Polynomial Gap Extensions of the Erdős–Pósa Theorem

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Abstract

Given a graph H, we denote by $\mathcal{M}(H)$ all graphs that can be contracted to H. The following extension of the Erdős–Pósa theorem holds: for every h-vertex planar graph H, there exists a function f_H such that every graph G, either contains k disjoint copies of graphs in $\mathcal{M}(H)$, or contains a set of $f_H(k)$ vertices meeting every subgraph of G that belongs in $\mathcal{M}(H)$. In this paper we prove that this is the case for every graph H of pathwidth at most 2 and, in particular, that $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$. As a main ingredient of the proof of our result, we show that for every graph H on h vertices and pathwidth at most 2, either G contains k disjoint copies of H as a minor or the treewidth of G is upper-bounded by $2^{O(h^2)} \cdot k^2 \cdot \log k$. We finally prove that the exponential dependence on h in these bounds can be avoided if $H = K_{2,r}$. In particular, we show that $f_{K_{2,r}} = O(r^2 \cdot k^2)$.

Keywords: Treewidth, Graph Minors, Erdős–Pósa Theorem

1 Introduction

In 1965, Paul Erdős and Lajos Pósa proved that every graph that does not contain k disjoint cycles, contains a set of $O(k \log k)$ vertices meeting all its cycles [9]. Moreover, they gave a construction asserting that this bound is tight. This classic result can be seen as a "loose" min-max relation between covering and packing of combinatorial objects. Various extensions of this result, referring to different notions of packing and covering, attracted the attention of many researchers in modern Graph Theory (see, *e.g.* [2, 14]).

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Given a graph H, we denote by $\mathcal{M}(H)$ the set of all graphs that can be contracted to H (*i.e.* if $H' \in \mathcal{M}(H)$, then H can be obtained from H' after contracting edges). We call the members of $\mathcal{M}(H)$ *models* of H. Then the notions of covering and packing can be extended as follows: we denote by $\mathbf{cover}_H(G)$ the minimum number of vertices that meet every model of H in G and by $\mathbf{pack}_H(G)$ the maximum number of mutually disjoint models of H in G. We say that a graph H *has the Erdős–Pósa Property* if there exists a function $f_H \colon \mathbb{N} \to \mathbb{N}$ such that for every graph G,

$$\text{if } k = \mathbf{pack}_H(G), \text{ then } k \leqslant \mathbf{cover}_H(G) \leqslant f_H(k) \tag{1}$$

We will refer to f_H as the *gap* of the Erdős–Pósa Property. Clearly, if $H = K_3$, then (1) holds for $f_{K_3} = O(k \log k)$ and the general question is to find, for each instantiation of H, the best possible estimation of the gap f_H, if it exists.

It turns out that H has the Erdős–Pósa Property if and only if H is a planar graph. This beautiful result appeared as a byproduct of the Graph Minors series of Robertson and Seymour. In particular, it is a consequence of the grid-exclusion theorem, proved in [20] (see also [6]).

Proposition 1. *There is a function $g \colon \mathbb{N} \to \mathbb{N}$ such that if a graph excludes an r-vertex planar graph R as a minor, then its treewidth is bounded by $g(r)$.*

In [20] Robertson, Seymour, and Thomas conjectured that g is a low degree polynomial function. Currently, the best known bound for g is $g(k) = 2^{O(k \log k)}$ and follows from [7] and [18] (see also [15, 20] for previous proofs and improvements). As the function g is strongly used in the construction of the function f_H in (1), the best, so far, estimation for f_H is far from being exponential in general. This initiated a quest for detecting instantiations of H where a polynomial gap f_H can be proved.

The first result in the direction of proving polynomial gaps for the Erdős–Pósa Property appeared in [12] where H is the graph θ_c consisting of two vertices connected by c multiple edges (also called *c-pumpkin graph*). In particular, in [12] it was proved that $f_{\theta_c}(k) = O(c^2 k^2)$. More recently Fiorini, Joret, and Sau optimally improved this bound by proving that $f_{\theta_c}(k) \leqslant c_t \cdot k \cdot \log k$ for some computable constant c_t depending on c [11]. In [21] Fiorini, Joret, and Wood proved that if T is a tree, then $f_T(k) \leqslant c_T \cdot k$ where c_T is some computable constant depending on T. Finally, very recently, Fiorini [10] proved that $f_{K_4} = O(k \log k)$.

Our main result is a polynomial bound on f_H for a broad family of planar graphs, namely those of pathwidth at most 2. We prove the following:

Theorem 1. *If H is an h-vertex graph of pathwidth at most* 2 *and $h > 5$, then* (1) *holds for $f_H(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$.*

Note that the contribution of h in f_H is exponential. However, such a dependence can be waived when we restrict H to be $K_{2,r}$. Our second result is the following:

Theorem 2. *If $H = K_{2,r}$, then* (1) *holds for $f_H(k) = O(r^2 \cdot k^2)$.*

Both results above are based on a proof of Proposition 1, with polynomial g, for the cases where R consists of k disjoint copies of H and H is either a graph of pathwidth at most 2 or $H = K_{2,3}$ (Theorems 3 and 4 respectively). For this, we follow an approach that makes strong use of the k-mesh structure introduced by Diestel *et al.* [7] in their proof of Proposition 1. Our proof indicates that, when excluding copies of some graph of pathwidth at most 2, the entangled machinery of [7] can be partially modified so that polynomial bounds on treewidth are possible. Finally, these bounds are then "translated" to polynomial bounds for the Erdős–Pósa gap using a technique developed in [13] (see also [12]).

2 Definitions and notations

2.1 Basics

In this paper, logarithms are binary.

Graphs and subgraphs A *graph* G is a pair (V, E) where V is called the set of vertices of G and E is called the set of edges of G and satisfies $E \subseteq V^2$. Two vertices v, u of G are said to be *adjacent* if $(u, v) \in E$. A *multigraph* is a graph where multiple edges between two vertices are allowed. In this paper, the graphs we consider are finite, undirected and without loops. Unless otherwise specified, graphs are assumed to be simple (*i.e.* multiedges are not allowed).

For any graph G, $V(G)$ (resp. $E(G)$) denotes the set of vertices (resp. edges) of G. A graph G' is a *subgraph* of a graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ and we write it $G' \subseteq G$. If X is a subset of $V(G)$, we note $G[X]$ the *subgraph of G induced by X*, *i.e.* the graph $(X, \{xy \in E(G),\ x \in X \text{ and } y \in X\})$.

When talking about graphs, unless otherwise stated, by *disjoint* we mean vertex-disjoint. We denote by K_n the complete graph on n vertices and by $\mathrm{K}_{p,q}$ the complete bipartite graph with partitions of size p and q. For any integer k and any graph G, the graph $k \cdot G$ is the disjoint union of k copies of the graph G. A pair $\{A, B\}$ is a separation of a graph G if $A \cup B = V(()G)$ and G has no edge between $A \setminus B$ and $B \setminus A$. The integer $|A \cap B|$ is the *order* of the separation $\{A, B\}$. We assume that the reader is familiar with the basic graph classes: paths, cycles, trees, *etc.*.

Neighbourhood and degree For any vertex $v \in V(G)$, the *neighbourhood* $\mathrm{N}_G(v)$ of v in G is the set of vertices that are adjacent to v in G. The *degree* of $v \in V(G)$ in G, denoted $\deg_G(v)$, is the cardinal of $\mathrm{N}_G(v)$. The minimum value taken by $\deg_G$ in $V(G)$ is called the *minimum degree* of G and denoted by $\delta(G)$. When dealing with multigraphs, the *multidegree* of a vertex v (written $\deg^{\mathrm{m}}(v)$) is the number of simple edges incident to v. In these notations, we drop the subscript when it is obvious. The average degree over all vertices of a graph G is written $\mathrm{ad}(G)$.

Contractions In a graph G, a *contraction* of the edge $e = (u, v) \in E(G)$ is the operation that transforms G into a graph H such that $V(H) = V(G) \setminus \{u, v\} \cup$

$\{v_e\}$ and $E(H) = \{(x,y) \in E(G),\ x \notin \{u,v\}$ and $y \notin \{u,v\}\} \cup \{(x,v_e),\ (x,u) \in E(G)$ or $(x,v) \in E(G)\}$. We say that a graph G can be *contracted* to a graph H if H is the result of a sequence of edge contractions on G.

Trees An acyclic connected graph is called a *tree*. The vertices of degree 1 of a tree are its *leaves* and its other vertices are called *internal vertices*. A tree whose every internal vertex has degree at most 3 is said to be *ternary*. A *binary tree* is a ternary tree whose one of the internal nodes, the *root*, is distinguished and has degree at most 2.

2.2 More definitions

Definition 1 (graph Ξ_r)**.** We define the graph Ξ_r as the graph of the following form (see figure 1).

$$\begin{cases} V(G) = \{x_0,\dots,x_{r-1},y_0,\dots,y_{r-1},z_0,\dots,z_{r-1}\} \\ E(G) = \{(x_i,x_{i+1}),(z_i,z_{i+1})\}_{i\in[\![0,r-2]\!]} \cup \{(x_i,y_i),(y_i,z_i)\}_{i\in[\![0,r-1]\!]} \end{cases}$$

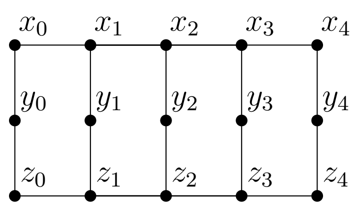

Figure 1: The graph Ξ_5

Definition 2 (minor model)**.** A *minor model* (sometimes abbreviated *model*) of a graph H in a graph G is a pair $(\mathcal{M},\varphi)$ where $\mathcal{M}$ is a collection of disjoint subsets of $V(G)$ such that $\forall X \in \mathcal{M}$, $G[X]$ is connected and $\varphi\colon V(H) \to \mathcal{M}$ is a bijection that satisfies $\forall\{u,v\} \in E(H), \exists u' \in \varphi(u), \exists v' \in \varphi(v),\ \{u',v'\} \in E(G)$. We say that a graph H is a *minor* of a graph G ($H \leqslant_{\mathrm{m}} G$) if there is a minor model of H in G. Notice that H is a minor of G if H can be obtained by a subgraph of G after contracting edges.

Definition 3 (degeneracies)**.** The *degeneracy* of G, written $\delta^*(G)$, is the maximum value taken by $\delta(G')$ over all subgraphs G' of G:

$$\delta^*(G) = \max_{G'\subseteq G} \delta(G')$$

Similarly, the *contraction degeneracy* of G, introduced in [3] and denoted $\delta_c(G)$, is the maximum value of $\delta(G')$ for all minors G' of G:

$$\delta_c(G) = \max_{G'\leqslant_{\mathrm{m}} G} \delta(G')$$

Remark that, as a subgraph is a minor, for all graph G we have the following inequality

$$\delta_c(G) \geqslant \delta^*(G)$$

These definitions remains the same on multigraphs (we do not take into account the potential multiplicities of the edges).

Definition 4 (tree decomposition and treewidth)**.** A *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}$ a family $(X_t)_{t\in V(T)}$ of subsets of $V(G)$ (called *bags*) indexed by elements of $V(T)$ and such that

(i) $\bigcup_{t\in V(T)} X_t = V(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition T is defined as equal to $\max_{t\in V(T)} |X_t| - 1$. The *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions.

Definition 5 (nice tree decomposition)**.** A tree decomposition $(T, \mathcal{V}$ of a graph G is said to be a *nice* tree decomposition if

1. every vertex of T has degree at most 3;
2. T is rooted on one of its vertices r whose bag is empty ($V_r = \emptyset$);
3. every vertex t of T is
   - either a *base node*, *i.e.* a leaf of T whose bag is empty ($V_t = \emptyset$) and different from the root;
   - or an *introduce node*, *i.e.* a vertex with only one child t' such that $V_{t'} = V_t \cup \{u\}$ for some $u \in V(G)$;
   - or a *forget node*, *i.e.* a vertex with only one child t' such that $V_t = V_{t'} \cup \{u\}$ for some $u \in V(G)$;
   - or a *join node*, *i.e.* a vertex with two child t_1 and t_2 such that $V_t = V_{t_1} = V_{t_2}$.

It is known that every graph has an optimal tree decomposition which is nice [16].

Definition 6 (path decomposition and pathwidth)**.** A *path decomposition* of a graph G is a tree decomposition T of G such that T is a path. Its width is the width of the tree decomposition T and the *pathwidth* of G, written $\mathbf{pw}(G)$, is the minimum width of any of its path decompositions.

Definition 7 (linked and externally k-connected)**.** Let k be a positive integer, G be a graph and X, Y be two subsets of $V(G)$.

X and Y are said to be *linked* by a path if there is a path in G from an element of X to an element of Y.

X and Y are said to be *k-connected in G* if for all disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that $|X'| = |Y'| \leqslant k$ there are $|X'|$ disjoint paths between X' and Y'

in G. If these paths have no internal vertices nor edges in $G[X \cup Y]$, then X and Y are said to be *externally k-connected in G*. If $X = Y$, X is said to be *(externally) k-connected in G*.

Definition 8 (k-mesh, [6])**.** An (ordered) pair (A, B) of subsets of $V(G)$ is a called a *k-mesh of order s in G* if $V(G) = A \cup B$ and $G[A]$ contains a ternary tree T such that

(i) $A \cap B \subseteq V(T)$ and $A \cap B \cap V(T)$ are nodes of degree at most 2 in T;

(ii) at least one leaf of T is in $A \cap B$;

(iii) $|A \cap B| = s$;

(iv) $A \cap B$ is externally k-connected in B.

3 Preliminaries

Proposition 2 ([6], (12.14.5))**.** *Let G be a graph and let $p \geqslant q \geqslant 1$ be integers. If G contains no q-mesh of order p then G has treewidth less than $p + q - 1$.*

Proposition 3 (follows from [6], (2.14.6))**.** *Let $k \geqslant 2$ be an integer. Let T be a tree of maximum degree at most 3 and $X \subseteq V(T)$. Then T has $\left\lfloor \frac{|X|}{2k-1} \right\rfloor - 1$ vertex-disjoint subtrees each containing at least k vertices of X.*

Proposition 4 ([4])**.** *For any integer $r \geqslant 1$ and any graph G,*

$$G \not\geqslant_{\mathrm{m}} K_{2,r} \Rightarrow \mathbf{tw}(G) < 2r - 2$$

Proposition 5 ([22])**.** *For any integer $k \geqslant 1$ and any graph G, there exist sets $V_1, \ldots, V_k$ partitioning $V(G)$ (*i.e. $\sqcup_{i \in [\![1,k]\!]} V_i = V(G)$*) such that*

$$\forall i \in [\![1, k]\!], \forall u \in V_i,\ \deg_{V_i}(v) \geqslant \frac{\deg_G(v)}{k} - 1$$

In particular, if $\delta(G) \geqslant p$ then $\forall i \in [\![1, k]\!],\ \delta(G[V_i]) \geqslant \frac{p}{k} - 1$

Proposition 6 (Erdős–Szekeres Theorem, [8])**.** *Let k and ℓ be two strictly positive integers. Then any sequence of $(\ell - 1)(k - 1) + 1$ distinct integers contains either an increasing subsequence of length k or a decreasing subsequence of length ℓ.*

Proposition 7 ([17], [23], [6] (7.2.3))**.** *There is a real constant c such that every graph of average degree more than a function $c(t) = (c + o(1))t\sqrt{\log t}$ contains K_t as minor. According to [17], $c(t) < 648 \cdot t\sqrt{\log t}$.*

4 Excluding packings of planar graphs

Theorems 1 and 2 follow combining the two following results with the machinery introduced in [13] (see also [12]). They have independent interest as they detect cases of Theorem 1 where g depends polynomially on k.

Theorem 3. *Let H be a graph of pathwidth at most 2 on $r > 5$ vertices. If G does not contain k disjoint copies of H as minors then* $\mathbf{tw}(G) \leqslant 2^{O(r^2)} \cdot k^2 \cdot \log 2k$.

Theorem 4. *For every positive integer r, if G does not contain k disjoint copies of $K_{2,r}$ as a minors then* $\mathbf{tw}(G) < 20k^2r^2 - 8k^2r + 2r - 1$.

4.1 Auxiliary results

Lemma 1. *Let G be a graph and let $p \geqslant q \geqslant 1$ be integers. If $\mathbf{tw}(G) \geqslant 5pq - 2q + 2p - 1$, then there exist $2q$ disjoint sets $X_1, \dots, X_{2q}$ of $V(G)$ and a set $\mathcal{P}$ of pq disjoint paths in G of length at least 2 and such that*

(i) $\forall i \in [\![1, 2q]\!]$, *$X_i$ is of size p and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$;*

(ii) any path in $\mathcal{P}$ has one of its ends in some X_i with $i \in [\![1, q]\!]$, the other end in some X_j with $j \in [\![q+1, 2q]\!]$ and its internal vertices are in none of the X_l, for all $l \in [\![1, 2q]\!]$, nor in A.

(iii) $\forall i, j \in [\![1, 2k]\!],\ i \neq j \Rightarrow T_i \cap T_j = \emptyset$

Proof. Let G be a graph, $p \geqslant q \geqslant 1$ two integers and assume that $\mathbf{tw}(G) \geqslant 5pq - 2q + 2p - 1$. According to Proposition 2, G contains a (pq)-mesh of order $(2p-1)(2q+1)$. Let (A, B) be this mesh, $X = A \cap B$ and let T be the tree related to A. By definition of a mesh, T is a tree of maximum degree 3 and $X \subseteq V(T)$.

Using Proposition 3, there exist $\left\lfloor \frac{|X|}{2p-1} \right\rfloor - 1 = 2q$ disjoint subtrees $T_1, \dots T_{2q}$ of $V(T)$ such that for all $i \in [\![1, 2q]\!]$, $|V(T_i) \cap X| \geqslant p$. For all $i \in [\![1, 2q]\!]$, let X_i be a subset of $V(T_i) \cap X$ such that $|X_i| = p$.

The set X is externally (pq)-connected in B (by definition of a mesh), *i.e.* any two subsets of X of size pq are linked by pq disjoint paths whose internally vertices are in B. Thus, the sets $Z_1 = \bigcup_{i \in [\![1,q]\!]} X_i$ and $Z_2 = \bigcup_{i \in [\![q+1,2q]\!]} X_i$ (whose each is of size pq) are externally connected in B. Let $\mathcal{P}$ be these pq paths between Z_1 and Z_2. We now check the conditions (i), (ii) and (iii) on $\{X_i\}_{i \in [\![1,2q]\!]}$ and $\mathcal{P}$.

(i) by definition of $\{X_i\}_{i \in [\![1,2q]\!]}$, for all $i \in [\![1, 2q]\!]$, $|X_i| = p$ and X_i belongs to $V(T_i)$, therefore X_i is connected in G by the tree T_i;

(ii) $\mathcal{P}$ contains disjoint paths such that

- they do not use elements of A (by definition);
- they are external to Z_1 and Z_2 (*i.e.* none of their internal vertices belongs to X_i, for all $i \in [\![1, 2q]\!]$);

- any $p \in \mathcal{P}$ links Z_1 to Z_2, thus p have one end in Z_1 and the other end in Z_2, put another way p have one end in some X_i for $i \in [\![1, 2q]\!]$ and the other end in some X_j for some $j \in [\![q+1, 2q]\!]$.

(iii) by definition the T_i's are all disjoint.

The sets $\{X_i\}_{i \in [\![1,2q]\!]}$ satisfies the properties (i), (ii) and (iii) so we found these sets we were looking for. □

Lemma 2. *For any integer $a \geqslant 1$ and for any graph G, $V(G)$ contains more than $(1-\frac{1}{a})\,|V(G)|$ vertices of degree strictly less than $2a\delta^* G$. In particular, $V(G)$ contains at least $\frac{|V(G)|}{2}$ vertices of degree strictly less than $\delta^*(G)$.*

Proof. Let $a \geqslant 1$ be an integer and let G be a graph.

Let n_h be the number of vertices of G with degree at least $h = 2a \times \delta^*(G)$, *i.e.* $n_h = |\{v \in V(G),\ \deg(v) \geqslant h\}|$ and n_{-h} the number of vertices of degree strictly less than h, *i.e.* $n_{-h} = |V(G)| - n_h$. Clearly, there is at least $\frac{1}{2}hn_h$ edges incident the n_h vertices of degree at least h. We thus have:

$$\begin{aligned}
\frac{1}{2}hn_h &\leqslant |E(G)| && \text{(because there may be other edges)}\\
&\leqslant \frac{1}{2}\sum_{v \in V(G)} \deg(v) && \text{(Handshaking lemma)}\\
\frac{hn_h}{|V(G)|} &\leqslant \frac{\sum_{v \in V(G)} \deg(v)}{|V(G)|}\\
&< 2\delta^*(G) && \left(\text{because } \frac{\sum_{v \in V(G)} \deg(v)}{|V(G)|} = \mathrm{ad}(G) < 2\delta^*(G)\right)\\
n_h &< |V(G)|\,\frac{2\delta^*(G)}{h}\\
n_{-h} &> |V(G)|\left(1 - \frac{2\delta^*(G)}{h}\right)\\
&> |V(G)|\left(1 - \frac{1}{a}\right) && \text{(by replacing } h \text{ by its value)}
\end{aligned}$$

Finally, we found that G contains more than $|V(G)|\left(1 - \frac{1}{a}\right)$ vertices of degree strictly less than $2a \times \delta^*(G)$, what we wanted to prove. □

Lemma 3. *Let k, r be two positive integers and G a graph such that $\delta_c(G) \geqslant 2kr$. Then G contains k disjoint copies of $\mathrm{K}_{2,r}$ as minors.*

Proof. Let k, r be two positive integers and G a graph of contraction degeneracy at least $2kr$. Then G has a minor G' such that $\delta(G') \geqslant 2kr$.

According to Proposition 5, there is a partition $\mathcal{V} = \{V_1, \ldots, V_k\}$ of $V(G')$ such that

$$\forall V_i \in \mathcal{V},\ \delta(G'[V_i]) \geqslant \frac{2kr}{k} - 1 = 2r - 1$$

The minimum degree of a graph is a lower bound for its treewidth, then any $V_i \in \mathcal{V}$ has treewidth at least $2r-1$, and thus by Proposition 4 V_i contains $\mathrm{K}_{2,r}$ as a minor. $\mathcal{V}$ is a partition of size k of $V(G')$ and each element of $\mathcal{V}$ contains $\mathrm{K}_{2,r}$ as a minor consequently G' contains k disjoint copies of $\mathrm{K}_{2,r}$ as minors. As G' is a minor of G, G contains k disjoint copies of $\mathrm{K}_{2,r}$ as minors, what we wanted to show.

□

Lemma 4. *Let T be a ternary tree and $X = \{v \in V(T),\ \deg_T(v) \leqslant 2\}$. Then*

(i) for any path P on l vertices in T, T has a partition $\mathcal{M}$ such that

a) every vertex of P belongs to a different element of $\mathcal{M}$;

b) every element of $\mathcal{M}$ contains an element of X.

(ii) T has diameter at least $2\log\frac{2}{3}|X|$.

Proof of (i). Let T, X, P be as in the statement of the lemma. For every $u \in V(P)$, we set M_u as the set of vertices of the connected component $G\backslash(P\backslash\{u\})$ that contains u. Let $\mathcal{M} = \{M_u\}_{u\in P}$. Clearly, for all $u, v \in V(P)$, if $u \neq v$ then $M_u \cap M_v = \emptyset$. Also, since T is connected, there is no vertex of $V(T)$ that is not in an element of . Therefore $\mathcal{M}$ is a partition of $V(T)$. By definition, for every $u \in V(P),\ u \in M_u$. Besides, every element M of $\mathcal{M}$ contains either exactly one element, which is necessarily a vertex of degree 2 in T, or more than one element ad in this case it induces in G a tree whose leaves are also leaves of G. In both cases M contains an element of X as required. □

Proof of (ii). Let $P = p_0 \ldots p_k$ be a longest path in T. In order to be able to use the notions of height and of child, we root T at node $n_{\left\lfloor \frac{k}{2} \right\rfloor}$ (which is clearly not a leave).

We prove the proposition for the case where T has no vertices of degree two. If this is not the case, we can just add a leaf as child of every vertex of degree two. As these vertices have an other child, there is at least one longest path that use none of the new vertices.

Let $\ell = |X|$. By contradiction, assume that $k < 2\log\frac{2}{3}\ell$.

Let T' be the full ternary tree of height $\left\lceil \frac{k'}{2} \right\rceil$. As T' is complete, it has $3{\cdot}2^{\left\lceil \frac{k}{2} \right\rceil - 1}$ leaves. The tree T' clearly contains T as subgraph because they have same height, thus T' has at most as much leaves as T, *i.e.* $l \leqslant 3 \cdot 2^{\left\lceil \frac{k}{2} \right\rceil - 1}$. If we use our first assumption, we get:

$$\begin{aligned} l &\leqslant 3 \cdot 2^{\left\lceil \frac{k}{2} \right\rceil - 1} \\ &< 3 \cdot 2^{\left\lceil \log\frac{2}{3}\ell \right\rceil - 1} \\ l &< l \end{aligned}$$

We obtain a contradiction, thus our assumption $k < 2\log\frac{2}{3}\ell$ was false: T has diameter at least $2\log\frac{2}{3}|X|$. □

Lemma 5. *Let k, r be two positive integers and $G = ((V_1, V_2), E)$ a bipartite multigraph such that*

$$|V_1| = |V_2| \geqslant 4k^2 r$$
$$\forall v \in V(G),\ \deg^{\mathrm{m}}(v) = 2kr^2$$
$$\delta^*(G) < 2kr$$

Then G has at least k (vertex-)disjoint multiedges of multiplicity at least r.

Proof. Let G be a graph that fill the conditions of the lemma. For $(u, v) \in E(G)$, let $\mathrm{mult}(u, v)$ denote the multiplicity of the edge (u, v). According to lemma 2, G contains at least $\frac{1}{2}V(G) \geqslant 4k^2 r$ vertices of degree strictly less than $\delta^*(G) < 2kr$. Then, one of V_1, V_2 contains at least $2k^2 r$ such vertices. We assume without loss of generality that this is V_1. Let L be a subset of V_1 of size $2k^2 r$ containing vertices of degree strictly less than $2kr$. For all $v \in L$, v has degree less than $2kr$ (by definition of L) and multidegree $2kr^2$ (by initial assumption) so there is a least one $u \in V_2$ such that $\mathrm{mult}(u, v) \geqslant r$.

We now define an auxiliary function. Let $f : L \to V_2$ a function such that $\forall v \in L,\ \mathrm{mult}(v, f(v)) \geqslant r$. According to the previous remark, such a function exists. For all $u \in f(L)$, the multidegree of u is by assumption $2kr^2$ thus u cannot be the image of more than $\frac{\deg^{\mathrm{m}}(u)}{r} = 2kr$ elements of L. Consequently, $f(L)$ has size at least $\frac{|L|}{2kr} \geqslant k$. Remark that for all $u_1, u_2 \in f(L)$ with $u_1 \neq u_2$, the preimages of u_1 and u_2 are disjoint.

We finally show k disjoint multiedges of multiplicity at least r in G. Choose k distinct elements $u_1, \ldots, u_k$ of $f(L)$ and for all $i \in [\![1, k]\!]$ let v_i be an element of L in the preimage of u_i (*i.e.* such that $f(v_i) = u_i$). As said before, the preimages of distinct elements of $f(L)$ are distinct so the v_i's are all distinct. By definition $\forall i \in [\![1, k]\!], f(v_i) = u_i$ so there is an edge of multiplicity r between u_i and v_i in G. Therefore, $\{(v_i, u_i)\}_{i \in [\![1,k]\!]}$ is the set of edges we were looking for.

□

In [19] we prove the following lemma.

Lemma 6 ([19]). *For all graph G, if $n = |V(G)|$, then $\mathbf{pw}(G) \leqslant 2 \Rightarrow G \leqslant_{\mathrm{m}} \Xi_n$.*

Lemma 7. *For all positive integers p, q and all graph G, if $\mathbf{tw}(G) \geqslant 20p^2q^2 - 8p^2q + 2q - 1$ and $\delta_c(G) < 2pq$ then G contains $2p$ disjoint subsets $X_1, \ldots, X_{2p}$ of $V(G)$ and a set $\mathcal{P}$ of pq disjoint paths of length at least 2 in G such that*

(i) $\forall i \in [\![1, 2p]\!]$, *$X_i$ is of size q and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$;*

(ii) *any path in $\mathcal{P}$ has one of its ends in some X_i with $i \in [\![1, p]\!]$, the other end in X_{2i} with $j \in [\![q + 1, 2p]\!]$ and its internal vertices are in none of the X_l, for all $l \in [\![1, 2p]\!]$, nor in A;*

(iii) $\forall i, j \in [\![1, 2p]\!],\ i \neq j \Rightarrow T_i \cap T_j = \emptyset$.

Proof. According to lemma 1, G contains $8p^2q$ disjoint sets $Y_1, \ldots, Y_{8p^2q}$ of $V(G)$ and a set $\mathcal{P}$ of $4p^2q^2$ disjoint paths in G of length at least 2 and such that

(i) $\forall i \in [\![1, 8p^2q]\!]$, Y_i is of size q and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$;

(ii) any path in $\mathcal{P}$ has one of its ends in some Y_i with $i \in [\![1, 4p^2q]\!]$, the other end in some Y_j with $j \in [\![4p^2q+1, 8p^2q]\!]$ and its internal vertices are in none of the Y_l, for all $l \in [\![1, 8p^2q]\!]$, nor in A;

(iii) $\forall i, j \in [\![1, 8p^2q]\!],\ i \neq j \Rightarrow T_i \cap T_j = \emptyset$.

Let us consider the bipartite multigraph H defined by

- $V(H) = \{Y_i\}_{i \in [\![1, 8p^2q]\!]}$;
- for all n integer and $i, j \in [\![1, 8p^2q]\!]$ there is an edge of multiplicity m between the two vertices Y_i and Y_j iff there is exactly m paths from a vertex of Y_i to a vertex of Y_j in P.

Clearly, H is a minor of G. Consequently $2pq > \delta_c(G) \geqslant \delta_c(H) \geqslant \delta^*(H)$.

The three conditions required on H by lemma 5 are filled, so H contains p disjoint multiedges of multiplicity q.

By construction of H, having an edge of multiplicity m in H is equivalent to having m distinct paths in P between two sets Y_i and Y_j, then having p disjoint multiedges of multiplicity q in H is equivalent to having p disjoint pairs $(X_i, X_{2i})_{i \in [\![1,p]\!]}$ of elements of $\{Y_i\}_{i \in [\![1, 4p^2q]\!]}$ and a set P of pq paths that contains q paths that links the two elements of each of the p pairs. The set $\{X_i\}_{i \in [\![1, 2p]\!]}$ is thus the one we were looking for. $\square$

4.2 Proof of Theorem 3

Proof of theorem 3. We prove the contrapositive. Let k be a integer, H a graph on $r > 5$ vertices and of pathwidth at most 2 and G a graph. From Proposition 6, $H \leqslant_{\mathrm{m}} \Xi_r$. If we show that G contains k disjoint copies of Ξ_r as minors then we are done. Let $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$g(k, r) = k^2 \log 2k \left(180 \cdot 2^{r(r-2)} - 24 \cdot 2^{\frac{1}{2}r(r-2)}\right) + 6 \cdot 2^{\frac{1}{2}r(r-2)} - 1$$

We prove the following statement: for all graph G, $\mathbf{tw}(G) \geqslant g(k, r)$ implies that $G \geqslant_{\mathrm{m}} k \cdot \Xi_r$. Let k and $r > 5$ be two positive integers and assume that $\mathbf{tw}(G) \geqslant g(k, r)$.

First case: $\delta_c(G) \geqslant c \cdot 3rk\sqrt{\log 3rk}$.

By definition of the contraction degeneracy, there is a graph G' minor of G and such that $\delta(G') \geqslant c \cdot 3rk\sqrt{\log 3rk}$. The average degree is at least the minimum degree, so $\mathrm{ad}(G') \geqslant c \cdot 3rk\sqrt{\log 3rk}$. According to Proposition 7, G' contains K_{3kr} as minor.

The graph Ξ_r have $3r$ vertices, therefore K_{3kr} contains $k \cdot \Xi_r$ as minor. We then have $k \cdot \Xi_r \leqslant_{\mathrm{m}} K_{3kr}$, $K_{3kr} \leqslant_{\mathrm{m}} G'$ and $G' \leqslant_{\mathrm{m}} G$, therefore by transitivity of the minor relation, G contains $k \cdot \Xi_r$ as minor, what we wanted to show.

Second case: $\delta_c(G) < c \cdot 3rk\sqrt{\log 3rk}$.

Observe that $c \cdot 3rk\sqrt{\log 3rk} < c \cdot 3r\sqrt{\log 6r} \cdot k\sqrt{\log 2k}$. Let $k_0 = k\sqrt{\log 2k}$ and $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$, and remark that $k_0 \geqslant k$ and, $r_0 \geqslant c \cdot 3r\sqrt{\log 6r}$ (remember that $c \leqslant 648$ and $r > 5$). With these notations, we have $\delta_c(G) < 2k_0 r_0$. We will show that $G \geqslant_{\mathrm{m}} k_0 \cdot K_{2,r}$ from which yields that $G \geqslant_{\mathrm{m}} k \cdot K_{2,r}$. By assumption, $\mathbf{tw}(G) \geqslant g(k,r)$. Therefore, by Lemma 7 (applied for $p := k_0$ and $q := r_0$), G contains $2k_0$ subsets $X_1, \ldots, X_{2k_0}$ of $V(G)$ and a set $\mathcal{P}$ of $k_0 r_0 = 3k_0 \cdot 2^{\frac{r(r-2)}{2}}$ disjoint paths of length at least 2 in G such that

(i) $\forall i \in [\![1, 2k_0]\!]$, X_i is of size $r_0 = 3 \cdot 2^{\frac{r(r-2)}{2}}$ and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$;

(ii) any path in $\mathcal{P}$ has one of its ends in some X_i with $i \in [\![1, k_0]\!]$, the other end in X_{2i} and its internal vertices are in none of the X_l, for all $l \in [\![1, 2k_0]\!]$, nor in A;

(iii) $\forall i, j \in [\![1, 2k_0]\!]$, $i \neq j \Rightarrow T_i \cap T_j = \emptyset$.

We assume that for all $i \in [\![1, 2k_0]\!]$, $X_i = \{v \in V(T_i),\ \deg_T(v) \leqslant 2\}$. It is easy to come down to this case by considering the minor of G obtained after deleting in T_i the leaves that are not in X_i and contracting one edge meeting a vertex of degree 2 which is not in X while such a vertex exists.

As T_i is a ternary tree, one can easily prove that for all $i \in [\![1, 2k_0]\!]$, T_i contains a path containing $2\log\frac{2}{3}|X_i| = (r-1)^2 + 1$ vertices of X_i. Let us call P_i such a path whose two ends are in X_i. Let us consider now the paths $\{P_i\}_{i \in [\![1,2k_0]\!]}$ and the paths that link the elements of different P_i's. For each path $i \in [\![1, 2k_0]\!]$, we choose in P_i one end vertex (remember that both are in X_i) that we name $p_{i,0}$. We follow P_i from this vertex and we denote the other vertices of $P_i \cap X_i$ by $p_{i,1}, p_{i,1}, \ldots, p_{i,(r-1)^2}$ in this order. The *corresponding vertex* of some vertex $p_{i,j}$ of $P_i \cap X_i$ (for $i \in [\![1, k_0]\!]$) is defined as the vertex of $P_{2i} \cap X_{2i}$ to which $p_{i,j}$ is linked to by a path of $\mathcal{P}$.

As said before, the sets $\{P_i \cap X_i\}_{i \in [\![1,2k_0]\!]}$ are of size $(r-1)^2 + 1$. According to Proposition 6, one can find for all $i \in [\![1, k_0]\!]$ a subsequence of length r in $p_{i,0}, p_{i,1}, \ldots, p_{i,(r-1)^2}$, such that the corresponding vertices in X_{2i} of this sequence are either in the same order (with respect to the subscripts of the names of the vertices), or in reverse order. For all $i \in [\![1, k_0]\!]$, this subsequence, its corresponding vertices and the vertices of the paths that link them together forms a Ξ_r model. We have thus k_0 models of Ξ_r in G, that gives us k disjoint models of Ξ_r in G (since $k \leqslant k_0$).

We showed that for all k and $r > 5$ positive integers, if a graph G has $\mathbf{tw}(G) \geqslant g(k,r)$, then $G \geqslant_{\mathrm{m}} k \cdot \Xi_r$. For every graph H on r vertices and of pathwidth at most 2, H is a minor of the subdivided grid Ξ_r (Proposition 6). Consequently, if G has treewidth at least $g(k,r)$, then G contains k disjoint copies of H and we are done. ☐

4.3 Proof of Theorem 4

Proof of theorem 4. We prove the contrapositive. Let k and r be two positive integers and G a graph such that $\mathbf{tw}(G) \geqslant 20k^2r^2 - 8k^2r + 2k - 1$. We want to show that G contains k disjoint copies of $\mathrm{K}_{2,r}$.

First case: $\delta_c(G) \geqslant 2kr$

According to lemma 3, G contains k disjoint copies of $\mathrm{K}_{2,r}$, what we wanted to show.

Second case: $\delta_c(G) < 2kr$

According to lemma 7, there exist $2k$ disjoint subsets $X_1, \dots, X_{2k}$ of $V(G)$ and a set $\mathcal{P}$ of disjoint paths of length at least 2 such that

(i) $\forall i \in [\![1, 2k]\!]$, X_i is of size r and is connected in G by a tree T_i using the elements of some set $A \subseteq V(G)$;

(ii) any path in $\mathcal{P}$ has one of its ends in some X_i with $i \in [\![1, k]\!]$, the other end in X_{2i} with $j \in [\![q+1, 2k]\!]$ and its internal vertices are in none of the X_l, for all $l \in [\![1, 2k]\!]$, nor in A;

(iii) $\forall i, j \in [\![1, 2k]\!],\ i \neq j \Rightarrow T_i \cap T_j = \emptyset$.

We then perform the following operations on G.

1. for all $i \in [\![1, 2k]\!]$, we contract the set X_i to a single vertex x_i (this is possible because X_i is connected by the tree T_i);
2. for all path $p \in \mathcal{P}$, we contract some edges of p until it have length exactly 2.

Because it has been obtained by contraction of edges, the graph G' we get by these operations is a minor of G. This new graph has the following properties.

1. for all $i \in [\![1, k]\!]$, the vertex x_i is linked to the vertex x_{2i} by r disjoint paths of length 2;
2. for all $i, j \in [\![1, k]\!]$ $i \neq j \Rightarrow x_i \neq x_j$ because the trees T_i and T_j contracted to obtain x_i and x_j are disjoint.

Remark that for all $i \in [\![1, k]\!]$, the subgraph of G' induced by the vertices x_i, x_{2i} and the r middle vertices of the paths of length 2 that links x_i and x_{2i} is the graph $\mathrm{K}_{2,r}$. We consequently found k disjoint copies of $\mathrm{K}_{2,r}$ in a minor of G, so G contains $k \times \mathrm{K}_{2,r}$ as minor, what we wanted to prove.

$\square$

5 From planar graph exclusion to Erdős–Pósa Property

In the section, we adapt to our needs the technique introduced in [13] (and also used in [12]) to translate a bound on the treewidth of a graph that does not contain a planar graph as minor to a gap for the Erdős–Pósa Property. We need two lemmata and a theorem in order to prove Theorems 1 and 2.

Lemma 8 (adapted from [13])**.** *Let H be a connected planar graph. Every graph G of treewidth w such that $\mathbf{pack}_H(G) = k$ has a separation (A, B) of order at most $w + 1$ satisfying $\mathbf{pack}_H(G[A \setminus B]) \leqslant \left\lfloor \frac{2k}{3} \right\rfloor$ and $A \cup B = V(G)$.*

Proof. Let H be a connected planar graph, G be a graph of treewidth w such that $\mathbf{pack}_H(G) = k$ and (T, V) be a nice optimal tree decomposition of G. For every $t \in V(T)$, we denote by G_t the subgraph of G equal to $G[\left(\cup_{u \in \mathrm{desc}_T(t)} V_u\right) \setminus V_t]$. We consider the function $p : V(T) \to \mathbb{N}$ defined by $\forall t \in V(T),\ p(t) = \mathbf{pack}_H(G_t)$. Let us now state some remarks about the function p.

Remark 1. For every two vertices $u, v \in V(T)$, if $v \in \mathrm{desc}_T(u)$ then p is non-decreasing along the (unique) path of T from v to u. To see this, it suffices to remark that if $t \in V(T)$ has child t', then $G_t \supseteq G_{t'}$ (what implies that $G_v \supseteq G_u$).

In particular, p is non-decreasing along the path from every vertex of T to the root of T.

Remark 2. As T is a nice decomposition of G, its vertices can be of four different kinds:

- Base node t : $p(t) = 0$ because as t has no descendant, $G_t = \emptyset$;
- Introduce node t with child t' : as the unique element of $V_t \setminus V_{t'}$ cannot appear in the elements of $\mathrm{desc}_T(t')$ (by definition of a tree decomposition), $G_t = G_{t'}$ and then $p(t) = p(t')$;
- Forget node t with child t' : in this case, the unique element of $G_t \setminus G_{t'}$ may be part of at most one model of H in G_t (because we want vertex-disjoint models) therefore either $p(t) = p(t')$ or $p(t) = p(t') + 1$;
- Join node t with children t_1 and t_2 : the graphs G_{t_1} and G_{t_2} are disjoint and $G_t = G_{t_1} \cup G_{t_2}$. As H is connected, there is no model of H in G_t that is simultaneously in G_{t_1} and in G_{t_2}, consequently $p(t) = p(t_1) + p(t_2)$.

Let t be a vertex of T such that $p(t) > \frac{2}{3}k$ and for every child t' of t, $p(t') \leqslant \frac{2}{3}k$. We make some claims about this vertex t:

(1) such t exists;

(2) t is unique;

(3) t is either a forget node or a join node.

Proof of Claim (1). The value of p on the root r of T is k (because $G_r = G$) and the value of p on every base nodes b is 0 (because G_b is the empty graph). As p is non decreasing on a path from a base node to the root (Remark 1), a vertex such t exists. $\diamondsuit$

Proof of Claim (2). To show that t is unique, we assume by contradiction that there is another $t' \in V(T)$ with $t' \neq t$ and $p(t') > \frac{2}{3}k$ and for every child t'' of t, $p(t'') \leqslant \frac{2}{3}k$. Three cases can occur:

- either t' is a descendant of t. However, p is non decreasing on a path from a vertex to the root (Remark 1) and $p(t') \geqslant \frac{2}{3}k$ whereas the value of p for each child of t is at most $\frac{2}{3}k$ (by definition of t): this is a contradiction.
- or t is a descendant of t' and the same argument applies (symmetric situation).

- or t and t' are not in the above situations. Let $v \in V(T) \setminus \{t, t'\}$ be the least common ancestor of t and t'. As p is non decreasing along any path from a vertex to the root (Remark 1), the child v_t (resp. $v_{t'}$) of v whose t (resp. t') is descendant of should be such $p(v_t) > \frac{2}{3}k$ (resp. $p(v_{t'}) > \frac{2}{3}k$). By definition of v, we have $v_t \neq v_{t'}$. As v is a join node, $p(v) = p(v_t) + p(v_{t'}) > \frac{4}{3}k$, what is impossible.

$\Diamond$

Proof of Claim (3). By definition the value of p on t is strictly positive and different from the value of p on every child of t. As this cannot occur with introduce nodes (where p take on t the same value it takes on the child of t) nor on base nodes (where p is null), t is either a join node or a forget node.

$\Diamond$

We now present a separation (A, B) of order at most $w + 1$ in G.
Case 1: t is a forget node with t' as child.
Let $A = V(G_t) \cup V_t$ and $B = V(G) \setminus V(G_t)$.
Case 2: t is a join node with t_1, t_2 as children.
By definition of t we have $p(t) \geqslant \frac{2k}{3}$. As $p(t) = p(t_1) + p(t_2)$ (according to Remark 2) there is a $i \in \{1, 2\}$ such that $p(t_i) \geqslant \frac{k}{3}$. Let $A = V(G_{t_i}) \cup V_t$ and $B = V(G) \setminus V(G_{t_i})$
In both cases, we have

(i) there is no edge between $A \setminus B$ and $B \setminus A$ therefore (A, B) is a separation;
(ii) $|A \cap B| \leqslant w + 1$ because $A \cap B = V_t$ and V_t is a bag in an optimal tree decomposition of G which have treewidth w, thus (A, B) is a separation of order at most $w + 1$;
(iii) $\mathbf{pack}_H(G[A \setminus B]) \leqslant \frac{2}{3}k$ by definition of A and t;
(iv) $A \cup B = V(G)$ by definition of B.

Consequently, the pair (A, B) is a separation of the kind we were looking for. $\square$

Lemma 9 (adapted from [13])**.** *Let H be a connected planar graph, let $\varepsilon > 0$ be a real, and let $g \colon \mathbb{N} \to \mathbb{N}$ be a function such that $g(n) = \Omega(n^{1+\varepsilon})$. For every integer $k > 0$ and graph G of treewidth less than $g(k)$, if G contains less than k disjoint models of H then G has a H-hitting set of size $O(g(k))$.*

Proof. We assume that $\mathbf{tw}(G) < g(k)$ and that $\mathbf{pack}_H(G) < k$. According to Lemma 8, there is in G a separation (A, B) of order at most $g(k)$ such that $\mathbf{pack}_H(G[A \setminus B]) \leqslant \left\lfloor \frac{2k}{3} \right\rfloor$ and $A \cup B = V(G)$.

Remark that as $\{A \setminus B, A \cap B, B \setminus A\}$ is a partition of $V(G)$ such that there is no edge between $A \setminus B$ and $B \setminus A$ (because (A, B) is a separation), every model of the connected graph H that use vertices of $A \setminus B$ and of $B \setminus A$ also use vertices of $A \cap B$. Consequently we have

$$\mathbf{cover}_H(G) \leqslant \mathbf{cover}_H(G[A \setminus B]) + \mathbf{cover}_H(G[B \setminus A]) + |A \cap B| \qquad (2)$$

As H is connected and $A \setminus B$ is disjoint from $B \setminus A$, we also have

$$\mathbf{pack}_H(G) \geqslant \mathbf{pack}_H(G[A \setminus B]) + \mathbf{pack}_H(G[B \setminus A])$$

Let $\alpha \in [0, 1]$ be a real such that

$$\mathbf{pack}_H(G[A \setminus B]) \leqslant \alpha \cdot \mathbf{pack}_H(G) \tag{3}$$

$$\mathbf{pack}_H(G[A \setminus B]) \leqslant (1 - \alpha) \cdot \mathbf{pack}_H(G) \tag{4}$$

We are looking for a function f satisfying the inequality $\mathbf{cover}_H(G) \leqslant f(\mathbf{pack}_H(G))$ for every graph G and for every planar connected graph H. A consequence of the grid-exclusion theorem (see [20] and Theorems 12.4.4 and 12.4.10 of [6]) is that every planar graph has the Erdős–Pósa Property, thus a function such f exists. We assume without loss of generality that

$$f(\mathbf{pack}_H(G)) \leqslant \mathbf{cover}_H(G[A \setminus B]) + \mathbf{cover}_H(G[B \setminus A]) + |A \cap B| \tag{5}$$

(to ensure this we can choose as value for $f(\mathbf{pack}_H(G))$ the minimum of the right part of the inequality on all graphs F such that $\mathbf{pack}_H(F) = \mathbf{pack}_H(G)$).

By combining the definition of f with (5), (2) and (3) and using the fact that (A, B) has order at most $g(k)$, we get

$$\begin{aligned}
f(\mathbf{pack}_H(G)) &\leqslant \mathbf{cover}_H(G[A \setminus B]) + \mathbf{cover}_H(G[B \setminus A]) + |A \cap B| \\
&\leqslant f(\mathbf{pack}_H(G[A \setminus B])) + f(\mathbf{pack}_H(G[B \setminus A])) + |A \cap B| \\
&\leqslant f(\mathbf{pack}_H(G[A \setminus B])) + f(\mathbf{pack}_H(G[B \setminus A])) + g(k) \\
f(\mathbf{pack}_H(G)) &\leqslant f(\alpha \cdot \mathbf{pack}_H(G)) + f((1 - \alpha) \cdot \mathbf{pack}_H(G)) + g(k)
\end{aligned}$$

By the Akra–Bazzi Theorem [1], the recurrence $h(p) = h(\alpha p) + h((1-\alpha)p) + g(p)$ where $g(p) = \Omega(p^{1+\varepsilon})$ is satisfied by a function $f(p) = O(g(p))$. Therefore we have $\mathbf{cover}_H(G) \leqslant f(k) = O(g(k))$, which means that G has a H-hitting set of size $O(g(k))$, what we wanted to prove. □

The proofs of Theorems 1 and 2 immediately follow from this theorem combined with lemmata 3 and 4.

Theorem 5 (adapted from [13])**.** *Let H be a connected planar graph, let $\varepsilon > 0$ be a real. Assume that there is a function $g \colon \mathbb{N} \to \mathbb{N}$ such that $g(n) = \Omega(n^{1+\varepsilon})$ and for all graph G, for all integer $k > 0$, $\mathbf{tw}(G) \geqslant g(k) \Rightarrow G \geqslant_{\mathrm{m}} k \cdot H$. Then H has the Erdős–Pósa Property with gap $f_H(k) = O(g(k))$.*

Proof. Let H, ε and g be as in the statement of the lemma. Let G be a graph.
Case 1: $\mathbf{tw}(G) \geqslant g(k)$

By definition of g, G contains $k \cdot H$.
Case 2: $\mathbf{tw}(G) < g(k)$

If G does not contain k disjoint models of H, it has a H-hitting set of size $O(g(k))$ according to Lemma 9.

Consequently, either G contains k disjoint models of H, or G has a H-hitting set of size $O(g(k))$, in other words: H has the Erdős–Pósa Property with gap $f_H(k) = O(g(k))$. □

Proof. Proofs of Theorems 1 and 2 According to Theorem 3, there is a function $f(k) = 2^{O(h^2)} \cdot k^2 \cdot \log k$ such that for every graph H on h vertices and of pathwidth at most 2, every graph G of treewidth more than $f(k)$ contains k disjoint copies of H. The application of Theorem 5 immediately yields that the graphs of pathwidth at most 2 have the Erdős–Pósa Property with gap at most f.

Similarly, since Theorem 4 ensure that every graph of treewidth more than some function $g(k, r) = O(k^2 r^2)$ contains k disjoint copies of $K_{2,r}$, the application of Theorem 5 gives that for every integer $r > 0$, the graph $K_{2,r}$ has the Erdős–Pósa Property with gap at most g. □

Postscript. Very recently, the general open problem of estimating $f_H(k)$ when H is a general planar graph has been tackled in [5]. Moreover, very recently, using the results of [18] we were able to improve both Theorems 3 and 2 by proving low degree polynomial (on both k and $|V(H)|$) bounds for more general instantiations of H [19].

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