every subset $A \subseteq V(G)$, and every integer $r \geq 0$, we have

$$\mathrm{pc}_r(G,A) \leq 1 + (2r+2)^2|A|.$$

In the following we assume that G is connected. To deal with disconnected graphs, we can apply separately the argument to every component that contains a vertex of A and sum the numbers of profiles in each. Since A contributes linearly to the bound this will give the desired result.

Following the proof of Theorem 19 of [BDF+18], we use the fact that outerplanar graphs can be represented in the plane in such a way that all vertices lie in the boundary of a circle (see [ST99]). Let $<$ be an ordering of $V(G)$ obtained by moving along this circle, starting at some vertex $a_1 \in A$.

Let d be the diameter of G. For each $1 \leq i \leq d$, let L_i be the set of vertices at distance exactly i from a_1. The following result from [BDF+18] is essential to the proof.

Lemma 15 ([BDF+18, Claim 19.B])**.** *Let $i \in \{1,\dots,d\}$, $a \in A$, and let y be a vertex of L_i which minimizes the distance to vertex a among all vertices of L_i. For $u,v \in L_i$, if $y < u < v$ or $v < u < y$, then we have $\mathrm{dist}(a,u) \leq \mathrm{dist}(a,v)$.*

For every $i \in \{1,\dots,d\}$, let A_i be the subset of those vertices of $A \setminus \{a_1\}$ whose r-neighbourhood intersects L_i. Observe that for every $a \in A \setminus \{a_1\}$ there are at most $2r+1$ values of i such that $N_r[a]$ intersects L_i (otherwise there would be a shortcut contradicting the definition of the L_i's). So we have

$$\sum_{i=1}^{d} |A_i| \leq (2r+1)|A|. \tag{3}$$

Lemma 15 implies that for every $a \in A_i$ there is a partition of L_i into at most $2r+2$ parts such that for every u,v belonging to the same part we have $\mathrm{Cap}_r(\mathrm{dist}(u,a)) = \mathrm{Cap}_r(\mathrm{dist}(v,a))$. Moreover, since such partition proceeds from $<$, we see that, together, the vertices of A_i partition L_i into at most $(2r+2)|A_i|$ parts where the r-profile of v to A is defined by the position of v in the partition. Thus (also counting the profile of a_1) we have

$$\begin{aligned}\mathrm{pc}_r(G,A) &\leq 1 + \sum_{i=1}^{d}(2r+2)|A_i| \\ &\leq 1 + (2r+2)^2|A|,\end{aligned}$$

where the second inequality comes from (3). ∎

5. Graphs excluding a fixed minor

In this section we give an affirmative answer to Problem 42 from [JR24] by showing that if G excludes K_h as a minor, then, for every integer r we have $\mathrm{pc}_r(G,k) \in h^{O(h)}(r+1)^{3(h-1)}k$.

Using Lemma 11, Theorem 14, and arguments from the paper of Reidl, Sánchez Villaamil and Stavropoulos [RVS19], Joret and Rambaud [JR24] proved that if G excludes K_h as a minor, we have $\mathrm{pc}_r(G,k) \in O(h^{O(h)} r^{h^2} k)$. As part of this proof, they construct guarding sets which were created in terms of the weak colouring numbers of G. We improve on these guarding sets by also considering the strong colouring numbers of G; the resulting guarding sets will have much smaller elements.

Theorem 16. *Let r, t, α, β be nonnegative integers, G a graph, and L a linear ordering of $V(G)$ with $\mathrm{scol}_{2r}(G,L) \le \alpha$ and $\mathrm{wcol}_r(G,L) \le \beta$. For every $A \subseteq V(G)$ there is an (r,α)-guarding set for A in G of size at most $\beta|A|$.*

Proof. Let $B = \cup_{a\in A} \mathrm{WReach}_r[G,L,a]$, and for every $b \in B$ let $S_b = \mathrm{SReach}_{2r}[G,L,b]$. We will show that the set $\mathcal{S} = \cup_{b\in B} S_b$ is the desired guarding set.

Since by hypothesis we have $|\mathcal{S}| \le \beta|A|$ and $|S| \le \alpha$ for every $S \in \mathcal{S}$, all we need to show is that for every vertex $x \in V(G)$ there exists $S \in \mathcal{S}$ which intersects every (x,A)-path of length at most r in G. Assume there is such a path P joining x to some $a \in A$ (otherwise there is nothing to prove), and let

$$\mu(x) = \max_L (B \cap \mathrm{WReach}_r[G,L,x]).$$

Notice that $B \cap \mathrm{WReach}_r[G,L,x]$ is nonempty and thus $\mu(x)$ does exist: for $y = \min_L V(P)$, the subpath P_1 of P from x to y witnesses $y \in \mathrm{WReach}_r[G,L,x]$, while the subpath P_2 from y to a witnesses $y \in B$.

We will show that $S_{\mu(x)}$ intersects P. If $y = \mu(x)$, then we are done, so we assume otherwise. Let Q be a path witnessing that $\mu(x) \in \mathrm{WReach}_r[G,L,x]$, and let w be the first vertex of P_1 (moving from x to y) which satifies $w <_L \mu(x)$. Note that w exists because, by the choice of $\mu(x)$ and since $y \in B \cap \mathrm{WReach}_r[G,L,x]$, we at least have $y <_L \mu(x)$. The concatenation of Q with the subpath of P_1 from x to w forms a path that witnesses that $w \in \mathrm{SReach}_{2r}[G,L,\mu(x)] = S_{\mu(x)}$. The result follows. □

Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, and Siebertz [HOdMQ+17] gave near-optimal upper bounds for the weak and strong colouring numbers of graphs excluding K_h as a minor. The proof for the weak colouring numbers provides an ordering which gives good bounds for all weak colouring numbers at once (a so-called uniform ordering). Moreover, that same ordering gives even better upper bounds for (all) the strong colouring numbers. This is made particularly clear in a recent work of Cortés, Kumar, Moore, Ossona de Mendez and Quiroz [CKM+25]. The following is implied by Lemma 4.9 and the proof of Lemma 4.8 from [CKM+25].

Lemma 17 ([CKM+25])**.** *Let $h \ge 4$ be an integer and G a graph excluding K_h as a minor. There is a linear ordering L of $V(G)$ such that for every integer $r \ge 0$ we have*

$$\mathrm{scol}_r(G,L) \le (h-3)(h-1)(2r+1),$$

and

$$\mathrm{wcol}_r(G,L) \le \binom{r+h-2}{h-2}(h-3)(2r+1).$$

Now we have all the ingredients for the main result of this section.

Theorem 18. *Let r, h be nonnegative integers with $h \geq 4$, G a graph excluding K_h as a minor. We have*

$$\mathrm{pc}_r(G,k) \leq 4^h(h-3)h^{2(h-1)}(r+1)^{3(h-1)}k.$$

In particular $\mathrm{pc}_r(G,k) \in h^{O(h)}(r+1)^{3(h-1)}k$.

Proof. Let A be a k-subset of $V(G)$. Since G excludes K_h as a minor, by Lemma 17, $V(G)$ has an ordering L such that

$$\begin{aligned} \mathrm{scol}_{2r}(G,L) &\leq (h-3)(h-1)(4r+1) \\ &\leq h^2(4r+1) \\ \text{and} \quad \mathrm{wcol}_r(G,L) &\leq \binom{r+h-2}{h-2}(h-3)(2r+1) \\ &\leq 2(h-3)(r+1)^{h-1}. \end{aligned}$$

Thus, by Theorem 16, there exists an $(r, h^2(4r+1))$-guarding set $\mathcal{S}$ for A in G of size at most $2(h-3)(r+1)^{h-1}k$. Then by Theorem 14 and Lemma 11 we obtain

$$\begin{aligned} \mathrm{pc}_r(G,k) &\leq (r+1)^{h-1}(h^2(4r+1))^{h-1}|\mathcal{S}| \\ &\leq (r+1)^{h-1}(h^2(4r+1))^{h-1} \cdot 2(h-3)(r+1)^{h-1}k \\ &\leq 2(h-3)(r+1)^{2(h-1)}h^{2(h-1)}(4r+1)^{h-1}k \\ &\leq 4^h(h-3)h^{2(h-1)}(r+1)^{3(h-1)}k. \end{aligned}$$

□

5.1. **Uniform orderings in other classes with bounded expansion.** Our bounds for graphs excluding a complete graph as a minor spring from the fact that these graphs have uniform orderings for the generalised colouring numbers. Kreutzer, Pilipczuk, Rabinovich and Siebertz [KPRS16] proved that graphs excluding a fixed subdivision also have uniform orderings. However, the bounds obtained for the weak colouring numbers do not (and cannot, see [GKR+18]) differ greatly from the bounds for the strong colouring numbers. So, in this case, the guarding sets obtained by Theorem 16 do not differ greatly from those that can be deduced from the techniques of Joret and Rambaud [JR24]. Indeed, while, in some sense, every class with bounded expansion allows for uniform orderings [HK21], Theorem 16 performs best with orderings that give substantially different bounds for weak and strong colouring numbers.

In [DPUY22], Dvořák, Pekárek, Ueckerdt and Yuditsky studied intersection graphs of various types of subsets of $\mathbb{R}^d$, and showed that if these objects are ordered in a non-increasing manner according to their diameter, then this ordering gives good (and different!) upper bounds for the weak and strong colouring numbers. As examples of the bounds that can be achieved through these orderings and Theorem 16, we study here intersection graphs of balls in $\mathbb{R}^d$. Similar results can be obtained for intersection graphs of scaled and translated copies of the same centrally symmetric compact convex subsets of $\mathbb{R}^d$ (like axis-aligned hypercubes), intersection graphs of ball-like subsets of $\mathbb{R}^d$ (see [DPUY22] for a definition), and intersection graphs of comparable axis-aligned boxes. For these graph classes, however, we use a rough alternative to Theorem 14, since this theorem is proved by using the fact that the class of graphs excluding K_h as a minor has bounded distance VC-dimension (we do not know if similar bounds hold for these geometric classes) and is closed under the addition of pendant vertices (which does not hold for these classes).

Let S be a set of balls in $\mathbb{R}^d$. Its *intersection graph* is the graph with vertex set S and an edge uv if and only if $u \cap v \neq \varnothing$. For an integer $t \geq 1$, S is t-*thin* if every point of $\mathbb{R}^d$ is contained in the interior of at most t balls of S.

Theorem 19. *Let t and d be positive integers. Let S be a t-thin finite set of balls in $\mathbb{R}^d$. Let G be the intersection graph of S, and L an ordering of S such that $u \leq_L v$ whenever $\operatorname{diam}(u) \geq \operatorname{diam}(v)$. There exists r_0 (depending on d) such that for every $r \geq r_0$ we have*

$$\operatorname{pc}_r(G,k) \leq 4^d (r+2)^{d+t(2r+1)^d} t \lceil \log_2 r \rceil \binom{r+2t+2}{2t+2} k.$$

Proof. Let A be some k-subset of $V(G)$. By [DPUY22, Lemma 1] the ordering L satisfies $\operatorname{scol}_r(G,L) \leq t(2r+1)^d$ for every r and by [DPUY22, Theorem 3] it satisfies $\operatorname{wcol}_r(G,L) \leq t\lceil \log_2 r \rceil (4r-1)^d \binom{r+2t+2}{2t+2}$ for r large enough. Hence, by Theorem 16, there is an $(r, t(2r+1)^d)$-guarding set $\mathcal{S}$ for A of size at most $t\lceil \log_2 r \rceil (4r-1)^d \binom{r+2t+2}{2t+2}$. Thus, by using Lemma 11 and the fact that $\operatorname{pc}_r(G,A') \leq (r+2)^{|A'|}$ for every $A' \subseteq V(G)$, we obtain

$$\begin{aligned}
\operatorname{pc}_r(G,A) &\leq (r+2)^{t(2r+1)^d} |\mathcal{S}| \\
&\leq (r+2)^{t(2r+1)^d} t\lceil \log_2 r \rceil (4r-1)^d \binom{r+2t+2}{2t+2} |A| \\
&\leq 4^d (r+2)^{d+t(2r+1)^d} t\lceil \log_2 r \rceil \binom{r+2t+2}{2t+2} k.
\end{aligned}$$

□

6. Graphs of bounded treelength

In this section we prove Theorem 9. We start with a lemma to deal with separators that have certain distance properties.

Lemma 20. *Let G be a graph, $r, \ell \in \mathbb{N}$, $A \subseteq V(G)$. Let $X \subseteq V(G)$ and suppose there is a subset $S \subseteq V(G)$ such that:*

(1) every path from X to $G - X$ intersects S; and

(2) the distance in G between any two vertices of S is at most ℓ,

then the number of r-profiles with respect to A of vertices from X is at most $(r+2)(\ell+1)^{|A|}$.

Actually, we will also need a variant of Lemma 20 where the separator consists of two sets of small diameter. Because the proofs are very similar, we first state and prove the (more complicated) variant and then explain how its proof can be modified to yield Lemma 20.

Lemma 21. *Let G be a graph, $r, \ell \in \mathbb{N}$, $A \subseteq V(G)$. Let $X \subseteq V(G) \setminus A$ and suppose there are two (possibly equal) subsets $S, S' \subseteq V(G)$ such that:*

(1) every path from X to $G - X$ intersects $S \cup S'$; and

(2) the distance in G between any two vertices of S (resp. two vertices of S') is at most ℓ,

then the number of r-profiles with respect to A of vertices from X is at most $(r+2)^2(\ell+1)^{|A|}$.

Proof. For every vertex $v \in V(G)$, we denote by d_v (resp. d'_v) the distance in G from v to some vertex of S (resp. S'). Let $v \in X$ and let us count the number of possible r-profiles with respect to A it may have.

Let $a \in A$. By assumption every shortest path from v to a intersects one of S and S'. Without loss of generality assume that such a path intersects S. Then we have $\operatorname{dist}(v, a) \geq d_v + d_a$. On the other hand, observe that $\operatorname{dist}(v, a) \leq d_v + d_a + \ell$. If $d_v > r$ there is no choice for the entry of the r-profile of v corresponding to a: $\operatorname{Cap}_r(\operatorname{dist}(v, a)) = +\infty$.

Thus

$$\operatorname{Cap}_r(d_v + d_a) \leq \operatorname{Cap}_r \operatorname{dist}(v, a) \leq \operatorname{Cap}_r(d_v + d_a + \ell).$$

The same inequality (with d'_a and d'_v instead of d_a and d_v, respectively) would hold if a shortest path from v to a met S'. The distances d_v and d'_v can each take a value in $\{0, \ldots, r\}$ or be larger than r, which makes $(r + 2)^2$ possible choices. Suppose that the values d_v and d'_v are fixed, then for every $a \in A$, by the inequalities above, $\operatorname{Cap}_r \operatorname{dist}(v, a)$ can take at most $\ell + 1$ different values. So, in total, the number of r-profiles with respect to A of the vertices of X is at most $(r + 2)^2(\ell + 1)^{|A|}$. □

The proof of Lemma 20 can be obtained by following the very same steps as in the proof of Lemma 21 for $S' = S$, with the following difference: as in this case $d_v = d'_v$, there are $r + 2$ possible choices for this value and not $(r + 2)^2$, which yields the bound $(r + 2)(\ell + 1)^{|A|}$.

Lemma 20 has the following special case when $\ell = 1$, that can be used to deal with graphs that have clique separators, typically chordal graphs.

Corollary 22. *Let G be a graph, $r, \ell \in \mathbb{N}$, $A \subseteq V(G)$. Let $X \subseteq V(G)$ and suppose there is a subset $S \subseteq V(G)$ such that:*

(1) every path from X to $G - X$ intersects S; and
(2) $G[X]$ is a clique,

then the number of r-profiles with respect to A of vertices from X is at most $(r + 2)2^{|A|}$.

The bound of the above result has the correct order of magnitude, as shown by the following construction. Start with a (split) graph consisting of a clique K and add $2^{|K|} - 1$ independent vertices, each neighbouring with a different nonempty subset of K. Finally, attach a new path of length r to every such vertex. It is not hard to observe that with $S = A = K$, such a graph shows that the contributions of r and $|A|$ to the bound of Corollary 22 cannot be substantially improved. It also gives the following lower bound.

Lemma 23. *For any integers $k, r \geq 1$ there is a chordal graph G such that $\operatorname{pc}_r(G, k) \geq (r + 1)(2^k - 1)$.*

We are now ready to prove the main result of this section.

Theorem 9. *Let G be a graph of treelength at most ℓ. Then $\operatorname{pc}_r(G, k) \in O\left(k \cdot (r^2(\ell + 1)^k)\right)$.*

Proof. Let $\mathcal{T} = (T, \{T_v\}_{v \in V(G)})$ be a tree representation of length at most ℓ of G. We start as in the proof of Lemma 13. That is, we root T at some node s and for every vertex $x \in V(G)$ we define s_x as the node of T_x the closest to the root. Let $B = \{s_a \mid a \in A\}$ and let B' be the LCA-closure of B in T with root s. By Lemma 12, $|B'| \leq 2|B| \leq 2|A|$ and every component C of $T - B'$ has at most two neighbours in B'. So we can deal separately with the following types of vertices:

(1) *Vertices in $\beta(t)$ for some $t \in B'$.* Consider a vertex $v \in \beta(t)$. By the condition on the length of $\mathcal{T}$, for every $a \in A$ we have

$$\operatorname{dist}_G(a, \beta(t)) \leq \operatorname{dist}_G(v, a) \leq \operatorname{dist}_G(a, \beta(t)) + \ell.$$

So in total the vertices in $\beta(t)$ may have at most $(\ell + 1)^{|A|}$ different r-profiles with respect to A.

(2) *Vertices in the bags of the components of $T - B'$ with the same unique neighbour t in B'.* Let C denote the set of nodes of the components of $T - B'$ that have t as unique neighbour in B'. Let $X = \bigcup_{c \in C} \beta(c)$. We now apply Lemma 20 to G, r, ℓ, A, X and with $S = \beta(t)$. As $\mathcal{T}$ has length at most ℓ and by definition of B' and X, the conditions of the lemma are indeed satisfied. So the number of r-profiles to A of the vertices in X is at most $(r+1)(\ell+1)^{|A|}$.

(3) *Vertices in the bags of a component of $T - B'$ that have two neighbours t and t' in B'.* Let C denote the set of nodes of this component and $X = \bigcup_{c \in C} \beta(c)$. We apply Lemma 21 to G, r, ℓ, A, X and with $S = \beta(t)$ and $S' = \beta(t')$ and obtain that the number of r-profiles with respect to A of vertices of X is at most $(r+2)^2(\ell+1)^{|A|}$.

As noted above, $|B'| \leq 2|A|$ so there are at most $2|A|$ sets of vertices to consider in each of the following three cases detailed above. Hence, the number of r-profiles with respect to A sums to $O\left(|A| \cdot \left(r^2(\ell+1)^{|A|}\right)\right)$, as claimed. □

7. Open problems

Graphs excluding a fixed minor. While Theorem 18 improves on the upper bound for the profile (and neighbourhood) complexity of graphs excluding a fixed complete minor, the following remains open.

Problem 24. *Up to a constant factor, what are the profile and neighbourhood complexity of graphs excluding K_t as a minor?*

More generally, it is an open problem to investigate the profile and neighborhood complexity of graphs excluding K_t as as subdivision. The approach used to obtain Theorem 19 and the vertex orderings obtained in [KPRS16] give that there is a constant c_t such that every such graph satisfies $\mathrm{pc}_r(G,k) \leq (r+2)^{(c_t r)^r}(c_t r)^r k$, but we believe this bound is far from optimal. Obtaining a good bound on the distance VC-dimension of these graphs would considerably improve this profile complexity bound.

Graphs with bounded treewidth. Theorem 6 gives an asymptotically sharp bound for the r-profiles of graphs with bounded treewidth. On the one hand, Theorem 6 implies that for every integers $r, k \geq 0$, every graph G with treewidth at most t we have $\mathrm{nc}_r(G,k) \in O_t(r^{t+1}k)$. On the other hand, by Corollary 37 of [JR24], there is a graph G of treewidth at most t and $A \subseteq V(G)$ such that $\mathrm{nc}_r(G,A) \in \Omega_t(r^t|A|)$. The following remains open.

Problem 25. *Up to a constant factor, what is the neighbourhood complexity of graphs with bounded treewidth?*

Graphs with bounded simple treewidth. A graph is a *k-tree* if it is either a clique of order $k+1$ or can be obtained from a smaller k-tree by adding a vertex and making it adjacent to k pairwise-adjacent vertices. A *simple k-tree* is a k-tree built with the restriction that when adding a new vertex, the clique to which we make it adjacent cannot have been used when adding some other vertex. The *simple treewidth*, $\mathrm{stw}(G)$, of a graph G is the smallest k such that G is a subgraph of a simple k-tree. Since the treewidth of a graph G can be equivalently defined as the smallest k such that G is a subgraph of a k-tree, we have $\mathrm{tw}(G) \leq \mathrm{stw}(G)$ for every graph G. Graphs with $\mathrm{stw}(G) \leq 1$ are the disjoint union of paths, and graphs with $\mathrm{stw}(G) \leq 2$ form exactly the class of outerplanar graphs. Recall that we obtained an $O(r^2k)$ bound on the profile complexity of outerplanar graphs in Theorem 7.

We conjecture that this result can be generalized to graphs of bounded simple treewidth as follows.

Conjecture 26. *Let t, r be two positive integers and let G be a graph of simple treewidth at most t. Then,* $\mathrm{pc}_r(G, k) \in O(t^{O(t)} r^t k)$.

This conjecture holds for $t = 2$ by Proposition 7, and holds trivially for $t = 1$: since a graph G with $\mathrm{stw}(G) = 1$ is the disjoint union of paths, for every vertex v in G we have $|N_r[v]| \leq 2r + 1$. This immediately gives $\mathrm{pc}_r(G, k) \leq (2r + 1)k$. Since $\mathrm{tw}(G) \leq \mathrm{stw}(G)$ for every graph G, the construction of Corollary 37 of [JR24] attests that, if true, this conjecture would be asymptotically tight.

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