UNIVERSAL NORMS AND DE RHAM REPRESENTATIONS

by

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Abstract. — The purpose of this note is to show that the computation of the universal norms in Iwasawa cohomology for the p-adic cyclotomic extension implies the same result for any p-adic extension that contains the cyclotomic extension.

Let K be a finite extension of $\mathbf{Q}_p$ (or, more generally, a finite extension of $W(k)[1/p]$, where k is a perfect field of characteristic p). Let $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ denote the absolute Galois group of K and let V be a p-adic representation of G_K. Let $K_\infty \subset K^{\mathrm{alg}}$ be an infinite extension of K. We consider the Iwasawa cohomology group $H^1_{\mathrm{Iw}}(K_\infty/K, V) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_L H^1(L,T)$ where T is a lattice of V, L runs through the set $E(K_\infty/K)$ of finite subextensions of K_∞/K, and the transition maps are the corestriction maps. If V is a de Rham representation, we consider the submodule $H^1_{\mathrm{Iw},g}(K_\infty/K, V)$ of $H^1_{\mathrm{Iw}}(K_\infty/K, V)$ consisting of those classes whose image in $H^1(L,V)$ is in the subspace $H^1_g(L,V)$ of $H^1(L,V)$ parametrizing de Rham extensions, for every $L \in E(K_\infty/K)$.

Let $\mathrm{Fil}^1 V$ be the largest subrepresentation of V whose Hodge–Tate weights are all ≥ 1. The space $H^1_{\mathrm{Iw}}(K_\infty/K, \mathrm{Fil}^1 V)$ is then contained in $H^1_{\mathrm{Iw},g}(K_\infty/K, V)$. When $K_\infty = K_{\mathrm{cyc}} := K(\mu_{p^\infty})$ is the cyclotomic extension of K, the main result of [**Ber05**] is that this inclusion is almost an equality. More precisely, Theorem A of ibid is the following (where $H_K = \mathrm{Gal}(K^{\mathrm{alg}}/K_{\mathrm{cyc}})$).

Theorem A. — *Let V be a de Rham representation of G_K and let K_{cyc}/K be the cyclotomic extension.*

1. *If V has no subquotient fixed by H_K, then $H^1_{\mathrm{Iw},g}(K_{\mathrm{cyc}}/K, V) = H^1_{\mathrm{Iw}}(K_{\mathrm{cyc}}/K, \mathrm{Fil}^1 V)$*
2. *In general, $H^1_{\mathrm{Iw}}(K_{\mathrm{cyc}}/K, \mathrm{Fil}^1 V) \subset H^1_{\mathrm{Iw},g}(K_{\mathrm{cyc}}/K, V)$ and the quotient is a finite-dimensional $\mathbf{Q}_p$-vector space.*

This result proved a conjecture of Nekovář (conjecture A of [**PR00**]), building on earlier work of Perrin-Riou [**PR00**] who used her big exponential map. The proof of Theorem A uses the theory of (φ,Γ)-modules. For other extensions K_∞/K, in the absence of a good theory of (φ,Γ)-modules, proving an analogue of this result seems difficult. There has however been recent encouraging progress by Ponsinet [**Pon24, Pon25**], using the Fargues–Fontaine curve instead of (φ,Γ)-modules.

The purpose of this note is to show that Theorem A, together with the methods used to prove it, implies the following result.

Theorem B. — *If V is a de Rham representation of G_K and if K_∞* contains *the cyclotomic extension K_{cyc} of K, then $H^1_{\mathrm{Iw},g}(K_\infty/K,V)/H^1_{\mathrm{Iw}}(K_\infty/K,\mathrm{Fil}^1 V)$ is a finite-dimensional $\mathbf{Q}_p$-vector space.*

The idea behind this theorem is quite simple. Recall that $E(K_\infty/K)$ denotes the set of all finite subextensions L of K_∞/K. First suppose, for simplicity, that V is such that for all $L \in E(K_\infty/K)$, we have $H^1_{\mathrm{Iw},g}(L_{\mathrm{cyc}}/L,V) = H^1_{\mathrm{Iw}}(L_{\mathrm{cyc}}/L,\mathrm{Fil}^1 V)$ (namely that for all L, we are in case (1) of Theorem A). Take $c \in H^1_{\mathrm{Iw},g}(K_\infty/K,V)$ and for $n \geq 0$, let $L_n = L(\mu_{p^n})$ so that $L_{\mathrm{cyc}} = \cup_{n\geq 0} L_n$. For every such L and every $n \geq 0$, we have $\mathrm{pr}_{L_n}(c) \in H^1_g(L_n,V)$. Thus, the compatible collection of classes $\{\mathrm{pr}_{L_n}(c)\}_{n\geq 0}$ defines an element of $H^1_{\mathrm{Iw},g}(L_{\mathrm{cyc}}/L,V)$. Since we are assuming that $H^1_{\mathrm{Iw},g}(L_{\mathrm{cyc}}/L,V) = H^1_{\mathrm{Iw}}(L_{\mathrm{cyc}}/L,\mathrm{Fil}^1 V)$, we have $\mathrm{pr}_L(c) \in H^1(L,\mathrm{Fil}^1 V)$. Since this holds for all $L \in E(K_\infty/K)$, we have $c \in H^1_{\mathrm{Iw}}(K_\infty/K,\mathrm{Fil}^1 V)$.

The only additional complication that arises when proving Theorem B in full generality is to take care of the possible finite-dimensional quotient in case (2) of Theorem A. The main result that we need in order to prove Theorem B is the following (Prop II.3.1 of [**Ber05**]; recall that $H^1_{\mathrm{Iw}}(K_{\mathrm{cyc}}/K,V) = \mathrm{D}^\dagger(V)^{\psi=1}$ in the notation of ibid).

Proposition. — *If $k \geq 0$ and if V is a de Rham representation of G_K whose Hodge–Tate weights are all $\geq -k$, and which has no subrepresentation all of whose Hodge–Tate weights are $\geq 1-k$, then $H^1_{\mathrm{Iw},g}(K_{\mathrm{cyc}}/K,V) \subset V^{H_K}$.*

Corollary. — *If $k \geq 0$ and if V is a de Rham representation of G_K whose Hodge–Tate weights are all $\geq -k$, and which has no subrepresentation all of whose Hodge–Tate weights are $\geq 1-k$, then $H^1_{\mathrm{Iw},g}(K_\infty/K,V)$ is a finite-dimensional $\mathbf{Q}_p$-vector space of dimension $\leq \dim(V)$.*

Proof. — We have $H^1_{\mathrm{Iw},g}(K_\infty/K,V) \subset \varprojlim_{L\in E(K_\infty/K)} H^1_{\mathrm{Iw},g}(L_{\mathrm{cyc}}/L,V)$ and each of the $H^1_{\mathrm{Iw},g}(L_{\mathrm{cyc}}/L,V)$ is a finite-dimensional $\mathbf{Q}_p$-vector space of dimension $\leq \dim(V)$ by the Proposition. This implies the result by the Lemma below. □

Lemma. — *If $d \geq 1$ and $\{V_1 \leftarrow V_2 \leftarrow \cdots\}$ is a projective system of $\mathbf{Q}_p$-vector spaces with* $\dim V_n \leq d$ *for all $n \geq 1$, then* $\dim \varprojlim V_n \leq d$.

Proof. — Take $c_1, \dots, c_{d+1} \in \varprojlim V_n$. Let $R_n \subset \mathbf{Q}_p^{d+1}$ be the set of $(a_1, \dots, a_{d+1})$ such that $\sum_{i=1}^{d+1} a_i \cdot \mathrm{pr}_{V_n}(c_i) = 0$ in V_n. We have $R_n \subset R_{n-1}$ by compatibility, and each R_n is nonzero since $\dim V_n \leq d$. The decreasing sequence of R_n's stabilizes to a nonzero subspace of $\mathbf{Q}_p^{d+1}$ and this gives a nontrivial linear relation between the c_i. □

Theorem B now follows from the same dévissage arguments as in [**Ber05**], which we now give. Let $\mathrm{Fil}^j V$ denote the largest subrepresentation of V whose Hodge–Tate weights are all $\geq j$. Note that if L/K is a finite extension and V is a representation of G_K, then the largest G_L-subrepresentation $\mathrm{Fil}^j_L V$ of V whose Hodge–Tate weights are all $\geq j$ is stable under G_K. Hence it is equal to $\mathrm{Fil}^j_K V$, so we can drop the subscript from the notation. Now take $k \geq 0$. We have an exact sequence

$$0 \to H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{1-k} V) \to H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{-k} V) \to H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{-k} V/\mathrm{Fil}^{1-k} V).$$

By Lemma II.3.3 of [**Ber05**], we have $\mathrm{Fil}^j(V/\mathrm{Fil}^j V) = \{0\}$ for all $j \in \mathbf{Z}$. Therefore the quotient $\mathrm{Fil}^{-k} V/\mathrm{Fil}^{1-k} V$ has no subrepresentation all of whose Hodge–Tate weights are $\geq 1-k$. The Corollary applies, so that $H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{-k} V/\mathrm{Fil}^{1-k} V)$ is a finite-dimensional $\mathbf{Q}_p$-vector space, and therefore so is $H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{-k} V)/H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^{1-k} V)$. Since $V = \mathrm{Fil}^{-k} V$ for $k \gg 0$, it follows by dévissage that

$$H^1_{\mathrm{Iw},g}(K_\infty/K, V)/H^1_{\mathrm{Iw},g}(K_\infty/K, \mathrm{Fil}^1 V)$$

is a finite-dimensional $\mathbf{Q}_p$-vector space. This proves Theorem B.

Tool and computational resource disclosure. ChatGPT was used to improve the style of writing. I also asked it for a reference for the above lemma, but it only provided a proof.

References

[Ber05] L. Berger, *Représentations de de Rham et normes universelles*, Bull. Soc. Math. France **133** (4), 2005, 601–618.

[PR00] B. Perrin-Riou, *Représentations p-adiques et normes universelles. I : Le cas cristallin*, J. Amer. Math. Soc., t. **13** (2000), pp. 533–551.

[Pon24] G. Ponsinet, *Bloch–Kato groups over perfectoid fields and Galois theory of p-adic periods*, preprint 2024.

[Pon25] G. Ponsinet, *Universal norms and the Fargues–Fontaine curve*, Math. Annalen, Vol. **392** (2025), pp. 2853–2912.

September 2, 2026

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