INTEGER-VALUED POLYNOMIALS AND p-ADIC FOURIER THEORY

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Abstract. The goal of this paper is to give a numerical criterion for an open question in p-adic Fourier theory. Let F be a finite extension of $\mathbf{Q}_p$. Schneider and Teitelbaum defined and studied the character variety $\mathfrak{X}$, which is a rigid analytic curve over F that parameterizes the set of locally F-analytic characters $\lambda : (o_F, +) \to (\mathbf{C}_p^\times, \times)$. Determining the structure of the ring $\Lambda_F(\mathfrak{X})$ of bounded-by-one functions on $\mathfrak{X}$ defined over F seems like a difficult question. Using the Katz isomorphism, we prove that if $F = \mathbf{Q}_{p^2}$, then $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$ if and only if the o_F-module of integer-valued polynomials on o_F is generated by a certain explicit set. Some computations in SageMath indicate that this seems to be the case.

Contents

1. Introduction

Let F be a finite extension of $\mathbf{Q}_p$ with ring of integers o_F. Let $\mathbf{C}_p$ denote the completion of an algebraic closure of $\mathbf{Q}_p$. In their work on p-adic Fourier theory [ST01], Schneider and Teitelbaum defined and studied the character variety $\mathfrak{X}$. This character variety is a rigid analytic curve over F that parameterizes the set of locally F-analytic characters $\lambda : (o_F, +) \to (\mathbf{C}_p^\times, \times)$. Let $\Lambda_F(\mathfrak{X})$ denote the ring of functions on $\mathfrak{X}$ defined over F whose norms are bounded above by 1. If $\mu \in o_F[\![o_F]\!]$ is a measure on o_F, then $\lambda \mapsto \mu(\lambda)$ gives rise to such a function $\mathfrak{X} \to \mathbf{C}_p$. The resulting map $o_F[\![o_F]\!] \to \Lambda_F(\mathfrak{X})$ is injective. We do not know of any example of an element of $\Lambda_F(\mathfrak{X})$ that is not in the image of the above map.

Question. *Do we have $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$?*

This question seems to be quite difficult, and is extensively studied in [AB24]. The goal of our paper is to give, for $F = \mathbf{Q}_{p^2}$, a simple criterion for the above question, that can be checked numerically. Numerical evidence then seems to indicate that the answer

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to the question is "yes" for $F = \mathbf{Q}_{p^2}$. We now formulate this criterion. For the time being, let $F \neq \mathbf{Q}_p$ be any finite proper extension of $\mathbf{Q}_p$, let π be a uniformizer of o_F and let LT denote the Lubin–Tate formal group attached to π. Once we have chosen a coordinate X on LT, we have a formal addition law $T \oplus U \in o_F[\![T,U]\!]$ and endomorphisms $[a](X) \in o_F[\![X]\!]$ for all $a \in o_F$.

Let Int denote the set of integer-valued polynomials on o_F, namely those polynomials $P(T) \in F[T]$ such that $P(o_F) \subset o_F$. If $f(X) \in o_F[\![X]\!]$, there exist polynomials $c_{f,n}(T) \in$ Int for all $n \geq 0$ such that $f([a](X)) = \sum_{n \geq 0} c_{f,n}(a) X^n$.

Definition. *If M is a subset of $o_F[\![X]\!]$, let* $\mathrm{Pol}(M)$ *denote the sub o_F-module of* Int *generated by the $c_{f,n}$ with $f \in M$ and $n \geq 0$.*

In particular, the module Pol defined in §1.5 of [AB24] is equal to $\mathrm{Pol}(\{1, X, X^2, \ldots\}) = \mathrm{Pol}(o_F[\![X]\!])$, and theorem 1.5.1 of [AB24] states that if $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$, then Pol = Int. However, the reverse implication is not true, and the goal of our paper is to provide an analogous "if and only if" statement, at least when $F = \mathbf{Q}_{p^2}$. The ring $o_F[\![X]\!]$ is equipped with an operator φ defined by $\varphi(f)(X) = f([\pi](X))$ and an operator ψ given by

$$\varphi(\psi(f(X))) = \frac{1}{\pi} \cdot \mathrm{Tr}_{o_F[\![X]\!]/\varphi(o_F[\![X]\!])}(f(X)).$$

Note that $\psi(f) = 0$ if and only if $\sum_{\eta \in \mathfrak{m}_{\mathbf{C}_p}, [\pi](\eta)=0} f(X \oplus \eta) = 0$.

The first result of this paper is the following criterion for the above question.

Theorem A. *If $F = \mathbf{Q}_{p^2}$, then* $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!] \Leftrightarrow \mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) = \mathrm{Int}$.

Let us remark that the question of whether $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) = \mathrm{Int}$ depends neither on the choice of a coordinate on LT nor on the choice of the uniformizer π used in the definition of LT (for instance, it is equivalent to $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$ by the above theorem). The appendix of [AB24] is devoted to checking numerically that Pol = Int for various fields F. We adapt these methods and give numerical evidence that $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) = \mathrm{Int}$ when $F = \mathbf{Q}_{p^2}$. This is the first compelling evidence in favor of the fact that $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$, at least for $F = \mathbf{Q}_{p^2}$. In addition, we prove theorem B below, which implies that $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ is p-adically dense in Int.

Theorem B. *For all F, we have* $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) + \pi \cdot \mathrm{Int} = \mathrm{Int}$.

The main ingedient for the proof of theorem A is the "Katz isomorphism" proved in [AB24] for $F = \mathbf{Q}_{p^2}$, which gives rise to an isomorphism $\mathrm{Hom}_{o_F}(o_\infty, o_F) \simeq o_F[\![X]\!]^{\psi=0}$ where o_∞ is (at least when $\pi = p$) the ring of integers of the field generated by the torsion points of LT. We prove theorem B by using some results of [SI09] on Mahler bases and coefficients of Lubin–Tate power series. Using these results, it is enough to show that $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ is stable under multiplication. In order to do this, we prove that in some sense, the coefficients of a power series $F(X,Y) \in k[\![X,Y]\!]$ (where k is the residue field of o_F) can be recovered from the coefficients of $F(X,[b](X))$ for sufficiently many $b \in o_F$.

We also sketch a completely different proof of theorem B, based on a similar unpublished argument of Ardakov for proving that $\mathrm{Pol}(o_F[\![X]\!]) + \pi \cdot \mathrm{Int} = \mathrm{Int}$.

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2. Notation

We use the notation of the introduction: F is a finite extension of $\mathbf{Q}_p$ of degree $d > 1$ and ramification index e, with ring of integers o_F. The residue field k of o_F has cardinality $q = p^f$ and π is a uniformizer of o_F. Let Int denote the set of integer-valued polynomials on o_F, namely those polynomials $P(T) \in F[T]$ such that $P(o_F) \subset o_F$. Let LT denote the Lubin–Tate formal group attached to π (see [LT65]) and let X be a coordinate on LT. We have a formal addition law $T \oplus U \in o_F[\![T, U]\!]$, endomorphisms $[a](X) \in o_F[\![X]\!]$ for all $a \in o_F$, a logarithm $\log_{\mathrm{LT}}(X) \in F[\![X]\!]$ and a Lubin–Tate character $\chi_\pi : \mathrm{Gal}(\overline{\mathbf{Q}}_p/F) \to o_F^\times$. Let χ_{cyc} denote the cyclotomic character, and let $\tau : G_L \to o_F^\times$ denote the character $\tau = \chi_{\mathrm{cyc}} \cdot \chi_\pi^{-1}$. If $F \neq \mathbf{Q}_p$, the image of τ is open in $o_F^\times$, compare [AB24, Lemma 2.6.3].

The monoid $(o_F, \times)$ acts on $o_F[\![X]\!]$ by $a \cdot f(X) = f([a](X))$. The map φ is defined by $\varphi(f)(X) = f([\pi](X))$ and ψ is given by $\varphi(\psi(f(X))) = 1/\pi \cdot \mathrm{Tr}_{o_F[\![X]\!]/\varphi(o_F[\![X]\!])}(f(X))$.

If $f(X) \in o_F[\![X]\!]$, there exist polynomials $c_{f,n}(T) \in \mathrm{Int}$ for all $n \geq 0$ such that $f([a](X)) = \sum_{n \geq 0} c_{f,n}(a) X^n$. If M is a subset of $o_F[\![X]\!]$, let $\mathrm{Pol}(M)$ denote the sub o_F-module of Int generated by the $c_{f,n}$ with $f \in M$ and $n \geq 0$. If $i \geq 0$, we let $c_{i,n} = c_{f,n}$ with $f(X) = X^i$. Note that $\mathrm{Pol}(\{1, X, X^2, \ldots\}) = \mathrm{Pol}(o_F[\![X]\!])$.

3. p-adic Fourier theory

Recall (see §3 and §4 of [ST01] for what follows) that $\mathrm{Hom}_{o_{\mathbf{C}_p}}(\mathrm{LT}, \mathbf{G}_{\mathrm{m}})$ is a free o_F-module of rank 1. Choosing a generator of this module gives a power series $G(X) \in X \cdot o_{\mathbf{C}_p}[\![X]\!]$ such that $G(X) = \Omega \cdot X + \cdots$, where $\Omega \in o_{\mathbf{C}_p}$ is such that $g(\Omega) = \tau(g) \cdot \Omega$ if $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and $\mathrm{val}_p(\Omega) = 1/(p-1) - 1/e(q-1)$. In particular, $1 + G(X) = \exp(\Omega \cdot \log_{\mathrm{LT}}(X)) = \sum_{n \geq 0} P_n(\Omega) X^n$ where $P_n(Y) \in F[Y]$ is a polynomial of degree n such that $P_n(\Omega \cdot o_F) \subset o_{\mathbf{C}_p}$.

Let $F_\infty = \mathbf{C}_p^{\ker \tau}$ and let o_∞ denote the ring of integers of F_∞. Note that by §2.7 of [AB24], we have $F_\infty = \widehat{F(\Omega)}$. We have (see §3.3 of [AB24] as well as [Kat77]) a map $\mathcal{K}_1^* : \mathrm{Hom}_{o_F}(o_\infty, o_F) \to o_F[\![X]\!]$ that sends $h \in \mathrm{Hom}_{o_F}(o_\infty, o_F)$ to $\sum_{n \geq 0} h(P_n(\Omega)) X^n$.

Theorem 3.1. *The map $\mathcal{K}_1^*$ is injective, its image is included in $o_F[\![X]\!]^{\psi=0}$, and if $F = \mathbf{Q}_{p^2}$ then it gives rise to an isomorphism $\mathrm{Hom}_{o_F}(o_\infty, o_F) \to o_F[\![X]\!]^{\psi=0}$.*

Proof. If $h \in \mathrm{Hom}_{o_F}(o_\infty, o_F)$, then h extends to a continuous F-linear map $h : F_\infty \to F$. If $h(P_n(\Omega)) = 0$ for all $n \geq 0$, then $h = 0$ on $F[\Omega]$. By prop 6.2 of [APZ98], $F[\Omega]$ is p-adically dense in $\widehat{F(\Omega)} = F_\infty$. This proves the injectivity of $\mathcal{K}_1^*$. The fact that the image of $\mathcal{K}_1^*$ is included in $o_F[\![X]\!]^{\psi=0}$ is lemma 3.3.8 of [AB24]. The last assertion is theorem 3.6.14 of [AB24]. □

We now assume that $F = \mathbf{Q}_{p^2}$. We use the map $\mathcal{K}_1^*$ to define a pairing $\langle \cdot, \cdot \rangle : o_\infty \times o_F[\![X]\!]^{\psi=0} \to o_F$, given by the formula $\langle z, f(X) \rangle = h(z)$ where $h \in \mathrm{Hom}_{o_F}(o_\infty, o_F)$ is such that $\mathcal{K}_1^*(h) = f(X)$. By definition, we have $\langle P_n(\Omega), \sum_{i \geq 0} f_i X^i \rangle = f_n$.

Lemma 3.2. *If $P(T) \in F[T]$ is such that $P(\Omega) \in o_\infty$, and $f(X) \in o_F[\![X]\!]^{\psi=0}$, then $\langle P(\Omega), f([a](X)) \rangle = \langle P(a\Omega), f(X) \rangle$.*

Proof. See §3.2 of [AB24], in particular equation (3) above definition 3.2.4. □

Let $B = F[\Omega] \cap o_\infty$ and pick a regular basis (definition 4.2.5 of [AB24]) $\{b_n(\Omega)\}_{n \geq 0}$ for B. Recall (lemma 4.2.8 of [AB24]) that the polynomials $\rho_{i,k}(T) \in \mathrm{Int}$ are defined by $P_k(a\Omega) = \sum_{i=0}^k \rho_{i,k}(a) b_i(\Omega)$. As in §4.2 of [AB24], let $\check{B} \subset \mathrm{Int}$ denote the o_F-span of the $\rho_{i,k}(T)$ with $i, k \geq 0$. We then have (corollary 4.2.19 of [AB24]) the following criterion.

Proposition 3.3. *We have $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$ if and only if $\check{B} = \mathrm{Int}$.*

Given this criterion, theorem A results from the following claim.

Proposition 3.4. *If $F = \mathbf{Q}_{p^2}$, then $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) = \check{B}$.*

Proof. Recall that if $f(X) \in o_F[\![X]\!]^{\psi=0}$, we write $f([a](X)) = \sum_{n \geq 0} c_{f,n}(a) X^n$. We first prove that each $c_{f,n}$ is in $\check{B}$. By lemma 3.2, $\langle P_k(\Omega), f([a](X)) \rangle = \langle P_k(a\Omega), f(X) \rangle$, so that

$$\begin{aligned} c_{f,k}(a) &= \langle P_k(\Omega), f([a](X)) \rangle = \langle P_k(a\Omega), f(X) \rangle \\ &= \langle \sum_{i=0}^k \rho_{ik}(a) b_i(\Omega), f(X) \rangle = \sum_{i=0}^k \rho_{ik}(a) \langle b_i(\Omega), f(X) \rangle, \end{aligned}$$

and each $\langle b_i(\Omega), f(X) \rangle$ belongs to o_F since $b_i(\Omega) \in o_\infty$. Hence $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) \subset \check{B}$.

To show equality, the above computation implies that it is enough to show that given $k \geq 0$ and $j \leq k$, there exists $f(X) \in o_F[\![X]\!]^{\psi=0}$ with $\langle b_i(\Omega), f(X) \rangle = \delta_{i,j}$ for $0 \leq i \leq k$.

Let $N = o_F \cdot b_0(\Omega) + \cdots + o_F \cdot b_k(\Omega) = F[\Omega]_k \cap o_\infty$. The o_F-module N is a finitely generated and pure submodule of the o_F-module o_∞, hence a direct summand (see §16 of [Kap69], in particular exercise 57). The map $N \to o_F$ that sends $b_i(\Omega)$ to $\delta_{i,j}$ therefore extends to an o_F-linear map $h : o_\infty \to o_F$. We can now take $f(X) = \mathcal{K}_1^*(h)$. □

Remark 3.5. *If $F \neq \mathbf{Q}_{p^2}$, let M denote the image of the map $\mathcal{K}_1^* : \mathrm{Hom}_{o_F}(o_\infty, o_F) \to o_F[\![X]\!]^{\psi=0}$. The proof of prop 3.4 shows that $\mathrm{Pol}(M) = \check{B}$ and hence that $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$ if and only if $\mathrm{Pol}(M) = \mathrm{Int}$. It would therefore be interesting to compute M in general. Another consequence of this is that if $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$, then $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) = \mathrm{Int}$.*

4. Pol and Int modulo π

In this §, we prove theorem B. Let F be a finite extension of $\mathbf{Q}_p$. Recall that k denotes the residue field of o_F. Let B be a finite subset of o_F and let $s : k[\![X,Y]\!] \to \prod_{b\in B} k[\![X]\!]$ be the map $F(X,Y) \mapsto \{F(X,[b](X))\}_{b\in B}$.

Lemma 4.1. *We have* $\ker s = \prod_{b\in B}(Y - [b](X)) \cdot k[\![X,Y]\!]$.

Proof. If $F(X,[b](X)) = 0$, then $F(X,Y) = F(X,Y) - F(X,[b](X)) = (Y - [b](X)) \cdot G(X,Y)$. This implies the claim by induction since $[b](X) \neq [b'](X)$ if $b \neq b'$. □

Let $d = |B|$ and let $I = (X,Y)^d$ so that I is an open neighborhood of $\ker s$ in $k[\![X,Y]\!]$ and we have a well-defined and injective map $s : k[\![X,Y]\!]/I \to k[\![X]\!]^d/s(I)$.

Lemma 4.2. *There exists $n = n(B)$ having the property that if $f \in k[\![X,Y]\!]$ is such that $f(X,[b](X)) \in X^n k[\![X]\!]$ for all $b \in B$, then $f \in I$.*

Proof. If there is no such n, then for all n there is an f_n contradicting the lemma. Since $k[\![X,Y]\!]/I$ is a finite set, there is an f not in I such that $f(X,[b](X)) \in X^n k[\![X]\!]$ for all $b \in B$ and infinitely many n, so that $f(X,[b](X)) = 0$ for all $b \in B$. Hence $f \in \ker s \subset I$ by lemma 4.1. □

Corollary 4.3. *The map $s : k[\![X,Y]\!]/I \to k[\![X]\!]^d/(X^n + s(I))$ is injective.*

Proof. If $f \in k[\![X,Y]\!]$ is such that $s(f) = X^n g + s(i) \in X^n k[\![X]\!]^d + s(I)$, the above lemma applied to $f - i$ shows that $f - i \in I$. □

If $h(X) \in k[\![X]\!]$, let $\langle h(X)|X^j\rangle \in k$ denote the coefficient of X^j in $h(X)$. If $d \geq 1$ and $m + \ell < d$, the coefficient of $X^m Y^\ell$ in $F(X,Y) \in k[\![X,Y]\!]/I$ is well defined.

Proposition 4.4. *If $d \geq 1$, and $B = \{b_1, \ldots, b_d\}$ and $n = n(B)$ is as above, and if $m + \ell < d$, there exist some $\mu_{i,j} \in k$ for $1 \leq i \leq d$ and $0 \leq j \leq n-1$, such that for all $F(X,Y) \in k[\![X,Y]\!]$, the coefficient of $X^m Y^\ell$ in $F(X,Y)$ is equal to*

$$\sum_{i=1}^{d} \sum_{j=0}^{n-1} \mu_{i,j} \cdot \langle F(X,[b_i](X))|X^j\rangle.$$

Proof. Let $M = k[\![X,Y]\!]/I$ and $N = k[\![X]\!]^d/(X^n + s(I))$; they are both finite dimensional k-vector spaces. Let $h_{m,\ell} : M \to k$ be the linear form giving the coefficient of $X^m Y^\ell$ in $F(X,Y) \bmod I$. Consider the injective map (lemma 4.3) $s : M \to N$. The linear form $h_{m,\ell} \circ s^{-1} : s(M) \to k$ extends to a linear form $\lambda : N \to k$ which in turn gives rise to a linear form $\mu : k[\![X]\!]^d/X^n \to k$ factoring through N.

There exist some $\mu_{i,j} \in k$ such that if $f = (f_1, \ldots, f_d) \in k[\![X]\!]^d/X^n$, then $\mu(f) = \sum_{i=1}^{d} \sum_{j=0}^{n-1} \mu_{i,j} \cdot \langle f_i(X)|X^j \rangle$. If $F \in k[\![X, Y]\!]/I$, we have

$$\begin{aligned} h_{m,\ell}(F) &= \lambda \circ s(F) \\ &= \mu\left(F(X, [b_1](X)), \ldots, F(X, [b_d](X))\right) \\ &= \sum_{i=1}^{d} \sum_{j=0}^{n-1} \mu_{i,j} \cdot \langle F(X, [b_i](X))|X^j \rangle. \end{aligned} \qquad \square$$

Lemma 4.5. *If $f(X) \in o_F[\![X]\!]^{\psi=0}$, $g(X) \in o_F[\![X]\!]$ and $b \in \pi \cdot o_F$, then $f(X)g([b](X)) \in o_F[\![X]\!]^{\psi=0}$.*

Proof. This follows from

$$\sum_{\eta \in \mathfrak{m}_{\mathbf{C}_p}, [\pi](\eta)=0} f(X \oplus \eta) g([b](X \oplus \eta)) = g([b](X)) \sum_{\eta \in \mathfrak{m}_{\mathbf{C}_p}, [\pi](\eta)=0} f(X \oplus \eta) = 0. \qquad \square$$

Theorem 4.6. *If $f, g \in o_F[\![X]\!]^{\psi=0}$ and $m, \ell \geq 0$ then $c_{f,m}(T) \cdot c_{g,\ell}(T) \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})/\pi$.*

Proof. If $a \in o_F$, then $c_{f,m}(a) \cdot c_{g,\ell}(a)$ is the coefficient of $X^m Y^\ell$ in $f([a](X)) \cdot g([a](Y))$. Choose $d > m + \ell$ and $B \subset \pi \cdot o_F$ with $|B| = d$.

If $H(X, Y) \in k[\![X, Y]\!]$ and $h_b(X) = H(X, [b](X))$ for $b \in o_F$, then

$$H([a](X), [a]([b](X))) = H([a](X), [b]([a](X))) = h_b([a](X)).$$

Take $H(X, Y) = f(X) \cdot g(Y)$ and let $F(X, Y) = f([a](X)) \cdot g([a](Y)) = H([a](X), [a](Y))$. Since $B \subset \pi \cdot o_F$, lemma 4.5 implies that $h_b(X) = f(X) \cdot g([b](X))$ belongs to $o_F[\![X]\!]^{\psi=0}$. By prop 4.4, the coefficient of $X^m Y^\ell$ in $F(X, Y)$ is $\sum_{i=1}^{d} \sum_{j=0}^{n-1} \mu_{i,j} \cdot \langle h_{b_i}([a](X))|X^j \rangle$, and hence $c_{f,m}(T) \cdot c_{g,\ell}(T) = \sum_{i=1}^{d} \sum_{j=0}^{n-1} \mu_{i,j} \cdot c_{h_{b_i},j}(T) \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})/\pi$. $\square$

Recall that $[a](X) = \sum_{n \geq 1} c_{1,n}(a) X^n$ with $c_{1,n}(T) \in \mathrm{Int}$, and that a Mahler basis for o_F is a regular basis of Int.

Lemma 4.7. *The functions c_{1,q^k} are part of a Mahler basis for o_F, and the o_F-algebra* Int *is generated by the c_{1,q^k} for $k \geq 0$.*

Proof. See theorem 3.1 of [SI09]. $\square$

Let us write $\psi_q := \frac{\pi}{q}\psi$, i.e. ψ_q is the unique operator $\psi_q \colon o_F[\![X]\!] \otimes_{o_F} F \to o_F[\![X]\!] \otimes_{o_F} F$ such that $\varphi \circ \psi_q = \frac{1}{q} \mathrm{Tr}$, where Tr denotes the trace of $o_F[\![X]\!] \otimes_{o_F} F$ over $\varphi(o_F[\![X]\!] \otimes_{o_F} F)$.

Lemma 4.8. *If X is a coordinate on* LT *such that $[\pi](X) = \pi X + X^q$, then*

$$\begin{aligned} \psi_q(1) &= 1 \\ \psi_q(X^i) &= 0 \text{ if } 1 \leq i \leq q-2 \\ \psi_q(X^{q-1}) &= \frac{\pi}{q} \cdot (1-q). \end{aligned}$$

Proof. See the proof of [FX13, Prop 2.2]. $\square$

Proof of theorem B. Note that $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ is independent of the choice of coordinate X on LT. We choose one such that $[\pi](X) = \pi X + X^q$. By lemma 4.8, $1-q/(\pi(1-q))\cdot X^{q-1}$ belongs to $o_F[\![X]\!]^{\psi=0}$ so that $1 \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$. Likewise, $X \in o_F[\![X]\!]^{\psi=0}$, so that we have $c_{1,q^k} \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ for all $k \geq 0$. Theorem B now follows from lemma 4.7, and from theorem 4.6. $\square$

Another proof of theorem B using completely different ideas is sketched below. It is based on arguments of Konstantin Ardakov for proving that $\mathrm{Pol}(o_F[\![X]\!]) + \pi \cdot \mathrm{Int} = \mathrm{Int}$. Let M be either $o_F[\![X]\!]$ or $o_F[\![X]\!]^{\psi=0}$.

Lemma 4.9. *If $b \in o_F^\times$, then $\mathrm{Pol}(M)$ is stable under $P(T) \mapsto P(b \cdot T)$.*

Proof. We have $c_{f\circ[b],n}(a) = c_{f,n}(ba)$ and if $f \in M$ and $b \in o_F^\times$, then $f \circ [b] \in M$. $\square$

Let $\partial = \log'_{\mathrm{LT}}(X)^{-1} \cdot d/dX$ be the normalized invariant differential on $F[\![X]\!]$. Recall (see §1 of [Kat81]) that if $f(X) \in F[\![X]\!]$, then $f(X \oplus H) = \sum_{n\geq 0} P_n(\partial)(f(X)) \cdot H^n$.

Lemma 4.10. *If $b \in o_F$, then $\mathrm{Pol}(M)$ is stable under $P(T) \mapsto P(T + b)$.*

Proof. We first check that M is stable under $P_n(\partial)$ for all $n \geq 0$. We have $f(X \oplus H) = \sum_{n\geq 0} P_n(\partial)(f(X)) \cdot H^n$ and $P_n(\partial)(f(X))$ belongs to $o_F[\![X]\!]$ as $f(X \oplus H) \in o_F[\![X, H]\!]$. Finally $\partial \circ \psi = \pi^{-1}\psi \circ \partial$ so that if $\psi(f) = 0$ then $\psi(P_n(\partial)f) = 0$ as well.

If $f(X) \in M$, then $f([a + b](X)) = \sum_{i\geq 0} c_{f,i}(a + b)X^i$. On the other hand,

$$f([a + b](X)) = f([a](X) \oplus [b](X)) = \sum_{n\geq 0} P_n(\partial)(f)([a](X)) \cdot [b](X)^n.$$

This implies that

$$c_{f,i}(T + b) = \sum_{\substack{\ell+m=i,\\ 0\leq n\leq m}} c_{P_n(\partial)(f),\ell}(T) c_{n,m}(b).$$ $\square$

Proposition 4.11.

(1) *The image of $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ in $\mathrm{Int}/\pi \cdot \mathrm{Int}$ is infinite dimensional.*
(2) *We have $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0}) + \pi \cdot \mathrm{Int} = \mathrm{Int}$.*

Proof. We have $X \in o_F[\![X]\!]^{\psi=0}$ and hence $c_{1,i} \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ for all $i \geq 1$. In particular, $c_{1,q^k} \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ for all $k \geq 0$. By lemma 4.7, these elements are part of a Mahler basis, hence linearly independent mod π. This implies (1).

We now sketch the proof of (2). We have $\mathrm{Int}/\pi = C^0(o_F, k)$ and its dual is $k[\![o_F]\!]$. Let $I \subset k[\![o_F]\!]$ be the orthogonal of the image P of $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})$ in $C^0(o_F, k)$. Since P is stable under $f(T) \mapsto f(b \cdot T)$ for $b \in o_F^\times$ by lemma 4.9, I is stable under the action of $o_F^\times$. Since P is also stable under $f(T) \mapsto f(T + b)$ for $b \in o_F$ by lemma 4.10, I is an ideal of $k[\![o_F]\!]$. By either §8.1 of [Ard12] or the main result of [HMS14], either $I = \{0\}$ or I is open in $k[\![o_F]\!]$. By item (1), I cannot be open, so that $I = \{0\}$ and hence $P = \mathrm{Int}/\pi$. $\square$

5. Numerical verification

In this §, we assume that $F = \mathbf{Q}_{p^2}$ and that $\pi = p$. We choose a coordinate X on LT with $[p](X) = pX + X^q$. Recall that $\psi_q = \pi/q \cdot \psi$.

For a non-negative integer n, we write $\mathrm{Int}_{\leq n}$ for the o_F-module of integer-valued polynomials on o_F of degree $\leq n$. Similarly, let us denote by $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq n}$ the o_F-module generated by the set

$$\{c_{f,m} : 0 \leq m \leq n,\ f \in o_F[\![X]\!]^{\psi=0}\}.$$

Recall from [SI09, Proposition 2.2], that a sequence $\{P_n\}_{n\geq 0}$ of integer-valued polynomials with $\deg(P_n) = n$ is a basis of Int if and only if $v_p(\mathrm{lc}(P_n)) = -w_q(n)$, where $w_q(n) = \sum_{k\geq 1} \lfloor n/q^k \rfloor$ and $\mathrm{lc}(P)$ denotes the leading coefficient of a polynomial P.

Definition 5.1. *For a fixed non-negative integer n, we define $s_0(n) = \inf\{N \geq n$ such that there exists $P \in \mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$ of degree n such that $v_p(\mathrm{lc}(P)) = -w_q(n)\}$.*

By Theorem A, we have $\Lambda_F(\mathfrak{X}) = o_F[\![o_F]\!]$ if and only if $s_0(n)$ is finite for every $n \in \mathbf{N}$. In this section, we explain how to compute $s_0(n)$ numerically. As a first step, let us give explicit generators of the o_F-module $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$.

Recall that $F = \mathbf{Q}_{p^2}$ and that $\pi = p$.

Lemma 5.2. *We have $o_F[\![X]\!]^{\psi=0} = (\oplus_{i=1}^{q-2} X^i \cdot \varphi(o_F[\![X]\!])) \oplus (pX^{q-1} - (1-q)) \cdot \varphi(o_F[\![X]\!])$.*

Proof. We have $o_F[\![X]\!] = \oplus_{i=0}^{q-1} X^i \varphi(o_F[\![X]\!])$. By lemma 4.8, $\psi_q(1) = 1$ and $\psi_q(X^i) = 0$ for $1 \leq i \leq q-2$ and $\psi_q(X^{q-1}) = (1-q)/p$. If $f = \sum_{i=0}^{q-1} X^i \varphi(f_i)$ then $\psi_q(f) = f_0 + (1-q)/p \cdot f_{q-1}$ so that $\psi_q(f) = 0$ if and only if $f_{q-1} = -p/(1-q) \cdot f_0$. □

Since $c_{f,m}$ for $0 \leq m \leq N$ does only depend on $f \in o_F[\![X]\!]/X^{N+1}[\![X]\!]$, we obtain the following corollary:

Corollary 5.3. *Let $b_0, \ldots, b_N$ be a basis of the o_F-submodule of $o_F[\![X]\!]/X^{N+1}[\![X]\!]$ generated by the images of*

$$(1)\quad \{(pX^{q-1} - (1-q))\varphi(X)^j : 0 \leq j \leq N\} \cup \{X^i \varphi(X)^j : 0 \leq i+j \leq N,\ 1 \leq i \leq q-2\}$$

in $o_F[\![X]\!]/X^{N+1}[\![X]\!]$, then the o_F-module $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$ is generated by

$$\{c_{b,m} : b = b_0, \ldots, b_N, 0 \leq m \leq N\}.$$

The basis $b_0, \ldots, b_N$ can easily be computed from the generators in (1) using Gaußian elimination. In order to compute $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$ efficiently, we need to compute for $b \in \{b_0, \ldots, b_N\}$ and $0 \leq m \leq N$ the polynomials $c_{b,m}$. For $0 \leq i \leq N$, let us write $b_i = b_{i,0} + b_{i,1}X + \ldots b_{i,N}X^N$ and define the matrices

$$B = \begin{pmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,N} \\ b_{1,0} & b_{1,1} & \cdots & b_{1,N} \\ \vdots & \ddots & & \vdots \\ b_{N,0} & b_{N,1} & \cdots & b_{N,N} \end{pmatrix} \in \mathrm{Mat}_{N+1,N+1}(o_F),$$

and

$$C = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,N} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,N} \\ \vdots & \ddots & & \vdots \\ c_{N,0} & c_{N,1} & \cdots & c_{N,N} \end{pmatrix} \in \mathrm{Mat}_{N+1,N+1}(\mathrm{Int}).$$

We have

$$B \cdot C = \begin{pmatrix} c_{b_0,0} & c_{b_0,1} & \cdots & c_{b_0,N} \\ c_{b_1,0} & c_{b_1,1} & \cdots & c_{b_1,N} \\ \vdots & \ddots & & \vdots \\ c_{b_N,0} & c_{b_N,1} & \cdots & c_{b_N,N} \end{pmatrix} \in \mathrm{Mat}_{N+1,N+1}(\mathrm{Int}).$$

Hence, in order to compute $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$ it remains to compute the polynomials $c_{i,j}$ efficiently. For all $i \in \mathbf{N}$, we have

$$[a](X)^i = \sum_{j \geq i} c_{i,j}(a) X^j = \exp_{\mathrm{LT}}(a \cdot \log_{\mathrm{LT}}(X))^i. \tag{2}$$

Let us denote by $D = \big(\langle \log_{\mathrm{LT}}(X)^j \mid X^k \rangle\big)_{0 \leq j,k \leq N}$ the (truncated) Carleman matrix of $\log_{\mathrm{LT}}$. Then (2) can be re-written as the matrix identity

$$D \cdot C = \begin{pmatrix} 1 & & & \\ & a & & \\ & & \ddots & \\ & & & a^N \end{pmatrix} \cdot D,$$

and we get $C = D^{-1}\mathrm{diag}(1, a, \ldots, a^N) D$. For the computation of the Carleman matrix of $\log_{\mathrm{LT}}$, we have the following efficient recursive formula for the coefficients of $\log_{\mathrm{LT}}(X)$. Write $\log_{\mathrm{LT}}(X) = \sum_{k \geq 1} h_k X^k$ with $h_1 = 1$.

Lemma 5.4. *We have* $h_n = 1/(p - p^n) \cdot \sum_{i=1}^{\lfloor n/q \rfloor} h_j \binom{j}{i} p^{j-i}$ *where* $j = n - i(q-1)$.

Proof. We have $\log_{\mathrm{LT}}(pX + X^q) = p \log_{\mathrm{LT}}(X)$. We can expand $\log_{\mathrm{LT}}(pX + X^q)$ as

$$\log_{\mathrm{LT}}(pX + X^q) = \log_{\mathrm{LT}}(pX) + \sum_{i \geq 1} X^{qi} \log_{\mathrm{LT}}^{[i]}(pX),$$

where $\log_{\mathrm{LT}}^{[i]}$ denotes the Hasse derivative of $\log_{\mathrm{LT}}$. Computing the coefficient of X^n on each side of $p \log_{\mathrm{LT}}(X) - \log_{\mathrm{LT}}(pX) = \sum_{i \geq 1} X^{qi} \log_{\mathrm{LT}}^{[i]}(pX)$, we get

$$(p - p^n) h_n = \sum_{i \geq 1} \langle \log_{\mathrm{LT}}^{[i]}(pX) \mid X^{n-qi} \rangle = \sum_{i=1}^{\lfloor n/q \rfloor} h_j \binom{j}{i} p^{j-i}, \text{ where } j = n - i(q-1). \qquad \square$$

Fix a positive integer N. We now describe an algorithm for computing $s_0(n)$ for all $n \leq N$ (it returns -1 if $s_0(n) > N$).

(1) Compute the Carleman matrix D of $\log_{\mathrm{LT}}$, see Lemma 5.4.
(2) Compute $(c_{i,j}(a))_{i,j} = C = D^{-1}\mathrm{diag}(1, \ldots, a^N) D$.
(3) Compute a basis $b_0, \ldots, b_N$ of $o_F[\![X]\!]^{\psi=0}$ modulo X^{N+1}, see Lemma 5.2, and store the coefficients of $b_0, \ldots, b_N$ in a matrix B.

(4) The matrix $B \cdot C = (c_{b_i,m})_{i,m}$ contains generators of $\mathrm{Pol}(o_F[\![X]\!]^{\psi=0})_{\leq N}$, see Corollary 5.3.
(5) For each n with $0 \leq n \leq N$ let $s_0(n)$ be the smallest $s \in \{1, \ldots, N\}$ such that the o_F-module spanned by $c_{b_i,m}$ with $0 \leq i, m \leq s$ contains a polynomial of leading coefficient $-w_q(n)$. If there is no s with this property set $s_0(n) = -1$.

For an implementation of this algorithm in SageMath, see Appendix A. The code is adapted from a similar program by Crisan and Yang, see the appendix of [AB24].

Here are some results, running SageMath 10.5 on an M1 iMac.

(1) For $p = 2$ and $N = 800$ and precision 6000 we find that $s_0(n)$ is finite for $n \leq 206$

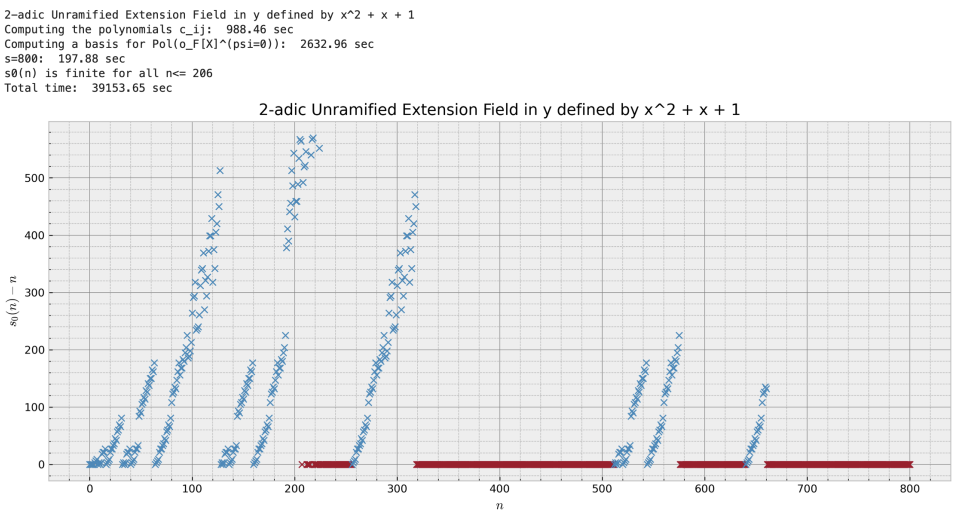

FIGURE 1. Plot of $s_0(n) - n$ for $p = 2$ and $N = 800$. Red points are the n's for which $s_0(n) = -1$.

(2) For $p = 3$ and $N = 800$ and precision 6000 we find that $s_0(n)$ is finite for $n \leq 226$

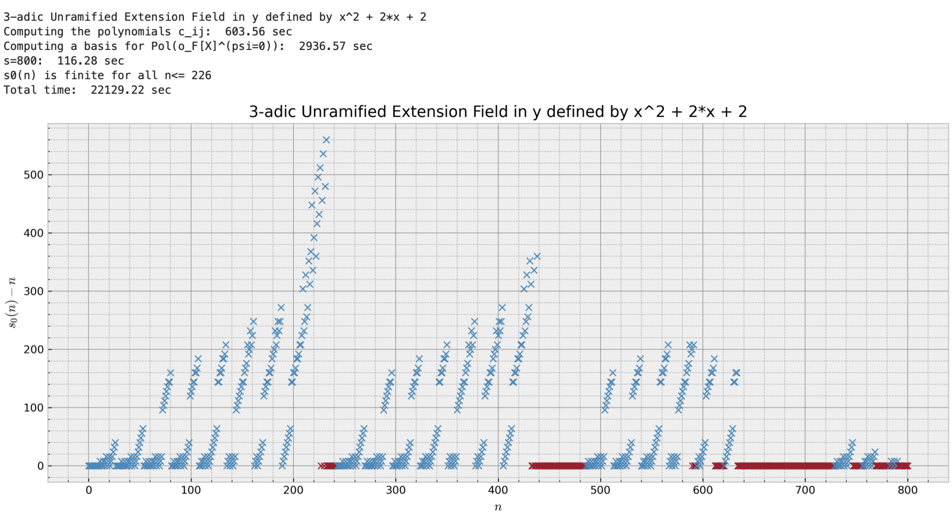

FIGURE 2. Plot of $s_0(n) - n$ for $p = 3$ and $N = 800$. Red points are the n's for which $s_0(n) = -1$.

Appendix A. SageMath code

```
p=3   # prime number
N=120   # cutoff
precision = 1000   # p-adic precision

q=p^2   # Degree of the residue field

# Python imports
from time import process_time
import matplotlib.pyplot as plt
import numpy as np

# Definitions
import sage.rings.padics.padic_extension_generic

# Define the p-adic field, its ring of integers and the valuation function v
o_F.<y> = Zq(ZZ(q),prec=precision)
F.<y> = Qq(ZZ(q),prec=precision)
v = F.valuation()
print(F)

# Define the generator of the unique maximal ideal in o_F.
Pi = o_F(p)

# Do linear algebra over the ring of polynomials F[X]
# in one variable X with coefficients in the field F:
F_X.<X> = F[]
F_Y.<Y> = F[]

# Define the rings of power series over F and o_F and the Frobenius lift phi=pT+T^q.
F_T.<T>=PowerSeriesRing(F,default_prec=N)
phi_F=Pi*T+T^q
o_F_tt.<t>=PowerSeriesRing(o_F,default_prec=N)
phi=o_F_tt(phi_F(t))

def LogLT(N):
    # Computes the logarithm of the Lubin-Tate formal group law with phi(T)=Pi*T+T^q up
        to precision O(T^(N+1))
    h=[0,1]
    for n in range(2,N+1):
        h.append(sum([F(h[n-q*i+i]*binomial(n-q*i+i,i)*p^(n-q*i)) for i in range(1,floor
            ((n)/q)+1)])/(p-p^n))
    log_LT= sum([h[n]*T^n for n in range(1,N)])+O(T^(N+1))
    return log_LT

def LogCarlemanMatrix(N):
    # Computes the Carleman matrix of log_LT which is used for computing c_(i,j)
    CMat = matrix(F,N+1,N+1)

    # Compute the logarithm of the Lubin-Tate formal group law
    log_LT=LogLT(N)

    log_LT_j=F_T(1)
    for j in range(0,N+1):
        # Stores the coefficients of the power series log_LT(T)^j in row j of the matrix
             CMat
        log_LT_dict_j=log_LT_j.dict()
        l=[0 for _ in range(0,N+1)]
        for key in log_LT_dict_j.keys():
```

```
            l[key]=log_LT_dict_j[key]
        for i in range(0,len(l)):
            CMat[j,i]=l[i]
        log_LT_j=log_LT_j*log_LT+O(T^(N+1))
    return CMat

def BMatrix(N):
    # BMatrix returns a square matrix B of size (N+1)x(N+1)
    # such that B[i,0]+X*B[i,1]+...+X^N*B[i,N] for i=0...N forms
    # an o_F-basis of the degree <=N part of o_F[X]^(psi=0)

    # Create a list (genList) of all generators of o_F[X]^(psi=0) of degree at most N
    genList=[]
    for i in range(0,q-1):
        phi_j=o_F_tt(1)
        for j in range(0,N-i+1):
            if i==0:
                genList.append((Pi*t^(q-1)-(1-q))*phi_j+O(t^(N+1)))
            else:
                genList.append(t^i*phi_j+O(t^(N+1)))
            phi_j=phi_j*phi

    # Store the coefficients of the polynomials in genList in the matrix B
    B=matrix(F,len(genList),N+1)
    i=0
    for gen in genList:
        l=gen.list()
        for j in range(0,len(l)):
            B[i,j]=l[j]
        i=i+1

    # Perform Gaussian elimination
    i0 = 0
    for k in range(B.ncols()):
        valuation_row_pairs = [
            (v(B[i,k]), i) for i in range(i0, B.nrows()) if B[i,k] != 0]

        if not valuation_row_pairs:
            raise ValueError("B is not full-rank")
        minv, i_minv = min(valuation_row_pairs)

        # Swap the row of minimum valuation with the first bad row
        B[i0, :], B[i_minv, :] = B[i_minv, :], B[i0, :]

        # Divide the top row by a unit in o_F
        u = B[i0, k] / Pi^int(v(B[i0, k]))
        B[i0, :] /= u

        # Cleave through the other rows
        for i in range(i0 + 1, B.nrows()):
            if v(B[i, k]) >= v(B[i0, k]):
                B[i, :] -= B[i, k]/B[i0, k] * B[i0, :]

        i0 += 1

    # Return the first B.ncols() rows of the matrix B
    return B[0:B.ncols(), :]

def CMatrix(N, D=None):
```

```
    # Computes a matrix containing the polynomials c_(ij) using the Carleman matrix of
        log_LT
    if D is None:
        D = LogCarlemanMatrix(N)

    # Define a diagonal matrix:
    Diag = matrix(F_X, N+1,N+1, lambda x,y: kronecker_delta(x,y) * X^x)

    # Compute the inverse of D:
    S = D.inverse()

    # Compute the matrix C:
    C = S * Diag * D

    return C

def w_q(n):
    return (n - sum(n.digits(base=q))) / (q-1)

def compute_s(N, filename=None):

    t_start = process_time()
    D = LogCarlemanMatrix(N)
    C = CMatrix(N, D)
    t_end = process_time()
    print(f"Computing␣the␣polynomials␣c_ij:␣{t_end-t_start␣:␣.2f}␣sec")

    t_start = process_time()
    # BPsi contains a basis of o_F[X]^(psi=0)_{<=N} of degree <=N
    BPsi=BMatrix(N)
    # Tau contains a basis of Pol(o_F[X]^{psi=0}_{<=N})
    Tau=BPsi*C
    t_end = process_time()
    print(f"Computing␣a␣basis␣for␣Pol(o_F[X]^(psi=0)):␣{t_end-t_start␣:␣.2f}␣sec")

    # s0_s[n] will store the minimal degree such that Int_n is contained in Pol(o_F[X]^{
        psi=0}_{<=s0_s[n]})
    s0_s = [-1 for _ in range(N+1)]

    B_old = Matrix(0,0)
    d = 0
    for s in range(N+1):
        t_start = process_time()

        # 1. Use the non-zero rows from previous calculations
        # 2. Add a 0 column to its left
        # 3. Add rows corresponding to entries from the j_th column of Tau_a
        B = Matrix(F, 2*s-d+1, s-d+1)
        B[0,0] = 1
        B[1:s-d+1, 1:] = B_old
        for i in [0 .. s-1]:
            coeffs = Tau[i, s].list()
            B[s-d+1+i, B.ncols()-len(coeffs)+d:] = vector(F, reversed(coeffs[d:]))

        # Perform Gaussian elimination
        i0 = 0
        ks = []
        for k in range(B.ncols()):
            valuation_row_pairs = [
                (v(B[i,k]), i) for i in range(i0, B.nrows()) if B[i,k] != 0]
```

```
            if not valuation_row_pairs:
                raise ValueError("B is not full-rank")
            minv, i_minv = min(valuation_row_pairs)
            ks.append(k)

            # Swap the row of minimum valuation with the first bad row
            B[i0, :], B[i_minv, :] = B[i_minv, :], B[i0, :]

            # Divide the top row by a unit in o_F
            u = B[i0, k] / Pi^int(v(B[i0, k]))
            B[i0, :] /= u

            # Cleave through the other rows
            for i in range(i0 + 1, B.nrows()):
                if v(B[i, k]) >= v(B[i0, k]):
                    B[i, :] -= B[i, k]/B[i0, k] * B[i0, :]

            i0 += 1

        d_is_updated = False
        for b in [d .. s]:
            n=b
            # if the valuation of the leading coefficient is for the first time -w_q
                then store s in s0_s
            if v(B[s-b, s-b]) == -w_q(n):
                if s0_s[b] == -1:
                    s0_s[b] = s
            else:
                if not d_is_updated:
                    d = b
                    d_is_updated = True
        B_old = B[:s-d+1, :s-d+1]

        t_end = process_time()
        print(f"s={s}: {t_end-t_start : .2f} sec", end='\r')
        if filename is not None:
            with open(filename, 'w') as f:
                f.write("n,s0\n")
                for n, s0 in enumerate(s0_s):
                    f.write(f"{n},{s0}\n")
    print()

    # plot the result
    plt.style.use('bmh')
    fig = plt.figure(figsize=(15,6), dpi=300)
    for n, s0 in enumerate(s0_s):
        if s0 != -1:
            plt.plot(n, s0-n, 'x', c='C0')
        else:
            plt.plot(n, 0, 'x', c='C1')
    plt.xlabel(r"$n$")
    plt.ylabel("$s_0(n) - n$")
    plt.title(str(F))
    plt.minorticks_on()
    plt.grid(which='both')
    plt.grid(which='major', linestyle='-', c='grey')

    return s0_s, fig
```

```
t_start_tot = process_time()
s0_s = compute_s(N);
print("s0(n)␣is␣finite␣for␣all␣n<=",(s0_s[0]+[-1]).index(-1)-1)
t_end_tot = process_time()
print(f"Total␣time:␣{t_end_tot-t_start_tot␣:␣.2f}␣sec")
```

References

[APZ98] V. Alexandru, N. Popescu and A. Zaharescu, *On the closed subfields of* $\mathbf{C}_p$, J. Number Theory **68** (1998), no. 2, 131–150.

[Ard12] K. Ardakov, *Prime ideals in nilpotent Iwasawa algebras*, Invent. Math. **190** (2012), no. 2, 439–503.

[AB24] K. Ardakov and L. Berger, *Bounded functions on the character variety*, Münster J. Math. (2024), to appear.

[FX13] L. Fourquaux and B. Xie, *Triangulable* $\mathcal{O}_F$*-analytic* (φ_q, Γ)*-modules of rank 2*, Algebra Number Theory **7** (2013), no. 10, 2545–2592.

[HMS14] Y. Hu, S. Morra and B. Schraen. *Sur la fidélité de certaines représentations de* $\mathrm{GL}_2(F)$ *sous une algèbre d'Iwasawa*. Rend. Semin. Mat. Univ. Padova **131** (2014), 49–65.

[Kap69] I. Kaplansky. *Infinite abelian groups.* Revised edition. University of Michigan Press, Ann Arbor, MI, 1969.

[Kat77] N. M. Katz, *Formal groups and p-adic interpolation*, Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976), Astérisque No. 41–42, Soc. Math. France, Paris, 1977, pp. 55–65.

[Kat81] N. M. Katz, *Divisibilities, congruences, and Cartier duality*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 667–678 (1982).

[LT65] J. Lubin and J. Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387.

[ST01] P. Schneider and J. Teitelbaum, *p-adic Fourier theory*, Doc. Math. **6** (2001), 447–481.

[SI09] E. de Shalit and E. Iceland, *Integer valued polynomials and Lubin-Tate formal groups*, J. Number Theory **129** (2009), no. 3, 632–639.

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