GALOIS MEASURES AND THE KATZ MAP

by

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Abstract. — The purpose of this paper is to explain the proofs of the results announced by Nick Katz in 1977, namely a description of “Galois measures for Tate modules of height two formal groups over the ring of integers of a finite unramified extension of $\mathbf{Q}_p$”.

Contents

Introduction 1
1. Preliminaries 3
2. The Katz map 5
3. Surjectivity 6
4. Injectivity 8
References 9

Introduction

The purpose of this paper is to explain the proofs of the results announced by Nick Katz in 1977, namely a theory of “Galois measures for Tate modules of height two formal groups over the ring of integers of a finite unramified extension of $\mathbf{Q}_p$”. As Katz writes in the introduction of [**Kat81**], these results were announced in [**Kat77**] but “the details of this general theory remain unpublished”.

In the author’s joint paper [**AB24**] with Konstantin Ardakov, Katz’ results were proved in the case of a Lubin–Tate formal group attached to a uniformizer of $\mathbf{Q}_{p^2}$ (theorem 1.6.1 of [**AB24**]). In the present paper, we simplify and clarify that proof, and extend it to all height two formal groups. We now describe our results in more detail.

Formal groups. — Let K be a finite unramified extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of K, let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$. Let G be a formal group of dimension 1 and height 2 over $\mathcal{O}_K$. Let $A(G) = \mathcal{O}_K[\![X]\!]$ denote the coordinate ring of G.

Let H be the Cartier dual of G (denoted by $G^\vee$ in [**Kat77**]), so that (cf [**Tat67**]) H is also a formal group of dimension 1 and height 2. The Tate module T_pH parameterizes all formal group homomorphisms $G \to \mathbf{G}_\mathrm{m}$ over $\mathcal{O}_{\mathbf{C}_p}$. They are given by power series $t(X) \in \mathcal{O}_{\mathbf{C}_p} \widehat{\otimes} A(G) = \mathcal{O}_{\mathbf{C}_p}[\![X]\!]$ such that $t(0) = 1$ and $t(X \oplus_G Y) = t(X) \cdot t(Y)$.

The covariant bialgebra of G. — Let $U(G)$ be the covariant bialgebra of G (we use $U(G)$ instead of Katz' algebra $\mathrm{Diff}(G)$ of translation-invariant differential operators on G. We have $U(G) \simeq \mathrm{Diff}(G)$, see lemma 2.1). The set $U(G)$ is the set of all $\mathcal{O}_K$-linear maps $A(G) \to \mathcal{O}_K$ that vanish on some power of the augmentation ideal. If $f, g \in U(G)$, their product is defined by $(f \cdot g)(a(X)) = (f \otimes g)a(X \oplus_G Y)$ if $a(X) \in A(G)$.

Let $\widehat{U}(G)$ denote the set of $\mathcal{O}_K$-linear maps $A(G) \to \mathcal{O}_K$ that are continuous for the (p, X)-adic topology, so that $\widehat{U}(G)$ is the p-adic completion of $U(G)$. Let $\langle \cdot, \cdot \rangle : \widehat{U}(G) \times A(G) \to \mathcal{O}_K$ denote the evaluation pairing.

The Katz map. — Let $C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ denote the $\mathcal{O}_K$-module of Galois continuous functions, namely those functions $f : T_pH \to \mathcal{O}_{\mathbf{C}_p}$ that are continuous and G_K-equivariant: $f(\sigma(t)) = \sigma(f(t))$ for all $t \in T_pH$ and $\sigma \in G_K$.

The evaluation pairing extends to $\langle \cdot, \cdot \rangle : \widehat{U}(G) \times (\mathcal{O}_{\mathbf{C}_p} \widehat{\otimes} A(G)) \to \mathcal{O}_{\mathbf{C}_p}$. If $u \in \widehat{U}(G)$, then $t \mapsto \langle u, t(X) \rangle$ is a Galois continuous function $T_pH \to \mathcal{O}_{\mathbf{C}_p}$. The Katz map is

$$\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p}),$$

defined by $\mathcal{K}(u)(t) = \langle u, t(X) \rangle$ (this is the map $(*)$ on page 59 of [**Kat77**]).

Galois measures. — Let S be a p-adically complete and separated flat $\mathcal{O}_K$-algebra. Applying the functor $\mathrm{Hom}_{\mathcal{O}_K}(\cdot, S)$ to $\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ gives an S-linear map (the map $(**)$ on page 59 of [**Kat77**])

$$\mathcal{K}^* : \mathrm{Hom}_{\mathcal{O}_K}(C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p}), S) \to S \widehat{\otimes} A(G).$$

If $f(X) \in S \widehat{\otimes} A(G)$, let $\psi f \in S[1/p] \widehat{\otimes} A(G)$ be defined by (compare with §III of [**Col79**])

$$(\psi f)([p]_G(X)) = \frac{1}{p^2} \cdot \sum_{\pi \in G[p]} f(X \oplus_G \pi).$$

We say that $f(X) \in S \widehat{\otimes} A(G)$ is ψ-integral if $\psi^n f \in S \widehat{\otimes} A(G)$ for all $n \geq 0$. The main result claimed by Katz (see page 60 of [**Kat77**]) is the following.

Theorem A. — *The map $\mathcal{K}^*$ is injective, and its image is the set of ψ-integral power series $f(X) \in S \widehat{\otimes} A(G)$.*

Moreover, a Galois measure μ is supported on $T_p^\times H$ if and only if $\psi(\mathcal{K}^*(\mu)) = 0$.

Main results. — The main result of this paper is a proof of theorem B below. Choose an element $t = (t_0, t_1, \ldots) \in T_p^\times H = T_p H \setminus p \cdot T_p H$ and let $\mathcal{K}_t : \widehat{U}(G) \to \mathcal{O}_{\mathbf{C}_p}$ denote the map $u \mapsto \mathcal{K}(u)(t)$. Let K_∞ denote the completion of the field generated over K by the t_n for $n \geq 1$. Since $\mathcal{K}(u)$ is G_K-equivariant, the Ax-Sen-Tate theorem implies that $\mathcal{K}(u)(t) \in \mathcal{O}_{K_\infty}$ for all $t \in T_p H$.

Theorem B. — *The map $\mathcal{K}$ is injective, and the map $\mathcal{K}_t : \widehat{U}(G) \to \mathcal{O}_{K_\infty}$ is surjective.*

Theorem A follows from theorem B by mostly formal arguments, that are carried out in detail in [**AB24**] for the Lubin–Tate case, see in particular coro 3.4.10 of ibid (the hypothesis "$\tau : G_L \to o_L^\times$ is surjective" is replaced here by "G_K acts transitively on $T_p^\times H$", cf lemma 1.1 below). To keep this paper short, we focus on proving theorem B.

Besides clarifying the arguments of §3 of [**AB24**], our proof also shows that $\widehat{U}(G)$ has a perfectoid-like nature. In particular, the injectivity of $\mathcal{K}$ is related to the following result in p-adic Hodge theory (cf 5.1.4 of [**Fon94**], and its Lubin–Tate generalizations such as prop 9.6 of [**Col02**]): $\{x \in \widetilde{\mathbf{A}}^+$ such that $\theta \circ \varphi^n(x) = 0$ for all $n \geq 0\} = \pi \cdot \widetilde{\mathbf{A}}^+$.

1. Preliminaries

In all this paper, we let $q = p^2$. The multiplication-by-p map on G is given by a power series $[p]_G(X) = \sum_{i \geq 1} r_i X^i$ with $r_1 = p$ and $r_i \in p\mathcal{O}_K$ for $i \leq q-1$ and $r_q \in \mathcal{O}_K^\times$.

Lemma 1.1. — *If $t, u \in T_p^\times H$, there exists $\sigma \in G_K$ such that $u = \sigma(t)$.*

Proof. — For all $n \geq 0$, in the Weierstrass factorization of $[p^{n+1}]_H(X)/[p^n]_H(X)$, the distinguished polynomial is Eisenstein, as its constant term is p and $K/\mathbf{Q}_p$ is unramified.

Its roots are therefore conjugate by G_K. □

Lemma 1.2. — *There is a change of variables such that $[p]_G(X) = pX + \mathrm{O}(X^q)$.*

Proof. — If $r(X) = \sum_{i=1}^{q-1} r_i X^i$, then $r(X) + X^q$ is a Lubin–Tate power series for the uniformizer p, so by lemma 1 of [**LT65**] there exists a reversible power series $h(X)$ such that $h^{-1} \circ (r(X) + X^q) \circ h = pX + X^q$. We then have $h^{-1} \circ [p]_G(X) \circ h = pX + \mathrm{O}(X^q)$. □

A coordinate satisying the above conditions is said to be clean. Let $\log_G(X) \in K[\![X]\!]$ denote the logarithm attached to the formal group G.

Lemma 1.3. — *If X is clean, then* $\log_G(X) = X + \alpha \cdot X^q/p + \mathrm{O}(X^{q+1})$ *for some* $\alpha \in \mathcal{O}_K^\times$.

Let t be an element of $T_pH = \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(G, \mathbf{G}_\mathrm{m})$, and let $t(X) = 1 + \sum_{i \geq 1} a_i(t) X^i \in \mathcal{O}_{\mathbf{C}_p}[\![X]\!]$ be the corresponding Hom from G to $\mathbf{G}_\mathrm{m}$.

Lemma 1.4. — *If* $\sigma \in G_K$ *then* $a_n(\sigma(t)) = \sigma(a_n(t))$ *for all* $n \geq 1$.

Proof. — The isomorphism $T_pH = \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(G, \mathbf{G}_\mathrm{m})$ is G_K-equivariant. □

Lemma 1.5. — *We have* $a_n(t) = a_1(t)^n/n!$ *if* $n \leq q-1$ *and* $a_q(t) = a_1(t)^q/q! + \alpha \cdot a_1(t)/p$.

Proof. — Since $t(X) = \exp(a_1(t) \cdot \log_G(X))$, the claim follows from lemma 1.3. □

Proposition 1.6. — *If* $t \in T_p^\times H$, *the abscissa of the breakpoints of the Newton polygon of* $t(X) - 1$ *are the* p^m *with* $m \geq 0$, *and* $\mathrm{val}_p(a_{p^m}(t)) = 1/p^{m-1}(q-1)$ *for all* $m \geq 0$.

Proof. — The Newton polygon of $t(X) - 1$ is independant of the choice of coordinate X, and we choose a clean coordinate on G. We first prove that $\mathrm{val}_p(a_1(t)) \leq p/(q-1)$. Let ζ be a primitive p-th root of 1. Since $t \in T_p^\times H$, there exists $\eta \in G[p]$ such that $t(\eta) = \zeta$. We have $\mathrm{val}_p(\eta) = 1/(q-1)$ and $\mathrm{val}_p(\zeta - 1) = 1/(p-1)$. Since X is a clean coordinate on G, we have $a_n(t) = a_1(t)^n/n!$ if $n \leq q-1$ by lemma 1.5. If $\mathrm{val}_p(a_1) > p/(q-1)$, then for $n < q$

$$\mathrm{val}_p(a_n(t)\eta^n) = \mathrm{val}_p\left(a_1(t)^n \frac{\eta^n}{n!}\right) > \frac{np+n}{q-1} - \frac{n - s_p(n)}{p-1} = \frac{s_p(n)}{p-1} \geq \frac{1}{p-1},$$

while $\mathrm{val}_p(\eta^n) = n/(q-1) > 1$ if $n \geq q$. It is therefore not possible to have $t(\eta) - 1 = \zeta - 1$, so that $\mathrm{val}_p(a_1(t)) \leq p/(q-1)$.

For every $m \geq 1$, $t(X) - 1$ has $p^m - p^{m-1}$ zeroes in $G[p^m] \setminus G[p^{m-1}]$ and they are all of valuation $1/(q^m - q^{m-1})$. Since $\sum_{m \geq 1} (p^m - p^{m-1})/(q^m - q^{m-1}) = p/(q-1)$, the theory of Newton polygons tells us that $\mathrm{val}_p(a_1(t)) = p/(q-1)$, and then that $t(X) - 1$ cannot have other zeroes.

The valuations of the $a_{p^m}(t)$ can then be read off the Newton polygon of $t(X) - 1$. □

Remark 1.7. —

1. Compare with coro 3.5.8 of [**AB24**];
2. Compare with the proof of lemma 13 of [**Box86**];
3. Prop 1.6 applied to $m = 0$ gives us lemma 1 on page 62 of [**Kat77**];
4. Lemma 1.5 and the fact that $\mathrm{val}_p(a_q(t)) = 1/p(q-1)$ imply that $\mathrm{val}_p(a_1(t) \cdot (a_1(t)^{q-1}/q! + \alpha/p)) = 1/p(q-1)$ and hence $\mathrm{val}_p(a_1(t)^{q-1}p/\alpha q! + 1) = 1 - 1/p$, compare with lemma 2 on page 63 of [**Kat77**].

2. The Katz map

Let $C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ denote the set of Galois continuous functions $f : T_pH \to \mathcal{O}_{\mathbf{C}_p}$, namely those maps that are continuous and G_K-equivariant: $f(\sigma(t)) = \sigma(f(t))$ for all $t \in T_pH$ and $\sigma \in G_K$. By lemma 1.1, any two elements of $p^n \cdot T_p^\times H$ are Galois conjugates, so if we fix $t \in T_p^\times H$, a function $f \in C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ is determined by $\{f(p^n t)\}_{n\geq 0}$. In addition, we have $f(p^n t) \to f(0)$ as $n \to +\infty$ by continuity, and $f(0) \in \mathcal{O}_K$ by the Ax-Sen-Tate theorem since $f(0)$ is fixed by G_K.

Write $t = (t_0, t_1, \ldots) \in T_pH$ and let K_n denote the extension of K generated by t_n and let K_∞ be the completion of $\cup_{n\geq 0} K_n$. By the Ax-Sen-Tate theorem, we have $f(p^n t) \in \mathcal{O}_{K_\infty}$ for all n. Let $\prod'_{n\geq 0} \mathcal{O}_{K_\infty}$ denote the $\mathcal{O}_K$-algebra of sequences $\{f_n\}_{n\geq 0}$ with $f_n \in \mathcal{O}_{K_\infty}$ and such that $\{f_n\}_{n\geq 0}$ converges to an element of $\mathcal{O}_K$ as $n \to +\infty$. The map $f \mapsto \{f(p^n t)\}_{n\geq 0}$ gives an isomorphism $C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p}) \to \prod'_{n\geq 0} \mathcal{O}_{K_\infty}$.

Recall that $U(G)$ is the covariant bialgebra of G, namely the set of all $\mathcal{O}_K$-linear maps $A(G) \to \mathcal{O}_K$ that vanish on some power of the augmentation ideal. If $f, g \in U(G)$, their product is defined by $(f \cdot g)(a(X)) = (f \otimes g)a(X \oplus_G Y)$. In [**Kat77**] and [**Kat81**], Katz introduces the algebra $\mathrm{Diff}(G)$ of translation-invariant differential operators on G.

Lemma 2.1. — *If $D \in \mathrm{Diff}(G)$, then $[f \mapsto D(f)(0)] \in U(G)$, and this map gives rise to an isomorphism between the $\mathcal{O}_K$-algebras $\mathrm{Diff}(G)$ and $U(G)$.*

Proof. — The proof is the same as that in §3.2 of [**AB24**] for the Lubin–Tate case: for $n \geq 0$, let $u_n \in U(G)$ be the map $u_n : A(G) \to \mathcal{O}_K$ given by $u_n(\sum_{i\geq 0} b_i X^i) = b_n$. We have $u_n \cdot u_m = \sum_{k\geq 0} s_{k,n,m} u_k$ where $(X \oplus_G Y)^k = \sum_{n,m\geq 0} s_{k,n,m} X^n Y^m$ for $k \geq 0$, and we get the same structure constants as in $\mathrm{Diff}(G)$, see for instance (1.2) of [**Kat81**]. □

Recall that $\widehat{U}(G)$ denotes the set of $\mathcal{O}_K$-linear maps $A(G) \to \mathcal{O}_K$ that are continuous for the (p, X)-adic topology, so that $\widehat{U}(G)$ is the p-adic completion of $U(G)$. The evaluation pairing $\langle\cdot,\cdot\rangle : \widehat{U}(G) \times A(G) \to \mathcal{O}_K$ extends to $\langle\cdot,\cdot\rangle : \widehat{U}(G) \times (\mathcal{O}_{\mathbf{C}_p} \widehat{\otimes} A(G)) \to \mathcal{O}_{\mathbf{C}_p}$.

Definition 2.2. — The Katz map is the map $\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ defined by $\mathcal{K}(u)(t) = \langle u, t(X)\rangle$.

The map $\mathcal{K}(u) : T_pH \to \mathcal{O}_{\mathbf{C}_p}$ is Galois continuous by lemma 1.4. The Katz map $\mathcal{K}$ is an $\mathcal{O}_K$-algebra homomorphism (the proof is the same as that of lemma 3.3.5 of [**AB24**]). Let $\varphi_C : C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p}) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ denote the map given by $(\varphi_C f)(t) = f(pt)$. We have a map $U(\varphi_G) : \widehat{U}(G) \to \widehat{U}(G)$ coming by duality from the map $\varphi_G : A(G) \to A(G)$ given by $a(X) \mapsto a([p]_G(X))$.

Lemma 2.3. — *We have $\mathcal{K} \circ U(\varphi_G) = \varphi_C \circ \mathcal{K}$.*

Proof. — The proof is the same as that of lemma 3.3.6 of [**AB24**], where $U(\varphi_G)$ is denoted by φ^*. □

Remark 2.4. — Since $\widehat{U}(G) = \{\sum_{n\geq 0} \lambda_n u_n$ with $\{\lambda_n\}_{n\geq 0} \in c^0(\mathcal{O}_K)\}$, and $u_n(t(X)) = a_n(t)$, we can reformulate theorem B as follows.

1. If $t \in T_p^\times H$, every $x \in \mathcal{O}_{K_\infty}$ can be written as $x = \sum_{n\geq 0} \lambda_n a_n(t)$ with $\{\lambda_n\} \in c^0(\mathcal{O}_K)$;
2. If $\{\lambda_n\} \in c^0(\mathcal{O}_K)$ and $\sum_{n\geq 0} \lambda_n a_n(t) = 0$ for all $t \in T_p H$, then $\lambda_n = 0$ for all n.

3. Surjectivity

We now prove that $\mathcal{K}_t : \widehat{U}(G) \to \mathcal{O}_{K_\infty}$ is surjective if $t = (t_0, t_1, \ldots) \in T_p^\times H$.

For $n \geq 0$, let $G_n = G[p^n]$ and $H_n = H[p^n]$ and recall that $K_n = K(t_n)$. The inclusion $G_n \to G_{n+1}$ is (§2.3 of [**Tat67**]) the Cartier dual of $[p]_H : H_{n+1} \to H_n$ (and vice versa). We have $A(G_n) = \mathcal{O}_K[\![X]\!]/\varphi_G^n(X)$. Let $U(G_n) = \mathrm{Hom}_{\mathcal{O}_K}(A(G_n), \mathcal{O}_K) \subset \widehat{U}(G)$. Cartier duality gives an isomorphism $U(G_n) = A(H_n)$, so that $U(G_n) = \mathcal{O}_K[\![Y_n]\!]/\varphi_H^n(Y_n)$. On $U(G_n)$, we have $U([p]_G) = [p]_H$ so that $U(\varphi_G) = \varphi_H$. The natural inclusion $U(G_n) \to U(G_{n+1})$ is the map $A(H_n) \to A(H_{n+1})$ that comes from $[p]_H : H_{n+1} \to H_n$. Its image is $\varphi_H(U(G_{n+1}))$. If $U(G_n)_K = K \otimes_{\mathcal{O}_K} U(G_n)$, then

$$U(G_n)_K = K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![Y_n]\!]/\varphi_H^n(Y_n) = K_n \times K_{n-1} \times \cdots \times K_1 \times K_0.$$

Fix $t \in T_p^\times H$. For $n \geq 0$, let $\kappa_n : \widehat{U}(G) \to \mathcal{O}_{K_\infty}$ denote the map $u \mapsto \mathcal{K}(u)(p^n t)$. Lemma 2.3 implies that $\kappa_n(\varphi_H(u)) = \kappa_{n+1}(u)$ if $u \in U(G_m)$ for some $m, n \geq 0$.

Proposition 3.1. — *For all $n \geq 1$, the image of the map $\kappa_0 : U(G_n)_K \to K_\infty$ is K_n.*

Proof. — We first prove that the image of $\kappa_0 : U(G_n)_K \to K_\infty$ is contained in K_n. By Cartier duality, we have $H_n = \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(G_n, \mathbf{G}_\mathrm{m})$, and the map $t_n(X) \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(G_n, \mathbf{G}_\mathrm{m})$ corresponding to t_n is given by $t(X) \bmod \varphi_G^n(X)$. We then have

$$t_n(X) \in \mathcal{O}_{\mathbf{C}_p} \otimes A(G_n) = \mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(\mathcal{O}_{\mathbf{C}_p} \otimes U(G_n), \mathcal{O}_{\mathbf{C}_p})$$

The resulting map $U(G_n)_K \to \mathbf{C}_p$ is given by $u \mapsto u(t(X) \bmod \varphi_G^n(X)) = u(t)$, namely the restriction of κ_0 to $U(G_n)_K$. For all $\sigma \in G_{K_n}$, $\sigma(t_n) = t_n$ and hence $\sigma(t_n(X)) = t_n(X)$ so that $\sigma(\kappa_0(u)) = \kappa_0(u)$ if $u \in U(G_n)_K$. By the Ax-Sen-Tate theorem, $\kappa_0(u) \in K_n$.

We now prove that the image of κ_0 is K_n. Let $[p^n]_G(X) = f(X)u(X)$ be the Weierstrass factorization of $[p^n]_G(X)$, where $f(X)$ is a distinguished polynomial of degree q^n and $u(X)$

is a unit. Thanks to Weierstrass division by $f(X)$, we can write

$$\mathcal{O}_K[\![X]\!] = (\oplus_{i=0}^{q^n-1}\mathcal{O}_K X^i) \oplus f(X)\mathcal{O}_K[\![X]\!].$$

For $0 \leq r \leq q^n - 1$, let $w_r \in U(G_n)_K$ be the $\mathcal{O}_K$-linear form that maps X^i to δ_{ir} for $i \leq q^n - 1$ and maps $f(X)\mathcal{O}_K[\![X]\!]$ to 0. It maps X^{j+q^n} to $p\mathcal{O}_K$ for all $j \geq 0$ since $f(X) = f_{q^n}X^{q^n} + pg(X)$ with $f_{q^n} \in \mathcal{O}_K^\times$ and hence $f_{q^n}X^{j+q^n} = X^j f(X) - pX^j g(X)$.

By prop 1.6, we have $\mathrm{val}_p(a_{p^m}(t)) = 1/p^{m-1}(q-1)$ for all $m \geq 0$. In particular, setting $m = 2n-1$, so that $p^m = q^n/p \leq q^n - 1$, we have $\mathrm{val}_p(a_{p^m}(t)) = 1/q^{n-1}(q-1)$. We have

$$\kappa_0(w_{p^{2n-1}}) = w_{p^{2n-1}}(t(X)) \equiv a_{p^m}(t) \bmod p,$$

so that $\kappa_0(U(G_n)_K)$ contains a uniformizer of K_n and is therefore equal to K_n. □

Remark 3.2. — Compare with (the proof of) prop 3.6.7 of [**AB24**].

Proposition 3.3. — *Take $n \geq 1$ and fix an isomorphism $U(G_n) = \mathcal{O}_K[\![Y]\!]/\varphi_H^n(Y)$.*

There exists $(u_n, u_{n-1}, \dots, u_1, u_0)$ with $u_k \in K_\infty$ and $[p]_H(u_k) = u_{k-1}$ for $1 \leq k \leq n$ and $u_1 \neq 0$ and $u_0 = 0$ such that the map $\kappa_j : U(G_n)_K \to K_\infty$ is given by $P(Y) \mapsto P(u_{n-j})$ if $j \leq n$, and by $P(Y) \mapsto P(0)$ if $j \geq n$.

Proof. — For all j, there exists a root u_{n-j} of $\varphi_H^n(Y)$ in $\mathfrak{m}_{K_\infty}$ such that the K-algebra map $\kappa_j : K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![Y]\!]/\varphi_H^n(Y) \to K_\infty$ is given by $P(Y) \mapsto P(u_{n-j})$. In addition, we have $\kappa_{j-1}(\varphi_H(Y)) = \kappa_j(Y)$ so that $[p]_H(u_{n-j+1}) = u_{n-j}$.

We have $\kappa_n(Y) = \kappa_0(\varphi_H^n(Y)) = 0$ so that $u_0 = 0$, and finally since $\kappa_0 : U(G_1)_K \to K_1$ is surjective by prop 3.1, and

$$\kappa_{n-1}(U(G_n)_K) = \varphi_C^{n-1} \circ \kappa_0(U(G_n)_K) = \kappa_0(\varphi_H^{n-1}(U(G_n))_K) = \kappa_0(U(G_1)_K),$$

the element u_1 generates K_1 over K and so $u_1 \neq 0$. □

Corollary 3.4. — *The map $\kappa_0 : U(G_n) \to \mathcal{O}_{K_n}$ is surjective for all $n \geq 0$.*

Proof. — The map $\kappa_0 : U(G_n) \to \mathcal{O}_{K_\infty}$ is given by $P(Y_n) \mapsto P(u_n)$ as in prop 3.3, so that its image is $\mathcal{O}_{K_n}$ since u_n is a uniformizer of K_n. □

Corollary 3.5. — *The map $\kappa_0 : \widehat{U}(G) \to \mathcal{O}_{K_\infty}$ is surjective.*

This proves the surjectivity part of Theorem B since $\kappa_0 = \mathcal{K}_t$.

4. Injectivity

We now prove the injectivity of the Katz map $\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$. Recall that $U(G_n) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\![X]\!]/\varphi_G^n(X), \mathcal{O}_K)$ and that we have an injection $U(G_n) \to \widehat{U}(G)$.

Lemma 4.1. — *The $\mathcal{O}_K$-module $\cup_{n\geq 1} U(G_n)$ is p-adically dense in $\widehat{U}(G)$.*

Proof. — Take $k, d \geq 0$ and $u \in \widehat{U}(G)$ such that $u(X^d\mathcal{O}_K[\![X]\!]) \subset p^k\mathcal{O}_K$, and $n \geq 1$ such that $[p^n]_G(X) \in (X^d, p^k)$. Let $[p^n]_G(X) = f(X)u(X)$ be the Weierstrass factorization of $[p^n]_G(X)$, where $f(X)$ is distinguished of degree q^n and $u(X)$ is a unit. Write $\mathcal{O}_K[\![X]\!] = (\oplus_{i=0}^{q^n-1}\mathcal{O}_K X^i) \oplus f(X)\mathcal{O}_K[\![X]\!]$. Define $w \in U(G_n)$ by $w = u$ on $1, X, \dots, X^{q^n-1}$ and $w = 0$ on $f(X)\mathcal{O}_K[\![X]\!]$. If $j \geq 0$, then $X^{j+q^n} \in f(X)\mathcal{O}_K[\![X]\!] + (\oplus_{i=d+j}^{q^n+j-1}\mathcal{O}_K X^i) + p^k\mathcal{O}_K[\![X]\!]$. By induction on j, we have $w(X^{j+q^n}) \in p^k\mathcal{O}_K$ for all $j \geq 0$ and therefore $w - u \in p^k\widehat{U}(G)$. □

The group G_K acts transitively on $T_p^\times H$ by lemma 1.1, so that in the isomorphisms $U(G_n) = \mathcal{O}_K[\![Y_n]\!]/\varphi_H^n(Y_n)$, it is possible to change the coordinates Y_n to get a common $u = (\dots, u_1, u_0)$ for all $U(G_n)_K$ in prop 3.3. We can then write $U(G_n) = \mathcal{O}_K[\![\varphi_H^{-n}(Y)]\!]/Y$ for each n, with each transition map $U(G_{n-1}) \to U(G_n)$ sending Y to Y and the map κ_j sending $\varphi_H^{-n}(Y)$ to u_{n-j}. Lemma 4.1 and the fact that $U(G_n) \cap p \cdot \widehat{U}(G) = p \cdot U(G_n)$ for all $n \geq 0$ imply that $\widehat{U}(G)$ is the p-adic completion of $\cup_{n\geq 0}\mathcal{O}_K[\![\varphi_H^{-n}(Y)]\!]/Y$.

Let $\mathbf{A}$ be the p-adic completion of $\cup_{n\geq 0}\mathcal{O}_K[\![\varphi_H^{-n}(Y)]\!]$ and let $\theta : \mathbf{A} \to \mathcal{O}_{K_\infty}$ be the $\mathcal{O}_K$-linear ring homomorphism that sends $\varphi_H^{-n}(Y)$ to u_n so that $\widehat{U}(G) = \mathbf{A}/Y$ and $\kappa_0 = \theta$.

Let $\mathbf{E}$ denote the ring $\cup_{n\geq 0} k[\![\varphi_H^{-n}(Y)]\!]$. The valuation val_Y is compatibly defined on each $k[\![\varphi_H^{-n}(Y)]\!]$ and hence on $\mathbf{E}$ (indeed, $\varphi_H(Y) \in Y^q \cdot k[\![Y]\!]^\times$ so that if $x \in k[\![Y_n]\!] \subset k[\![Y_{n+1}]\!]$ with $Y_n = \varphi_H(Y_{n+1})$, then $\mathrm{val}_{Y_{n+1}}(x) = q \cdot \mathrm{val}_{Y_n}(x)$).

Lemma 4.2. — *If $x \in \mathbf{E}$, then $x \in Y \cdot \mathbf{E}$ if and only if $\mathrm{val}_Y(x) \geq 1$.*

Let $\theta : \mathbf{E} \to \mathcal{O}_{K_\infty}/p$ be the k-linear ring homomorphism that sends $\varphi_H^{-n}(Y)$ to u_n so that $\mathbf{E} = \mathbf{A}/p\mathbf{A}$ with compatible θ.

Lemma 4.3. — *We have $\ker(\theta : \mathbf{E} \to \mathcal{O}_{K_\infty}/p) = Y/\varphi_H^{-1}(Y) \cdot \mathbf{E}$.*

Proof. — It is enough to prove that for all $n \geq 1$, $\ker(\theta : k[\![\varphi_H^{-n}(Y)]\!] \to \mathcal{O}_{K_\infty}/p)$ is generated by $Y/\varphi_H^{-1}(Y)$. Let $Q_n(X)$ be the minimal polynomial of u_n over K, so that $Q_n(X) \in \mathcal{O}_K[X]$ is monic of degree $q^{n-1}(q-1)$. If $P(X) \in \mathcal{O}_K[\![X]\!]$, we can write it as $P = SQ_n + R$ with $\deg R < q^{n-1}(q-1)$. If $P(u_n) \in p \cdot \mathcal{O}_{K_n}$, then $R(u_n) \in p \cdot \mathcal{O}_{K_n}$ and since $\mathrm{val}_p(u_n) = 1/q^{n-1}(q-1)$, this implies that $R(X) \in p \cdot \mathcal{O}_K[X]$. The claim now follows from this, and from the fact that $\theta(\varphi_H^{-n}(Y)) = u_n$ so that $\ker(\theta)$ is generated by $\varphi_H^{-n}(Y)^{q^{n-1}(q-1)}$ and hence by $Y/\varphi_H^{-1}(Y)$ since $\varphi_H(Y) = Y^q \cdot f(Y)$ with $f(Y) \in k[\![Y]\!]^\times$. □

Proposition 4.4. — *We have* $\ker(\theta : \mathbf{A} \to \mathcal{O}_{K_\infty}) = Y/\varphi_H^{-1}(Y) \cdot \mathbf{A}$.

Proof. — If $x \in \ker(\theta)$, then $\overline{x} \in \mathbf{E}$ is also killed by θ and is divisible by $Y/\varphi_H^{-1}(Y)$ in $\mathbf{E}$ by lemma 4.3. We can therefore write $x = Y/\varphi_H^{-1}(Y) \cdot x_1 + py_1$ with $x_1, y_1 \in \mathbf{A}$ and $\theta(y_1) = 0$. By induction, we can write $x = Y/\varphi_H^{-1}(Y) \cdot x_k + p^k y_k$ with $x_k, y_k \in \mathbf{A}$ and $\theta(y_k) = 0$ for all $k \geq 1$. Since $\mathbf{A}$ is p-adically complete, this implies the claim. □

Proposition 4.5. — *We have* $\{x \in \mathbf{A},\ \theta \circ \varphi_H^n(x) = 0$ *for all* $n \geq 0\} = Y \cdot \mathbf{A}$.

Proof. — One inclusion is clear, since $\theta \circ \varphi_H^n(Y) = 0$ for all $n \geq 0$. We now prove the reverse inclusion. Prop 4.4 shows that $\ker(\theta) = Y/\varphi_H^{-1}(Y) \cdot \mathbf{A}$ and therefore that for all $j \geq 0$, $\ker(\theta \circ \varphi_H^j) = \varphi_H^{-j}(Y)/\varphi_H^{-j-1}(Y) \cdot \mathbf{A}$. For $n \geq 0$, let I_n denote the set of $x \in \mathbf{A}$ such that $(\theta \circ \varphi_H^j)(x) = 0$ for $0 \leq j \leq n$. Since $(\theta \circ \varphi_H^\ell)(\varphi_H^{-j}(Y)/\varphi_H^{-j-1}(Y)) \neq 0$ if $\ell < j$, and $Y/\varphi_H^{-1}(Y) \cdot \varphi_H^{-1}(Y)/\varphi_H^{-2}(Y) \cdots \varphi_H^{-(n-1)}(Y)/\varphi_H^{-n}(Y) = Y/\varphi_H^{-n}(Y)$, we have $I_n = Y/\varphi_H^{-n}(Y) \cdot \mathbf{A}$. Let $I = \cap_{n \geq 0} I_n = \{x \in \mathbf{A},\ \theta \circ \varphi_H^n(x) = 0 \text{ for all } n \geq 0\}$.

The above reasoning and lemma 4.2 imply that in $\mathbf{E}$, we have $\overline{I} = Y \cdot \mathbf{E}$. Hence if $x \in I$, then $\overline{x} \in Y \cdot \mathbf{E}$. We can therefore write $x = Y \cdot x_1 + py_1$ with $x_1 \in \mathbf{A}$ and $y_1 \in I$. By induction, we can write $x = Y \cdot x_k + p^k y_k$ with $x_k \in \mathbf{A}$ and $y_k \in I$ for all $k \geq 1$. Since $\mathbf{A}$ is p-adically complete, this implies the claim. □

Remark 4.6. — Compare with prop 9.6 of [**Col02**] (and 5.1.4 of [**Fon94**]), noting however that our $\mathbf{E}$ is not complete for the Y-adic topology. We could actually replace $\mathbf{E}$ and $\mathbf{A}$ above with their Y-adic completions, since in any case $\widehat{U}(G) = \mathbf{A}/Y$. In the Lubin–Tate case, we would then have $\mathbf{E} = \widetilde{\mathbf{E}}_K^+$ and $\mathbf{A} = \widetilde{\mathbf{A}}_K^+$ in the notation of ibid.

In general, $\varphi_H(Y) = f(Y^q)$ in $k[\![Y]\!]$ for some reversible $f(Y)$ so that $\mathbf{E}$ is still perfect.

Corollary 4.7. — *The map* $\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ *is injective.*

Proof. — If we write $C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p}) = \prod'_{n \geq 0} \mathcal{O}_{K_\infty}$, then $\mathcal{K} : \widehat{U}(G) \to C^0_{\mathrm{Gal}}(T_pH, \mathcal{O}_{\mathbf{C}_p})$ comes from $\{\theta \circ \varphi_H^n\}_{n \geq 0} : \mathbf{A} \to \prod'_{n \geq 0} \mathcal{O}_{K_\infty}$ and the claim results from prop 4.5 above and the fact that $\widehat{U}(G) = \mathbf{A}/Y$ by lemma 4.1. □

This finishes the proof of Theorem B.

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