Theory and Metatheory of Elimination in Rocq

Master 2 Internship Defense, Lyon

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[3]LMF & Inria Saclay, Saclay, France

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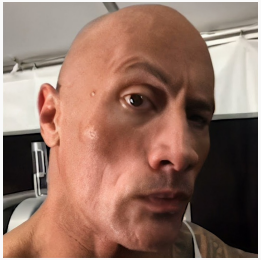

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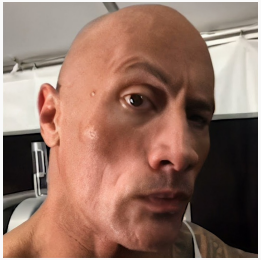

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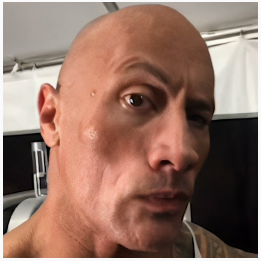

Liberté, égalité, prouvabilité

Avoid duplication in proof assistants.

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Proof assistants?

- Model your problem.
- Try to prove things about it.
- Get a cold, mechanical, answer.[1]

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Rocq is a proof assistant doing things ~~mostly~~ right.

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Le duché de Tiffe

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Concept used in every-day life:

- Program instructions.
- Paths in a graph (or free categories).

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What could *possibly* go wrong?

Comme une sensation de déjà vu...

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In Rocq, terms live in sorts (`Prop`, `Type`).

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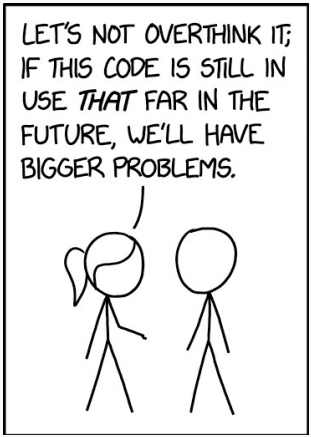

Rare image of the Rocq’s development team, 2005 (xkcd.com/2730)

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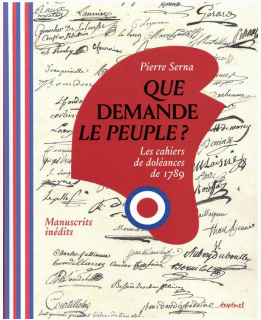

Users ask for more!

- Strict propositions,
- Fibrant or strict universes (HoTT),
- Pure or impure universes (RETT),
- Proof relevance with computational irrelevance (GTT),
- etc...

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Still need to declare 2^n induction principles...

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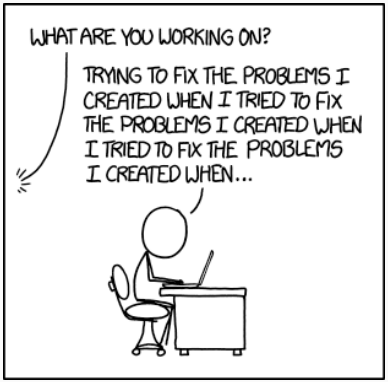

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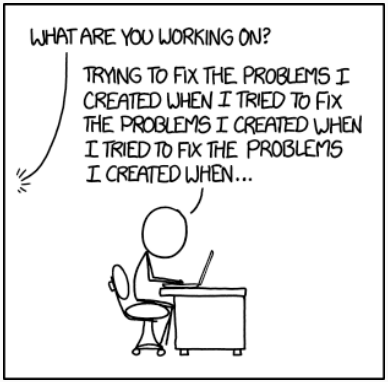

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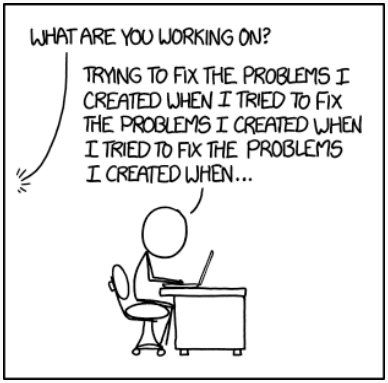

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With ~~sort polymorphism~~, no way to express that.

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Elle est fraiche, elle est fraiche ma théorie !

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Introduce elimination constraints.

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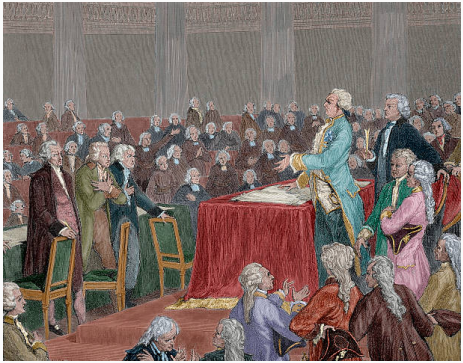
Cool, but… is that it?

Les doléances du peuple



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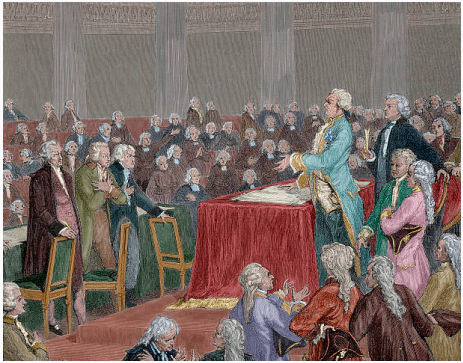

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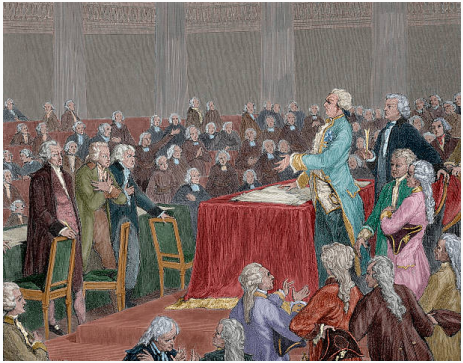

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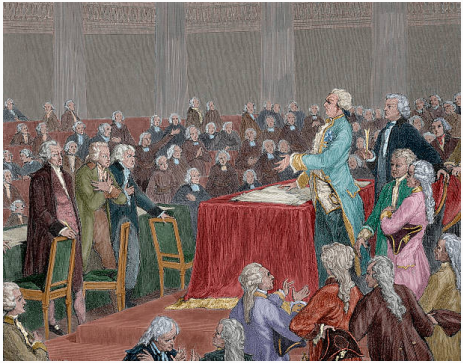

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This is pretty hard.

L’arnaqueur arnaqué

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In the end, what did we get?

- Implemented in Rocq, in the process of being merged.
- Consistency of $\mathrm{SortPoly}^{\vec{\forall}}$ under closure condition.
- Type-checking probably decidable under stronger conditions.

No general theorem, but works for Rocq's Type Theory.

“Au revoir.”

Avoid duplication in proof assistants by using $\mathrm{SortPoly}^{\overrightarrow{\sim}}$.

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Recap:

- Implemented in Rocq (8k lines).
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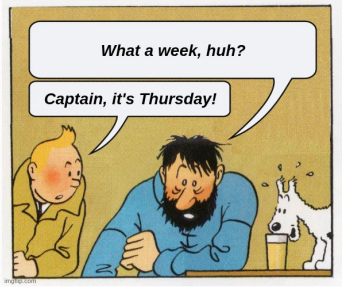

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