Internship report
Random walks in random environments

Julien Allasia
ENS de Lyon - Master 1

Supervisor: Jessica Lin
McGill University, Montreal, Canada

1 Introduction

Random walks in random environments (RWREs) are a quite recent area of research in applied probability theory and mathematical physics, born in the 1970s. Random walks have been a useful tool to model transport processes, for example the movements of particles in a media or the diffusion of heat. However, the relevant characteristics of the media to study the random walk are often highly inhomogeneous. This is why it is relevant to use a random environment, chosen from a space of environments with a certain probability measure.

The choice of assumptions on this probability measure can help us get results and are natural hypotheses in the physical world. One of them, which will be essential, is ellipticity (meaning the random walk has a positive probability of moving to every direction). Another assumption that is often useful is that environments are constructed in an i.i.d. fashion, meaning that the spacial inhomogeneities of the media are i.i.d. However, general results can be obtained in even more general settings where inhomogeneities in points that are far away from each other are almost independent, and the most general assumption for that matter is ergodicity.

One of the most prominent issue in the study of RWREs has been their asymptotic behaviour when rescaling space and time. In lots of quite general cases, the random walks converge to a deterministic Brownian motion for almost every environment. This is a very strong generalization of Donsker's invariance principle. Since RWREs emerged in mathematical research, assumptions to get this convergence have got weaker and weaker, so that it is now known that this convergence holds even in quite surprising cases.

Another major interest of this area of research is that it is closely linked to stochastic homogenization, a subject in partial differential equations which is also very active at the moment.

In this internship, I worked with Paul Melloni (ENS de Lyon) under the supervision of Jessica Lin (McGill University, Montreal). Our goal was to generalize the invariance principle for RWREs to a weaker case that has been studied on the PDE side. This report presents the important results on invariance principles that have been shown for the past decades, sometimes using arguments inspired by other papers. Then we introduce another proof of those results, which will also give a proof for the weaker case we have studied.

2 The invariance principle for RWREs

The setup of this theorem mostly follows the notations and arguments of Guo & Zeitouni in [3], but most ideas were already present in a seminal paper by Lawler [1], which was itself a rewriting of a paper by Papanicolaou & Varadhan [2].

2.1 Definitions

Let's now define properly what a random walk in a random environment is. In order to do this, we have to define two levels of randomness. The first one is that of the environments, the second one is that of the random walk in a fixed environment.

2.1.1 First level of randomness

Definition. Let $d \geq 1$. Let $S = \{(p_1, ..., p_d, p_{-1}, ..., p_{-d}) \in \mathbb{R}^{2d}, \forall i,\, p_i \geq 0, \sum_{i=1}^{d}(p_i + p_{-i}) = 1\}$. We define the set of environments on $\mathbb{Z}^d$ to be $\Omega = S^{\mathbb{Z}^d}$. Take the topology induced from $\mathbb{R}^d$ on S, and take the product topology on Ω. Let $\mathcal{F}$ be the Borel σ-algebra on Ω. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.

S can also be seen as the set of probability measures on the set of the nearest neighbors of the origin in the lattice $\mathbb{Z}^d$. Therefore, choosing $\omega \in \Omega$ is simply assigning to each $x \in \mathbb{Z}^d$ a probability distribution on its nearest neighbors. In other words, if $\omega(x) = (p_1, ..., p_d, p_{-1}, ..., p_{-d})$, and if $1 \leq i \leq d$, p_i is the probability of moving from x to $x + e_i$, and p_{-i} is the probability of moving from x to $x - e_i$. That is why, we will also note, for $1 \leq i \leq d$, $\omega(x, e_i) = \omega_i(x)$ and $\omega(x, -e_i) = \omega_{d+i}(x)$ (and we will often consider ω as the vector $(\omega_i)_{i\in\{1,...,d\}} \in \mathbb{R}^d$).

Assumptions. Define θ to be the shift operator on the environments, that is

$$\forall x_0 \in \mathbb{Z}^d,\, \forall \omega \in \Omega,\, \forall x \in \mathbb{Z}^d, \quad \theta^{x_0}\omega(x) = \omega(x + x_0).$$

We make the following assumptions on the environments for the rest of this section :

1. Balancedness: *for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, $\forall x \in \mathbb{Z}^d$, $\forall i \in \{1, ..., d\}$, $\omega(x, e_i) = \omega(x, -e_i)$.
2. Ellipticity: *for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, $\forall x \in \mathbb{Z}^d$, $\forall i \in \{1, ..., d\}$, $\omega(x, \pm e_i) > 0$.
3. Stationarity: $\forall x_0 \in \mathbb{Z}^d$, $\forall F \in \mathcal{F}$, $\mathbb{P}(\theta^{x_0}F) = \mathbb{P}(F)$.
4. Ergodicity: $\forall F \in \mathcal{F}$, $(\forall x_0 \in \mathbb{Z}^d,\, \theta^{x_0}F = F) \Rightarrow \mathbb{P}(F) \in \{0, 1\}$.

N.B. In the rest of this report, we will often skip the ω subscript for the sake of simplicity, but bear in mind that in what follows, everything is defined after fixing an environment ω.

2.1.2 Second level of randomness

Now let us fix an environment $\omega \in \Omega$ and a point $x \in \mathbb{Z}^d$. We take a random walk in ω to be an element of the following probability space.

Definition. Consider the space $(\mathbb{Z}^d)^{\mathbb{N}}$, equipped with the product σ-algebra $\mathcal{G}$. Take X_n to be the n^{th} projection from $(\mathbb{Z}^d)^{\mathbb{N}}$ to $\mathbb{Z}^d$. Define the probability $\mathbb{P}_\omega^x$ on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ recursively by

$$\begin{cases} \mathbb{P}_\omega^x(X_0 = x) = 1 \\ \mathbb{P}_\omega^x(X_{n+1} = y \pm e_i \,|\, X_n = y) = \omega(y, \pm e_i). \end{cases}$$

Under $\mathbb{P}_\omega^x$, a sequence $(X_n)_n \in (\mathbb{Z}^d)^{\mathbb{N}}$ is called a random walk in the environment ω, starting at x.

Remark. The previous formulas are sufficient to define a unique probability measure on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ thanks to Carathéodory's extension theorem.

Proposition. Let Q_ω be the transition kernel on $\mathbb{Z}^d$ defined by

$$\forall y, z \in \mathbb{Z}^d, \quad Q_\omega(y,z) = \begin{cases} \omega(y, \pm e_i) & \text{if } z = y \pm e_i \\ 0 & \text{otherwise} \end{cases}$$

Then, *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ and under $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G}, \mathbb{P}_\omega^x)$,

- $(X_n)_n$ is an irreducible Q_ω-Markov chain starting at x.
- $(X_n)_n$ is a martingale with respect to the canonical filtration $\mathcal{G}_n = \sigma(X_0, ..., X_n)$.

Proof. The fact that it is a Markov chain is a direct consequence of the definition. If ω is elliptic (assumption 2.), then $(X_n)_n$ is clearly irreducible. If ω is taken to be balanced (assumption 1.), then for each $n \geq 0$, $X_{n+1} - X_n$ takes values in $\{\pm e_i, 1 \leq i \leq d\}$ and

$$\mathbb{P}_\omega^x(X_{n+1} - X_n = e_i \,|\, \mathcal{G}_n) = \mathbb{P}_\omega^x(X_{n+1} - X_n = -e_i \,|\, \mathcal{G}_n) = \omega_i(X_n).$$

Therefore $\mathbb{E}_\omega^x[X_{n+1} - X_n \,|\, \mathcal{G}_n] = 0$, and $(X_n)_n$ is a martingale. □

In the following, we will denote by $L_\omega = Q_\omega - I$ the generator of the random walk, defined by

$$\forall f : \mathbb{Z}^d \to \mathbb{R}, \forall x \in \mathbb{Z}^d, \quad L_\omega f(x) = \mathbb{E}_\omega^x \left[f(X_1) \right] - f(x).$$

2.2 Invariance principle

We now state the invariance principle we are after under two different assumptions, a weaker one and a stronger one. Before that, we need a definition of a new space in which we will work.

Definition. Let $C = \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^d)$. Equip C with the coarsest σ-algebra for which the following projections are continuous for $t \geq 0$:

$$ev_t : \left| \begin{array}{ccc} C & \longrightarrow & (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \\ f & \longmapsto & f(t) \end{array} \right.$$

$(C, \mathcal{C})$ is often referred to as the Wiener space.

Notation. If $b \in (\mathbb{R}_+^*)^d$, let B^b be the d-dimensional Brownian motion started at 0 with covariance matrix diag(b). In other words, $\left(\frac{B^{(1)}}{\sqrt{b_1}}, ..., \frac{B^{(d)}}{\sqrt{b_d}} \right)$ is a standard Brownian motion in $\mathbb{R}^d$.

In order to work in the Wiener space, the random walk has to be interpolated. If $(x_k)_{k \in \mathbb{N}} \in (\mathbb{Z}^d)^{\mathbb{N}}$, we define $(\widetilde{x_t})_{t \geq 0}$ to be the linear interpolation of the $(x_k)_{k \in \mathbb{N}}$, that is, for $t \geq 0$,

$$x_t = x_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) \, (x_{\lfloor t+1 \rfloor} - x_{\lfloor t \rfloor}).$$

Assumptions. For the following theorem to hold, we need to control the transition probabilities of our random walks properly. Ellipticity is actually not sufficient, we need one of the two following additional assumptions. Note that the first one implies the second one, but the proof will be easier in the first case.

4. Uniform ellipticity: $\exists\, \alpha > 0$, *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, $\forall x \in \mathbb{Z}^d$, $\forall i$, $\omega(x, e_i) \geq \alpha$.
5. Moment condition: $\exists\, p > d$, $\mathbb{E}\left[\epsilon^{-p}(0) \right] < \infty$, where $\epsilon(x) = \epsilon_\omega(x) = \prod_{i=1}^{d} \omega(x, e_i)^{1/d}$.

Theorem 2.1. ***(invariance principle)***
There exists $b \in (\mathbb{R}_+^*)^d$ *such that* $\sum_{i=1}^d b_i = 1$ *and*

$$\textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega, \qquad \left(\frac{\widetilde{X}_{nt}}{\sqrt{n}} \right)_{t \geq 0} \underset{n\to\infty}{\longrightarrow} B^b$$

In other words, by rescaling space and time properly, almost all our random walks converge in distribution to the same Brownian motion, which has a deterministic covariance matrix diag(b).

Remark. In terms of weak convergence of measures, this means the following. Let $\mathbb{W}^b$ be the Wiener measure on $(C, \mathcal{C})$ associated to the Brownian motion B^b. For $n \geq 0$, let

$$\phi_n : \left| \begin{array}{ccc} ((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G}, \mathbb{P}_\omega^0) & \longrightarrow & (C, \mathcal{C}) \\ (x_k)_{k\in\mathbb{N}} & \longmapsto & \left(\frac{\widetilde{x}_{nt}}{\sqrt{n}} \right)_{t\geq 0} \end{array} \right.$$

Then if we denote $\mathbb{W}_{\omega,n} = \phi_{n\,*}\mathbb{P}_\omega^0$ the pushforward measure of $\mathbb{P}_\omega^0$ on the Wiener space, the convergence in the theorem is in fact the weak convergence

$$\textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega, \qquad \mathbb{W}_{\omega,n} \underset{n\to\infty}{\longrightarrow} \mathbb{W}^b.$$

2.3 Ergodic theorems

The key idea to use here is that, although the random walks do not have i.i.d increments, almost surely they are martingales. So we can use a generalization of Donsker's theorem using this. Such a theorem can be found in [9] (theorem 4.1.). The idea is the same as in Donsker's theorem, except that we have to take account of the fact that the increments are not i.i.d. by replacing the rescaling variance factor $\sqrt{n\,\sigma^2}$ by $\left(\sum_{j=1}^n \mathbb{E}_\omega^0 \left[(X_j - X_{j-1})^2 \,|\, \mathcal{G}_{j-1} \right] \right)^{1/2}$.

Theorem 2.2. *For* $j \geq 1$*, let* $Z_j = X_j - X_{j-1}$ *and for* $i \in \{1, ..., d\}$*, let* $V_n^i = \sum_{j=1}^n \mathbb{E}_\omega^0 \left[(Z_j^i)^2 \,|\, \mathcal{G}_{j-1} \right]$.
Then, if B *is a standard Brownian motion on* $\mathbb{R}^d$*,*

$$\left(\frac{\widetilde{X}_{nt}}{\sqrt{V_n}} \right)_{t\geq 0} \underset{n\to\infty}{\longrightarrow} B.$$

A simple computation gives that for $\mathbb{P}$-*a.e.* ω, $\mathbb{E}_\omega^0 \left[(Z_j^i)^2 \,|\, \mathcal{G}_{j-1} \right] = 2\,\omega_i(X_{j-1})$, so that in fact $V_n = 2\sum_{j=0}^{n-1} \omega(X_j)$. Therefore, in order to get the result we want, by Slutsky's theorem, it suffices to show the following theorem :

Theorem 2.3. *There exists* $b \in (\mathbb{R}_+^*)^d$ *such that* $\sum_{i=1}^d b_i = 1$ *and*

$$\textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega, \qquad \frac{1}{n} \sum_{j=0}^{n-1} \omega(X_j) \overset{\mathbb{P}_\omega^0\textit{-a.s.}}{\longrightarrow} \frac{1}{2}\, b.$$

This theorem looks like an ergodic theorem because it is a result of convergence of the space average of the $w(x)$'s along the random walk. But it is not quite a standard ergodic theorem yet, because the function is ω itself. The idea is to change the point of view in order to write it as a standard ergodic theorem.

Studying the random walk in a fixed environment under $\mathbb{P}_\omega^0$ amounts to studying the movement of a particle starting from 0. Now let's imagine the particle is fixed at location 0, and the environments move around it ; in other words, let's study the environment viewed from the point of view of the particle.

Definition. Let $\omega \in \Omega$ and $(X_n)_{n\in\mathbb{N}}$ be a random walk in the environment ω. Let's define the dual random walk $(\bar{\omega}_n)_{n\in\mathbb{N}}$ by

$$\forall n \in \mathbb{N}, \quad \bar{\omega}_n = \theta^{X_n}\omega.$$

Remark. If we want to be thorough, we can see the dual random walk as being an element of a dual probability space. Let

$$\psi_\omega : \left| \begin{array}{ccc} ((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G}, \mathbb{P}_\omega^0) & \longrightarrow & \Omega^{\mathbb{N}} \\ (x_n)_{n\in\mathbb{N}} & \longmapsto & (\theta^{x_n}\omega)_{n\in\mathbb{N}} \end{array} \right.$$

Let $\mathcal{H}_\omega$ be the coarsest σ-algebra on $\Omega^{\mathbb{N}}$ that makes ψ_ω measurable: $\mathcal{H}_\omega = \{H \subseteq \Omega^{\mathbb{N}},\ \psi_\omega^{-1}(H) \in \mathcal{G}\}$. The dual random walk of $(X_n)_n$ is an element of the probability space $(\Omega^{\mathbb{N}}, \mathcal{H}_\omega, \psi_{\omega*}\mathbb{P}_\omega^0)$. In the rest of the proof, we still denote by $\mathbb{P}_\omega^0$ the pushforward measure for the sake of simplicity.

Proposition. *For $\mathbb{P}$-a.e. $\omega \in \Omega$ and under $(\Omega^{\mathbb{N}}, \mathcal{H}_\omega, \mathbb{P}_\omega^0)$, $(\bar{\omega}_n)_n$ is Markov chain starting at ω whose transition kernel R is defined by*

$$\forall \omega_1, \omega_2 \in \Omega, \quad R(\omega_1, \omega_2) = \begin{cases} \omega_1(0, \pm e_i) & \text{if } \omega_2 = \theta^{\pm e_i}\omega_1 \\ 0 & \text{otherwise} \end{cases}$$

Now let's rewrite the ergodic theorem we are after in terms of the dual random walk. Let

$$g_0 : \left| \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^d \\ \omega & \longmapsto & (\omega(0, e_i))_{i\in\{1,\ldots,d\}} \end{array} \right.$$

Theorem 2.4. *There exists $b \in (\mathbb{R}_+^*)^d$ such that $\sum_{i=1}^d b_i = 1$ and*

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad \frac{1}{n}\sum_{j=0}^{n-1} g_0(\bar{\omega}_n) \overset{\mathbb{P}_\omega^0\text{-a.s.}}{\longrightarrow} \frac{1}{2} b.$$

Now this is a standard ergodic theorem for Markov chains. In order to prove it, it suffices to construct a probability measure $\mathbb{Q}$ on Ω that is invariant and ergodic with respect to $(\bar{\omega}_n)_n$, so that

$$\mathbb{Q}\text{-a.s.} \quad \frac{1}{n}\sum_{j=0}^{n-1} g_0(\bar{\omega}_j) \to \int_\Omega g_0 \, \mathrm{d}\mathbb{Q}.$$

Also, we need $\mathbb{P}$ to be absolutely continuous with respect to $\mathbb{Q}$, so that having convergence $\mathbb{Q}$-*a.s.* gives the result $\mathbb{P}$-*a.s.* This is actually possible under our assumptions :

Theorem 2.5. *There exists a probability measure $\mathbb{Q}$ on Ω such that :*

1. *$\mathbb{Q}$ is mutually absolutely continuous with respect to $\mathbb{P}$;*
2. *$\mathbb{Q}$ is invariant and ergodic with respect to the Markov chain $(\bar{\omega}_n)_n$.*

2.4 Proof of theorem 2.5

The proof presented here mixes arguments of [1], [3] and [5], completed where it needs to be.
We have to prove the ergodic theorem above. The idea is to construct $\mathbb{Q}$ as a limit of invariant

measures in the case where the environments are periodic. This is motivated by two ideas which will be developed in the proof :

- A stationary measure (assumption 3.) can be approximated by measures on finite spaces.
- Periodicity will allow us to work on finite state spaces, and an irreducible Markov chain on a finite state space always has a (unique) invariant probability measure.

Notation. Let $N \geq 1$.

- Let $\Delta_N = \{x \in \mathbb{Z}^d,\ |x|_\infty \leq N\}$ be the hypercube of size N. Note that $|\Delta_N| = (2N+1)^d$.
- Define, for $x \in \mathbb{Z}^d$, $\hat{x} = x + (2N+1)\,\mathbb{Z}^d \in \mathbb{Z}^d/(2N+1)\,\mathbb{Z}^d$. In the rest of this report, we'll do the identification $\mathbb{Z}^d/(2N+1)\,\mathbb{Z}^d = \Delta_N$.
- Define the periodic environment associated to $\omega \in \Omega$ by

$$\omega^N(x) = \begin{cases} \omega(x) \text{ if } x \in \Delta_N \\ \omega(\bar{x}) \text{ where } \bar{x} \in \Delta_N \text{ and } \hat{x} = \hat{\bar{x}}. \end{cases}$$

  Let $\Omega^N = \{\omega^N,\ \omega \in \Omega\}$ be the set of periodic environments.

- Let $(X_n^N)_{n\in\mathbb{N}}$ be a random walk under ω^N, $(\bar{\omega}_n^N) = (\theta^{X_n^N}\omega^N)$ be the associated random walk on Ω^N. Note that the transition kernel of $(X_n^N)_n$ is Q_{ω^N} and that of $(\bar{\omega}_n^N)_n$ is still R.

2.4.1 Constructing invariant measures on the Ω^N spaces

For a fixed environment $\omega \in \Omega$, consider the random walk $(\hat{X}_n^N)_{n\in\mathbb{N}}$ on Δ_N. Denote by $\hat{Q}_{\omega^N}$ its transition kernel. It is, just as in the previous cases, an irreducible Markov chain. Since Δ_N is a finite state space, the Markov chain has an invariant probability measure that we shall denote μ_N.

Let Φ_N be the density of μ_N with respect to the uniform measure on Δ_N, so that

$$\mu_N = \frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} \Phi_N(x)\,\delta_x.$$

We naturally define a probability measure on Ω^N by

$$\mathbb{Q}_N = \mathbb{Q}_{\omega,N} = \frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} \Phi_N(x)\,\delta_{\theta^x\omega^N}.$$

Then $\mathbb{Q}_N$ is an invariant probability measure for the Markov chain $(\bar{\omega}_n^N)_n$, because $\forall x \in \Delta_N$,

$$\begin{aligned} \mathbb{Q}_N\,R\,(\theta^x\omega^N) &= \sum_{y\in\Delta_N} \mathbb{Q}_N(\theta^y\omega^N)\,R(\theta^y\omega^N, \theta^x\omega^N) \\ &= \sum_{y\in\Delta_N} \frac{\Phi_N(y)}{|\Delta_N|}\,\hat{Q}_{\omega^N}(y,x) \\ &= \sum_{y\in\Delta_N} \mu_N(y)\,\hat{Q}_{\omega^N}(y,x) \\ &= \mu_N(x) \\ &= \mathbb{Q}_N(\theta^x\omega^N). \end{aligned}$$

2.4.2 Compactness arguments for $N \to \infty$

The $\mathbb{Q}_N$ measures can be trivially extended to probability measures on Ω. Now, Ω is a compact space (by Tykhonov's theorem). It is also metrizable, and the product metric gives the same topology as the product topology we put on Ω. It is therefore separable, and by a corollary of the algebraic Stone-Weierstrass theorem, $\mathcal{C}^0(\Omega, \mathbb{R}) = \mathcal{C}^0_b(\Omega, \mathbb{R})$ is separable too.
According to Banach Alaoglu's theorem, the unit ball of $\mathcal{C}^0(\Omega, \mathbb{R})^*$ is weakly-* compact, and because of separability it is also weakly-* metrizable. Therefore it is sequentially compact. Now the probability measures on Ω can be seen as elements of this unit ball, through

$$\phi : \left| \begin{array}{ccc} \{\text{probability measures on } \Omega\} & \longrightarrow & \mathcal{C}^0(\Omega, \mathbb{R})^* \\ \mu & \longmapsto & \phi_\mu : f \mapsto \int f \mathrm{d}\mu \end{array} \right.$$

Therefore we can extract a subsequence of $(\phi_{\mathbb{Q}_N})_{N \geq 1}$ that converges weakly-* to a certain element of $\mathcal{C}^0(\Omega, \mathbb{R})^*$. Now, by applying Riesz-Markov-Kakutani's representation theorem, this limit is itself a $\phi_{\mathbb{Q}}$. Moreover, the weak-* convergence is equivalent to the weak convergence of the associated measures. Therefore, there exists a subsequence of $(\mathbb{Q}_N)_N$ which converges weakly to a probability measure $\mathbb{Q}$.

2.4.3 The limit measure has the good properties

We make the following statement, which we will prove later (there will be a simpler proof in the uniformly elliptic case but we will also show it in the finite moment case).

Lemma 2.6. *There exists $q > 1$ and $C > 0$ (depending only on d and $\mathbb{P}$) such that for every continuous bounded function g on Ω,*

$$\mathbb{P}\text{-}a.s. \qquad \left| \int_\Omega g \, \mathrm{d}\mathbb{Q} \right| \leq C \left(\int_\Omega |g|^{q'} \mathrm{d}\mathbb{P} \right)^{1/q'}.$$

Using this, let us show the properties that we need for $\mathbb{Q}$.

- $\mathbb{Q} \ll \mathbb{P}$.
  This is not the absolute continuity that we need, but it is useful to show the reciprocal. Note that if the lemma was true for indicator functions, we would directly deduce that $\mathbb{Q} \ll \mathbb{P}$ almost surely. Let $A \in \mathcal{F}$ be an event such that $\mathbb{P}(A) = 0$. We have to show that $\mathbb{Q}(A) = 0$. Because Ω is a Polish space, it is a Radon space and it suffices to check that for all closed subset $F \subseteq A$, $\mathbb{Q}(F) = 0$. For $\delta > 0$, let us define $g_F^\delta = \left(1 - \frac{d(\cdot, F)}{\delta}\right)^+$ where $d(\cdot, F)$ is the distance to F. This is a continuous bounded function on Ω, whose support is contained in $F_\delta = \{\pi \in \Omega, \, d(\pi, F) < \delta\}$. Because $\mathbb{1}_F \leq g_F^\delta \leq \mathbb{1}_{F_\delta}$, we have

  $$\mathbb{Q}(F) \leq \int_\Omega g_F^\delta \, \mathrm{d}\mathbb{Q} \leq C \left(\int_\Omega (g_F^\delta)^{q'} \, \mathrm{d}\mathbb{P} \right)^{1/q'} \leq C \left(\mathbb{P}(F_\delta) \right)^{1/q'}$$

  and letting $\delta \to 0$ we get $\mathbb{Q}(F) \leq C \, \mathbb{P}(F)^{1/q'} \leq C \, \mathbb{P}(A)^{1/q'} = 0$.

- Invariance. The invariance of $\mathbb{Q}$ is a consequence of the invariance of the $\mathbb{Q}_N$'s, but the proof remains to be written.

- $\mathbb{P} \ll \mathbb{Q}$.
  Because $\mathbb{Q} \ll \mathbb{P}$, all we need to show is that the set $E = \{\omega \in \Omega, \, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega) = 0\}$ satisfies $\mathbb{P}(E) = 0$.
  - First let's prove that $\mathbb{P}$-$a.s.$ $R \mathbb{1}_E \leq \mathbb{1}_E$ (recall R denotes the transition kernel of our Markov

chain). Because $\mathbb{Q}$ is invariant with respect to R, using the Fubini theorem,

$$\mathbb{E}_{\mathbb{Q}}[R\mathbb{1}_E] = \mathbb{Q}(E) = 0.$$

As a result $\mathbb{Q}$-*a.s.* $R\mathbb{1}_E = 0$, and

$$\mathbb{E}_{\mathbb{P}}[(R\mathbb{1}_E)\mathbb{1}_{E^c}] = \mathbb{E}_{\mathbb{Q}}\left[(R\mathbb{1}_E)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\mathbb{1}_{E^c}\right] = 0$$

where $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ can be inverted on E^c. Therefore, for $\mathbb{P}$-*a.e.* $\omega \in E^c$, $R\mathbb{1}_E \leq \mathbb{1}_E$. Of course, the inequality also holds on E, since $R\mathbb{1}_E(\cdot) = R(\cdot, E) \leq 1$.

– From this, let's show that E is invariant under shifts. For every $i \in \{1, ..., d\}$, we have

$$\mathbb{P}\text{-}a.s. \quad \mathbb{1}_E(\omega) \geq R\mathbb{1}_E(\omega) = \sum_{i=1}^{d} \omega(0, \pm e_i)\left(\mathbb{1}_E(\theta^{e_i}\omega) + \mathbb{1}_E(\theta^{-e_i}\omega)\right) \geq \epsilon(\omega)\,\mathbb{1}_E(\theta^{\pm e_i}\omega)$$

where $\epsilon(\omega) > 0$ by ellipticity. Therefore $\mathbb{1}_E \geq \mathbb{1}_E \circ \theta^{\pm e_i}$, which means that $\mathbb{P}$-*a.s.* $\theta^{\mp e_i}E \subseteq E$, and by stationarity, $\mathbb{P}(E \subseteq \theta^{\mp e_i}E) = \mathbb{P}(\theta^{\pm e_i}E \subseteq E) = 1$, so $\mathbb{P}$-*a.s.* $\theta^{\pm e_i}E = E$, which yields that $\mathbb{P}$-*a.s.* E is invariant under shifts.

– Let $A = \cap_{x\in\mathbb{Z}^d}\theta^x E$. By construction A is invariant under shifts, so by ergodicity, $\mathbb{P}(A) \in \{0, 1\}$. Now $\mathbb{P}(A) = \mathbb{P}(E)$, so, because $\mathbb{Q}(E) = 0$ and $\mathbb{Q} \ll \mathbb{P}$, necessarily $P(A) = 0$, so $P(E) = 0$.

- Ergodicity. Note that since $\mathbb{P}$ is ergodic and $\mathbb{Q} \ll \mathbb{P}$, $\mathbb{Q}$ is also ergodic (with respect to the shifts). Now, ergodicity with respect to the shifts gives ergodicity with respect to the Markov chain R, that is

$$\forall A \in \mathcal{F},\ (\forall \omega \in A,\ R(\omega, A^c) = 0) \Rightarrow \mathbb{Q}(A) \in \{0, 1\}.$$

Indeed, take $A \in \mathcal{F}$ such that $\forall \omega \in A$, $R(\omega, A^c) = 0$. Then, consider the event $F = [\theta^{e_i}A \nsubseteq A]$. If $\omega_0 \in F$, write $\omega_0 = \theta^{e_i}\omega_1$ with $\omega_1 \in A$. Because $\omega_0 \notin A$, we get $R(\omega_1, \omega_0) = 0$. But, by ellipticity, $\mathbb{P}$-*a.s.* $R(\omega_1, \omega_0) \neq 0$. So $\mathbb{P}(F) = 0$, and because of stationarity, $\mathbb{P}(\forall x \in \mathbb{Z}^d,\ \theta^x A = A) = 1$. Because $\mathbb{Q} \ll \mathbb{P}$, $\mathbb{Q}(\forall x \in \mathbb{Z}^d,\ \theta^x A = A) = 1$. Letting $G = \cap_{x\in\mathbb{Z}^d}\theta^x A$, by ergodicitiy $\mathbb{Q}(G) \in \{0, 1\}$, so $\mathbb{Q}(A) \in \{0, 1\}$.

Remark.
- It appears that the limit $\mathbb{Q}$ depends on ω and that what we have truly shown is that $\mathbb{P}$-*a.s.* $\mathbb{Q} \ll \mathbb{P}$. But in fact, the ergodicity of $\mathbb{Q}$, together with the fact that $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous, ensures that there exists a unique possible limit $\mathbb{Q}$ (because two distinct ergodic measures are always mutually singular).
- An interesting thing to notice is that actually $\mathbb{Q}_N \rightharpoonup \mathbb{Q}$ is true without having to extract a subsequence. Indeed, in a metric space, a sequence $(u_n)_n$ converges to l if and only if from every subsequence of $(u_n)_n$ we can extract a subsubsequence converging to l. This property together with the fact that the only possible limit for a subsequence of $\mathbb{Q}_N$ is $\mathbb{Q}$ ensures that $\mathbb{Q}_N \rightharpoonup \mathbb{Q}$.

2.5 Proof of lemma 2.6

The proof of this lemma relies on a discrete version of the maximum principle for PDEs, widely known as the Aleksandrov-Bakelman-Pucci estimate (ABP). This maximum principle uses our assumptions, so we will state in a different way in the two setups. A proof of this theorem can be found in [5], p.274.

Remember Q_{ω^N} denotes the transition kernel of the random walk $(X_n^N)_n$ in the environment ω^N, and $L_{\omega^N} = Q_{\omega^N} - I$ is the generator of this Markov chain, that is

$$L_{\omega^N} f(x) = \sum_{i=1}^{d} \omega^N(x, e_i)\,[f(x+e_i) + f(x-e_i) - 2f(x)].$$

Note that this is a discrete analog of an elliptic operator of the form $Lf(x) = \mathrm{Tr}(A(x,\omega)\,D^2 f)$ where $A(x,\omega) = \mathrm{diag}(\omega(x,e_1),...,\omega(x,e_d))$, which is why we use a PDE-type argument here.
For $g : \Delta_N \to \mathbb{R}$, we define the q-norms to be $||g||_{L^q(\Delta_N)} = \left(\frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} |g(x)|^q \right)^{1/q}$.

Lemma 2.7. ***(ABP maximum principle)***
Let $E \subseteq \mathbb{Z}^d$ be a bounded set. Let's define $\partial E = \{y \in E^c, \exists\, x \in E, |x-y|_\infty = 1\}$, $\bar{E} = E \cup \partial E$, and $\mathrm{diam}\,\bar{E} = \max\{|x-y|_\infty,\ x, y \in \bar{E}\}$. Let $u : \bar{E} \to \mathbb{R}$ and $g : E \to \mathbb{R}$.
Let $\omega \in \Omega$ be such that the assumptions on the environments are true for ω.
Suppose that $L_\omega u \geq -g$ in E.

- *(Uniformly elliptic case) There exists $C_\alpha > 0$ (depending on d and α) such that*

$$\max_E u \leq \max_{\partial E} u + C_\alpha\,(\mathrm{diam}\,\bar{E})\,|E|^{1/d}\ \|g\|_{L^d(E)}$$

- *(General case) There exists $C_0 > 0$ (depending only on d) such that*

$$\max_E u \leq \max_{\partial E} u + C_0\,(\mathrm{diam}\,\bar{E})\,|E|^{1/d}\ \left\| \frac{g}{\epsilon} \right\|_{L^d(E)}$$

2.5.1 The uniformly elliptic case

This part of the proof is inspired by the Lawler paper [1]. However, here it is presented using the maximum principle rather than elementary but unpleasant lemmas.

Definition. Define the resolvent $\mathcal{R}_N$ of the Markov chain to be the following operator :

$$\forall g : \Delta_N \to \mathbb{R}, \qquad \mathcal{R}_N g(x) = \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \mathbb{E}_\omega^x \left[g(\hat{X}_j^N) \right].$$

Lemma 2.8. *Define the scalar product $\langle\cdot,\cdot\rangle$ by $\langle f, g\rangle = \frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} f(x)\,g(x)$ for $f, g : \Delta_N \to R$. Then the adjoint of the resolvent and the density Φ_N satisfy*

$$\mathcal{R}_N^* \,\Phi_N = N^2\,\Phi_N.$$

Proof. Let $g : \Delta_N \to \mathbb{R}$. Remember $\hat{Q}_{\omega^N}$ is the transition kernel of $(\hat{X}_n^N)_{n\in\mathbb{N}}$.

$$\begin{aligned}
\langle \mathcal{R}_N^* \Phi_N,\, g\rangle &= \langle \Phi_N,\, \mathcal{R}_N g\rangle \\
&= \frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} \Phi_N(x) \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \hat{Q}^j_{\omega^N}\, g(x) \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \sum_{x\in\Delta_N} \frac{\Phi_N(x)}{|\Delta_N|}\, \hat{Q}^j_{\omega^N}\, g(x) \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \sum_{x\in\Delta_N} \mu_N(x)\, \hat{Q}^j_{\omega^N}\, g(x) \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \sum_{x\in\Delta_N} \sum_{x_1,\ldots,x_j\in\Delta_N} \mu_N(x)\, \hat{Q}_{\omega^N}(x, x_1) \ldots \hat{Q}_{\omega^N}(x_{j-1}, x_j)\, g(x_j) \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \sum_{x\in\Delta_N} \mu_N(x)\, g(x) \\
&= N^2\, \langle \Phi_N, g\rangle.
\end{aligned}$$

□

Lemma 2.9. *There exists a constant $C > 0$ (depending only on d and α) such that for every $\omega \in \Omega$ and $f : \Delta_N \to \mathbb{R}_+$,*

$$\|\mathcal{R}_{\omega^N}\, f\|_\infty \le C\, N^2\, \|f\|_{L_d(\Delta_N)}.$$

Proof. We want to use the ABP estimate, so we compute, for all $x \in \Delta_N$,

$$\begin{aligned}
L_{\omega^N}(\mathcal{R}f)(x) = L_{\omega^N}(\mathcal{R}_{\omega^N} f)(x) &= \mathbb{E}^x_{\omega^N}\left[\mathcal{R}f(\hat{X}_1^N)\right] - \mathcal{R}f(x) \\
&= \mathbb{E}^x_{\omega^N}\, \mathbb{E}^{\hat{X}_1^N}_{\omega^N}\left[\sum_{j=0}^{\infty}\left(1 - \frac{1}{N^2}\right)^j f(\hat{X}_j^N)\right] - \mathcal{R}f(x) \\
&= \mathbb{E}^x_{\omega^N}\, \mathbb{E}^x_{\omega^N}\left[\sum_{j=0}^{\infty}\left(1 - \frac{1}{N^2}\right)^j f(\hat{X}_{j+1}^N) \mid \mathcal{G}_1\right] - \mathcal{R}f(x) \\
&= \mathbb{E}^x_{\omega^N}\left[\sum_{j=1}^{\infty}\left(1 - \frac{1}{N^2}\right)^{j-1} f(\hat{X}_j^N)\right] - \mathcal{R}f(x) \\
&= \mathbb{E}^x_{\omega^N}\left[\sum_{j=1}^{\infty}\left(1 - \frac{1}{N^2}\right)^j \left[\frac{1}{1 - 1/N^2} - 1\right] f(\hat{X}_j^N)\right] - f(x) \\
&= \frac{1}{N^2 - 1}\, \mathcal{R}f(x) - \frac{N^2}{N^2 - 1}\, f(x) \\
&\ge -2\, f(x).
\end{aligned}$$

Therefore, applying the maximum principle, we get that $\max_{\Delta_N} \mathcal{R}_{\omega^N} f \le 2\, C_\alpha\, N^2\, \|f\|_{L^d(\Delta_N)}$. □

Let $\mathbb{P}_N = \mathbb{P}_{\omega,N}$ be the uniform probability measure on the $\{\theta^x \omega^N,\, x \in \Delta_N\}$, that is

$$\mathbb{P}_N = \frac{1}{|\Delta_N|} \sum_{x\in\Delta_N} \delta_{\theta^x \omega^N}.$$

To compute the density of $\mathbb{Q}_N$ with respect to $\mathbb{P}_N$, let us write $\{\theta^x\omega^N,\, x\in\Delta_N\} = \{\omega_i^N\}_{i\in\{1,\dots,m\}}$ where the ω_i^N are distinct. For each i, let $C_i^N = \{x\in\Delta_N,\, \theta^x\omega^N = \omega_i^N\}$. Then we have

$$\mathbb{P}_N = \frac{1}{|\Delta_N|}\sum_{i=1}^m |C_i^N|\,\delta_{\omega_i^N}$$

$$\mathbb{Q}_N = \frac{1}{|\Delta_N|}\sum_{i=1}^m \left(\sum_{x\in C_i^N}\Phi_N(x)\right)\delta_{\omega_i^N}$$

$$\frac{\mathrm{d}\mathbb{Q}_N}{\mathrm{d}\mathbb{P}_N} = \sum_{i=1}^m \left(\frac{1}{|C_i^N|}\sum_{x\in C_i^N}\Phi_N(x)\right)\delta_{\omega_i^N}.$$

Let q be defined as in the lemma, and let q' be its conjugate exponent ($1/q + 1/q' = 1$). Therefore we have, using Hölder's inequality twice, for every continuous bounded function g on Ω,

$$\begin{aligned}\left|\int_\Omega g\,\mathrm{d}\mathbb{Q}_N\right| &\le \left(\int_\Omega \left(\frac{\mathrm{d}\mathbb{Q}_N}{\mathrm{d}\mathbb{P}_N}\right)^q \mathrm{d}\mathbb{P}_N\right)^{1/q}\left(\int_\Omega |g|^{q'}\mathrm{d}\mathbb{P}_N\right)^{1/q'}\\ &\le \left(\frac{1}{|\Delta_N|}\sum_{x\in\Delta_N}\Phi_N(x)^q\right)^{1/q}\left(\int_\Omega |g|^{q'}\mathrm{d}\mathbb{P}_N\right)^{1/q'}\\ &= ||\Phi_N||_{L^q(\Delta_N)}\left(\int_\Omega |g|^{q'}\mathrm{d}\mathbb{P}_N\right)^{1/q'} \qquad (1)\end{aligned}$$

Now, up to some extraction, the left term converges to $\left|\int_\Omega g\,\mathrm{d}\mathbb{Q}\right|$. We also have that almost surely $\mathbb{P}_N \rightharpoonup \mathbb{P}$: because $\mathbb{P}$ is assumed to be invariant and ergodic, the multidimensional Birhoff ergodic theorem ensures that for every continuous bounded function g on Ω,

$$\begin{aligned}\int_\Omega g\,\mathrm{d}\mathbb{P}_N &= \sum_{x\in\Delta_N} g(\theta^x\omega^N)\,\mathbb{P}_N(\theta^x\omega^N)\\ &= \frac{1}{|\Delta_N|}\sum_{x\in\Delta_N} g(\theta^x\omega^N)\\ &\underset{N\to\infty}{\longrightarrow}\int_\Omega g\,\mathrm{d}\mathbb{P} \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega\in\Omega.\end{aligned}$$

Therefore we only need to get an estimate on $||\Phi_N||_{L^q(\Delta_N)}$. Now we showed that $\mathcal{R}_N^*\Phi_N = N^2\,\Phi_N$. Besides, lemma 2.9 ensures that the operator $\mathcal{R}_N : L^d(\Delta_N)\to L^\infty(\Delta_N)$ is bounded by $C\,N^2$, so the adjoint operator $\mathcal{R}_N^* : L^1(\Delta_N)\to L^{d/d-1}(\Delta_N)$ is also bounded by $C\,N^2$. So

$$N^2\,\|\Phi_N\|_{L^{d/d-1}(\Delta_N)} \le C\,N^2\,\|\Phi_N\|_{L^1(\Delta_N)} = C\,N^2$$

so that $\|\Phi_N\|_{L^{d/d-1}(\Delta_N)} \le C$. Taking $q = d/d-1$, and using this in equation (1), we get that for every continuous bounded function on Ω,

$$\mathbb{P}\text{-}a.s. \qquad \left|\int_\Omega g\,\mathrm{d}\mathbb{Q}\right| \le C\left(\int_\Omega |g|^{q'}\mathrm{d}\mathbb{P}\right)^{1/q'}.$$

2.5.2 The finite moment case

This more general case was studied by Guo & Zeitouni in [3] and rephrased in [5]. The proof presented here hopefully clears up some issues in both papers using arguments from [6].

Let $\tau_0 = 0$, and $\tau = \tau_1 = \inf\{j \geq 1, |X_j^N - X_0^N|_\infty > N\}$ be the first hitting time of the boundary of the discrete ball of radius N around the starting point. For $m \geq 1$, let

$$\tau_{m+1} = \inf\{j > \tau_m, |X_j^N - X_{\tau_m}^N|_\infty > N\}.$$

Lemma 2.10. *There exists $C \in]0,1[$ such that for every $\omega \in \Omega$ and $x \in \mathbb{Z}^d$,*

$$\mathbb{E}_{\omega^N}^x \left[\left(1 - \frac{1}{N^2}\right)^\tau \right] \leq C.$$

Proof. First, let's prove that we have $\mathbb{E}_{\theta^x\omega^N}^0 \left[X_k^N(i)^2\right] \leq k$ for all $k \geq 1$ and $i \in \{1, ..., d\}$.
Denote $Z_j^N = X_j^N - X_{j-1}^N$ for $j \geq 1$. Then the $(Z_j^N(i))_j$ are uncorrelated, because for all x and ω,

$$\forall j_1 < j_2,\ \mathbb{E}_\omega^x \left[Z_{j_1}^N(i)\, Z_{j_2}^N(i)\right] = \mathbb{E}_\omega^x \left[\mathbb{E}_\omega^x \left[Z_{j_1}^N(i)\, Z_{j_2}^N(i) \,|\, \mathcal{G}_{j_1}\right]\right] = \mathbb{E}_\omega^x \left[Z_{j_1}^N(i)\, \mathbb{E}_\omega^x \left[Z_{j_2}^N(i) \,|\, \mathcal{G}_{j_1}\right]\right] = 0.$$

So we get

$$\mathbb{E}_{\theta^x\omega^N}^0 \left[X_k^N(i)^2\right] = \sum_{j=1}^k \mathbb{E}_{\theta^x\omega^N}^0 \left[Z_j^N(i)^2\right] + \sum_{j_1 \neq j_2} \mathbb{E}_{\theta^x\omega^N}^0 \left[Z_{j_1}^N(i)\, Z_{j_2}^N(i)\right] \leq k.$$

Now $(X_n^N)_{n\in\mathbb{N}}$ is a martingale, so, if $k \geq 1$, by Doob's inequality and Schwarz's inequality,

$$\begin{aligned}
\mathbb{P}_{\theta^x\omega^N}^0(\tau \leq k) &\leq 2\sum_{i=1}^d \mathbb{P}_{\theta^x\omega^N}^0 \left(\sup_{n\leq k} X_n^N(i) \geq N+1\right) \\
&\leq \frac{2}{N+1} \sum_{i=1}^d \mathbb{E}_{\theta^x\omega^N}^0 \left[X_k^N(i)^+\right] \\
&\leq \frac{2}{N+1} \sum_{i=1}^d \sqrt{\mathbb{E}_{\theta^x\omega^N}^0 [X_k^N(i)^2]} \\
&\leq \frac{2d}{N+1}\sqrt{k}.
\end{aligned}$$

Therefore we get $\mathbb{E}_{\theta^x\omega^N}^0 \left[\left(1 - \frac{1}{N^2}\right)^\tau\right] \leq \left(1 - \frac{1}{N^2}\right)^k + \frac{2d}{N+1}\sqrt{k}$.
Taking $k = \lfloor \frac{N^2}{8d^2} \rfloor$, we get the result for $\mathbb{E}_{\theta^x\omega^N}^0 \left[\left(1 - \frac{1}{N^2}\right)^\tau\right]$, which by the definition of τ, is equal to $\mathbb{E}_{\omega^N}^x \left[\left(1 - \frac{1}{N^2}\right)^\tau\right]$. □

End of the proof

Fix $\omega \in \Omega$. Just like in the uniformly elliptic case, the idea is to use the resolvent, but here we'll work with the random walk on the environments.

Let g be a continuous bounded non-negative function on Ω. Note that because $\mathbb{Q}_N$ is invariant for $\bar{\omega}_n^N$, using the same computation as in the adjoint equation for the resolvent, we have for all $j \geq 0$,

$$\int_{\Omega^N} g \,\mathrm{d}\mathbb{Q}_N = \int_{\Omega^N} R^j g \,\mathrm{d}\mathbb{Q}_N.$$

To simplify notations, if $\pi \in \Omega^N$, let g_π be the function on $\mathbb{Z}^d$ defined by $g_\pi(x) = g(\theta^x \pi)$. Also, denote by $\Omega^N_\omega = \{\theta^x \omega^N,\, x \in \Delta_N\}$ (it is the support of the measure $\mathbb{Q}^N$). Then we get, using the resolvent,

$$\begin{aligned}
N^2 \int_{\Omega^N} g \,\mathrm{d}\mathbb{Q}_N &= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \int_{\Omega^N} R^j g \,\mathrm{d}\mathbb{Q}_N \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j \sum_{\pi \in \Omega^N_\omega} \mathbb{E}^0_\pi \left[g_\pi(X^N_j) \right] \mathbb{Q}_N(\pi) \\
&\leq \sup_{\pi \in \Omega^N_\omega} \mathbb{E}^0_\pi \left[\sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j g_\pi(X^N_j) \right] \\
&\leq \sup_{\pi \in \Omega^N_\omega} \sum_{m=0}^{\infty} \mathbb{E}^0_\pi \left[\sum_{\tau_m \leq j < \tau_{m+1}} \left(1 - \frac{1}{N^2}\right)^j g_\pi(X^N_j) \right] \\
&\leq \sup_{\pi \in \Omega^N_\omega} \sum_{m=0}^{\infty} \mathbb{E}^0_\pi \left[\left(1 - \frac{1}{N^2}\right)^{\tau_m} \mathbb{E}^{X^N_{\tau_m}}_\pi \left[\sum_{j=0}^{\tau - 1} g_\pi(X^N_j) \right] \right] \\
&\leq \sup_{\pi \in \Omega^N_\omega} \sup_{y \in \mathbb{Z}^d} \mathbb{E}^y_\pi \left[\sum_{j=0}^{\tau - 1} g_\pi(X^N_j) \right] \sum_{m=0}^{\infty} \mathbb{E}^0_\pi \left[\left(1 - \frac{1}{N^2}\right)^{\tau_m} \right] \\
&\leq \sup_{\pi \in \Omega^N_\omega} \sup_{y \in \mathbb{Z}^d} \mathbb{E}^y_\pi \left[\sum_{j=0}^{\tau - 1} g_\pi(X^N_j) \right] \sum_{m=0}^{\infty} \left(\sup_{y \in \mathbb{Z}^d} \mathbb{E}^y_\pi \left[\left(1 - \frac{1}{N^2}\right)^{\tau} \right] \right)^m \\
&\leq \frac{1}{1 - C} \sup_{\pi \in \Omega^N_\omega} \sup_{y \in \mathbb{Z}^d} \mathbb{E}^y_\pi \left[\sum_{j=0}^{\tau - 1} g_\pi(X^N_j) \right] \\
&= \frac{1}{1 - C} \sup_{\pi \in \Omega^N_\omega} \mathbb{E}^0_\pi \left[\sum_{j=0}^{\tau - 1} g_\pi(X^N_j) \right]
\end{aligned}$$

Note that because of the definition of τ, we have for any function ψ,

$$\mathbb{E}^x_\pi[\psi(\tau)] = \mathbb{E}^0_{\theta^x \pi}[\psi(\tau)] = \mathbb{E}^0_{\theta^x \pi}[\psi(T)].$$

where $T = \inf\{j \geq 0,\, |X^N_j|_\infty > N\}$.

Now we want to use the maximum principle. Suppose ω satisfies the assumptions made on the environments (the statements will therefore be for $\mathbb{P}$-*a.e.* ω). Now define, for $x \in \mathbb{Z}^d$ and $\pi \in \Omega^N_\omega$,

$$f_\pi(x) = \mathbb{E}^x_\pi \left[\sum_{j=0}^{T-1} g_\pi(X^N_j) \right].$$

Let $(\mathcal{G}^N_n)_{n \in \mathbb{N}}$ denote the canonical filtration associated with the random walk $(X^N_n)_{n \in \mathbb{N}}$. Observe that under $\mathbb{P}^x_\pi$ with $x \in \Delta_N$, $T \geq 1$ and so $T((X^N_n)_{n \geq 0}) = T((X^N_n)_{n \geq 1}) + 1$. We can therefore apply the strong Markov property : for every $x \in \Delta_N$,

$$\begin{aligned}
L_\pi f_\pi(x) &= \mathbb{E}_\pi^x[f_\pi(X_1^N)] - f_\pi(x) \\
&= \mathbb{E}_\pi^x\,\mathbb{E}_\pi^{X_1^N}\left[\sum_{j=0}^{T-1} g_\pi(X_j^N)\right] - f_\pi(x) \\
&= \mathbb{E}_\pi^x\,\mathbb{E}_\pi^{x}\left[\sum_{j=1}^{T-1} g_\pi(X_j^N) \mid \mathcal{G}_1^N\right] - f_\pi(x) \\
&= \mathbb{E}_\pi^x\left[\sum_{j=1}^{T-1} g_\pi(X_j^N)\right] - f_\pi(x) \\
&= -g_\pi(x).
\end{aligned}$$

Observe now that $T = 0$ so $f_\pi = 0$ on $\partial\Delta_N$. So, by applying the discrete maximum principle to f_π on $E = \Delta_N$, we get $f_\pi(0) \leq \mathrm{cst}\, N^2 \left\|\frac{g_\pi}{\epsilon_\pi}\right\|_{L^d(\Delta_N)}$. Now, we have $\left\|\frac{g_\pi}{\epsilon_\pi}\right\|_{L^d(\Delta_N)} = \left\|\frac{g}{\epsilon(0)}\right\|_{L^d(\mathbb{P}_{N,\pi})}$ and this right-hand side is in fact the same for all $\pi \in \Omega_\omega^N$ and is equal to $\left\|\frac{g}{\epsilon(0)}\right\|_{L^d(\mathbb{P}_{N,\omega^N})}$.

Therefore,

$$\sup_{y\in\mathbb{Z}} \mathbb{E}_\pi^y\left[\sum_{j=0}^{\tau-1} g_\pi(X_j^n)\right] = \sup_{y\in\Delta_N} \mathbb{E}_{\theta^y\pi}^0\left[\sum_{j=0}^{T-1} g_\pi(X_j^n)\right] = \sup_{y\in\Delta_N} f_{\theta^y\pi}(0) \leq \mathrm{cst}\, N^2 \left\|\frac{g}{\epsilon(0)}\right\|_{L^d(\mathbb{P}_{N,\omega^N})}$$

and so we get

$$\int_{\Omega^N} g\,\mathrm{d}\mathbb{Q}_N \leq \mathrm{cst}\left\|\frac{g}{\epsilon(0)}\right\|_{L^d(\mathbb{P}_{N,\omega^N})}$$

Now, because $p > d$, using the Hölder inequality, we have, for a certain $q > 1$,

$$\left\|\frac{g}{\epsilon(0)}\right\|_{L^d(\mathbb{P}_N)} \leq \|g\|_{L^q(\mathbb{P}_N)}\ \|\epsilon^{-1}(0)\|_{L^p(\mathbb{P}_N)}.$$

Because $\mathbb{Q}_N \rightharpoonup \mathbb{Q}$ along some subsequence and $\mathbb{P}$-*a.s.* $\mathbb{P}_N \rightharpoonup \mathbb{P}$, we get

$$\mathbb{P}\text{-}a.s.\ \int_\Omega g\,\mathrm{d}\mathbb{Q} \leq \mathrm{cst}\,\|g\|_{L^q(\mathbb{P})}\,\|\epsilon^{-1}(0)\|_{L^p(\mathbb{P})} = C\,\|g\|_{L^q(\mathbb{P})},$$

which yields lemma 2.6.

3 Another proof using stopping times

The arguments presented here are inspired by ideas given to us by J. Lin and J.D. Deuschel. We rephrase the proof of Lawler [1] and Guo & Zeitouni [3] using stopping times instead of periodization. We still assume we have balancedness, stationarity and ergodicity of $\mathbb{P}$, as well as one of the following conditions :

1. Uniform ellipticity : $\exists\,\alpha > 0$, *for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, $\forall x \in \mathbb{Z}^d$, $\forall i$, $\omega(x, e_i) \geq \alpha$.
2. Finite p^{th}-moment $(p > d)$: $\mathbb{E}\left[\epsilon^{-p}(0)\right] < \infty$.

We use the same notations as above. Fix $\omega \in \Omega$ and $N \geq 1$. Again define

$$T = T_N = \inf\{j \geq 0,\ |X_n|_\infty > N\}.$$

Now we define a probability measure on $(\Omega, \mathcal{F})$ by

$$Q_N = Q_{\omega,N} = \frac{1}{\mathbb{E}_\omega^0[T]}\, \mathbb{E}_\omega^0 \left[\sum_{j=0}^{T-1} \delta_{\theta^{X_j}\omega} \right].$$

Note that we now use the notation Q instead of $\mathbb{Q}$ (be careful not to mistake it with the transition kernel Q_ω).

For the exact same reasons as previously, we can extract a subsequence of Q_N converging to a probability measure Q on $(\Omega, \mathcal{F})$.

Now we need to show that $Q \ll \mathbb{P}$. Here we can actually show that we have an inequality directly on the measures. We show the following lemma :

Lemma 3.1. *There exists a constant $C > 0$ (depending only on d and $\mathbb{P}$) such that,*

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad Q \leq C\, \mathbb{P}^{1/q}.$$

where $\begin{cases} q = d & \textit{in case 1.} \\ \frac{1}{q} = \frac{1}{d} - \frac{1}{p} & \textit{in case 2.} \end{cases}$

Proof. Let $N \geq 1$. Note that the support of Q_N is still the set $\{\theta^x\omega,\ x \in \Delta_N\}$ because for $k < T$, $X_k \in \Delta_N$. Again we define $\mathbb{P}_N$ as the uniform probability measure on this set.

We use the same notations and computations as in 2.5.1: $\{\theta^x\omega,\ x \in \Delta_N\} = \{\omega_i^N\}_{i\in\{1,\dots,m\}}$ where the ω_i^N are distinct. For each i, let $C_i^N = \{x \in \Delta_N,\ \theta^x\omega = \omega_i^N\} = \{x_k^{(i)}, k \in \{1, \dots, |C_i^N|\}\}$ and $l_i^N = |C_i^N|$. Fix i from now on. We have

$$\frac{Q_N}{\mathbb{P}_N^{1/q}}\,(\omega_i^N) = \frac{|\Delta_N|^{1/q}}{(l_i^N)^{1/q}\ \mathbb{E}_\omega^0[T]}\ \mathbb{E}_\omega^0 \left[\sum_{j=0}^{T-1} \sum_{k=1}^{l_i^N} \mathbb{1}_{X_j = x_k^{(i)}} \right]$$

Just as in the end of the proof in 2.5.2, this can be estimated applying the maximum principle 2.7 in Δ_N to the function

$$f_i(x) = \mathbb{E}_\omega^x \left[\sum_{j=0}^{T-1} g_i(X_j) \right] \quad \text{where} \quad g_i(y) = \sum_{k=1}^{l_i^N} \mathbb{1}_{y = x_i^{(k)}} = \mathbb{1}_{C_i^N}(y).$$

Because $\|g_i\|_{L^q(\Delta_N)} = \dfrac{(l_i^N)^{1/q}}{|\Delta_N|^{1/q}}$, we get (using the Hölder inequality in case 2.),

$$\max_{\Delta_N} f_i \leq \begin{cases} C_\alpha\, N^2\, \dfrac{(l_i^N)^{1/d}}{|\Delta_N|^{1/d}} & \text{in case 1.} \\ C_0\ \mathbb{E}[\epsilon^{-p}(0)]^{1/p}\ N^2\ \dfrac{(l_i^N)^{1/q}}{|\Delta_N|^{1/q}} & \text{in case 2.} \end{cases}$$

Therefore we get

$$\frac{Q_N}{\mathbb{P}_N^{1/d}}\,(\omega_i^N) \leq C\, \frac{N^2}{\mathbb{E}_\omega^0[T]}.$$

In order to end the proof, let us show that we have $\mathbb{E}_\omega^0[T] \geq N^2$ (this actually requires no ellipticity or moment condition whatsoever). Let's define for $n \geq 0$, $M_n = |X_n|^2 - n$ (where $|\cdot|$ denotes the Euclidian norm). Then under $\mathbb{P}_\omega^0$ (for $\mathbb{P}$-*a.e.* ω), $(M_n)_{n\geq 0}$ is a martingale for the canonical filtration $(\mathcal{G}_n)_n$. Indeed, let's define for $j \geq 1$ the increments $Z_j = X_j - X_{j-1}$.

$$\begin{aligned}
\mathbb{E}_\omega^0\left[M_{n+1} - M_n \,|\, \mathcal{G}_n\right] &= \mathbb{E}_\omega^0\left[\sum_{k,l=1}^{n+1}\langle Z_k,\, Z_l\rangle - \sum_{k,l=1}^{n}\langle Z_k,\, Z_l\rangle - 1 \,|\, \mathcal{G}_n\right] \\
&= \mathbb{E}_\omega^0\left[2\sum_{k=1}^{n}\langle Z_k,\, Z_{n+1}\rangle + \langle Z_{n+1},\, Z_{n+1}\rangle - 1 \,|\, \mathcal{G}_n\right] \\
&= 2\sum_{k=1}^{n}\langle Z_k,\, \mathbb{E}_\omega^0[Z_{n+1} \,|\, \mathcal{G}_n]\rangle \\
&= 0
\end{aligned}$$

because of the balanced condition and the fact that $\|Z_{n+1}\|^2 = 1$. By Doob's stopping theorem, $(M_{n\wedge T})_{n\in\mathbb{N}}$ is also a martingale, and we have $\mathbb{E}_\omega^0[M_{n\wedge T}] = \mathbb{E}_\omega^0[X_0] = 0$, so that

$$\mathbb{E}_\omega^0\left[|X_{n\wedge T}|^2\right] = \mathbb{E}_\omega^0[n \wedge T].$$

Now, by the monotone convergence theorem, $\mathbb{E}_\omega^0[n\wedge T] \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\omega^0[T]$, and by the dominated convergence theorem $\mathbb{E}_\omega^0\left[|X_{n\wedge T}|^2\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\omega^0\left[|X_T|^2\right]$ (because $|X_{n\wedge T}|^2 \leq d\,N^2$). So we get

$$\mathbb{E}_\omega^0[T] = \mathbb{E}_\omega^0\left[|X_T|^2\right] \geq N^2.$$

□

The next step is to prove the invariance of Q with respect to the Markov chain $(\bar{\omega}_n)_{n\in\mathbb{N}}$. Afterwards, using the same arguments as above, we can derive from $Q \ll \mathbb{P}$ and the invariance of Q that $\mathbb{P} \ll Q$ and that Q is ergodic for the Markov chain $(\bar{\omega}_n)_{n\in\mathbb{N}}$, which ends the proof.

Remark. Note that using the same arguments of uniqueness as before, we actually have that Q does not depend on ω and that $Q = \mathbb{Q}$. This could be used to show the invariance of Q directly, but it would mean using periodicity and the assumption with $p > d$ to have a complete proof.

4 Generalization to $p = d$

The case studied here has actually been studied on the PDE side by Armstrong & Smart [8], but there has been no written literature about it in the RWRE field yet. The arguments presented here are inspired by ideas given to us by J. Lin and J.D. Deuschel.
We now relax the assumption made by Guo & Zeitouni [3] to the case when $p = d$, that is

$$\epsilon(x) := \prod_{i=1}^{d}\omega(x, e_i)^{1/d}, \quad \mathbb{E}\left[\epsilon(0)^{-d}\right] < \infty.$$

We define our measures Q_N in the same way as in the previous section, so that

$$Q_N = \frac{1}{\mathbb{E}_\omega^0[T]}\sum_{i=1}^{m}\mathbb{E}_\omega^0\left[\sum_{j=0}^{T-1}\mathbb{1}_{C_i^N}(X_j)\right]\delta_{\omega_i^N}.$$

Recall that along some subsequence we have $Q_N \rightharpoonup Q$, and again, what we want to show is that $Q \ll \mathbb{P}$. If we let $p \to d$ in lemma 3.1, we get $q \to \infty$, so it seems impossible to get an inequality directly on the measures. This is why we want to have an inequality as in lemma 2.6. In order to do so, we compute, for g bounded continuous function on Ω,

$$\begin{aligned}
\int_\Omega g \,\mathrm{d}Q_N &= \frac{1}{\mathbb{E}^0_\omega[T]} \sum_{i=1}^m g(\omega_i^N)\, \mathbb{E}^0_\omega \left[\sum_{j=0}^{T-1} \mathbb{1}_{C_i^N}(X_j) \right] \\
&= \frac{1}{\mathbb{E}^0_\omega[T]}\, \mathbb{E}^0_\omega \left[\sum_{j=0}^{T-1} \left(\sum_{i=1}^m g(\omega_i^N)\, \mathbb{1}_{C_i^N}(X_j) \right) \right] \\
&= \frac{1}{\mathbb{E}^0_\omega[T]}\, \mathbb{E}^0_\omega \left[\sum_{j=0}^{T-1} h(X_j) \right],
\end{aligned}$$

where, if $x \in \mathbb{Z}^d$, $h(x) = \sum_{i=1}^m g(\omega_i^N)\, \mathbb{1}_{C_i^N}(x)$. Now, using the maximum principle, we get that

$$\begin{aligned}
\mathbb{E}^0_\omega \left[\sum_{j=0}^{T-1} h(X_j) \right] &\le C_0\, N^2 \left\| \frac{h}{\epsilon} \right\|_{L^d(\Delta_N)} \\
&= C_0\, N^2 \left(\frac{1}{|\Delta_N|} \sum_{x \in \Delta_N} \epsilon^{-d}(x) \left(\sum_{i=1}^m g(\omega_i^N)\, \mathbb{1}_{C_i^N}(x) \right)^d \right)^{1/d} \\
&= C_0\, N^2 \left(\frac{1}{|\Delta_N|} \sum_{x \in \Delta_N} \epsilon^{-d}(x) \sum_{i=1}^m g(\omega_i^N)^d\, \mathbb{1}_{C_i^N}(x) \right)^{1/d} \\
&= C_0\, N^2 \left(\frac{1}{|\Delta_N|} \sum_{i=1}^m \sum_{x \in C_i^N} \epsilon^{-d}(x)\, g(\omega_i^N)^d \right)^{1/d} \\
&= C_0\, N^2 \left(\frac{1}{|\Delta_N|} \sum_{x \in \Delta_N} \epsilon^{-d}_{\theta^x\omega}(0)\, g(\theta^x\omega)^d \right)^{1/d} \\
&= C_0\, N^2 \left\| \frac{g}{\epsilon(0)} \right\|_{L^d(\mathbb{P}_N)}.
\end{aligned}$$

Now, because $\mathbb{E}^0_\omega[T] \ge N^2$, we get that

$$\int_\Omega g \,\mathrm{d}Q_N \le C_0 \left\| \frac{g}{\epsilon(0)} \right\|_{L^d(\mathbb{P}_N)}.$$

Now, because g is bounded and $\epsilon^{-d}(0)$ is in $L^1(\mathbb{P})$, we can apply Birkhoff's ergodic theorem to conclude that

$$\int_\Omega g \,\mathrm{d}Q \le C_0 \int_\Omega g^d\, \epsilon^{-d}(0)\, \mathrm{d}\mathbb{P}.$$

Now, we can end the proof as we did in the proof of lemma 2.6. The only thing to say is that if we have an event A such that $\mathbb{P}(A) = 0$, then $\int_A \epsilon^{-d}(0)\,\mathrm{d}\mathbb{P} = 0$.

5 Quantitative results

This section is based on a recent paper by Guo, Peterson & Tran [7]. We want to get rates of convergence in the invariance principle. In order to have quantitative results, quite strong assumptions have to be made. In the following, we still assume that we have balancedness, stationarity and ergodicity of $\mathbb{P}$, as well as :

- Uniform ellipticity : $\exists\, \alpha > 0$, *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, $\forall x \in \mathbb{Z}^d$, $\forall i$, $\omega(x, e_i) \geq \alpha$.
- I.i.d. environments : $\{\omega(x),\, x \in \mathbb{Z}^d\}$ are i.i.d. random variables under $\mathbb{P}$.

Remark. Be careful that the last assumption does not actually mean that when fixing an environment ω, a random walk in ω has i.i.d. increments. Physically speaking, the i.i.d. assumption simply means that we have i.i.d. causes of inhomogeneity in the media.

5.1 Quantifying the ergodicity of $\mathbb{Q}$

Remember that the only information we have on the covariance matrix $\mathrm{diag}(b)$ is given by the ergodic theorem 2.4. It is therefore useful to know the rate of this convergence to approximate b properly. More generally, we can use Birkhoff's ergodic theorem to approximate the measure $\mathbb{Q}$: for any measurable function $\psi : \Omega \to \mathbb{R}$,

$$\mathbb{P}\text{-}a.s. \quad \frac{1}{n}\sum_{j=0}^{n-1} \psi(\bar{\omega}_j) \overset{\mathbb{P}^0_\omega\text{-}a.s.}{\longrightarrow} \mathbb{E}_{\mathbb{Q}}[\psi] = \int_\Omega \psi \, \mathrm{d}\mathbb{Q}.$$

This convergence is almost sure, so, if ψ is bounded, we also have convergence in $L^1(\mathbb{P}^0_\omega)$:

$$\mathbb{P}\text{-}a.s. \quad \frac{1}{n}\sum_{j=0}^{n-1} \Big[\mathbb{E}^0_\omega[\psi(\bar{\omega}_j)] - \mathbb{E}_{\mathbb{Q}}[\psi]\Big] \longrightarrow 0.$$

In the following, we will restrict to ψ being a bounded function depending only on $\omega(0)$, as is the case in theorem 2.4.

We need a PDE lemma shown in [7], whose proof uses geometrical tools that we won't present here. Let $E \subset \mathbb{Z}^d$ be a bounded domain. Consider the discrete Dirichlet problem

$$(S) \quad \begin{cases} L_\omega f = \psi_\omega - \mathbb{E}_{\mathbb{Q}}[\psi] & \text{in } E \\ f = 0 & \text{on } \partial E \end{cases}$$

where we denote $\psi_\omega(x) = \psi(\theta^x \omega)$. Note that, by the ABP, there exists a unique solution to this problem and it is given by

$$\forall x \in \bar{E}, \;\; f(x) = -\,\mathbb{E}^x_\omega\left[\sum_{j=0}^{T-1}(\psi_\omega(X_j) - \mathbb{E}_{\mathbb{Q}}[\psi])\right] = -\,\mathbb{E}^x_\omega\left[\sum_{j=0}^{T-1}(\psi(\bar{\omega}_j) - \mathbb{E}_{\mathbb{Q}}[\psi])\right].$$

where T denotes the first hitting time of ∂B.

Lemma 5.1. *Let $R > 0$ and $0 < p < d$. There exist c, $C > 0$ and $a > 0$ (depending on d, α and p), such that for any $E \subseteq \{|x|_\infty \leq R\}$, the solution of (S) satisfies*

$$\mathbb{P}\left(\sup_E |u| \geq C\, \|\psi\|_\infty\, R^{2-a}\right) \leq C \exp(-c\, R^p).$$

Now, using this lemma, we show the following rate of convergence in the ergodic theorem :

Theorem 5.2. *Let $0 < p < d$. There exist $c, C > 0$ and $a > 0$ (depending on d, α and p), such that for every τ stopping time of the random walk $(X_k)_{k\in\mathbb{N}}$,*

$$\mathbb{P}\left(\left|\frac{1}{n}\,\mathbb{E}^0_\omega\left[\sum_{j=0}^{\tau\wedge n-1}(\psi(\bar\omega_j)-\mathbb{E}_{\mathbb{Q}}[\psi])\right]\right|\geq C\,\|\psi\|_\infty\,n^{-a}\right)\leq C\exp(-c\,n^{p/2}).$$

Proof. Because the inequality we want to show only uses $\tau\wedge n$, suppose that $\tau\leq n$. For now, suppose also that $\mathbb{E}_{\mathbb{Q}}[\psi]=0$, and without loss of generality, that $\|\psi\|_\infty=1$.
Let $n\geq 1$ and $R=\sqrt{n}$. Let $B_R(x)=B_{\mathbb{R}^d}(x,R)\cap\mathbb{Z}^d$ and $B_R=B_R(0)$. Let

$$T(x)=\inf\{j\geq 0,\ X_j\in\partial B_R(x)\}.$$

Again, define the stopping times $\tau_0=0$ and for all $k\geq 0$,

$$\tau_{k+1}=\inf\{j>\tau_k,\ X_j-X_{\tau_k}\in\partial B_R\}.$$

Then, applying the lemma, for all $x\in\mathbb{Z}^d$,

$$\mathbb{P}\left(\max_{y\in B_R(x)}\left|\mathbb{E}^y_\omega\left[\sum_{j=0}^{T(x)-1}\psi(\bar\omega_j)\right]\right|<C\,n^{1-a/2}\right)\geq 1-C\,e^{-c\,n^{p/2}}.$$

In fact, we actually have (up to changing C):

$$\mathbb{P}\left(\bigcap_{x\in B_n}\max_{y\in B_R(x)}\left|\mathbb{E}^y_\omega\left[\sum_{j=0}^{T(x)-1}\psi(\bar\omega_j)\right]\right|<C\,n^{1-a/2}\right)\geq 1-C\,n^d\,e^{-c\,n^{p/2}}.$$

Therefore, using the Markov property, we get that with $\mathbb{P}$-probability at least $1-C\,n^d\,e^{-c\,n^{p/2}}$,

$$\begin{aligned}\left|\mathbb{E}^0_\omega\left[\sum_{j=0}^{\tau-1}\psi(\bar\omega_j)\right]\right| &= \left|\sum_{k=0}^{\infty}\mathbb{E}^0_\omega\left[\sum_{j=\tau_k}^{\tau_{k+1}-1}\psi(\bar\omega_j)\,\mathbb{1}_{\tau>\tau_k}-\sum_{j=\tau}^{\tau_{k+1}}\psi(\bar\omega_j)\,\mathbb{1}_{\tau_k<\tau\leq\tau_{k+1}}\right]\right|\\ &= \left|\sum_{k=0}^{\infty}\mathbb{E}^0_\omega\left[\mathbb{1}_{\tau>\tau_k}\,\mathbb{E}^{X_{\tau_k}}_\omega\left[\sum_{j=0}^{T(X_{\tau_k})-1}\psi(\bar\omega_j)-\sum_{j=\tau}^{T(X_{\tau_k})-1}\psi(\bar\omega_j)\,\mathbb{1}_{T(X_{\tau_k})\geq\tau}\right]\right]\right|\\ &\leq 2\,C\,n^{1-a/2}\sum_{k=0}^{\infty}\mathbb{P}^0_\omega(\tau>\tau_k).\end{aligned}$$

where we used that the last sum can be estimated in the same way because

$$\mathbb{E}^{X_{\tau_k}}_\omega\left[\sum_{j=\tau}^{T(X_{\tau_k})-1}\psi(\bar\omega_j)\,\mathbb{1}_{T(X_{\tau_k})\geq\tau}\right]=\mathbb{E}^{X_{\tau_k}}_\omega\left[\mathbb{1}_{T(X_{\tau_k})\geq\tau}\,\mathbb{E}^{X_\tau}_\omega\left[\sum_{j=0}^{T(X_{\tau_k})-1}\psi(\bar\omega_j)\right]\right].$$

Now, using lemma 2.10, there exists a constant $c_0 < 1$ such that for every R and every ω,

$$\mathbb{E}_\omega^0\left[\left(1-\frac{1}{R^2}\right)^{\tau_1}\right] \le c_0 \quad \text{and actually } \forall\, k \ge 1,\ \ \mathbb{E}_\omega^0\left[\left(1-\frac{1}{R^2}\right)^{\tau_k}\right] \le c_0^k.$$

Therefore, using the Markov inequality, we get

$$\mathbb{P}_\omega^0(\tau_k < \tau) \le \mathbb{P}_\omega^0(\tau_k < n) \le \mathbb{P}_\omega^0\left(\left(1-\frac{1}{R^2}\right)^{\tau_k} > \left(1-\frac{1}{R^2}\right)^{R^2}\right) \le \text{cst}\ c_0^k.$$

Up to changing the constants c, C and a, we get that the probability in the theorem is bounded by $C\,e^{-c\,n^{p/2}}$, which ends the proof if $\mathbb{E}_\mathbb{Q}[\psi]=0$.

In general, apply the previous result to $\psi - \mathbb{E}_\mathbb{Q}[\psi]$. Then with probability at most $C\,e^{-c\,R^p}$, we get that

$$\left|\frac{1}{n}\,\mathbb{E}_\omega^0\left[\sum_{j=0}^{\tau\wedge n-1}(\psi(\bar\omega_j)-\mathbb{E}_\mathbb{Q}[\psi])\right]\right| \ge C\,\|\psi-\mathbb{E}_\mathbb{Q}[\psi]\|_\infty\, n^{-a}.$$

Now $\|\psi - \mathbb{E}_Q[\psi]\|_\infty \le 2\,\|\psi\|_\infty$, so the event above contains the event

$$\left|\frac{1}{n}\,\mathbb{E}_\omega^0\left[\sum_{j=0}^{\tau\wedge n-1}(\psi(\bar\omega_j)-\mathbb{E}_\mathbb{Q}[\psi])\right]\right| \ge 2\,C\,\|\psi\|_\infty\, n^{-a},$$

which, up to changing C, yields the result. □

5.2 Rate of convergence for the projected invariance principle

Now let's go back to the invariance principle. Looking at $t=1$, we have that $\frac{X_n}{\sqrt{n}} \rightharpoonup B^b(1) = \mathcal{N}(0,\mathrm{diag}(b))$, and so, if $l\in\mathbb{R}^d$, $\frac{X_n}{\sqrt{n}}\cdot l \rightharpoonup \mathcal{N}(0, {}^t l\ \mathrm{diag}(b)\, l)$, which gives that $\frac{X_n\cdot l}{\sqrt{n}\sqrt{{}^t l\ \mathrm{diag}(b)\, l}}$ converges weakly to a standard normal distribution. As a consequence of the previous theorem, we can now state the following theorem.

Theorem 5.3. *Let $0<p<d$. There exist $\gamma>0$ (depending on d, α and p), such that for every $l\in\mathbb{R}^d$, there exists $C_0>0$ such that*

$$\mathbb{P}\left(\sup_{r\in\mathbb{R}}\left|\mathbb{P}_\omega^0\left(\frac{X_n\cdot l}{\sqrt{n}} \le r\,\sqrt{{}^t l\ \mathrm{diag}(b)\, l}\right) - \phi(r)\right| \le C_0\, n^{-\gamma}\right) \ge 1 - C_0\, e^{-n^{p/2}}.$$

where ϕ is the CDF of a standard normal distribution, i.e. $\phi(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^r e^{-x^2/2}\,\mathrm{d}x$.

Proof. Again, let ψ be a bounded function of $\omega(0)$ such that $\mathbb{E}_\mathbb{Q}[\psi]=0$ and $\|\psi\|_\infty = 1$. By theorem 5.2, we have, for all $0\le m\le n$, with $\mathbb{P}$-probability at least $1-C\,e^{-c\,n^{p/2}}$,

$$\left|\frac{1}{n}\,\mathbb{E}_\omega^0\left[\sum_{j=0}^{m-1}\psi(\bar\omega_j)\right]\right| \le C\,n^{-a}.$$

By stationarity of $\mathbb{P}$, we therefore have that for all $x \in \mathbb{Z}^d$, with $\mathbb{P}$-probability at least $1 - C\, e^{-c\, n^{p/2}}$,

$$\left| \mathbb{E}_\omega^x \left[\sum_{j=0}^{m-1} \psi(\bar{\omega}_j) \right] \right| \leq C\, n^{1-a}.$$

Therefore, with $\mathbb{P}$-probability at least $1 - C\, e^{-c\, n^{p/2}}$,

$$\begin{aligned}
\mathbb{E}_\omega^0 \left[\left(\sum_{k=0}^{n-1} \psi(\bar{\omega}_k) \right)^2 \right] &\leq 2 \sum_{i=0}^{n-1} \mathbb{E}_\omega^0 \left[\psi(\bar{\omega}_i) \sum_{j=0}^{n-i-1} \psi(\bar{\omega}_{i+j}) \right] \\
&= 2 \sum_{i=0}^{n-1} \mathbb{E}_\omega^0 \left[\psi(\bar{\omega}_i)\, \mathbb{E}_\omega^{X_i} \left[\sum_{j=0}^{n-i-1} \psi(\bar{\omega}_j) \right] \right] \\
&\leq 2\, C \sum_{i=0}^{n-1} n^{1-a} = 2\, C\, n^{2-a}.
\end{aligned}$$

and so, up to changing C,

$$\mathbb{E}_\omega^0 \left[\left(\frac{1}{n} \sum_{k=0}^{n-1} \psi(\bar{\omega}_k) \right)^2 \right] \leq C\, n^{-a}. \tag{2}$$

Now, let

$$\psi(\omega) = \mathbb{E}_\omega^0 \left[(X_1 \cdot l)^2 \right] = 2\, {}^t l\, \omega(0)\, l.$$

ψ is indeed a bounded function of $\omega(0)$, and $\mathbb{E}_\mathbb{Q}[\psi] = {}^t l\, b\, l$ (because by the ergodic theorem, $\int_\Omega \omega(0)\, \mathrm{d}\mathbb{Q}(\omega) = \frac{1}{2} b$). Then, using the Markov property,

$$\begin{aligned}
\sum_{k=0}^{n-1} \psi(\bar{\omega}_k) &= \sum_{k=0}^{n-1} \mathbb{E}_{\theta^{X_k}\omega}^0 \left[(X_1 \cdot l)^2 \right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}_\omega^{X_k} \left[((X_1 - X_0) \cdot l)^2 \right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}_\omega^0 \left[((X_{k+1} - X_k) \cdot l)^2 \,|\, \mathcal{G}_k \right].
\end{aligned}$$

And so, using (2), with $\mathbb{P}$-probability at least $1 - C\, e^{-c\, n^{p/2}}$,

$$\mathbb{E}_\omega^0 \left[\left(\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_\omega^0 \left[((X_{k+1} - X_k) \cdot l)^2 \,|\, \mathcal{G}_k \right] - {}^t l\, b\, l \right)^2 \right] \leq C\, n^{-a}.$$

We want to use a quantitative result on the one-dimensional martingale central limit theorem, as can be found in [10, theorem 1.1.]. In order to do so, we need to have an estimate on

$$\mathbb{E}_\omega^0 \left[\left(\frac{1}{S_n^2} \sum_{k=0}^{n-1} \mathbb{E}_\omega^0 \left[((X_{k+1} - X_k) \cdot l)^2 \,|\, \mathcal{G}_k \right] - 1 \right)^2 \right]$$

where $S_n^2 = \mathbb{E}_\omega^0\left[(X_n \cdot l)^2\right]$. Now, by a similar computation as before, using that the $(X_{k+1} - X_k)_k$ are uncorrelated, we get that $S_n^2 = 2\,{}^t l\, M_n\, l$ where $M_n = \mathbb{E}_\omega^0\left[\sum_{j=0}^{n-1} \bar{\omega}_j(0)\right]$. This, using theorem 5.2 with the function $\psi(\omega) = \omega(0)$, gives that with $\mathbb{P}$-probability at least $1 - C\,e^{-c\,n^{p/2}}$,

$$\left|\frac{1}{n}\,S_n^2 - {}^t l\, b\, l\right| \leq C\,n^{-a},$$

and so, up to changing C,

$$\mathbb{E}_\omega^0\left[\left(\frac{1}{n\,{}^t l\, b\, l}\sum_{k=0}^{n-1}\mathbb{E}_\omega^0\left[(X_{k+1} - X_k)\cdot l)^2\,|\,\mathcal{G}_k\right] - 1\right)^2\right] \leq C\,n^{-a} \quad \text{and} \quad \left|\frac{1}{n\,{}^t l\, b\, l}\,S_n^2 - 1\right| \leq C\,n^{-a}.$$

Now, note that, because the $(X_{k+1} - X_k)_k$ are uncorrelated,

$$\mathbb{E}_\omega^0\left[\sum_{k=0}^{n-1}\mathbb{E}_\omega^0\left[((X_{k+1} - X_k)\cdot l)^2\,|\,\mathcal{G}_k\right]\right] = \mathbb{E}_\omega^0\left[(X_n \cdot l)^2\right] = S_n^2,$$

and so with $\mathbb{P}$-probability at least $1 - C\,e^{-c\,n^{p/2}}$,

$$\mathbb{E}_\omega^0\left[\left(\frac{1}{S_n^2}\sum_{k=0}^{n-1}\mathbb{E}_\omega^0\left[((X_{k+1} - X_k)\cdot l)^2\,|\,\mathcal{G}_k\right] - 1\right)^2\right] \leq Cn^{-\beta}$$

for a certain $\beta > 0$. Now, using [10], we get that with $\mathbb{P}$-probability at least $1 - C\,e^{-c\,n^{p/2}}$,

$$\sup_{r\in\mathbb{R}}\left|\mathbb{P}_\omega^0\left(\frac{X_n \cdot l}{\sqrt{n}} \leq r\,\sqrt{{}^t l\, b\, l}\right) - \phi(r)\right| \leq C\,(n^{-\beta} + n^{-1})^{1/5} \leq C\,n^{-\gamma}.$$

□

Wheter we can generalize these quantitative results to the non-uniformly elliptic case with $p \geq d$ remains an open problem.

Bibliography

1. G. Lawler, *Weak convergence of a Random Walk in a Random Environment*, Springer-Verlag, Commun. Math. Phys. 87, 81-87 (1982).

2. G. Papanicolaou & S. Varadhan, *Diffusions with random coefficients*, North-Holland Publishing Compagny, 547-552 (1982).

3. X. Guo & O. Zeitouni, *Quenched invariance principle for random walks in balanced random environment*, arXiv (2011).

4. G. Lawler, *Random Walk and the Heat Equation*, Department of Mathematics, University of Chicago, Chicago, IL 60637 (2010).

5. S. Tavaré & O. Zeitouni, *Lectures on Probability Theory and Statistics*, Ecole d'été de probabilités de Saint-Flour XXXI (2001).

6. J.D. Deuschel, X. Guo & A. Ramirez, *Quenched invariance principle for random walk in time-dependent balanced random environment*, arXiv (2018).

7. X. Guo, J. Peterson & H. Tran, *Quantitative homogenization in a balanced random environment*, arXiv (2019).

8. S. Armstrong & C. Smart, *Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity*, *The Annals of Probability*, vol. 42, No. 6, 2558-2594 (2014).

9. P. Hall, & C. Heyde, *Martingale Limit Theory and Its Application*, New York: Academic Press (1980).

10. J.-C. Mourrat, *On the rate of convergence in the martingale central limit theorem*, Bernouilli. 19, no. 2, 633-645 (2013).