# RANDOM WALKS IN DYNAMIC RANDOM ENVIRONMENTS

Internship report
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## 1 Introduction

Random walks in random environments (RWREs) are a quite recent area of research in probability theory, born around the 1970s. At first, researchers in the subject were interested in static environments, meaning that each site is allocated a transition probability measure on the set of its neighbours, which determines the laws of the jumps of a particle at this site. The case of RWREs on $\mathbb{Z}$ has been thoroughly studied, but the case of higher dimensions remains quite challenging. Some results such as laws of larges numbers (LLNs) and even central limit theorems (CLTs) can be shown using assumptions that ensure some mixing properties of the environment seen from the particle. For instance, it is the case for balanced ergodic environments when the moments of the transition probability measures are properly controlled. This example was the object of my M1 internship under the tutorship of Jessica Lin (McGill University, Montréal, Canada).

Then came interest in a dynamic version of the problem, for instance on $\mathbb{Z}$. The environments can depend on time and so the laws of the jumps can depend both on localisation and time. To get asymptotic results, strong assumptions often have to be made on the dynamics of the environment. For instance, CLTs can be shown for environments that change independently in time. Weaker assumptions are usually made up of strong mixing conditions, which ensure that the dependency on events that are far away in the past vanishes - for instance, uniformly on said events. Unfortunately, lots of natural examples of environments do not satisfy these strong mixing conditions. However they are likely to satisfy weaker decoupling conditions such as those on which we will focus in this report.

In this internship, I worked under the supervision of Oriane Blondel (ICJ, Lyon) with help from Augusto Teixeira (IMPA, Rio de Janeiro, Brazil). The starting point of the internship was to understand the tools used in [2], which shows a LLN for nearest-neighbour random walks on $\mathbb{Z}$ on dynamic environments satisfying a polynomial decoupling property. Showing a LLN in this case was already a challenge on account of the generality of the framework and the relative weakness of the assumptions made. Further results such as CLTs are hard to reach in general, however they can be shown for specific environments, such as the simple symmetric exclusion process, as was done in [4]. The goal for me was to generalize the LLN from [2] to the finite-range case, where particles may jump further than their nearest neighbours. Although it is natural to think that the LLN still holds, using the tools of [2] is actually not straightforward at all, for they highly rely on an argument of monotonic coupling which collapses when not in the nearest-neighbour setting.

Be warned that the proof presented here is actually incomplete. We believe we can make it work with further thinking. This will be one of the goals of my PhD thesis under the direction of Oriane Blondel and Augusto Teixeira.

**Structure of this report** Section 2 of this report aims at presenting the model and showing a LLN in the nearest-neighbour setting, following [2] (note that the figures are all extracted from this article). The notations, ideas and examples in this section will be used in the rest of the report as well. In section 3, we will try to lift the nearest-neighbour assumption in order to show a LLN in the finite-range case, by generalizing ideas of section 2. Note that some propositions, theorems and methods are framed; this merely aims at highlighting the main ideas, the core propositions and the structure of our proofs.

**Notations**

- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively denote the set of natural integers (starting from 0), relative integers and real numbers. $\mathbb{N}^*$ denotes $\mathbb{N} \setminus \{0\}$. $[a, b)$ is the interval of real numbers $\{x \in \mathbb{R},\ a \leqslant x < b\}$, and $[\![n, m]\!]$ is the set of integers $[n, m] \cap \mathbb{Z}$.
- $c$ will be a constant that can change throughout this report and even from line to line. Constants that are used again later in the paper will be denoted with an index (for instance $c_0$, $c_1$...).

# 2 LLN in the nearest-neighbor setting

## 2.1 Setup of the problem and main result

Set $\mathbb{L} = \mathbb{Z} \times \mathbb{R}_+$: we consider a discrete-space continuous-time setting. The main idea of this section is the following (see remark 2.7 for more details).

Method 2.1.

We want to study the asymptotic behaviour of a particle moving randomly in a dynamic random environment. More precisely, we want to have an asymptotic speed for our particle (i.e. we want to show a law of large numbers). In order to do this, it is very useful to couple particles starting at different sites, in the sense that if a particle starts at the left of another particle, it will remain at its left. This way, particles can block each other, forcing some asymptotic speed behaviour. Therefore, we are going to define the setting by allocating time jumps and information on the jumps to each $x \in \mathbb{Z}$, and imagine that we have particles starting from all sites $(x, t) \in \mathbb{L}$.

We now define a random walk in a dynamic random environment in our setting. Note that we allocate jumping times to sites instead of particles for practical reasons (as was explained in method 2.1), but this is actually not so different (see example 2.4).

Definition 2.2. Let $S$ be a countable space and $\ell \in \mathbb{N}^*$. We denote by $\mathbb{P}$ the probability measure defined on a probability space where all of the random variables we introduce below are defined.

- An *environment* is a random variable $\eta$ taking values in the Skorokhod space $\mathcal{D}(\mathbb{R}_+, S^{\mathbb{Z}})$ (i.e. the space of càdlàg functions for the classical topology on $\mathbb{R}_+$ and the product topology on $S^{\mathbb{Z}}$).
- The *jumping times* at location $x \in \mathbb{Z}$ are an increasing sequence of random variables $(T_i^x)_{i\in\mathbb{N}^*}$ in $\mathbb{R}_+$. It is equivalent to be given for each $x \in \mathbb{Z}$ the counting process $(N_t^x)_{t\geqslant 0}$ defined by

$$\forall t \geqslant 0, \quad N_t^x = \sum_{i\in\mathbb{N}^*} \mathbf{1}_{T_i^x \leqslant t}. \tag{2.1}$$

- The *jumping function* is a deterministic function $g : S^{[\![-\ell,\ell]\!]} \times [0, 1] \to \{-1, 0, -1\}$.
- We also define a family of independent uniform random variables in $[0, 1]$ denoted by $(U_i^x)_{i\in\mathbb{N}^*,x\in\mathbb{Z}}$, independent from $\eta$ and the $(T_i^x)_{i\in\mathbb{N}^*,x\in\mathbb{Z}}$.
- Let $y = (x, s) \in \mathbb{L}$. A *random walk* (in the environment $\eta$, with jumping times $(T_i^x)$) starting at $y$ is a random variable $t \geqslant 0 \mapsto Y_t^y = (X_t^y, t + s) \in \mathbb{L}$ taking values in the Skorokhod space $\mathcal{D}(\mathbb{R}_+, \mathbb{L})$ such that

$$\mathbb{P}\text{-almost surely,} \quad \begin{cases} X_0^y = x \\ X_t^y = X + g(\eta_t(X-\ell), \dots, \eta_t(X+\ell), U_i^X) \text{ if } t = T_i^X \text{ and } X_{t^-}^y = X \end{cases}$$

Let us explain this with words. We can think of $\eta_t(x)$ as the state of the environment at location $x$ and time $t$, and the $(T_i^x)$ as the times when a particle located at $x$ may jump. The function $g$ determines the jump of the random walk at time $T_i^x$ and location $x$ when applying it to $(\eta_{T_i^x}(x-\ell), \dots, \eta_{T_i^x}(x+\ell), U_i^x)$. In other words, if the random walk is at location $x$ just before time $t = T_i^x$, then the law of its jump at time $t$ depends only on the state of the environment at time $t$ and locations $x - \ell, \dots, x + \ell$, and on an additional random variable independent of the environment. The fact that $g$ has values in $\{-1,0,1\}$ expresses the fact that our particles can only jump to nearest-neighbour sites.

Remark 2.3. In most cases, the jumping times will be chosen independently of the environment $(\eta_t)_{t\geqslant 0}$ (see example 2.4 for a canonical choice of jumping times). However, this is not necessary in our proofs and so we do not ask for it. Actually in some cases, environments are constructed using clocks that we can take as jumping times as well.

Example 2.4. Suppose that the jumping times $(T_i^x)_{i\in\mathbb{N}^*}$ are given by a Poisson process of parameter 1, denoted by $(N_t^x)_{t\geqslant 0}$, drawn independently for each site $x\in\mathbb{Z}$. Let us focus on the trajectory of the particle starting at $y=(0,0)$. Its first jumping time is $T_1^0$, so it is exponential with parameter 1. Say the particle has jumped to site $x_1\in\mathbb{Z}$ at time $t=T_1^0$. Its second jumping time is $T^{x_1}_{N_t^{x_1}+1}$, so it has to wait $T^{x_1}_{N_t^{x_1}+1}-t$ to jump. Due to the properties of exponential clocks, this is also exponential with parameter 1, and it is independent of $T_1^0$. And so on. In the end, we recover the case in which one particle jumps at rate 1.

Now we need to introduce the assumptions that are going to be crucial to get a law of large numbers. First we introduce assumptions on the $(T_i^x)$, which are not very restrictive : for instance, the i.i.d. Poisson construction of example 2.4 satisfies the assumptions below (see example 2.8). These are assumptions on the possible paths that a random walker could follow if we only considered the $(T_i^x)$.

Definition 2.5. An allowed path on $[0,T]$ is a function $t\in[0,T]\mapsto\gamma(t)=(\gamma_1(t),\gamma_2(t))\in\mathbb{L}$, càdlàg, with jumps in $\{-1,0,1\}$, such that

$$\begin{cases}\forall s\geqslant 0,\ \ \gamma_2(t+s)-\gamma_2(t)=s;\\ \text{If } \gamma(t)=(x,s), \text{ then } \gamma_1(t+r)=x \text{ for every } r<\min_i\{T_i^x-s,\ T_i^x>s\}.\end{cases}$$

We define allowed paths on any time interval of the form $[T,T']$ in the same way.

Mind that in the rest of this report, we will sometimes use allowed paths (always denoted by $\gamma$) without specifying that they are *allowed* paths.

Assumptions 2.6.

- *Coupling property* : for every $x\in\mathbb{Z}$, $\mathbb{P}$-almost surely, $\{T_i^x,\ i\in\mathbb{N}^*\}$ and $\{T_i^{x+1},\ i\in\mathbb{N}^*\}$ are disjoint.
- *Limiting speed*: for every $v>1$,

$$\liminf_{T\to\infty}\mathbb{P}\begin{pmatrix}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } [0,T)\times\{0\}\\ \text{and such that } \gamma_1(T)-\gamma_1(0)\geqslant vT\end{pmatrix}=0\ ; \tag{2.2}$$

$$\liminf_{T\to\infty}\mathbb{P}\begin{pmatrix}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } [0,T)\times\{0\}\\ \text{and such that } \gamma_1(T)-\gamma_1(0)\leqslant -vT\end{pmatrix}=0. \tag{2.3}$$

- *Large deviation bound* : there exists $c_0>0$ such that for all $T>0$,

$$\mathbb{P}\begin{pmatrix}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting at } 0,\\ \exists\, s\in[0,T],\ [\gamma_1(s)-\ell,\gamma_1(s)+\ell]\nsubseteq[-2T,2T]\end{pmatrix}\leqslant c_0^{-1}\,e^{-c_0T}. \tag{2.4}$$

Remark 2.7.

- Note that in the limiting speed assumption, the choice of $\pm1$ as limiting speeds is arbitrary, as is the choice of $2T$ in (2.4) - they are chosen to be consistent with example 2.4 (see example 2.8).
- Let us now comment on the so-called coupling property. The idea is that we want to couple the paths of the particles starting at $(x,t)\in\mathbb{L}$ and $(x+1,t)$. The assumption made here ensures that almost

surely, those two particles cannot cross paths without meeting: either the particle which started at $x$ remains strictly at the left of the particle which started at $x+1$, or they meet and their paths will be the same afterwards. Indeed, the paths of the particles are deterministic given $\eta$, the $(T_i^x)$ and the $(U_i^x)$. This is why we decided to allocate the time jumps and the rules of the jumps to sites, and not to particles. In the end, we have the very important monotonicity property:

$$\text{if } x \leqslant x' \in \mathbb{Z} \text{ and } s \geqslant 0, \text{ then } X_t^{(x,s)} \leqslant X_t^{(x',s)} \text{ for every } t \geqslant 0. \tag{2.5}$$

Example 2.8. Let us look again at example 2.4 and show that it satisfies assumptions 2.6.

- Coupling property. For every $x \in \mathbb{Z}$, by independence of the Poisson processes and the fact that the law of $T_j^{x+1}$ is continuous with respect to the Lebesgue measure,

$$\begin{aligned}\mathbb{P}(\exists\, i,j \in \mathbb{N}^*,\ T_i^x = T_j^{x+1}) &\leqslant \sum_{i,j\in\mathbb{N}^*} \iint_{\mathbb{R}_+^2} \mathbf{1}_{t=s}\, \mathrm{d}\mathbb{P}_{T_i^x}(t)\, \mathrm{d}\mathbb{P}_{T_j^{x+1}}(s) \\ &= \sum_{i,j\in\mathbb{N}^*} \int_{\mathbb{R}_+} \underbrace{\mathbb{P}(T_j^{x+1} = t)}_{=0}\, \mathrm{d}\mathbb{P}_{T_i^x}(t) = 0.\end{aligned}$$

- Limiting speed. Let us fix $v > 1$. Before showing (2.2) and (2.3), let us show the following estimate for a Poisson variable $N_T$ of parameter $T$: there exists $\lambda = \lambda(v) > 0$ such that for any $T > 0$,

$$\mathbb{P}(N_T \geqslant vT) \leqslant e^{-\lambda T}. \tag{2.6}$$

To do this, we use a Chernoff bound: let $a > 0$, we have, by the Markov inequality,

$$\begin{aligned}\mathbb{P}(N_T \geqslant vT) &\leqslant e^{-avT}\, \mathbb{E}\left[e^{aN_T}\right] \\ &= e^{-avT}\, e^{-T} \sum_{n\in\mathbb{N}} \frac{e^{an}T^n}{n!} \\ &= e^{-(av+1-e^a)T}.\end{aligned}$$

Now, $av + 1 - e^a \underset{a\to 0}{\sim} a(v-1) > 0$, since $v > 1$. Choosing a small enough $a$ yields (2.6).

Now we can show that

$$\mathbb{P}\left(\begin{array}{c}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } [0,T)\times\{0\} \\ \text{and such that } \gamma_1(T) - \gamma_1(0) \geqslant vT\end{array}\right) \xrightarrow[T\to\infty]{} 0,$$

which yields (2.2) (showing (2.3) can be done in a similar way). Note that, for every $y \in [0,T)\times\{0\}\cap\mathbb{L}$, on the event

$$\left\{\begin{array}{c}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } y \\ \text{and such that } \gamma_1(T) - \pi_1(y) \geqslant vT\end{array}\right\},$$

the path $\hat\gamma$ which starts at $y$ and always jumps to the right when a clock rings, has to satisfy

$$\hat\gamma_1(T) - \pi_1(y) \geqslant vT.$$

Now, let us denote by $\tau_i$ the $i^{th}$ clock to ring along path $\hat\gamma$ (with $\tau_0 = 0$). Because of the properties of exponential clocks that we already used in example 2.4, the number $N_T$ of those clocks is a Poisson variable with parameter $T$. Moreover, we have $\hat\gamma_1(T) - \pi_1(y) = N_T$. Thus, using a union bound on the possible starting points,

$$\mathbb{P}\left(\begin{array}{c}\exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } [0,T)\times\{0\} \\ \text{and such that } \gamma_1(T) - \gamma_1(0) \geqslant vT\end{array}\right) \leqslant T\,\mathbb{P}(N_T \geqslant vT) \leqslant Te^{-\lambda T} \xrightarrow[T\to\infty]{} 0.$$

- Large deviation bound. By the same kinds of arguments as before, we have

$$
\begin{aligned}
&\mathbb{P}\begin{pmatrix} \exists\,\gamma \text{ allowed path on } [0,T], \text{ starting at } 0, \\ \exists\, s\in[0,T],\ [\gamma_1(s)-\ell,\gamma_1(s)+\ell]\nsubseteq[-2T,2T] \end{pmatrix} \\
&\leqslant 2\,\mathbb{P}(N_T\geqslant 2T-\ell) \\
&\leqslant 2\,\mathbb{P}\left(N_T\geqslant \frac{3}{2}T\right) && \text{pour } T\geqslant 2\ell \\
&= 2\,e^{-\lambda(3/2)T} && \text{by applying (2.6) with } v=3/2>1
\end{aligned}
$$

Then it simply remains to adjust $c_0$ to also take account of the small values of $T$.

Now we introduce assumptions where the whole setting appears. Our main assumption will be a decoupling property which ensures that events depending on the environment and the $(T_i^x)$ in boxes far away from each other in time are well decorrelated. For this, we define a box to be a set $B=[a,b]\times[c,d]\subseteq\mathbb{R}^2$, we call its side length $\max(b-a,d-c)$, and we say that an event $A$ is measurable with respect to $B$ if it is measurable with respect to $\{\eta_t(x),(x,t)\in B\cap\mathbb{L}\}$ and $\{T_i^x$ such that $(x,T_i^x)\in B\cap\mathbb{L}\}$. If $B_1=[a,b]\times[c,d]$ and $B_2=[a',b']\times[c',d']$ are two boxes, we say they are separated in time by at least $H$ if $c'-d\geqslant H$ or $c-d'\geqslant H$.

Assumptions 2.9.

- *Space-time translation invariance* : for every $(z,s)\in\mathbb{L}$, $\big((\eta_t(x))_{(x,t)\in\mathbb{L}},\ (T_i^x)_{x\in\mathbb{Z},\,i\in\mathbb{N}^*}\big)$ and $\Big((\eta_{s+t}(z+x))_{(x,t)\in\mathbb{L}},\ (T_i^{z+x}-s)_{x\in\mathbb{Z},\,i\,|\,T_i^{z+x}>s}\Big)$ have same law.
- *Decoupling property*: there exist $c_1$, $\alpha>0$ such that, for every $H\geqslant 1$ and every box $B_1$, $B_2$ having both side lengths at most $5H$ and separated in time by at least $H$, for every event $A_1$ resp. $A_2$ measurable with respect to $B_1$ resp. $B_2$,

$$\mathbb{P}(A_1\cap A_2)-\mathbb{P}(A_1)\,\mathbb{P}(A_2)\leqslant c_1\,H^{-\alpha}. \tag{2.7}$$

Remark 2.10. Note that although these assumptions concern the process $(\eta,(T_i^x))$, they are in fact true for the whole process $(\eta,(T_i^x),(U_i^x))$, because the $U_i^x$ are i.i.d. and independent of the rest. Also, note that by a standard linearity and density argument, inequality (2.7) implies that for all measurable functions $f_1$ and $f_2$ with values in $[-1,1]$,

$$\mathbb{E}[f_1(\eta_1,\mathcal{T}_1)\,f_2(\eta_2,\mathcal{T}_2)]-\mathbb{E}[f_1(\eta_1,\mathcal{T}_1)]\,\mathbb{E}[f_2(\eta_2,\mathcal{T}_2)]\leqslant c_1\,H^{-\alpha}, \tag{2.8}$$

where $\eta_1$, $\eta_2$ denote the environment inside $B_1$, $B_2$, and $\mathcal{T}_1$, $\mathcal{T}_2$ denote the jumping times inside $B_1$, $B_2$.

Remark 2.11. When the jumping times $(T_i^x)$ are independent of the environment $\eta$ (which is the case in most common examples), it suffices to check both properties in assumptions 2.9 separately for $\eta$ and for the jumping times $(T_i^x)$. More precisely, it suffices to check:

$$\forall (z,s)\in\mathbb{L},\ \ (T_i^x)_{x\in\mathbb{Z},\,i\in\mathbb{N}^*}\sim(T_i^{z+x}-s)_{x\in\mathbb{Z},\,i\,|\,T_i^{z+x}>s} \tag{2.9}$$

$$\forall (z,s)\in\mathbb{L},\ \ (\eta_t(x))_{(x,t)\in\mathbb{L}}\sim(\eta_{s+t}(z+x))_{(x,t)\in\mathbb{L}} \tag{2.10}$$

$$\begin{aligned}&\exists\, c_1',\ \alpha>0,\ \forall H\geqslant 1,\ \forall B_1,\ B_2 \text{ as before},\ \forall A_1,\ A_2 \text{ measurable w.r.t.}\\ &\text{the jumping times inside } B_1,\ B_2,\ \mathbb{P}(A_1\cap A_2)-\mathbb{P}(A_1)\,\mathbb{P}(A_2)\leqslant c_1'\,H^{-\alpha}\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\exists\, c_1'',\ \alpha>0,\ \forall H\geqslant 1,\ \forall B_1,\ B_2 \text{ as before},\ \forall A_1,\ A_2 \text{ measurable w.r.t.}\\ &\eta \text{ inside } B_1,\ B_2,\ \mathbb{P}(A_1\cap A_2)-\mathbb{P}(A_1)\,\mathbb{P}(A_2)\leqslant c_1''\,H^{-\alpha}\end{aligned} \tag{2.12}$$

For the invariance property, it is clear that (2.9) and (2.10) imply the invariance property for the joint process. For the decoupling property, suppose that both (2.11) and (2.12) are satisfied with the same $\alpha>0$. Then, let $B_1$, $B_2$ be two boxes as in (2.7), and $A_1$, $A_2$ be measurable with respect to $B_1$, $B_2$. There exist measurable functions $f_1$, $f_2$ with values in $[0,1]$ such that $\mathbf{1}_{A_1} = f_1(\eta_1, \mathcal{T}_1)$ and $\mathbf{1}_{A_2} = f_2(\eta_2, \mathcal{T}_2)$. Set $g_1(\sigma) = \mathbb{E}[f_1(\sigma, \mathcal{T}_1)]$ and $g_2(\sigma) = \mathbb{E}[f_2(\sigma, \mathcal{T}_2)]$. We have

$$\begin{aligned}
\mathbb{E}[f_1(\eta_1, \mathcal{T}_1)\, f_2(\eta_2, \mathcal{T}_2)] &= \mathbb{E}\left[\mathbb{E}\left[f_1(\eta_1, \mathcal{T}_1)\, f_2(\eta_2, \mathcal{T}_2) \,|\, \mathcal{T}_1,\, \mathcal{T}_2\right]\right] \\
&= \mathbb{E}\left[\int_\sigma f_1(\sigma_1, \mathcal{T}_1)\, f_2(\sigma_2, \mathcal{T}_2)\, \mathrm{d}\mathbb{P}_\eta(\sigma)\right] && \text{by independence} \\
&= \int_\sigma \mathbb{E}\left[f_1(\sigma_1, \mathcal{T}_1)\, f_2(\sigma_2, \mathcal{T}_2)\right] \mathrm{d}\mathbb{P}_\eta(\sigma) && \text{by Fubini's theorem} \\
&\leqslant \int_\sigma \mathbb{E}[f_1(\sigma_1, \mathcal{T}_1)]\, \mathbb{E}[f_2(\sigma_2, \mathcal{T}_2)]\, \mathrm{d}\mathbb{P}_\eta(\sigma) + c_1' H^{-\alpha} && \text{by (2.11) and remark 2.10} \\
&= \mathbb{E}[g_1(\eta_1)\, g_2(\eta_2)] + c_1' H^{-\alpha} \\
&\leqslant \mathbb{E}[g_1(\eta_1)]\, \mathbb{E}[g_2(\eta_2)] + (c_1' + c_1'')H^{-\alpha} && \text{by (2.12) and remark 2.10} \\
&= \mathbb{E}[f_1(\eta_1, \mathcal{T}_1)]\, \mathbb{E}[f_2(\eta_2, \mathcal{T}_2)] + (c_1' + c_1'')H^{-\alpha}.
\end{aligned}$$

Hence we do get (2.7), with constant $c_1 = c_1' + c_1''$.

Example 2.12. If we only focus on jumping times, the construction from example 2.4 does satisfy the invariance assumption (2.9), on account of properties of exponential clocks. It also satisfies the decoupling property (2.11): in fact, in this case, we have independence of the jumping times in boxes $B_1$ and $B_2$, because of the properties of Poisson point processes and the equivalence between Poisson point processes and Poisson processes constructed through exponential clocks.

The goal of this section is to show the following theorem, under assumptions 2.6 and 2.9.

Theorem 2.13.

*Suppose $\alpha > 8$. Then there exists $v \in [-1, 1]$ such that*

$$\mathbb{P}\text{-almost surely,} \quad \frac{X_t^{(0,0)}}{t} \xrightarrow[t\to\infty]{} v. \tag{2.13}$$

*Moreover we have a polynomial rate of convergence:*

$$\forall \varepsilon > 0,\ \exists\, c = c(\epsilon) > 0,\ \forall t > 0, \quad \mathbb{P}\left(\left|\frac{X_t^{(0,0)}}{t} - v\right| \geqslant \varepsilon\right) \leqslant c\, t^{-\alpha/4}. \tag{2.14}$$

## 2.2 Examples of environment

We now present two important families of examples that satisfy the assumptions on the environments, namely (2.10) and (2.12). More examples can be found in **[2]**. Here, we focus on those that will also satisfy stronger assumptions that will be required in section 3. The first example, which uses Markov processes, is quite useful and will be presented in details.

### 2.2.1 Markov processes with positive spectral gap

In this example, we suppose that we are given an environment $(\eta_t)_{t\geqslant 0}$ such that:

- $(\eta_t)_{t\geqslant 0}$ is a càdlàg Markov process on $S^{\mathbb{Z}}$, which is space-invariant;
- $(\eta_t)_{t\geqslant 0}$ has a stationary measure $\nu$;
- The family given for $t \geqslant 0$, $x \in S^{\mathbb{Z}}$ and $f \in L^2(\nu)$ by $P_t f(x) = \mathbb{E}_x[f(\eta_t)]$ is a strongly continuous semi-group, which means in particular that $t \geqslant 0 \mapsto P_t f$ is right-continuous and $f \in L^2(\nu) \mapsto P_t f$ is continuous;

- The generator $\mathcal{L}$ of $\eta$ has a positive spectral gap, that is

$$\lambda = \inf_{\substack{f \in D(\mathcal{L}) \\ \nu(f)=0}} \frac{\langle -\mathcal{L}f, f\rangle_\nu}{\|f\|_\nu^2} > 0,$$

where $D(\mathcal{L})$ is the domain of $\mathcal{L}$, $v(f) = \int f \,\mathrm{d}\nu$ and $\langle \cdot, \cdot \rangle_\nu$ is the canonical scalar product on $L^2(\nu)$.

By definition, the environment $\eta$ under $\mathbb{P}_\nu$ satisfies the invariance assumption (2.10). It suffices to check the decoupling property (2.12).

Lemma 2.14. *For all $f \in L^2(\nu)$ and $t \geqslant 0$,*

$$\|P_t f - \nu(f)\|_\nu \leqslant e^{-\lambda t} \|f - \nu(f)\|_\nu. \tag{2.15}$$

Proof. First, note that $D(\mathcal{L})$ is dense in $L^2(\nu)$. Indeed, if $f \in L^2(\nu)$, then by right-continuity of $s \mapsto P_s f$, we have

$$\frac{1}{t}\int_0^t P_s f \,\mathrm{d}s \xrightarrow[t\to 0]{L^2(\nu)} f.$$

Now, for every $t > 0$, $\frac{1}{t}\int_0^t P_s f \,\mathrm{d}s \in D(\mathcal{L})$, because if $h > 0$,

$$\begin{aligned}
\frac{1}{h}\left(P_h \int_0^t P_s f \,\mathrm{d}s - \int_0^t P_s f \,\mathrm{d}s\right) &= \frac{1}{h}\left(\int_0^t P_{h+s} f \,\mathrm{d}s - \int_0^t P_s f \,\mathrm{d}s\right) \\
&= \frac{1}{h}\left(\int_h^{t+h} P_s f \,\mathrm{d}s - \int_0^t P_s f \,\mathrm{d}s\right) \\
&\overset{h\leqslant t}{=} \frac{1}{h}\left(\int_t^{t+h} P_s f \,\mathrm{d}s - \int_0^h P_s f \,\mathrm{d}s\right) \\
&= P_t\left(\frac{1}{h}\int_0^h P_s f \,\mathrm{d}s\right) - \frac{1}{h}\int_0^h P_s f \,\mathrm{d}s \\
&\xrightarrow[h\to 0^+]{L^2(\nu)} P_t f - f,
\end{aligned}$$

by right-continuity of $s \mapsto P_s f$ and continuity of $g \in L^2(\nu) \mapsto P_t g$. Therefore, still by continuity of $g \mapsto P_t g$, it suffices to show (2.15) for $f \in D(\mathcal{L})$, and up to subtracting a constant, it suffices to show it when $\nu(f) = 0$. In this case, $t \geqslant 0 \mapsto \|P_t f\|_\nu^2$ is differentiable and its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_t f\|_\nu^2 = 2\langle \mathcal{L}P_t f, P_t f\rangle_\nu \leqslant -2\lambda \|P_t f\|_\nu^2,$$

because by stationarity, $\nu(P_t f) = \nu(f) = 0$. Therefore, by Gronwall's inequality,

$$\forall t \geqslant 0,\ \|P_t f\|_\nu \leqslant e^{-\lambda t}\|f\|_\nu.$$

$\square$

We can now show that the decoupling property (2.12) is satisfied. In fact, we show a stronger version, which requires nothing on the size of the boxes, that is a control in absolute value, and whose decoupling rate is exponential. This will be crucial in section 3.

Proposition 2.15. *Let $H \geqslant 1$. Let $B_1$, $B_2$ be two boxes separated in time by at least $H$. Then for every event $A_1$ resp. $A_2$ measurable with respect to the environment inside $B_1$ resp. $B_2$,*

$$|\mathbb{P}_\nu(A_1 \cap A_2) - \mathbb{P}_\nu(A_1)\,\mathbb{P}_\nu(A_2)| \leqslant e^{-\lambda H}. \tag{2.16}$$

PROOF. Without loss of generality, suppose that $B_1$ is before $B_2$ in time. Let $T \geqslant 0$ be such that $\pi_2(B_1)$ is included in $[0,T]$. There exist measurable functions $f_1$ and $f_2$ such that $\mathbf{1}_{A_1} = f_1(\eta|_{[0,T]})$ and $\mathbf{1}_{A_2} = f_2(\eta|_{[T+H,+\infty)})$. Up to subtracting their expectations and changing the definitions of $f_1$ and $f_2$, we can assume that $\mathbb{E}_\nu[f_1(\eta|_{[0,T]})] = \mathbb{E}_\nu[f_2(\eta|_{[T+H,+\infty)})] = 0$ and that $f_1$, $f_2$ take values in $[-1,1]$. Let us define, for $\sigma \in S^{\mathbb{Z}}$, $\tilde{f}_2(\sigma) = \mathbb{E}_\sigma[f_2(\eta)]$. Note that

$$\nu(\tilde{f}_2) = \mathbb{E}_\nu[f_2(\eta)] = \mathbb{E}_\nu[f_2(\eta|_{[T+H,+\infty)})] = 0.$$

Therefore, by appling the Markov property twice,

$$\begin{aligned}
\left|\mathbb{E}_\nu\left[f_1(\eta|_{[0,T]})\, f_2(\eta|_{[T+H,+\infty)})\right]\right| &= \left|\mathbb{E}_\nu\left[f_1(\eta|_{[0,T]})\, P_H\tilde{f}_2(\eta_T)\right]\right| \\
&\leqslant \mathbb{E}_\nu\left[(P_H\tilde{f}_2(\eta_T))^2\right]^{1/2} && \text{by the Cauchy-Schwarz inequality} \\
&= \left\|P_H\tilde{f}_2\right\|_\nu && \text{because } \nu \text{ is invariant} \\
&\leqslant e^{-\lambda H}\,\|\tilde{f}_2\|_\nu && \text{by lemma 2.14} \\
&\leqslant e^{-\lambda H}.
\end{aligned}$$

□

### 2.2.2 Independent renewal chains

In this example, the state space $S$ is $\mathbb{N}$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers for which there exists $c > 0$ such that

$$\forall n \in \mathbb{N}^*, \quad 0 < a_n \leqslant c e^{-\log^2 n}.$$

This ensures that $(a_n)_{n\in\mathbb{N}}$ is summable, as well as $\left(\sum_{j\geqslant n} a_j\right)_{n\in\mathbb{N}}$. Let $Z = \sum_{n\in\mathbb{N}} a_n$ and $p_n = \frac{1}{Z}a_n$. A renewal chain of coefficients $(a_n)_{n\in\mathbb{N}}$ is a Markov chain on $\mathbb{N}$ whose transition kernel $Q$ is given by

$$\begin{cases} Q(n,n-1) = 1 & \text{if } n > 0 \\ Q(0,n) = p_n & \text{for } n \in \mathbb{N} \end{cases}$$

It is straightforward to check that this Markov chain has a stationary distribution given by

$$q_n = \frac{1}{Z'}\sum_{j\geqslant n} a_j, \quad \text{where} \quad Z' = \sum_{n\in\mathbb{N}}\sum_{j\geqslant n} a_j.$$

The example we are interested in is that of an environment given by renewal chains chosen independently for each site $x \in \mathbb{Z}$ and started from their invariant measure. It is clear that this environment satisfies the invariance assumption. As far as the decoupling property (2.12) is concerned, **[3]** shows the following property (see (3.44) in **[3]**), which is even stronger.

PROPOSITION 2.16. *Let $H \geqslant 1$. There exists $c_2 > 0$ such that, for any $B_1$, $B_2$ boxes separated in time by at least $H$ (with $B_1$ before $B_2$ in time), and for every event $A_1$ resp. $A_2$ measurable with respect to the environment inside $B_1$ resp. $B_2$,*

$$|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\,\mathbb{P}(A_2)| \leqslant c_2^{-1}\,\mathrm{per}(B_2)\, e^{-c_2 H^{1/8}}, \tag{2.17}$$

*where* $\mathrm{per}(B_2)$ *denotes the perimeter of box* $B_2$.

Note that this decoupling inequality does not require any restriction of size for $B_1$. Now, if $B_2$ has both side lengths at most $5H$, then $\mathrm{per}(B_2) \leqslant 20H$. Therefore, there exists $c_3 > 0$ such that for any $B_1$, $B_2$, $A_1$ and $A_2$ as before,

$$|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| \leqslant c_3^{-1}\, e^{-c_3 H^{1/8}}. \tag{2.18}$$

## 2.3 Strategy of the proof

The idea is to introduce two limiting speeds $v_-$ and $v_+$ that control the asymptotic behaviour of the random walk - roughly speaking, the probability that the random walk has average speed below $v_-$ or beyond $v_+$ between times 0 and $H$ will go to 0 as $H$ goes to infinity (see section 2.4). Then, it remains to show that $v_+ = v_-$, which will give our limiting speed $v$ (see section 2.5.4). In order to do this, we will formalize the idea of particles blocking each other from method 2.1 by introducing notions of trapped points (see section 2.5.1).

Notations 2.17. Let $w \in \mathbb{R}^2$, $y \in \mathbb{L}$ and $H \geqslant 1$.

- We denote by $\mathrm{V}^y_{t,t+s} = \frac{X^y_{t+s} - X^y_t}{s}$ the average speed of the random walk $Y^y$ between $t$ and $t+s$.
- We will very often consider the horizontal interval $I_H(w)$ and the box $B_H(w)$ defined by

$$I_H(w) = w + [0,H) \times \{0\} \quad \text{and} \quad B_H(w) = w + [-2H, 3H) \times [0,H) \subseteq \mathbb{R}^2.$$

- We define

$$A^+_{H,w}(v) = \left\{ \exists\, y \in I_H(w) \cap \mathbb{L},\ \mathrm{V}^y_{0,H} \geqslant v \right\}; \quad p^+_H(v) = \sup_{w \in \mathbb{R}^2} \mathbb{P}\left( A^+_{H,w}(v) \right);$$

$$A^-_{H,w}(v) = \left\{ \exists\, y \in I_H(w) \cap \mathbb{L},\ \mathrm{V}^y_{0,H} \leqslant v \right\}; \quad p^-_H(v) = \sup_{w \in \mathbb{R}^2} \mathbb{P}\left( A^-_{H,w}(v) \right).$$

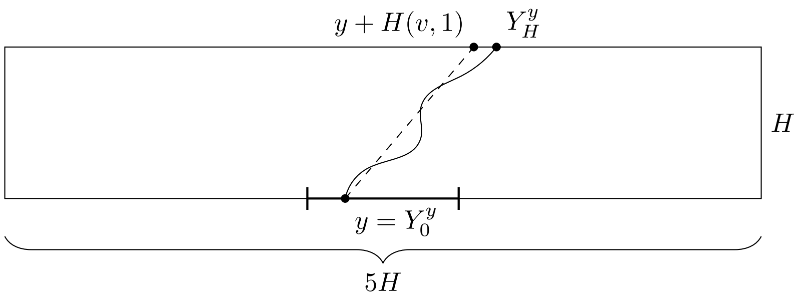


Figure 1: Illustration of the event $A^+_{H,w}(v)$. Starting from the point $y \in I_H(w) \cap \mathbb{L}$ the particle attains an average speed larger than $v$ during the time interval $[0,H]$.

Note that here, taking the supremum over $w \in \mathbb{R}^2$ is equivalent to taking it over $w \in [0,1) \times \{0\}$ by translation invariance in assumptions 2.9. Also, see figure 1 for an illustration of those notations.

Remark 2.18.

Let $w \in \mathbb{R}^2$ and $H \geqslant 1$. In the rest of this report, it will be very useful to consider the box $B_H(w)$. Indeed, consider the event

$$F_H(w) = \left\{ \begin{array}{c} \forall \gamma = (\gamma(t))_{t \in [0,H]} \text{ allowed path starting at } I_H(w) \cap \mathbb{L}, \\ \{\gamma(t),\, t \in [0,H]\} + [-\ell, \ell] \times \{0\} \subseteq B_H(w) \end{array} \right\}. \tag{2.19}$$

Thanks to (2.4), we have

$$\begin{aligned} \mathbb{P}\left( F_H(w)^c \right) &\leqslant c_0^{-1} H e^{-c_0 H} \\ &\leqslant c_5^{-1} e^{-c_5 H}, \end{aligned} \tag{2.20}$$

for $c_5 > 0$ a well chosen constant. Therefore, up to an event of exponentially small probability, our random walks stay in boxes whose dimensions are consistent with those used in the decoupling assumption (2.7).

Definition 2.19. We define

$$v_+ = \inf\left\{v \in \mathbb{R},\ \liminf_{H\to\infty} p_H^+(v) = 0\right\};$$
$$v_- = \sup\left\{v \in \mathbb{R},\ \liminf_{H\to\infty} p_H^-(v) = 0\right\}.$$

Remark 2.20.

- Note that by assumptions (2.2) and (2.3), $v_- \geqslant -1$ and $v_+ \leqslant 1$. Also, $v_- \leqslant v_+$ but this is not obvious *a priori*, see corollary 2.22.
- Note that the function $v \mapsto p_H^+(v)$ is non-increasing. Therefore $v \mapsto \liminf_{H\to\infty} p_H^+(v)$ is too. This ensures that for all $v > v_+$, $\liminf_{H\to\infty} p_H^+(v) = 0$. Using a symmetric argument, for all $v < v_-$, $\liminf_{H\to\infty} p_H^-(v) = 0$.

The proof of theorem 2.13 comprises two main parts. The first part consists in showing deviation estimates in the form of (2.14) for velocities that are below $v_-$ or beyond $v_+$. The second part aims at proving that $v_- = v_+$, which yields the desired velocity $v$.

## 2.4 First part: bounds for deviations below $v_-$ and beyond $v_+$

The goal of this section is to show the following estimates:

Proposition 2.21.

*Assume $\alpha > 5$. For any $\varepsilon > 0$, there exists $c_4 = c_4(\varepsilon) > 0$ such that*

$$p_H^+(v_+ + \varepsilon) \leqslant c_4\, H^{-\alpha/4}; \tag{2.21}$$
$$p_H^-(v_- - \varepsilon) \leqslant c_4\, H^{-\alpha/4}. \tag{2.22}$$

Corollary 2.22. $v_- \leqslant v_+$.

We start by noticing that if we replace the jumping function $g$ by $-g$, we obtain a random walk whose upper speed is $-v_-$. This allows us to show only (2.21) for $p^+$.

All throughout this report, we will use the following scales, constructed to work well with the polynomial decoupling given by (2.7).

Definition 2.23. We set $L_0 = 10^{10}$ and, for $k \geqslant 0$,

$$L_{k+1} = l_k\, L_k, \quad \text{where} \quad l_k = \lfloor L_k^{1/4} \rfloor.$$

The proof of proposition 2.21 is based on the fundamental idea of renormalization schemes, which will be useful in other proofs later.

Method 2.24.

Suppose we want to show an estimate for the probability of a certain family of "bad" events.

**Step 1** We start by focusing on a certain subsequence $(A_k)_{k\in\mathbb{N}}$. We set $p_k = \mathbb{P}(A_k)$. We show that on $A_{k+1}$, two events of probability $\mathbb{P}(A_k)$ occur, with the constraint that these two events are supported by boxes separated in time by $R_k$.

**Step 2** We deduce the desired estimate for $(p_k)_{k\in\mathbb{N}}$.

- Using our decorrelation assumption (2.7) and a union bound, we get an inequality of the form

$$p_{k+1} \leqslant C_k\,(p_k^2 + c_1 R_k^{-\alpha}),$$

where $C_k$ is a certain integer.

- From this inequality we deduce the desired estimate of $p_k$ by induction on $k$. For this to work, the subsequence and the bound to show have to be chosen properly.

**Step 3** We conclude using an interpolation argument.

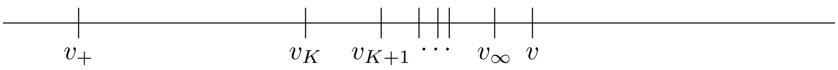


Figure 2: The sequence of velocities $(v_k)_{k\geqslant K}$

Proof of proposition 2.21. We follow method 2.24 step by step. We fix $v > v_+$.

**Step 1** We start by linking what happens at scales $k$ and $k+1$. The kea idea of the proof is roughly that, choosing a sequence of velocities $(v_k)$ properly, on the event that there exists a trajectory going faster than $v_{k+1}$ at scale $k+1$, there exists two well-separated boxes at scale $k$ on which there exists trajectories going faster than $v_k$. Let $K$ be an integer large enough so that

$$\sum_{k\geqslant K} \frac{8}{l_k} < \frac{v - v_+}{2}.$$

Let $v_K = \frac{v+v_+}{2}$ and $v_{k+1} = v_k + \frac{8}{l_k}$ for every $k \geqslant K$ (see figure 2). We have

$$v_k \xrightarrow[k\to\infty]{\nearrow} v_\infty = v_K + \sum_{k\geqslant K} \frac{8}{l_k} < v.$$

Let $k \geqslant K$ and $w \in \mathbb{R}^2$. Also let $h \geqslant 1$ (a parameter to be fixed later). Denote by $\mathcal{C} = \mathcal{C}_{h,k,w} \subseteq \mathbb{R}^2$ the set of cardinal $5\, l_k^2$ such that

$$\bigcup_{w'\in\mathcal{C}} I_{hL_k}(w') = B_{hL_{k+1}}(w) \cap \left(\mathbb{R} \times (\pi_2(w) + hL_k\mathbb{Z})\right). \tag{2.23}$$

Claim: on the event $A^+_{hL_{k+1},w}(v_{k+1}) \cap \left(\bigcap_{w'\in\mathcal{C}} F_{hL_k}(w')\right)$, there exists two points $w_1'$, $w_2' \in \mathcal{C}$ such that

$$\begin{cases} A^+_{hL_k,w_1'}(v_k) \text{ and } A^+_{hL_k,w_2'}(v_k) \text{ occur;} \\ d(B_{hL_k}(w_1'), B_{hL_k}(w_2')) \geqslant hL_k. \end{cases}$$

It is enough to show that there exist $w_1'$, $w_2'$, $w_3' \in \mathcal{C}$ such that $A^+_{hL_k,w_i'}(v_k)$ occurs for $i = 1, 2, 3$ and $(\pi_2(B_{hL_k}(w_i'))_{i=1,2,3}$ are disjoint. Assume by contradiction that this does not hold. It means that there exists $j_1$, $j_2 \in \{0, \ldots, l_k - 1\}$ such that

$$\text{for all } j \notin \{j_1, j_2\},\ (A^+_{hL_k,w'}(v_k))^c \text{ occurs for all } w' = (x, \pi_2(w) + jhL_k) \in \mathcal{C}. \tag{2.24}$$

Fix $y \in I_{hL_{k+1}}(w)$. Remark that $\bigcap_{w'\in\mathcal{C}} F_{hL_k}(w') \subseteq F_{hL_{k+1}}(w)$, so on this event, for every $j \in [\![0, l_k - 1]\!]$, $Y^y_{jhL_k}$ is on one of the intervals $I_{hL_k}(w')$ for $w' \in \mathcal{C}$. Therefore, by (2.24), for $j \notin \{j_1, j_2\}$, we can upper-bound the displacement between time $jhL_k$ and $(j+1)hL_k$ by $v_k hL_k$. For $j_1$ and $j_2$, using that we are on $\bigcap_{w'\in\mathcal{C}} F_{hL_k}(w')$, we can upper-bound the displacement by $3hL_k$. In the end,

$$\begin{aligned} X^y_{hL_{k+1}} - \pi_1(y) &= \sum_{j=0}^{l_k-1} \left( X^{Y^y_{jhL_k}}_{hL_k} - X^y_{jhL_k} \right) \\ &\leqslant (l_k - 2) v_k hL_k + 2 \cdot 3hL_k \end{aligned}$$

$$
\begin{aligned}
&= v_k h L_{k+1} + \frac{6-2v_k}{l_k}\, hL_{k+1} \\
&\overset{v_k>-1}{<} \left(v_k + \frac{8}{l_k}\right) hL_{k+1} \\
&= v_{k+1} hL_{k+1},
\end{aligned}
$$

so $A^+_{hL_k,w}(v_{k+1})$ does not occur. This proves the claim.

**Step 2** We now show an estimate in the form of (2.21) along a subsequence of the form $(hL_k)_{k\in\mathbb{N}}$. More precisely, we prove by induction on $k$ that there exists $c_6 = c_6(v) \geqslant 1$ and $k_0 = k_0(v) \geqslant 1$ such that for every $k \geqslant k_0$,

$$
p^+_{c_6 L_k}(v) \leqslant L_k^{-\alpha/2}. \tag{2.25}
$$

For now, let us fix $k_0 \geqslant K$, which will be chosen later.

- Base case. Using remark 2.20, note that since $v_{k_0} > v_+$,

$$
\liminf_{h\to\infty} p^+_{hL_{k_0}}(v_{k_0}) = 0.
$$

  Therefore there exists $c_6 \geqslant 1$ such that

$$
p^+_{c_6 L_{k_0}}(v_{k_0}) \leqslant L_{k_0}^{-\alpha/2}.
$$

  Therefore, since $v_{k_0} \leqslant v$ and $v \mapsto p^+_{hL_{k_0}}(v)$ is non-increasing,

$$
p^+_{c_6 L_{k_0}}(v) \leqslant L_{k_0}^{-\alpha/2}.
$$

- Induction step. Because of what we have shown in step 1, we have, setting $\mathcal{C} = \mathcal{C}_{c_6,k,w}$,

$$
\begin{aligned}
&\mathbb{P}\left(A^+_{c_6 L_{k+1},w}(v_{k+1})\right) \\
&\leqslant \mathbb{P}\left(A^+_{c_6 L_{k+1},w}(v_{k+1}) \cap \left(\bigcap_{w'\in\mathcal{C}} F_{c_6 L_k}(w')\right)\right) + \mathbb{P}\left(\bigcup_{w'\in\mathcal{C}} F_{c_6 L_k}(w')^c\right) \\
&\leqslant \mathbb{P}\left(\bigcup_{w_1',w_2'\in\mathcal{C},\ |\pi_2(w_1')-\pi_2(w_2')|\geqslant 2c_6 L_k} \begin{array}{c}(A^+_{c_6 L_k,w_1'}(v_k) \cap F_{c_6 L_k}(w_1')) \\ \cap\, (A^+_{c_6 L_k,w_2'}(v_k) \cap F_{c_6 L_k}(w_2'))\end{array}\right) + \mathbb{P}\left(\bigcup_{w'\in\mathcal{C}} F_{c_6 L_k}(w')^c\right) \\
&\leqslant 25\, l_k^4 \sup_{w_1',w_2'\in\mathcal{C},\ |\pi_2(w_1')-\pi_2(w_2')|\geqslant 2c_6 L_k} \mathbb{P}\left(\begin{array}{c}(A^+_{c_6 L_k,w_1'}(v_k) \cap F_{c_6 L_k}(w_1')) \\ \cap\, (A^+_{c_6 L_k,w_2'}(v_k) \cap F_{c_6 L_k}(w_2'))\end{array}\right) + 5\, l_k^2 \sup_{w'} \mathbb{P}\left(F_{c_6 L_k}(w')^c\right) \\
&\leqslant 25\, l_k^4\, \left(p^+_{c_6 L_k}(v_k)^2 + c_1 (c_6 L_k)^{-\alpha}\right) + 5\, l_k^2 c_5^{-1} e^{-c_5 c_6 L_k} \qquad \text{by (2.20)} \\
&\leqslant 25\, l_k^4\, \left(p^+_{c_6 L_k}(v_k)^2 + c (c_6 L_k)^{-\alpha}\right),
\end{aligned}
$$

  and so

$$
p^+_{c_6 L_k}(v_{k+1}) \leqslant 25\, l_k^4\, \left(p^+_{c_6 L_k}(v_k)^2 + c (c_6 L_k)^{-\alpha}\right).
$$

  Suppose that (2.25) is satisfied for $k$ fixed. Then

$$
\begin{aligned}
\frac{p^+_{c_6 L_{k+1}}(v_{k+1})}{L_{k+1}^{-\alpha/2}} &\leqslant 25\, L_{k+1}^{\alpha/2}\, l_k^4\, \left(p^+_{c_6 L_k}(v_k)^2 + c (c_6 L_k)^{-\alpha}\right) \\
&\overset{c_6\geqslant 1}{\leqslant} 25\, L_k^{5\alpha/8+1}\left(L_k^{-\alpha} + c L_k^{-\alpha}\right) \\
&\leqslant c\, L_k^{1-3\alpha/8} \\
&\overset{\alpha>5}{\leqslant} c\, L_k^{-7/8}.
\end{aligned}
$$

  This last term is less than 1 when $k$ is sufficiently large, which enables us to choose $k_0$ properly (independently of $c_6$). This concludes the induction. Now, because $v \mapsto p^+_{hL_k}(v)$ is non-decreasing for every $k$, we do get estimate (2.25) for every $k \geqslant k_0$.

**Step 3** We now interpolate estimate (2.25) to get something valid for any $H$. To do this, we let $v > v_+$, $v' = \frac{v_+ + v}{2}$ and $c_6 = c_6(v')$, $k_0 = k_0(v')$ be as in estimate (2.25). Let $H > (c_6 L_{k_0})^{11/10}$ and let $\bar{k}$ be such that

$$(c_6 L_{\bar{k}})^{11/10} \leqslant H < (c_6 L_{\bar{k}+1})^{11/10}. \tag{2.26}$$

Since $\bar{k} \geqslant k_0$ and $v' > v_+$, (2.25) ensures that

$$p_{c_6 L_{\bar{k}}}(v') \leqslant L_{\bar{k}}^{-\alpha/2}. \tag{2.27}$$

Now, consider the box $B_H(w)$ and pave it using boxes of the form $B_{c_6 L_{\bar{k}}}(w')$ where $w' \in c_6 L_{\bar{k}} \mathbb{Z}^2$. The set $\bar{\mathcal{C}}$ of small boxes needed in this paving satisfies

$$|\bar{\mathcal{C}}| \leqslant c \left( \frac{H}{c_6 L_{\bar{k}}} \right)^2 \leqslant c \left( \frac{(c_6 L_{\bar{k}+1})^{11/10}}{c_6 L_{\bar{k}}} \right)^2 \leqslant c\, L_{\bar{k}}^{3/4}. \tag{2.28}$$

Now, on the event $\bigcap_{w' \in \bar{\mathcal{C}}} \left[ \left( A^+_{c_6 L_{\bar{k}}, w'}(v') \right)^c \cap F_{c_6 L_{\bar{k}}}(w') \right]$ and for any $y \in I_H(w)$, we have

$$\begin{aligned} X^y_{\lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}} - \pi_1(y) &= \sum_{j=0}^{\lfloor H/c_6 L_{\bar{k}} \rfloor - 1} \left( X_{c_6 L_{\bar{k}}}^{Y^y_{j c_6 L_{\bar{k}}}} - X^y_{j c_6 L_{\bar{k}}} \right) \\ &\leqslant v' \lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}} \\ &\leqslant v' H. \end{aligned} \tag{2.29}$$

Now, let us look at what happens between time $\lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}$ and $H$. Using assumption (2.4) and translation invariance, we have

$$\begin{aligned} &\mathbb{P}\left( X^y_H - X^y_{\lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}} \geqslant (v' - v) H \right) \\ &\leqslant 4H\, \mathbb{P} \begin{pmatrix} \exists\, \gamma = (\gamma(t))_{t \in [0, H - \lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}]} \text{ such that} \\ \gamma(0) = 0 \text{ and } \gamma(H - \lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}) \geqslant (v' - v) H \end{pmatrix} + c_0^{-1} e^{-c_0 \lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}} \\ &\leqslant c^{-1} H e^{-c\, L_{\bar{k}}}, \end{aligned} \tag{2.30}$$

as long as $(v' - v) H \geqslant 2 c_6 L_{\bar{k}}$ (in the first inequality above, we used a union bound on the possible positions of $X^y_{\lfloor H/c_6 L_{\bar{k}} \rfloor c_6 L_{\bar{k}}}$). Assume that $H$ is large enough in the rest of this proof. By (2.29), we have

$$\begin{aligned} \mathbb{P}(A^+_{H,w}(v)) &= \mathbb{P}(\exists\, y \in I_H(w) \cap \mathbb{L},\ \mathrm{V}^y_{0,H} \geqslant v) \\ &\leqslant \mathbb{P}\left( \bigcup_{w' \in \bar{\mathcal{C}}} A^+_{c_6 L_{\bar{k}}, w'}(v') \cup F_{c_6 L_{\bar{k}}}(w')^c \right) + \mathbb{P}\left( X^y_H - X^y_{\lfloor H/c_6 L_{\bar{k}} \rfloor\, c_6 L_{\bar{k}}} \geqslant (v' - v) H \right). \end{aligned}$$

Now, using (2.26), (2.27), (2.28) and (2.30), as well as (2.20), we get that

$$\begin{aligned} \mathbb{P}(A^+_{H,w}(v)) &\leqslant c\, |\bar{\mathcal{C}}|\, L_{\bar{k}}^{-\alpha/2} + c^{-1} H e^{-c\, L_{\bar{k}}} \\ &\leqslant c\, L_{\bar{k}}^{3/4} L_{\bar{k}}^{-\alpha/2} + c^{-1} L_{\bar{k}+1}^{11/10} e^{-c\, L_{\bar{k}}} \\ &\leqslant c\, L_{\bar{k}}^{-7\alpha/20} \\ &\leqslant c\, H^{-\alpha/4}. \end{aligned}$$

To conclude, it suffices to change $c$ in order for the last inequality to hold also for small values of $H$. □

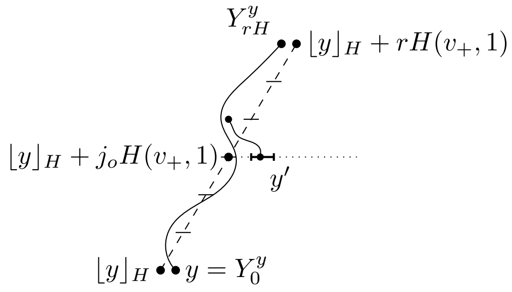


Figure 3: The point $\lfloor y \rfloor_H$ is $(H,r)$-threatened, since $\lfloor y \rfloor_H + j_o H(v_+,1)$ is $H$-trapped.

## 2.5 Second part: $v_+ = v_-$

Now that we have the estimates given by proposition 2.21, we want to prove that $v_- = v_+$. We argue by contradiction and assume that $v_- < v_+$. Now, the idea is to show that on a certain subsequence of times, the particles will be delayed to the left (*i.e.* their average speed will be close to $v_-$), therefore they will accumulate a delay with respect to $v_+$. Now this delay cannot be caught up because our particles cannot go faster than $v_+$, because of proposition 2.21. This will yield a contradiction. In the rest of this section, we set

$$\delta = \frac{v_+ - v_-}{4} \in (0, 1/2]. \tag{2.31}$$

### 2.5.1 Trapped and threatened points

The crucial idea of our proof is the property of monotonicity (2.5), which implies that a particle can be "trapped" by another particle. We want to ensure that trapped particles will experience a delay with respect to $v_+$, which motivates the first definition below. The problem is that the probability of being trapped may be very small. But actually, we will see that it suffices that there exist a trapped point along a line segment of slope $v_+$ in order to experience the delay, which motivates the second definition below (see figure 3 for an illustration of threatened points).

Definition 2.25.

Let $H \geqslant 1$ and $r \in \mathbb{N}^*$.

- $w \in \mathbb{R}^2$ is said to be $H$-trapped if there exists $y \in (w + [\delta H, 2\delta H) \times \{0\}) \cap \mathbb{L}$ such that
$$\mathrm{V}^y_{0,H} \leqslant v_- + \delta.$$
- $w$ is said to be $(H,r)$-threatened if one of the points $w_j := w + jH(v_+,1)$, where $j \in [\![0, r-1]\!]$, is $H$-trapped.

Remark 2.26. The first definition ensures that any particle starting near a trapped point $w$ will experience a delay with respect to $v_+$:

$$\forall w' \in (w + (-\infty, \delta H) \times \{0\}) \cap \mathbb{L}, \quad X^{w'}_H - \pi_1(w) \leqslant X^y_H - \pi_1(y) + 2\delta H \overset{(2.31)}{\leqslant} (v_+ - \delta)H.$$

The following proposition specifies the behavior of a random walk starting near a threatened point: if its speed is not too large with respect to $v_+$, it will have to experience a delay with respect to $v_+$.

Notation 2.27. Denote by $\lfloor y \rfloor_H$ the closest point to the left of $y$ that is in $\lfloor \delta H/4 \rfloor \mathbb{L}$ (this is because we do not want to consider too many points, and will become clearer later on).

Proposition 2.28. *Let $H \geqslant 1$ and $r \in \mathbb{N}^*$. Assume that $\lfloor y \rfloor_H$ is $(H,r)$-threatened. Then*

1. *Either there exists $j \in [\![0, r-1]\!]$ such that $\mathrm{V}^y_{jH,(j+1)H} > v_+ + \dfrac{\delta}{2r}$.*

2. *Or $\mathrm{V}^y_{0,rH} \leqslant v_+ - \dfrac{\delta}{2r}$.*

Proof. Denote by $j_0$ the integer such that $\lfloor y \rfloor_H + j_0 H(v_+, 1)$ is trapped: there exists $y'$ in $(\lfloor y \rfloor_H + j_0 H(v_+, 1) + [\delta H, 2\delta H) \times \{0\}) \cap \mathbb{L}$ such that $\mathrm{V}^{y'}_{0,H} \leqslant v_- + \delta = v_+ - 3\delta$. If condition 1. is not satisfied, then we can bound the speed over all time intervals $[jH, (j+1)H]$, and we get

$$
\begin{aligned}
X^y_{j_0 H} &\leqslant \pi_1(y) + \left(v_+ + \frac{\delta}{2r}\right) j_0 H \\
&\leqslant \pi_1(\lfloor y \rfloor_H) + \frac{\delta H}{4} + \left(v_+ + \frac{\delta}{2r}\right) j_0 H \\
&\leqslant \pi_1(\lfloor y \rfloor_H) + \left(1 - \frac{j_0}{2r}\right) \delta H + \left(v_+ + \frac{\delta}{2r}\right) j_0 H \\
&= \pi_1(\lfloor y \rfloor_H) + j_0 H v_+ + \delta H \\
&\leqslant \pi_1(y').
\end{aligned}
\tag{2.32}
$$

Therefore, by monotonicity,

$$
\begin{aligned}
X^y_{(j_0+1)H} &\leqslant \pi_1(y') + (v_- + \delta)H \\
&\leqslant \pi_1(\lfloor y \rfloor_H) + j_0 H v_+ + 2\delta H + (v_+ - 3\delta)H \\
&\leqslant \pi_1(y) + (j_0+1)v_+ H - \delta H.
\end{aligned}
\tag{2.33}
$$

Note that here we used the same reasoning as in remark 2.26. Now, by bounding the speed over all remaining time intervals $[jH, (j+1)H]$, we get

$$
\begin{aligned}
X^y_{rH} - \pi_1(y) &\leqslant \left(X^y_{rH} - X^y_{(j_0+1)H}\right) + \left(X^y_{(j_0+1)H} - \pi_1(y)\right) \\
&\leqslant (r - j_0 + 1) \left(v_+ + \frac{\delta}{2r}\right) H + (j_0+1)v_+ H - \delta H \\
&\leqslant r v_+ H - \frac{\delta}{2} H = \left(v_+ - \frac{\delta}{2r}\right) rH.
\end{aligned}
$$

$\square$

### 2.5.2 Probability of being threatened

When $r$ (the vertical length of the line segment along which we look for trapped points) increases, it is clear that the probability that $w$ is threatened increases. We now show that it goes to 1 when $r \to \infty$, and quantify the convergence using $\alpha$.

Proposition 2.29.

*Assume $\alpha \geqslant 1$. There exists $c_7 = c_7(\delta) > 0$ and $H_0 \geqslant 1$ such that for every $H \geqslant H_0$ and $r \in \mathbb{N}^*$,*

$$
\sup_{w \in \mathbb{R}^2} \mathbb{P}(w \text{ is not } (H,r)\text{-threatened}) \leqslant c_7\, r^{-\alpha}.
\tag{2.34}
$$

Proof. We follow again the ideas of method 2.24.

**Step 1** We start by showing that there exists $H_0$ such that

$$\inf_{H\geqslant H_0}\inf_{w\in\mathbb{R}^2}\mathbb{P}(w \text{ is } H\text{-trapped})>0. \tag{2.35}$$

By definition of $v_-$, we have

$$\liminf_{H\to\infty}\sup_{w\in[0,1)\times\{0\}}\mathbb{P}\left(\exists\, y\in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}\leqslant v_-+\delta\right)>0,$$

so there exists $H_0$ (which can be chosen arbitrarily large) such that

$$\inf_{H\geqslant H_0}\sup_{w\in[0,1)\times\{0\}}\mathbb{P}\left(\exists\, y\in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}\leqslant v_-+\delta\right)>0. \tag{2.36}$$

Now, for a fixed $H\geqslant H_0$, we have

$$\begin{aligned}
&\sup_{w\in\mathbb{R}^2}\mathbb{P}(w \text{ is not } H\text{-trapped})\\
&=\sup_{w\in\mathbb{R}\times\{0\}}\mathbb{P}(w \text{ is not } H\text{-trapped}) && \text{by translation invariance}\\
&=\sup_{w\in\mathbb{R}\times\{0\}}\mathbb{P}\left(\forall\, y\in I_{\delta H}(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}>v_-+\delta\right) && \text{by translation invariance}\\
&=\sup_{w\in[-1,0)\times\{0\}}\mathbb{P}\left(\forall\, y\in I_{\delta H}(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}>v_-+\delta\right) && \text{by translation invariance}\\
&\leqslant\inf_{w\in[0,1)\times\{0\}}\mathbb{P}\left(\forall\, y\in I_{\delta H/2}(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}>v_-+\delta\right) && (*)\\
&\leqslant 1-\lfloor 2/\delta\rfloor^{-1}\sup_{w\in[0,1)\times\{0\}}\mathbb{P}\left(\exists\, y\in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H}\leqslant v_-+\delta\right). && (**)
\end{aligned}$$

In $(*)$, we used that if $H\geqslant H_0$, for any $w\in[-1,0)\times\{0\}$ and $w'\in[0,1)\times\{0\}$, $w'+[0,\delta H/2)\times\{0\}$ is included in $w+[0,\delta H)\times\{0\}$, up to taking a larger $H_0$. In $(**)$, we split $[0,H]$ in intervals of length $\delta H/2$ and used a union bound. Combining these inequalities with (2.36) yields the claim (2.35).

**Step 2** We start by considering only the indexes $3^k$ for $k\in\mathbb{N}$, before interpolating. We set

$$q_k=\sup_{w\in\mathbb{R}^2}\mathbb{P}(w \text{ is not } (H,3^k)\text{-threatened}).$$

To do this, we begin by showing that $q_k$ converges to 0 when $k\to\infty$, uniformly in $H$ large enough. More precisely, we show that

$$\exists\, C\in[1/3,1),\ \exists\, k_0\in\mathbb{N},\ \exists\, H_0\geqslant 1,\ \forall k\geqslant 2,\ \forall H\geqslant H_0,\ q_{k_0+k}\leqslant C^k. \tag{2.37}$$

To do this, we use induction on $k\geqslant 2$. Let us fix $k_0\in\mathbb{N}$ (we will choose it at the end of the proof).

- Base case: Note that if a point is not threatened, in particular it is not trapped, so, by (2.35),

  $$\sup_{H\geqslant H_0} q_{k_0+2}\leqslant\sup_{H\geqslant H_0}\sup_{w\in\mathbb{R}^2}\mathbb{P}(w \text{ is not } H\text{-trapped})<1$$

  Therefore there exists $C\in[1/3,1)$ such that the case $k=2$ in (2.37) is satisfied: $q_{k_0+2}\leqslant C^2$ for all $H\geqslant H_0$.

- Induction step: fix $H\geqslant H_0$, $k\geqslant 2$ and $w\in\mathbb{R}^2$. The event $\{w \text{ is not } (H,3^{k_0+k+1})\text{-threatened}\}$ is included in the events

  $$A_k^{(1)}=\bigcap_{j=0}^{3^{k_0+k}-1}\{w_j \text{ is not } H\text{-trapped}\}\quad\text{and}\quad A_k^{(2)}=\bigcap_{j=2\cdot 3^{k_0+k}}^{3^{k_0+k+1}-1}\{w_j \text{ is not } H\text{-trapped}\}.$$

These events are measurable with respect to horizontal strips separated in time by $3^{k_0+k}H$. In order to replace those strips by boxes of side lengths at most $5\cdot 3^{k_0+k}H$, similarly as before, we introduce the following events, with $w'\in\mathbb{R}^2$:

$$F_{H,k,w'}(w)=\left\{\begin{array}{c}\forall\,\gamma \text{ allowed path on } [0,H] \text{ starting in } (w'+[\delta H,2\delta H)\times\{0\})\cap\mathbb{L},\\ \{\gamma(t),\,t\in[0,H]\}+[-\ell,\ell]\times\{0\}\subseteq B_{3^{k_0+k}H}(w)\end{array}\right\}$$

$$F_k^{(1)}=\bigcap_{j=0}^{3^{k_0+k}-1}F_{H,k,w_j}(w)\ \text{ and }\ F_k^{(2)}=\bigcap_{j=2\cdot 3^{k_0+k}}^{3^{k_0+k+1}-1}F_{H,k,w_j}(w_{2\cdot 3^{k_0+k}}).$$

This way, $A_k^{(1)}\cap F_k^{(1)}$ is measurable with respect to the box $B_{3^{k_0+k}H}(w)$, and $A_k^{(2)}\cap F_k^{(2)}$ is measurable with respect to the box $B_{3^{k_0+k}H}(w_{2\cdot 3^{k_0+k}})$. Note that, following notation (2.19), we have

$$\begin{cases}F_{H,k,w_j}(w)\supseteq F_{3^{k_0+k}H}(w+(0,jH)) & \forall j\in[\![0,3^{k_0+k}-1]\!]\\ F_{H,k,w_j}(w_{2\cdot 3^{k_0+k}})\supseteq F_{3^{k_0+k}H}(w_{2\cdot 3^{k_0+k}}+(0,jH)) & \forall j\in[\![2\cdot 3^{k_0+k},3^{k_0+k+1}-1]\!]\end{cases}$$

So, by inequality (2.20),

$$\begin{aligned}q_{k_0+k+1}&\leqslant \underbrace{\mathbb{P}\Big((A_k^{(1)}\cap F_k^{(1)})\cap(A_k^{(2)}\cap F_k^{(2)})\Big)}_{\leqslant q^2_{k_0+k}+c_1(3^{k_0+k}H)^{-\alpha}}+\underbrace{\mathbb{P}\big((F_k^{(1)})^c\cup(F_k^{(2)})^c\big)}_{\leqslant 3^{k_0+k+1}c_5^{-1}e^{-c_5 3^{k_0+k}H}}\\ &\overset{H\geqslant 1}{\leqslant} q^2_{k_0+k}+c_1\,3^{-(k_0+k)\alpha}+c\,3^{-(k_0+k)\alpha}.\end{aligned}$$

In the end, since $C\geqslant 1/3$, $\alpha\geqslant 1$ and $k\geqslant 2$,

$$\frac{q_{k_0+k+1}}{C^{k+1}}\leqslant C^{k-1}+(c_1+c)\,C^{k_0-1}\leqslant 1\ \text{ if } k_0 \text{ is chosen large enough.}$$

**Step 3** We now prove the desired estimate on the subsequence; more precisely, we prove that

$$\exists\,k_1\in\mathbb{N}^*,\ \forall k\in\mathbb{N}^*,\ \forall H\geqslant H_0,\ q_{k_1+k}\leqslant\frac{1}{2}\,3^{-\alpha k}. \tag{2.38}$$

We use exactly the same method as in step 2 : fix $k_1\in\mathbb{N}$ a large enough integer (to be chosen later) such that $\forall H\geqslant H_0,\ q_{k_1+1}\leqslant\frac{1}{2}\,3^{-\alpha}$, which is possible because, as we have just shown, $q_k\to 0$ uniformly in $H\geqslant H_0$. We show by induction on $k\geqslant 1$ that $q_{k_1+k}\leqslant\frac{1}{2}\,3^{-\alpha k}$. For the induction step, we have

$$\frac{q_{k_1+k+1}}{\frac{1}{2}3^{-\alpha(k+1)}}\leqslant 2\cdot 3^{\alpha(k+1)}\left(\frac{1}{4}3^{-2\alpha k}+(c_1+c)\,3^{-\alpha(k_1+k)}\right)\leqslant 1\ \text{ if } k_1 \text{ is chosen large enough,}$$

which yields (2.38).

**Step 4** We conclude by interpolation. Let $r\geqslant 3^{k_1+1}$ and $k\in\mathbb{N}^*$ such that $3^{k_1+k}\leqslant r<3^{k_1+k+1}$. Then

$$\mathbb{P}(w\text{ is not }(H,r)\text{-threatened})\leqslant\mathbb{P}(w\text{ is not }(H,3^{k_1+k})\text{-threatened})\leqslant\frac{1}{2}3^{-\alpha k}\overset{(2.38)}{\leqslant}\frac{3^{\alpha(k_1+1)}}{2}\,r^{-\alpha}.$$

It remains to adapt the constant in order for (2.34) to hold for every $r\in\mathbb{N}^*$. □

### 2.5.3 Density of threatened points

We now know that when a particle passes near a threatened point, it will be delayed to the left, and that each point has a high probability of being threatened. We now show that with high probability, every particle meats a lot of threatened points along the way.

From now on, we will work with a fixed integer $k_2$ (see lemma 2.30 for its proper definition) and with

$H = hL_k$, $r = l_k$, $rH = hL_{k+1}$, where $k > k_2$ and $h \geqslant 1$.

LEMMA 2.30. *Assume that $\alpha \geqslant 8$. There exists $k_2 = k_2(\delta) \in \mathbb{N}^*$ and $c_8 = c_8(\delta) > 0$ such that the following conditions are satisfied:*

1. $L_{k_2} > H_0$ ;
2. *For every $w \in \mathbb{R}^2$ and $h \geqslant 1$,*

$$\mathbb{P}\left(\exists\, y \in I_{hL_{k_2+1}}(w) \cap \mathbb{L},\ \lfloor y \rfloor_{hL_{k_2}} \text{ is not } (hL_{k_2}, l_{k_2})\text{-threatened}\right) \leqslant c_8\, L_{k_2+1}^{-(\alpha-1)/5}. \tag{2.39}$$

3. $\forall k > k_2$,

$$25\,(c_8^2 + c_1) L_k^{-(3\alpha-23)/20} + 25\, c_5^{-1}\, L_k^{(\alpha+3)/4} e^{-c_5 L_k} \leqslant c_8. \tag{2.40}$$

Note that (2.40) is just a technical requirement that will appear naturally later in the proof.

PROOF. Let $w \in \mathbb{R}^2$, $h \geqslant 1$ and $k_2 \in \mathbb{N}^*$ (to be chosen later) such that $L_{k_2} > H_0$. Using proposition 2.29, we have

$$\begin{aligned} &\mathbb{P}\left(\exists\, y \in I_{hL_{k_2+1}}(w) \cap \mathbb{L},\ \lfloor y \rfloor_{hL_{k_2}} \text{ is not } (hL_{k_2}, l_{k_2})\text{-threatened}\right) \\ &\leqslant \left\lceil \frac{hL_{k_2+1}}{\left\lfloor \frac{\delta h L_{k_2}}{4} \right\rfloor} \right\rceil c_7\, l_{k_2}^{-\alpha} \leqslant c(\delta)\, L_{k_2+1}^{-\frac{\alpha-1}{5}}. \end{aligned}$$

Therefore, we do get inequality (2.39) with a certain constant $c_8 > 0$. Note that considering only rounded points $\lfloor y \rfloor$ was crucial here, to obtain a bound which is uniform in $h$. Now that $c_8$ is fixed, it suffices to take $k_2$ even larger so that inequality (2.40) holds as well, which is possible because $3\alpha - 23 > 0$. □

DEFINITION 2.31. Let $k_2$ be defined as in lemma 2.30. Let $k > k_2$ and $\gamma = (\gamma(t))_{t\in[0,hL_k]}$ an allowed path. We define its threatened density as

$$D^h(\gamma) = \frac{L_{k_2+1}}{L_k}\, \#\left\{0 \leqslant j < \frac{L_k}{L_{k_2+1}},\ \lfloor \gamma(jhL_{k_2+1}) \rfloor_{hL_{k_2}} \text{ is } (hL_{k_2}, l_{k_2})\text{-threatened}\right\}. \tag{2.41}$$

PROPOSITION 2.32. *Let $\alpha \geqslant 8$. For every $k > k_2$, $w \in \mathbb{R}^2$ and $h \geqslant 1$,*

$$\mathbb{P}(\exists\, \gamma = (\gamma(t))_{t\in[0,hL_k]} \text{ starting in } I_{hL_k}(w) \cap \mathbb{L},\ D^h(\gamma) < 1/2) \leqslant c_8\, L_k^{-(\alpha-1)/5}.$$

PROOF. The proof resembles that of proposition 2.21. Again, we are going to use method 2.24, along with a sequence of densities $(\rho_k)_{k\geqslant k_2}$. We define $\rho_{k_2} = 1$ and for $k \geqslant k_2$, $\rho_{k+1} = \rho_k - \frac{2}{l_k}$. Because $\sum_{k\geqslant 1} \frac{2}{l_k} \leqslant \frac{1}{2}$, we have $\rho_k \geqslant 1/2$ for every $k \geqslant k_2$. We define

$$S_{h,k}(w) = \{\exists\, \gamma = (\gamma(t))_{t\in[0,hL_k]} \text{ starting in } I_{hL_k}(w) \cap \mathbb{L},\ D^h(\gamma) \leqslant \rho_k\}.$$

Since $\rho_k \geqslant 1/2$, it suffices to show the desired estimate for $s_{h,k} := \sup_{w\in\mathbb{R}^2} \mathbb{P}(S_{h,k}(w))$. To do this, we use induction on $k > k_2$.

- The case $k = k_2 + 1$ follows directly from definition 2.31 and lemma 2.30.
- Induction step: assume that the desired estimate has been shown for a fixed $k > k_2$. Recall the definition of $\mathcal{C}$ from (2.23).

  <u>Claim</u>: on the event $S_{h,k+1}(w) \cap \bigcap_{w'\in\mathcal{C}} F_{hL_k}(w')$, there exist $w_1', w_2' \in \mathcal{C}$ such that

$$\begin{cases} S_{h,k}(w_1') \text{ and } S_{h,k}(w_2') \text{ occur} \\ d(B_{hL_k}(w_1'), B_{hL_k}(w_2')) \geqslant hL_k. \end{cases}$$

Indeed, assume that there are *not* three points $w'_1$, $w'_2$, $w'_3 \in \mathcal{C}$, for which the $(\pi_2(B_{hL_k}(w'_i))_{i=1,2,3}$ are distinct and $S_{h,k}(w'_i)$ occurs for every $i \in \{1,2,3\}$. Then, if $\gamma = (\gamma(t))_{t\in[0,hL_{k+1}]}$ is an allowed path starting in $I_{hL_{k+1}}(w) \cap \mathbb{L}$, on the event $\bigcap_{w'\in\mathcal{C}} F_{hL_k}(w')$, $\gamma$ passes by intervals of the form $I_{hL_k}(w')$ with $w' \in \mathcal{C}$ for each time in $\pi_2(w) + [jhL_k, (j+1)hL_k]$. Since $S_{h,k}(w')$ occurs at most twice along the way, we have

$$\begin{aligned}
D^h(\gamma) &= \frac{L_{k_2+1}}{L_{k+1}} \# \left\{ 0 \leqslant j < \frac{L_{k+1}}{L_{k_2+1}},\ \lfloor \gamma(jhL_{k_2+1}) \rfloor_{hL_{k_2}} \text{ is } (hL_{k_2}, l_{k_2})\text{-threatened} \right\} \\
&\geqslant \frac{L_{k_2+1}}{L_{k+1}} (l_k - 2) \frac{L_k}{L_{k_2+1}} \rho_k = \frac{l_k - 2}{l_k} \rho_k > \rho_{k+1},
\end{aligned}$$

which contradicts the event $S_{h,k+1}(w)$. Therefore, since $h \geqslant 1$,

$$s_{h,k+1} \leqslant 25\, l_k^4\, (s_{h,k}^2 + c_1 L_k^{-\alpha}) + 5\, l_k^2\, c_5^{-1} e^{-c_5 L_k}.$$

Therefore

$$\begin{aligned}
\frac{s_{h,k+1}}{L_{k+1}^{-(\alpha-1)/5}} &\leqslant L_k^{(\alpha-1)/4} \left( 25\, l_k^4\, (s_{h,k}^2 + c_1 L_k^{-\alpha}) + 5\, l_k^2\, c_5^{-1} e^{-c_5 L_k} \right) \\
&\leqslant 25\, l_k^4\, L_k^{(\alpha-1)/4} \left( s_{h,k}^2 + c_1 L_k^{-\alpha} + c_5^{-1} e^{-c_5 L_k} \right) \\
&\leqslant 25\, L_k^{(\alpha+3)/4} \left( c_8^2 L_k^{-2(\alpha-1)/5} + c_1\, L_k^{-\alpha} + c_5^{-1} e^{-c_5 L_k} \right) \\
&\leqslant 25\, (c_8^2 + c_1) L_k^{-(3\alpha-23)/20} + 25\, c_5^{-1}\, L_k^{(\alpha+3)/4} e^{-c_5 L_k} \underset{(2.40)}{\leqslant} c_8.
\end{aligned}$$

□

### 2.5.4 Proof of $v_+ = v_-$

Proof of estimate (2.14). We now prove that $v_- = v_+$, which will yield estimate (2.14), from which we will show theorem 2.13. To do this, we argued by contradiction that $v_- < v_+$, therefore $\delta = \frac{v_+ - v_-}{4} > 0$. Let $\eta = \frac{\delta}{4 l_{k_2}}$ where $k_2$ is defined as in lemma 2.30. We are going to show that

$$p^+_{L_k^2} \left( v_+ - \frac{\eta}{3} \right) \xrightarrow[k\to\infty]{} 0, \tag{2.42}$$

which contradicts the definition of $v_+$. Therefore we are now working with $h = h_k = L_k$, which is why it was important for our previous estimates to hold for any $h \geqslant 1$. In order to prove (2.42), we let $k > k_2$ and consider the large box $B_{L_k^2}(w)$ and the small sub-boxes $B_{L_k L_{k_2}}(y)$ for $y \in \hat{\mathcal{C}} = \hat{\mathcal{C}}_{k,w}$, where $\hat{\mathcal{C}}$ is such that

$$\bigcup_{w'\in\hat{\mathcal{C}}} I_{L_k L_{k_2}}(w') = B_{L_k^2}(w) \cap (\mathbb{R} \times (\pi_2(w) + L_k L_{k_2} \mathbb{Z})).$$

We define three events that are very likely to happen:

$$\begin{aligned}
F(w) &= \left\{ \forall y \in I_{L_k^2}(w) \cap \mathbb{L},\ (Y_t^y)_{t\in[0,L_k^2]} \text{ stays in } B_{L_k^2}(w) \right\}; \\
G_1(w) &= \left\{ \forall w' \in \hat{\mathcal{C}},\ \forall y \in I_{L_k L_{k_2}}(w') \cap \mathbb{L},\ \mathrm{V}^y_{0,L_k L_{k_2}} \leqslant v_+ + \eta \right\}; \\
G_2(w) &= \left\{ \forall \gamma = (\gamma(t))_{t\in[0,L_k^2]} \text{ allowed path starting in } I_{L_k^2}(w) \cap \mathbb{L},\ D^{L_k}(\gamma) \geqslant 1/2 \right\}.
\end{aligned}$$

First, by (2.20), we have

$$\sup_{w\in\mathbb{R}^2} \mathbb{P}(F(w)^c) \leqslant c_5^{-1} e^{-c_5 L_k^2} \xrightarrow[k\to\infty]{} 0.$$

Furthermore, using proposition 2.21, we have

$$\sup_{w\in\mathbb{R}^2} \mathbb{P}(G_1(w)^c) \leqslant c\, c_4(\eta) \left(\frac{L_k}{L_{k_2}}\right)^2 (L_k L_{k_2})^{-\alpha/4} \xrightarrow[k\to\infty]{\alpha>8} 0.$$

Using lemma 2.32, we also have

$$\sup_{w\in\mathbb{R}^2} \mathbb{P}(G_2(w)^c) \leqslant c_8\, L_k^{-(\alpha-1)/5} \xrightarrow[k\to\infty]{\alpha>8} 0.$$

Now, given $w \in \mathbb{R}^2$ and $y \in I_{L_k^2}(w) \cap \mathbb{L}$, we denote by $J^y$ the set of indices $j \in \left[\!\left[0, \frac{L_k}{L_{k_2+1}} - 1\right]\!\right]$ such that the point $\lfloor Y^y_{jL_kL_{k_2+1}} \rfloor_{L_kL_{k_2}}$ is $(L_kL_{k_2}, l_{k_2})$-threatened. By definition 2.31, considering the allowed path $(Y^y_t)_{t\in[0,L_k^2]}$, we have the inclusion of events

$$G_2(w) \subseteq \left\{ |J^y| \geqslant \frac{L_k}{2L_{k_2+1}} \right\}. \tag{2.43}$$

Suppose now that we are on the event $F(w) \cap G_1(w) \cap G_2(w)$. Then, given $y \in I_{L_k^2}(w) \cap \mathbb{L}$, we have

$$\begin{aligned}
X^y_{L_k^2} - \pi_1(y) &= \sum_{j=0}^{L_k/L_{k_2+1}-1} \left( X^y_{(j+1)L_kL_{k_2+1}} - X^y_{jL_kL_{k_2+1}} \right) \\
&= \sum_{j\in J^y} \left( X^y_{(j+1)L_kL_{k_2+1}} - X^y_{jL_kL_{k_2+1}} \right) + \sum_{j\notin J^y} \left( X^y_{(j+1)L_kL_{k_2+1}} - X^y_{jL_kL_{k_2+1}} \right) \\
&\overset{(*)}{\leqslant} |J^y| \left(v_+ - \delta/2l_{k_2}\right) L_kL_{k_2+1} + \left( \frac{L_k}{L_{k_2+1}} - |J^y| \right) (v_+ + \eta)\, L_kL_{k_2+1} \\
&= v_+L_k^2 - |J^y| \left(\delta/2l_{k_2}\right) L_kL_{k_2+1} + \left( \frac{L_k}{L_{k_2+1}} - |J^y| \right) \eta L_kL_{k_2+1} \\
&\overset{(2.43)}{\leqslant} v_+L_k^2 - \left(\delta/4l_{k_2} - \eta/2\right) L_k^2 \\
&= \left(v_+ - \eta/2\right) L_k^2 \\
&< \left(v_+ - \eta/3\right) L_k^2.
\end{aligned} \tag{2.44}$$

In $(*)$, we used definition 2.31 with $H = L_kL_{k_2}$, $r = l_{k_2}$ and the fact that $\frac{\delta}{4l_{k_2}} < \frac{\delta}{2l_{k_2}}$. See figure 4 for an illustration of the above bounds.
In the end, $p^+_{L_k^2}\left(v_+ - \frac{\eta}{3}\right)$

$$\begin{aligned}
&= \sup_{w\in\mathbb{R}^2} \mathbb{P}\left( \exists\, y \in I_{L_k^2}(w) \cap \mathbb{L},\ V^y_{0,L_k^2} \geqslant v_+ - \frac{\eta}{3} \right) \\
&\leqslant \sup_{w\in\mathbb{R}^2} \underbrace{\mathbb{P}\left( \begin{array}{c} F(w) \cap G_1(w) \cap G_2(w) \cap \\ \left\{ \exists\, y \in I_{L_k^2}(w) \cap \mathbb{L},\ V^y_{0,L_k^2} \geqslant v_+ - \frac{\eta}{3} \right\} \end{array} \right)}_{=0 \text{ by } (*)} + \mathbb{P}(F(w)^c) + \mathbb{P}(G_1(w)^c) + \mathbb{P}(G_2(w)^c) \\
&\xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

which gives a contradiction. Therefore, $v_- = v_+$. Setting $v = v_- = v_+$, proposition 2.21 yields estimate 2.14. □

Proof of theorem 2.13. We define $o = (0, 0)$. Let $\varepsilon > 0$. We now know from the above that the

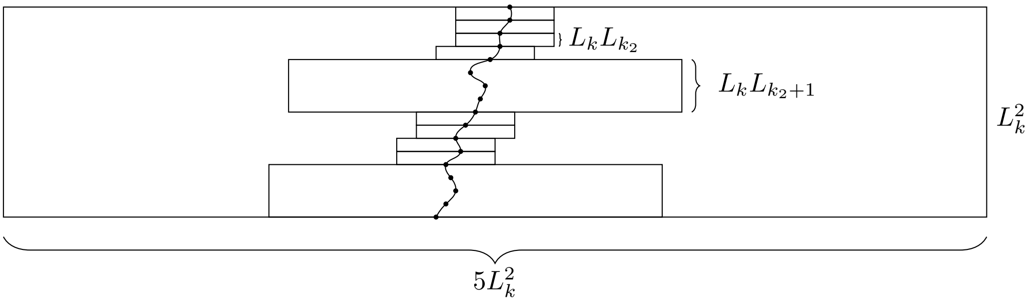


Figure 4: The final bound in the proof of (2.14). The large boxes correspond to the displacements for $j \in J^y$.

following estimate holds for every $t > 0$:

$$\mathbb{P}\left(\left|\frac{X_t^o}{t} - v\right| \geqslant \varepsilon\right) \leqslant 2\, c_4(\varepsilon)\, t^{-\alpha/4}.$$

Using this estimate along the subsequence $t = t_n = n \in \mathbb{N}^*$, we get that the series $\sum_{n\in\mathbb{N}} \mathbb{P}\left(\left|\frac{X_n^o}{n} - v\right| \geqslant \varepsilon\right)$ converges, since it is assumed that $\alpha > 8$. The Borel-Cantelli lemma therefore ensures that

$$\frac{X_n^o}{n} \xrightarrow[n\to\infty]{a.s.} v.$$

Now, if $t \geqslant 0$, let $n = \lfloor t \rfloor$ and write

$$\left|\frac{X_t^o}{t} - v\right| = \underbrace{|X_n^o|\left(\frac{1}{n} - \frac{1}{t}\right)}_{(1)} + \underbrace{\frac{1}{t}\,|X_t^o - X_n^o|}_{(2)} + \underbrace{\left|\frac{X_n^o}{n} - v\right|}_{\xrightarrow[t\to\infty]{a.s.} 0}.$$

Now, note that $\mathbb{P}\left(|X_n^o| > 2n\right) \leqslant \mathbb{P}(F_n(o)^c) \leqslant c_5^{-1}\, e^{-c_5 n}$ using (2.20). This bound is summable in $n$ so, using the Borel-Cantelli lemma again, almost surely for $n$ large enough, $|X_n^o| \leqslant 2n$. Therefore

$$(1) \leqslant 2 - \frac{2n}{t} \xrightarrow[t\to\infty]{} 0.$$

Finally, let $a \in (0,1)$ and remark that

$$\begin{aligned}
\mathbb{P}\left(\exists\, t \in [n, n+1),\, |X_t^o - X_n^o| > 2t^a\right) &\leqslant \mathbb{P}\left(\exists\, t \in [n, n+1),\, |X_{t-n}^o| > 2n^a\right) \quad \text{by translation invariance}\\
&\leqslant \mathbb{P}\left(\exists\, t \in [0, n^a],\, |X_t^o| > 2n^a\right)\\
&\leqslant \mathbb{P}(F_{n^a}(o)^c) \leqslant c_5^{-1}\, e^{-c_5 n^a},
\end{aligned}$$

which is again summable. Therefore, almost surely for $n$ large enough, $\forall t \in [n, n+1)$, $|X_t^o - X_n^o| \leqslant 2t^a$, and

$$(2) \leqslant \frac{2}{t^{1-a}} \xrightarrow[t\to\infty]{} 0.$$

□

# 3 Generalization to the finite-range case

We now keep the model presented in the previous section, except that we allow jumps to be in the set $[\![-R, R]\!]$, where $R \in \mathbb{N}_{\geqslant 2}$. In other words, our jumping function is now $g : S^{[\![-\ell,\ell]\!]} \times [0,1] \to [\![-R, R]\!]$. Our particles may therefore jump further than their nearest-neighbour sites. The key of the proof of the LLN in the nearest-neighbour setting was the monotonicity property (2.5), which does not hold anymore in the finite range framework. Therefore the idea is to control the probability of events where we do have monotonicity, by adapting the notions of traps and threats that were previously defined. Fundamentally, what we want is that when a particle meets a trapped point, its trajectory is delayed to the left with positive probability.

**N.B.** In this section, some constants will bear the same name as constants from section 2, when they replace them and play the exact same role in the proof.

## 3.1 New assumptions

Because we can now jump to range $R$, it would seem natural to choose $R$ instead of 1 as a limiting speed (in the sense of (2.2) and (2.3)). In fact, in order to have an exponential decay as in (2.4), it seems reasonable to choose $R^2$ as a limiting speed (see remark 3.6). Similarly, it would be natural to replace the term $2T$ in (2.4) by something like $2R^2T$. In order not to make all these changes, we choose to keep 1 as a limiting speed, and instead ask our jumping clocks to compensate by ringing on average after time $R^2$.

The assumptions we are going to work with in this new framework are the following:

Assumptions 3.1.

- *Jumping times*: the $(T_i^x)_{i\in\mathbb{N}^*,x\in\mathbb{Z}}$ are given by independent Poisson processes $(N_t^x)_{t\geqslant 0,x\in\mathbb{Z}}$ of parameter $1/R^2$, independent of the environment $\eta$.
- *Space-time translation invariance* : for every $(z,s)\in\mathbb{L}$, $(\eta_t(x))_{(x,t)\in\mathbb{L}}$ and $(\eta_{s+t}(z+x))_{(x,t)\in\mathbb{L}}$ have same law.
- *Markovian property*: the process $\eta$ is a Markov process.
- *Uniform ellipticity*: there exists $\gamma>0$ such that for any uniform random variable $U$ in $[0,1]$,

$$\inf_{x\in[\![-R,R]\!]}\ \inf_{\sigma_{-\ell},\dots,\sigma_\ell\in S}\mathbb{P}(g(\sigma_{-\ell},\dots,\sigma_\ell,U)=x)\geqslant\gamma. \tag{3.1}$$

- *Strong decoupling property*: there exists $c_1$, $\kappa$, $\beta>0$ such that, for every $H\geqslant 1$ and $r\in\mathbb{N}^*$, for every box $B_1,\dots,B_r$ having both side lengths at most $6H$ and 2-by-2 separated in time by at least $H$, for every event $A_1,\dots,A_r$ respectively measurable with respect to $\eta$ inside $B_1,\dots,B_r$,

$$\left|\mathbb{P}\left(\bigcap_{i=1}^r A_i\right)-\prod_{i=1}^r\mathbb{P}(A_i)\right|\leqslant c_1\,r\,e^{-\kappa\,H^\beta}. \tag{3.2}$$

Remark 3.2. The Markovian property and the assumptions made on the jumping times imply that actually the whole process $(\eta_t,(N_t^x)_{x\in\mathbb{Z}},(X_t^y)_{y=(x,0),x\in\mathbb{Z}})_{t\geqslant 0}$ is a Markov process. Also, space-time invariance and the strong decoupling property are actually true for the whole process $(\eta,(T_i^x),(U_i^x))$ (because of remark 2.11 and example 2.12).

Remark 3.3. Note that the new decoupling property is stronger than property (2.12) in several ways:

- It requires to control the correlation between $r$ events and not only two.
- It requires a control in absolute value while property (2.12) only required an upper bound.
- It requires a sub-exponential decay instead of a polynomial one.
- We ask for the maximum size of the boxes considered to be $6H$ instead of $5H$. This is because in the proof of propositions 3.11 and 3.20, we need to consider trajectories starting from intervals greater than $I_H(w)$ and therefore boxes larger than $B_H(w)$. See remark 3.7 for more details.

Remark 3.4.

Note that, for every $\alpha, \beta, \kappa > 0$, there exists $c = c(\alpha, \beta, \kappa) > 0$ such that for every $H \geqslant 1$, we have $e^{-\kappa H^\beta} \leqslant cH^{-\alpha}$. Now, in the following, the exponential decay is only used at one specific point of our proof (namely, in (3.13)), and we believe we may get around using it. Therefore we have chosen to write everything else (including assumptions on $\alpha$) as if we only had a polynomial decay.

Examples 3.5. Let us take a second look at examples given in section 2.2, especially decoupling inequalities (2.16) and (2.18). They both satisfy a decoupling assumption that is in absolute value and that is at worst sub-exponential with parameter $\beta > 1/9$, which is required in lemma 3.24. The only thing left to check is that this decoupling can control the correlation between $r$ events.

- For Markov processes with positive spectral gaps, note that in (2.16), no assumption is made on the size of the boxes. So a simple induction on $r$ will yield (3.2) with $c_1 = \beta = 1$ and $\kappa = \lambda$. It suffices to group $r-1$ boxes into a single box and write

$$\begin{aligned} &\left|\mathbb{P}_\nu\left(\bigcap_{i=1}^{r} A_i\right) - \prod_{i=1}^{r} \mathbb{P}_\nu(A_i)\right| \\ &\leqslant \left|\mathbb{P}_\nu\left(\bigcap_{i=1}^{r-1} A_i \cap A_r\right) - \mathbb{P}_\nu\left(\bigcap_{i=1}^{r-1} A_i\right)\mathbb{P}_\nu(A_r)\right| + \mathbb{P}_\nu(A_r)\left|\mathbb{P}_\nu\left(\bigcap_{i=1}^{r-1} A_i\right) - \prod_{i=1}^{r-1} \mathbb{P}_\nu(A_i)\right| \\ &\leqslant e^{-\lambda H} + (r-1)\, e^{-\lambda H} \\ &= re^{-\lambda H}. \end{aligned} \tag{3.3}$$

- For independent renewal chains, the same line of reasoning applies, using (2.18). Mind that in (2.17), the size of box $B_2$ does appear, contrary to the previous example, but this does not impede the result, for it suffices that there be no restriction of size for box $B_1$ only (in (3.3)). We obtain (3.2) with $c_1 = c_3^{-1}$, $\kappa = c_3$ and $\beta = 1/8 > 1/9$.

As for invariance and the Markovian property, they are clearly satisfied for both these environments.

## 3.2 New notions of traps or threats

Recall notations 2.17 and definition 2.19 for $v_-$ and $v_+$. Also, recall the scales from definition 2.23.

Remark 3.6. We first note that the following properties we used in section 2 remain:

- Limiting speed properties (2.2) and (2.3) still hold, with an argument similar to that used in example 2.8. Let us fix $y \in [0, T) \times \{0\} \cap \mathbb{L}$. We define a sequence of clocks $(\tau_i)_{i\in\mathbb{N}}$ and a path $\hat\gamma$ by the following. We set for $j \geqslant 0$,

$$\begin{cases} \tau_0 = 0 \\ \hat\gamma(0) = y \\ \tau_{j+1} = \min\left\{T^x_{N^x_{\tau_j}+1},\ x \in \hat\gamma_1(\tau_j^-) + [\![-R+1, 0]\!]\right\} \\ \hat\gamma_1(t) = \hat\gamma_1(\tau_j^-) + R \quad \forall t \in [\tau_j, \tau_{j+1}). \end{cases}$$

Now, we claim that on the event

$$\left\{ \begin{array}{c} \exists\, \gamma \text{ allowed path on } [0,T], \text{ starting in } y \\ \text{and such that } \gamma_1(T) - \pi_1(y) \geqslant vT \end{array} \right\},$$

the path $\hat\gamma$ has to satisfy

$$\hat\gamma_1(T) - \pi_1(y) \geqslant vT.$$

Indeed, let $\bar\gamma$ be an allowed path on $[0, T]$ starting at $y$ such that $\bar\gamma_1(T) - \pi_1(y) \geqslant vT$, and suppose that $\hat\gamma_1(T) - \pi_1(y) < vT$. In particular, $\hat\gamma_1(T) < \bar\gamma_1(T)$. Let $t_0 = \inf\{t \in [0, T],\ \hat\gamma_1(t) < \bar\gamma_1(t)\}$. By construction, $t_0$ is a jumping time for $\bar{\gamma}$, but not for $\hat{\gamma}$. Because $\bar{\gamma}$ can only jump to range $R$, we must have $\bar{\gamma}(t_0^-) \in \hat{\gamma}(t_0^-) + [\![-R+1, 0]\!]$. Therefore, by definition of the $\tau_i$'s, $t_0$ must be a jumping time for $\hat{\gamma}$, which is a contradiction.

Now, $N_T$ is a Poisson variable with parameter $T/R$ (recall that the minimum of two independent exponential variables is an exponential variable whose parameter is the sum of the two parameters). Moreover, by definition, we have $\hat{\gamma}_1(T) - \pi_1(y) = R\,N_T$. Thus, using a union bound on the possible starting points,

$$\mathbb{P}\left( \begin{array}{c} \exists\,\gamma \text{ allowed path on } [0,T], \text{ starting in } [0,T) \times \{0\} \\ \text{and such that } \gamma_1(T) - \gamma_1(0) \geqslant vT \end{array} \right) \leqslant T\,\mathbb{P}(R\,N_T \geqslant vT) \leqslant T e^{-\lambda(v)T/R} \xrightarrow[T\to\infty]{} 0.$$

- Same holds for the large deviation bound (2.4): the proof is similar to that of example 2.8 with the definition of $\hat{\gamma}$ introduced in the previous point.
- The fact that $v_+ \leqslant 1$ and $v_- \geqslant -1$ is a direct consequence of the limiting speed property.
- Proposition 2.21 also holds. Indeed, we never used the nearest-neighbour assumption in the proof. Consequently, the rest of this section will be devoted to showing that $v_- = v_+$.

Remark 3.7. We will need a slightly different version of remark 2.18. Let $w \in \mathbb{R}^2$ and $H \geqslant 1$. We define

$$\tilde{B}_H(w) = w + [-3H, 3H) \times [0, H) \quad \text{and} \quad \tilde{I}_H(w) = w + (-H, H) \times \{0\}.$$

Consider the event

$$\tilde{F}_H(w) = \left\{ \begin{array}{c} \forall \gamma = (\gamma(t))_{t\in[0,H]} \text{ allowed path starting at } \tilde{I}_H(w) \cap \mathbb{L}, \\ \{\gamma(t),\, t \in [0,H]\} + [-\ell, \ell] \times \{0\} \subseteq \tilde{B}_H(w) \end{array} \right\}.$$

Thanks to (2.4), we have

$$\begin{aligned} \mathbb{P}\left(\tilde{F}_H(w)^c\right) &\leqslant 2H\,c_0^{-1} e^{-c_0 H} \\ &\leqslant \tilde{c}_5^{-1} e^{-\tilde{c}_5 H}, \end{aligned} \tag{3.4}$$

for $\tilde{c}_5 > 0$ a well chosen constant.

We present two methods that we have tried to use in order to account for the lack of the nearest-neighbour assumption. Both work quite well but have not yet yielded a complete proof.

- In the first attempt, the idea is to say that when a particle is trapped, it will stay on the left of the particle that "blocks" it with positive probability. To recover a proof, it thus suffices to strengthen the notion of threatened points by demanding that there be lots of traps along the way. The problem is a lack of independence between the event {a certain point is trapped} and the uniform variables that dictate the jumps of our particles, in conjecture 3.9.
- In the second attempt, we try to get around the latter issue by strengthening the definition of traps itself, in order to have a lot of particles to "block" our random walk. But another issue arises: we cannot seem to show that with this new definition, a point is trapped with a uniformly (in $H$ large enough) positive probability (which was crucial in our proof, see (2.35)). See conjecture 3.16 for more details.

### 3.2.1 First attempt

The idea is to keep the notion of trapped points unchanged but to strengthen that of threatened points, by demanding that there be a certain number of trapped points - instead of only one - along a line segment of slope $v_+$. Also, in order to have nicely decorrelated events, we refine the notion of threats by looking only at the $w_j$'s with even indices $j$.

Definition 3.8. Let $H \geqslant 1$ and $r \in \mathbb{N}^*$. Let $q = q(r) \in [\![0, r]\!]$. Let $I_r = \{j \text{ even}, j \in [\![0, r-1]\!]\}$. $w$ is said to be $(H, I_r, q)$-threatened if

$$\#\{j \in I_r,\ w_j \text{ is } H\text{-trapped}\} \geqslant q.$$

If we take a close look at the proof of proposition 2.28, what was crucial in (2.33) was the use of remark 2.26. That remark was a deterministic fact, which will now be true with high probability.

Conjecture 3.9.

*There exists $\gamma_0 > 0$ such that*

$$\inf_{H \geqslant 1} \inf_{w \in \mathbb{R}^2} \inf_{w' \in (w + (-\infty, \delta H) \times \{0\}) \cap \mathbb{L}} \mathbb{P}\left( X_H^{w'} \leqslant \pi_1(w) + (v_+ - \delta) H \,\middle|\, w\ H\text{-trapped} \right) \geqslant \gamma_0 > 0. \tag{3.5}$$

Ideas and problems. We delay the idea of the proof to the proof of proposition 3.19, which is very similar. The problem here is that the event $\{w \text{ is } H\text{-trapped}\}$ does not identify a trajectory that has speed less than $v_- + \delta$. Therefore, we cannot identify a uniform variable $U_i^x$ which is independent of the event $\{w \text{ is } H\text{-trapped}\}$.

Of course, proposition 2.28 does not hold anymore: a particle starting near a threatened point could indeed meet trapped points along its trajectory but overstep them all. However, when $q$ is large enough, the probability of this bad event will be small, which is the idea of the next proposition.

Notation 3.10. In the rest of the report, we choose

$$q(r) = \lfloor \sqrt{r} \rfloor.$$

Proposition 3.11.

*There exists $c_9 > 0$ such that for every $H \geqslant 1$, $r \in \mathbb{N}^*$ and $y \in \mathbb{L}$,*

$$\mathbb{P}\begin{pmatrix} \mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r}; \\ \lfloor y \rfloor_H \text{ is } (H, I_r, q)\text{-threatened}; \\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r} \end{pmatrix} \leqslant (1-\gamma_0)^q + c_9\, r\, 4^r e^{-\kappa H^\beta}$$

Proof. Let us write $w = \lfloor y \rfloor_H$ and $w_j = w + jH(v_+, 1)$ for every $j \in [\![0, r-1]\!]$. Let $H \geqslant 1$. Note that, after partitioning on the set of points which are trapped, the same arguments as in the proof of proposition 2.28 imply that

$$\begin{aligned} &\mathbb{P}\left( \mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r};\ \lfloor y \rfloor_H \text{ is } (H, I_r, q)\text{-threatened};\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r} \right) \\ &\leqslant \sum_{\substack{J \subseteq I_r \\ |J| \geqslant q}} \mathbb{P}\begin{pmatrix} \{j \in I_r,\ w_j \text{ is } H\text{-trapped}\} = J \\ \forall j \in J,\ X_H^{Y^y_{jH}} > \pi_1(w_j) + (v_+ - \delta) H \\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r} \end{pmatrix} \end{aligned}$$

Now, using (2.32), we know that on the event

$$\forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r}, \tag{3.6}$$

the position of $Y^y$ at time $jH$ satisfies $X^y_{jH} < \pi_1(w_j) + \delta H$. Moreover, on the event (3.6) and the event

$$\forall j \in J,\ X_H^{Y^y_{jH}} > \pi_1(w_j) + (v_+ - \delta) H,$$

the position of $Y^y$ at time $jH$ also satisfies

$$\begin{aligned} X^y_{jH} &= X^y_{(j+1)H} - V^y_{jH,(j+1)H} H \\ &> \pi_1(w_j) + (v_+ - \delta)H - \left(v_+ + \frac{\delta}{2r}\right) H \\ &= \pi_1(w_j) - \delta\left(1 + \frac{1}{2r}\right) H \\ &> \pi_1(w_j) - H. \end{aligned}$$

In particular, $Y^y_{jH} \in \tilde{I}_H(w_j)$. It is here that having replaced the maximum size $5H$ to $6H$ in assumption (3.2) is crucial. We can now write

$$\begin{aligned} &\mathbb{P}\left(\mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r};\ \lfloor y \rfloor_H \text{ is } (H, I_r, q)\text{-threatened};\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r}\right) \\ &\leqslant \sum_{\substack{J \subseteq I_r \\ |J| \geqslant q}} \sup_{\substack{x_0,\dots,x_{r-1} \in \mathbb{Z} \\ \pi_1(w_j) - H < x_j < \pi_1(w_j) + \delta H}} \underbrace{\mathbb{P}\begin{pmatrix} \forall j \in J,\ w_j \text{ is } H\text{-trapped and } X_H^{(x_j, \pi_2(w_j))} > \pi_1(w_j) + (v_+ - \delta)H \\ \forall j \in I_r \setminus J,\ w_j \text{ is not } H\text{-trapped} \end{pmatrix}}_{=:A = A_{J, x_0, \dots, x_r}}. \end{aligned}$$

Now, using remark 3.7, we get

$$\begin{aligned} \mathbb{P}(A) &\leqslant \mathbb{P}\left(A \cap \bigcap_{j \in [\![0, r-1]\!]} \tilde{F}_H(w_j)\right) + \mathbb{P}\left(\bigcup_{j \in [\![0, r-1]\!]} \tilde{F}_H(w_j)^c\right) \\ &\overset{(3.2),(3.4)}{\leqslant} \begin{array}{c} \prod_{j \in J} \mathbb{P}\begin{pmatrix} w_j \text{ is } H\text{-trapped and} \\ X_H^{(x_j, \pi_2(w_j))} > \pi_1(w_j) + (v_+ - \delta)H \end{pmatrix} \\ \cdot \prod_{j \in I_r \setminus J} \mathbb{P}(w_j \text{ is not } H\text{-trapped}) \end{array} + c_1 r e^{-\kappa H^\beta} + \tilde{c}_5^{-1}\, r\, e^{-\tilde{c}_5 H} \\ &\overset{(3.5)}{\leqslant} (1 - \gamma_0)^{|J|} \prod_{j \in J} \mathbb{P}(w_j \text{ is } H\text{-trapped}) \prod_{j \in I_r \setminus J} \mathbb{P}(w_j \text{ is not } H\text{-trapped}) + c\, r e^{-\kappa H^\beta}. \end{aligned}$$

Now, we want to use decoupling again to regroup the factors inside a unique event:

$$\begin{aligned} &\prod_{j \in J} \mathbb{P}(w_j \text{ is } H\text{-trapped}) \prod_{j \in I_r \setminus J} \mathbb{P}(w_j \text{ is not } H\text{-trapped}) \\ &\leqslant \prod_{j \in J} [\mathbb{P}(\{w_j \text{ trapped}\} \cap F_H(w_j)) + \mathbb{P}(F_H(w_j)^c)] \cdot \prod_{j \in I_r \setminus J} [\mathbb{P}(\{w_j \text{ not trapped}\} \cap F_H(w_j)) + \mathbb{P}(F_H(w_j)^c)] \\ &\overset{(2.20)}{\leqslant} \prod_{j \in J} (\mathbb{P}(\{w_j \text{ trapped}\} \cap F_H(w_j)) \cdot \prod_{j \in I_r \setminus J} \mathbb{P}(\{w_j \text{ not trapped}\} \cap F_H(w_j)) + 2^r\, c_5^{-1}\, e^{-c_5 H} \\ &\overset{(3.2)}{\leqslant} \mathbb{P}(\{j \in I_r,\ w_j \text{ is } H\text{-trapped}\} = J) + c_1\, r\, e^{-\kappa H^\beta} + 2^r\, c_5^{-1}\, e^{-c_5 H}. \end{aligned}$$

In the end,

$$\begin{aligned} &\mathbb{P}\left(\mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r};\ \lfloor y \rfloor_H \text{ is } (H, I_r, q)\text{-threatened};\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r}\right) \\ &\leqslant \sum_{\substack{J \subseteq I_r \\ |J| \geqslant q}} \left[(1 - \gamma_0)^q\, \mathbb{P}(\{j \in I_r,\ w_j \text{ is } H\text{-trapped}\} = J) + c\, r\, 2^r\, e^{-\kappa H^\beta}\right] \\ &\leqslant (1 - \gamma_0)^q + c\, r\, 4^r\, e^{-\kappa H^\beta} \end{aligned}$$

$\square$

### 3.2.2 Second attempt

We start by defining new velocities $\tilde{v}_-$ and $\tilde{v}_+$, in order to have a uniform lower bound on the probability of being trapped, *i.e.* something similar to (2.35).

Definition 3.12. We define

$$\tilde{v}_+ = \inf\left\{v \in \mathbb{R},\ \limsup_{H\to\infty} \inf_{w\in\mathbb{R}^2} \mathbb{P}\left(\#\left\{y \in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H} \geqslant v\right\} \geqslant \frac{H}{2}\right) = 0\right\}.$$

$$\tilde{v}_- = \sup\left\{v \in \mathbb{R},\ \limsup_{H\to\infty} \inf_{w\in\mathbb{R}^2} \mathbb{P}\left(\#\left\{y \in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H} \leqslant v\right\} \geqslant \frac{H}{2}\right) = 0\right\};$$

Remark 3.13. Note that, by arguments similar to the second point of remark 2.20, we have, for every $v < \tilde{v}_-$,

$$\limsup_{H\to\infty} \inf_{w\in\mathbb{R}^2} \mathbb{P}\left(\#\left\{y \in I_H(w)\cap\mathbb{L},\ \mathrm{V}^y_{0,H} \leqslant v\right\} \geqslant \frac{H}{2}\right) = 0.$$

The same holds for $v > \tilde{v}_+$.

Proposition 3.14. *We have*

$$v_- \leqslant \tilde{v}_- \leqslant \tilde{v}_+ \leqslant v_+.$$

Proof. To show that $\tilde{v}_- \leqslant \tilde{v}_+$, argue by contradiction and use the definitions of $\tilde{v}_-$ and $\tilde{v}_+$ (note that here, the $\limsup$ in the definitions of $\tilde{v}_-$ and $\tilde{v}_+$ is crucial). To show that $v_- \leqslant \tilde{v}_-$, argue by contraction and use both the definition of $\tilde{v}_-$ and the deviation estimate (2.21) for $v_-$. A similar argument leads to $\tilde{v}_+ \leqslant v_+$. □

A direct consequence of this property is that assuming $v_- < v_+$, then either $v_- < \tilde{v}_+$ or $\tilde{v}_- < v_+$. In the rest of this section, we will assume that we are in the second case. If we are not, it suffices to adapt the arguments presented by swapping the roles of $v_-$ and $v_+$, and delaying particles to the right instead of the left. Note that setting $v_0 = \frac{1}{2}(\tilde{v}_- + v_+)$, we have

$$\limsup_{H\to\infty} \inf_{w\in\mathbb{R}^2} \mathbb{P}\left(\#\left\{y \in I_H(w)\cap\mathbb{L},\ V^y_{0,H} \leqslant v_0\right\} \geqslant \frac{H}{2}\right) > 0.$$

The idea is now to adapt the initial proof of the LLN from section 2 replacing $v_- + \delta$ by $v_0$. We define

$$\delta = \frac{v_+ - v_0}{4} \in (0, 1/2].$$

Definition 3.15. Let $H \geqslant 1$. $w$ is said to be $H$-trapped if

$$\#\left\{y \in (w + [\delta H, 2\delta H)\times\{0\})\cap\mathbb{L},\ \mathrm{V}^y_{0,H} \leqslant v_0\right\} \geqslant \frac{\delta H}{2}.$$

We define the notion of threats in the same way as in definition 3.8.

With those new definitions, we now have three things to focus on:

- Showing that we have a uniform lower bound on the probability of being trapped, as in (2.35), in order to show lemma 3.21 later. This we are not sure of, see conjecture 3.16.
- Showing something similar to conjecture 3.9, see proposition 3.19. Actually our new notion of traps is constructed specifically for this to work, contrary to our first attempt.
- Showing something similar to proposition 3.11 with our new notion of traps, see proposition 3.20.

Conjecture 3.16. *There exists $H_0 \geqslant 1$ such that*

$$\inf_{H \geqslant H_0} \inf_{w \in \mathbb{R}^2} \mathbb{P}(w \text{ is } H\text{-trapped}) > 0.$$

Ideas and problems. Note that by definition of trapped points and translation invariance, we have

$$\limsup_{H \to \infty} \inf_{w \in \mathbb{R}^2} \mathbb{P}(w \text{ is } H\text{-trapped}) > 0.$$

As a result, there exists a subsequence $(H_k)_{k \in \mathbb{N}^*}$ such that

$$\inf_{k \in \mathbb{N}^*} \inf_{w \in \mathbb{R}^2} \mathbb{P}(w \text{ is } H_k\text{-trapped}) > 0.$$

A natural idea would be an interpolation argument similar to step 3 in the proof of proposition 2.21, in order for this to hold for any $H$ large enough. But here interpolation does not work quite as well, because we demand that there be lots of trajectories satisfying a certain condition, instead of only one.

Definition 3.17. Let $(Z_{w,H})_{w \in \mathbb{R}^2,\, H \geqslant 1}$ be independent random variables such that

$$Z_{w,H} \sim \mathcal{U}\big((w + [\delta H, 2\delta H) \times \{0\}) \cap \mathbb{L}\big).$$

There exists a probability space supporting all random variables introduced until now ($(Z_{w,H})$, $\eta$, $(T_i^x)$ and $(U_i^x)$), equipped with a probability measure $\tilde{\mathbb{P}}$ such that under $\tilde{\mathbb{P}}$, the variables $(Z_{w,H})$ are independent of all the rest. For the sake of simplicity, in the rest of this section, we keep notation $\mathbb{P}$ instead of $\tilde{\mathbb{P}}$.

Remark 3.18. Set $\mathbb{P}_{cond} = \mathbb{P}\left(\cdot \mid (\eta_t)_{t \geqslant 0},\, (T_i^x)_{i \in \mathbb{N}^*, x \in \mathbb{Z}},\, (U_i^x)_{i \in \mathbb{N}^*, x \in \mathbb{Z}}\right)$. We actually have the equality of events

$$\{w \text{ is } H\text{-trapped}\} = \left\{\mathbb{P}_{cond}\left(\mathrm{V}_{0,H}^{Z_{w,H}} \leqslant v_0\right) \geqslant 1/2\right\}. \tag{3.7}$$

**Proposition 3.19.**

*There exists $\gamma_0 > 0$ such that*

$$\inf_{H \geqslant 1} \inf_{w \in \mathbb{R}^2} \inf_{w' \in (w + (-\infty, \delta H) \times \{0\}) \cap \mathbb{L}} \mathbb{P}\left(X_H^{w'} \leqslant \pi_1(w) + (v_+ - \delta)H \mid \mathrm{V}_{0,H}^{Z_{w,H}} \leqslant v_0\right) \geqslant \gamma_0.$$

Proof. Let $H \geqslant 1$ and $w \in \mathbb{R}^2$; we assume that $\pi_2(w) = 0$, which does not change anything and simplifies notations. Let $w' \in ((-\infty, \delta H) \times \{0\}) \cap \mathbb{L}$. We also set $Z = Z_{w,H}$. Let $T$ be the first time when $Y^{w'}$ is at distance less than $R$ of $Y^Z$. It is clear that if $T > H$, then $Y^{w'}$ stays at the left of $Y^Z$ up to time $H$. In this case, on the event that $\mathrm{V}_{0,H}^Z \leqslant v_0$, we have

$$\begin{aligned} X_H^{w'} \leqslant X_H^Z &\leqslant \pi_1(Z) + v_0 H \\ &\leqslant \pi_1(w) + 2\delta H + v_0 H \\ &\leqslant \pi_1(w) + (v_+ - \delta)H, \end{aligned}$$

by definition of $\delta$. In the case where $T \leqslant H$, the same still holds if $Y^{w'}$ coalesces with $Y^Z$ before time $H$. Let

$$\tau_1 = \inf\{t \geqslant T,\, Y^{w'} \text{ jumps}\} \quad \text{and} \quad \tau_2 = \inf\{t \geqslant T,\, Y^Z \text{ jumps}\}.$$

The above considerations yield

$$\mathbb{P}\left(X_H^{w'} \leqslant \pi_1(w) + (v_+ - \delta)H,\ \mathrm{V}_{0,H}^Z \leqslant v_0\right)$$

$$\geqslant \mathbb{P}\left(T > H,\, V^Z_{0,H} \leqslant v_0\right) + \mathbb{P}\left(T \leqslant H,\, \tau_1 < \tau_2,\, Y^{w'} \text{ jumps to } Y^Z_{\tau_1} \text{ at time } \tau_1,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right).$$

Consider the uniform, denoted by $U$, that dictates the jump of $Y^{w'}$ at time $\tau_1$. Note that, by construction, $U$ is independent of $\left\{T \leqslant H,\, \tau_1 < \tau_2,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right\}$. This independence is crucial and is what did not work in our first attempt, in conjecture 3.9 - here it works because we have *fixed* a trajectory, namely $Y^Z$. Let us denote by $\Lambda$ the process given by $\eta$, the $(T^x_i)_{i,x}$ and all the uniform variables $(U^x_i)$ except $U$. We can write, with $f_1$ and $f_2$ two non-negative measurable functions,

$$\begin{aligned}
&\mathbb{P}\left(T \leqslant H,\, \tau_1 < \tau_2,\, Y^{w'} \text{ jumps to } Y^Z_{\tau_1} \text{ at time } \tau_1,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right) \\
&= \mathbb{E}\left[f_1(\Lambda)\, f_2(\Lambda, U)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f_1(\Lambda)\, f_2(\Lambda, U)\, |\Lambda\right]\right] \\
&= \mathbb{E}\int_\lambda f_1(\lambda)\, f_2(\lambda, U)\, \mathrm{d}\mathbb{P}_\Lambda(\lambda) \\
&= \int_\lambda f_1(\lambda)\, \mathbb{E}[f_2(\lambda, U)]\, \mathrm{d}\mathbb{P}_\Lambda(\lambda) \\
&\geqslant \inf_\lambda \mathbb{E}[f_2(\lambda, U)]\, \mathbb{E}[f_1(\Lambda)] \\
&\geqslant \inf_{\sigma_{-\ell},\dots,\sigma_\ell} \inf_{x \in [\![1,R]\!]} \mathbb{P}\left(g(\sigma_{-\ell},\dots,\sigma_\ell, U) = x\right) \mathbb{P}\left(T \leqslant H,\, \tau_1 < \tau_2,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right) \\
&\geqslant \gamma\, \mathbb{P}\left(T \leqslant H,\, \tau_1 < \tau_2,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right).
\end{aligned}$$

Now, it would be natural to say that this last term is equal to $\frac{\gamma}{2}\ \mathbb{P}\left(T \leqslant H,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right)$, because $\mathbb{P}(\tau_1 < \tau_2) = 1/2$. However, the event $\{\tau_1 < \tau_2\}$ is not independent from $\{\mathrm{V}^Z_{0,H} \leqslant v_0\}$, so we cannot conclude straight away. To get around this issue, we apply the strong Markov property.

$$\begin{aligned}
&\mathbb{P}\left(T \leqslant H,\, \tau_1 < \tau_2,\, \mathrm{V}^Z_{0,H} \leqslant v_0\right) \\
&= \sum_{x_1, x_2 \in \mathbb{Z}} \mathbb{E}\left[\mathbf{1}_{T \leqslant H}\, \mathbf{1}_{X^{w'}_T = x_1}\, \mathbf{1}_{X^Z_T = x_2}\, \mathbb{P}_{\eta_T}\left(T^{x_1}_1 < T^{x_2}_1,\ X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right)\right] \\
&= \sum_{x_1, x_2 \in \mathbb{Z}} \mathbb{E}\left[\mathbf{1}_{T \leqslant H}\, \mathbf{1}_{X^{w'}_T = x_1}\, \mathbf{1}_{X^Z_T = x_2}\, \mathbb{E}_{\eta_T}\left[\mathbf{1}_{T^{x_1}_1 < T^{x_2}_1}\, \mathbb{P}_{\eta_{T^{x_1}_1}}\left(X^{(x_2,0)}_{H-T-T^{x_1}_1} - \pi_1(Z) \leqslant v_0 H\right)\right]\right]
\end{aligned}$$

Now, note that the three random variables $(T^{x_1}_1,\, T^{x_2}_1,\, \eta_{T^{x_1}_1})$ are independent. In order to justify that $T^{x_1}_1 = T$ is independent of $\eta_T$, we take $f_1,\, f_2$ two non-negative measurable functions and write

$$\begin{aligned}
\mathbb{E}\left[f_1(T)\, f_2(\eta_T)\right] &= \int_{(t,\sigma)} f_1(t)\, f_2(\sigma_t)\, \mathrm{d}\mathbb{P}_{(T,\eta)}(t,\sigma) \\
&= \int f_1(t)\, f_2(\sigma_t)\, \mathrm{d}\mathbb{P}_T(t)\, \mathrm{d}\mathbb{P}_\eta(\sigma) && \text{by independence} \\
&= \int f_1(t)\, \mathbb{E}[f_2(\eta_t)]\, \mathrm{d}\mathbb{P}_T(t) \\
&= \mathbb{E}[f_2(\eta_0)]\, \mathbb{E}[f_1(T)] && \text{by invariance} \\
&= \mathbb{E}[f_2(\eta_T)]\, \mathbb{E}[f_1(T)] && \text{(just take } f_1 = 1 \text{ to check this)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_{\eta_T}\left[\mathbf{1}_{T^{x_1}_1 < T^{x_2}_1}\, \mathbb{P}_{\eta_{T^{x_1}_1}}\left(X^{(x_2,0)}_{H-T-T^{x_1}_1} - \pi_1(Z) \leqslant v_0 H\right)\right] \\
&= \int_{t_1 < t_2,\, \sigma \in S^{\mathbb{Z}}} \mathbb{P}_\sigma\left(X^{(x_2,0)}_{H-T-t_1} - \pi_1(Z) \leqslant v_0 H\right) \mathbb{P}_{\eta_T}\left(T^{x_1}_1 = \mathrm{d}t_1,\, T^{x_2}_1 = \mathrm{d}t_2,\, \eta_{t_1} = \mathrm{d}\sigma\right) \\
&= \int_{t_1 < t_2,\, \sigma \in S^{\mathbb{Z}}} \mathbb{P}_\sigma\left(X^{(x_2,0)}_{H-T-t_1} - \pi_1(Z) \leqslant v_0 H\right) \mathbb{P}_{\eta_T}(T^{x_1}_1 = \mathrm{d}t_1)\, \mathbb{P}_{\eta_T}(T^{x_2}_1 = \mathrm{d}t_2)\, \mathbb{P}_{\eta_T}(\eta_{t_1} = \mathrm{d}\sigma)
\end{aligned}$$

$$= \int_{t\geqslant 0} \frac{e^{-2t/R}}{R}\, \mathbb{E}_{\eta_T}\left[\mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right)\right] \mathrm{d}t$$
$$= \mathbb{E}_{\eta_T} \int_{t\geqslant 0} \frac{e^{-2t/R}}{R}\, \mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right) \mathrm{d}t.$$

Let $\varepsilon > 0$. We have

$$\begin{aligned} &\int_{t\geqslant 0} \frac{e^{-2t/R}}{R}\, \mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right) \mathrm{d}t \\ &\geqslant \frac{\varepsilon\, e^{-2\varepsilon/R}}{R}\, \underbrace{\frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right) \mathrm{d}t}_{\xrightarrow[\epsilon\to 0]{} \mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z)\leqslant v_0 H\right),\ \text{uniformly in } H} \end{aligned} \tag{3.8}$$

The convergence above comes from the fact that $t \geqslant 0 \mapsto \mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right)$ is continuous at 0, uniformly in $H$. Indeed, if we set $A_t = \left\{Y^{(x_2,0)} \text{ ne saute pas entre 0 et } t\right\}$, the Markov property gives

$$\mathbb{P}_{\eta_t}\left(X^{(x_2,0)}_{H-T-t} - \pi_1(Z) \leqslant v_0 H\right) = \mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H \mid A_t\right).$$

Now

$$\begin{aligned} &\left|\mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H \mid A_t\right) - \mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right)\right| \\ &= \frac{1}{\mathbb{P}(A_t)} \left|\mathbb{P}\left(\{X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\} \cap A_t\right) - \mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right) \mathbb{P}(A_t)\right| \quad (*) \end{aligned}$$

where $\mathbb{P}(A_t) \xrightarrow[t\to 0]{} 1$ and

$$\mathbb{P}\left(\{X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\} \cap A_t\right) \xrightarrow[t\to 0]{\text{uniform in } H} \mathbb{P}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right),$$

by monotone continuity of $\mathbb{P}$ and the fact that the jumping times have a continuous law. Therefore $(*)$ goes to 0 when $t \to 0$, uniformly in $H$, hence the result (3.8). Therefore, for an $\varepsilon$ small enough, which does not depend on $H$,

$$\mathbb{E}_{\eta_T}\left[\mathbf{1}_{T_1^{x_1} < T_1^{x_2}}\, \mathbb{P}_{\eta_{T_1^{x_1}}}\left(X^{(x_2,0)}_{H-T-T_1^{x_1}} - \pi_1(Z) \leqslant v_0 H\right)\right] \geqslant \frac{\varepsilon e^{-2\varepsilon/R}}{2R} \mathbb{P}_{\eta_T}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right).$$

Consequently,

$$\begin{aligned} &\mathbb{P}\left(T \leqslant H,\ \tau_1 < \tau_2,\ \mathrm{V}^Z_{0,H} \leqslant v_0\right) \\ &\geqslant \frac{\varepsilon e^{-2\varepsilon/R}}{2R} \sum_{x_1,\, x_2\in\mathbb{Z}} \mathbb{E}\left[\mathbf{1}_{T\leqslant H}\, \mathbf{1}_{X^{w'}_T = x_1}\, \mathbf{1}_{X^Z_T = x_2}\, \mathbb{P}_{\eta_T}\left(X^{(x_2,0)}_{H-T} - \pi_1(Z) \leqslant v_0 H\right)\right] \\ &= \frac{\varepsilon e^{-2\varepsilon/R}}{2R}\, \mathbb{P}\left(T \leqslant H,\ \mathrm{V}^Z_{0,H} \leqslant v_0\right). \end{aligned}$$

At the end of the day, setting $\gamma_0 = \gamma\, \frac{\varepsilon e^{-2\varepsilon/R}}{2R}$ (which does not depend on $H$, $w$ and $w'$), we do get what we wanted:

$$\mathbb{P}\left(X^{w'}_H \leqslant \pi_1(w) + (v_+ - \delta)H,\ \mathrm{V}^Z_{0,H} \leqslant v_0\right) \geqslant \gamma_0\, \mathbb{P}\left(\mathrm{V}^Z_{0,H} \leqslant v_0\right).$$

$\square$

The only thing left to do is to check that we still have a bound for the probability of overstepping all traps when starting near a threatened point, similar to proposition 3.11 in our first attempt. Actually, we obtain a very similar bound.

**Proposition 3.20.**

*There exists a constant $c_{10} \in (0,1)$ such that for every $H \geqslant 1$, $r \in \mathbb{N}^*$ and $y \in \mathbb{L}$,*

$$\mathbb{P}\begin{pmatrix} \mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r}; \\ \lfloor y \rfloor_H \text{ est } (H, I_r, q)\text{-threatened}; \\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r} \end{pmatrix} \leqslant 2\, c_{10}^q + c_9\, r\, 4^r\, e^{-\kappa\, H^\beta}.$$

Proof. Set $w = \lfloor y \rfloor_H$, $w_j = w + jH(v_+, 1)$ and $Z_j = Z_{w_j,H}$.

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r};\ \lfloor y \rfloor_H\ (H, I_r, q)\text{-threatened};\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r}\right) \\
&\leqslant \mathbb{P}\left(w \text{ threatened},\ \#\left\{j \in I_r,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} < \frac{q}{3}\right) \qquad (1) \\
&+ \mathbb{P}\begin{pmatrix} \mathrm{V}^y_{0,rH} > v_+ - \frac{\delta}{2r}; \\ \#\left\{j \in I_r,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} \geqslant \frac{q}{3}; \\ \forall j \in [\![0, r-1]\!],\ \mathrm{V}^y_{jH,(j+1)H} \leqslant v_+ + \frac{\delta}{2r} \end{pmatrix} \qquad (2)
\end{aligned}$$

Because of property 3.19, with the same line of reasoning as in the proof of proposition 3.11, we get

$$(2) \leqslant (1-\gamma_0)^{q/3} + c_9\, r\, 4^r\, e^{-\kappa H^\beta}.$$

Now, for the first term,

$$\begin{aligned}
(1) &= \mathbb{P}\left(w \text{ threatened},\ \#\left\{j \in I_r,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} < \frac{q}{3}\right) \\
&\leqslant \sum_{|J| \geqslant q} \mathbb{P}\left(\{j :\ w_j \text{ trapped}\} = J,\ \#\left\{j \in J,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} < \frac{|J|}{3}\right) \\
&= \sum_{|J| \geqslant q} \mathbb{E}\left[\mathbf{1}_{\{\{j :\, w_j \text{ trapped}\} = J\}}\ \mathbb{P}_{cond}\left(\#\left\{j \in J,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} < \frac{|J|}{3}\right)\right].
\end{aligned}$$

Now, let us work on the event that $\{j :\ w_j \text{ trapped}\} = J$. We have, using the Markov inequality and independence of the $(Z_j)$,

$$\begin{aligned}
\mathbb{P}_{cond}\left(\#\left\{j \in J,\ \mathrm{V}^{Z_j}_{0,H} \leqslant v_0\right\} < \frac{|J|}{3}\right) &= \mathbb{P}_{cond}\left(\sum_{j \in J} \mathbf{1}_{\{\mathrm{V}^{Z_j}_{0,H} > v_0\}} > \frac{2|J|}{3}\right) \\
&\leqslant e^{-\frac{2|J|}{3}} \prod_{j \in J} \underbrace{\mathbb{E}_{cond}\left[\exp\left(\mathbf{1}_{\{\mathrm{V}^{Z_j}_{0,H} > v_0\}}\right)\right]}_{=(e-1)\,\mathbb{P}_{cond}(\mathrm{V}^{Z_j}_{0,H} > v_0) + 1 \leqslant \frac{e+1}{2} \text{ by } (3.7)} \\
&\leqslant \left(\frac{e+1}{2}\, e^{-2/3}\right)^{|J|} \\
&\leqslant C^q,
\end{aligned}$$

where $C = \frac{e+1}{2}\, e^{-2/3} \in (0,1)$. We obtain the desired result by setting $c_{10} = \max\left(C, (1-\gamma_0)^{1/3}\right)$. □

## 3.3 End of the proof

The end of the proof is the same in both attempts; it is based on two things:

- A uniformly positive lower bound for the probability of being trapped. It has already been shown for the definition of traps that we use in our first attempt (see step 1 in the proof of proposition 2.29).

For the second attempt, it is conjecture 3.16.

- A proper bound for the probability of overstepping all traps when starting near a threatened point, see propositions 3.11 and 3.20. Both bounds are very similar, so we will work with that of proposition 3.11.

### 3.3.1 Probability of being threatened

We nom prove an estimate of the probability of being threatened in the new sense that is very similar to proposition 2.29 in the nearest-neighbour setting.

First, remark that the same estimate as in proposition 2.29 holds for the new notion of threats with $q = 1$, for the proof can be adapted without any difficulty, once we have a uniformly positive lower bound for the probability of being trapped. Let $H_0$ and $c_7$ as in proposition 2.29.

LEMMA 3.21. *Assume $\alpha \geqslant 1$. For every $H \geqslant H_0$ and $r \in \mathbb{N}^*$,*

$$\sup_{w\in\mathbb{R}^2} \mathbb{P}(w \textit{ is } (H, I_r, 1)\textit{-threatened}) \leqslant c_7\, r^{-\alpha}. \tag{3.9}$$

Now it is easy to deduce the following estimate.

PROPOSITION 3.22. *Assume $\alpha > 1$. There exists $c_{11} = c_{11}(\delta) > 0$ such that for every $H \geqslant H_0$ and $r \in \mathbb{N}^*$,*

$$\sup_{w\in\mathbb{R}^2} \mathbb{P}(w \textit{ is not } (H, I_r, q)\textit{-threatened}) \leqslant c_{11}\, r^{\frac{1-\alpha}{2}}. \tag{3.10}$$

PROOF. Let $w \in \mathbb{R}^2$ and $H \geqslant H_0$. Let

$$s = s(r) = \sup\left\{ j \in I_r,\ j \leqslant \frac{r}{q} \right\},$$

the last even integer before $r/q$. We have

$$\begin{aligned}
&\mathbb{P}(w \text{ is not } (H, I_r, q)\text{-threatened}) \\
&\leqslant \mathbb{P}\left(\exists\, j \in [\![0, q-1]\!],\ w + sjH(v_+, 1) \text{ is not } (H, I_s, 1)\text{-threatened}\right) \\
&\leqslant q \sup_{w\in\mathbb{R}^2} \mathbb{P}(w \text{ is not } (H, I_s, 1)\text{-threatened}) && \text{by translation invariance} \\
&\leqslant q\, c_7\, s^{-\alpha} && \text{by lemma 3.21} \\
&\leqslant c_{11}\, r^{\frac{1-\alpha}{2}} && \text{because } q = \lfloor \sqrt{r} \rfloor.
\end{aligned}$$

□

### 3.3.2 Density of delayed points

We now introduce a notion stronger than threats, which ensures that when starting near such a point, our particles are delayed. With this new notion, we will be able to adapt the end of the proof of the LLN from section 2 without having too many things to change.

In words, we will say that a point is *delayed* if a particle starting near this point can only avoid being delayed by going faster that $v_+ + \frac{\delta}{2r}$ on some time sub-interval of length $H$.

DEFINITION 3.23. Let $w \in \mathbb{L}$. $w$ is said to be $(H, I_r, q)$-delayed if

- $w$ is $(H, I_r, q)$-threatened ;
- For every $y \in \mathbb{R}^2$ such that $\lfloor y \rfloor_H = w$,
  - Either there exists $j \in [\![0, r-1]\!]$ such that $\mathrm{V}^y_{jH,(j+1)H} > v_+ + \frac{\delta}{2r}$ ;

– Or $\mathrm{V}^y_{0,rH} \leqslant v_+ - \frac{\delta}{2r}$.

Now, we would like to have something similar to lemma 2.30, where we simply replace the notion of threatened points by that of delayed points. The problem is that we have a new term in the computation (term (2) in the proof) in which $h$ would not simplify. This is why we need to work with a fixed $h$, and we choose $h = L_{k_2+1}$ for now.

Lemma 3.24. *Assume that $\beta > 1/9$ and $\alpha \geqslant 17$. Let $q = q_k = \lfloor \sqrt{l_k} \rfloor$. There exists $k_2 = k_2(\delta) \in \mathbb{N}^*$ and $c_8 = c_8(\delta) > 0$ such that the following conditions are satisfied:*

1. *$L_{k_2} > H_0$ ;*
2. *For every $w \in \mathbb{R}^2$,*

$$\mathbb{P}\left(\exists\, y \in I_{L^2_{k_2+1}}(w) \cap \mathbb{L},\ \lfloor y \rfloor_{L_{k_2} L_{k_2+1}} \text{ is not } (L_{k_2} L_{k_2+1},\, I_{l_{k_2}}, q_{k_2})\text{-delayed}\right) \leqslant c_8\, L_{k_2+1}^{-(\alpha-3)/10}. \tag{3.11}$$

3. *$\forall k > k_2$,*

$$25\,(c_8^2 + c_1) L_k^{-(3\alpha-49)/40} + 25\, c_0^{-1}\, L_k^{(\alpha+5)/8} e^{-c_0 L_k^2} \leqslant c_8. \tag{3.12}$$

Proof. Let $w \in \mathbb{R}^2$ and $k_2 \in \mathbb{N}^*$ such that $L_{k_2} > H_0$ (to be chosen later).

$$\begin{aligned}
&\mathbb{P}\left(\exists\, y \in (w + [0, L^2_{k_2+1}) \times \{0\}) \cap \mathbb{L},\ \lfloor y \rfloor_{L_{k_2} L_{k_2+1}} \text{ is not } (L_{k_2} L_{k_2+1},\, I_{l_{k_2}}, q_{k_2})\text{-delayed}\right) \\
&\leqslant \mathbb{P}\left(\exists\, y \in (w + [0, L^2_{k_2+1}) \times \{0\}) \cap \mathbb{L},\ \lfloor y \rfloor_{L_{k_2} L_{k_2+1}} \text{ is not } (L_{k_2} L_{k_2+1},\, I_{l_{k_2}}, q_{k_2})\text{-threatened}\right) \qquad (1) \\
&+ \mathbb{P}\begin{pmatrix} \exists\, y \in (w + [0, L^2_{k_2+1}) \times \{0\}) \cap \mathbb{L},\ \lfloor y \rfloor_{L_{k_2} L_{k_2+1}} \text{ is } (L_{k_2} L_{k_2+1},\, I_{l_{k_2}}, q_{k_2})\text{-threatened}, \\ \forall j,\ \mathrm{V}^y_{jL_{k_2} L_{k_2+1}, (j+1) L_{k_2} L_{k_2+1}} \leqslant v_+ + \dfrac{\delta}{2 l_{k_2}} \text{ but } \mathrm{V}^y_{0, L^2_{k_2+1}} > v_+ - \dfrac{\delta}{2 l_{k_2}} \end{pmatrix} \qquad (2)
\end{aligned}$$

Now, using proposition 3.22,

$$(1) \leqslant \left\lceil \frac{L^2_{k_2+1}}{\left\lfloor \frac{\delta L_{k_2} L_{k_2+1}}{4} \right\rfloor} \right\rceil c_{11}\, l_{k_2}^{\frac{1-\alpha}{2}} \leqslant c(\delta)\, L_{k_2+1}^{-\frac{\alpha-3}{10}}.$$

For the second term, we use proposition 3.11 to conclude that

$$(2) \leqslant L^2_{k_2+1}\,(1-\gamma_0)^{q_{k_2}} + c_9 L^2_{k_2+1} 4^{l_{k_2}} l_{k_2} e^{-\kappa (L_{k_2} L_{k_2+1})^\beta}. \tag{3.13}$$

Now, this last term is less than $L_{k_2+1}^{-\frac{\alpha-3}{10}}$ if $k_2$ is large enough, by definition of $q_{k_2}$ (recall notation 3.10) and the fact that $\beta > 1/9$. In the end, up to changing $c$, we do get inequality (3.11) with a certain constant $c_8 > 0$. Now that $c_8$ is fixed, it suffices to take $k_2$ even larger so that inequality (3.12) holds as well, which is possible because $3\alpha - 49 > 0$, since $\alpha \geqslant 17$. □

Definition 3.25. Let $k_2$ be defined as in lemma 3.24. Let $k > k_2$ and $\gamma = (\gamma(t))_{t \in [0, L_k^2]}$ be an allowed path. We define its delayed density as

$$\tilde{D}_k(\gamma) = \frac{L_{k_2+1}}{L_k} \# \left\{ 0 \leqslant j < \frac{L_k}{L_{k_2+1}},\ \lfloor \gamma(j L_k L_{k_2+1}) \rfloor_{L_k L_{k_2}} \text{ is } (L_k L_{k_2}, I_{l_{k_2}}, q_{k_2})\text{-delayed} \right\}. \tag{3.14}$$

Conjecture 3.26. *Let $\alpha \geqslant 17$. For every $k > k_2$ and $w \in \mathbb{R}^2$,*

$$\mathbb{P}(\exists\, \gamma = (\gamma(t))_{t \in [0, L_k^2]} \text{ starting in } I_{L_k^2}(w) \cap \mathbb{L},\ \tilde{D}_k(\gamma) < 1/2) \leqslant c_8\, L_k^{-(\alpha-3)/10}.$$

Ideas and problems. It is very tempting to say that we can follow the lines of the proof of proposition 2.32 only by replacing $h$ by $L_k$. But in the induction step, we cannot go from $h = L_k$ to $h = L_{k+1}$: the proof of proposition 2.32 highly relies on the fact that when changing scales in the induction step, we are however looking at the same points : the $\gamma(jhL_{k_2+1})$, where $h$ is fixed. If we managed to get around this issue, we could obtain an inequality of the form

$$s_{k+1} \leqslant 25\, l_k^4\, (s_k^2 + c_1 L_k^{-2\alpha}) + 5\, l_k^2\, c_0^{-1} e^{-c_0 L_k^2},$$

where

$$s_k = \sup_{w \in \mathbb{R}^2} \mathbb{P}\left( \exists\, \gamma = (\gamma(t))_{t \in [0, L_k^2]} \text{ starting in } I_{L_k^2}(w) \cap \mathbb{L},\ \tilde{D}(\gamma) \leqslant \rho_k \right).$$

Therefore

$$\begin{aligned}
\frac{s_{k+1}}{L_{k+1}^{-(\alpha-3)/10}} &\leqslant L_k^{(\alpha-3)/8} \left( 25\, l_k^4\, (s_k^2 + c_1 L_k^{-2\alpha}) + 5\, l_k^2\, c_0^{-1} e^{-c_0 L_k^2} \right) \\
&\leqslant 25\, l_k^4\, L_k^{(\alpha-3)/8} \left( s_k^2 + c_1 L_k^{-2\alpha} + c_0^{-1} e^{-c_0 L_k^2} \right) \\
&\leqslant 25\, L_k^{(\alpha+5)/8} \left( c_8^2 L_k^{-(\alpha-3)/5} + c_1\, L_k^{-2\alpha} + c_0^{-1} e^{-c_0 L_k^2} \right) \\
&\leqslant 25\, (c_8^2 + c_1) L_k^{-(3\alpha-49)/40} + 25\, c_0^{-1}\, L_k^{(\alpha+5)/8} e^{-c_0 L_k^2} \underset{(3.12)}{\leqslant} c_8.
\end{aligned}$$

Proof of theorem 2.13. If we assume that conjecture 3.26 is true, then the end of the proof is almost exactly the same as in section 2. The condition $\alpha \geqslant 17$ is stronger than $\alpha > 8$, which was required in the proof, and the only thing that changes is the estimate for $\mathbb{P}(G_2(w)^c)$. Here, using conjecture 3.26,

$$\sup_{w \in \mathbb{R}^2} \mathbb{P}(G_2(w)^c) \leqslant c_8\, L_k^{-(\alpha-3)/10} \xrightarrow[k \to \infty]{\alpha \geqslant 17} 0.$$

□

# References


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