Do Integers Really Exist?

And Yet Another Talk about Gödel's Incompleteness Theorem

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LIRMM's Semindoc

Introduction

In Search of a Beautiful Theory

Dreams

- A physician's dream – a theory of everything
- A mathematician's dream – a theory to define mathematics

A Successful Endeavor?

That's what we are gonna talk about today.

A First Try

Frege

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- End of 19^{th} century.
- Invention of predicate logic (also called 1^{st} order logic).
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Russel's Paradox (1901)

$$y := \{x \mid x \notin x\}$$

y is the set of sets that are not member of themselves. Then:

$$y \in y \Longleftrightarrow y \notin y$$

Hilbert

Subsequent Formalisations

- Principia Mathematica (Russel – Whitehead)
- Zermelo-Frænkel's set theory.

Foundational Crisis

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Hilbert's program (1930)

- "We must know. We will know."
- Prove that arithmetic isn't contradictory.
- Entscheidungsproblem (Hilbert's decision problem).

Gödel

Theorem (Gödel, 1931)

We will never know.

Some Vocabulary

Predicate Logic

Terms and Formulas

- $x, y, a, b, c, f(x, y, z), \ldots$ are terms.
- Let P be an n-ary relational symbol and $t_1, \ldots, t_n$ be n terms. $P(t_1, \ldots, t_n)$ is a formula. Furthermore, if F_1 and F_2 are formulas, $\neg F_1$, $F_1 \wedge F_2$, $F_1 \vee F_2$, $F_1 \to F_2$, $\forall x\, F_1$ and $\exists x\, F_1$ are also formulas.

Examples

$$\forall x \forall y\, (x + y = y + x)$$
$$\forall x \forall y\, (x \leq y \to \exists z\, (x + z = y))$$

Theories (1)

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A formula F is provable in a theory T iff F is true in the theory. It is denoted $T \vdash F$ and read "T proves F".

Example

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Definition (consistency)

A theory T is consistent iff it doesn't prove false, i.e., there doesn't exist F such that $T \vdash F$ and $T \vdash \neg F$.

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Definition (decidability of a theory)

A theory T is decidable iff for every formula F, we can compute whether $T \vdash F$ or $T \vdash \neg F$.

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Corollary

Given an integer n, it is easy to compute if there exists F such that $\#F = n$.

Gödel's Incompleteness Theorem

Peano's Arithmetic

Language

- The constant 0.
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- $+$ and $\times$, binary function symbols.

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Axioms of $\mathcal{P}_0$

(A_1) 0 is not a successor.

(A_2) If x is not 0, then there exists y such that $x = s(y)$.

(A_3) The successor function s is injective.

(A_4) Forall x, $x + 0 = x$.

(A_5) Forall x, y, $x + s(y) = s(x + y)$.

(A_6) Forall x, $x \times 0 = 0$.

(A_7) Forall x, y, $x \times s(y) = (x \times y) + x$.

Representation Theorem

Survey

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Theorem (admitted)

Every computable function can be represented by a formula.
Furthermore, if a characteristic function of a set A, $\mathbf{1}_A$, is represented by a formula F then for any p-uple $(n_1, \ldots, n_p)$:

- $(n_1, \ldots, n_p) \in A$ iff $\mathcal{P}_0 \vdash F(n_1, \ldots, n_p)$
- $(n_1, \ldots, n_p) \notin A$ iff $\mathcal{P}_0 \nvdash F(n_1, \ldots, n_p)$

Example

Let $A := 2\mathbb{N}$. The following formula F represents A :

$$F(y) := \exists x\,(y = 2 \times x)$$

Gödel's Incompleteness Theorem (1)

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Proof

$$\theta := \{(m, n) \mid m = \#F(x) \wedge T \vdash F(n)\}$$

If T is decidable, then $\mathbf{1}_\theta$ is computable.

$$B := \{n \in \mathbb{N} \mid (n, n) \notin \theta\}$$

As $\mathbf{1}_\theta$ is computable, $\mathbf{1}_B$ is also computable. Let $G(x)$ be the formula that represents B and $a := \#G(x)$. Thus:

- $a \in B \implies (a, a) \notin \theta \implies T \nvdash G(a)$ but $\mathcal{P}_0 \vdash G(a) \implies T \vdash G(a)$.
- $a \notin B \implies (a, a) \in \theta \implies T \vdash G(a)$ but $a \notin B$ thus $\mathcal{P}_0 \vdash \neg G(a)$ and so $T \vdash \neg G(a)$. But T is consistent, so T cannot prove $G(a)$ and $\neg G(a)$.

Gödel's Incompleteness Theorem (2)

Theorem (Gödel, 1931)

Let T be a consistent theory with $\mathbf{1}_T$ computable. If $\mathcal{P}_0 \subseteq T$, then T is incomplete, i.e. there exists a formula F such that $T \nvdash F$ and $T \nvdash \neg F$.

Gödel's Incompleteness Theorem (2)

Theorem (Gödel, 1931)

Let T be a consistent theory with $\mathbf{1}_T$ computable. If $\mathcal{P}_0 \subseteq T$, then T is incomplete, i.e. there exists a formula F such that $T \nvdash F$ and $T \nvdash \neg F$.

Proof

If T was consistent, complete and with $\mathbf{1}_T$ computable, it would be decidable. However, T is consistent and contains $\mathcal{P}_0$ so T is undecidable and thus incomplete.

Consequences

Philosophy

For every system containing a definition of integers, there exists a true sentence that cannot be proved. Hence the entirety of mathematics might be inconsistent and we cannot know if, at some point, a contradiction will occur and render all the mathematics invalid.

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There is a rumor that Gödel’s incompleteness theorem lead Von Neumann to give up mathematics entirely.

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Mathematical Beliefs

- The gold duplication paradox of Banach-Tarski (the axiom of choice).
- Do you know others?

The End

C'est la fin !

Thanks for your attention, do you have questions?