Handling Bound Variables via De Bruijn's Indices

A Fast Overview of the Definition and Properties

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1 Introduction

Whenever bound variables appear in an object during a computer formalization, we wish to consider α-equality instead of the simple syntactical equality. A way of doing so is by using de Bruijn's indices instead of plain string variables. For instance,

$$\lambda x.x \qquad \text{is instead written} \qquad \lambda.0.$$

In turn, it means that the substitution $M[N/k]$ should be adapted (i) to avoid capturing free-variables of N under quantifiers in M, and (ii) to relevantly update the index of bound variables (i.e., take into account the removal of a binder).
In this document, we formally define the substitution operation for a simple language: untyped minimal λ-calculus. This language is generated by the following grammar:

$$M, N \quad ::= \quad n \mid \lambda.M \mid (M)N$$

with $n \in \mathbb{N}$. We then (formally) show multiple useful[1] substitution lemmas.

2 Substitution

In this section, we start by giving the definition of the substitution. It needs an auxiliary definition: the lifting of a term. Then, we enumerate lemmas about (i) lifting and (ii) substitution. The goal is to be able to browse through this document and quickly find the relevant statement. Note that the formal proof of each statement is given in Sec. 2.4.

2.1 Definitions

First, we shall handle the way of avoiding the capture of free-variables of N. We define the lifting operation $\uparrow_i^j M$ by induction on M as follows:

$$\uparrow_i^j n = \begin{cases} n & n < i \\ n + j & n \geqslant i \end{cases} \qquad \uparrow_i^j (\lambda.M) = \lambda. \uparrow_{i+1}^j M \qquad \uparrow_i^j ((M)N) = (\uparrow_i^j M)(\uparrow_i^j N)$$

We can then define the substitution $M[N/k]$ by induction on M:

$$n[N/k] = \begin{cases} n & n < k \\ \uparrow_0^k N & n = k \\ n - 1 & n > k \end{cases} \qquad (\lambda.M)[N/k] = \lambda.M[N/k+1] \qquad ((M)P)[N/k] = (M[N/k])P[N/k]$$

[1]Useful in the sense that I've needed to implement them at least once during my formalizations.

2.2 Lifting Lemmas

We have ways of simplifying a double-lifting operation.

Lemma 2.1:

For $i, j, k, \ell \in \mathbb{N}$ and M a λ-term, if $j \leqslant \ell \leqslant i + j$ then

$$\uparrow^k_\ell \uparrow^i_j M = \uparrow^{k+i}_j M$$

Lemma 2.2:

For $i, j, k \in \mathbb{N}$ and M a λ-term, if $j \leqslant i + j$ then

$$\uparrow^k_j \uparrow^i_j M = \uparrow^k_{i+j} \uparrow^i_j M$$

Lemma 2.3:

For $i, j, k, \ell \in \mathbb{N}$ and M a λ-term, if $\ell \leqslant i$ then

$$\uparrow^k_\ell \uparrow^j_i M = \uparrow^j_{i+k} \uparrow^k_\ell M$$

2.3 Substitution Lemmas

The first substitution lemma describes what happens when we want to chain two substitutions, i.e., what $M[N/k][P/\ell]$ is equal to. We start by two auxiliary lemmas about lifting and substitutions.

Lemma 2.4:

For $i, j, k \in \mathbb{N}$ and M, N two λ-terms, if $i + k \leqslant j$ then:

$$(\uparrow^k_i M)[N/j] = \uparrow^k_i M[N/j - k].$$

Lemma 2.5:

For $i, j, k \in \mathbb{N}$ and M, N two λ-terms, if $i \leqslant j < k + i$ then:

$$(\uparrow^k_i M)[N/j] = \uparrow^{k-1}_i M.$$

Lemma 2.6: First Substitution Lemma

For $k, \ell \in \mathbb{N}$ and M, N, P three λ-terms, if $k \leqslant \ell$ then:

$$M[N/k][P/\ell] = M[P/\ell + 1][N[P/\ell - k]/k].$$

Another interesting situation happens when we lift *inside* the substitution.

Lemma 2.7: Second Substitution Lemma

For $i, j, k \in \mathbb{N}$ and M, N two λ-terms,

$$(\uparrow^j_{i+k+1} M)[\uparrow^j_i N/k] = \uparrow^j_{i+k} M[N/k].$$

2.4 Proofs

Proof (of Lem. 2.1): By induction on M.

- If $M = n$, then there are two cases:
  - if $n < j$, then $\uparrow^k_\ell \uparrow^i_j n = \uparrow^k_\ell n = n$ as $n < j \leqslant \ell$ and $\uparrow^{k+i}_j n = n$.
  - if $n \geqslant j$, then $\uparrow^k_\ell \uparrow^i_j n = \uparrow^k_\ell n + i = n + i + k$ as $n \geqslant j \implies n + i \geqslant j + i \geqslant \ell$, and $\uparrow^{k+i}_j n = n + k + i$.
- If $M = \lambda.N$ then:

$$\begin{aligned}\uparrow^k_\ell \uparrow^i_j (\lambda.N) &= \lambda.\uparrow^k_{\ell+1} \uparrow^i_{j+1} N \\ &= \lambda.\uparrow^{k+i}_{j+1} N \\ &= \uparrow^{k+i}_j (\lambda.N)\end{aligned}$$

- If $M = (N)P$ then:

$$\begin{aligned}\uparrow^k_\ell \uparrow^i_j ((N)P) &= (\uparrow^k_\ell \uparrow^i_j N)(\uparrow^k_\ell \uparrow^i_j P) \\ &= (\uparrow^{k+i}_j N)(\uparrow^{k+i}_j P) \\ &= \uparrow^{k+i}_j ((N)P)\end{aligned}$$

□

Proof (of Lem. 2.2): By induction on M.

- If $M = n$ then there are two cases:
  - if $n < j$ then $\uparrow^k_j \uparrow^i_j n = n$ and likewise for $\uparrow^k_{i+j} \uparrow^i_j n$ as $n < j \leqslant i + j$.
  - if $n \geqslant j$ then $\uparrow^k_j \uparrow^i_j n + i = \uparrow^k_j n + i = n + i + k$ and $\uparrow^k_{i+j} \uparrow^i_j n = \uparrow^k_{i+j} n + i = n + i + k$ as $n + i \geqslant j + i$.
- If $M = \lambda.N$, then:

$$\begin{aligned}\uparrow^k_j \uparrow^i_j (\lambda.N) &= \lambda.\uparrow^k_{j+1} \uparrow^i_{j+1} N \\ &= \lambda.\uparrow^k_{i+j+1} \uparrow^i_{j+1} N \\ &= \uparrow^k_{i+j} \uparrow^i_j (\lambda.N)\end{aligned}$$

- If $M = (N)P$, then:

$$\begin{aligned}\uparrow^k_j \uparrow^i_j ((N)P) &= (\uparrow^k_j \uparrow^i_j N)(\uparrow^k_j \uparrow^i_j P) \\ &= (\uparrow^k_{i+j} \uparrow^i_j N)(\uparrow^k_{i+j} \uparrow^i_j P) \\ &= \uparrow^k_{i+j} \uparrow^i_j ((N)P)\end{aligned}$$

□

Proof (of Lem. 2.3): By induction on M.

- If $M = n$, then there are two cases:
  - if $n < \ell$ then $\uparrow^k_\ell \uparrow^j_i n = \uparrow^k_\ell n = n$ and $\uparrow^j_{i+k} \uparrow^k_\ell n = \uparrow^j_{i+k} n = n$ as $n < \ell \leqslant i$.
  - if $n \geqslant \ell$, then there are two cases:
    - ⋆ if $n < i$, then $\uparrow^k_\ell \uparrow^j_i n = \uparrow^k_\ell n = n + k$ and $\uparrow^j_{i+k} \uparrow^k_\ell n = \uparrow^j_{i+k} n + k = n + k$ as $n < i$ thus $n + k < i + k$.
    - ⋆ if $n \geqslant i$, then $\uparrow^k_\ell \uparrow^j_i n = \uparrow^k_\ell n + j = n + j + k$ and $\uparrow^j_{i+k} \uparrow^k_\ell n = \uparrow^j_{i+k} n + k = n + k + j$.

- If $M = \lambda.N$, then:

$$\begin{aligned}\uparrow_\ell^k \uparrow_i^j (\lambda.N) &= \lambda. \uparrow_{\ell+1}^k \uparrow_{i+1}^j N \\ &= \lambda. \uparrow_{i+k+1}^j \uparrow_{\ell+1}^k N \\ &= \uparrow_{i+k}^j \uparrow_\ell^k (\lambda.N)\end{aligned}$$

- If $M = ((N)P)$, then:

$$\begin{aligned}\uparrow_\ell^k \uparrow_i^j ((N)P) &= (\uparrow_\ell^k \uparrow_i^j N)(\uparrow_\ell^k \uparrow_i^j P) \\ &= (\uparrow_{i+k}^j \uparrow_\ell^k N)(\uparrow_{i+k}^j \uparrow_\ell^k P) \\ &= \uparrow_{i+k}^j \uparrow_\ell^k ((N)P)\end{aligned}$$

□

Proof (of Lem. 2.4): By induction on M.

- If $M = n$, then there are two cases:
  - if $n < i$, then $(\uparrow_i^k n)[N/j] = n[N/j]$ and as $i + k \leqslant j$, $n < j$ and hence $n[N/j] = n$. Likewise, $\uparrow_i^k n[N/j-k] = \uparrow_i^k n$ as $n < i \leqslant j-k$ and $\uparrow_i^k n = n$.
  - if $n \geqslant i$, then $(\uparrow_i^k n)[N/j] = (n+k)[N/j]$ and there are three other cases:
    - ⋆ if $n+k < j$, then $(n+k)[N/j] = n+k$ and $\uparrow_i^k n[N/j-k] = \uparrow_i^k n$ as $n+k < j \implies n < j-k$; and $\uparrow_i^k n = n+k$.
    - ⋆ if $n+k = j$, then $(n+k)[N/j] = \uparrow_0^j N$ and $\uparrow_i^k n[N/j-k] = \uparrow_i^k \uparrow_0^{j-k} N$. By Lem. 2.1, $\uparrow_i^k \uparrow_0^{j-k} N = \uparrow_0^{k+j-k} N = \uparrow_0^j N$.
    - ⋆ if $n+k > j$, then $(n+k)[N/j] = n+k-1$, $\uparrow_i^k n[N/j-k] = \uparrow_i^k n-1$ as $n > j-k$ and $\uparrow_i^k n-1 = n+k-1$.
- If $M = \lambda.M'$, then:

$$\begin{aligned}(\uparrow_i^k \lambda.M')[N/j] &= (\lambda. \uparrow_{i+1}^k M')[N/j] \\ &= \lambda.(\uparrow_{i+1}^k M')[N/j+1] \\ &= \lambda. \uparrow_{i+1}^k (M'[N/j+1-k]) \\ &= \uparrow_i^k (\lambda.M'[N/j+1-k]) \\ &= \uparrow_i^k ((\lambda.M')[N/j-k])\end{aligned}$$

- If $M = (M_1)M_2$, then:

$$\begin{aligned}(\uparrow_i^k ((M_1)M_2))[N/j] &= ((\uparrow_i^k M_1)[N/j])(\uparrow_i^k M_2)[N/j] \\ &= (\uparrow_i^k (M_1[N/j-k]))(\uparrow_i^k (M_2[N/j-k])) \\ &= \uparrow_i^k ((M_1)M_2)[N/j-k]\end{aligned}$$

□

Proof (of Lem. 2.5): By induction on M.

- If $M = n$, then there are 2 cases:
  - if $n < i$, then $(\uparrow_i^k n)[N/j] = n[N/j] = n$ as $n < i \leqslant j$. Likewise, $\uparrow_i^{k-1} n = n$.
  - if $n \geqslant i$, then $(\uparrow_i^k n)[N/j] = (n+k)[N/j] = n+k-1$ as $n \geqslant i \implies n+k \geqslant i+k > j$. Moreover, $\uparrow_i^{k-1} n = n+k-1$.

- If $M = \lambda.M'$, then:

$$\begin{aligned}(\uparrow_i^k \lambda.M')[N/j] &= (\lambda. \uparrow_{i+1}^k M')[N/j] \\ &= \lambda.(\uparrow_{i+1}^k M')[N/j+1] \\ &= \lambda. \uparrow_{i+1}^{k-1} M' \\ &= \uparrow_i^{k-1} (\lambda.M')\end{aligned}$$

- If $M = (M_1)M_2$, then:

$$\begin{aligned}(\uparrow_i^k ((M_1)M_2))[N/j] &= ((\uparrow_i^k M_1)[N/j])(\uparrow_i^k M_2)[N/j] \\ &= (\uparrow_i^{k-1} M_1)(\uparrow_i^{k-1} M_2) \\ &= \uparrow_i^{k-1} ((M_1)M_2)\end{aligned}$$

□

Proof (of Lem. 2.6): By induction on M.

- If $M = n$, then there are three cases:
  - if $n < k$, then $n[N/k][P/\ell] = n$ as $n < k \leqslant \ell$, and $n[P/\ell+1][N[P/\ell-k]/k] = n$ for the same reason.
  - if $n = k$, then $n[N/k][P/\ell] = (\uparrow_0^k N)[P/\ell] = \uparrow_0^k N[P/\ell-k]$ by Lem. 2.4, and $n[P/\ell+1][N[P/\ell-k]/k] = n[N[P/\ell-k]/k] = \uparrow_0^k N[P/\ell-k]$.
  - if $n > k$, then $n[N/k][P/\ell] = (n-1)[P/\ell]$ and there are three other cases:
    - ⋆ if $n-1 < \ell$, then $(n-1)[P/\ell] = n-1$ and $n[P/\ell+1][N[P/\ell-k]/k] = n[N[P/\ell-k]/k] = n-1$.
    - ⋆ if $n-1 = \ell$, then $(n-1)[P/\ell] = \uparrow_0^\ell P$ and $n[P/\ell+1][N[P/\ell-k]/k] = (\uparrow_0^{\ell+1} P)[N[P/\ell-k]/k]$. By Lem. 2.5, $(\uparrow_0^{\ell+1} P)[N[P/\ell-k]/k] = \uparrow_0^{\ell+1-1} P = \uparrow_0^\ell P$.
    - ⋆ if $n-1 > \ell$, then $(n-1)[P/\ell] = n-2$ and $n[P/\ell+1][N[P/\ell-k]/k] = n-1[N[P/\ell-k]/k] = n-2$.
- If $M = \lambda.M'$, then:

$$\begin{aligned}(\lambda.M')[N/k][P/\ell] &= \lambda.M'[N/k+1][P/\ell+1] \\ &= \lambda.M'[P/\ell+2][N[P/\ell+1-k-1]/k+1] \\ &= (\lambda.M')[P/\ell+1][N[P/\ell-k]/k]\end{aligned}$$

- If $M = (M_1)M_2$, then:

$$\begin{aligned}((M_1)M_2)[N/k][P/\ell] &= (M_1[N/k][P/\ell])M_2[N/k][P/\ell] \\ &= (M_1[P/\ell+1][N[P/\ell-k]/k])M_2[P/\ell+1][N[P/\ell-k]/k] \\ &= ((M_1)M_2)[P/\ell+1][N[P/\ell-k]/k]\end{aligned}$$

□

Proof (of Lem. 2.7): By induction on M.

- If $M = n$, then there are two cases:
  - if $n < i+k+1$, then $(\uparrow_{i+k+1}^j n)[\uparrow_i^j N/k] = n[\uparrow_i^j N/k]$ and there are three further cases:
    - ⋆ if $n < k$, then $n[\uparrow_i^j N/k] = n$ and $\uparrow_{i+k}^j n[N/k] = \uparrow_{i+k}^j n = n$.
    - ⋆ if $n = k$, then $n[\uparrow_i^j N/k] = \uparrow_0^k \uparrow_i^j N = \uparrow_{i+k}^j \uparrow_0^k N$ by Lem. 2.3, and $\uparrow_{i+k}^j n[N/k] = \uparrow_{i+k}^j \uparrow_0^k N$.

    - ⋆ if $n > k$, then $n[\uparrow_i^j N/k] = n - 1$ and $\uparrow_{i+k}^j n[N/k] = \uparrow_{i+k}^j (n-1) = n - 1$ as $n < i + k + 1$.
  - ∘ if $n \geqslant i + k + 1$, then $(\uparrow_{i+k+1}^j n)[\uparrow_i^j N/k] = (n + j)[\uparrow_i^j N/k] = n + j - 1$ and $\uparrow_{i+k}^j n[N/k] = \uparrow_{i+k}^j n - 1 = n + j - 1$ as $n \geqslant i + k + 1 \implies n - 1 \geqslant i + k$.

- If $M = \lambda.M'$, then:

$$\begin{aligned}(\uparrow_{i+k+1}^j (\lambda.M'))[\uparrow_i^j N/k] &= \lambda.((\uparrow_{i+k+2}^j M')[\uparrow_i^j /k+1]) \\ &= \lambda.(\uparrow_{i+k+1}^j M'[N/k+1]) \\ &= \uparrow_{i+k}^j (\lambda.M')[N/k]\end{aligned}$$

- If $M = (M_1)M_2$, then:

$$\begin{aligned}(\uparrow_{i+k+1}^j (M_1)M_2)[\uparrow_i^j N/k] &= ((\uparrow_{i+k+1}^j M_1)[\uparrow_i^j N/k])(\uparrow_{i+k+1}^j M_2)[\uparrow_i^j N/k] \\ &= (\uparrow_{i+k}^j M_1[N/k])(\uparrow_{i+k}^j M_2[N/k]) \\ &= \uparrow_{i+k}^j ((M_1)M_2)[N/k]\end{aligned}$$

□