SUPER-HÖLDER VECTORS AND THE FIELD OF NORMS

by

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Abstract. — Let E be a field of characteristic p. In a previous paper of ours, we defined and studied super-Hölder vectors in certain E-linear representations of $\mathbf{Z}_p$. In the present paper, we define and study super-Hölder vectors in certain E-linear representations of a general p-adic Lie group. We then consider certain p-adic Lie extensions K_∞/K of a p-adic field K, and compute the super-Hölder vectors in the tilt of K_∞. We show that these super-Hölder vectors are the perfection of the field of norms of K_∞/K. By specializing to the case of a Lubin-Tate extension, we are able to recover $E(\!(Y)\!)$ inside the Y-adic completion of its perfection, seen as a valued E-vector space endowed with the action of $\mathcal{O}_K^\times$ given by the endomorphisms of the corresponding Lubin-Tate group.

Contents

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Introduction

Let E be a field of characteristic p, for example a finite field. In our paper [**BR22**], we defined and studied super-Hölder vectors in certain E-linear representations of the p-adic Lie group $\mathbf{Z}_p$. These vectors are a characteristic p analogue of locally analytic vectors. They allowed us to recover $E(\!(X)\!)$ inside the X-adic completion of its perfection, seen as a valued E-vector space endowed with the action of $\mathbf{Z}_p^\times$ given by $a \cdot f(X) = f((1+X)^a - 1)$.

In the present paper, we define and study super-Hölder vectors in certain E-linear representations of a general p-adic Lie group. We then consider certain p-adic Lie extensions K_∞/K of a p-adic field K, and compute the super-Hölder vectors in the tilt of K_∞. We show that these super-Hölder vectors are the perfection of the field of norms of K_∞/K. By specializing to the case of a Lubin-Tate extension, we are able to recover $E(\!(Y)\!)$ inside the Y-adic completion of its perfection, seen as a valued E-vector space endowed with the action of $\mathcal{O}_K^\times$ given by the endomorphisms of the corresponding Lubin-Tate group.

We now give more details about the contents of our paper. Let Γ be a p-adic Lie group. It is known that Γ always has a uniform open pro-p subgroup G. Let G be such a subgroup, and let $G_i = G^{p^i}$ for $i \geqslant 0$. Let M be an E-vector space, endowed with a valuation val_M such that $\mathrm{val}_M(xm) = \mathrm{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M-adic topology. We say that a function $f : G \to M$ is super-Hölder if there exist constants $e > 0$ and $\lambda, \mu \in \mathbf{R}$ such that $\mathrm{val}_M(f(g) - f(h)) \geqslant p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geqslant 0$. If M is now endowed with an action of G by isometries, and $m \in M$, we say that m is a super-Hölder vector if the orbit map $g \mapsto g \cdot m$ is a super-Hölder function $G \to M$. We let $M^{G\text{-}e\text{-sh},\lambda}$ denote the space of super-Hölder vectors for given constants e and λ as in the definition above. The space of vectors of M that are super-Hölder for a given e is independent of the choice of the uniform subgroup G, and denoted by $M^{e\text{-sh}}$. When $G = \mathbf{Z}_p$ and $e = 1$, we recover the definitions of [**BR22**]. If Γ is a p-adic Lie group and $e = 1$, we get an analogue of locally $\mathbf{Q}_p$-analytic vectors. If K is a finite extension of $\mathbf{Q}_p$, Γ is the Galois group of a Lubin-Tate extension of K, and $e = [K : \mathbf{Q}_p]$, we seem to get an analogue of locally K-analytic vectors.

From now on, assume that $p \neq 2$. Let K be a p-adic field and let K_∞/K be an almost totally ramified p-adic Lie extension, with Galois group Γ of dimension $d \geqslant 1$. The tilt of K_∞ is the fraction field $\widetilde{\mathbf{E}}_{K_\infty}$ of $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$. It is a perfect complete valued field of characteristic p, endowed with an action of Γ by isometries. The field $\widetilde{\mathbf{E}}_{K_\infty}$ naturally contains the field of norms $X_K(K_\infty)$ of the extension K_∞/K, and it is known that $\widetilde{\mathbf{E}}_{K_\infty}$ is the completion of the perfection of $X_K(K_\infty)$. We have the following result (theorem 2.2.3).

Theorem A. — *We have* $\widetilde{\mathbf{E}}_{K_\infty}^{d\text{-sh}} = \cup_{n \geqslant 0} \varphi^{-n}(X_K(K_\infty))$.

Assume now that K is a finite extension of $\mathbf{Q}_p$, with residue field k, and let LT be a Lubin-Tate formal group attached to K. Let K_∞ be the extension of K generated by the torsion points of LT, so that $\mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathcal{O}_K^\times$. The field of norms $X_K(K_\infty)$ is isomorphic to $k(\!(Y)\!)$, and $\mathcal{O}_K^\times$ acts on this field by the endomorphisms of the Lubin-Tate group: $a \cdot f(Y) = f([a](Y))$. Let $d = [K : \mathbf{Q}_p]$. The following (theorem 3.2.1) is a more precise version of theorem A in this situation.

Theorem B. — *If* $j \geqslant 1$*, then* $\widetilde{\mathbf{E}}_{K_\infty}^{1+p^j \mathcal{O}_K\text{-}d\text{-sh},dj} = k(\!(Y)\!)$.

If $K = \mathbf{Q}_p$ and K_∞/K is the cyclotomic extension, theorem B was proved in [**BR22**]. A crucial ingredient of the proof of this theorem was Colmez' analogue of Tate traces for $\widetilde{\mathbf{E}}_{K_\infty}$. If the Lubin-Tate group if of height $\geqslant 2$, there are no such traces (we state and prove a precise version of this assertion in §3.2). Instead of Tate traces, we a theorem of Ax and a precise characterization of the field of norms $X_K(K_\infty)$ inside $\widetilde{\mathbf{E}}_{K_\infty}$ in order to prove theorem A.

As an application of theorem B, we compute the perfectoid commutant of Aut(LT). If $b \in \mathcal{O}_K^\times$ and $n \in \mathbf{Z}$, then $u(Y) = [b](Y^{q^n})$ is an element of $\widetilde{\mathbf{E}}_{K_\infty}^+$ that satisfies the functional equation $u \circ [g](Y) = [g] \circ u(Y)$ for all $g \in \mathcal{O}_K^\times$. Conversely, we prove the following (theorem 3.3.1).

Theorem C. — *If* $u \in \widetilde{\mathbf{E}}_{K_\infty}^+$ *is such that* $\mathrm{val}_Y(u) > 0$ *and* $u \circ [g] = [g] \circ u$ *for all* $g \in \mathcal{O}_K^\times$*, there exists* $b \in \mathcal{O}_K^\times$ *and* $n \in \mathbf{Z}$ *such that* $u(Y) = [b](Y^{q^n})$.

In the last section, we give a characterization of super-Hölder functions on a uniform pro-p group in terms of their Mahler expansions (theorem 4.3.4). In order to do so, we prove some results of independent interest on the space of continuous functions on $\mathcal{O}_K^d$ with values in a valued E-vector space M as above.

At the end of [**BR22**], we suggested an application of super-Hölder vectors for the action of $\mathbf{Z}_p$ to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. We hope that this general theory of super-Hölder vectors, especially in the Lubin-Tate case, will have applications to the p-adic local Langlands correspondence for other fields than $\mathbf{Q}_p$.

1. Super-Hölder functions and vectors

In this section, we define Super-Hölder vectors inside a valued E-vector space M endowed with an action of a p-adic Lie group Γ. The definition is very similar to the one that we gave for $\Gamma = \mathbf{Z}_p$ in our paper [**BR22**]. The main new technical tool is the existence of uniform open subgroups of Γ. These uniform subgroups look very much like $\mathbf{Z}_p^d$ in a sense that we make precise.

1.1. Uniform pro-p groups. — Uniform pro-p groups are defined at the beginning of §4 of [**DdSMS99**]. We do not recall the definition, nor the notion of rank of a uniform pro-p group, but rather point out the following properties of uniform pro-p groups. A coordinate (below) is simply a homeomorphism.

Proposition 1.1.1. — *If G is a uniform pro-p group of rank d, then*

1. $G_i = \{g^{p^i},\, g \in G\}$ *is an open normal (and uniform) subgroup of G for $i \geqslant 0$*
2. *We have $[G_i : G_{i+1}] = p^d$ for $i \geqslant 0$*
3. *There is a coordinate $c : G \to \mathbf{Z}_p^d$ such that $c(G_i) = (p^i\mathbf{Z}_p)^d$ for $i \geqslant 0$*
4. *If $g, h \in G$, then $gh^{-1} \in G_i$ if and only if $c(g) - c(h) \in (p^i\mathbf{Z}_p)^d$*

Proof. — Properties (1-4) are proved in §4 of [**DdSMS99**]. Alternatively, a uniform pro-p group G has a natural integer valued p-valuation ω such that (G, ω) is saturated (remark 2.1 of [**Klo05**]). Properties (1-4) are then proved in §26 of [**Sch11**]. □

For example, the pro-p group $\mathbf{Z}_p^d$ is uniform for all $d \geqslant 1$.

Lemma 1.1.2. — *If G is a uniform pro-p group, and H is a uniform open subgroup of G, there exists $j \geqslant 0$ such that $G_{i+j} \subset H_i$ for all $i \geqslant 0$.*

Proof. — This follows from the fact that $\{G_i\}_{i \geqslant 0}$ forms a basis of neighborhoods of the identity in G. □

A p-adic Lie group is a p-adic manifold that has a compatible group structure. For example, $\mathrm{GL}_n(\mathbf{Z}_p)$ and its closed subgroups are p-adic Lie groups. We refer to [**Sch11**] for a comprehensive treatment of the theory. Every uniform pro-p group is a p-adic Lie group. Conversely, we have the following.

Proposition 1.1.3. — *Every p-adic Lie group Γ has a uniform open subgroup G, and the rank of G is the dimension of Γ.*

Proof. — See Interlude A (pages 97–98) of [**DdSMS99**]. □

Proposition 1.1.4. — *Let G be a pro-p group of finite rank, and N a closed normal subgroup of G. There exists an open subgroup G' of G such that G', $G' \cap N$ and $G'/G' \cap N$ are all uniform.*

Proof. — This is stated and proved on page 64 of [**DdSMS99**] (their H is our G'). □

1.2. Super-Hölder functions and vectors. — Let M be an E-vector space, endowed with a valuation val_M such that $\mathrm{val}_M(xm) = \mathrm{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M-adic topology. Throughout this §, G denotes a uniform pro-p group.

Definition 1.2.1. — We say that $f : G \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbf{R}$ and $e > 0$ such that $\mathrm{val}_M(f(g) - f(h)) \geqslant p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geqslant 0$.

Remark 1.2.2. — If $G = \mathbf{Z}_p$ and $e = 1$, we recover the functions defined in §1.1 [**BR22**] (see also remark 1.12 of ibid).

In the above definition, e will usually be equal to either 1 or $\dim(G)$.

We let $\mathcal{H}_e^{\lambda,\mu}(G, M)$ denote the space of functions such that $\mathrm{val}_M(f(g) - f(h)) \geqslant p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geqslant 0$, and $\mathcal{H}_e^\lambda(G, M) = \cup_{\mu \in \mathbf{R}} \mathcal{H}_e^{\lambda,\mu}(G, M)$ and $\mathcal{H}_e(G, M) = \cup_{\lambda \in \mathbf{R}} \mathcal{H}_e^\lambda(G, M)$.

If M, N are two valued E-vector spaces, and $f : M \to N$ is an E-linear map, we say that f is Hölder-continuous if there exists $c > 0$, $d \in \mathbf{R}$ such that $\mathrm{val}_N(f(x)) \geqslant c \cdot \mathrm{val}_M(x) + d$ for all $x \in M$.

Proposition 1.2.3. — *If $\pi : M \to N$ is a Hölder-continuous linear map, we get a map $\mathcal{H}_e(G, M) \to \mathcal{H}_e(G, N)$.*

Proof. — Take $c, d \in \mathbf{R}$ of Hölder continuity for π, $f \in \mathcal{H}_e^{\lambda,\mu}(G, M)$, and $g, h \in G$ with $gh^{-1} \in G_i$. We have $\mathrm{val}_N(\pi(f(g)) - \pi(f(h))) \geqslant c \cdot \mathrm{val}_M(f(g) - f(h)) + d \geqslant cp^\lambda \cdot p^{ei} + (\mu + d)$, so that $\pi \circ f \in \mathcal{H}_e^{\lambda',\mu'}(G, N)$ with $p^{\lambda'} = cp^\lambda$, and $\mu' = \mu + d$. $\square$

Proposition 1.2.4. — *If $\alpha : G \to H$ is a group homomorphism, we get a map $\alpha^* : \mathcal{H}_e(H, M) \to \mathcal{H}_e(G, M)$.*

Proof. — By definition of the subgroups G_i and H_i, we have $\alpha(G_i) \subset H_i$ for all i. Take $f \in \mathcal{H}_e^{\lambda,\mu}(H, M)$, and $g, h \in G$ with $gh^{-1} \in G_i$. We have $\mathrm{val}_M(f(\alpha(g)) - f(\alpha(h))) \geqslant p^\lambda \cdot p^{ei} + \mu$ as $\alpha(g)\alpha(h)^{-1} \in H_i$, so that $\alpha^*(f) = f \circ \alpha \in \mathcal{H}_e^{\lambda,\mu}(G, M)$. $\square$

Proposition 1.2.5. — *Suppose that M is a ring, and that $\mathrm{val}_M(mm') \geqslant \mathrm{val}_M(m) + \mathrm{val}_M(m')$ for all $m, m' \in M$. If $c \in \mathbf{R}$, let $M_c = M^{\mathrm{val}_M \geqslant c}$.*

1. *If $f \in \mathcal{H}_e^{\lambda,\mu}(G, M_c)$ and $g \in \mathcal{H}_e^{\lambda,\nu}(G, M_d)$, and $\xi = \min(\mu + d, \nu + c)$, then $fg \in \mathcal{H}_e^{\lambda,\xi}(G, M_{c+d})$.*
2. *If $\lambda, \mu \in \mathbf{R}$, then $\mathcal{H}_e^{\lambda,\mu}(G, M_0)$ is a subring of $C^0(G, M)$.*
3. *If $\lambda \in \mathbf{R}$, then $\mathcal{H}_e^\lambda(G, M)$ is a subring of $C^0(G, M)$.*

Proof. — Items (2) and (3) follow from item (1), which we now prove. If $x, y \in G$, then

$$(fg)(x) - (fg)(y) = (f(x) - f(y))g(x) + (g(x) - g(y))f(y),$$

which implies the claim. □

We now assume that M is endowed with an E-linear action by isometries of G. If $m \in M$, let $\mathrm{orb}_m : G \to M$ denote the function defined by $\mathrm{orb}_m(g) = g \cdot m$.

Definition 1.2.6. — Let $M^{G\text{-}e\text{-sh},\lambda,\mu}$ be those $m \in M$ such that $\mathrm{orb}_m \in \mathcal{H}_e^{\lambda,\mu}(G, M)$, and let $M^{G\text{-}e\text{-sh},\lambda}$ and $M^{G\text{-}e\text{-sh}}$ be the corresponding sub-E-vector spaces of M.

Remark 1.2.7. — We assume that G acts by isometries on M, but not that G acts continuously on M, namely that $G \times M \to M$ is continuous. However, let M^{cont} denote the set of $m \in M$ such that $\mathrm{orb}_m : G \to M$ is continuous. It is easy to see that M^{cont} is a closed sub-E-vector space of M, and that $G \times M^{\mathrm{cont}} \to M^{\mathrm{cont}}$ is continuous (compare with §3 of [**Eme17**]). We then have $M^{\mathrm{sh}} \subset M^{\mathrm{cont}}$.

Lemma 1.2.8. — *If $m \in M$, then $m \in M^{G\text{-}e\text{-sh},\lambda,\mu}$ if and only if for all $i \geqslant 0$, we have* $\mathrm{val}_M(g \cdot m - m) \geqslant p^\lambda \cdot p^{ei} + \mu$ *for all $g \in G_i$.*

Proof. — If $m \in M$, then $m \in M^{G\text{-}e\text{-sh},\lambda,\mu}$ if and only if the function orb_m is in $\mathcal{H}_e^{\lambda,\mu}(G, M)$, that is, for all g, h with $gh^{-1} \in G_i$, we have $\mathrm{val}_M(g \cdot m - h \cdot m) \geqslant p^\lambda \cdot p^{ei} + \mu$. As G acts by isometries, we have $\mathrm{val}_M(g \cdot m - h \cdot m) = \mathrm{val}_M(h^{-1}g \cdot m - m)$. The result follows, as $h^{-1}g = h^{-1} \cdot gh^{-1} \cdot h \in G_i$. □

Lemma 1.2.9. — *The space $M^{G\text{-}e\text{-sh},\lambda,\mu}$ is a closed sub-E-vector space of M.*

Lemma 1.2.10. — *If $i_0 \geqslant 0$, and $m \in M$ is such that* $\mathrm{val}_M(g \cdot m - m) \geqslant p^\lambda \cdot p^{ei} + \mu$ *for all $g \in G_i$ with $i \geqslant i_0$, then $m \in M^{G\text{-}e\text{-sh},\lambda}$.*

Proof. — Take $i < i_0$, and let R_i be a set of representatives of $G_{i_0} \backslash G_i$. This is a finite set, so there exists $\mu_i \in \mathbf{R}$ such that $\mathrm{val}_M(r \cdot m - m) \geqslant p^\lambda \cdot p^{ei} + \mu_i$ for all $r \in R_i$. If $g \in G_i$, it can be written as $g = hr$ for some $h \in G_{i_0}$ and $r \in R_i$. We then have $g \cdot m - m = hr \cdot m - h \cdot m + h \cdot m - m$, so that $\mathrm{val}_M(g \cdot m - m) \geqslant \min(\mathrm{val}_M(r \cdot m - m), \mathrm{val}_M(h \cdot m - m))$ (recall that G acts by isometries), so $\mathrm{val}_M(g \cdot m - m) \geqslant \min(p^\lambda \cdot p^{ei} + \mu_i, p^\lambda \cdot p^{ei_0} + \mu) \geqslant p^\lambda \cdot p^{ei} + \min(\mu, \mu_i)$ as $i_0 > i$. If μ' is the min of μ and the μ_i for $0 \leqslant i < i_0$, then $m \in M^{G\text{-}e\text{-sh},\lambda,\mu'}$. □

Recall that if $k \geqslant 0$, then G_k is also a uniform pro-p group.

Lemma 1.2.11. — *If $k \geqslant 0$ then $M^{G\text{-}e\text{-sh},\lambda} = M^{G_k\text{-}e\text{-sh},\lambda+k}$.*

Proof. — Note that $(G_k)_i = G_{i+k}$. The inclusion $M^{G\text{-}e\text{-sh},\lambda} \subset M^{G_k\text{-}e\text{-sh},\lambda+k}$ is obvious, and the reverse inclusion follows from lemma 1.2.10. □

Proposition 1.2.12. — *The space $M^{H\text{-}e\text{-sh}}$ does not depend on the choice of a uniform open subgroup $H \subset G$.*

Proof. — Let H and H' be uniform open subgroups of G. The group $H \cap H'$ contains an open uniform subgroup by prop 1.1.3, so to prove the proposition, we can further assume that $H' \subset H$. We then have $H'_i \subset H_i$ for all i, so that if $m \in M^{H\text{-}e\text{-sh},\lambda,\mu}$, then $m \in M^{H'\text{-}e\text{-sh},\lambda,\mu}$. This implies that $M^{H\text{-}e\text{-sh},\lambda} \subset M^{H'\text{-}e\text{-sh},\lambda}$. Conversely, by lemma 1.1.2, there exists j such that $H_j \subset H'$. The previous reasoning implies that $M^{H'\text{-}e\text{-sh},\lambda} \subset M^{H_j\text{-}e\text{-sh},\lambda}$. Lemma 1.2.11 now implies that $M^{H_j\text{-}e\text{-sh},\lambda} = M^{H\text{-}e\text{-sh},\lambda-j}$.

These inclusions imply the proposition. □

Definition 1.2.13. — If Γ is a p-adic Lie group that acts by isometries on M, we let $M^{e\text{-sh}} = M^{G\text{-}e\text{-sh}}$ where G is any uniform open subgroup of Γ.

Remark 1.2.14. — If $e \leqslant f$, then $M^{f\text{-sh}} \subset M^{e\text{-sh}}$.

Recall that G is a uniform pro-p group. If a closed normal subgroup N of G acts trivially on M, then G/N acts on M.

Proposition 1.2.15. — *If a closed normal subgroup N of G acts trivially on M, then $M^{G\text{-}e\text{-sh}} = M^{G/N\text{-}e\text{-sh}}$.*

Proof. — By prop 1.1.4, G has an open subgroup G' such that G' and G'/N' are uniform (where $N' = G' \cap N$). By prop 1.2.12, we have $M^{G\text{-}e\text{-sh}} = M^{G'\text{-}e\text{-sh}}$ and $M^{G/N\text{-}e\text{-sh}} = M^{G'/N'\text{-}e\text{-sh}}$. Let $\pi : G' \to G'/N'$ denote the projection. We have $\pi(G'_i) = (G'/N')_i$ for all i. Hence if $m \in M$, then $\mathrm{val}_M(g \cdot m - m) \geqslant p^\lambda \cdot p^{ei} + \mu$ for all $g \in G'_i$ if and only if $\mathrm{val}_M(\pi(g) \cdot m - m) \geqslant p^\lambda \cdot p^{ei} + \mu$ for all $\pi(g) \in (G'/N')_i$. □

Proposition 1.2.16. — *Suppose that M is a ring, and that $g(mm') = g(m)g(m')$ and $\mathrm{val}_M(mm') \geqslant \mathrm{val}_M(m) + \mathrm{val}_M(m')$ for all $m, m' \in M$ and $g \in G$.*

1. *If $v \in \mathbf{R}$ and $m, m' \in M^{G\text{-}e\text{-sh},\lambda,\mu} \cap M^{\mathrm{val}_M \geqslant v}$, then $m \cdot m' \in M^{G\text{-}e\text{-sh},\lambda,\mu+v}$.*
2. *If $m \in M^{G\text{-}e\text{-sh},\lambda,\mu} \cap M^\times$, then $1/m \in M^{G\text{-}e\text{-sh},\lambda,\mu-2\,\mathrm{val}_M(m)}$.*

Proof. — Item (1) follows from prop 1.2.5 and lemma 1.2.8. Item (2) follows from

$$g\left(\frac{1}{m}\right) - \frac{1}{m} = \frac{m - g(m)}{g(m)m}.$$

□

2. The field of norms

Let K be a p-adic field, and let K_∞ be an algebraic Galois extension of K, whose Galois group G is a p-adic Lie group of dimension $\geqslant 1$. We assume that K_∞/K is almost totally ramified, namely that the inertia subgroup of G is open in G. Let $d = \dim(G)$ and let $\ell = p^d$. Let $\widetilde{\mathbf{E}}^+_{K_\infty}$ denote the ring $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/p$. This is a perfect domain of characteristic p, which has a natural action of G. The map $(y_j)_{j \geqslant 0} \mapsto (y_{di})_{i \geqslant 0}$ gives an isomorphism between $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$ and $\widetilde{\mathbf{E}}^+_{K_\infty}$, so that $\widetilde{\mathbf{E}}^+_{K_\infty}$ is the ring of integers of the tilt of $\hat{K}_\infty$ (see §3 of [**Sch12**]).

If $x = (x_i)_{i \geqslant 0}$, and $\hat{x}_i$ is a lift of x_i to $\mathcal{O}_{K_\infty}$, then $\ell^i \operatorname{val}_p(\hat{x}_i)$ is independent of $i \geqslant 0$ such that $x_i \neq 0$. We define a valuation on $\widetilde{\mathbf{E}}^+_{K_\infty}$ by $\operatorname{val}_{\mathrm{E}}(x) = \lim_{i \to +\infty} \ell^i \operatorname{val}_p(\hat{x}_i)$.

The aim of this section is to compute $(\widetilde{\mathbf{E}}^+_{K_\infty})^{d\text{-sh}}$. Given definition 1.2.13, we assume from now on (replacing K by a finite subextension if necessary) that G is uniform and that K_∞/K is totally ramified. Let k denote the common residue field of K and K_∞.

2.1. The field of norms. — Let $\mathcal{E}(K_\infty)$ denote the set of finite extensions E of K such that $E \subset K_\infty$. Let $X_K(K_\infty)$ denote the set of sequences $(x_E)_{E \in \mathcal{E}(K_\infty)}$ such that $x_E \in E$ for all $E \in \mathcal{E}(K_\infty)$, and $\mathrm{N}_{F/E}(x_F) = x_E$ whenever $E \subset F$ with $E, F \in \mathcal{E}(K_\infty)$.

If $n \geqslant 0$, let $K_n = K_\infty^{G_n}$ so that $[K_{n+1} : K_n] = \ell$, $\{K_n\}_{n \geqslant 0}$ is a cofinal subset of $\mathcal{E}(K_\infty)$, and $X_K(K_\infty) = \varprojlim_{\mathrm{N}_{K_n/K_{n-1}}} K_n$. If $x = (x_n)_{n \geqslant 0} \in X_K(K_\infty)$, let $\operatorname{val}_{\mathrm{E}}(x) = \operatorname{val}_p(x_0)$.

Theorem 2.1.1. — *Let K and K_∞ be as above.*

1. *If $x, y \in X_K(K_\infty)$, then $\{\mathrm{N}_{K_{n+j}/K_n}(x_{n+j} + y_{n+j})\}_{j \geqslant 0}$ converges for all $n \geqslant 0$.*
2. *If we set $(x+y)_n = \lim_{j \to +\infty} \mathrm{N}_{K_{n+j}/K_n}(x_{n+j} + y_{n+j})$, then $x + y \in X_K(K_\infty)$, and the set $X_K(K_\infty)$ with this addition law, and componentwise multiplication, is a field of characteristic p.*
3. *The function* $\operatorname{val}_{\mathrm{E}}$ *is a valuation on $X_K(K_\infty)$, for which it is complete*
4. *If $\varpi = (\varpi_n)_{n \geqslant 0}$ is a norm compatible sequence of uniformizers of $\mathcal{O}_{K_n}$, the valued field $X_K(K_\infty)$ is isomorphic to $k(\!(\varpi)\!)$ (with* $\operatorname{val}(\varpi) = \operatorname{val}_p(\varpi_0)$*).*

Proof. — By a result of Sen [**Sen72**], K_∞/K is strictly APF in the terminology of §1.2 of [**Win83**] (see 1.2.2 of ibid). The theorem is then proved in §2 of ibid. □

Let $X^+_K(K_\infty) = \varprojlim_{\mathrm{N}_{K_n/K_{n-1}}} \mathcal{O}_{K_n}$ be the ring of integers of the valued field $X_K(K_\infty)$.

If $c > 0$, let $I^c_n = \{x \in \mathcal{O}_{K_n}$ such that $\operatorname{val}_p(x) \geqslant c\}$. If $m, n \geqslant 0$, the map $\mathcal{O}_{K_n}/I^c_n \to \mathcal{O}_{K_{m+n}}/I^c_{m+n}$ is well-defined and injective.

Proposition 2.1.2. — *There exists* $c(K_\infty/K) \leqslant 1$ *such that if* $0 < c \leqslant c(K_\infty/K)$*, then* $\mathrm{val}_p(\mathrm{N}_{K_{n+k}/K_n}(x)/x^{[K_{n+k}:K_n]} - 1) \geqslant c$ *for all* $n, k \geqslant 0$ *and* $x \in \mathcal{O}_{K_{n+k}}$.

Proof. — See [**Win83**] as well as §4 of [**CD15**]. The result follows from the fact (see 1.2.2 of [**Win83**]) that the extension K_∞/K is strictly APF. One can then apply 1.2.1, 4.2.2 and 1.2.3 of [**Win83**]. □

Using prop 2.1.2, we get a map $\iota : X_K^+(K_\infty) \to \varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ given by $(x_n)_{n\geqslant 0} \in \varprojlim_{\mathrm{N}_{K_n/K_{n-1}}} \mathcal{O}_{K_n} \mapsto (\overline{x}_n)_{n\geqslant 0}$. Let $\varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$ denote the set of $(x_n)_{n\geqslant 0} \in \varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ such that $x_n \in \mathcal{O}_{K_n}/I_n^c$ for all $n \geqslant 0$.

Proposition 2.1.3. — *Let* $0 < c \leqslant c(K_\infty/K)$ *be as in prop 2.1.2.*

1. *the natural map* $\widetilde{\mathbf{E}}^+_{K_\infty} \to \varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ *is a bijection*
2. *the map* $\iota : X_K^+(K_\infty) \to \varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c = \widetilde{\mathbf{E}}^+_{K_\infty}$ *is injective and isometric*
3. *the image of* ι *is* $\varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$.

Proof. — See [**Win83**] and §4 of [**CD15**]. We give a few more details for the convenience of the reader. Item (1) is classical (see for instance prop 4.2 of [**CD15**]).The map ι is obviously injective and isometric. For (3), choose $x = (x_n)_{n\geqslant 0} \in \varprojlim_{x\mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$, and choose a lift $\hat{x}_n \in \mathcal{O}_{K_n}$ of x_n. One proves that $\{\mathrm{N}_{K_{n+j}/K_n}(\hat{x}_{n+j})\}_{j\geqslant 0}$ converges to some $y_n \in \mathcal{O}_{K_n}$, and that $(y_n)_{n\geqslant 0} \in X_K^+(K_\infty)$ is a lift of $(x_n)_{n\geqslant 0}$. See §4 of [**CD15**] for details, for instance the proof of lemma 4.1. □

Prop 2.1.3 allows us to see $X_K^+(K_\infty)$, and hence $\varphi^{-n}(X_K^+(K_\infty))$ for all $n \geqslant 0$, as a subring of $\widetilde{\mathbf{E}}^+_{K_\infty}$.

Proposition 2.1.4. — *The ring* $\cup_{n\geqslant 0}\varphi^{-n}(X_K^+(K_\infty))$ *is dense in* $\widetilde{\mathbf{E}}^+_{K_\infty}$.

Proof. — See §4.3 of [**Win83**]. □

2.2. Decompleting the tilt. — We now compute $(\widetilde{\mathbf{E}}^+_{K_\infty})^{d\text{-sh}}$. Since prop 2.2.1 below is vacuous if $p = 2$, we assume in this § that $p \neq 2$.

Proposition 2.2.1. — *If* $0 < c \leqslant 1 - 1/(p-1)$*, and* $x \in \mathcal{O}_{K_\infty}$ *is such that* $\mathrm{val}_p(g(x) - x) \geqslant 1$ *for all* $g \in G_n$*, then the image of* x *in* $\mathcal{O}_{K_\infty}/I_\infty^c$ *belongs to* $\mathcal{O}_{K_n}/I_n^c$.

Proof. — If $\mathrm{val}_p(g(x) - x) \geqslant 1$ for all $g \in \mathrm{Gal}(K^{\mathrm{alg}}/K_n)$, then by theorem 1.7 of [**LB10**] (an optimal version of a theorem of Ax), there exists $y \in K_n$ such that $\mathrm{val}_p(x - y) \geqslant 1 - 1/(p-1)$. This implies the proposition. □

Proposition 2.2.2. — *If* $c = p^\gamma$ *is as above, then* $X_K^+(K_\infty) \subset (\widetilde{\mathbf{E}}^+_{K_\infty})^{G\text{-}d\text{-sh},\gamma,0}$.

Proof. — Take $x = (x_n)_{n \geqslant 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$. If $g \in G_i$, then $g(x_n) = x_n$ for $n \leqslant i$, so that $\mathrm{val}_{\mathrm{E}}(gx - x) \geqslant p^{di} p^\gamma$. □

Theorem 2.2.3. — *We have*

1. $(\widetilde{\mathbf{E}}^+_{K_\infty})^{G\text{-}d\text{-sh},0,0} \subset X_K^+(K_\infty)$
2. $(\widetilde{\mathbf{E}}^+_{K_\infty})^{d\text{-sh}} = \cup_{n \geqslant 0} \varphi^{-n}(X_K^+(K_\infty))$ *and* $\widetilde{\mathbf{E}}^{d\text{-sh}}_{K_\infty} = \cup_{n \geqslant 0} \varphi^{-n}(X_K(K_\infty))$

Proof. — Take $c \leqslant \min(c(K_\infty/K), 1 - 1/(p-1))$. Take $x = (x_n)_{n \geqslant 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/p$. If $n \geqslant 0$ and $x \in (\widetilde{\mathbf{E}}^+_{K_\infty})^{G\text{-}d\text{-sh},0,0}$, then $\mathrm{val}_{\mathrm{E}}(g(x) - x) \geqslant p^{dn}$ if $g \in G_n$. This implies that $\mathrm{val}_p(g(x_n) - x_n) \geqslant 1$ if $g \in G_n$. By prop 2.2.1, the image of x_n in $\mathcal{O}_{K_\infty}/I_\infty^c$ belongs to $\mathcal{O}_{K_n}/I_n^c$. Hence the image of x in $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ belongs to $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$. By prop 2.1.3, x belongs to $X_K^+(K_\infty)$. This proves (1).

Since $\mathrm{val}_{\mathrm{E}}(\varphi(x)) = p \cdot \mathrm{val}_{\mathrm{E}}(x)$, item (2) follows from (1) and props 2.2.2 and 1.2.16. □

Remark 2.2.4. — We have $\widetilde{\mathbf{E}}^{d\text{-sh}}_{K_\infty} \subset \widetilde{\mathbf{E}}^{1\text{-sh}}_{K_\infty}$. The field $\widetilde{\mathbf{E}}^{1\text{-sh}}_{K_\infty}$ contains the field of norms $X_K(L_\infty)$ of any p-adic Lie extension L_∞/K contained in K_∞. Indeed, $\widetilde{\mathbf{E}}_{L_\infty} \subset \widetilde{\mathbf{E}}_{K_\infty}$ and if $e = \dim \mathrm{Gal}(L_\infty/K)$, then $X_K(L_\infty) \subset \widetilde{\mathbf{E}}^{e\text{-sh}}_{L_\infty} \subset \widetilde{\mathbf{E}}^{1\text{-sh}}_{K_\infty}$ (see prop 1.2.15).

Can one give a description of $\widetilde{\mathbf{E}}^{1\text{-sh}}_{K_\infty}$, for example along the lines of §5 of [**Ber16**]?

3. The Lubin-Tate case

We now specialize the constructions of the previous section to the case when K_∞ is generated over K by the torsion points of a Lubin-Tate formal group.

3.1. Lubin-Tate formal groups. — Let K be a finite extension of $\mathbf{Q}_p$ of degree d, with ring of integers $\mathcal{O}_K$, inertia index f, ramification index e, and residue field k. Let $q = p^f = \mathrm{Card}(k)$ and let π be a uniformizer of $\mathcal{O}_K$. Let LT be the Lubin-Tate formal $\mathcal{O}_K$-module attached to π (see [**LT65**]). We choose a coordinate Y on LT. For each $a \in \mathcal{O}_K$ we get a power series $[a](Y) \in \mathcal{O}_K[\![Y]\!]$, that we now see as an element of $k[\![Y]\!]$. In particular, $[\pi](Y) = Y^q$. Let $S(T,U) \in k[\![T,U]\!]$ denote the reduction mod π of the power series giving the addition law in LT in that coordinate. Recall that $S(T,0) = T$ and $S(0,U) = U$.

Lemma 3.1.1. — *If $a, b \in \mathcal{O}_K$ and $i \geqslant 0$, then* $\mathrm{val}_Y([a + p^i b](Y) - [a](Y)) \geqslant p^{di}$.

Furthermore, $[1 + \pi^i](Y) = Y + Y^{q^i} + \mathrm{O}(Y^{q^i+1})$.

Proof. — We have $[\pi](Y) = Y^q$, so $\mathrm{val}_Y([\pi](Y)) \geqslant p^f$. Writing $p = u\pi^e$ for a unit u, we see that $\mathrm{val}_Y([p^i b](Y)) \geqslant p^{di}$ if $b \in \mathcal{O}_K$. If $a, b \in \mathcal{O}_K$ and $i \geqslant 0$, then $[a + bp^i](Y) =$

$S([a](Y),[bp^i](Y))$. We have $S(T,U)=T+U+TU\cdot R(T,U)$, so that $[a+bp^i](Y)-[a](Y)=S([a](Y),[bp^i](Y))-[a](Y)\in[bp^i](Y)\cdot k[\![Y]\!]$. This implies the first result.

The second claim follows likewise from the fact that $[1+\pi^i](Y)=S(Y,[\pi^i](Y))=Y+[\pi^i](Y)+Y\cdot[\pi^i](Y)\cdot R(Y,[\pi^i](Y))$. □

Let $\mathbf{E}=k(\!(Y)\!)$. Let $\mathbf{E}_n=k(\!(Y^{1/q^n})\!)$ and let $\mathbf{E}_\infty=\cup_{n\geqslant 0}\mathbf{E}_n$. These fields are endowed with the Y-adic valuation val_Y, and we let $\mathbf{E}_\star^+$ denote the ring of integers of $\mathbf{E}_\star$. The group $\mathcal{O}_K^\times$ acts on $\mathbf{E}_n$ by $a\cdot f(Y^{1/q^n})=f([a](Y^{1/q^n}))$.

Lemma 3.1.2. — *If $j\geqslant 1$ ($j\geqslant 2$ if $p=2$), then $1+p^j\mathcal{O}_K$ is uniform, and $(1+p^j\mathcal{O}_K)_i=1+p^{i+j}\mathcal{O}_K$.*

Proof. — The map $1+p^j\mathcal{O}_K\to\mathcal{O}_K$, given by $x\mapsto p^{-j}\cdot\log_p(x-1)$, is an isomorphism of pro-p groups taking $1+p^{i+j}\mathcal{O}_K$ to $p^i\mathcal{O}_K$. □

Recall that $d=[K:\mathbf{Q}_p]$, that $f=[k:\mathbf{F}_p]$, and that $q=p^f$.

Proposition 3.1.3. — *We have $\mathbf{E}_n^+=(\mathbf{E}_n^+)^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-fn,0}$.*

Proof. — If $b\in\mathcal{O}_K$ and $i,j\geqslant 0$, then by lemma 3.1.1, we have

$$\mathrm{val}_Y([1+p^{i+j}b](Y^{1/q^n})-Y^{1/q^n})\geqslant 1/q^n\cdot p^{d(i+j)}=p^{dj-fn}\cdot p^{di}.$$

Lemma 3.1.2 then implies that $Y^{1/q^n}\in(\mathbf{E}_n^+)^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-fn,0}$. The lemma now follows from prop 1.2.16 and lemma 1.2.9. □

Corollary 3.1.4. — *We have $\mathbf{E}=\mathbf{E}^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj}$.*

Proof. — This follows from prop 3.1.3 with $n=0$, and prop 1.2.16. □

Proposition 3.1.5. — *If $\varepsilon>0$, then $k[\![Y]\!]^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj+\varepsilon}\subset k[\![Y^p]\!]$.*

Proof. — Take $f(Y)\in k[\![Y]\!]$. There is a power series $h(T,U)\in k[\![T,U]\!]$ such that

$$f(T+U)=f(T)+U\cdot f'(T)+U^2\cdot h(T,U).$$

If $m\geqslant 0$, lemma 3.1.1 implies that $[1+\pi^m](Y)=Y+Y^{q^m}+\mathrm{O}(Y^{q^m+1})$. Therefore,

$$f([1+\pi^m](Y))=f(Y)+(Y^{q^m}+\mathrm{O}(Y^{q^m+1}))\cdot f'(Y)+\mathrm{O}(Y^{2q^m}).$$

If $f(Y)\notin k[\![Y^p]\!]$, then $f'(Y)\neq 0$. Let $\mu=\mathrm{val}_Y(f'(Y))$. The above computations imply that $\mathrm{val}_Y(f([1+\pi^{ei+ej}](Y))-f(Y))=p^{dj}\cdot p^{di}+\mu$ for $i\gg 0$.

This implies the claim, since $\pi^e\mathcal{O}_K=p\mathcal{O}_K$. □

Corollary 3.1.6. — *We have $\mathbf{E}_\infty^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-fn}=\mathbf{E}_n$.*

Proof. — We prove that, more generally, $\mathbf{E}_\infty^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-\ell} = k(\!(Y^{1/p^\ell})\!)$. Take $f(Y^{1/p^m}) \in (\mathbf{E}_\infty^+)^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-\ell}$ where $f(Y) \in k[\![Y]\!]$. Since $\mathrm{val}_Y(h^p) = p \cdot \mathrm{val}_Y(h)$ for all $h \in \widetilde{\mathbf{E}}^+$, we have $f^{p^m}(Y) \in (\mathbf{E}_\infty^+)^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj-\ell+m}$, where $f^{p^m}(Y) \in E[\![Y]\!]$ is $f^{p^m}(Y) = f(Y^{1/p^m})^{p^m}$. If $m \geqslant \ell + 1$, then prop 3.1.5 implies that $f^{p^m}(Y) \in E[\![Y^p]\!]$, so that $f(Y) = g(Y^p)$, and $f(Y^{1/p^m}) = g(Y^{1/p^{m-1}})$. This implies the claim. □

3.2. Decompletion of $\widetilde{\mathbf{E}}$. — Since we use the results of §2.2, we once more assume that $p \neq 2$. Let $\widetilde{\mathbf{E}}$ denote the Y-adic completion of $\mathbf{E}_\infty$.

Theorem 3.2.1. — *We have* $\widetilde{\mathbf{E}}^{1+p^j\mathcal{O}_K\text{-}d\text{-sh},dj} = \mathbf{E}$, *and* $\widetilde{\mathbf{E}}^{d\text{-sh}} = \mathbf{E}_\infty$.

Proof. — Let $K_\infty = K(\mathrm{LT}[\pi^\infty])$ denote the extension of K generated by the torsion points of LT, and let $\Gamma = \mathrm{Gal}(K_\infty/K)$. The Lubin-Tate character χ_π gives rise to an isomorphism $\chi_\pi : \Gamma \to \mathcal{O}_K^\times$. For $n \geqslant 1$, let $K_n = K(\mathrm{LT}[\pi^n])$. If $(\pi_n)_{n\geqslant 1}$ is a compatible sequence of primitive π^n-torsion points of LT, then π_n is a uniformizer of $\mathcal{O}_{K_n}$, $\varpi = (\pi_n)_{n\geqslant 0}$ belongs to $\varprojlim_{\mathrm{N}_{K_n/K_{n-1}}} \mathcal{O}_{K_n}$, and $X_K(K_\infty) = k(\!(\varpi)\!)$ by theorem 2.1.1. If $g \in \Gamma$, then $g(\varpi) = [\chi_\pi(g)](\varpi)$, so that if we identify Γ and $\mathcal{O}_K^\times$, then $X_K(K_\infty) = \mathbf{E}$ with its action of $\mathcal{O}_K^\times$. Prop 2.1.4 implies that $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_{K_\infty}$ as valued fields with an action of (an open subgroup of) $\mathcal{O}_K^\times$. We can therefore apply theorem 2.2.3, and get $(\widetilde{\mathbf{E}}^+)^{d\text{-sh}} = \mathbf{E}_\infty^+$. This implies the second statement. The first one then follows from coro 3.1.6. □

Remark 3.2.2. — In the above proof, note that $K_\infty^{1+p^n\mathcal{O}_K} = K_{ne}$, so that the numbering is not the same as in §2.1.

Remark 3.2.3. — We can define Lubin-Tate Γ-modules over $\mathbf{E}$ as in §3.2 of [**BR22**]. The results proved in that section carry over to the Lubin-Tate setting without difficulty.

In theorem 2.9 of [**BR22**], we proved theorem 3.2.1 above in the cyclotomic case, using Tate traces. There are no such Tate traces in the Lubin-Tate case if $K \neq \mathbf{Q}_p$. We now explain why this is so. More precisely, we prove that there is no Γ-equivariant k-linear projector $\widetilde{\mathbf{E}} \to \mathbf{E}$ if $K \neq \mathbf{Q}_p$. Choose a coordinate T on LT such that $\log_{\mathrm{LT}}(T) = \sum_{n\geqslant 0} T^{q^n}/\pi^n$, so that $\log'_{\mathrm{LT}}(T) \equiv 1 \bmod \pi$. Let $\partial = 1/\log'_{\mathrm{LT}}(T) \cdot d/dT$ be the invariant derivative on LT. Let $\varphi_q = \varphi^f$ where $q = p^f$.

Lemma 3.2.4. — *We have* $d\gamma(Y)/dY \equiv \chi_\pi(\gamma)$ *in* $\mathbf{E}$ *for all* $\gamma \in \Gamma$.

Proof. — Since $\log'_{\mathrm{LT}} \equiv 1 \bmod \pi$, we have $\partial = d/dY$ in $\mathbf{E}$. Applying $\partial \circ \gamma = \chi_\pi(\gamma)\gamma \circ \partial$ to Y, we get the claim. □

Lemma 3.2.5. — *If* $\gamma \in \Gamma$ *is nontorsion, then* $\mathbf{E}^{\gamma=1} = k$.

Proposition 3.2.6. — *If $K \neq \mathbf{Q}_p$, there is no Γ-equivariant map $R : \mathbf{E} \to \mathbf{E}$ such that $R(\varphi_q(f)) = f$ for all $f \in \mathbf{E}$.*

Proof. — Suppose that such a map exists, and take $\gamma \in \Gamma$ nontorsion and such that $\chi_\pi(\gamma) \equiv 1 \bmod \pi$. We first show that if $f \in \mathbf{E}$ is such that $(1-\gamma)f \in \varphi_q(\mathbf{E})$, then $f \in \varphi_q(\mathbf{E})$. Write $f = f_0 + \varphi_q(R(f))$ where $f_0 = f - \varphi_q(R(f))$, so that $R(f_0) = 0$ and $(1-\gamma)f_0 = \varphi_q(g) \in \varphi_q(\mathbf{E})$. Applying R, we get $0 = (1-\gamma)R(f_0) = g$. Hence $g = 0$ so that $(1-\gamma)f_0 = 0$. Since $\mathbf{E}^{\gamma=1} = k$ by lemma 3.2.5, this implies $f_0 \in k$, so that $f \in \varphi_q(\mathbf{E})$.

However, lemma 3.2.4 and the fact that $\chi_\pi(\gamma) \equiv 1 \bmod \pi$ imply that $\gamma(Y) = Y + f_\gamma(Y^p)$ for some $f_\gamma \in \mathbf{E}$, so that $\gamma(Y^{q/p}) = Y^{q/p} + \varphi_q(g_\gamma)$. Hence $(1-\gamma)(Y^{q/p}) \in \varphi_q(\mathbf{E})$ even though $Y^{q/p}$ does not belong to $\varphi_q(\mathbf{E})$. Therefore, no such map R can exist. □

Corollary 3.2.7. — *If $K \neq \mathbf{Q}_p$, there is no Γ-equivariant k-linear projector $\varphi_q^{-1}(\mathbf{E}) \to \mathbf{E}$. A fortiori, there is no Γ-equivariant k-linear projector $\widetilde{\mathbf{E}} \to \mathbf{E}$.*

Proof. — Given such a projector Π, we could define R as in prop 3.2.6 by $R = \Pi \circ \varphi_q^{-1}$. □

3.3. The perfectoid commutant of Aut(LT)**.** — In §3.1 of [**BR22**], we computed the perfectoid commutant of $\mathrm{Aut}(\mathbf{G}_{\mathrm{m}})$. We now use theorem 3.2.1 to do the same for Aut(LT). We still assume that $p \neq 2$.

Theorem 3.3.1. — *If $u \in \widetilde{\mathbf{E}}^+$ is such that $\mathrm{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, there exists $b \in \mathcal{O}_K^\times$ and $n \in \mathbf{Z}$ such that $u(Y) = [b](Y^{q^n})$.*

Recall that a power series $f(Y) \in k[\![Y]\!]$ is separable if $f'(Y) \neq 0$. If $f(Y) \in Y \cdot k[\![Y]\!]$, we say that f is invertible if $f'(0) \in k^\times$, which is equivalent to f being invertible for composition (denoted by $\circ$). We say that $w(Y) \in Y \cdot k[\![Y]\!]$ is nontorsion if $w^{\circ n}(Y) \neq Y$ for all $n \geqslant 1$. If $w(Y) = \sum_{i \geqslant 0} w_i Y^i \in k[\![Y]\!]$ and $m \in \mathbf{Z}$, let $w^{(m)}(Y) = \sum_{i \geqslant 0} w_i^{p^m} Y^i$. Note that $(w \circ v)^{(m)} = w^{(m)} \circ v^{(m)}$.

Proposition 3.3.2. — *Let $w(Y) \in Y + Y^2 \cdot k[\![Y]\!]$ be a nontorsion series, and let $f(Y) \in Y \cdot k[\![Y]\!]$ be a separable power series. If $w^{(m)} \circ f = f \circ w$ for some $m \in \mathbf{Z}$, then f is invertible.*

Proof. — This is a slight generalization of lemma 6.2 of [**Lub94**]. Write

$$\begin{aligned} f(Y) &= f_n Y^n + \mathrm{O}(Y^{n+1}) \\ f'(Y) &= g_j Y^j + \mathrm{O}(Y^{j+1}) \\ w(Y) &= Y + w_r Y^r + \mathrm{O}(Y^{r+1}), \end{aligned}$$

with $f_n, g_j, w_r \neq 0$. Since w is nontorsion, we can replace w by $w^{\circ p^\ell}$ for $\ell \gg 0$ and assume that $r \geqslant j+1$. We have

$$\begin{aligned} w^{(m)} \circ f &= f(Y) + w_r^{(m)} f(Y)^r + \mathrm{O}(Y^{n(r+1)}) \\ &= f(Y) + w_r^{(m)} f_n^r Y^{nr} + \mathrm{O}(Y^{nr+1}). \end{aligned}$$

If $j=0$, then $n=1$ and we are done, so assume that $j \geqslant 1$. We have

$$\begin{aligned} f \circ w &= f(Y + w_r Y^r + \mathrm{O}(Y^{r+1})) \\ &= f(Y) + w_r Y^r f'(Y) + \mathrm{O}(Y^{2r}) \\ &= f(Y) + w_r g_j Y^{r+j} + \mathrm{O}(Y^{r+j+1}). \end{aligned}$$

This implies that $nr = r+j$, hence $(n-1)r = j$, which is impossible if $r > j$ unless $n=1$. Hence $n=1$ and f is invertible. $\square$

Lemma 3.3.3. — *If $u \in \widetilde{\mathbf{E}}^+$ is such that $\mathrm{val}_X(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, then $u \in (\widetilde{\mathbf{E}}^+)^{d\text{-sh}}$.*

Proof. — The group $\mathcal{O}_K^\times$ acts on $\widetilde{\mathbf{E}}^+$ by $g \cdot u = u \circ [g]$. By lemmas 3.1.1 and 3.1.2, the function $g \mapsto [g] \circ u$ is in $\mathcal{H}_d^\lambda(1 + p\mathcal{O}_K, \widetilde{\mathbf{E}}^+)$, where $p^\lambda = \mathrm{val}_Y(u)$. $\square$

Proof of theorem 3.3.1. — Take $u \in \widetilde{\mathbf{E}}$ such that $\mathrm{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$. By lemma 3.3.3 and theorem 3.2.1, there is an $m \in \mathbf{Z}$ such that $f(Y) = u(Y)^{p^m}$ belongs to $Y \cdot k[\![Y]\!]$ and is separable. Take $g \in 1 + \pi\mathcal{O}_K$ such that g is nontorsion, and let $w(Y) = [g](Y)$ so that $u \circ w = w \circ u$. We have $f \circ w = w^{(m)} \circ f$. By prop 3.3.2, f is invertible. In addition, $f \circ w = w^{(m)} \circ f$ if $w(Y) = [g](Y)$ for all $g \in \mathcal{O}_K^\times$. Hence $f_0 \cdot \overline{g} = \overline{g}^{p^m} \cdot f_0$, so that $a^{p^m} = a$ for all $a = \overline{g} \in k$. This implies that $\mathbf{F}_q \subset \mathbf{F}_{p^{|m|}}$, so that $m = fn$ for some $n \in \mathbf{Z}$. Hence $w^{(m)} = w$, and $f \circ [g] = [g] \circ f$ for all $g \in \mathcal{O}_K^\times$. Theorem 6 of [**LS07**] implies that $f \in \mathrm{Aut}(\mathrm{LT})$. Hence there exists $b \in \mathcal{O}_K^\times$ such that $u(Y) = [b](Y^{q^n})$. $\square$

4. Mahler expansions and super-Hölder functions

In §1.3 of [**BR22**], we proved an analogue of Mahler's theorem for continuous functions $\mathbf{Z}_p \to M$, and then gave a characterization of super-Hölder functions in terms of their Mahler expansions. We now indicate how these results generalize to functions $G \to M$ for a uniform pro-p group G. Given the definition of super-Hölder functions and the existence of a coordinate $c : G \to \mathbf{Z}_p^d$ as in prop 1.1.1, it is enough to study functions $\mathbf{Z}_p^d \to M$. We generalize the setting a little bit, and study functions $\mathcal{O}_K^d \to M$ where K

is a finite extension of $\mathbf{Q}_p$. Let K be such a field, fix a uniformizer π of $\mathcal{O}_K$ and let k be the residue field of K. Let $q = \mathrm{Card}(k)$.

4.1. Good bases and wavelets. — Let $X = \mathcal{O}_K^d$, which we endow with the valuation $\mathrm{val}_X(x_1, \dots, x_d) = \min_i \mathrm{val}_\pi(x_i)$. For $n \geqslant 0$, let $X_n = \pi^n X = \{x \in X, \mathrm{val}_X(x) \geqslant n\}$.

We endow X with the val_X-adic topology. For any set Y, we denote by $\mathrm{LC}(X,Y)$ the set of locally constant functions $X \to Y$. For $n \geqslant 0$ we denote by $\mathrm{LC}_n(X,Y)$ the subset of elements of $\mathrm{LC}(X,Y)$ that factor through X/X_n. Let $I = \cup_{n\geqslant 0} I_n$ be a set of indices, where $I_n \subset I_{n+1}$ for all $n \geqslant 0$, and $\mathrm{Card}(I_n) = \mathrm{Card}(X/X_n) = q^{nd}$. Let E be a field of characteristic p.

Definition 4.1.1. — A family $\{h_i\}_{i\in I}$ is a good basis of $\mathrm{LC}(X,E)$ if it is a basis of the E-vector space $\mathrm{LC}(X,E)$ such that for all $n \geqslant 0$, $\{h_i\}_{i\in I_n}$ is a basis of $\mathrm{LC}_n(X,E)$.

Let M be (as usual) an E-vector space with a valuation val_M, such that $\mathrm{val}_M(ax) = \mathrm{val}_M(x)$ for all $a \in E^\times$ and $x \in M$. We assume that M is separated and complete for the val_M-adic topology.

Proposition 4.1.2. — *Every $f \in \mathrm{LC}_n(X,M)$ can be written uniquely as $\sum_{i\in I_n} h_i \cdot m_i$ for some elements $m_i \in M$. Moreover, $\inf_{x\in X} \mathrm{val}_M(f(x)) = \inf_{i\in I_n} \mathrm{val}_M(m_i)$.*

Proof. — Let $\{\delta_x\}_{x\in X/X_n}$ be the basis of $\mathrm{LC}_n(X,E)$ defined as follows: δ_x is the characteristic function of $x + X_n$. Then $f \in \mathrm{LC}_n(X,M)$ is equal to $\sum_{x\in X/X_n} \delta_x \cdot f(x)$.

As $\{h_i\}_{i\in I_n}$ is also a basis of $\mathrm{LC}_n(X,E)$, we can write $\delta_x = \sum_{i\in I_n} a_{i,x} h_i$ for some elements $a_{i,x} \in E$. We now have $f = \sum_{i\in I_n} h_i \cdot m_i$ where $m_i = \sum_{x\in X/X_n} a_{i,x} f(x)$. This formula implies that $\inf_{i\in I_n} \mathrm{val}_M(m_i) \geqslant \inf_{x\in X} \mathrm{val}_M(f(x))$.

On the other hand we can also write $h_i = \sum_{x\in X/X_n} b_{x,i}\delta_x$ for some elements $b_{x,i} \in E$, so that $f(x) = \sum_{i\in I_n} b_{x,i} m_i$. This implies that $\inf_{i\in I_n} \mathrm{val}_M(m_i) \leqslant \inf_{x\in X} \mathrm{val}_M(f(x))$. □

We now give an example of a particularly nice good basis of $\mathrm{LC}(X,E)$, the basis of wavelets (see §I.3 of [**Col10**] and §2.1 of [**dS16**]). Let $\mathcal{T}$ be a set of representatives of X/X_1 in X, chosen so that the representative of 0 is 0. For each $n \geqslant 0$, let $\mathcal{R}_n$ be the set of representatives of X/X_n defined as follows: $\mathcal{R}_0 = \{0\}$, and for $n \geqslant 1$, $\mathcal{R}_n = \{\sum_{i=0}^{n-1} \pi^i x_i$, $x_i \in \mathcal{T}$ for all $i\}$. We have $\mathcal{R}_1 = \mathcal{T}$, and $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ for all n. Let $\mathcal{R} = \cup_{n\geqslant 0}\mathcal{R}_n$. If $r \in \mathcal{R}$ let $\ell(r)$ be the smallest n such that $r \in \mathcal{R}_n$. For $r \in \mathcal{R}$, let χ_r be the characteristic function of the closed disc $r + X_{\ell(r)} = \{x \in X, \mathrm{val}_X(x - r) \geqslant \ell(r)\}$.

Proposition 4.1.3. — *The set $\{\chi_r\}_{r\in\mathcal{R}}$ is a good basis of* $\mathrm{LC}(X,E)$.

Proof. — We prove that for all $n \geqslant 0$, the set $\{\chi_r\}_{r \in \mathcal{R}_n}$ is a basis of $\mathrm{LC}_n(X,E)$. Consider the basis $\{\delta_r\}_{r\in\mathcal{R}_n}$ of $\mathrm{LC}_n(X,E)$, where δ_r is the characteristic function of $r+X_n$. We have

$$\chi_r = \sum_{r' \in \mathcal{R}_{n-\ell(r)}} \delta_{r+\pi^{\ell(r)}r'}.$$

This implies that if we write $\mathcal{R}_n = (\mathcal{R}_n \setminus \mathcal{R}_{n-1}) \sqcup \ldots \sqcup (\mathcal{R}_1 \setminus \mathcal{R}_0) \sqcup \mathcal{R}_0$ and we express the family $\{\chi_r\}_{r\in\mathcal{R}_n}$ in terms of the basis $\{\delta_r\}_{r\in\mathcal{R}_n}$, we get a unipotent matrix. This shows that $\{\chi_r\}_{r\in\mathcal{R}_n}$ is also a basis of $\mathrm{LC}_n(X,E)$. $\square$

4.2. Expansions of continuous functions. — We show that every continuous function $X \to M$ has a convergent expansion along a good basis of X, and prove some continuity estimates in terms of the coefficients of the expansion. If $\{m_i\}_{i\in I}$ is a family of M, we say that $m_i \to 0$ if $\inf_{i \notin I_n} \mathrm{val}_M(m_i) \to +\infty$ as $n \to +\infty$.

Theorem 4.2.1. — *Let $\{h_i\}_{i\in I}$ be a good basis of* $\mathrm{LC}(X,E)$.

If $\{m_i\}_{i\in I}$ is a family of M such that $m_i \to 0$, the function $f : X \to M$ given by $f = \sum_{i\in I} h_i \cdot m_i$ belongs to $C^0(X,M)$, and $\inf_{x\in X} \mathrm{val}_M(f(x)) = \inf_{i\in I} \mathrm{val}_M(m_i)$.

Conversely, if $f \in C^0(X,M)$, there exists a unique family $\{m_i(f)\}_{i\in I}$ of elements of M such that $m_i(f) \to 0$ and such that $f = \sum_{i\in I} h_i \cdot m_i(f)$.

Proof. — Let $\{m_i\}_{i\in I}$ be a family of M such that $m_i \to 0$. If $f_n = \sum_{i\in I_n} h_i \cdot m_i$, then $f_n \in C^0(X,M)$, and f is the uniform limit of the f_n. We have $\inf_X \mathrm{val}_M(f_n(x)) = \inf_{i\in I_n} \mathrm{val}_M(m_i)$ by prop 4.1.2. Since $m_i \to 0$, we have $\inf_{i\in I} \mathrm{val}_M(m_i) = \inf_{i\in I_n} \mathrm{val}_M(m_i)$ for $n \gg 0$. Hence $\inf_X \mathrm{val}_M(f_n(x)) = \inf_{i\in I} \mathrm{val}_M(m_i)$ for $n \gg 0$. Since $\inf_{x\in X} \mathrm{val}_M(f(x)) = \lim_n \inf_x \mathrm{val}_M(f_n(x))$, we have $\inf_{x\in X} \mathrm{val}_M(f(x)) = \inf_{i\in I} \mathrm{val}_M(m_i)$.

We now prove the converse. Let $M_n = \{m \in M, \mathrm{val}_M(m) \geqslant n\}$, let $\pi_n : M \to M/M_n$ be the projection, and for each n, fix a lift $\psi_n : M/M_n \to M$. Take $f \in C^0(X,M)$, and let $f_n = \psi_n \circ \pi_n \circ f$. As f and f_n coïncide modulo M_n, f is the uniform limit of the f_n. On the other hand, $\pi_n \circ f$ is locally constant, and therefore so is f_n. As X is compact, there exists some $k(n) \geqslant 0$ such that $f_n \in \mathrm{LC}_{k(n)}(X,M)$. By prop 4.1.2, we can write $f_n = \sum_{i\in I} h_i \cdot m_{i,n}$, where $m_{i,n} = 0$ if $i \notin I_{k(n)}$. We have $\mathrm{val}_M(m_{i,n} - m_{i,n'}) \geqslant \min(n,n')$ by construction, so that for each i, the sequence $\{m_{i,n}\}_n$ converges to some $m_i \in M$. Moreover, if $i \notin I_{k(n)}$, then $\mathrm{val}_M(m_i) \geqslant n$, so that $m_i \to 0$. The continuous function $\sum_{i\in I} h_i \cdot m_i$ is the uniform limit of the f_n, so that finally $f = \sum_{i\in I} h_i \cdot m_i$. $\square$

Proposition 4.2.2. — *Take $f \in C^0(X,M)$ and $t \in \mathbf{Z}_{\geqslant 0}$. If $\{h_i\}_{i\in I}$ is a good basis of* $\mathrm{LC}(X,E)$*, and we write $f = \sum_i h_i \cdot m_i$ with $m_i \to 0$, then* $\inf_{i\notin I_t} \mathrm{val}_M(m_i)$ *depends only on f and not on the choice of the good basis.*

Proof. — Fix two good bases $\{h_i\}_{i\in I}$ and $\{h'_i\}_{i\in I}$ of $\mathrm{LC}(X,E)$. There exists a family $\{\lambda_{i,j}\}_{(i,j)\in I\times I}$ of elements of E such that $h_i = \sum_j \lambda_{i,j} h'_j$ for all i. Moreover, if $i\in I_c$ then $\lambda_{i,j}=0$ for all $j\notin I_c$. Now write $f=\sum_{i\in I} h_i\cdot m_i(f)=\sum_{i\in I} h'_i\cdot m'_i(f)$. We also have

$$f=\sum_i\Big(\sum_j \lambda_{i,j}h'_j\Big)\cdot m_i(f)=\sum_j h'_j\cdot\Big(\sum_i \lambda_{i,j}m_i(f)\Big),$$

so that $m'_j(f)=\sum_i \lambda_{i,j}m_i(f)$. If $j\notin I_t$, then $m'_j(f)=\sum_{i\notin I_t}\lambda_{i,j}m_i(f)$, as $\lambda_{i,j}=0$ if $i\in I_t$ and $j\notin I_t$. This implies that $\inf_{j\notin I_t}\mathrm{val}_M(m'_j(f))\geqslant \inf_{i\notin I_t}\mathrm{val}_M(m_i(f))$.

By symmetry, we get that $\inf_{j\notin I_t}\mathrm{val}_M(m'_j(f))=\inf_{i\notin I_t}\mathrm{val}_M(m_i(f))$. □

Theorem 4.2.3. — *Take $f\in C^0(X,M)$ and $t\in\mathbf{Z}_{\geqslant 0}$.*

If $\{h_i\}_{i\in I}$ is a good basis of $\mathrm{LC}(X,E)$, and we write $f=\sum_i h_i\cdot m_i$ with $m_i\to 0$, then

$$\inf_{i\notin I_t}\mathrm{val}_M(m_i)=\inf_{\substack{x,y\in X\\ \mathrm{val}_X(x-y)\geqslant t}}\mathrm{val}_M(f(x)-f(y)).$$

Proof. — Let $C_t(f)=\inf_{x,y\in X,\mathrm{val}_X(x-y)\geqslant t}\mathrm{val}_M(f(x)-f(y))$ and $B_t(f)=\inf_{i\notin I_t}\mathrm{val}_M(m_i)$.

If $x\in X$ and $z\in X_t$, then $f(x+z)-f(x)=\sum_{i\in I}(h_i(x+z)-h_i(z))\cdot m_i(f)$. As $h_i\in\mathrm{LC}_t(X,E)$ for $i\in I_t$, the above equality gives us

$$f(x+z)-f(x)=\sum_{i\notin I_t}(h_i(x+z)-h_i(z))\cdot m_i(f).$$

This implies that $C_t(f)\geqslant B_t(f)$.

We now prove the converse inequality. By prop 4.2.2, $B_t(f)$ is independent of the choice of a good basis, and we choose the wavelet basis of prop 4.1.3. Write $f=\sum_{r\in\mathcal{R}}\chi_r\cdot m_r(f)$, so that we want to show that $\mathrm{val}_M(m_r(f))\geqslant C_t(f)$ for all $r\notin\mathcal{R}_t$. If $x\in X$, define $g_x: X\to M$ by $g_x(z)=f(x+\pi^t z)-f(x)$, and write $g_x=\sum_{r\in\mathcal{R}}\chi_r\cdot m_r(g_x)$. For each $r\in\mathcal{R}$, we can write uniquely $r=r_t+\pi^t s$ with $r_t\in\mathcal{R}_t$, where $s=0$ if $r\in\mathcal{R}_t$, and $s\neq 0\in\mathcal{R}_{\ell(r)-t}$ if $r\notin\mathcal{R}_t$. For $x\in\mathcal{R}_t$ and $r\notin\mathcal{R}_t$, the map $z\mapsto\chi_r(x+\pi^t z)-\chi_r(x)$ is the zero function if $r_t\neq x$, and is χ_s if $r_t=x$. This implies that if $x\in\mathcal{R}_t$, then

$$\begin{aligned} g_x(z) &= \sum_{r\in\mathcal{R}}\Big(\chi_r(x+\pi^t z)-\chi_r(x)\Big)\cdot m_r(f)\\ &= \sum_{r\notin\mathcal{R}_t}\Big(\chi_r(x+\pi^t z)-\chi_r(x)\Big)\cdot m_r(f)\\ &= \sum_{s\notin\mathcal{R}_0}\chi_s(z)\cdot m_{x+\pi^t s}(f).\end{aligned}$$

Therefore if $x\in\mathcal{R}_t$, then $m_0(g_x)=0$ and $m_s(g_x)=m_{x+\pi^t s}(f)$ if $s\neq 0$. We have $\inf_{s\in\mathcal{R}}\mathrm{val}_M(m_s(g_x))=\inf_{z\in X}\mathrm{val}_M(g_x(z))\geqslant C_t(f)$, so that $\mathrm{val}_M(m_s(g_x))\geqslant C_t(f)$ for all $x\in X$ and $s\in\mathcal{R}$. This implies that for all $x\in\mathcal{R}_t$ and $s\neq 0$, $\mathrm{val}_M(m_{x+\pi^t s}(f))\geqslant C_t(f)$. Hence for all $r\notin\mathcal{R}_t$, we have $\mathrm{val}_M(m_r(f))\geqslant C_t(f)$. □

4.3. Mahler bases. — We now construct some other examples of good bases. For $n \geqslant 0$, let $\mathrm{Int}_n(\mathcal{O}_K)$ denote the set of polynomials $f(T) \in K[T]$ such that $\deg(P) \leqslant n$ and $f(\mathcal{O}_K) \subset \mathcal{O}_K$. Recall (see for instance §1.2 of [**dS16**]) that a Mahler basis for $\mathcal{O}_K$ is a sequence $\{h_n\}_{n\geqslant 0}$ with $h_n(T) \in K[T]$ of degree n, and such that $\{h_0, \dots, h_n\}$ is a basis of the free $\mathcal{O}_K$-module $\mathrm{Int}_n(\mathcal{O}_K)$ for all $n \geqslant 0$. For example, if $K = \mathbf{Q}_p$, we can take $h_n(T) = \binom{T}{n}$. Let $\{h_n\}_{n\geqslant 0}$ be a Mahler basis for $\mathcal{O}_K$. Each h_n defines a function $\mathcal{O}_K \to \mathcal{O}_K$ and hence $\mathcal{O}_K \to k$. Let $I = \mathbf{Z}_{\geqslant 0}$ and let $I_n = \{0, \dots, q^n - 1\}$ for $n \geqslant 0$.

Proposition 4.3.1. — *If $\{h_n\}_{n\geqslant 0}$ is a Mahler basis for $\mathcal{O}_K$, then $\{h_i\}_{i\in I}$ is a good basis of $\mathrm{LC}(\mathcal{O}_K, k)$.*

Proof. — By theorem 1.2 of [**dS16**], $\{h_0, \dots, h_{q^m-1}\}$ is a basis of the k-vector space $\mathrm{LC}_m(\mathcal{O}_K, k)$ for all $m \geqslant 0$. This implies the claim. □

We now specialize to $K = \mathbf{Q}_p$. Write $\mathbf{N}$ for $\mathbf{Z}_{\geqslant 0}$ and $\mathbf{n}$ for an element $(n_1, \dots, n_d) \in \mathbf{N}^d$. For each $\mathbf{n} \in \mathbf{N}^d$, we denote by $h_{\mathbf{n}}$ the function $\mathbf{Z}_p^d \to E$ given by $(x_1, \dots, x_d) \mapsto \binom{x_1}{n_1} \cdots \binom{x_d}{n_d}$. For $m \in \mathbf{Z}_{\geqslant 0}$, let $I_m = \{\mathbf{n} \in \mathbf{N}^d$ such that $\max(n_1, \dots, n_d) \leqslant p^m - 1\}$.

Proposition 4.3.2. — *The functions $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{N}^d}$ form a good basis of $\mathrm{LC}(\mathbf{Z}_p^d, \mathbf{F}_p)$.*

Proof. — The claim follows from prop 4.3.1 for $K = \mathbf{Q}_p$, and lemma 4.3.3 below. □

Lemma 4.3.3. — *If X and X' are as in §4.1, and $\{h_i\}_{i\in I}$ and $\{h'_j\}_{j\in J}$ are good bases of $\mathrm{LC}(X, E)$ and $\mathrm{LC}(X', E)$, then $\{h_i \otimes h'_j\}_{(i,j)\in I\times J}$ is a good basis of $\mathrm{LC}(X \times X', E)$, with $(I \times J)_n = I_n \times J_n$.*

Let G be a uniform pro-p group, and let $c : G \to \mathbf{Z}_p^d$ be a coordinate as in prop 1.1.1. The theorem below follows from prop 4.3.2, theorem 4.2.1, and theorem 4.2.3.

Theorem 4.3.4. — *If $\{m_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{N}^d}$ is a sequence of M such that $m_{\mathbf{n}} \to 0$, the function $f : G \to M$ given by $f(g) = \sum_{\mathbf{n}\in\mathbf{N}^d} \binom{c_1(g)}{n_1} \cdots \binom{c_d(g)}{n_d} m_{\mathbf{n}}$ belongs to $C^0(G, M)$. We have $\inf_{g\in G} \mathrm{val}_M(f(g)) = \inf_{\mathbf{n}\in\mathbf{N}^d} \mathrm{val}_M(m_{\mathbf{n}})$.*

Conversely, if $f \in C^0(G, M)$, there exists a unique sequence $\{m_{\mathbf{n}}(f)\}_{\mathbf{n}\in\mathbf{N}^d}$ such that $m_{\mathbf{n}}(f) \to 0$ and such that $f(g) = \sum_{\mathbf{n}\in\mathbf{N}^d} \binom{c_1(g)}{n_1} \cdots \binom{c_d(g)}{n_d} m_{\mathbf{n}}(f)$.

We have $f \in \mathcal{H}_e^{\lambda,\mu}(G, M)$ if and only if for all $i \geqslant 0$, we have $\mathrm{val}_M(m_{\mathbf{n}}(f)) \geqslant p^\lambda \cdot p^{ei} + \mu$ whenever $\max(n_1, \dots, n_d) \geqslant p^i$.

Remark 4.3.5. — The first two assertions in the above theorem also follow from theorem 1.2.4 in §III of [**Laz65**] (we thank Konstantin Ardakov for pointing this out).

We finish by considering the case $G = \mathcal{O}_K$ for K a finite extension of $\mathbf{Q}_p$, and working with a Mahler basis for $\mathcal{O}_K$. Let K be a finite extension of $\mathbf{Q}_p$ as before. Assume that E is an extension of k. Let $\{h_n\}_{n\geqslant 0}$ be a Mahler basis for $\mathcal{O}_K$. If $f \in C^0(\mathcal{O}_K, M)$, write $f = \sum_{n\geqslant 0} h_n m_n(f)$ with $m_n(f) \to 0$. Let e denote the ramification index of K.

Proposition 4.3.6. — *If $f = \sum_{n\geqslant 0} h_n m_n(f)$ as above, then $f \in \mathcal{H}_t^{\lambda,\mu}(\mathcal{O}_K, M)$ if and only if $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^{ti} + \mu$ whenever $n \geqslant p^{di}$.*

Proof. — This follows from theorem 4.2.3, since $\mathrm{val}_p(x-y) \geqslant i$ if and only if $\mathrm{val}_\pi(x-y) \geqslant ei$, and since $q^e = p^d$. □

In this situation we can also define a slightly different version of super-Hölder functions. We say that a function $f : \mathcal{O}_K \to M$ is in $\mathcal{H}_{K,t}^{\lambda,\mu}(\mathcal{O}_K, M)$ if $\mathrm{val}_M(f(x) - f(y)) \geqslant p^\lambda \cdot p^{ti} + \mu$ whenever $\mathrm{val}_\pi(x - y) \geqslant i$. We then have

$$\mathcal{H}_{te}^{\lambda+t(e-1),\mu}(\mathcal{O}_K, M) \subset \mathcal{H}_{K,t}^{\lambda,\mu}(\mathcal{O}_K, M) \subset \mathcal{H}_{te}^{\lambda,\mu}(\mathcal{O}_K, M).$$

In particular, $\mathcal{H}_{K,t}(\mathcal{O}_K, M) = \mathcal{H}_{te}(\mathcal{O}_K, M)$. If $K/\mathbf{Q}_p$ is unramified then $\mathcal{H}_{K,t}^{\lambda,\mu}(\mathcal{O}_K, M) = \mathcal{H}_t^{\lambda,\mu}(\mathcal{O}_K, M)$. Moreover we have the following criterion:

Proposition 4.3.7. — *If $f = \sum_{n\geqslant 0} h_n m_n(f)$ as above, then $f \in \mathcal{H}_{K,t}^{\lambda,\mu}(\mathcal{O}_K, M)$ if and only if $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^{ti} + \mu$ whenever $n \geqslant q^i$.*

Example 4.3.8. — For all $n \geqslant 0$, there exists $c_n(T) \in \mathrm{Int}_n(\mathcal{O}_K)$ such that $[a](Y) = \sum_{n\geqslant 0} c_n(a) Y^n$. This implies that $\mathrm{val}_Y\,(m_n(a \mapsto [a](Y))) \geqslant n$, so that the function $a \mapsto [a](Y)$ is in $\mathcal{H}_d^{0,0}(\mathcal{O}_K, E[\![Y]\!])$, and in $\mathcal{H}_{K,f}^{0,0}(\mathcal{O}_K, E[\![Y]\!])$ where $q = p^f$.

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