BOUNDED FUNCTIONS ON THE CHARACTER VARIETY FOR QUADRATIC EXTENSIONS OF $\mathbb{Q}_p$

LAURENT BERGER, JOHANNES SPRANG, AND RUSTAM STEINGART

Abstract. For any finite extension L of $\mathbb{Q}_p$, Schneider–Teitelbaum defined and studied the character variety in p-adic Fourier theory. It is a rigid analytic curve over L that parameterizes the set of locally L-analytic characters of o_L. It is conjectured that all bounded functions on the character variety of Schneider–Teitelbaum come from p-adic measures. We prove this conjecture for all quadratic extensions $L/\mathbb{Q}_p$.

1. Introduction

Let L be a finite extension of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ denote the completion of an algebraic closure of $\mathbb{Q}_p$. In their work on p-adic Fourier theory [ST01], Schneider and Teitelbaum defined and studied the character variety $\mathfrak{X}$. This character variety is a rigid analytic curve over L that parameterizes the set of locally L-analytic characters $\lambda : (o_L, +) \to (\mathbb{C}_p^\times, \times)$.

The character variety plays an important role in the theory of p-adic representations and in the construction of p-adic L-functions. Schneider and Teitelbaum show that the space of functions $\mathcal{O}(\mathfrak{X})$ on the character variety identifies with the space of locally L-analytic distributions $D(o_L, L)$. However, we have so far a fairly poor understanding of the bounded-by-1 functions $\mathcal{O}(\mathfrak{X})^\circ$ on the character variety. Since every p-adic measure gives rise to a unique bounded locally L-analytic distribution, we have an injective map of o_L-algebras

$$o_L[\![o_L]\!] \hookrightarrow \mathcal{O}(\mathfrak{X})^\circ. \tag{1}$$

A natural question first raised by Teitelbaum and studied extensively in [AB24] is the following:

Question 1.1. Do we have $o_L[\![o_L]\!] = \mathcal{O}(\mathfrak{X})^\circ$?

Some numerical evidence for a positive answer to this question has been given in [BS26] for quadratic unramified extensions. In this paper, we show that Question 1.1 has a positive answer for any quadratic extension L of $\mathbb{Q}_p$. More precisely:

Theorem 1.2. *If $L/\mathbb{Q}_p$ is a quadratic extension, the canonical injective map $o_L[\![o_L]\!] \hookrightarrow \mathcal{O}(\mathfrak{X})^\circ$ is an isomorphism.*

Before we discuss some applications of Theorem 1.2, let us briefly summarize the key ideas of its proof. For simplicity, assume that L is a quadratic unramified extension. The proof in the ramified case is only a minor variation of the argument in the unramified case. In the quadratic unramified case, the locally L-analytic character variety can be cut out from the

Date: November 17, 2026.

locally $\mathbb{Q}_p$-analytic character variety $\mathbb{D}^\circ \otimes_{\mathbb{Z}_p} \mathrm{Hom}(o_L, \mathbb{Z}_p) \cong (\mathbb{D}^\circ)^2$ by a single equation H. On the level of affinoid polydiscs of radius $\rho_n \in (0,1)$, this gives a surjection

$$B_n = L\langle \rho_n^{-1} x_1, \rho_n^{-1} x_2 \rangle \twoheadrightarrow A_n := B_n/(H).$$

The key observation of this paper is that for suitably chosen radii ρ_n, we get a surjection $B_n^\circ \twoheadrightarrow A_n^\circ$ for the rings of bounded functions. In other words, we can lift any bounded function in A_n to a bounded function in B_n. We prove this by comparing the quotient norm on A_n induced by $B_n \twoheadrightarrow A_n$ to the spectral norm on A_n. Using this, Theorem 1.2 is essentially a consequence of the isomorphism $\mathcal{O}(\mathfrak{X})^\circ \cong \varprojlim_{(0,1)\ni\rho_n\to 1} A_n^\circ$ and the fact that we can describe the bounded functions B_n° on a polydisc of radius ρ_n explicitly. In section 6, we explain the limitations of our method. In particular, we show that the quotient norm on A_n is in general not the spectral norm of A_n.

In [AB24], Ardakov and the first-named author have established several connections between Question 1.1 and Katz's theory of Galois-continuous measures as well as with questions about integer-valued polynomials. For example, they show that a positive answer to Question 1.1 implies the following conjecture about integer-valued polynomials:

Conjecture 1.3. For any finite extension $L/\mathbb{Q}_p$ we have $\mathbf{Int} = \mathbf{Pol}$.

Here, $\mathbf{Int} = \{P(X) \in L[X] \text{ such that } P(o_L) \subset o_L\}$ is the space of all integer-valued polynomials on o_L, and $\mathbf{Pol}$ denotes the o_L-module spanned by the polynomials $c_{n,i}(X)$ for all $i \geq n \geq 0$, where $[a](T)^n = \sum_{i\geq n} c_{n,i}(a)T^i$, where $[a](T)$ denotes the multiplication-by-a-map on a Lubin–Tate formal o_L-module attached to a uniformizer of o_L. Hence, our main theorem also proves Conjecture 1.3 for all quadratic extensions of $\mathbb{Q}_p$. Note that a stronger implication is proved in remark 3.5 of [BS26].

Another implication of Theorem 1.2 concerns associated orders in a Lubin–Tate extension. Let $L = \mathbb{Q}_{p^2}$ and let L_n be the extension of L generated by the p^n-torsion points of the Lubin–Tate group attached to L and p. Let $G_n = \mathrm{Gal}(L_n/L)$ and let $\mathcal{A}_n = \{f \in L[G_n] \text{ such that } f(o_n) \subset o_n\}$. There are restriction maps $\pi_n : \mathcal{A}_{n+m} \to \mathcal{A}_n$ for all $m, n \geq 0$. In §5, we prove the following result.

Theorem 1.4. *For all $n \geq 0$, there exists $m = m(n)$ such that $\pi_n(\mathcal{A}_{n+m}) = o_L[G_n]$.*

At about the same time that we proved Theorem 1.2, we learned that Uroš Colović had answered Question 1.1 for all finite extensions $L/\mathbb{Q}_p$ using a completely different method.

2. Notation and Preliminaries

Let $L/\mathbb{Q}_p$ be a finite extension of degree $d = ef$, where f denotes the inertia degree and e the ramification index of $L/\mathbb{Q}_p$. The field L has valuation ring o_L, uniformiser π, and residue field k_L. We write $q := |k_L| = p^f$ for the cardinality of the residue field. We denote by $o_L[\![o_L]\!]$ the completed group ring of o_L. Fixing a basis $(v_1, \ldots, v_d)$ of o_L over $\mathbb{Z}_p$, we obtain an isomorphism $o_L[\![x_1, \ldots, x_d]\!] \cong o_L[\![o_L]\!]$, by sending x_i to $\delta_{v_i} - 1$, where δ_x denotes the Dirac measure which evaluates a continuous function at x. We denote by $\mathbb{D}^\circ$ the rigid analytic open unit disc over L.

From the above isomorphism we can see that the rigid analytic generic fibre $\mathfrak{Y}$ of $\mathrm{Spf}(o_L[\![o_L]\!])$ is isomorphic to the open unit polydisc $(\mathbb{D}^\circ)^d$. For a complete field extension C of L, the

C-valued points of $\mathfrak{Y}$ parametrize locally $\mathbb{Q}_p$-analytic characters $o_L \to C^\times$. In [ST01], the authors construct a closed rigid analytic subvariety $\mathfrak{X} \subseteq \mathfrak{Y}$ which parametrizes the locally L-analytic characters. Since our proof makes use of the explicit description of $\mathcal{O}(\mathfrak{X})$, we recall the construction of $\mathfrak{X}$ in the following. The ring $\mathcal{O}(\mathfrak{Y})$ is isomorphic to the algebra of locally $\mathbb{Q}_p$-analytic distributions $D^{\mathbb{Q}_p}(o_L, L)$. Recall that the Lie-algebra $\mathfrak{g} \cong L$ of o_L can be embedded into $D^{\mathbb{Q}_p}(o_L, L)$ and a locally $\mathbb{Q}_p$-analytic character $o_L \to V$ valued in a Banach space over L is locally L-analytic if and only if its derivative at 0 is L-linear. Let us write $\iota\colon \mathfrak{g} \to D^{\mathbb{Q}_p}(o_L, L)$. By the construction of Schneider–Teitelbaum $\mathfrak{X}$ is the rigid analytic variety corresponding to the ideal $\mathcal{I}$ generated by elements of the form $(t\iota(\mathfrak{x}) - \iota(t\mathfrak{x}))$, where $t \in L$ and $\mathfrak{x} \in \mathfrak{g}$. Because $\mathfrak{g}$ is one-dimensional over L and $(v_1, \dots, v_d)$ is a basis of L over $\mathbb{Q}_p$ one has for any non-zero element $\mathfrak{x} \in \mathfrak{g}$

$$\mathcal{I} = (\iota(v_1\mathfrak{x}) - v_1\iota(\mathfrak{x}), \dots, \iota(v_d\mathfrak{x}) - v_d\iota(\mathfrak{x})).$$

Note that by choosing $v_1 := 1$ we can always eliminate the first equation. By [ST02, Lemma 4.3] the isomorphism

$$D^{\mathbb{Q}_p}(o_L, L) \cong \mathcal{O}(\mathfrak{Y}) \cong \mathcal{O}((\mathbb{D}^\circ)^d)$$

is compatible with the logarithm map $\log_{o_L} : o_L \to \mathfrak{g}$ in the sense that $\log_{o_L}(v_i)$ is sent to $\log(1 + x_i)$. We denote by I the ideal of $B := \mathcal{O}((\mathbb{D}^\circ)^d)$ generated by the image of $\mathcal{I}$. By choosing $v_1 := 1$ so that $v_j = v_j v_1$ we see that this is precisely the ideal generated by the elements

$$H_i(x_1, \dots, x_d) := \log(1 + x_i) - v_i \log(1 + x_1).$$

For any increasing sequence ρ_n of elements of $p^{\mathbb{Q}} \cap (0, 1)$ converging to 1 we can write

$$(\mathbb{D}^\circ)^d = \varinjlim_n \mathrm{Sp}(B_n)$$

with the Tate-algebras $B_n := L\langle \rho_n^{-1}x_1, \dots, \rho_n^{-1}x_d\rangle$. By Theorem B for Stein spaces one has $\mathcal{O}(\mathfrak{X}) \cong \mathcal{O}(\mathfrak{Y})/I \cong \varprojlim B_n/I$. From this one obtains an explicit exhaustion of $\mathfrak{X}$ by affinoid subdomains $X_n := \mathrm{Sp}(A_n)$ with $A_n := B_n/I$. We can equip A_n with the quotient norm $\|\cdot\|_{n,\mathrm{q}}$ induced by the Gauß norm $\|\cdot\|_n$ on B_n. Alternatively A_n can be equipped with its spectral semi norm $\|\cdot\|_{n,\mathrm{sp}}$. We denote by

$$A_n^\circ := \{f \in A_n \mid \|f\|_{n,\mathrm{sp}} \leq 1\}$$

the subring of power-bounded elements in A_n.

3. The unramified case

We specialize to the situation where $L/\mathbb{Q}_p$ is a quadratic unramified extension.

Choose a $\mathbb{Z}_p$-basis $(v_1 = 1, v_2)$ of o_L. We write x_1, x_2 for the coordinates on $\mathbb{D}^\circ \times \mathbb{D}^\circ$, so that the character variety of Schneider–Teitelbaum $\mathfrak{X}$ is locally cut out by the equation

$$H(x_1, x_2) := \log(1 + x_2) - v_2 \log(1 + x_1).$$

For us, it will be convenient to choose the sequence

$$\rho_n := p^{-\frac{1}{p^n - p^{n-1}}}.$$

of critical radii of the logarithm series, i.e. the increasing sequence $(\rho_n)_n$ consists exactly of those numbers $\rho \in (0, 1)$ where the function $\mathbb{Z}_{\geq 1} \to \mathbb{R}, \quad m \mapsto \rho^m/|m|_p$ does not have a

unique maximum. Indeed, for $\rho = \rho_n$ the maximum is attained for the two values $m = p^{n-1}$ and $m = p^n$. As before we have

$$\mathbb{D}^\circ \times \mathbb{D}^\circ = \varinjlim_n \mathrm{Sp}(B_n)$$

with the Tate algebra $B_n = L\langle \rho_n^{-1} x_1, \rho_n^{-1} x_2 \rangle$. The character variety of Schneider and Teitelbaum is given by $\mathfrak{X} = \varinjlim_n \mathfrak{X}_n$ where $\mathfrak{X}_n = \mathrm{Sp}(A_n)$ with

$$A_n = B_n/(H).$$

The goal of this section is to prove that the quotient norm on A_n induced by $B_n \twoheadrightarrow A_n$ is exactly the spectral norm on A_n. As a consequence, any bounded function on A_n can be lifted to a bounded function on B_n of the same norm. The computations are very much inspired by §3 of Tobias Schmidt's PhD thesis, see [Sch06]. The only difference is that we prefer to work with critical radii, while the computations in loc. cit. are mainly under the assumption of non-critical radii. Finally, we deduce that the bounded-by-1 functions $\mathcal{O}(\mathfrak{X})^\circ$ on the character variety are all given by measures.

Let us write $\|\cdot\|_n$ for the Gauß norm on B_n and consider for $s \in \mathbb{R}$ the filtration

$$F^s B_n := \{F \in B_n : \|F\|_n \leq p^{-s}\},$$
$$F^{s,+} B_n := \{F \in B_n : \|F\|_n < p^{-s}\}.$$

This filtration induces on L the usual filtration by the absolute value. Let us write $\mathrm{gr}^\bullet B_n$ and $\mathrm{gr}^\bullet L$ for the associated graded rings and $\sigma \colon B_n \to \mathrm{gr}^\bullet B_n$ for the principal symbol map, which is defined for $f \in F^s B_n \setminus F^{s,+} B_n$ as

$$\sigma(f) := f + F^{s,+} B_n \in F^s B_n / F^{s,+} B_n \subseteq \mathrm{gr}^\bullet B_n.$$

Let us write $\epsilon := \sigma(p) \in \mathrm{gr}^\bullet L$. Then $\mathrm{gr}^\bullet L = k_L[\epsilon, \epsilon^{-1}]$ and we have

$$\mathrm{gr}^\bullet B_n = \mathrm{gr}^\bullet L[X_1, X_2],$$

with $X_i = \sigma(x_i)$, cf. [ST03, Theorem 4.5].

As explained above, the dominating terms of the logarithm series $\log(1+T) = \sum_{k \geq 1} (-1)^{k-1} \frac{T^k}{k}$ for the critical radius ρ_n are $\frac{T^{p^n}}{p^n}$ and $\frac{T^{p^{n-1}}}{p^{n-1}}$. Hence

$$\sigma(H) = \pm \epsilon^{-n} \left[\epsilon \left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}} \right) + \left(X_2^{p^n} - \overline{v}_2 X_1^{p^n} \right) \right].$$

For later reference, and since it is crucial to the argument, we note that $\sigma(H)$ is irreducible:

Lemma 3.1. *The element $\sigma(H) \in \mathrm{gr}^\bullet B_n$ is irreducible.*

Proof. Since $\epsilon^{-n} \in \mathrm{gr}^\bullet B_n$ is a unit, it suffices to show that

$$\epsilon \left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}} \right) + \left(X_2^{p^n} - \overline{v}_2 X_1^{p^n} \right) \in k_L[\epsilon, \epsilon^{-1}, X_1, X_2]$$

is irreducible. We first prove that $X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}}$ and $X_2^{p^n} - \overline{v}_2 X_1^{p^n}$ are coprime in $k_L[X_1, X_2]$: We have

$$\left(X_2^{p^n} - \overline{v}_2 X_1^{p^n} \right) - \left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}} \right)^p = (\overline{v}_2^p - \overline{v}_2) X_1^{p^n}.$$

Since $\overline{v}_2 \in k_L \setminus \mathbb{F}_p$ we have $(\overline{v}_2^p - \overline{v}_2) \neq 0$. We deduce that any common irreducible factor must be associated to X_1. However, neither $X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}}$ and $X_2^{p^n} - \overline{v}_2 X_1^{p^n}$ is divisible by X_1. This shows that $X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}}$ and $X_2^{p^n} - \overline{v}_2 X_1^{p^n}$ are co-prime in $k_L[X_1, X_2]$. Now view

$$\epsilon \left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}} \right) + \left(X_2^{p^n} - \overline{v}_2 X_1^{p^n} \right) \in k_L[X_1, X_2][\epsilon]$$

as a degree 1 polynomial in ϵ over $k_L[X_1, X_2]$. Its coefficients are co-prime, hence Gauß's Lemma implies that it is irreducible in $k_L[\epsilon, X_1, X_2]$. Since $\epsilon \left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}} \right) + \left(X_2^{p^n} - \overline{v}_2 X_1^{p^n} \right)$ is not associated to ϵ, localization at ϵ preserves irreducibility. □

By the above Lemma, $(\sigma(H)) \subseteq \mathrm{gr}^\bullet B_n$ is a prime ideal. Let us now equip A_n with the quotient norm

$$\|f\|_{n,\mathrm{q}} := \inf_{\substack{F \in B_n \\ F \bmod (H) = f}} \|F\|_n.$$

Here, note that the ideal (H) is closed, so the quotient seminorm is actually a norm. Since the (additive) value group of the Gauß norm $\|\cdot\|_n$ on B_n is discrete, we can find for any non-zero $f \in A_n$ a lift $F \in B_n$ with $\|F\|_n = \|f\|_{n,\mathrm{q}}$.

Lemma 3.2. *Take $0 \neq f \in A_n$ and let $F \in B_n$ be a lift of f. The following are equivalent:*

(1) $\|F\|_n = \|f\|_{n,\mathrm{q}}$,
(2) $\sigma(F) \notin (\sigma(H))$.

Proof. For any lift, we have $\|F\|_n \geq \|f\|_{n,\mathrm{q}}$. Suppose we have $\|F\|_n \neq \|f\|_{n,\mathrm{q}}$, then $\|F\|_n > \|f\|_{n,\mathrm{q}}$. Since we can always find a representative in $F + (H)$ of minimal norm $\|f\|_{n,\mathrm{q}}$, there exists $G \in B_n$ such that $\|F - GH\|_n = \|f\|_{n,\mathrm{q}} < \|F\|_n$. This implies that the dominating terms of F and GH cancel, i.e.,

$$\sigma(F) = \sigma(GH) = \sigma(G)\sigma(H) \in (\sigma(H)).$$

Conversely, suppose we have $\sigma(F) \in (\sigma(H))$. Then we can find $G \in B_n$ such that $\sigma(F) = \sigma(H)\sigma(G)$. This implies $\|F - GH\|_n < \|F\|_n$. Hence we found a lift of strictly smaller norm and deduce $\|F\|_n \neq \|f\|_{n,\mathrm{q}}$. □

Corollary 3.3. *We have $\|\cdot\|_{n,\mathrm{q}} = \|\cdot\|_{n,\mathrm{sp}}$ on A_n.*

Proof. According to [BGR84, 6.2.3., Proposition 3] it suffices to show that $\|\cdot\|_{n,\mathrm{q}}$ is (power-) multiplicative. For non-zero $f, g \in A_n$, choose minimal lifts $F, G \in B_n$, i.e. $\|F\|_n = \|f\|_{n,\mathrm{q}}$ and $\|G\|_n = \|g\|_{n,\mathrm{q}}$. Lemma 3.2 implies $\sigma(F) \notin (\sigma(H))$ and $\sigma(G) \notin (\sigma(H))$. Since $(\sigma(H))$ is a prime ideal, we conclude $\sigma(FG) = \sigma(F)\sigma(G) \notin (\sigma(H))$. Again by Lemma 3.2, we get $\|FG\|_n = \|fg\|_{n,\mathrm{q}}$, i.e. we have

$$\|f\|_{n,\mathrm{q}} \cdot \|g\|_{n,\mathrm{q}} = \|F\|_n \cdot \|G\|_n = \|FG\|_n = \|fg\|_{n,\mathrm{q}}.$$ □

Proof of Theorem 1.2 in the unramified case. Let us first recall (Lemma 1.15 of [BSX20]) that the map $o_L[\![o_L]\!] \to \mathcal{O}(\mathfrak{X})$ is injective and has image in $\mathcal{O}(\mathfrak{X})^\circ$. In the following, we prove that the image is all of $\mathcal{O}(\mathfrak{X})^\circ$. For n a positive integer, write A_n for the ring of functions on the critical radius ρ_n as before. We have

$$\mathcal{O}(\mathfrak{X})^\circ = \varprojlim_n A_n^\circ,$$

with $A_n^\circ = \{f \in A_n : \|f\|_{n,\mathrm{sp}} \leq 1\}$. Let us write Λ_n for the image of $\Lambda := o_L[\![o_L]\!]$ in A_n°. Fix now $n \geq 1$ and consider $N > n$. We claim that $\mathrm{im}(A_N^\circ \to A_n^\circ) \subseteq \Lambda_n + p^{p^{N-n}-1} A_n^\circ$.
Proof of the Claim: Take $f \in A_N^\circ$. Choose a lift $F \in B_N$ with $\|F\|_N = \|f\|_{N,\mathrm{q}}$. By Corollary 3.3, we have $\|f\|_{N,\mathrm{q}} = \|f\|_{N,\mathrm{sp}}$. Hence $F \in B_N^\circ$. If we write $F = \sum_{\underline{k} \in \mathbb{N}_0^2} c_{\underline{k}} \underline{x}^{\underline{k}}$ (with the notation $\underline{x}^{\underline{k}} := x_1^{k_1} x_2^{k_2}$), this implies that for all $\underline{k} \in \mathbb{N}_0^2$ we have $|c_{\underline{k}}|_p \leq \rho_N^{-|\underline{k}|}$.

We can approximate F by the polynomial

$$P_N(x_1, x_2) := \sum_{|\underline{k}| < p^N - p^{N-1}} c_{\underline{k}} \underline{x}^{\underline{k}} \in L[x_1, x_2].$$

Note that $P_N(x_1, x_2)$ actually has integral coefficients because for $|\underline{k}| < p^N - p^{N-1}$, we have

$$|c_{\underline{k}}|_p \leq \rho_N^{-|k|} < \rho_N^{-(p^N - p^{N-1})} = p.$$

Since L has (additive) value group $\mathbb{Z}$, this implies $c_{\underline{k}} \in o_L$. Hence, we have

$$P_N(x_1, x_2) \in o_L[x_1, x_2].$$

We define

$$R_N := F - P_N = \sum_{|\underline{k}| \geq p^N - p^{N-1}} c_{\underline{k}} \underline{x}^{\underline{k}} \in B_N^\circ.$$

Using restriction, let us view R_N as an element of B_n:

$$\|R_N\|_n = \sup_{\underline{k}} |c_{\underline{k}}| \left(\frac{\rho_n}{\rho_N}\right)^{|\underline{k}|} \rho_N^{|\underline{k}|} \leq \left(\frac{\rho_n}{\rho_N}\right)^{p^N - p^{N-1}} \|R_N\|_N \leq \left(\frac{\rho_n}{\rho_N}\right)^{p^N - p^{N-1}}.$$

With $\frac{\rho_n}{\rho_N} = p^{-\frac{1}{p^n - p^{n-1}} + \frac{1}{p^N - p^{N-1}}}$ we get

$$\|R_N\|_n \leq \left(\frac{\rho_n}{\rho_N}\right)^{p^N - p^{N-1}} \leq p^{-p^{N-n}+1},$$

i.e. $R_N \in p^{p^{N-n}-1} B_n^\circ$, i.e.

$$F \in o_L[x_1, x_2] + p^{p^{N-n}-1} B_n^\circ.$$

Passing to the quotient $B_n^\circ \twoheadrightarrow A_n^\circ$ shows that $f \in \Lambda_n + p^{p^{N-n}-1} A_n^\circ$. This finishes the proof of the claim.

Since A_N is an integral domain for every $N \geq 1$ (see the above discussion), the restriction maps $A_N \to A_n$ for $N \geq n$ are injective, cf. [ABB18, Proposition 4.2]. It follows that the induced map

$$\mathcal{O}(\mathfrak{X}) = \varprojlim_N A_N \hookrightarrow A_n$$

is injective for every $n \geq 1$. Given $f \in \mathcal{O}(\mathfrak{X})^\circ$, write $f_n := f|_{\mathfrak{X}_n} \in \mathcal{O}(\mathfrak{X}_n)^\circ = A_n^\circ$. By the claim, we have

$$f_n \in \bigcap_{N \geq n} \Lambda_n + p^{p^{N-n}-1} A_n^\circ = \Lambda_n.$$

Here, the last equality follows since Λ_n is closed as the image of the compact set $o_L[\![o_L]\!]$ under the continuous map $o_L[\![o_L]\!] \to A_n$. We deduce $f \in o_L[\![o_L]\!]$ as the restriction map $\mathcal{O}(\mathfrak{X}) \hookrightarrow A_n$ is injective and $o_L[\![o_L]\!]$ maps isomorphically to Λ_n under this restriction map. □

4. The ramified case

The proof in the ramified case is quite similar to the unramified case with one small modification. Let us sketch the proof in the ramified case. Let $L/\mathbb{Q}_p$ be a ramified quadratic extension. If π is a uniformizer of L, then $v_1 = 1$ and $v_2 = \pi$ generate o_L as a $\mathbb{Z}_p$-module. Before we proceed, let us briefly discuss why the above argument needs a modification. The character variety is again locally cut out by the equation

$$H(x_1, x_2) := \log(1 + x_2) - \pi \log(1 + x_1),$$

however, for the previous choice of B_n with radius $\rho_n = p^{-1/(p^n - p^{n-1})}$, we run into the following problem. Since $\sigma(\pi)$ is not of degree zero, we have

$$\sigma(H) = \pm\epsilon^{-n}\left(\epsilon X_2^{p^{n-1}} + X_2^{p^n}\right),$$

which is not prime, and for $n > 1$ the quotient by the ideal $(\sigma(H))$ is not even reduced. However, the above proof can be fixed by considering two different radii. For $n \geq 1$, define

$$\alpha_n := p^{-\frac{1}{p^n - p^{n-1}} - \frac{1}{4p^{n-1}}}, \quad \beta_n := p^{-\frac{1}{p^n - p^{n-1}} - \frac{3}{4p^{n-1}}}.$$

These families of radii satisfy

$$p^{-1/2}\alpha_n^{p^{n-1}} = \beta_n^{p^{n-1}}. \tag{2}$$

The importance of this equation (and hence why we have chosen α_n and β_n as above) will become apparent in a second, when we compute the initial form of H. Let us write

$$B_n := L\langle \alpha_n^{-1} x_1, \beta_n^{-1} x_2 \rangle$$

and $A_n := B_n/(H)$. The dominant term of $\log(1 + x_2)$ on B_n is $\frac{x_2^{p^{n-1}}}{p^{n-1}}$, while the dominant term of $\pi \log(1 + x_1)$ is $\pi\frac{x_1^{p^{n-1}}}{p^{n-1}}$. Note that both terms have the same Gauß norm because of (2). Let us write $\epsilon := \sigma(\pi)$. Then $\mathrm{gr}^\bullet L = \mathbb{F}_p[\epsilon, \epsilon^{-1}]$ and

$$\mathrm{gr}^\bullet B_n = (\mathrm{gr}^\bullet L)[X_1, X_2]$$

with $X_1 = \sigma(x_1)$ and $X_2 = \sigma(x_2)$. The ideal generated by the initial form of H is

$$(\sigma(H)) = \left(X_2^{p^{n-1}} - \epsilon X_1^{p^{n-1}}\right) \subseteq \mathrm{gr}^\bullet B_n.$$

As in the unramified case, $X_2^{p^{n-1}} - \epsilon X_1^{p^{n-1}}$ is a polynomial of degree 1 in ϵ with co-prime coefficients and hence irreducible. This shows that $(\sigma(H))$ is a prime ideal which was the essential ingredient for proving that the spectral norm on A_n coincides with the quotient norm. We can now use essentially the same argument as in the unramified case to prove the ramified case.

Proof of Theorem 1.2 in the ramified case. Let n be a positive integer and write A_n for the ring of functions of radius (α_n, β_n) as above. We have

$$\mathcal{O}(\mathfrak{X})^\circ = \varprojlim_n A_n^\circ,$$

with $A_n^\circ = \{f \in A_n : \|f\|_{n,\mathrm{sp}} \leq 1\}$. Let us write Λ_n for the image of $\Lambda := o_L[\![o_L]\!]$ in A_n°. Fix now $n \geq 1$ and consider $N > n$.

Claim: $\mathrm{im}(A_N^\circ \to A_n^\circ) \subseteq \Lambda_n + p^{m_{N-n}} A_n^\circ$ with $m_{N-n} := \left\lfloor \frac{(p+3)}{8} \cdot \left(p^{N-n-1} - \frac{1}{p}\right) \right\rfloor$.

Proof of the Claim: Let $f \in A_N^\circ$. Choose a lift $F \in B_N$ with $\|F\|_N = \|f\|_{N,\mathrm{q}}$. By the analogue of Corollary 3.3, we have $\|f\|_{N,\mathrm{q}} = \|f\|_{N,\mathrm{sp}}$. Hence $F \in B_N^\circ$. If we write $F = \sum_{\underline{k} \in \mathbb{N}_0^2} c_{\underline{k}} \underline{x}^{\underline{k}}$ (with the notation $\underline{x}^{\underline{k}} := x_1^{k_1} x_2^{k_2}$), this implies for all $\underline{k} \in \mathbb{N}_0^2$

$$|c_{\underline{k}}|_p \leq \alpha_N^{-k_1} \beta_N^{-k_2}.$$

We can approximate F by the polynomial

$$P_N(x_1, x_2) := \sum_{|k| < \frac{p^{N-1} - p^{N-2}}{2}} c_{\underline{k}} \underline{x}^{\underline{k}} \in L[x_1, x_2].$$

Note that $P_N(x_1, x_2)$ has actually integral coefficients because for $|\underline{k}| < \frac{p^{N-1} - p^{N-2}}{2}$, we have

$$|c_{\underline{k}}|_p \leq \alpha_N^{-k_1} \beta_N^{-k_2} = p^{\frac{k_1(4+p-1)}{4p^{N-1}(p-1)} + \frac{k_2(4+3(p-1)}{4p^{N-1}(p-1)}} \leq p^{\frac{k_1+k_2}{p^{N-2}(p-1)}} < p^{1/2}.$$

Since L has (additive) value group $\frac{1}{2}\mathbb{Z}$, this implies $c_{\underline{k}} \in o_L$. Hence, we have

$$P_N(x_1, x_2) \in o_L[x_1, x_2].$$

We define

$$R_N := F - P_N = \sum_{|\underline{k}| \geq \frac{p^{N-1} - p^{N-2}}{2}} c_{\underline{k}} \underline{x}^{\underline{k}} \in B_N^\circ.$$

Using restriction, let us view R_N as an element of B_n:

$$\begin{aligned} \|R_N\|_n &= \sup_{\underline{k}} |c_{\underline{k}}| \left(\frac{\alpha_n}{\alpha_N}\right)^{k_1} \left(\frac{\beta_n}{\beta_N}\right)^{k_2} \alpha_N^{k_1} \beta_N^{k_2} \\ &\leq \left(\frac{\alpha_n}{\alpha_N}\right)^{\frac{p^{N-1} - p^{N-2}}{2}} \|R_N\|_N \\ &\leq \left(\frac{\alpha_n}{\alpha_N}\right)^{\frac{p^{N-1} - p^{N-2}}{2}} \end{aligned}$$

Here, we have used $0 < \frac{\beta_n}{\beta_N} \leq \frac{\alpha_n}{\alpha_N} < 1$ for $n < N$ in the first inequality. Let us write $\log_p$ for the logarithm with base p. We have

$$\begin{aligned} \log_p \left(\frac{\alpha_n}{\alpha_N}\right) &= -\frac{1}{p^n - p^{n-1}} - \frac{1}{4p^{n-1}} + \frac{1}{p^N - p^{N-1}} + \frac{1}{4p^{N-1}} \\ &= -\frac{(p+3)}{4} \cdot \frac{p^{N-n-1} - \frac{1}{p}}{p^{N-1} - p^{N-2}} \end{aligned}$$

we get

$$\log_p\|R_N\|_n \leq \frac{p^{N-1}-p^{N-2}}{2}\cdot\log_p\left(\frac{\alpha_n}{\alpha_N}\right) = -\frac{(p+3)}{8}\cdot\left(p^{N-n-1}-\frac{1}{p}\right).$$

i.e. we get

$$F \in o_L[x_1,x_2] + p^{m_{N-n}}B_n^\circ.$$

Passing to the quotient $B_n^\circ \twoheadrightarrow A_n^\circ$ shows that $f \in \Lambda_n + p^{m_{N-n}}A_n^\circ$. This finishes the proof of the claim.

Since $m_{N-n} \to \infty$ as $N \to \infty$, one can argue as in the unramified case to deduce that $o_L[\![o_L]\!] \to \mathcal{O}(\mathfrak{X})^\circ$ is an isomorphism. □

5. Associated orders

We now prove Theorem 1.4. Let $L = \mathbb{Q}_{p^2}$ and $\pi = p$, and let L_n be the extension of L generated by the p^n-torsion points of the Lubin–Tate group attached to L and p. Let $G_n = \mathrm{Gal}(L_n/L)$ and let $\Gamma_L = \varprojlim G_n$. Let $\mathcal{A}_n = \{f \in L[G_n] \text{ such that } f(o_n) \subset o_n\}$. There are restriction maps $\pi_n : \mathcal{A}_{n+m} \to \mathcal{A}_n$ for all $m, n \geq 0$.

Theorem 5.1. *For all $n \geq 0$, there exists m such that $\pi_n(\mathcal{A}_{n+m}) = o_L[G_n]$.*

Proof. Let o_∞ be the p-adic completion of $\cup_{n\geq 1} o_n$. By Corollary 5.1.7 of [AB24], we have $\Lambda(X) = o_L[\![o_L]\!]$ if and only if every Γ_L-equivariant o_L-linear map $o_\infty \to o_\infty$ comes from an element of $o_L[\![\Gamma_L]\!]$. We now prove that this condition implies the theorem.

Since L_n/L is abelian, the normal basis theorem implies that $L[G_n]$ is the set of Γ_L-equivariant L-linear maps $L_n \to L_n$. If $f \in \mathrm{End}_{o_L}^{\Gamma_L}(o_\infty)$ then $f(o_n) \subset o_n$ by the Ax-Sen-Tate theorem. The image of such an f in $L[G_n]$ therefore lies in $\mathcal{A}_n$. This gives rise to an isomorphism $\mathrm{End}_{o_L}^{\Gamma_L}(o_\infty) = \varprojlim \mathcal{A}_n$. Suppose that $\mathrm{End}_{o_L}^{\Gamma_L}(o_\infty) = o_L[\![\Gamma_L]\!]$. Let $\mathcal{M}_n = o_L[G_n] \subset \mathcal{A}_n$ and let $\mathcal{Q}_n = \mathcal{A}_n/\mathcal{M}_n$ which is a finite group. We have an exact sequence $0 \to \mathcal{M}_n \to \mathcal{A}_n \to \mathcal{Q}_n \to 0$. The maps $\pi_n : \mathcal{M}_{n+m} \to \mathcal{M}_n$ are surjective, so that the Mittag–Leffler condition holds and the sequence $0 \to \varprojlim \mathcal{M}_n \to \varprojlim \mathcal{A}_n \to \varprojlim \mathcal{Q}_n \to 0$ is exact. If the map $\varprojlim \mathcal{M}_n \to \varprojlim \mathcal{A}_n$ is an isomorphism, then $\varprojlim \mathcal{Q}_n = \{0\}$. Since each $\mathcal{Q}_n$ is a finite group, this implies that for all $n \geq 0$, there exists m such that the image of $\mathcal{Q}_{m+n}$ in $\mathcal{Q}_n$ is zero. □

6. Concluding remarks

For $d := [L : \mathbb{Q}_p] > 2$, a direct generalization of the above arguments is not possible. Let us explain this in the case where L is an unramified extension of degree 3. Let us choose a $\mathbb{Z}_p$-basis $v_1 = 1, v_2, v_3$ of o_L. We keep the notation introduced in section 2 with the choice of radii

$$\rho_n := p^{-\frac{1}{p^n-p^{n-1}}}.$$

In this case, we have

$$A_n = B_n/I$$

with $I = (H_2, H_3)$, where

$$H_2 = \log(1+x_2) - v_2 \log(1+x_1), \quad H_3 = \log(1+x_3) - v_3 \log(1+x_1).$$

The principal symbols $\sigma(H_i) \in \mathrm{gr}^\bullet B_n = k_L[\epsilon, \epsilon^{-1}, X_1, X_2, X_3]$ are given by

$$\sigma(H_2) = \pm\epsilon^{-n}\left[\epsilon\left(X_2^{p^{n-1}} - \overline{v}_2 X_1^{p^{n-1}}\right) + \left(X_2^{p^n} - \overline{v}_2 X_1^{p^n}\right)\right]$$
$$\sigma(H_3) = \pm\epsilon^{-n}\left[\epsilon\left(X_3^{p^{n-1}} - \overline{v}_3 X_1^{p^{n-1}}\right) + \left(X_3^{p^n} - \overline{v}_3 X_1^{p^n}\right)\right].$$

Let us write $\sigma(I)$ for the ideal generated by all principal symbols of all elements of I. Since $\sigma(H_2), \sigma(H_3)$ forms a regular sequence in $\mathrm{gr}^\bullet B_n$, we have by [NVO79, Ch. III, Proposition 3.10] that

$$\sigma(I) = (\sigma(H_2), \sigma(H_3)).$$

Choose elements $\overline{w}_2, \overline{w}_3 \in k_L$ such that $\overline{w}_2^{p^{n-1}} = \overline{v}_2$ and $\overline{w}_3^{p^{n-1}} = \overline{v}_3$, and define for $i = 2, 3$:

$$L_i := X_i - \overline{w}_i X_1$$
$$K_i := X_i^p - \overline{w}_i X_1^p.$$

Then

$$\sigma(I) = \left(\epsilon L_2^{p^{n-1}} + K_2^{p^{n-1}}, \epsilon L_3^{p^{n-1}} + K_3^{p^{n-1}}\right).$$

For $n \geq 2$, the element

$$L_3 K_2 - K_3 L_2 \in \mathrm{gr}^\bullet B_n$$

is not contained in $\sigma(I)$, but

$$(L_3 K_2 - K_3 L_2)^{p^{n-1}} = L_3^{p^{n-1}} K_2^{p^{n-1}} - K_3^{p^{n-1}} L_2^{p^{n-1}} \in \sigma(I).$$

Note that this implies that the quotient norm on A_n is not multiplicative. Indeed, choose any $F \in B_n$ with

$$\sigma(F) = L_3 K_2 - K_3 L_2 \notin \sigma(I).$$

Write $f \in A_n$ for the image of F under the quotient map. The same argument as in Lemma 3.2 shows that $\|F\|_n = \|f\|_{n,q}$. On the other hand, we have

$$\sigma(F^{p^{n-1}}) \in \sigma(I),$$

which implies $\|f^{p^{n-1}}\|_{n,q} < \|f\|_{n,q}^{p^{n-1}}$. This shows that the quotient norm is not power-multiplicative and hence it does not coincide with the spectral norm.

7. AI usage

The article was entirely written by the authors. All of the key ideas entering the proof were of a human nature (except for §5). ChatGPT was used for proofreading and checking mathematical correctness. The specific choices of radii (α_n and β_n) in the ramified case were suggested by ChatGPT in the process of proofreading of an early version of the manuscript.

References

[AB24] Konstantin Ardakov and Laurent Berger, *Bounded functions on the character variety*, Münster J. Math. **17** (2024), no. 1, 1–83.

[ABB18] Konstantin Ardakov and Oren Ben-Bassat, *Bounded linear endomorphisms of rigid analytic functions*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 5, 881–900.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin, 1984.

[BS26] Laurent Berger and Johannes Sprang, *Integer-valued polynomials and p-adic Fourier theory*, Rend. Semin. Mat. Univ. Padova (2026), 1–18, to appear.

[BSX20] Laurent Berger, Peter Schneider, and Bingyong Xie, *Rigid character groups, Lubin-Tate theory, and* (φ, Γ)*-modules*, Mem. Amer. Math. Soc. **263** (2020), no. 1275, v+79.

[NVO79] Constantin Năstăsescu and Freddy Van Oystaeyen, *Graded and filtered rings and modules*, Lecture Notes in Mathematics, vol. 758, Springer, Berlin, 1979. MR 551625

[Sch06] Tobias Schmidt, *Auslander Regularity of p-adic Distribution Algebras*, PhD thesis, Westfälische Wilhelms-Universität Münster, 2006, Available at `https://webdoc.sub.gwdg.de/ebook/dissts/Muenster/Schmidt2006.pdf`.

[ST01] Peter Schneider and Jeremy Teitelbaum, *p-adic Fourier theory*, Doc. Math. **6** (2001), 447–481.

[ST02] ———, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc. **15** (2002), no. 2, 443–468.

[ST03] ———, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), no. 1, 145–196.

UMPA, ENS de Lyon, UMR 5669 du CNRS, 69007 Lyon
Email address: `laurent.berger@ens-lyon.fr`

Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, D-45127 Essen
Email address: `johannes.sprang@uni-due.de`

Ruprecht-Karls-Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 205, D-69120 Heidelberg
Email address: `rsteingart@mathi.uni-heidelberg.de`