

# Twin-width and generalized coloring numbers

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## Abstract

In this paper, we prove that a graph  $G$  with no  $K_{s,s}$ -subgraph and twin-width  $d$  has  $r$ -admissibility and  $r$ -coloring numbers bounded from above by an exponential function of  $r$  and that we can construct graphs achieving such a dependency in  $r$ .

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## 1 Introduction

In this paper we consider the twin-width graph parameter, defined by Bonnet, Kim, Thomassé and Watrigant [5] as a generalization of a width invariant for classes of permutations defined by Guillemot and Marx [10]. This parameter was intensively studied recently in the context of many structural and algorithmic questions such as FPT model checking [5], graph enumeration [1], graph coloring [3], matrices and ordered graphs [4], and transductions of permutations [2]. (We postpone the formal definition of twin-width to Section 2.1.)

It is known that a graph class with bounded twin-width excludes some biclique as a subgraph if and only if it has bounded expansion [1]. Recall that a class  $\mathcal{C}$  has *bounded expansion* if, for each integer  $r$  the class of all the minors of graphs of  $\mathcal{C}$  obtained by contracting vertex disjoint connected subgraphs with radius at most  $r$  and deleting some edges and vertices have bounded average degree, which may depend on  $r$ . (We refer the interested reader to [12] for an in-depth study of classes with bounded expansion.) Among the numerous characterizations of classes with bounded expansion, three relate to the *generalized colouring numbers*  $wcol_r$  and  $scol_r$  introduced by Kierstead and Yang [11] and to the  *$r$ -admissibility*  $adm_r$  introduced by Dvořák [7]. Indeed, as proved by Zhu [13], the following are equivalent for a class  $\mathcal{C}$ :

1.  $\mathcal{C}$  has bounded expansion;
2.  $\sup\{wcol_r(G) : G \in \mathcal{C}\} < \infty$  for every integer  $r$ ;
3.  $\sup\{scol_r(G) : G \in \mathcal{C}\} < \infty$  for every integer  $r$ .

Moreover, using the inequality  $\text{adm}_r(G) \leq \text{scol}_r(G) \leq \text{wcol}_r(G) \leq \frac{\text{adm}_r(G)^{r+1}-1}{\text{adm}_r(G)-1}$  (see [7]), we get yet another equivalent property.

4.  $\sup\{\text{adm}_r(G) : G \in \mathcal{C}\} < \infty$  for every integer  $r$ .

One can show [1] that for every integer  $r$  there exists a function  $f_r : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that if  $G$  is a graph with twin-width  $t$  and no  $K_{s,s}$ -subgraph, then we have  $\text{wcol}_r(G) \leq f_r(t, s)$ . Similar bounds also exist for  $\text{scol}_r$  and  $\text{adm}_r$ . However, the proof given in [1] that biclique-free graphs with bounded twin-width have bounded expansion does not indicate how to compute such bounding functions.

In this paper, we prove that a graph  $G$  with no  $K_{s,s}$ -subgraph and twin-width  $d$  has  $\text{adm}_r$ ,  $\text{scol}_r$  and  $\text{wcol}_r$  bounded from above by an exponential function of  $r$ , and that we can construct graphs achieving such a dependency in  $r$ . In particular,  $\text{scol}_r(G) \leq (d^r + 3)s$  (Theorem 2). On the other hand, one can choose  $G$  such that  $\text{scol}_r(G) \geq (\frac{d-4}{8})^r s$  (Corollary 13).

## 2 Definitions and Notations

### 2.1 Twin-width

We define twin-width with the help of *trigraphs*. The notion of trigraphs used in this work is slightly different from the notion used in [5]. Both notions are nevertheless equivalent up to isomorphism. A *trigraph*  $\mathbf{G}$  on a graph  $G = (V, E)$  is a binary structure with two binary relations, the *black adjacency*  $E$  and the *red adjacency*  $R$ , whose domain is a partition of  $V$ , and whose black and red adjacencies are exclusive (that is: no two elements of  $\mathbf{G}$  can be adjacent in both relations). Thus, the elements of  $\mathbf{G}$  are subsets of the vertices of  $G$ . To distinguish the elements of  $\mathbf{G}$  from the vertices of  $G$ , we will call them *nodes* and denote them by capital letters, like  $X, Y, Z$ . The set of nodes of  $\mathbf{G}$  is denoted by  $V(\mathbf{G})$ . The elements of  $E(\mathbf{G})$  and  $R(\mathbf{G})$  are respectively called *black edges* and *red edges*. The *set of neighbours*  $N_{\mathbf{G}}(X)$  of a node  $X$  in a trigraph  $\mathbf{G}$  consists of all the nodes adjacent to  $X$  by a black or red edge; the *set of E-neighbours*  $N_{\mathbf{G}}^E(X)$  consists of all nodes adjacent to  $X$  by a black edge and the *set of R-neighbours*  $N_{\mathbf{G}}^R(X)$  consists of all nodes adjacent to  $X$  by a red edge. A *d-trigraph* is a trigraph  $\mathbf{G}$  with maximum red degree at most  $d$ , i.e.,  $|N_{\mathbf{G}}^R(X)| \leq d$  for all  $X \in V(\mathbf{G})$ .

Let  $\mathbf{G}$  be a trigraph on a graph  $G$  and let  $X$  and  $Y$  be (non-necessarily adjacent) nodes of  $\mathbf{G}$ . We say a trigraph  $\mathbf{G}'$  on  $G$  is obtained from  $\mathbf{G}$  by *contracting*  $X$  and  $Y$  if  $V(\mathbf{G}') = V(\mathbf{G}) \setminus \{X, Y\} \cup \{X \cup Y\}$ ,  $N_{\mathbf{G}'}(X \cup Y) = N_{\mathbf{G}}(X) \cup N_{\mathbf{G}}(Y)$ ,  $N_{\mathbf{G}'}^E(X \cup Y) = N_{\mathbf{G}}^E(X) \cap N_{\mathbf{G}}^E(Y)$  (and  $N_{\mathbf{G}'}^R(X \cup Y) = N_{\mathbf{G}'}(X \cup Y) \setminus N_{\mathbf{G}'}^E(X \cup Y)$ ), and the red and black adjacencies between all other nodes of  $\mathbf{G}'$  are as in  $\mathbf{G}$ .

A *d-contraction sequence* of a graph  $G = (V, E)$  with  $n$  vertices is a sequence  $\mathbf{G}_n, \dots, \mathbf{G}_1$  of  $d$ -trigraphs on  $G$ , where  $\mathbf{G}_n$  is the trigraph isomorphic to  $G$  defined by  $V(\mathbf{G}_n) = \{\{v\} : v \in V\}$ ,  $E(\mathbf{G}_n) = \{\{\{u\}, \{v\}\} : (u, v) \in E(G)\}$ , and  $R(\mathbf{G}_n) = \emptyset$ ,  $\mathbf{G}_1$  is the trigraph with single node  $V$ , and  $\mathbf{G}_i$  is obtained from  $\mathbf{G}_{i+1}$  by performing a single contraction. The minimum  $d$  such that there exists a  $d$ -contraction sequence of a graph  $G$  is the *twin-width* of  $G$ , denoted by  $\text{tw}(G)$ . For a contraction sequence  $\mathbf{G}_n, \dots, \mathbf{G}_1$ , we define the *universe*  $\mathcal{U} = \bigcup_{i=1}^n V(\mathbf{G}_i)$  to be the union of all node sets.

A given contraction sequence  $\mathbf{G}_n, \dots, \mathbf{G}_1$  on a graph  $G = (V, E)$  (with universe  $\mathcal{U}$ ) can also be reversed to  $\mathbf{G}_1, \dots, \mathbf{G}_n$  and seen as an *uncontraction sequence* where we start with a single node (the trigraph  $\mathbf{G}_1$ ) and a node  $Z$  of  $\mathbf{G}_i$  is split into two nodes  $X$  and  $Y$  with no edge, black edge or red edge between them in  $\mathbf{G}_{i+1}$ . With this picture in mind, we define for every  $X \in \mathcal{U}$ , the *birth time*  $\text{bt}(X)$  as the minimum integer  $i$  with  $X \in V(\mathbf{G}_i)$

and the *split time*  $\text{st}(X)$  as the maximum integer  $i$  with  $X \in V(\mathbf{G}_i)$ . Observe that for every  $i \in \{1, \dots, n-1\}$ , there is a unique  $X \in \mathcal{U}$  with  $\text{st}(X) = i$ ; the subsets  $X \in \mathcal{U}$  with  $\text{st}(X) = n$  are the nodes of  $\mathbf{G}_n$ , that is the singletons  $\{v\}$  with  $v \in V(G)$ . If  $X \in \mathcal{U} \setminus \{V\}$ , the *parent* of  $X$  is the minimal set  $Y \in \mathcal{U}$  with  $Y \supsetneq X$ . Conversely, if  $|X| > 1$ , the *children* of  $X$  are the two maximal sets  $Y$  and  $Z$  in  $\mathcal{U}$  with  $Y \subsetneq X$  and  $Z \subsetneq X$ . Note that  $\{Y, Z\}$  is a partition of  $X$  and that  $\text{bt}(Y) = \text{bt}(Z) = \text{st}(X) + 1$ .

## 2.2 Generalized Colouring Numbers and Admissibility

Let  $\Pi(G)$  be the set of all linear orders of the vertices of the graph  $G$ , and let  $L \in \Pi(G)$ . (We denote by  $\leq_L$  the corresponding binary relation for better readability.) Let  $u, v \in V(G)$ , and let  $r$  be a positive integer.

We say that  $u$  is *weakly  $r$ -reachable* from  $v$  with respect to  $L$ , if there exists a path  $P$  of length at most  $r$  between  $u$  and  $v$  such that  $u \leq_L w$  for all vertices  $w$  of  $P$ . Let  $\text{WReach}_r[L, v]$  be the set of vertices that are weakly  $r$ -reachable from  $v$  with respect to  $L$ . Note that  $v \in \text{WReach}_r[L, v]$ .

We say that  $u$  is *strongly  $r$ -reachable* from  $v$  with respect to  $L$ , if there is a path  $P$  of length at most  $r$  connecting  $u$  and  $v$  such that  $u \leq_L v$  and all inner vertices  $w$  of  $P$  satisfy  $v <_L w$ . Let  $\text{Sreach}_r[L, v]$  be the set of vertices that are strongly  $r$ -reachable from  $v$  with respect to  $L$ . Note that again we have  $v \in \text{Sreach}_r[L, v]$ .

The  *$r$ -backconnectivity*  $b_r(L, v)$  of a vertex  $v$  is the maximum number of paths of length at most  $r$  in  $G$  that start in  $v$ , share no other vertices except  $v$ , and end at vertices that lie before  $v$  in the ordering  $L$ .

The *weak  $r$ -colouring number*  $\text{wcol}_r(G)$  of  $G$  is defined as

$$\text{wcol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{WReach}_r[L, v]|,$$

and the *strong  $r$ -colouring number*  $\text{scol}_r(G)$  of  $G$  is defined as

$$\text{scol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{Sreach}_r[L, v]|.$$

The  *$r$ -admissibility* of  $G$  is defined as

$$\text{adm}_r(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} b_r(L, v).$$

## 3 From Strong Colouring to Weak Colouring

It is known that the weak and strong colouring numbers are related by  $\text{scol}_r(G) \leq \text{wcol}_r(G) \leq \text{scol}_r(G)^r$  [11]. However it is possible to improve the upper bound in the case where the strong coloring numbers increase at least at an exponential rate.

► **Lemma 1.** *For every graph  $G$  and every positive integer  $r$  we have*

$$\text{wcol}_r(G) \leq 2^{r-1} \max_{1 \leq k \leq r} \text{scol}_k(G)^{r/k}.$$

**Proof.** Let  $r$  be a positive integer and let  $L$  be a linear order on  $V(G)$  that minimizes

$$\max_{v \in V(G)} |\text{WReach}_r[L, v]|.$$

Let  $u$  be a vertex of  $G$ ,  $v \in \text{WReach}_r[L, u]$ , and consider a path  $P$  certifying that  $v$  is weakly  $r$ -reachable from  $u$ ; in particular  $P$  has length at most  $r$ . Let  $C(r)$  be the set of

all *compositions* of  $r$ , that is of all tuples  $(r_1, \dots, r_k)$  with  $r_i > 0$  (for  $1 \leq i \leq k$ ) and  $\sum_{1 \leq i \leq k} r_i = r$ . A *milestone* of  $P$  is a vertex  $v$  of  $P$  such that all the vertices of  $P$  from  $u$  (included) to  $v$  (excluded) are greater than  $v$ . Let  $v_1, \dots, v_k = v$  be the milestones of  $P$  other than  $u$ , and let  $r_1, \dots, r_{k-1}$  be the lengths of the paths from  $v_0 = u$  to  $v_1, \dots, v_{k-2}$  to  $v_{k-1}$ , and let  $r_k = r - \sum_{i=1}^{k-1} r_i$ , so that  $(r_1, \dots, r_k) \in C(r)$ . The subpath of  $P$  from  $v_{i-1}$  to  $v_i$  witnesses that  $v_i$  is strongly  $r_i$ -reachable from  $v_{i-1}$ . Note that strong  $r_k$ -reachability requires the existence of a witness path of length at most  $r_k$ , hence it is safe to consider  $r_k$  instead of the length of the subpath of  $P$  linking  $v_{k-1}$  and  $v_k$ . We deduce that

$$\text{WReach}_r[L, u] \subseteq \bigcup_{(r_1, \dots, r_k) \in C(r)} \bigcup_{v_1 \in \text{Sreach}_{r_1}[L, u]} \cdots \bigcup_{v_{k-1} \in \text{Sreach}_{r_{k-1}}[L, v_{k-2}]} \text{Sreach}_{r_k}[L, v_{k-1}].$$

(Note that we actually have equality, the reverse inclusion following from the concatenation of paths witnessing  $v_1 \in \text{Sreach}_{r_1}[L, v]$ ,  $\dots$ ,  $u \in \text{Sreach}_{r_k}[L, v_{k-1}]$ .) Thus we have

$$\text{wcol}_r(G) \leq \sum_{(r_1, \dots, r_k) \in C(r)} \prod_{i=1}^k \text{scol}_{r_i}(G).$$

Let  $z = \max_{1 \leq k \leq r} \text{scol}_k(G)^{1/k}$ . Then  $\text{scol}_{r_i}(G) \leq z^{r_i}$ . Thus

$$\text{wcol}_r(G) \leq \sum_{(r_1, \dots, r_k) \in C(r)} \prod_{i=1}^k z^{r_i} = |C(r)| z^r = 2^{r-1} z^r.$$

◀

## 4 Upper bounds

Let  $\text{b}\omega(G)$  denote the maximum integer  $s$  such that  $K_{s,s}$  is a subgraph of  $G$ .

► **Theorem 2.** *For every graph  $G$  and every positive integer  $r$  we have*

$$\text{scol}_r(G) \leq \left( 3 + \text{tw}(G) \sum_{i=0}^{r-1} (\text{tw}(G) - 1)^i \right) \text{b}\omega(G) \leq (\text{tw}(G)^r + 3) \text{b}\omega(G). \quad (1)$$

**Proof.** Let  $d = \text{tw}(G)$  and  $s = \text{b}\omega(G)$ . Without loss of generality, we can assume that  $G$  is connected and contains more than  $s$  vertices. We consider a  $d$ -uncontraction sequence  $\mathbf{G}_1, \dots, \mathbf{G}_n$  of  $G$  with universe  $\mathcal{U}$ . For every  $i \in \{1, \dots, n\}$  we say a node of  $\mathbf{G}_i$  is *small* if it contains at most  $s$  vertices and it is *big*, otherwise. A set  $X \in \mathcal{U}$  is *nice at step  $i$*  with  $\text{bt}(X) \leq i \leq \text{st}(X)$  if  $X$  is small and some black edge is incident to it in  $\mathbf{G}_i$ . Note that if  $X$  is nice at step  $i$ , it is nice at step  $j$  for all  $i \leq j \leq \text{st}(X)$ . The set  $X$  is *nice* if it is nice at some step (equivalently, at step  $\text{st}(X)$ ). For every nice set  $X$  we define  $\rho(X)$  as the minimum  $i$  such that  $X$  is nice at step  $i$ . (Note that  $\rho(X) > 1$  as  $\mathbf{G}_1$  is edgeless.) As  $G$  is connected, it is clear that every  $X \in \mathcal{U}$  has a subset  $Y \in \mathcal{U}$  that is nice. Also, if  $X, Y \in \mathcal{U}$ ,  $X \subseteq Y$  and  $Y$  is nice, then  $X$  is also nice. It follows that the family  $\mathcal{N}$  of all the maximal nice sets in  $\mathcal{U}$  form a partition of  $V$ . We order the elements of  $\mathcal{N}$  as  $N_1, \dots, N_k$  in such a way that for all  $i < j$ , either  $\rho(N_i) < \rho(N_j)$  holds or  $\rho(N_i) = \rho(N_j)$  and  $\text{bt}(N_i) \geq \text{bt}(N_j)$ . We now fix any linear ordering  $L$  of  $V$  such that for all  $v \in N_i$ ,  $w \in N_j$  with  $i < j$  holds  $v <_L w$ . See also Remark 6 for an equivalent algorithmic way to define the order  $L$ . We will use this ordering to bound the strong coloring numbers of  $G$ .

For  $1 \leq i \leq n$ , we define  $\mathcal{B}_i$  to be the set of all nodes of  $\mathbf{G}_i$  that are not nice at step  $i$ . We first establish some easy properties of  $\mathcal{B}_i$ .

▷ **Claim 3.** No small node in  $\mathcal{B}_i$  is incident to a black edge in  $\mathbf{G}_i$ .

*Proof of the claim.* Assume  $X \in \mathcal{B}_i$  is small. Then it is not adjacent to a black edge as it is not nice at step  $i$ . ◁

▷ **Claim 4.** No two nodes in  $\mathcal{B}_i$  are adjacent in  $\mathbf{G}_i$  by a black edge.

*Proof of the claim.* Assume for contradiction that  $X$  and  $Y$  are nodes in  $\mathcal{B}_i$  that are adjacent by a black edge in  $\mathbf{G}_i$ . Hence,  $G[X \cup Y]$  includes  $K_{|X|,|Y|}$  as a subgraph. According to Claim 3, both  $X$  and  $Y$  are big, contradicting the assumption  $\text{b}\omega(G) = s$ . ◁

▷ **Claim 5.** For every node  $X \in \mathcal{B}_i$  holds  $|\bigcup N_{\mathbf{G}_i}^E(X)| \leq s$ .

*Proof of the claim.* Let  $X \in \mathcal{B}_i$  and let  $Y = \bigcup N_{\mathbf{G}_i}^E(X)$ . Then  $X$  and  $Y$  induce a biclique in  $G$  thus  $\min(|X|, |Y|) \leq s$ . As only big nodes in  $\mathcal{B}_i$  are adjacent to black edges (by Claim 3), we deduce  $|X| > s$  and therefore  $|Y| \leq s$ . ◁

Let us consider a vertex  $v \in V$ . In the remainder of the proof, we will bound the number of vertices in  $G$  that are strongly  $r$ -reachable from  $v$  with respect to  $L$ . Let  $a \in \{1, \dots, k\}$  be such that  $v \in N_a$  and let  $t = \rho(N_a) - 1$ . Let  $S$  be the unique node of  $\mathbf{G}_t$  with  $\text{st}(S) = t$ , and let  $Y, Z$  be the two children of  $S$ .

Let  $\mathcal{L} = \{X \in V(\mathbf{G}_t) : (\exists i < a), N_i \supseteq X\}$ . Then  $\mathcal{L}$  is the set of all nodes of  $\mathbf{G}_t$  that are nice at step  $t$ . By definition of  $L$ , all the vertices of  $G$  that belong to these nodes appear before  $v$  in  $L$ . If we set  $\mathcal{R} = V(\mathbf{G}_t) \setminus \mathcal{L}$ , then  $\mathcal{R} = \mathcal{B}_t$ .

**Case 1:**  $v \in S$ .

Note that if a vertex  $u \in V(G)$  is strongly  $r$ -reachable from  $v$ , then  $u$  belongs either to  $S$  or to a set in  $\mathcal{L}$ . We consider a BFS-tree  $T$  in  $\mathbf{G}_t$ , starting at  $S$ , following only red edges, with depth  $r$ , and stopping each time it reaches a node in  $\mathcal{L}$ . We further remove from  $T$  any node with no descendant (in  $T$ ) belonging to  $\mathcal{L}$ . This way we get a tree  $T$  rooted at  $S$ , with internal nodes in  $\mathcal{R}$ , with depth at most  $r$ , and with leaves in  $\mathcal{L}$ . Let  $\mathcal{I}$  denote the sets of all internal nodes of  $T$  and let  $\mathcal{E}$  be the sets of all leaves of  $T$ . Then  $|\mathcal{I}| \leq 1 + \sum_{\ell=0}^{r-2} d(d-1)^\ell$  and  $|\mathcal{E}| \leq d(d-1)^{r-1}$ . Consider any vertex  $u$  that is strongly  $r$ -reachable from  $v$ , and let  $P$  be a path from  $v$  to  $u$  in  $G$  witnessing this. We can project  $P$  onto  $\mathbf{G}_t$  by mapping every vertex to the node of  $\mathbf{G}_t$  containing it. The projection is a walk from  $S$  to a node  $X_u$  containing  $u$ . From this walk we extract a path  $Q$  of length at most  $r$  from  $S$  to  $X_u$ . All the internal nodes of  $Q$  as well as  $S$  belong to  $\mathcal{R}$ , hence all the edges of  $Q$  (but maybe the last one) are red (according to Claim 4). Moreover,  $X_u$  is either  $S$  or it belongs to  $\mathcal{L}$ . So, either  $u \in S$ , or  $X_u$  has been reached by a black edge from some internal node of  $T$ , or  $X_u$  is a leaf of  $T$ . It is easily checked that at most  $|\mathcal{I}|s$  vertices of  $G$  can be of the second type (according to Claim 5), and at most  $|\mathcal{E}|s$  are of the last type (as leaves belong to  $\mathcal{L}$ , so they are small). Regarding the first type, we assume without loss of generality that  $v \in Z$  and observe that either  $u \in Z$ , so there are at most  $s$  choices for  $u$  (as  $Z = N_a$  is nice hence small at time  $t+1$ ), or  $u \in Y$  but then, as  $u \leq_L v$ ,  $Y$  is nice as well at time  $t+1$  so  $|Y| \leq s$ . Thus at most  $2s$  vertices of  $G$  can be of the first type. Altogether, we get

$$\begin{aligned} |\text{Sreach}[G, L, v]| &\leq (2 + 1 + d + \dots + d(d-1)^{r-2} + d(d-1)^{r-1}) s \\ &\leq \left( 3 + d \sum_{\ell=0}^{r-1} (d-1)^\ell \right) s. \end{aligned}$$

**Case 2:**  $v \notin S$ .

Note that if  $u$  is strongly  $r$ -reachable from  $v$ , then  $u$  belongs either to  $S$  or to  $N_a$ , or to a set in  $\mathcal{L}$ . We consider a BFS-tree  $T$  in  $\mathbf{G}_t$ , starting at  $N_a$ , following only red edges, with depth  $r$ , and stopping each time it reaches a node in  $\mathcal{L}$ . We further remove from  $T$  any node with no descendant (in  $T$ ) belonging to  $\mathcal{L} \cup \{S\}$ . This way we get a tree  $T$  rooted at  $N_a$ , with internal nodes in  $\mathcal{R}$ , with depth at most  $r$ , and with leaves in  $\mathcal{L}$ . Let  $\mathcal{I}$  denote the sets of all internal nodes of  $T$  and let  $\mathcal{E}$  be the sets of all leaves of  $T$ . Then  $|\mathcal{I}| \leq 1 + d + \dots + d(d-1)^{r-2}$  and  $|\mathcal{E}| \leq d(d-1)^{r-1}$ . Consider any vertex  $u$  that is strongly  $r$ -reachable from  $v$ , and let  $P$  be a path from  $v$  to  $u$  witnessing this. The path  $P$  projects on  $\mathbf{G}_t$  as a walk with length at most  $r$  from  $N_a$  to the vertex  $X_u$  containing  $u$ . From this walk we extract a path  $Q$  with length at most  $r$  from  $N_a$  to  $X_u$ . All the internal nodes of  $Q$  belong to  $\mathcal{R}$  hence all the edges of  $Q$  (but maybe the last one) are red (according to Claim 4). Moreover,  $X_u$  is either  $N_a$ , or  $S$ , or it belongs to  $\mathcal{L}$ . So, either  $u \in N_a$ , or  $u \in S$ , or  $X_u$  has been reached by a black edge from some internal node of  $T$ , or  $X_u$  is a leaf of  $T$ . The first type correspond to at most  $s$  vertices. The second type correspond to at most  $2s$  vertices because in this case,  $Y$ ,  $Z$ , or both, have been ordered by  $L$  before  $N_a$  which mean they are nice at step  $t+1$ , hence small. . The third type correspond to at most  $(|\mathcal{I}| - 1)s$ , as the root  $N_a$  is small hence adjacent to no black edge. The last type correspond to at most  $|\mathcal{E}|s$  vertices. Altogether, we get

$$\begin{aligned} |\text{Sreach}[G, L, v]| &\leq (1 + 2 + (1 + d + \dots + d(d-1)^{r-2} - 1) + d(d-1)^{r-1}) s \\ &\leq \left( 3 + d \sum_{\ell=0}^{r-1} (d-1)^\ell \right) s. \end{aligned}$$

Thus in both cases we have that every graph  $G$  with  $\text{tw}(G) = d$  and  $\text{bw}(G) = s$  satisfies (1).  $\blacktriangleleft$

► **Remark 6.** We also describe an algorithmic procedure that also yields the order  $L$  defined in Theorem 2. We are given an uncontraction sequence  $\mathbf{G}_n, \dots, \mathbf{G}_1$ . For each  $i \in \{1, \dots, n-1\}$  we define a function  $\text{origin}_i: V(\mathbf{G}_{i+1}) \rightarrow V(\mathbf{G}_i)$  that, informally, assigns each node in  $\mathbf{G}_{i+1}$  to the node in  $\mathbf{G}_i$  that it originates from. Let us be more precise: assume  $\mathbf{G}_{i+1}$  is constructed from  $\mathbf{G}_i$  by splitting a node  $Z$  into two nodes  $X$  and  $Y$ , then  $\text{origin}_i(X) = \text{origin}_i(Y) = Z$  and  $\text{origin}_i(W) = W$  for every other node  $W$  of  $\mathbf{G}_{i+1}$ .

Remember that a node of  $\mathbf{G}_i$  is *small* if it contains at most  $s$  vertices and is *big*, otherwise. A node  $X \in V(\mathbf{G}_i)$  is a *nice* node of  $\mathbf{G}_i$  if  $X$  is small and some black edge is incident to  $X$  in  $\mathbf{G}_i$ . We incrementally construct for all  $i$  an ordering  $<_i$  on the nice nodes of  $G_i$ . Since all nodes of  $\mathbf{G}_n$  are nice and correspond to singletons, the ordering  $<_n$  then corresponds to an ordering of the vertices of  $G$ . This order will be equivalent to the ordering  $L$  defined in Theorem 2 (up to non-determinism).

Since  $G_1$  has no nice nodes,  $<_1$  is the empty ordering. Assuming that  $<_i$  is already constructed, we construct  $<_{i+1}$  such that it satisfies for all nice  $X, Y \in V(\mathbf{G}_{i+1})$  the following conditions.

1. If  $\text{origin}_i(X)$  and  $\text{origin}_i(Y)$  are nice in  $\mathbf{G}_i$  and  $\text{origin}_i(X) <_i \text{origin}_i(Y)$  then  $X <_{i+1} Y$ .
  2. If  $\text{origin}_i(X)$  is nice in  $\mathbf{G}_i$  and  $\text{origin}_i(Y)$  is not nice in  $\mathbf{G}_i$  then  $X <_{i+1} Y$ .
  3. If both  $\text{origin}_i(X)$  and  $\text{origin}_i(Y)$  are not nice in  $\mathbf{G}_i$  and  $\text{bt}(X) > \text{bt}(Y)$  then  $X \leq_{i+1} Y$ .
- Each order  $<_i$  represents a partial order on  $V$  that is refined over time as  $i$  increases, until we reach a total order on  $V$ . Rule 1. states that the old order is preserved when possible, rule 2. states that new nice sets are appended at the end and rule 3. makes sure that we append new nice sets in order of their birth.

In the proof of Theorem 2, we fix a vertex  $v \in V$  and pick  $t$  maximal such that in  $\mathbf{G}_t$  the node  $N$  containing  $v$  is not nice. We then partition the nodes of  $\mathbf{G}_t$  into sets  $\mathcal{L}$  and  $\mathcal{R}$ . One can show that  $\mathcal{L}$  contains precisely those nodes of  $\mathbf{G}_t$  that are strictly smaller than  $N$  with respect to  $<_t$ .

► **Corollary 7.** *For every graph  $G$  and every positive integer  $r$  we have*

$$\text{scol}_r(G) \leq \begin{cases} 2 \text{b}\omega(G) & \text{if } \text{tww}(G) = 0, \\ 3 \text{b}\omega(G) & \text{if } \text{tww}(G) = 1, \\ 5 \text{b}\omega(G) & \text{if } \text{tww}(G) = 2, \\ 3(\text{tww}(G) - 1)^r \text{b}\omega(G) & \text{if } \text{tww}(G) \geq 3. \end{cases}$$

**Proof.** If  $\text{tww}(G) \geq 3$  we have

$$\begin{aligned} \text{scol}_r(G) &\leq \left( 3 + \text{tww}(G) \frac{(\text{tww}(G) - 1)^r - 1}{\text{tww}(G) - 2} \right) \text{b}\omega(G) \\ &\leq (3 + 3((\text{tww}(G) - 1)^r - 1)) \text{b}\omega(G) \\ &\leq 3(\text{tww}(G) - 1)^r \text{b}\omega(G). \end{aligned}$$

The cases where  $\text{tww}(G) = 1$  or  $2$  follow from the theorem. If  $\text{tww}(G) = 0$  then  $G$  is a cograph. Let us then show that for every cograph  $G$  it holds that  $\text{scol}_r(G) \leq 2 \text{b}\omega(G)$ .

The proof is by induction on the number of vertices. The base case  $|G| = 1$  is trivial so we consider a cograph with at least two vertices and assume that the desired bound holds for all cographs on less vertices. Being a cograph,  $G$  can be obtained from two cographs  $G_1$  and  $G_2$  by disjoint union or complete join [6]. Without loss of generality we assume  $|G_1| \leq |G_2|$ . By induction, for every  $i \in \{1, 2\}$  there is an ordering  $L_i$  of  $V(G_i)$  such that  $\text{scol}_r(G_i) \leq 2 \text{b}\omega(G_i)$ .

Then the order  $L$  is obtained by putting first  $L_1$ , then  $L_2$ . If  $G$  is the disjoint union of  $G_1$  and  $G_2$  then  $\text{scol}_r(G, L) = \max(\text{scol}_r(G, L_1), \text{scol}_r(G, L_2))$  and the result follows; if  $G$  is the complete join of  $G_1$  and  $G_2$  then  $\text{scol}_r(G, L) \leq \text{scol}_r(G_2, L_2) + |G_1|$  and  $\text{b}\omega(G) \geq \text{b}\omega(G_2) + |G_1|/2$ . As  $\text{scol}_r(G_2, L_2) \leq 2 \text{b}\omega(G_2)$ , we get  $\text{scol}_r(G_2, L_2) \leq 2 \text{b}\omega(G) - |G_1|$  hence  $\text{scol}_r(G, L) \leq 2 \text{b}\omega(G)$ . ◀

Combining Lemma 1 with Theorem 2 we get the following.

► **Corollary 8.** *For every graph  $G$  and every positive integer  $r$  we have*

$$\text{wcol}_r(G) \leq \frac{1}{2} ((2 \text{tww}(G) + 6) \text{b}\omega(G))^r.$$

Note that the base of the exponential comes from the degeneracy of  $G$ . In order to improve this upper bound it is thus natural to try to improve the degeneracy bound. Hence the following problem:

► **Problem 9.** *What is the maximum degeneracy of a  $K_{s+1, s+1}$ -free graph with twin-width at most  $d$ ?*

Recall that a *depth  $r$  minor* of a graph  $G$  is a graph  $H$  obtained from  $G$  by taking a subgraph and contracting vertex disjoint subgraphs of radius at most  $r$ . The *greatest reduced average density* ( $\text{grad}$ ) of  $G$  with rank  $r$  is the maximum ratio  $|E(H)|/|V(H)|$  over all (non-empty) depth  $r$  minors of a graph  $G$ ; it is denoted by  $\nabla_r(G)$ . Hence, by definition, a class  $\mathcal{C}$



has bounded expansion if, for each positive integer  $r$ , we have  $\sup\{\nabla_r(G) : G \in \mathcal{C}\} < \infty$ . It is known that  $\nabla_r(G) \leq \text{wcol}_{2r+1}(G)$  [13]. Hence the next corollary directly follows from Corollary 8.

► **Corollary 10.** *For every graph  $G$  and every positive integer  $r$  we have*

$$\nabla_r(G) \leq \frac{1}{2}((2 \text{tww}(G) + 6) \text{b}\omega(G))^{2r+1}. \quad (2)$$

In particular, every class of graphs of bounded clique-width that exclude a biclique as a subgraph has (at most) exponential expansion.

## 5 Lower bounds

It is known that high-girth graphs have large strong coloring numbers [9]. On the other hand, there exist expander graphs with high girth and small twinwidth [1]. We combine both results to construct graphs with small twinwidth whose strong  $r$ -coloring numbers grow exponentially in  $r$ .

► **Proposition 11** ([9, Theorem 5.1]). *Let  $G$  be a  $d$ -regular graph of girth at least  $4g + 1$ , where  $d \geq 7$ . Then for every  $r \leq g$ ,*

$$\text{scol}_r(G) \geq \frac{d}{2} \left( \frac{d-2}{4} \right)^{2^{\lceil \log_2 r \rceil} - 1}.$$

► **Lemma 12.** *For every integer  $\Delta \geq 7$  and every integers  $r$  and  $g \geq 4r + 1$  there exists a  $\Delta$ -regular graph  $G$  with girth at least  $g$ ,  $2\Delta - 1 \leq \text{tww}(G) \leq 2\Delta$ , and*

$$\text{scol}_r(G) \geq \frac{\Delta}{2} \left( \frac{\Delta-2}{4} \right)^{2^{\lceil \log_2 r \rceil} - 1} \geq \frac{\text{tww}(G)}{4} \left( \frac{\text{tww}(G) - 4}{8} \right)^{2^{\lceil \log_2 r \rceil} - 1}.$$

**Proof.** We will construct a sequence  $G_0, G_1, \dots$  of  $\Delta$ -regular graphs of twin-width at most  $2\Delta$  and increasing girth. Once we reach a graph with girth at least  $g \geq 4r + 1$ , the result of this lemma follows from Proposition 11. Note that the twin-width of a  $\Delta$ -regular graph with girth at least 5 is at least  $2\Delta - 1$  (because of the first contraction).

We define  $G_0$  to be the complete graph with  $\Delta + 1$  vertices. We fix a graph  $G_{k-1}$  with edges  $e_1, \dots, e_m$  and describe how to construct  $G_k$ . For every edge  $e_i$  of  $G_{k-1}$ , we define a mapping  $\theta_{e_i} : \{0, 1\}^m \rightarrow \{0, 1\}^m$  that flips the  $i$ th coordinate and preserves all other coordinates, i.e.,  $\theta_{e_i}((x_1, \dots, x_m)) = (y_1, \dots, y_m)$  with

$$y_j = \begin{cases} 1 - x_j & \text{if } j = i \\ x_j & \text{otherwise.} \end{cases}$$

We define  $G_k$  to be the graph with  $V(G_k) = V(G_{k-1}) \times \{0, 1\}^m$  and  $E(G_k) = \{(u, \mathbf{x}), (v, \theta_{uv}(\mathbf{x})) : uv \in E(G_{k-1}) \text{ and } \mathbf{x} \in \{0, 1\}^m\}$ .

A *2-lift* of a graph  $G$  is a graph obtained by adding for every vertex  $v$  of  $G$  two vertices  $v_1$  and  $v_2$ , and adding for every edge  $uv$  of  $G$  either the edges  $u_1v_1$  and  $u_2v_2$  (parallel edges) or the edges  $u_1v_2$  and  $u_2v_1$  (crossing edges). The graph  $G_{k+1}$  can be obtained by a sequence of 2-lifts from  $G_k$  and therefore also by a sequence of 2-lifts from  $G_0 = K_{\Delta+1}$ . We can construct a contraction sequence that “undoes“ these 2-lifts by repeatedly contracting all pairs of duplicates. Once we reach  $K_{\Delta+1}$ , we simply contract the remaining vertices one



by one. While doing so, the red degree never exceeds  $2\Delta$  (see also [1, Lemma 26]). Hence  $\text{tw}(G_k) \leq 2\Delta$ .

It remains to show that the girth of  $G_k$  is higher than the girth of  $G_{k-1}$ . Let  $\gamma$  be a shortest cycle of  $G_k$ . Let  $p_V : V(G_k) \rightarrow V(G_{k-1})$  be the standard projection, and let  $p_E : E(G_k) \rightarrow E(G_{k-1})$  be the associated projection. It is easily checked that applying  $p_E$  to a cyclic graph yields a cyclic graph, and thus  $p_E(\gamma)$  includes a cycle. If we apply the composition of all the mappings  $\theta_{p_E(e)}$  for  $e \in \gamma$  then the starting vertex of  $\gamma$  is fixed. It follows that each  $\theta_e$  is applied an even number of times. Thus the length of  $\gamma$  is at least twice the length of  $p_E(\gamma)$ . Hence, the girth of  $G_k$  is at least twice the girth of  $G_{k-1}$ . ◀

► **Corollary 13.** *For every integer  $d \geq 14$ , every positive integer  $s$ , and every integer  $r$  of the form  $2^k$ , there exists a graph  $G$  with  $\text{tw}(G) \leq d$ ,  $\text{bw}(G) = s$ , and*

$$\text{scol}_r(G) \geq \frac{ds}{4} \left( \frac{d-4}{8} \right)^{r-1} \geq 2 \left( \frac{\text{tw}(G) - 4}{8} \right)^r \text{bw}(G).$$

**Proof.** Take the lexicographic product of a graph obtained by Lemma 12 and  $K_s$ . This way we get a graph with twin-width at most  $d \geq 14$  and no  $K_{s+1, s+1}$ . ◀

► **Remark 14.** The 2-lift construction used in the proof of Lemma 12 was used in [1] to prove that there exist cubic expander graphs with twin-width at most 6. It follows from this result and the characterization of classes with polynomial expansion [8] that for  $d \geq 6$ , the value  $\nabla_r(G)$  is not bounded on the  $K_{2,2}$ -free graphs with twin-width at most  $d$  by a polynomial function of  $r$ . We leave as a question whether  $\sup\{\nabla_r(G) : \text{tw}(G) \leq d \text{ and } \text{bw}(G) \leq s\}$  increases exponentially with  $r$  for sufficiently large  $d$ .

Admissibility being a lower bound for strong coloring numbers, the above results do not provide any lower bound for admissibility. We show below how to construct classes of graphs that have no  $K_{2,2}$ -subgraph with low twin-width and high admissibility.

► **Lemma 15** ([1, Proposition 28]). *For  $d \geq 0$  and  $k > 0$ , if the clique  $K_n$  subdivided  $k$  times has twin-width less than  $d$ , then  $k \geq \log_d(n-1) - 1$ .*

► **Lemma 16** ([1, Proposition 31]). *For any  $c > 0$ , the class of cliques  $K_n$  subdivided at least  $\frac{\log n}{c}$  times has twin-width at most  $f(c)$  for some triple exponential function  $f$ .*

► **Lemma 17.** *For every integers  $d$  and  $r \geq 4$  there is a graph  $G$  such that*

- $G$  has no  $K_{2,2}$  subgraph;
- $\text{tw}(G) \leq f(2 \log d)$ ;
- $\text{adm}_r(G) \geq d^{2(r-1)}$ ,

where  $f$  is the function of Lemma 16.

In particular, the above lemma implies that for every  $d$ , there is a graph class of bounded twin-width, whose members contain no  $K_{2,2}$ , but which has  $r$ -admissibility (and thus  $r$ -weak and strong coloring numbers) at least  $d^{2(r-1)}$ .

**Proof.** Let  $d, r \in \mathbb{N}$  and  $n = d^{2(r-1)}$ . We define  $G_r^d$  as the graph obtained by subdividing  $r-1$  times each edge of  $K_n$ . By construction,  $G_r^d$  has no  $K_{2,2}$  subgraph. Let  $c = 2 \log d$  and notice that  $r-1 = \frac{\log n}{c}$ . According to Lemma 16,  $G_r^d$  has twin-width at most  $f(c)$ . On the other hand, as  $r > 3$  we have

$$\begin{aligned} r-1 &< 2(r-1) - 2 \\ &< \log_d \left( d^{2(r-1)} \right) - 2 \\ &< \log_d (n-1) - 1. \end{aligned}$$

In order to prove the bound on admissibility, let us now consider an arbitrary ordering  $\sigma$  of  $V(G_r^d)$ . Notice that  $G_r^d$  has two types of vertices:  $n$  vertices of degree  $n - 1$ , which correspond to the vertices of the  $n$ -clique that was used to construct  $G_r^d$ , and vertices of degree 2, which have been introduced by subdivisions. Let  $x$  denote the vertex of degree  $n - 1$  that appears the latest in  $\sigma$ . Notice that there are  $n - 1$  paths of length  $r$  that start in  $x$  and are otherwise disjoint and end at the  $n - 1$  other vertices of degree  $n - 1$  of  $G$ . By definition of  $x$ , all these vertices appear before in the ordering. This implies  $\text{adm}_r(G_r^d, \sigma) \geq n = d^{2(r-1)}$ . As  $\sigma$  was chosen arbitrarily, the same bound holds for the  $r$ -admissibility of  $G_r^d$ . ◀

► **Corollary 18.** *For all integers  $d, r \geq 4$ , there is a constant  $\varepsilon > 0$  such that for all positive integers  $r$  and  $n$  there is a  $K_{s+1, s+1}$ -free graph  $G$  with  $|G| \geq n$ ,  $\text{bw}(G) = s$ , and*

$$\text{adm}_r(G) \geq (\log \log \text{tw}(G))^\varepsilon \text{bw}(G).$$

**Proof.** We first consider the case where  $s = 1$ . Let  $G_0$  be the graph given by Lemma 17. If  $|G_0| \geq n$  then  $G_0$  is the desired graph. Otherwise, we denote by  $G$  the disjoint union of  $n$  copies of  $G_0$ . Clearly this does not create any  $K_{2,2}$  thus  $\text{bw}(G_0) = 1$ . We have  $\text{adm}_r(G) \geq d^{2(r-1)}$ , as otherwise any ordering of  $G$  with smaller  $r$ -admissibility would give an ordering with smaller  $r$ -admissibility for  $G_0$ . Finally, as the twin-width of the disjoint union of two graphs is the maximum of the twin-width of each of them, we have  $\text{tw}(G) = \text{tw}(G_0)$ , so  $d \leq \text{tw}(G) \leq f(2 \log d)$ . The existence of the constant  $C$  then follows from the fact that  $f$  is a triple exponential function.

The case where  $s > 1$  then follows by considering the lexicographic product of the graphs obtained above by  $K_s$ . ◀

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