

# Young domination on Hamming rectangles

Janko Gravner<sup>1</sup>, Matjaž Krnc<sup>2</sup>, Martin Milanič<sup>2</sup>, and Jean-Florent Raymond<sup>3</sup>

<sup>1</sup>Department of Mathematics, University of California, Davis  
 gravner@math.ucdavis.edu

<sup>2</sup>FAMNIT and IAM, University of Primorska, Koper, Slovenia  
 {matjaz.krnc,martin.milanic}@upr.si

<sup>3</sup>Univ. Lyon, CNRS, ENS de Lyon, Université Claude Bernard Lyon 1, LIP UMR5668,  
 Lyon, France, jean-florent.raymond@cnrs.fr

## Abstract

In the neighborhood growth dynamics on a Hamming rectangle  $[0, m - 1] \times [0, n - 1] \subseteq \mathbb{Z}_+^2$ , the decision to add a point is made by counting the currently occupied points on the horizontal and the vertical line through it, and checking whether the pair of counts lies outside a fixed Young diagram. After the initially occupied set is chosen, the synchronous rule is iterated. The Young domination number with a fixed latency  $L$  is the smallest cardinality of an initial set that covers the rectangle by  $L$  steps, for  $L = 0, 1, \dots$ . We compute this number for some special cases, including  $k$ -domination for any  $k$  when  $m = n$ , and devise approximation algorithms in the general case. These results have implications in extremal graph theory, via an equivalence between the case  $L = 1$  and bipartite Turán numbers for families of double stars. Our approach is based on a variety of techniques including duality, algebraic formulations, explicit constructions, and dynamic programming.

## 1 Introduction

In many out-of-equilibrium natural processes, activity is initiated by nuclei, which are formed at random in the most favorable circumstances, i.e., those that satisfy some minimality conditions. In discrete mathematical counterparts that model nucleation and growth, the simplest minimality condition is the cardinality of the initial set, which in many (but not all) stochastic models maximizes its probability. Aside from this motivation, looking for the smallest set of vertices in a graph with a given property is a natural and fundamental problem in extremal combinatorics and algorithmic graph theory. While we begin with a more general setup, we address this

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question for a growth dynamics on Cartesian products of two complete graphs, arguably the simplest nontrivial class. The parameter in our growth rule is a Young diagram, which ensures monotonicity; without this property, nucleation questions become much less tractable (see, e.g., [GG2]).

For an integer  $c \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ , we denote by  $[0, c]$  the set  $\{0, 1, \dots, c\}$ . For integers  $a, b \geq 1$ , we let  $R_{a,b} = [0, a-1] \times [0, b-1]$ , be the discrete  $a \times b$  rectangle. A set  $\mathcal{Z} = \bigcup_{(a,b) \in I} R_{a,b}$ , given by a union of rectangles over some set  $I \subseteq \mathbb{N}^2$ , is called a (discrete) *zero-set*. We allow the trivial case  $\mathcal{Z} = \emptyset$  and also infinite  $\mathcal{Z}$ . Observe that zero-sets are exactly *downward-closed* sets in the sense that if  $(x, y) \in \mathcal{Z}$  and  $0 \leq x' \leq x$ ,  $0 \leq y' \leq y$  then  $(x', y') \in \mathcal{Z}$ . Therefore, zero-sets are *Young diagrams* in the French notation [Rom]. Each zero-set determines a type of domination on Cartesian products of two graphs as follows.

Consider two finite graphs  $G_1$  and  $G_2$ . The *Cartesian product* of  $G_1$  and  $G_2$  is the graph  $G_1 \square G_2$  with vertex set  $V(G_1) \times V(G_2)$  in which two vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  are adjacent if and only if  $u_1 = v_1$  and  $u_2 v_2 \in E(G_2)$  or  $u_2 = v_2$  and  $u_1 v_1 \in E(G_1)$ . Each vertex  $v = (v_1, v_2) \in V(G_1 \square G_2)$  has neighbors divided into two disjoint sets:  $\mathcal{N}^r(v)$ , the *row neighborhood* of  $v$ , defined as the set of all vertices  $(v_1, y)$  such that  $y v_2$  is an edge of  $G_2$ , and  $\mathcal{N}^c(v)$ , the *column neighborhood* of  $v$ , defined as the set of all vertices  $(x, v_2)$  such that  $x v_1$  is an edge of  $G_1$ .

Given a zero-set  $\mathcal{Z}$ , the *transformation*  $\mathcal{T} = \mathcal{T}(\mathcal{Z})$ , defined on sets of vertices of  $G_1 \square G_2$ , assigns to every set  $A \subseteq V(G_1 \square G_2)$  a set  $\mathcal{T}(A) \subseteq V(G_1 \square G_2)$  as follows:

- the input set is enlarged, i.e., if  $v \in A$  then  $v \in \mathcal{T}(A)$ ;
- if  $v \in V(G_1 \square G_2) \setminus A$ , then  $v \in \mathcal{T}(A)$  if and only if  $(|\mathcal{N}^r(v) \cap A|, |\mathcal{N}^c(v) \cap A|) \notin \mathcal{Z}$ .

Thus, the iteration  $\mathcal{T}^t(A)$  yields a growth dynamics with initial set  $A = \mathcal{T}^0(A)$ , and we define  $\mathcal{T}^\infty(A) = \bigcup_{t \geq 0} \mathcal{T}^t(A)$ . Observe that  $\mathcal{T}$  is *monotone*:  $A_1 \subseteq A_2$  implies  $\mathcal{T}(A_1) \subseteq \mathcal{T}(A_2)$ ; in fact,  $\mathcal{T}$  is monotone if and only if its zero-set  $\mathcal{Z}$  is downward closed. We call the transformation  $\mathcal{T}$ , or equivalently the resulting growth dynamics the *neighborhood growth*.

For  $L = 0, 1, 2, \dots, \infty$ , we say that  $A$  is a  $\mathcal{Z}$ -*dominating set with latency*  $L$  if  $\mathcal{T}^L(A) = V(G_1 \square G_2)$ . Our focus are smallest such sets, so we define the  $\mathcal{Z}$ -*domination number* of  $(\mathcal{Z}, G_1, G_2)$  with latency  $L$  as

$$\gamma^L = \gamma^L(\mathcal{Z}, G_1, G_2) = \min\{|A| : \mathcal{T}^L(A) = V(G_1 \square G_2)\}.$$

It is trivially true that  $\gamma^0(\mathcal{Z}, G_1, G_2) = |V(G_1 \square G_2)|$ . When the latency is not specified, we assume the default choice  $L = 1$ , often omit it from the notation and call  $\gamma = \gamma(\mathcal{Z}, G_1, G_2) = \gamma^1(\mathcal{Z}, G_1, G_2)$  the  $\mathcal{Z}$ -*domination number* of  $(\mathcal{Z}, G_1, G_2)$ . In particular, if  $a \in \mathbb{N}$  and  $\mathcal{Z}$  is the triangle

$$(1.1) \quad T_a = \{(x, y) \in \mathbb{Z}_+^2 : x + y \leq a - 1\},$$

then  $\gamma(T_a, G_1, G_2)$  is exactly the  $a$ -domination number of the product graph  $G_1 \square G_2$  [HV]. Furthermore,  $\gamma^L(T_1, G_1, G_2)$  is exactly the distance- $L$  domination number of the product graph

$G_1 \square G_2$ , for any finite  $L$  (see, e.g., [Hen]). In our general discussions, or if  $\mathcal{Z}$  is clear from the context, we will refer to  $\mathcal{Z}$ -domination as *Young domination*.

Perhaps the most natural special case is where the two factors are complete graphs,  $G_1 = K_m$  and  $G_2 = K_n$  on vertex sets  $[0, m - 1]$  and  $[0, n - 1]$ , in which case we write

$$\gamma^L = \gamma^L(\mathcal{Z}, m, n) = \gamma^L(\mathcal{Z}, K_m, K_n).$$

The graph  $K_m \square K_n$  is known as a *Hamming rectangle*, and neighborhood growth on this graph with general  $\mathcal{Z}$  was studied in [GSS, GPS]. This dynamics with  $\mathcal{Z} = T_a$  given by (1.1) is also known as *bootstrap percolation* [GHPS], while a rectangular zero-set  $\mathcal{Z} = R_{b,a}$  induces *line growth* [GSS], also known as *line percolation* [BBLN]. In these and related papers on growth processes on graphs, points in the set  $\mathcal{T}^t(A)$  are called *occupied* (coded as 1s) and points outside of it *empty* (coded as 0s); a set  $A$  which makes every point eventually occupied is precisely a Young dominating set with  $L = \infty$  and is also called a *spanning* [GSS] or *percolating* [BBLN] set. For  $m = n$ ,  $K_n \square K_n$  is often referred to as the *rook's graph*. The  $a$ -domination number for this case was considered in [BLL], but domination and its variants on Hamming graphs remain relatively unexplored. See Figure 2.1 for an example.

Young domination is one way to extend  $a$ -domination on graphs [HV]. Another is vector domination (see, e.g., [CMV, HPV]) where every vertex has its own requirement for the number of neighbors in the dominating set. In a further generalization in the same fashion as in the present paper, Cicalese et al. [CCGMV] considered vector domination with latency. In social network theory, designing small initial sets that yield large final sets due to vector domination with infinite latency is referred to as *target set selection* (see [Chen]).

Domination with infinite latency, that is, spanning or percolation, with random initial sets has played a crucial role in spatial growth processes, starting with the foundational paper [AL]. By far the most common setting, with many deep and surprising results (see, e.g., the survey [Mor]), is the Cartesian product of  $d$  path graphs on  $n$  vertices, and thus with standard nearest neighbor lattice connectivity. Associated extremal quantities have also received attention, most of all the minimal size of a spanning set, that is,  $\gamma^\infty$  in our notation. For the bootstrap percolation on this graph with  $\mathcal{Z} = T_a$ , this quantity equals  $n^{d-1}$  for  $a = d$  [PS] and  $\lfloor d(n - 1)/2 \rfloor + 1$  for  $a = 2$  [BBM]. For other values of  $a$ , a sharp asymptotic formula for the hypercube ( $n = 2$ ) as  $d \rightarrow \infty$  was given in [MN], and an exact formula for  $d = a = 3$  in [DNR]. Smallest spanning sets have also been studied for bootstrap percolation on trees [Rie, BKN], two-dimensional lattice with large neighborhoods [GG1], and strong products [BH] and direct products [BHH] of graphs. Among many papers that study graph bootstrap percolation, a dynamics akin to ours when viewed as an edge-addition rule in bipartite graphs (see Section 3), we mention [HHQ] as it also discusses a method suited for handling product graphs. For Hamming graphs, [BBLN] and [BMT] study  $\gamma^\infty$  for line growth and bootstrap percolation, while [GSS] gives bounds on  $\gamma^\infty$  for general zero-sets. In this context, sets with a small latency number  $L$  represent fast spanning.

We now proceed to stating our main results. We begin with three special cases where Young domination number is given by simple expressions. The easiest are the  $L$ -shaped Young

diagrams, given by zero-sets  $V_{a,b}$ , where  $a, b \geq 0$  and

$$(1.2) \quad (V_{a,b})^c = \mathbb{Z}_+^2 \setminus V_{a,b} = [a, \infty) \times [b, \infty),$$

i.e., a point requires at least  $a$  occupied row neighbors and at least  $b$  occupied column neighbors to become occupied.

**Proposition 1.1.** *Assume that  $\mathcal{Z} = V_{a,b}$ , where  $0 \leq a \leq n$ ,  $0 \leq b \leq m$ . Then  $\gamma(\mathcal{Z}, m, n) = \max(am, bn)$ .*

Next we consider rectangular zero-sets, where  $\gamma$  is given by the following optimization procedure.

**Theorem 1.2.** *Assume that  $\mathcal{Z} = R_{b,a}$ , where  $1 \leq a \leq n$ ,  $1 \leq b \leq m$ . Then*

$$(1.3) \quad \gamma(\mathcal{Z}, m, n) = \min_{\substack{b \leq x \leq m \\ a \leq y \leq n}} ((m-x)(n-y) + \max(ax, by)),$$

which can be computed in time  $\mathcal{O}(\min(a, b)/\gcd(a, b))$ . In particular, for  $a = b$  and  $m = n$ ,

$$(1.4) \quad \gamma(\mathcal{Z}, n, n) = \begin{cases} an - a^2/4, & \text{if } a \leq 2n/3 \text{ and } a \text{ is even,} \\ an - (a^2 - 1)/4, & \text{if } a \leq (2n + 1)/3 \text{ and } a \text{ is odd,} \\ a^2 + (n - a)^2, & \text{otherwise.} \end{cases}$$

Our most substantial exact result gives a complete solution to Young domination for triangular Young diagrams  $T_a$  from (1.1) with  $m = n$ . This case was considered in [BLL], where various partial results were given. In particular, the formula (1.5) for even  $a$  confirms Conjecture 1 in that paper. Note that for  $a \geq 2n - 1$ , the only Young dominating set is  $[0, n - 1]^2$ , so we exclude this case from consideration.

**Theorem 1.3.** *Assume that  $n \geq 2$  and let  $\mathcal{Z} = T_a$  be given by (1.1), where  $1 \leq a \leq 2n - 2$ . If  $a$  is even, then*

$$(1.5) \quad \gamma(\mathcal{Z}, n, n) = \frac{an}{2}.$$

If  $a$  is odd and  $a \leq n$ , then

$$(1.6) \quad \gamma(\mathcal{Z}, n, n) = \frac{a+1}{2}n - \frac{a-1}{2}.$$

Finally, if  $a$  is odd and  $a > n$ , then

$$(1.7) \quad \gamma(\mathcal{Z}, n, n) = \frac{a-1}{2}n + \left\lceil \frac{n(2n-a+1)}{2(2n-a)} \right\rceil.$$

For general zero-sets, the complexity of computing the Young domination number is open for any  $L \geq 1$  (and is for  $L = 1$  connected to certain bipartite Turán numbers, see Section 3). We provide polynomial-time approximation algorithms, the most precise of which is for latency  $L = 1$ .

**Theorem 1.4.** *There exists an algorithm polynomial in  $n$  that takes as input  $m, n \in \mathbb{N}$  with  $m \leq n$  and a zero-set  $\mathcal{Z}$  with  $\mathcal{Z} \subseteq R_{m,n}$ , and returns a number in the interval  $[\gamma, 3\gamma]$  where  $\gamma = \gamma(\mathcal{Z}, m, n)$ .*

For an arbitrary fixed finite latency number, we give a constant factor approximation algorithm. In its statement, we allow  $\mathcal{Z}$  to be included in a rectangle much smaller than the universe  $R_{m,n}$ .

**Theorem 1.5.** *For any fixed finite  $L$ , there exist a constant  $C \geq 1$  and an algorithm that takes as input  $m, n \in \mathbb{N}$  and a zero-set  $\mathcal{Z}$  with  $\mathcal{Z} \subseteq R_{a,b} \subseteq R_{m,n}$ , returns a number in the interval  $[\gamma^L, C\gamma^L]$ , where  $\gamma^L = \gamma^L(\mathcal{Z}, m, n)$ , and runs in time that is polynomial in  $ab$ .*

In fact, the algorithms given by Theorems 1.4 and 1.5 can be easily adapted so that they return a  $\mathcal{Z}$ -dominating set  $A$  with  $|A| \leq 3\gamma$  and a  $\mathcal{Z}$ -dominating set  $A$  with latency  $L$  with  $|A| \leq C\gamma^L$ , respectively. These results place the optimization problems of computing the Young domination numbers  $\gamma^L(\mathcal{Z}, m, n)$  in the complexity class APX (see, e.g., [ACGKMP]).

Observe that  $\gamma^{L+1}(\mathcal{Z}, m, n) \leq \gamma^L(\mathcal{Z}, m, n)$  for all  $L \geq 0$ . In light of this and Theorem 1.5, it is natural to ask how different  $\gamma^L$  are for different  $L$ . The next result addresses this to some extent, showing in particular that at least the first four are not within a constant factor of each other.

**Theorem 1.6.** *Assume that  $\mathcal{Z} \subseteq [0, a-1] \times [0, b-1]$  for some  $a, b \geq 1$ . If  $2 \leq L \leq \infty$  and  $m, n > ab$ , then  $\gamma^L(\mathcal{Z}, m, n) = \gamma^L(\mathcal{Z}, \infty, \infty)$ . If, in addition,  $L \geq 2ab + 5$ , then*

$$\gamma^L(\mathcal{Z}, m, n) = \gamma^\infty(\mathcal{Z}, \infty, \infty) \in [|\mathcal{Z}|/4, |\mathcal{Z}|].$$

Moreover,

$$(1.8) \quad \sup_{\mathcal{Z}, m, n} \frac{\gamma^L(\mathcal{Z}, m, n)}{\gamma^{L+1}(\mathcal{Z}, m, n)} = \infty,$$

for  $L = 0, 1, 2$ .

We prove the above theorems, in order, in Sections 4–7. In Section 2 we introduce some tools and prove a few preliminary lemmas. Section 3 establishes a relationship between our framework and extremal problems in bipartite graphs. This connection is based on duality between Young diagrams and provides further motivation for study of Young domination.

For our exact results (Theorem 1.2 and 1.3), we use two methods to prove lower bounds in different regimes: one is standard bookkeeping, while the other is a novel approach using an algebraic formulation and appropriate inequalities. The upper bounds result from explicit constructions, which, as (1.7) suggests, may be somewhat involved. As already mentioned, Theorem 1.3 completely resolves the  $a$ -domination for rook graphs, significantly extending the results from [BLL]. The first approximation result, Theorem 1.4, is proved by comparison with a simpler growth dynamics, which for latency 1 yields a discrete optimization problem, which in turn can be solved by dynamic programming. The second approximation result, Theorem 1.5, works for general finite latency, and is proved by appropriately restricting the number of choices

of the initial configuration for the simplified dynamics (Lemma 7.2). Theorem 1.6 mostly follows from results from [GSS, GPS], but we do add an exact result for  $L = 2$  (Lemma 8.1).

## 2 Preliminaries

Let  $m, n \in \mathbb{N} = \{1, 2, 3, \dots\}$ . The vertex set of our graph  $K_m \square K_n$  will be  $R_{m,n} = [0, m-1] \times [0, n-1]$ . Also, we shorten  $R_n = R_{n,n}$ . Thus, two vertices  $(i, j)$  and  $(k, \ell)$  are adjacent if and only if they differ in precisely one coordinate. For each  $i \in [0, m-1]$  the  $i$ -th row of  $R_{m,n}$  is the set of all vertices of  $R_{m,n}$  with first coordinate  $i$ ; similarly, for each  $j \in [0, n-1]$  the  $j$ -th column is the set of all vertices of  $R_{m,n}$  with second coordinate  $j$ . Given a vertex  $(i, j) \in R_{m,n}$ , its row neighborhood  $\mathcal{N}^r(i, j)$  then consists of all vertices distinct from  $(i, j)$  with first coordinate  $i$  and its column neighborhood  $\mathcal{N}^c(i, j)$  of all vertices distinct from  $(i, j)$  with second coordinate  $j$ . Note that we imagine the first factor, the interval  $[0, m-1]$ , to be vertical and the second factor, the interval  $[0, n-1]$ , to be horizontal; also, each row neighborhood has  $n-1$  vertices and each column neighborhood  $m-1$  vertices. We will also encounter infinite  $m$  and  $n$ :  $R_{m,\infty} = [0, m-1] \times [0, \infty)$ , and analogously  $R_{\infty,n}$  and  $R_{\infty,\infty}$ . Recall that every zero-set  $\mathcal{Z}$  defines the corresponding transformation  $\mathcal{T} = \mathcal{T}(\mathcal{Z})$  on subsets of  $R_{m,n}$ ; note that we may always replace  $\mathcal{Z}$  by  $\mathcal{Z} \cap R_{m,n}$  without affecting the dynamics.

For a Young diagram  $\mathcal{Y}$ , we call a point  $(z_1, z_2) \in \mathbb{Z}_+^2$  its

- *convex corner* if  $(z_1, z_2) \in \mathcal{Y}$  and  $(z_1 + 1, z_2), (z_1, z_2 + 1) \notin \mathcal{Y}$ ;
- *concave corner* if  $(z_1, z_2) \notin \mathcal{Y}$  and  $(z_1 - 1, z_2), (z_1, z_2 - 1) \in \mathcal{Y} \cup (\mathbb{Z}^2 \setminus \mathbb{Z}_+^2)$ .

Note that the concave corners of a zero-set  $\mathcal{Z}$  are, under the natural partial order on  $\mathbb{Z}_+^2$ , the minimal points of  $\mathbb{Z}_+^2 \setminus \mathcal{Z}$  (i.e., the minimal neighborhood counts that induce occupation). Similarly, the convex corners are maximal points of  $\mathcal{Z}$ . In general, the set  $\mathcal{I}_{\mathcal{Y}}$  of concave corners of a Young diagram  $\mathcal{Y}$  is minimal in the following sense:  $\mathcal{Y} = \bigcup_{(a-1, b-1) \in \mathcal{I}_{\mathcal{Y}}} R_{a,b}$ , and  $\mathcal{Y} = \bigcup_{(a-1, b-1) \in \mathcal{I}} R_{a,b}$  implies  $\mathcal{I}_{\mathcal{Y}} \subseteq \mathcal{I}$ . Figure 2.1 provides an example of a  $\mathcal{Z}$ -dominating set on  $K_4 \square K_5$ . (In our figures, we often represent a point  $z \in \mathbb{Z}_+^2$  as a unit square with  $z$  in its center.)

We define the following operation on Young diagrams.

**Definition 2.1.** Fix  $m, n \in \mathbb{N}$ . For a Young diagram  $\mathcal{Y} \subseteq R_{m,n}$ , we define the *dual* of  $\mathcal{Y}$  as the unique Young diagram  $\check{\mathcal{Y}} \subseteq R_{m,n}$  such that the set of convex corners of  $\check{\mathcal{Y}}$  is given by

$$\{(m-b-1, n-a-1) : (b, a) \in \mathcal{I}_{\mathcal{Y}}, 0 \leq a \leq n-1, 0 \leq b \leq m-1\}.$$

For a set  $A \subseteq R_{m,n}$ , let  $\rho(A)$  be the reflection of  $A$  through the centroid  $((m-1)/2, (n-1)/2) \in \mathbb{R}_+^2$ . The next lemma, which follows easily from the definition, observes that the dual operation coincides with the one given in, e.g., [Roh]. See Fig. 2.2 for an example.

**Lemma 2.2.** For a given  $m, n$ , and a Young diagram  $\mathcal{Y} \subseteq R_{m,n}$ ,  $\check{\check{\mathcal{Y}}} = \rho(R_{m,n} \setminus \mathcal{Y})$ . Therefore, the transformation  $\mathcal{Y} \mapsto \check{\mathcal{Y}}$  is an involution, i.e., it is its own inverse.

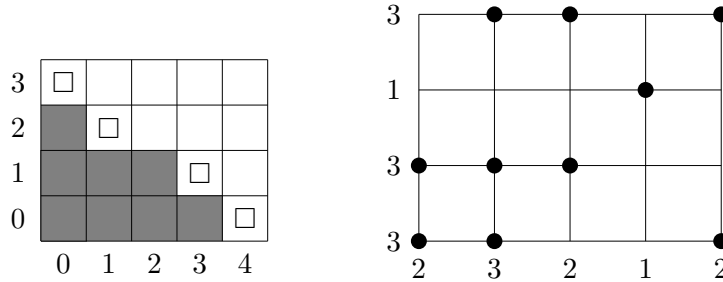


Figure 2.1: Left: a zero-set  $\mathcal{Z}$ ; the points of  $\mathcal{Z}$  are the centers of the dark squares. Its concave corners (see Section 2) are  $(0, 3)$ ,  $(1, 2)$ ,  $(3, 1)$ , and  $(4, 0)$  and are labeled by  $\square$ . Right, a  $\mathcal{Z}$ -dominating set  $D$  (with its members this time indicated by  $\bullet$ ) on the graph  $K_4 \square K_5$  with vertex set  $R_{4,5} = [0, 3] \times [0, 4]$ , of (minimal) size 10. Note that the vertices of the graph are drawn with the first coordinate vertical and the second coordinate horizontal, in line with the definitions of rows and columns, and row and column neighborhoods of a vertex. Accordingly, the row counts and column counts are given, respectively, along the vertical and horizontal axis. For every point in  $z \in R_{4,5} \setminus D$  the point  $(r, c)$  with coordinates given by its row and column counts lies outside  $\mathcal{Z}$ , therefore  $(r, c)$  is greater than or equal to a concave corner of  $\mathcal{Z}$  in the partial order of  $\mathbb{Z}_+^2$ .

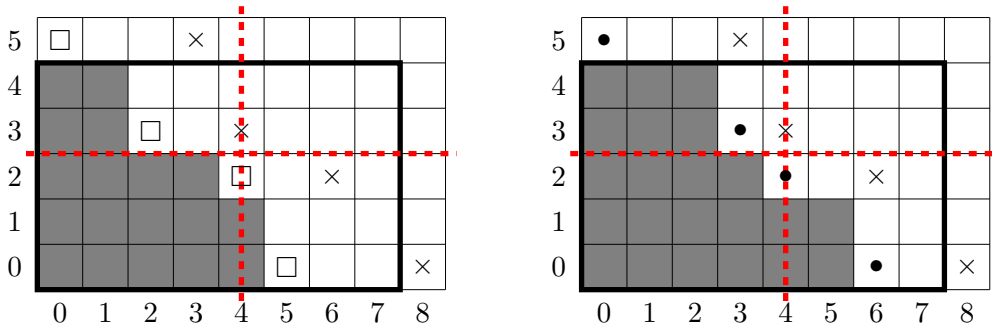


Figure 2.2: A Young diagram  $\mathcal{Y}$  (dark gray squares) within  $R_{5,8}$  (outlined) is given on the left. Its four concave corners, labeled by  $\square$ , are reflected through both of the bisecting lines, giving the four sites labeled by  $\times$ . These are reproduced on the right, and determine  $\check{\mathcal{Y}}$ , whose four concave corners are labeled by  $\bullet$ . Observe that the double reflection of the four  $\bullet$ 's gives the three points diagonally adjacent to the convex corners of  $\mathcal{Y}$ , as required because the dual map is an involution. Observe also that  $\check{\mathcal{Y}}$  equals  $R_{5,8} \setminus \mathcal{Y}$ , reflected through the centroid of  $R_{5,8}$ , as defined in [Roh].

We routinely use the following lemma, whose simple proof we omit.

**Lemma 2.3.** *Let  $\mathcal{S}$  be an operation that, given a subset  $A \subseteq R_{m,n}$ , returns the subset of  $R_{m,n}$  obtained from  $A$  by permuting rows  $k$  and  $k'$ , for some distinct  $k, k' \in \{1, \dots, m\}$ . Then, for any*

transformation  $\mathcal{T}$  given by some zero-set,  $\mathcal{T}(\mathcal{S}(A)) = \mathcal{S}(\mathcal{T}(A))$ . This also holds for operations permuting columns.

We now review a variant of the transformation  $\mathcal{T}$ , which adds fixed amounts to each row an column count [**GSS**, **GPS**]. The advantage is that one can get a nontrivial growth dynamics even if the initial set is  $A = \emptyset$  in which case the behavior of iterates is significantly more regular. Our transformation  $\mathcal{T}$  is given additional two parameters  $\vec{r} = (r_0, \dots, r_{m-1}) \in \mathbb{Z}_+^m$  and  $\vec{c} = (c_0, \dots, c_{n-1}) \in \mathbb{Z}_+^n$ , which are sequences of nonnegative integers called *enhancements*. Then the *enhanced neighborhood growth* is given by the triple  $(\mathcal{Z}, \vec{r}, \vec{c})$  and is defined as follows:

$$\mathcal{T}(A) = A \cup \{(i, j) \in R_{m,n} : (|\mathcal{N}^r(i, j) \cap A| + r_i, |\mathcal{N}^c(i, j) \cap A| + c_j) \notin \mathcal{Z}\}.$$

We do not introduce a new notation as the usual neighborhood growth given by  $\mathcal{Z}$  is the same as its enhancement given by  $(\mathcal{Z}, \vec{0}, \vec{0})$ , and we will not distinguish between the two. We denote by  $|\cdot|$  the  $\ell^1$ -norm, thus  $|\vec{r}| = r_0 + \dots + r_{m-1}$  and  $|\vec{c}| = c_0 + \dots + c_{n-1}$ .

The next lemma, whose proof is a simple verification (see [**GPS**, Lemma 2.3]) gives the key regularity property of the enhanced rule.

**Lemma 2.4.** *Let  $m, n \in \mathbb{N}$  and let  $\mathcal{T}$  be the dynamics defined by the triple  $(\mathcal{Z}, \vec{r}, \vec{c})$  of zero-set  $\mathcal{Z}$  and nonincreasing enhancements  $\vec{r}$  and  $\vec{c}$ . If  $A \subseteq R_{m,n}$  is a Young diagram, then so is  $\mathcal{T}(A)$ . Consequently,  $\mathcal{T}^k(\emptyset)$  is a Young diagram for every  $k \in \mathbb{Z}_+$ .*

We conclude this section with two lemmas about the connection between regular and enhanced dynamics. For  $m, n \in \mathbb{N}$ ,  $A \subseteq R_{m,n}$ , and  $i \in [0, m-1]$ ,  $j \in [0, n-1]$ , we denote by  $\text{row}_i(A)$  and  $\text{col}_j(A)$  the number of elements in the  $i$ th row and  $j$ th column of  $A$ , respectively.

**Lemma 2.5.** *Let  $m, n \in \mathbb{N}$  and the zero-set  $\mathcal{Z}$  be fixed. Assume  $A \subseteq R_{m,n}$  and the enhancements  $\vec{r}$  and  $\vec{c}$  are chosen so that: for each  $i \in [0, m-1]$ ,  $\text{row}_i(A) \leq r_i$ ; and for each  $j \in [0, n-1]$ ,  $\text{col}_j(A) \leq c_j$ . Let  $\mathcal{T}$  be the regular dynamics (with zero enhancements) and let  $\widehat{\mathcal{T}}$  be the dynamics defined by  $(\mathcal{Z}, \vec{r}, \vec{c})$ . Then, for every  $k$ ,  $\mathcal{T}^k(A) \setminus \widehat{\mathcal{T}}^k(\emptyset) \subseteq A$ .*

*Proof.* We prove the equivalent claim that  $\mathcal{T}^k(A) \setminus A \subseteq \widehat{\mathcal{T}}^k(\emptyset)$  by induction on  $k$ . For  $k = 0$ , both sets in the claim are empty. Assume now the claim is true for some  $k$ . Then,  $\mathcal{T}^k(A) \setminus A \subseteq \widehat{\mathcal{T}}^k(\emptyset) \subseteq \widehat{\mathcal{T}}^{k+1}(\emptyset)$ . Furthermore, for any point  $z = (i, j) \in \mathcal{T}^{k+1}(A) \setminus \mathcal{T}^k(A)$ ,

$$|\mathcal{N}^r(z) \cap \mathcal{T}^k(A)| = |\mathcal{N}^r(z) \cap A| + |\mathcal{N}^r(z) \cap (\mathcal{T}^k(A) \setminus A)| \leq r_i + |\mathcal{N}^r(z) \cap \widehat{\mathcal{T}}^k(\emptyset)|,$$

by the assumption and by the inductive hypothesis. Analogously,

$$|\mathcal{N}^c(z) \cap \mathcal{T}^k(A)| \leq c_j + |\mathcal{N}^c(z) \cap \widehat{\mathcal{T}}^k(\emptyset)|.$$

Since  $z \in \mathcal{T}^{k+1}(A)$ , the definition of  $\mathcal{T}$  yields that  $(|\mathcal{N}^r(z) \cap \mathcal{T}^k(A)|, |\mathcal{N}^c(z) \cap \mathcal{T}^k(A)|) \notin \mathcal{Z}$ . Hence, also

$$(r_i + |\mathcal{N}^r(z) \cap \widehat{\mathcal{T}}^k(\emptyset)|, c_j + |\mathcal{N}^c(z) \cap \widehat{\mathcal{T}}^k(\emptyset)|) \notin \mathcal{Z}$$

and consequently  $z \in \widehat{\mathcal{T}}^{k+1}(\emptyset)$ . □



Fix  $m, n \in \mathbb{N}$ , and enhancement vectors  $\vec{r} \in \mathbb{Z}_+^m$ ,  $\vec{c} \in \mathbb{Z}_+^n$ . For every  $A \subseteq R_{m,n}$ , we define its *enhancement*  $\hat{A}$  as an arbitrary but fixed subset of  $R_{m,n}$  such that:  $A \subseteq \hat{A}$ ; for every row  $i \in [0, m-1]$ ,  $\text{row}_i(\hat{A}) \geq \min(\text{row}_i(A) + r_i, n)$ ; and for every column  $j \in [0, n-1]$ ,  $\text{col}_j(\hat{A}) \geq \min(\text{col}_j(A) + c_j, m)$ . Thus, we obtain the enhancement by adding  $r_i$  elements in every row  $i$ , up to capacity, and analogously for columns.

**Lemma 2.6.** *Let  $\mathcal{T}$  be the standard dynamics given by  $\mathcal{Z}$ ,  $m$ , and  $n$ , and let  $\hat{\mathcal{T}}$  be the enhanced dynamics given by some  $(\mathcal{Z}, \vec{r}, \vec{c})$  with  $\vec{r} \in \mathbb{Z}_+^m$ ,  $\vec{c} \in \mathbb{Z}_+^n$ . Then, each  $A \subseteq R_{m,n}$  satisfies  $\hat{\mathcal{T}}(A) \subseteq \mathcal{T}(\hat{A})$ .*

*Proof.* Since  $A \subseteq \hat{A} \subseteq \mathcal{T}(\hat{A})$ ; it suffices to show that  $\hat{\mathcal{T}}(A) \setminus A \subseteq \mathcal{T}(\hat{A})$ . Consider an arbitrary  $z = (i, j) \in \hat{\mathcal{T}}(A) \setminus A$ . If either  $\text{row}_i(\hat{A}) = n$  or  $\text{col}_j(\hat{A}) = m$ , then  $z \in \hat{A} \subseteq \mathcal{T}(\hat{A})$ . Otherwise, using the partial order on  $\mathbb{Z}_+^2$ ,

$$\begin{aligned} (|\mathcal{N}^r(z) \cap \hat{A}|, |\mathcal{N}^c(z) \cap \hat{A}|) &= (\text{row}_i(\hat{A}), \text{col}_j(\hat{A})) \\ &\geq (\text{row}_i(A) + r_i, \text{col}_j(A) + c_j) \\ &= (|\mathcal{N}^r(z) \cap A| + r_i, |\mathcal{N}^c(z) \cap A| + c_j) \notin \mathcal{Z}, \end{aligned}$$

since  $z \in \hat{\mathcal{T}}(A) \setminus A$ . Therefore,  $(|\mathcal{N}^r(z) \cap \hat{A}|, |\mathcal{N}^c(z) \cap \hat{A}|) \notin \mathcal{Z}$ , and, hence,  $z \in \mathcal{T}(\hat{A})$ .  $\square$

### 3 A connection with extremal graph theory

For a family of graphs  $\mathcal{F}$ , let  $\text{ex}(n, \mathcal{F})$  denote the maximum number of edges in a graph on  $n$  vertices that does not contain any member of  $\mathcal{F}$  as a subgraph. The numbers  $\text{ex}(n, \mathcal{F})$  are called *Turán numbers*. Their determination is a classical problem in extremal graph theory, even in the special case when  $\mathcal{F}$  consists of a single graph  $H$  (see, e.g., [Bol, AKSV]).

Similarly, the *bipartite Turán problem* considers two positive integers  $m$  and  $n$  and a family of graphs  $\mathcal{F}$  such that each  $H \in \mathcal{F}$  is a bipartite graph equipped with a bipartition  $(X_H, Y_H)$ . The quantity of interest is then the *bipartite Turán number*  $\text{ex}(m, n, \mathcal{F})$ , the maximum number of edges in a bipartite graph  $G$  with bipartition  $(X, Y)$  with  $|X| = m$  and  $|Y| = n$  that does not contain any  $H \in \mathcal{F}$  as a subgraph that respects the given bipartitions (see, e.g., [FS] where the bipartite Turán number is denoted by  $\text{ex}^*(m, n, \mathcal{F})$ ). Most research in this direction addressed the case when  $\mathcal{F}$  contains a single graph  $H$ . When  $H$  is a complete bipartite graph  $K_{s,t}$ , the problem is known as the *Zarankiewicz problem* (see [Zar]). While the determination of Zarankiewicz numbers  $\text{ex}(m, n, K_{s,t})$  is still an open question (see, e.g., [CHM, CHM2, Con, ST] for some recent developments), the bipartite Turán problem was completely solved for paths (see [GyRS]) and partially for cycles (see, e.g., [LN]).

Our results lead to a resolution of the bipartite Turán problem for some families of double stars, including those containing a single double star. Given two integers  $p, q \geq 0$ , a *double star*  $S_{p,q}$  is the graph obtained from the disjoint union of complete bipartite graphs  $K_{1,p}$  and  $K_{1,q}$  by adding an edge between a vertex  $u$  of degree  $p$  in  $K_{1,p}$  and a vertex  $v$  of degree  $q$  in  $K_{1,q}$  (note that the vertex  $u$  is uniquely determined if  $p \neq 1$  and similarly for  $v$ ). We assume that  $S_{p,q}$  is

equipped with a bipartition  $(X_{p,q}, Y_{p,q})$  such that  $u \in X_{p,q}$  and  $v \in Y_{p,q}$ . A family  $\mathcal{F}$  of double stars is *minimal* if, for all distinct  $(p, q), (p', q')$  such that  $S_{p,q}, S_{p',q'} \in \mathcal{F}$ , it is not true that  $(p, q) \leq (p', q')$  (using the partial order in  $\mathbb{Z}_+^2$ ). We will show that the problems of computing the bipartite Turán number for a minimal family of double stars and the Young domination number are dual to each other, in the sense specified below.

Assume that  $m$  and  $n$  are positive integers,  $X$  and  $Y$  disjoint sets with  $|X| = m$  and  $|Y| = n$ , and that  $\mathcal{F}$  is a family of double stars, with fixed bipartitions. Observe that, for a given  $S_{p,q} \in \mathcal{F}$ , a given bipartite graph  $G$  with parts  $X$  and  $Y$  contains  $S_{p,q}$  as a subgraph that respects the bipartitions if and only if there exist vertices  $x \in X, y \in Y$ , so that  $xy$  is an edge of  $G$ , and the degrees of  $x$  and  $y$  in  $G$  are at least  $p + 1$  and  $q + 1$ , respectively. Recall that we denote by  $\text{ex}(m, n, \mathcal{F})$  the corresponding bipartite Turán number, the maximum number of edges in a bipartite graph  $G$  with bipartition  $(X, Y)$  that does not contain any  $S_{p,q} \in \mathcal{F}$  as a subgraph that respects the bipartitions. Note that we may replace  $\mathcal{F}$  by an equivalent minimal family by keeping only those  $S_{p,q}$  for which  $(p, q)$  is minimal.

Let us now connect these notions to Young domination in Hamming rectangles. Recall that, if  $\mathcal{Z}$  is a zero-set, a point  $(x, y)$  belongs to  $\mathbb{Z}_+^2 \setminus \mathcal{Z}$  if and only if  $(a, b) \leq (x, y)$  for some concave corner  $(a, b)$  of  $\mathcal{Z}$ . To any minimal family  $\mathcal{F}$  of double stars, we associate a Young diagram  $\mathcal{Y}_{\mathcal{F}}$  whose concave corners are all points  $(q, p)$  such that  $S_{p,q} \in \mathcal{F}$ .

**Theorem 3.1.** *Fix  $m, n \in \mathbb{N}$ . Assume that a zero-set  $\mathcal{Z}$  and a minimal family  $\mathcal{F}$  of double stars are such that  $\mathcal{Z}$  and  $\mathcal{Y}_{\mathcal{F}}$  are both subsets of  $R_{m,n}$  and duals of each other, in the sense of Definition 2.1. Then,*

$$\gamma(\mathcal{Z}, m, n) + \text{ex}(m, n, \mathcal{F}) = mn.$$

*Proof.* By definition,  $\gamma(\mathcal{Z}, m, n)$  is the minimum cardinality of a set  $A \subseteq V(K_m \square K_n)$  such that every vertex  $v \in V(K_m \square K_n) \setminus A$  satisfies  $(|\mathcal{N}^r(v) \cap A|, |\mathcal{N}^c(v) \cap A|) \notin \mathcal{Z}$ , that is, there exists a concave corner  $z = (a, b)$  of  $\mathcal{Z}$  such that  $|\mathcal{N}^r(v) \cap A| \geq a$  and  $|\mathcal{N}^c(v) \cap A| \geq b$ . Let us identify the vertices of  $K_m \square K_n$  with the edges of the complete bipartite graph  $K_{m,n}$  with bipartition  $(X, Y) = (V(K_m), V(K_n))$ . Then,  $A$  corresponds to a set of edges of  $K_{m,n}$  such that for every edge  $(i, j)$  of  $K_{m,n}$  that is not in  $A$  there exists a concave corner  $(a, b) \in \mathcal{I}_{\mathcal{Z}}$  such that vertex  $i \in X$  is incident with at least  $a$  edges in  $A$  and vertex  $j \in Y$  is incident with at least  $b$  edges in  $A$ . Let  $A^c$  denote the bipartite complement of  $A$  in  $K_{m,n}$ , that is, the bipartite graph with bipartition  $(X, Y)$  in which two vertices  $i \in X$  and  $j \in Y$  are adjacent if and only if  $(i, j) \notin A$ . The above condition on  $A$  can be equivalently phrased in terms of  $A^c$ , as follows: for every edge  $(i, j)$  of  $A^c$  there exists a concave corner  $(a, b) \in \mathcal{I}_{\mathcal{Z}}$  such that  $i$  is incident with at most  $n - a$  edges in  $A^c$  and  $j$  is incident with at most  $m - b$  edges in  $A^c$ . This is in turn equivalent to the following condition: for all  $p, q \in \mathbb{Z}_+$ , if a double star  $S_{p,q}$  is a subgraph of  $A^c$  such that the vertex of degree  $p + 1$  belongs to  $X$  and the vertex of degree  $q + 1$  belongs to  $Y$ , then there exists a concave corner  $(a, b) \in \mathcal{I}_{\mathcal{Z}}$  such that  $p \leq n - a - 1$  and  $q \leq m - b - 1$  (note that since  $p \geq 0$ , we have that  $a \leq n - 1$  and, similarly,  $b \leq m - 1$ ). In other words, the double stars that can appear as subgraphs of  $A^c$  are precisely the double stars  $S_{p,q}$  such that  $(q, p)$  is an element of the dual  $\check{\mathcal{Z}}$  of  $\mathcal{Z}$ . Equivalently, the minimal double stars that are forbidden for  $A^c$  are precisely the stars in  $\mathcal{F}$ . Hence, the maximum possible number of edges in  $A^c$  is given by the value of

$\text{ex}(m, n, \mathcal{F})$ . Since the number of edges in  $A^c$  equals  $mn - |A|$ , minimizing the cardinality of  $A$  is equivalent to maximizing the cardinality of  $A^c$ , which implies  $\gamma(\mathcal{Z}, m, n) = mn - \text{ex}(m, n, \mathcal{F})$ , as claimed.  $\square$

Our results on exact evaluation of  $\gamma$  then immediately give the following results. In particular, the next corollary is given by Proposition 1.1. Note that the double star  $S_{a,0}$  is isomorphic to the complete bipartite graph  $K_{1,a+1}$  with parts  $X_{a,0}$  and  $Y_{a,0}$  such that  $X_{a,0}$  consists of a single vertex of degree  $a + 1$  (and, hence, every vertex in  $Y_{a,0}$  has degree 1). A similar observation holds for the double star  $S_{0,b}$ . Therefore, the quantity  $\text{ex}(m, n, \mathcal{F})$  counts the maximum number of edges in a bipartite graph with parts  $X$  and  $Y$  of sizes  $m$  and  $n$  respectively such that each vertex in  $X$  has degree at most  $a$  and each vertex in  $Y$  has degree at most  $b$ .

**Corollary 3.2.** *For  $0 \leq a \leq n$ ,  $0 \leq b \leq m$ , we have  $\check{R}_{b,a} = V_{m-b,n-a}$ . Consequently, for the minimal family  $\mathcal{F} = \{S_{a,0}, S_{0,b}\}$ ,*

$$\text{ex}(m, n, \mathcal{F}) = \min\{am, bn\}.$$

*Proof.* The equality  $\check{R}_{b,a} = V_{m-b,n-a}$  follows directly from the definitions. By Proposition 1.1,  $\text{ex}(m, n, \mathcal{F}) = mn - \gamma(V_{m-b,n-a}, m, n) = mn - \max(m(n-a), n(m-b)) = \min\{am, bn\}$ .  $\square$

**Corollary 3.3.** *For  $0 \leq a \leq n$ ,  $0 \leq b \leq m$ , we have  $\check{V}_{b,a} = R_{m-b,n-a}$ . Consequently, for the minimal family  $\mathcal{F} = \{S_{a,b}\}$ ,*

$$\text{ex}(m, n, \mathcal{F}) = mn - \gamma(R_{m-b,n-a}, m, n),$$

given by Theorem 1.2.

**Corollary 3.4.** *For  $1 \leq a \leq m + n - 2$ , the dual of  $T_a \cap R_{m,n}$ , in the sense of Definition 2.1, is  $T_{m+n-2-a} \cap R_{m,n}$ . Consequently, for  $m = n$ ,  $a \leq 2n - 2$ , and the minimal family  $\mathcal{F} = \{S_{i,j} : i + j = a\}$ , we have*

$$\text{ex}(n, n, \mathcal{F}) = n^2 - \gamma(T_{2n-2-a}, n, n),$$

given by Theorem 1.3.

## 4 Young domination with $L$ -shaped and rectangular zero-sets

**Lemma 4.1.** *Fix integers  $n \geq a \geq 1$  and  $m \geq 1$ , and let  $R_{m,n}$  be the vertex set of the graph  $K_m \square K_n$ . Then, there exist a set  $S \subseteq R_{m,n}$  such that each row of  $R_{m,n}$  contains exactly  $a$  elements of  $S$ , and every column contains at least  $\lfloor am/n \rfloor$  and at most  $\lceil am/n \rceil$  elements of  $S$ .*

*Proof.* On each row,  $S$  contains an interval of  $a$  sites, starting at the horizontal position of the first site after the interval of the previous row ends, with wraparound boundary. More formally, on each row  $i \in [0, m-1]$  the interval is at column indices  $ia, ia+1, \dots, ia+a-1$ , taken modulo  $n$ . Every time one of these column indices reaches  $n-1$ , the minimal number of elements in  $S$  in

every column increases by 1, and every time one of these column indices reaches 0, the maximal number of elements in every column increases by 1. Therefore, the final minimal (resp. maximal) number sites in every column is  $\lfloor am/n \rfloor$  (resp.  $\lceil am/n \rceil$ ).  $\square$

We can now prove Proposition 1.1, which determines the minimum cardinality  $\gamma$  of a  $\mathcal{Z}$ -dominating set where the requirement is that each unoccupied vertex has at least  $a$  occupied row neighbors and at least  $b$  occupied column neighbors.

*Proof of Proposition 1.1.* We may assume that  $am \geq bn$  and  $a \geq 1$ . It is clear that every row needs at least  $a$  occupied sites, so  $\gamma \geq am$ . Now we need to construct a configuration in which every row contains exactly  $a$  sites, and also every column contains  $b$  or more sites. As  $\lfloor ma/n \rfloor \geq b$ , the above lemma provides this.  $\square$

For the rest of this section we assume that  $\mathcal{Z} = R_{b,a}$ , where  $1 \leq a \leq n$ ,  $1 \leq b \leq m$ , and prove Theorem 1.2. We recall that, in this case, a  $\mathcal{Z}$ -dominating set  $S$  is such that for every  $z \in R_{m,n} \setminus S$  there are at least  $a$  points of  $S$  in the row of  $z$  or at least  $b$  points of  $S$  in the column of  $z$ .

*Proof of (1.3).* To prove the lower bound, assume  $S$  is a set that realizes  $\gamma$ . For some  $x \in [0, m]$ , exactly  $m - x$  rows contain at most  $a - 1$  elements of  $S$ , and we may assume those are the first  $m - x$  rows. For some  $y \in [0, n]$ , exactly  $n - y$  columns do not lack any of the elements of  $S$  among the first  $m - x$  row positions, and we may assume those are the leftmost columns. It follows that  $S$  contains the  $(m - x) \times (n - y)$  block in the top left corner, that each of the rows  $m - x + 1, \dots, m$  contains at least  $a$  elements of  $S$  (by definition of  $x$ ), and all the columns  $n - y + 1, \dots, n$  contain at least  $b$  elements of  $S$  (as  $S$  is  $\mathcal{Z}$ -dominating). This gives the lower bound of the same form as (1.3), except the lower bounds on  $x$  and  $y$  are 1 instead of  $a$  and  $b$ . We next show that the minimum is achieved for  $x \geq b$  and  $y \geq a$ .

If  $x < b$ , then we may, without changing the size of  $S$  or affecting its Young dominance, move points of  $S$  within the last  $y$  columns downwards until  $S$  contains the entire  $x \times y$  block. Analogously, we may also assume that  $S$  contains this block when  $y < a$ .

Assume first that  $x < b$  and  $y \geq a$ . Then, in the last  $y$  rows,  $S$  has exactly  $b - x$  points outside the  $x \times y$  block in each column. Therefore

$$\begin{aligned} |S| &= (m - x)(n - y) + ax + y(b - x) \\ &= (a - n)x + (b - m)y + mn, \end{aligned}$$

which is nonincreasing in  $x$ , and so we can construct a new  $\mathcal{Z}$ -dominating set  $S'$ , with  $x$  replaced by  $x + 1$ , such that  $|S'| \leq |S|$ . The case  $x \geq b$ ,  $y < a$  is eliminated similarly.

If  $x < b$ ,  $y < a$ , then  $|S|$  has a similar form:

$$\begin{aligned} |S| &= (m - x)(n - y) + xy + y(b - x) + x(a - y) \\ &= (a - n)x + (b - m)y + mn, \end{aligned}$$

so the size of  $S$  decreases if we increase either  $x$  or  $y$ . This establishes the  $\geq$  part of (1.3).

To prove the  $\leq$  part, we construct a  $\mathcal{Z}$ -dominating set of size  $(m-x)(n-y) + \max(ax, by)$  whenever  $b \leq x \leq m$  and  $a \leq y \leq n$ . We may assume that  $ax \geq by$ . Our set will have the block structure

$$B = \begin{bmatrix} B_{11}^{(m-x) \times (n-y)} & B_{12}^{(m-x) \times y} \\ B_{21}^{x \times (n-y)} & B_{22}^{x \times y} \end{bmatrix},$$

with the blocks  $B_{12}$  and  $B_{21}$  comprised of 0s, the block  $B_{11}$  comprised of 1s, and the block  $B_{22}$  chosen so that there are exactly  $a$  1s in each row and at least  $b$  1s in each column. This is possible by Lemma 4.1 and clearly the resulting  $B$  gives a  $\mathcal{Z}$ -dominating set.  $\square$

*Proof of (1.4).* Let  $\varphi(y) = (n-y)^2 + ay$  for all  $y \in \mathbb{Z}$ . We may assume that  $x \leq y$  in (1.3), which, by minimizing over  $x$  first, gives  $\gamma = \min_{a \leq y \leq n} \varphi(y)$ . But  $\varphi$  achieves its global minimum at  $\lceil (2n-a-1)/2 \rceil$ . Therefore,  $\gamma = \varphi(\max(a, \lceil (2n-a-1)/2 \rceil))$ , which can be checked to equal the expression in (1.4).  $\square$

*Proof of the upper bound on the computation time of (1.3).* We may assume that  $a \leq b$ . For a fixed  $y$ , optimization over  $x$  in the two cases (when the maximum equals its first or its second argument) amounts to minimizing a linear function, and thus is achieved at  $x = \lfloor by/a \rfloor$  or  $x = \lceil by/a \rceil$  or at an expression that does not depend on  $y$ . Let  $a_1 = a/\gcd(a, b)$  and  $b_1 = b/\gcd(a, b)$ . If we write  $y = ja_1 + r$ , where  $0 \leq r < a_1$ , then  $\lfloor by/a \rfloor = b_1 \cdot j + \lfloor b_1 r/a_1 \rfloor$  and  $\lceil by/a \rceil = b_1 \cdot j + \lceil b_1 r/a_1 \rceil$ . We then perform optimization over  $j$  first, for a fixed  $r$ . In every case, we need to compute the minimum of a function that is at most quadratic in  $j$ , which we can solve explicitly. Finally, we perform minimum over  $r$ , which we can do in  $\mathcal{O}(a_1)$  steps.  $\square$

## 5 Exact results for $a$ -domination on Hamming squares

Throughout this section, we consider  $a$ -domination on the Hamming square, that is, we assume  $m = n$ , that  $\mathcal{Z}$  is the triangular set  $T_a$  given by (1.1), and study  $\gamma = \gamma(\mathcal{Z}, n, n)$ . We prove Theorem 1.3 in the three cases in order in the next three subsections.

### 5.1 Even $a$

**Lemma 5.1.** *For all positive integers  $a$  and  $n$  such that  $a \leq 2n$ , it holds that  $\gamma \geq an/2$ .*

*Proof.* To prove that  $\gamma \geq an/2$ , we begin with a reformulation of the claim: if  $x_{ij} \in \{0, 1\}$ ,  $i, j = 1, \dots, n$ , are such that for every  $i, j$

$$(5.1) \quad \sum_{k, k \neq j} (1 - x_{ij})x_{ik} + \sum_{k, k \neq i} (1 - x_{ij})x_{kj} \geq a(1 - x_{ij}),$$

then

$$(5.2) \quad \sigma := \sum_{i,j} x_{ij} \geq \frac{na}{2}.$$

Let  $r_i = \sum_j x_{ij}$  and  $c_j = \sum_i x_{ij}$  be the row and column sums. Summing (5.1) over all  $i, j$ , the right-hand side sums to  $a(n^2 - \sigma)$ . The first terms of the left-hand side sum to

$$(5.3) \quad \begin{aligned} \sum_i \sum_{j,k,k \neq j} (1 - x_{ij})x_{ik} &= \sum_i |\{(k, j) : x_{ik} = 1 \text{ and } x_{ij} = 0\}| \\ &= \sum_i r_i(n - r_i) \\ &= n\sigma - \sum_i r_i^2. \end{aligned}$$

Similarly, the second terms on the left-hand side of (5.1) similarly sum to

$$(5.4) \quad n\sigma - \sum_j c_j^2.$$

Note that, by the Cauchy–Bunyakovsky–Schwarz inequality applied to the vectors  $(1, \dots, 1) \in \mathbb{R}^n$  and  $(r_1, \dots, r_n)$ , we have that  $\sigma^2 = (\sum_i r_i)^2 \leq n \cdot \sum_i r_i^2$ . Therefore

$$\sum_i r_i^2 \geq \sigma^2/n, \quad \sum_j c_j^2 \geq \sigma^2/n,$$

so that (5.3, 5.4) yield the inequality

$$2n\sigma - \frac{2\sigma^2}{n} \geq an^2 - a\sigma,$$

that is,

$$(5.5) \quad 2\sigma^2 - n(2n + a)\sigma + an^3 \leq 0,$$

and, as  $a \leq 2n$ , the left zero of the quadratic function is at  $an/2$ . Thus, (5.2) follows.  $\square$

*Proof of Theorem 1.3 for even  $a$ .* Here, we need to prove (1.5). The inequality  $\gamma \leq an/2$  is proved in Corollary 11 of [BLL], but also follows from observing that the set in Lemma 4.1, with  $m = n$  and  $a$  replaced by  $a/2$ , is  $a$ -dominating. The opposite inequality follows from Lemma 5.1.  $\square$

## 5.2 Odd $a \leq n$

We first state a lemma on a common design of dominating sets.

**Lemma 5.2.** *Assume that a set  $D \subseteq R_n$  contains at least  $b$  elements in each row and column of  $R_n$ . Furthermore, assume that  $(i, j) \in D$  for every  $(i, j)$  such that both row  $i$  and column  $j$  contain exactly  $b$  elements of  $D$ . Then  $D$  is  $(2b + 1)$ -dominating.*

*Proof.* By permuting rows and columns, we may assume that, for some  $(k, \ell)$ ,  $[0, k-1] \times [0, \ell-1] \subseteq D$ , that the first  $k$  rows and the first  $\ell$  columns each contain exactly  $b$  elements of  $D$ , and that the remaining rows and columns contain at least  $b + 1$  elements of  $D$  each.

To see that  $D$  is an  $(2b+1)$ -dominating set in  $R_n$ , consider an arbitrary vertex  $(i, j) \in R_n \setminus D$ . If  $i \leq k-1$ , then  $j \geq \ell$ , and vertex  $(i, j)$  has  $b$  row neighbors in  $D$  and  $b+1$  column neighbors in  $D$ ; hence, it has  $2b+1$  neighbors in  $D$ . The argument is similar if  $j \leq \ell-1$ . Suppose now that  $i \geq k$  and  $j \geq \ell$ . Then,  $(i, j)$  has  $b+1$  row neighbors in  $D$ , as well as  $b+1$  column neighbors in  $D$ , for a total of  $2b+2$  neighbors in  $D$ .  $\square$

**Lemma 5.3.** *If  $a = 2b + 1$ , for some integer  $b \in [0, \lfloor (n-1)/2 \rfloor]$ , then it holds that*

$$\gamma \leq (b+1)n - b.$$

*Proof.* We construct a  $(2b+1)$ -dominating set  $D$  of  $K_n \square K_n$  with cardinality  $(b+1)n - b$  as follows. Let  $B = [0, b-1]$  and  $\overline{B} = [0, n-1] \setminus B$ . Let  $D = (B \times B) \cup C$ , where  $C$  is an arbitrary subset of the set  $\overline{B} \times \overline{B}$  such that each row and column of  $\overline{B} \times \overline{B}$  contains precisely  $b+1$  elements of  $C$ . Note that such a set exists by Lemma 4.1, since  $b+1 \leq n-b = |\overline{B}|$ . By Lemma 5.2,  $D$  is a  $(2b+1)$ -dominating set. Since  $|D| = b^2 + (b+1)(n-b) = (b+1)n - b$ , we obtain that  $\gamma \leq (b+1)n - b$ , as required.  $\square$

We next provide a matching lower bound to the upper bound given by Lemma 5.3.

**Lemma 5.4.** *If  $a = 2b + 1$ , for some integer  $b \in [0, \lfloor (n-1)/2 \rfloor]$ , then it holds that*

$$\gamma \geq (b+1)n - b.$$

*Proof.* Let  $D$  be a  $(2b+1)$ -dominating set of  $K_n \square K_n$ . We show that  $|D| \geq (b+1)n - b$ .

Suppose for a contradiction that  $|D| < (b+1)n - b$ . For  $i \in [0, n-1]$ , let  $d_i$  be the number of vertices in  $D$  in row  $i$  (i.e., with first coordinate  $i$ ), and let  $\ell = \min_{1 \leq i \leq n} d_i$ .

We prove the lemma via a sequence of claims established in steps (1)–(4) below.

(1)  $\ell \leq b$ .

If  $\ell \geq b+1$ , then  $|D| \geq (b+1)n$ , a contradiction. This proves (1).

(2)  $\ell = b$ .

Call a subset of  $R_n$   $k$ -thick if every row and every column contains at least  $k$  points of the set. Suppose we know, for some integer  $k \geq 0$ , that  $D$  is  $k$ -thick. Suppose also that  $k^2 + (n-k)(a-k) \geq bn + n - b$ . Then we claim that  $D$  is also  $(k+1)$ -thick. Assume not. Then we may without loss of generality assume that the first row (i.e., row 0) contains exactly  $k$  points of  $D$  in the

leftmost  $k$  positions. Since  $D$  is  $k$ -thick, each of the leftmost  $k$  columns contains at least  $k$  points of  $D$ . As  $D$  is  $a$ -dominating, each of the other  $n - k$  columns contains at least  $a - k$  points of  $D$ . Therefore,  $|D| \geq k^2 + (n - k)(a - k) \geq bn + n - b$ , which contradicts our assumption on the size of  $D$ .

It remains to show that, for  $2b + 1 \leq n$  and  $0 \leq k \leq b - 1$ ,

$$k^2 + (n - k)(2b + 1 - k) \geq bn + n - b.$$

In fact the above inequality holds for  $k \in [0, b]$ . To see this, observe that the derivative of the left hand side with respect to  $k$  is  $4k - (n + 2b + 1) \leq 2b - 1 - n < 0$ , and that at  $k = b$  the two sides are equal. This show that, since  $D$  is trivially 0-thick, that  $D$  is also  $b$ -thick, concluding the proof of (2).

Let us call a row index  $i \in [0, n - 1]$  *light* if  $D$  contains exactly  $b$  vertices from the  $i$ -th row. Let  $I$  be the set of all light row indices.

$$(3) |I| \geq b + 1.$$

Suppose for a contradiction that  $|I| \leq b$ . Summing up the elements of  $D$  row by row, we obtain  $|D| \geq |I| \cdot b + (n - |I|) \cdot (b + 1) \geq (b + 1)n - b$ , a contradiction. This proves (3).

By (3) and by permuting the rows if necessary, we may assume without loss of generality that  $[0, b] \subseteq I$ .

$$(4) D \text{ contains at least } b + 1 \text{ vertices from each column.}$$

Consider a column index  $j \in [0, n - 1]$ . If its first  $b + 1$  vertices belong to  $D$ , that is,  $(i, j) \in D$  for all  $i \in [0, b]$ , then we are done. So we may assume that this is not the case, that is, there exists a row index  $i \in [0, b]$  such that  $(i, j) \notin D$ . Note that  $i \in I$ , since  $[0, b] \subseteq I$ . Hence, the vertex  $(i, j)$  has exactly  $b$  row neighbors in  $D$ . Since  $D$  is a  $(2b + 1)$ -dominating set in  $K_n \square B_n$ , we infer that the vertex  $(i, j)$  has at least  $b + 1$  column neighbors in  $D$ . This proves (4).

Summing up the elements of  $D$  column by column, we obtain using (4) that  $|D| \geq (b + 1) \cdot n$ , a contradiction.  $\square$

*Proof of Theorem 1.3 for odd  $a \leq n$ .* This follows immediately from Lemmas 5.3 and 5.4.  $\square$

### 5.3 Odd $a > n$

Observe that (1.7) can be rewritten as

$$(5.6) \quad \gamma = \frac{a - 1}{2} n + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2(2n - a)} - \frac{1}{2} \cdot \mathbb{1}_{n \text{ odd}} \right\rceil,$$

and we call the third term the *correction term*. The next lemma provides a lower bound, which turns out to be sharp in this case (as well as in the case  $a = n$ ), but not for  $a < n$  (where the approach in the proof of Lemma 5.4 is superior). For the upper bound, we use a block construction, which is similar to the one in Lemma 5.3, but a bit more involved.



**Lemma 5.5.** For  $s \in \mathbb{Z}$ , let

$$\nu(s) = -2n \cdot \lfloor s/n \rfloor^2 + 2(2s - n) \cdot \lfloor s/n \rfloor + an^2 + (2 - a - 2n).$$

Then, for any  $a \leq 2n - 2$ ,

$$(5.7) \quad \gamma \geq \min\{s \in [\lceil an/2 \rceil, \lceil a/2 \rceil n] : \nu(s) \leq 0\}.$$

*Proof.* For a fixed integer  $n \geq 1$  and integer  $s \geq 0$  we define

$$\mu(s) = \min \left\{ r_1^2 + \dots + r_n^2 : r_1, \dots, r_n \in \mathbb{Z}_+, \sum_{i=1}^n r_i = s \right\}.$$

We first claim that the numbers  $r_i$  at which the minimum is achieved may have at most two distinct values. Indeed, if, say,  $r_2 \geq r_1 + 2$ , then

$$(r_1 + 1)^2 + (r_2 - 1)^2 = r_1^2 + r_2^2 + 2(1 - r_2 + r_1) < r_1^2 + r_2^2$$

and so replacing  $r_1$  and  $r_2$  with  $(r_1 + 1)$  and  $(r_2 - 1)$ , respectively, decreases the sum of squares. It follows that, with  $r = \lfloor s/n \rfloor$  and  $t = s - rn$ ,

$$\mu(s) = (n - t)r^2 + t(r + 1)^2 = nr^2 + t(2r + 1).$$

From the proof of Lemma 5.1, we get that any feasible solution  $(x_{ij})$ , with  $\sigma = \sum_{ij} x_{ij}$ , must satisfy

$$2\mu(\sigma) \leq -an^2 + (a + 2n)\sigma.$$

Furthermore, for the optimal solution,  $\sigma \geq \lceil an/2 \rceil$  (Lemma 5.1) and  $\sigma \leq \lceil a/2 \rceil n$  (Corollaries 11 and 12 in [BLL]). Simple algebra yields (5.7).  $\square$

*Proof of Theorem 1.3 when  $a > n$ ,  $a$  is odd, and  $n$  is even.* Write  $a = 2b + 1$  and  $n = 2m$ . We first design a dominating set with its size given by (5.6). The  $a$ -dominating set will be constructed as a block matrix with four blocks (where as usual, vertices in the set are encoded by 1s):

$$B = \begin{bmatrix} B_{11}^{(m-k) \times (m-k)} & B_{12}^{(m-k) \times (m+k)} \\ B_{21}^{(m+k) \times (m-k)} & B_{22}^{(m+k) \times (m+k)} \end{bmatrix}$$

The dimensions depend on the number  $k$  which will be specified below. The matrix  $B$  will be symmetric, and will have  $b$  1s in each of its first  $m - k$  rows and  $b + 1$  1s in each of its remaining  $m + k$  rows. Therefore, the number of 1s is  $2mb + m + k$ , so  $k$  will be exactly the correction term in (5.6). All entries of  $B_{11}$  are set to 1. Note that  $B$  determines an  $a$ -dominating set by Lemma 5.2.

We start by making all entries in  $B_{22}$  1 as well, but this will later change. We now concentrate on  $B_{12}$ , so all rows and columns are of that block only, unless otherwise specified. The goal is to define nonnegative integers  $c$  and  $d$ , and design a configuration in  $B_{12}$  which has  $d$  1s in every row and either  $c$  or  $c + 1$  1s in every column; we now decide what  $c$  and  $d$  must be. First,  $c$

is the number that, together with the  $m + k$  1s in the same column of  $B_{22}$  makes for  $b + 1$  1s, that is,  $c = b - m - k + 1$ . (We will choose  $k$  so that  $k \leq b - m + 1$ ; this will assure that  $c$  is nonnegative.) Furthermore,  $d$  is the number that, together with the  $m - k$  1s in the same row of  $B_{11}$  makes for  $b$  1s, that is,  $d = b - m + k$ . In order for this construction to be possible,

$$(5.8) \quad (m - k)d \geq (m + k)c,$$

which works out to be

$$m(4k - 1) \geq (2b + 1)k = ak,$$

and so  $k$  needs to be at least the correction term in (5.6).

We next show that the claimed construction in  $B_{12}$  is possible if  $k$  is the smallest positive integer such that (5.8) holds. We first claim that we then have

$$(5.9) \quad (m - k)d \leq (m + k)(c + 1).$$

If  $k = 1$ ,  $d = c + 1$  and the claim holds. Assume now that  $k \geq 2$ . In this case, we know that (5.8) does not hold if we replace  $k$  by  $k - 1$ , which yields

$$b > \frac{4k - 3}{2(k - 1)}m - \frac{1}{2}.$$

Equation (5.9) is equivalent to

$$b \geq \frac{2k - 1}{k}m - 1,$$

so it holds if

$$\frac{4k - 5}{2(k - 1)} \geq \frac{2k - 1}{k},$$

which holds when  $k \geq 2$ , establishing (5.9).

The next step is to use Lemma 4.1 (with  $a = d$ ,  $m = m - k$ , and  $n = m + k$ ), which provides a configuration in  $B_{12}$  with the required row sums, and such that each column sum is between  $\lfloor (m - k)d/(m + k) \rfloor \geq c$  (by (5.8)) and  $\lceil (m - k)d/(m + k) \rceil \leq c + 1$  (by (5.9)). Note that this last inequality implies that  $c \geq 0$ , as promised. Recall also that, since  $B$  is symmetric,  $B_{21}$  is uniquely determined by  $B_{12}$ .

In the final step, we eliminate some 1s from  $B_{22}$ . Namely, for every column  $j$  of  $B$  which has  $(c + 1)$  1s in  $B_{12}$ , the row  $j$  of  $B$  also has  $(c + 1)$  1s in  $B_{21}$ . In this case, we replace the 1 at the diagonal position  $jj$  of  $B$  (which must fall in  $B_{22}$ ) by 0. It is clear from the construction that we have produced an  $a$ -dominating set with claimed row and column sums. See Figure 5.1 for an example.

We have therefore established the  $\leq$  part in (5.6). In particular, this implies that  $\gamma < n(b + 1)$ . We now proceed to verify the  $\geq$  part. Recall the expression  $\nu(s)$  from Lemma 5.5. Note that in our case, the interval  $[\lceil an/2 \rceil, \lceil a/2 \rceil n]$  from the lower bound given by Lemma 5.5 equals to the interval  $[bn + n/2, bn + n]$ . However, since  $\gamma < n(b + 1)$ , the right endpoint of the interval can be safely excluded from consideration. We next observe that  $\nu$  is decreasing for

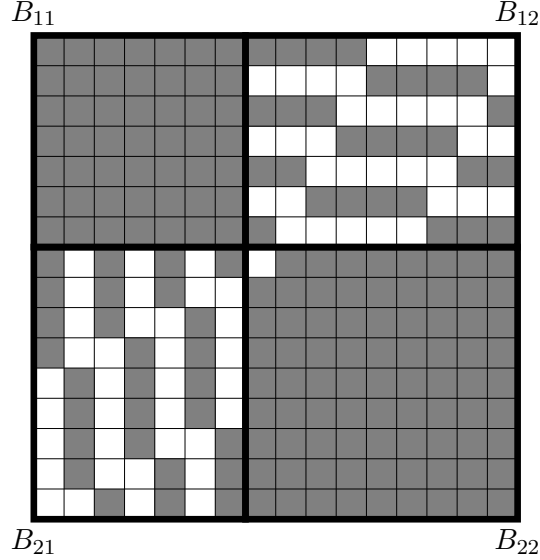


Figure 5.1: An example of the construction of  $B$  for  $m = 8$ ,  $k = 1$ ,  $c = 3$ ,  $d = 4$ , which correspond to  $b = 11$  and  $a = 23$ , the largest one with  $m = 8$  for which  $k = 1$ . The first column of  $B_{12}$  is the only one with 4 1s, so the corresponding diagonal element of  $B_{22}$  switches to 0. Points in the  $a$ -dominating set are represented by dark squares.

$s \in [bn + n/2, bn + n - 1]$ . Indeed, on this interval  $\lfloor s/n \rfloor = b$  and so  $\nu$  is a linear function with coefficient  $4b+2-a-2n = a-2n < 0$ . We therefore only need to check that  $\nu(bn+n/2+k-1) > 0$ . After some algebra, we get

$$\nu(bn + n/2 + k - 1) = \frac{5}{2}n - a - (2n - a)k = (2n - a) \left( 1 + \frac{n}{2(2n - a)} - k \right) > 0,$$

which establishes (5.6).  $\square$

*Proof of Theorem 1.3 when  $a > n$ , and both  $a$  and  $n$  are odd.* Now write  $a = 2b + 1$  and  $n = 2m + 1$ . The argument is quite similar to the previous case, but differs in the details, so we go through it again. Now

$$B = \begin{bmatrix} B_{11}^{(m-k) \times (m-k)} & B_{12}^{(m-k) \times (m+k+1)} \\ B_{21}^{(m+k+1) \times (m-k)} & B_{22}^{(m+k+1) \times (m+k+1)} \end{bmatrix},$$

and  $B$  is symmetric, all entries of  $B_{11}$  are 1, the first  $m - k$  rows each have  $b$  1s, and remaining  $m + k + 1$  rows each have  $b + 1$  1s. Thus, the number of 1s is  $(2m + 1)b + m + 1 + k$ , so that again  $k$  is the correction term. Domination again follows from Lemma 5.2. Further, now  $c = b - m - k$ ,  $d = b - m + k$ , and the consistency requirement is

$$(5.10) \quad (m - k)d \geq (m + k + 1)c,$$

which is equivalent to

$$a(2k+1) \leq n(4k+1),$$

and again we can solve the inequality for  $k$  to get the correction term as the smallest  $k$  that satisfies it. The construction is concluded as for even  $n$ , provided

$$(5.11) \quad (m-k)d \leq (m+k+1)(c+1),$$

which is equivalent to

$$b \geq \frac{2k(n-1)}{2k+1} - \frac{1}{2k+1}.$$

By minimality of  $k$ ,

$$(2b+1)(2k-1) > n(4k-3),$$

which gives a lower bound on  $b$ ,

$$b > \frac{4k-3}{2(2k-1)}n - \frac{1}{2},$$

which, after a bit of algebra, shows that (5.11) is satisfied if  $k \geq 2$ . We therefore need to address the case  $k = 1$  separately. First, note that, since  $d = c + 2k$ , for  $k = 1$  the inequality (5.11) is satisfied if we replace the factor  $(c+1)$  by  $(c+2)$ . It follows that each column has at most  $c+2$  1s, and by construction in Lemma 4.1, when a column does have  $c+2$  1s, all columns have at least  $c+1$  1s. Suppose first that there is a single column (of  $B_{12}$ ) with  $c+2$  1s. Then, we may assume it is the first one and we make the  $2 \times 2$  block in the upper left corner of  $B_{22}$  to be

$$\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}$$

and put 0s on all diagonals of  $B_{22}$  outside of this block. Suppose now that the number of columns with  $c+2$  1s is  $\ell \geq 2$ . Then, the  $\ell \times \ell$  block in the upper corner of  $B_{22}$  is a matrix with exactly  $\ell - 2$  1s in its every row and column. Such a matrix is the  $2 \times 2$  matrix will all entries equal to 0 if  $\ell = 2$ , while for  $\ell \geq 3$ , we can take the matrix of 1s minus the adjacency matrix of the cycle graph on  $\ell$  vertices. The diagonals of  $B_{22}$  outside of this block are switched to 0. This again results in the matrix  $B$  with required row and column counts.

Again, it remains to verify the  $\geq$  part in (5.6). Exactly as for even  $n$ , we verify that  $\nu$  from Lemma 5.5 is decreasing on the same interval so that we only need to check that  $\nu(bn + (n+1)/2 + k - 1) > 0$ . Now this yields

$$\nu(bn + (n+1)/2 + k - 1) = \frac{3}{2}n - \frac{1}{2}a - (2n-a)k = (2n-a) \left( \frac{1}{2} + \frac{n}{2(2n-a)} - k \right) > 0,$$

finishing the proof of (5.6) for odd  $n$ . □

## 6 A 3-approximation to Young domination

The first step in our proof of Theorem 1.4 is to compare  $\gamma^L$  to one of its enhanced versions; another version is used in Section 7. In this comparison, we convert points in the initial set  $A$

used in the standard dynamics into enhancements as indicated by Lemma 2.5. The following definition accounts for the fact that the resulting enhanced dynamics may not cover points in  $A$ .

$$\begin{aligned} \widehat{\gamma}^L = \widehat{\gamma}^L(\mathcal{Z}, m, n) = \min\{|\vec{r}| + |\vec{c}| + |R_{m,n} \setminus \mathcal{Y}| : \vec{r} \in \mathbb{Z}_+^m, \vec{c} \in \mathbb{Z}_+^n \text{ are nonincreasing,} \\ \mathcal{Y} \subseteq R_{m,n} \text{ is a Young diagram, and,} \\ \text{for } \mathcal{T} \text{ given by } (\mathcal{Z}, \vec{r}, \vec{c}), \mathcal{T}^L(\emptyset) = \mathcal{Y}\}. \end{aligned}$$

The inequalities in the next lemma follow from Lemma 2.5 and a repeated application of Lemma 2.6.

**Lemma 6.1.** *For any zero-set  $\mathcal{Z}$  and positive integers  $m, n$ ,*

$$\frac{1}{3} \widehat{\gamma}^L \leq \gamma^L \leq L \widehat{\gamma}^L.$$

*Proof.* Fix a smallest set  $A \subseteq R_{m,n}$  that occupies  $R_{m,n}$  in  $L$  steps and order the rows and columns of  $R_{m,n}$  so that the row sums  $\vec{r} = (r_0, r_1, \dots)$  and column sums  $\vec{c} = (c_0, c_1, \dots)$  of  $A$  are non-increasing. Starting from  $\emptyset$ , the dynamics given by  $(\mathcal{Z}, \vec{r}, \vec{c})$  occupies in  $L$  steps a Young diagram  $\mathcal{Y} \subseteq R_{m,n}$  (by Lemma 2.4) such that  $R_{m,n} \setminus \mathcal{Y} \subseteq A$  (by Lemma 2.5). Then  $\widehat{\gamma}^L \leq |\vec{r}| + |\vec{c}| + |R_{m,n} \setminus \mathcal{Y}| \leq 3|A| = 3\gamma^L$ , which proves the left inequality.

To prove the right inequality, assume that nonincreasing  $\vec{r}, \vec{c}$ , with  $\vec{r} \in \mathbb{Z}_+^m, \vec{c} \in \mathbb{Z}_+^n$ , define, together with  $\mathcal{Z}$ , the enhanced dynamics  $\widehat{\mathcal{T}}$ , and thus also give  $\mathcal{Y} = \widehat{\mathcal{T}}^L(\emptyset)$ . Assume also that these enhancements realize  $\widehat{\gamma}^L$ . Let  $\mathcal{T}$  be the regular dynamics given by  $\mathcal{Z}$ .

Recall the enhancement operation defined on sets before Lemma 2.6. Inductively define the following sets:  $B_0 = \widehat{\emptyset}$ ,  $B_k = \widehat{\mathcal{T}(B_{k-1})}$ , for all  $k = 1, \dots, L-1$ . We first prove by induction that

$$(6.1) \quad \widehat{\mathcal{T}}^k(\emptyset) \subseteq \mathcal{T}(B_{k-1}),$$

for  $k = 1, \dots, L$ . By Lemma 2.6, this holds for  $k = 1$ . Assume now  $2 \leq k \leq L$  and that (6.1) holds for  $k-1$ . Then

$$\widehat{\mathcal{T}}^k(\emptyset) = \widehat{\mathcal{T}}(\widehat{\mathcal{T}}^{k-1}(\emptyset)) \subseteq \widehat{\mathcal{T}}(\mathcal{T}(B_{k-2})) \subseteq \mathcal{T}(\widehat{\mathcal{T}(B_{k-2})}) = \mathcal{T}(B_{k-1}),$$

where we used, in order, the definition of  $\widehat{\mathcal{T}}^k$ , the induction hypothesis and the monotonicity of  $\widehat{\mathcal{T}}$ , Lemma 2.6, and the definition of  $B_k$ . This establishes (6.1).

Define

$$A = (R_{m,n} \setminus \mathcal{Y}) \cup B_0 \cup \bigcup_{k=1}^{L-1} (B_k \setminus \mathcal{T}(B_{k-1})).$$

We now prove by induction that

$$(6.2) \quad \mathcal{T}(B_{k-1}) \subseteq \mathcal{T}^k(A),$$

for  $k = 1, \dots, L$ . For  $k = 1$  this is true as  $B_0 \subseteq A$  and  $\mathcal{T}$  is monotone. Assume now that  $k \geq 2$  and that (6.2) is true for  $k - 1$ , i.e., that  $\mathcal{T}(B_{k-2}) \subseteq \mathcal{T}^{k-1}(A)$ . As we have  $B_{k-1} \setminus \mathcal{T}(B_{k-2}) \subseteq A \subseteq \mathcal{T}^{k-1}(A)$ , it follows that  $B_{k-1} \subseteq \mathcal{T}^{k-1}(A)$  and so  $\mathcal{T}(B_{k-1}) \subseteq \mathcal{T}^k(A)$ , establishing (6.2).

It immediately follows from (6.1) and (6.2) that  $\mathcal{Y} = \widehat{\mathcal{T}}^L(\emptyset) \subseteq \mathcal{T}^L(A)$ . Clearly,

$$R_{m,n} \setminus \mathcal{Y} \subseteq A \subseteq \mathcal{T}^L(A).$$

We thus conclude that  $\mathcal{T}^L(A) = R_{m,n}$  and, hence,  $\gamma^L \leq |A|$ . Finally, we estimate the size of  $A$ . Observe that  $|B_0| \leq |\vec{r}| + |\vec{c}|$  and  $|B_k \setminus \mathcal{T}(B_{k-1})| \leq |\vec{r}| + |\vec{c}|$  for all  $k = 1, \dots, L - 1$ . It follows that

$$|A| \leq |R_{m,n} \setminus \mathcal{Y}| + L(|\vec{r}| + |\vec{c}|) \leq L\widehat{\gamma}^L,$$

and therefore  $\gamma^L \leq |A| \leq L\widehat{\gamma}^L$ , as desired.  $\square$

**Theorem 6.2.** *There exists an algorithm polynomial in  $n$  that takes as input  $m, n \in \mathbb{N}$  with  $m \leq n$  and a zero-set  $\mathcal{Z}$  with  $\mathcal{Z} \subseteq R_{m,n}$ , and computes the value of  $\widehat{\gamma} = \widehat{\gamma}(\mathcal{Z}, m, n)$ .*

*Proof.* In this proof, the enhancement vectors will always be assumed to be nonincreasing.

We will show how to compute  $\widehat{\gamma}^1$  in polynomial time using a dynamic programming approach. In order to explain it, we fix some notation and preliminary observations. Consider a given zero-set  $\mathcal{Z}$  and  $m, n$  with  $m \leq n$  and  $\mathcal{Z} \subseteq R_{m,n}$ . The set  $\mathcal{Z}$  has a non-empty set of concave corners  $\{z^0, z^1, \dots, z^p\}$ , for some  $p \geq 0$ ; we may assume that they are ordered so that, writing  $z^i = (a_i, b_{p-i})$  for all  $i = 0, 1, \dots, p$ , we have  $a_0 = b_0 = 0$ ,  $a_1 < \dots < a_p$ , and  $b_1 < \dots < b_p$  (see Fig. 6.1).

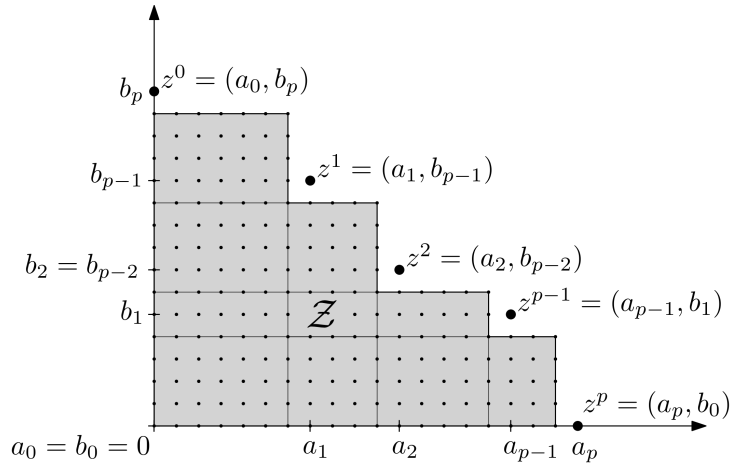


Figure 6.1: A zero-set  $\mathcal{Z}$  and its concave corners

Note that any pair of enhancements  $\vec{r} \in \mathbb{Z}_+^m$  and  $\vec{c} \in \mathbb{Z}_+^n$  defines a feasible solution for  $\widehat{\gamma}^1$  by taking  $\mathcal{Y} = \mathcal{T}(\emptyset)$  for  $\mathcal{T}$  given by  $(\mathcal{Z}, \vec{r}, \vec{c})$ . Indeed, it suffices to apply Lemma 2.4 to infer that  $\mathcal{Y}$  is a Young diagram.

Feasible solutions for  $\hat{\gamma}^1$  can thus be identified with enhancement pairs  $(\vec{r}, \vec{c})$ . Any such pair defines a set of points  $P_{\vec{r}, \vec{c}} := \{(r_u, c_v) : 0 \leq u \leq m-1, 0 \leq v \leq n-1\}$  in  $\mathbb{Z}_+^2$ . Let  $Q_{\vec{r}, \vec{c}} = P_{\vec{r}, \vec{c}} \setminus \mathcal{Z}$ . We claim that in every optimal solution  $(\vec{r}, \vec{c})$  for  $\hat{\gamma}^1$ , the set of minimal points in  $Q_{\vec{r}, \vec{c}}$ , under the natural partial order on  $\mathbb{Z}_+^2$ , is a subset of the set  $\{z^0, \dots, z^p\}$  of concave corners of  $\mathcal{Z}$ . Indeed, suppose this is not the case and that there exists an optimal solution  $(\vec{r}, \vec{c})$  for  $\hat{\gamma}^1$  such that some minimal point, say  $(r_s, c_t)$ , of  $Q_{\vec{r}, \vec{c}}$  is not a concave corner of  $\mathcal{Z}$ . Then, there is a concave corner  $(a_q, b_{p-q})$  that is strictly below  $(r_s, c_t)$ , say  $a_q < r_s$  and  $b_{p-q} \leq c_t$ . By optimality and the definition of  $\hat{\gamma}^1$ , we have  $\hat{\gamma}^1(\mathcal{Z}, m, n) = |\vec{r}| + |\vec{c}| + mn - |\{(u, v) \in R_{m,n} : (r_u, c_v) \notin \mathcal{Z}\}|$ . Since the condition  $(r_u, c_v) \notin \mathcal{Z}$  is equivalent to the condition that  $(a_j, b_{p-j}) \leq (r_u, c_v)$  for some  $j \in [0, p]$  (cf. Fig. 6.1), replacing  $\vec{r}$  with row enhancement  $\vec{r}'$  defined by

$$r'_u = \begin{cases} a_q, & \text{if } u = s; \\ r_u, & \text{otherwise.} \end{cases}$$

results in a feasible solution  $(\vec{r}', \vec{c})$  for  $\hat{\gamma}^1$  such that  $(r_u, c_v) \notin \mathcal{Z}$  if and only if  $(r'_u, c_v) \notin \mathcal{Z}$ . This implies that the objective function value at  $(\vec{r}', \vec{c})$  is smaller than the one at  $(\vec{r}, \vec{c})$ , contradicting the optimality of  $(\vec{r}, \vec{c})$ .

The above observation implies that in order to compute the value of  $\hat{\gamma}^1$ , we may restrict our attention to enhancement pairs  $(\vec{r}, \vec{c})$  such that  $r_u \in \{a_0, \dots, a_p\}$  for all  $u \in [0, m-1]$  and  $c_v \in \{b_0, \dots, b_p\}$  for all  $v \in [0, n-1]$ . We call such enhancement pairs *tame*. Any tame enhancement pair  $(\vec{r}, \vec{c})$  can be represented by a pair of vectors  $\vec{x} = (x_0, \dots, x_p) \in \mathbb{Z}_+^{p+1}$  and  $\vec{y} = (y_0, \dots, y_p) \in \mathbb{Z}_+^{p+1}$  given by

$$\begin{aligned} x_i &= |\{u : 0 \leq u \leq m-1, r_u = a_i\}|, \\ y_j &= |\{v : 0 \leq v \leq n-1, c_v = b_{p-j}\}|. \end{aligned}$$

Clearly, we have  $|\vec{x}| = m$  and  $|\vec{y}| = n$ , and, since enhancements are assumed to be nonincreasing sequences, any pair of vectors  $(\vec{x}, \vec{y}) \in (\mathbb{Z}_+^{p+1})^2$  with  $|\vec{x}| = m$  and  $|\vec{y}| = n$  represents a unique tame enhancement pair.

Given such a vector pair  $(\vec{x}, \vec{y})$ , the tame enhancement pair  $(\vec{r}, \vec{c})$  represented by  $(\vec{x}, \vec{y})$  satisfies  $|\vec{r}| = \sum_{i=0}^p a_i x_i$  and  $|\vec{c}| = \sum_{j=0}^p b_{p-j} y_j$ . Moreover, letting  $\mathcal{Y} = \mathcal{T}(\emptyset)$  for  $\mathcal{T}$  given by  $(\mathcal{Z}, \vec{r}, \vec{c})$ , we have:

$$\begin{aligned} |R_{m,n} \setminus \mathcal{Y}| &= |\{(u, v) \in R_{m,n} : (r_u, c_v) \in \mathcal{Z}\}| \quad (\text{by the definition of } \mathcal{Y}) \\ &= \sum \{x_i y_j : 0 \leq i \leq p, 0 \leq j \leq p, (a_i, b_{p-j}) \in \mathcal{Z}\} \quad (\text{by the definition of } (\vec{x}, \vec{y})) \\ &= \sum_{0 \leq i < j \leq p} x_i y_j \quad (\text{since } (a_i, b_{p-j}) \in \mathcal{Z} \text{ if and only if } i < j, \text{ cf. Fig. 6.1}). \end{aligned}$$

Therefore, we can redefine  $\hat{\gamma}^1(\mathcal{Z}, m, n)$  equivalently in terms of  $(\vec{x}, \vec{y})$  as follows:

$$(6.3) \quad \hat{\gamma}^1(\mathcal{Z}, m, n) = \min \left\{ \sum_{i=0}^p a_i x_i + \sum_{j=0}^p b_{p-j} y_j + \sum_{0 \leq i < j \leq p} x_i y_j : \vec{x}, \vec{y} \in \mathbb{Z}_+^{p+1}, |\vec{x}| = m, |\vec{y}| = n \right\}.$$

We now have everything ready to explain our algorithm for computing  $\hat{\gamma}^1(\mathcal{Z}, m, n)$ . Given  $\mathcal{Z}$ ,  $m$ ,  $n$ , the  $a_i$ s and the  $b_j$ s as above, we will use a dynamic programming approach to compute optimal values to polynomially many instances of a problem generalizing  $\hat{\gamma}^1$ . To define the set of inputs for this more general problem, we let, for  $q \in [0, p]$ ,

$$\mathcal{V}_q = \{(k, \ell, C) \in \mathbb{Z}_+^3 : k \in [0, m], \ell \in [0, n], C \in [0, (p-q)n]\}.$$

The task is then to compute, for all  $q \in [0, p]$ , and all  $(k, \ell, C) \in \mathcal{V}_q$ , the value  $f_q(k, \ell, C)$  given by the following expression:

$$f_q(k, \ell, C) = \min \left\{ \sum_{i=0}^q (a_i + C)x_i + \sum_{j=0}^q b_{p-j}y_j + \sum_{0 \leq i < j \leq q} x_i y_j : \vec{x}, \vec{y} \in \mathbb{Z}_+^{q+1}, |\vec{x}| = k, |\vec{y}| = \ell \right\}.$$

Computing all the  $f_q(k, \ell, C)$  values will suffice, since the value of  $\hat{\gamma}^1(\mathcal{Z}, m, n)$  is given by  $\hat{\gamma}^1(\mathcal{Z}, m, n) = f_p(m, n, 0)$ .

The values  $f_q(k, \ell, C)$  for all  $q \in [0, p]$  and  $(k, \ell, C) \in \mathcal{V}_q$  can be computed recursively as follows.

*Step 1.* Compute  $f_0(k, \ell, C)$  for all  $(k, \ell, C) \in \mathcal{V}_0$ .

For all  $(k, \ell, C) \in \mathcal{V}_0$ , we have

$$f_0(k, \ell, C) = \min \{(a_0 + C)x_0 + b_p y_0 : \vec{x}, \vec{y} \in \mathbb{Z}_+, |\vec{x}| = k, |\vec{y}| = \ell\}.$$

As the only feasible solution to the above problem is  $(\vec{x}, \vec{y}) = (k, \ell)$ , we have  $f_0(k, \ell, C) = (a_0 + C)k + b_p \ell$ .

*Step 2.* Suppose that  $q \geq 1$  and we that we have already computed the values  $f_{q-1}(k', \ell', C')$  for all  $(k', \ell', C') \in \mathcal{V}_{q-1}$ . Then we compute all the values  $f_q(k, \ell, C)$  for  $(k, \ell, C) \in \mathcal{V}_q$ .

The values of  $k, \ell, C, q$  are considered fixed throughout Step 2, so the dependency of various quantities on them will be suppressed from the notation. We partition the set  $\mathcal{D}$  of feasible solutions  $(\vec{x}, \vec{y})$  for  $f_q(k, \ell, C)$  according to the values of the last coordinates,  $x_q$  and  $y_q$ : for  $A \in [0, k]$  and  $B \in [0, \ell]$ , we set  $\mathcal{D}_{A,B} = \{(\vec{x}, \vec{y}) \in \mathcal{D} : x_q = A, y_q = B\}$ . Then

$$(6.4) \quad f_q(k, \ell, C) = \min_{\substack{0 \leq A \leq k \\ 0 \leq B \leq \ell}} g(A, B)$$

where

$$g(A, B) = \min \left\{ \sum_{i=0}^q (a_i + C)x_i + \sum_{j=0}^q b_{p-j}y_j + \sum_{0 \leq i < j \leq q} x_i y_j : (\vec{x}, \vec{y}) \in \mathcal{D}_{A,B} \right\}.$$

By (6.4), it follows that in order to compute the value  $f_q(k, \ell, C)$  it suffices to compute the  $\mathcal{O}(|k||\ell|)$  values  $g(A, B)$ .



Since every feasible solution  $(\vec{x}, \vec{y}) \in \mathcal{D}_{A,B}$  for  $g(A, B)$  satisfies  $x_q = A$  and  $y_q = B$ , the objective function value simplifies to

$$\begin{aligned} & \sum_{i=0}^{q-1} (a_i + C)x_i + (a_q + C) \cdot A + \sum_{j=0}^{q-1} b_{p-j}y_j + b_{p-q} \cdot B + \sum_{0 \leq i < j \leq q-1} x_i y_j + \left( \sum_{i=0}^{q-1} x_i \right) \cdot B \\ &= \left( \sum_{i=0}^{q-1} (a_i + B + C)x_i + \sum_{j=0}^{q-1} b_{p-j}y_j + \sum_{0 \leq i < j \leq q-1} x_i y_j \right) + ((a_q + C) \cdot A + b_{p-q} \cdot B). \end{aligned}$$

Therefore, by considering the vectors  $\vec{x}'$  and  $\vec{y}'$  obtained from  $\vec{x}$  and  $\vec{y}$  by removing the last coordinates,  $\vec{x}' = (x_0, \dots, x_{q-1})$  and  $\vec{y}' = (y_0, \dots, y_{q-1})$ , we infer that

$$\begin{aligned} & g(A, B) \\ &= ((a_q + C) \cdot A + b_{p-q} \cdot B) + \min \left\{ \sum_{i=0}^{q-1} (a_i + B + C)x'_i + \sum_{j=0}^{q-1} b_{p-j}y'_j + \sum_{0 \leq i < j \leq q-1} x'_i y'_j : \right. \\ & \quad \left. \vec{x}', \vec{y}' \in \mathbb{Z}_+^{q-1}, |\vec{x}'| = k - A, |\vec{y}'| = \ell - B \right\}, \end{aligned}$$

which simplifies further to

$$(6.5) \quad g(A, B) = ((a_q + C) \cdot A + b_{p-q} \cdot B) + f_{q-1}(k - A, \ell - B, B + C).$$

Note that since  $C \leq (p - q)n$  and  $B \leq \ell \leq n$ , we have  $B + C \leq (p - q + 1)n$ , hence, by our assumption, the value of  $f_{q-1}(k - A, \ell - B, B + C)$  was already computed. Using (6.4) and (6.5), this shows that the value  $f_q(k, \ell, C)$  can be computed in time  $\mathcal{O}(|k||\ell|)$  from the previously computed values.

The number of values  $f_q(k, \ell, C)$  for all  $q \in [0, p]$  and  $(k, \ell, C) \in \mathcal{V}_q$  is  $\mathcal{O}(mn^2p^2)$  and they can therefore all be computed in time  $\mathcal{O}(m^2n^3p^2)$ . Recalling that  $\hat{\gamma}^1(\mathcal{Z}, m, n) = f_p(m, n, 0)$ , this completes the proof.  $\square$

*Proof of Theorem 1.4.* The statement follows immediately from Lemma 6.1 and Theorem 6.2.  $\square$

## 7 A constant approximation to Young domination with fixed latency

In this section we make use of a simpler version of enhanced domination numbers, namely the minimum norm of enhancements such that the empty set spans in  $L$  steps. While it gives a less accurate approximation, it can be used to handle arbitrary latency. Let

$$\bar{\gamma}^L = \bar{\gamma}^L(\mathcal{Z}, m, n) = \min\{|\vec{r}| + |\vec{c}| : \text{for } \mathcal{T} \text{ given by } (\mathcal{Z}, \vec{r}, \vec{c}), \mathcal{T}^L(\emptyset) = R_{m,n}\}.$$

**Lemma 7.1.** *For every  $L < \infty$  there is a constant  $\delta > 0$  such that*

$$\delta \bar{\gamma}^L \leq \hat{\gamma}^L \leq \bar{\gamma}^L.$$

*Proof.* Without loss of generality we may assume that  $m \leq n$ .

The right inequality is immediate from the definitions. The proof of the left inequality is divided into three cases.

*Case 1:*  $m \leq 100$ .

Let  $a$  be the length of the longest row of  $\mathcal{Z}$ . Then, by Lemma 6.1,

$$L \hat{\gamma}^L \geq \gamma^L \geq \gamma^\infty \geq a,$$

as a set  $A$  consisting of any  $a - 1$  fully occupied columns is a fixed point:  $\mathcal{T}(A) = A$ . Also,  $\bar{\gamma}^L \leq \bar{\gamma}^1 \leq ma \leq 100a$ , as constant  $\vec{r}$  with  $m$  components equal to  $a$  spans in one step. Therefore  $\bar{\gamma}^L \leq 100L \hat{\gamma}^L$ .

For the remaining two cases, choose  $s$  to be the minimal positive integer such that  $(s, \lfloor ms/n \rfloor) \notin \mathcal{Z}$ .

*Case 2:*  $m > 100$ ,  $s \leq n/8$ .

The constant enhancements  $\vec{r} = (s, \dots, s) \in \mathbb{Z}_+^m$ ,  $\vec{c} = (\lfloor ms/n \rfloor, \dots, \lfloor ms/n \rfloor) \in \mathbb{Z}_+^n$  span in one step, so we have  $\mathcal{Y} = R_{m,n}$ , and therefore

$$\hat{\gamma}^1 \leq ms + n \lfloor ms/n \rfloor \leq 2ms,$$

and so we may restrict to  $\mathcal{Y}$  with  $|R_{m,n} \setminus \mathcal{Y}| \leq 2ms$  in our objective function for  $\hat{\gamma}^L$ . Assume that some enhancement pair  $(\vec{r}, \vec{c})$  together with a Young diagram  $\mathcal{Y}$  that satisfies this restriction, realize  $\hat{\gamma}^L$ . Then consider the largest  $t$  such that

$$(m - \lceil mt/n \rceil, n - t) \in R_{m,n} \setminus \mathcal{Y}.$$

Then

$$t \cdot \lceil mt/n \rceil \leq |R_{m,n} \setminus \mathcal{Y}| \leq 2ms$$

and so  $t^2 \leq 2ns \leq n^2/4$  and  $t \leq n/2$ . As

$$(m - \lceil m(t+1)/n \rceil, n - t - 1) \in \mathcal{Y},$$

$\mathcal{Y}$  includes a rectangle whose number of columns is  $n - t \geq n/2$  and the number of rows at least  $m - mt/n - 1 \geq m/2 - 1 \geq m/3$ .

It follows that the dynamics with enhancements  $\vec{r}$  and  $\vec{c}$  occupy in time  $L$  a rectangle with  $\lceil m/3 \rceil$  rows and  $\lceil n/2 \rceil$  columns. By permuting the enhancements we may assume this occupied rectangle is placed anywhere in our universe  $R_{m,n}$ . Clearly, we can cover  $R_{m,n}$  with six properly placed such rectangles, each with its own enhancement vector. Summing the six row and six column enhancement vectors therefore produces row and column enhancement vectors  $\vec{r}_0$  and  $\vec{c}_0$  that occupy the entire  $R_{m,n}$  at time  $L$ , and  $|\vec{r}_0| \leq 6|\vec{r}|$  and  $|\vec{c}_0| \leq 6|\vec{c}|$ . Therefore, in this case,

$$\bar{\gamma}^L \leq |\vec{r}_0| + |\vec{c}_0| \leq 6(|\vec{r}| + |\vec{c}|) \leq 6\hat{\gamma}^L.$$

Case 3:  $m > 100$ ,  $s > n/8$ .

Using Lemma 6.1 and [GSS, Theorems 1 and 2],

$$\begin{aligned}
L\widehat{\gamma}^L &\geq \gamma^L \geq \gamma^\infty \geq |\mathcal{Z}|/4 \\
&\geq (s-1)(m(s-1)/n-1)/4 \\
&\geq (n/9) \cdot (m/9-1)/4 \\
&\geq mn/360
\end{aligned}$$

as  $n \geq m > 100$ . On the other hand, we trivially have  $\bar{\gamma}^L \leq \bar{\gamma}^1 \leq mn$ .

From the three cases, we see that we can take  $\delta = 1/(360L)$ . □

A polynomial algorithm for  $\bar{\gamma}^L$  will be a consequence of the following lemma.

**Lemma 7.2.** *For any  $L \geq 1$ , every minimal  $\vec{r}$  and  $\vec{c}$  for  $\bar{\gamma}^L$  each have at most  $2^L$  different values.*

*Proof.* Assume that  $\vec{r}$  and  $\vec{c}$  are some minimal enhancements for  $\bar{\gamma}^L$  and that they are both nonincreasing. These determine the enhanced dynamics  $\mathcal{T}$  and will be fixed for the rest of the proof.

A Young diagram  $\mathcal{Y}$  is an *trigger* for a point  $(a, b) \in R_{m,n}$ , if one of the following holds:

- $\mathcal{Y} = R_{a+1, b+1}$ ; or
- $\mathcal{Y}$  has exactly two convex corners: one at  $(a, b_1)$  for  $0 \leq b_1 < b$  and one at  $(a_1, b)$  for  $0 \leq a_1 < a$  and  $(r_b + a_1 + 1, c_a + b_1 + 1)$  is a concave corner of  $\mathcal{Z}$ ; or
- $\mathcal{Y}$  has exactly one convex corner at  $(a_1, b)$  for  $0 \leq a_1 < a$  and  $(r_b + a_1 + 1, c_a)$  is a concave corner of  $\mathcal{Z}$ ; or
- $\mathcal{Y}$  has exactly one convex corner at  $(a, b_1)$  for  $0 \leq b_1 < b$  and  $(r_b, c_a + b_1 + 1)$  is a concave corner of  $\mathcal{Z}$ ; or
- $\mathcal{Y} = \emptyset$  and  $(r_b, c_a)$  is a concave corner of  $\mathcal{Z}$ .

Observe that, if  $\mathcal{Y}$  is a trigger for  $(a, b)$ , then  $(a, b) \in \mathcal{T}(\mathcal{Y})$ . Conversely, if  $(a, b) \in \mathcal{T}(\mathcal{Y}')$ , for some Young diagram  $\mathcal{Y}'$ , then there is a trigger  $\mathcal{Y} \subseteq \mathcal{Y}'$  for  $(a, b)$ . Indeed, the five items in the definition of the trigger correspond to the cases when  $(a, b) \in \mathcal{Y}'$ , or  $(a, b) \in \mathcal{T}(\mathcal{Y}') \setminus \mathcal{Y}'$  and uses, in addition to the enhancements, both  $\mathcal{N}^r(a, b) \cap \mathcal{Y}'$  and  $\mathcal{N}^c(a, b) \cap \mathcal{Y}'$ , one of them, or neither, to become occupied. See Figure 7.1 for an example.

Note also that any convex corner of  $\mathcal{Y}$  shares at least one coordinate with  $(a, b)$ , and when  $\mathcal{Y}$  has two corners, each shares a different coordinate with  $(a, b)$ .

We will define, by backward recursion, Young diagrams  $\mathcal{Y}_L \supseteq \mathcal{Y}_{L-1} \supseteq \dots \supseteq \mathcal{Y}_0 = \emptyset$  such that  $\mathcal{Y}_i \subseteq \mathcal{T}^i(\emptyset)$  for all  $i = 0, 1, \dots, L$ . We start with  $\mathcal{Y}_L = R_{m,n}$ . The set  $\mathcal{Y}'_{L-1} = \mathcal{T}^{L-1}(\emptyset)$  of

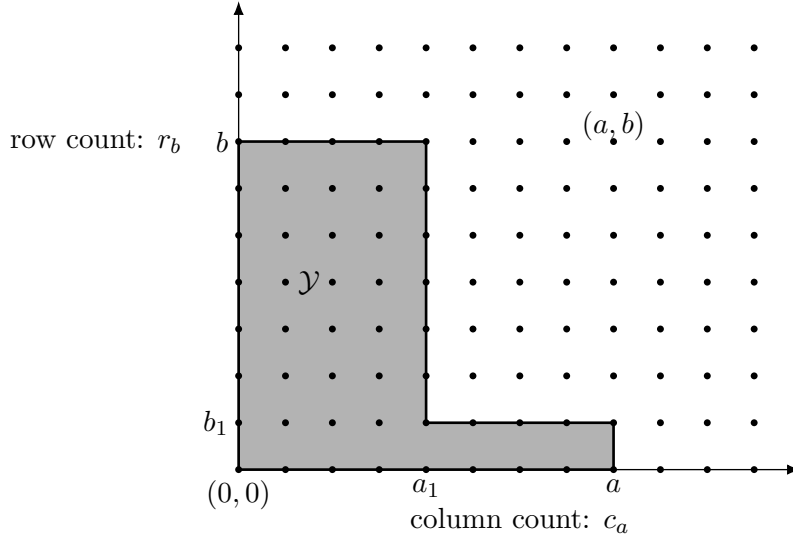


Figure 7.1: A trigger  $\mathcal{Y}$  for the point  $(a, b) = (8, 7)$ . For this example, we assume that  $(r_7 + 5, c_8 + 2)$  is a concave corner of  $\mathcal{Z}$ . Vertices that touch the shaded area comprise  $\mathcal{Y}$ .

occupied sites of the enhanced dynamics at time  $L - 1$  is a Young diagram (Lemma 2.4). As the enhancements span in  $L$  steps,  $(m - 1, n - 1) \in \mathcal{T}(\mathcal{Y}'_{L-1})$  and therefore there exists a trigger  $\mathcal{Y}_{L-1} \subseteq \mathcal{Y}'_{L-1}$  for  $(m - 1, n - 1)$  (arbitrarily chosen in case it is not unique).

In general, assume that we have chosen  $\mathcal{Y}_i$ , for some  $i \in \{1, \dots, L\}$ . Let  $\mathcal{Y}'_{i-1} = \mathcal{T}^{i-1}(\emptyset)$ . By the induction hypothesis,  $\mathcal{Y}_i \subseteq \mathcal{T}^i(\emptyset) = \mathcal{T}(\mathcal{Y}'_{i-1})$ . In particular, for any convex corner  $z \in \mathcal{Y}_i$ , we have  $z \in \mathcal{T}(\mathcal{Y}'_{i-1})$  and therefore there exists a trigger  $\mathcal{Y}_{i-1,z} \subseteq \mathcal{Y}'_{i-1}$  for  $z$ . Now let

$$(7.1) \quad \mathcal{Y}_{i-1} = \bigcup_{z \text{ convex corner of } \mathcal{Y}_i} \mathcal{Y}_{i-1,z}.$$

Then  $\mathcal{Y}_{i-1} \subseteq \mathcal{Y}_i$  and  $\mathcal{Y}_{i-1} \subseteq \mathcal{T}^{i-1}(\emptyset)$ , which completes the inductive step in the construction. Note that  $\mathcal{Y}_0 \subseteq \mathcal{T}^0(\emptyset) = \emptyset$ .

For all  $i = 0, 1, \dots, L$ , let  $Q_i$  be the set of all convex corners of  $\mathcal{Y}_i$  and  $Q = \bigcup_{i=0}^L Q_i$ . Let  $\pi_x$  and  $\pi_y$  be the projections onto the  $x$ -axis and the  $y$ -axis, respectively. For all  $j = 0, 1, \dots, L$ , let  $P_j = \bigcup_{i=j}^L \pi_x(Q_i)$ . By definition of a trigger and (7.1),

$$|P_j| \leq 2|P_{j-1}|$$

for every  $i = 1, \dots, L$ , and therefore  $|\pi_x(Q)| \leq 2^L$ . Analogously,  $|\pi_y(Q)| \leq 2^L$ .

Now assume that  $i_1 < \dots < i_A = n - 1$  and  $j_1 < \dots < j_B = m - 1$ ,  $A, B \leq 2^L$  are exactly the elements of  $\pi_x(Q)$  and  $\pi_y(Q)$ , respectively. Assume that  $\vec{r}'$  and  $\vec{c}'$  are any nonincreasing enhancements whose values agree at these indices, that is,  $c'_{i_a} = c_{i_a}$ ,  $a = 1, \dots, A$ , and  $r'_{j_b} = r_{j_b}$ ,  $b = 1, \dots, B$ . Denote the dynamics with these enhancements by  $\widehat{\mathcal{T}}$ . Then we claim that  $\widehat{\mathcal{T}}^L(\emptyset) = R_{m,n}$ .

To verify the claim, we prove by induction on  $i$  that  $\mathcal{Y}_i \subseteq \widehat{\mathcal{T}}^i(\emptyset)$ . This is clearly true for  $i = 0$ , and the induction step follows from the construction of  $\mathcal{Y}_i$ .

As  $\vec{r}$  and  $\vec{c}$  are assumed minimal, the above claim implies that any of their values must equal a value at one of the above indices  $i_a$  or  $j_b$ . Namely, for any  $j = 0, \dots, m-1$ ,  $r_j = r_{j_b}$  for the smallest  $j_b \geq j$  and, for any  $i = 0, \dots, n-1$ ,  $c_i = c_{i_a}$  for the smallest  $i_a \geq i$ . As the number of these indices is at most  $2^L$ , this concludes the proof.  $\square$

**Theorem 7.3.** *Assume  $a, b \in \mathbb{N}$ ,  $2 \leq L < \infty$ , and  $\mathcal{Z} \subseteq R_{a,b} \subseteq R_{m,n}$ . Then  $\bar{\gamma}^L \leq ab$  and  $\bar{\gamma}^L$  can be computed in time  $\mathcal{O}((\min(m, ab) \cdot \min(n, ab))^{2^L+1} (ab)^{2^L})$ .*

*Proof.* First observe that there exist enhancements  $\vec{r}$  and  $\vec{c}$  with  $|\vec{r}| + |\vec{c}| = ab$  that span at time 2: for example, we can let  $r_0 = \dots = r_{b-1} = a$  and make all remaining enhancements 0 (in particular,  $c_j = 0$  for all  $j$ ). Therefore, since  $L \geq 2$ , we obtain that  $\bar{\gamma}^L \leq ab$  and we may restrict our enhancements to those with at most  $ab$  nonzero components.

Let  $m' = \min(m, ab + 1)$ ,  $n' = \min(n, ab + 1)$ . With our restriction, as we can assume that the enhancements are nonincreasing, we only need to choose first  $m'$  row, and first  $n'$  column enhancements. By Lemma 7.2, it suffices to choose at most  $2^L - 1$  dividing points between different row enhancements, and the same for column enhancements. This can be done in  $\mathcal{O}((m'n')^{2^L-1})$  many ways. Next, note that we can replace any row (resp. column) enhancement larger than  $a$  by  $a$  (resp. larger than  $b$  by  $b$ ) without affecting the spanning time. Therefore, for any fixed choices of the  $2^L - 1$  dividing points for row and column indices, the number of candidates for minimal enhancement vectors  $\vec{r}$  and  $\vec{c}$  is  $\mathcal{O}((ab)^{2^L})$ . For every such candidate, we only need to run the dynamics on  $R_{m',n'}$  to verify if it spans in  $L$  steps (by part (3) of Lemma 2.3 in [GPS]). For a single update, we need to compute row and column sums and check against every concave corner of  $\mathcal{Z}$ . Since  $\mathcal{Z}$  has at most  $\min(a, b)$  concave corners, a single update can be computed in time  $\mathcal{O}(m'n'(m' + n' + \min(a, b))) = \mathcal{O}((m'n')^2)$ ; therefore, since  $L$  is fixed,  $L$  steps can be computed in time  $\mathcal{O}((m'n')^2)$ . Multiplying the number of candidates by the verification time for each gives the claimed upper bound.  $\square$

*Proof of Theorem 1.5.* By Theorem 7.3, there exists a constant  $C > 1$  such that for every fixed  $L$ , there exists a  $C$ -approximation algorithm for  $\bar{\gamma}^L$  that runs in time polynomial in  $ab$ . Lemmas 6.1 and 7.1 now finish the proof.  $\square$

## 8 Comparison between different latencies

In the next two lemmas, we assume that  $a$  and  $b$  are the sizes of the longest row and column of  $\mathcal{Z}$ , respectively.

**Lemma 8.1.** *If  $m, n > ab$ , then*

$$\gamma^2(\mathcal{Z}, m, n) = \min(bx + ay - xy),$$

where the minimum is over all concave corners  $(x, y)$  of  $\mathcal{Z}$ .

*Proof.* Assume that  $D$  is a set that realizes  $\gamma^2$ . As  $R_{b,a}$  clearly occupies  $R_{m,n}$  in two steps,  $|D| \leq ab$ . By permuting rows and columns, we may assume that  $D \subseteq [0, ab - 1] \times [0, ab - 1]$ , and therefore the last row and the last column of  $R_{m,n}$  contain no elements of  $D$ . In order for the point  $(m - 1, n - 1)$  to become occupied at time 2, there have to be at time 1 some  $x \in [0, a]$  occupied points in the last row and some  $y \in [0, b]$  occupied points in the last column, with  $(x, y)$  a concave corner of  $\mathcal{Z}$ . Each of the  $x$  points must have at least  $b$  points in  $D$  in its column, while each of the  $y$  points must have at least  $a$  points in its row.

Thus, we may assume that the first  $x$  columns each contain  $b$  points in  $D$ ; then, some  $y$  rows each contain  $a$  points in  $D$ , at least  $a - x$  of them not within the first  $x$  columns. Therefore,  $\gamma^2 = |D| \geq bx + (a - x)y$ .

To show the opposite inequality, take  $(x, y)$  to be the concave corner at which the minimum is achieved, then take

$$D = R_{y,a} \cup R_{b,x},$$

which occupies all points on the first  $x$  columns and first  $y$  rows of  $R_{m,n}$  at time 1 and then the entire  $R_{m,n}$  at time 2.  $\square$

**Lemma 8.2.** *If  $L \geq 2$  and  $m, n > ab$ , then  $\gamma^L(\mathcal{Z}, m, n) = \gamma^L(\mathcal{Z}, \infty, \infty)$ .*

*Proof.* Consider a set  $D$  that realizes  $\gamma^L(\mathcal{Z}, m, n)$ . Then observe that every time a point  $(i, j)$  outside  $[0, ab - 1] \times [0, ab - 1]$  becomes occupied: if  $i \leq ab - 1$ , then the entire  $i$ th row gets occupied at the same time; if  $j \leq ab - 1$ , then the entire  $j$ th column gets occupied at the same time; and, therefore, when both  $i, j \geq ab$ , then the entire  $R_{m,n}$  becomes occupied. As the argument is independent on the exact values of  $m$  and  $n$ , provided that they are large enough, the claim remains true if we extend  $m$  and  $n$  to infinity.  $\square$

*Proof of Theorem 1.6.* The first statement is Lemma 8.2. The second statement follows from [GPS, Theorem 1.1] and [GSS, Theorem 1]. We now address the three cases of the third statement, which can already be achieved with  $m = n$ , so we will assume so for the rest of the proof.

When  $L = 0$  or  $L = 1$ , (1.8) follows by taking any fixed  $\mathcal{Z}$  and large  $n$ , but for concreteness we assume  $\mathcal{Z}$  is  $T_a$  given by (1.1) for even  $a \ll n$ , so that  $\gamma^1(\mathcal{Z}, n, n) = an/2$  by (1.5). Then  $\gamma^0(\mathcal{Z}, n, n) = n^2 \gg an/2$  and  $\gamma^2(\mathcal{Z}, n, n) \leq a^2 \ll an/2$ , where the inequality  $\gamma^2(\mathcal{Z}, n, n) \leq a^2$  follows from the fact that the  $a \times a$  square spans in two steps.

For  $L = 2$ , consider

$$\mathcal{Z} = [0, a - 1]^2 \cup (\{0\} \times [0, b - 1]) \cup ([0, b - 1] \times \{0\}),$$

with  $1 \ll a \ll b \ll \sqrt{n}$ . Then we know from by Lemma 8.1 that  $\gamma^2(\mathcal{Z}, n, n) \geq ab$ . On the other hand, if  $\mathcal{Z}$  is itself the initial set, we get

$$\mathcal{T}^1(\mathcal{Z}) \supseteq ([0, a - 1] \times [0, b - 1]) \cup ([0, b - 1] \times [0, a - 1]),$$

$$\mathcal{T}^2(\mathcal{Z}) \supseteq ([0, a - 1] \times [0, n - 1]) \cup ([0, n - 1] \times [0, a - 1]),$$

and so  $\mathcal{T}^3(\mathcal{Z}) = [0, n-1]^2$ , that is,  $\mathcal{Z}$  is  $\mathcal{Z}$ -dominating with latency 3. Therefore,  $\gamma^3(\mathcal{Z}) \leq |\mathcal{Z}| < a^2 + 2b \ll ab$ .  $\square$

## 9 Open problems and possible further directions

We begin with a few questions that are left open by the present work.

1. It is well-known that computing the domination number of a graph is, in general, NP-hard. This remains true for distance- $L$  domination for any fixed finite  $L$  (see, e.g., [Hen]). Consequently, for any fixed finite  $L$ , the problem of computing  $\gamma^L(T_1, G_1, G_2)$  is NP-hard (where  $T_1 = \{(0, 0)\}$ ), even if one of the two factors  $G_1$  and  $G_2$  is a one-vertex graph. (Note that the problem of computing  $\gamma^\infty(T_1, G_1, G_2)$  is trivial.) In particular, for any fixed finite  $L$ , the problem of computing  $\gamma^L(\mathcal{Z}, G_1, G_2)$  is NP-hard. However, these considerations tell nothing about the special case when both factors are complete, for any  $L$  (finite or infinite).

What is the computational complexity of computing  $\gamma^L(\mathcal{Z}, m, n)$ ?

The complexity of the special case  $L = m = n = \infty$  is the final open problem in [GSS].

2. Recall from [GSS, Theorems 1 and 2] that the value of  $\gamma^\infty$  can be 4-approximated in polynomial time, while for every finite  $L$ , Theorem 1.5 gives a polynomial-time algorithm that approximates  $\gamma^L$  up to a constant factor in polynomial time, where both the approximation ratio and the exponent of the polynomial depend on  $L$ . These results suggest the following questions.

Is there a constant  $\delta > 0$  such that for every  $L$ , there is an algorithm that approximates  $\gamma^L$  up to factor  $\delta$  in polynomial time?

For finite  $L$ , is there an algorithm that approximates  $\gamma^L$  up to a constant factor in polynomial time with powers independent of  $L$ ?

3. Theorem 1.6 showed that for  $L = 0, 1, 2$ , the Young domination number with latency  $L$  is not bounded from above by any constant multiple of the Young domination number with any higher latency. This suggests the following question.

Is (1.8) true for all  $L \geq 0$ ?

There are two natural directions for exploration of Young domination outside of Cartesian products of two graphs. For the first one, observe that our setting works on any graph with two-coloring of edges, i.e., on any ordered pair of two edge-disjoint graphs on the same vertex-set. The second direction is a multidimensional version. The zero-set is now a  $d$ -dimensional Young diagram, i.e., a downward-closed set  $\mathcal{Z} \subseteq \mathbb{Z}_+^d$ , and the dynamics proceeds on any graph with a  $d$ -coloring of edges, for example on a product graph  $G_1 \square \cdots \square G_d$  of  $d$  graphs, with the natural coloring.

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