A p-ADIC FAMILY OF DIHEDRAL (φ, Γ)-MODULES

by

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Abstract. — The goal of this article is to construct explicitly a p-adic family of representations (which are dihedral representations), to construct their attached (φ, Γ)-modules by writing down explicit matrices for φ and for the action of Γ, and finally to determine which of these are trianguline.

***Résumé* (Une famille p-adique de (φ, Γ)-modules diédraux).** — L'objet de cet article est de construire explicitement une famille p-adique de représentations (qui sont des représentations diédrales), de construire les (φ, Γ)-modules qui leurs sont associés en écrivant des matrices explicites pour φ et pour l'action de Γ, et finalement de déterminer lesquelles sont triangulines.

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Introduction

A fundamental tool in the theory of p-adic representations is Fontaine's construction in [**Fon90**] of the (φ, Γ)-modules attached to p-adic representations. These are modules over a ring of power series, and are very explicit objects which contain all of the information about the representations they are attached to. It is however not always easy to extract that information. The work of Cherbonnier-Colmez [**CC98**] and Kedlaya [**Ked05**] has done much to clarify the situation, and in particular has allowed us to understand

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the structure of the (φ, Γ)-modules attached to semistable representations as in [**Ber02**] and [**Ber08**]. Inspired by these constructions and the p-adic Langlands correspondence, Colmez has defined in [**Col08b**] the notion of trianguline representation, and reinterpreted the first main result of [**Kis03**] as saying that the p-adic representations coming from overconvergent modular eigenforms of finite slope are trianguline.

The goal of this short article is to construct explicitly a p-adic family of representations, which are dihedral representations, to construct their (φ, Γ)-modules by writing down explicit matrices for φ and for the action of Γ, and finally to determine which of these are trianguline. This can be seen as a very first step towards constructing a universal family of (φ, Γ)-modules corresponding to the universal deformation space (in the sense of [**Maz89**]) of a mod p representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Our results extend without trouble to potentially abelian representations, but the general case will require new ideas.

We now give a more precise description of our results. Let $\chi_2 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2}) \to \mathbf{Z}_{p^2}^\times$ be the character attached to the Lubin-Tate module over $\mathbf{Z}_{p^2}$ for the uniformizer p of $\mathbf{Q}_{p^2}$. Every element $x \in \mathbf{Z}_{p^2}^\times$ can be written in a unique way as $x = \omega(x)\langle x \rangle$ where $\omega(x)^{p^2-1} = 1$ and $\langle x \rangle \in 1 + p\mathbf{Z}_{p^2}$. If $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$, then we define $\omega_2(g) = \omega(\chi_2(g))$ and $\langle g \rangle_2 = \langle \chi_2(g) \rangle$. Since $\langle g \rangle_2 \in 1 + p\mathbf{Z}_{p^2}$, the expression $\langle g \rangle_2^s$ makes sense if $s \in \mathbf{Z}_p$ and the representations $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r \langle \cdot \rangle_2^s)$ with $r \in \mathbf{Z}/(p^2-1)\mathbf{Z}$ and $s \in \mathbf{Z}_p$ interpolate p-adically the $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^h)$ with $h \in \mathbf{Z}$. Our first result is an explicit construction of the (φ, Γ)-modules attached to these representations, and we briefly recall what these objects are.

The Robba ring is the ring $\mathcal{R}$ consisting of power series $f(X) = \sum_{k \in \mathbf{Z}} a_k X^k$ with $a_k \in \mathbf{Q}_p$ and such that $f(X)$ converges on an annulus $r_f \leqslant |X|_p < 1$ where r_f depends on f. This ring is endowed with a Frobenius φ given by $\varphi(f)(X) = f((1+X)^p - 1)$, and with an action of $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ given by $\gamma(f)(X) = f((1+X)^{\chi(\gamma)} - 1)$ where $\chi : \Gamma \to \mathbf{Z}_p^\times$ is the cyclotomic character. A (φ, Γ)-module over $\mathcal{R}$ is a free module of finite rank over $\mathcal{R}$ endowed with a semilinear Frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{R})$ and a semilinear continuous action of Γ commuting with φ. By combining the aforementioned constructions of Fontaine, Cherbonnier-Colmez and Kedlaya, we get a functor $V \mapsto \mathrm{D}(V)$ which to every p-adic representation attaches a (φ, Γ)-module over $\mathcal{R}$. This functor gives an equivalence of categories between p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and étale (φ, Γ)-modules over $\mathcal{R}$. Finally, let $Q = \varphi(X)/X = ((1+X)^p - 1)/X$ and let $Q_2 = \varphi(Q)$ so that in some suitable sense, $Q_2/Q^p = 1 \bmod p$ and the expression $(Q_2/Q^p)^u$ makes sense if $u \in \mathbf{Z}_p$.

Theorem A. — *If $r \in \mathbf{Z}$ and $u \in \mathbf{Z}_p$ and $s = r + (p+1)u$, then the (φ,Γ)-module attached to $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ has a basis in which*

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & 1 \\ Q_2^r(Q_2/Q^p)^u & 0 \end{pmatrix}$$

and

$$\mathrm{Mat}(\gamma) = \begin{pmatrix} \chi(\gamma)^r\langle\chi(\gamma)\rangle^u(1+\mathrm{O}(X)) & 0 \\ 0 & \chi(\gamma)^r\langle\chi(\gamma)\rangle^u(1+\mathrm{O}(X)) \end{pmatrix},$$

where the $1+\mathrm{O}(X)$ are two power series belonging to $1 + X\mathbf{Z}_p[\![X]\!]$.

The proof of this result (theorem 4.3) is by p-adic interpolation. If $h \in \mathbf{Z}$, then the representation $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^h)$ is crystalline and we can compute its (φ,Γ)-module by using the theory of Wach modules of [**Ber04**]. One then only needs to change the basis so that the matrices of φ and $\gamma \in \Gamma$ become continuous functions of h.

One can then work concretely with the representation $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ and our next result (theorem 5.2) tells us exactly when it is trianguline. A p-adic representation is said to be trianguline if its attached (φ,Γ)-module over $\mathcal{R}$ is an iterated extension of (φ,Γ)-modules of rank 1, after possibly extending scalars.

Theorem B. — *The representation $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ is trianguline if and only if $s \in \mathbf{Z}$ and $r = s \bmod p+1$.*

In particular, by combining theorem 6.3 of [**Kis03**] and proposition 4.3 of [**Col08b**], we see that if $s \notin \mathbf{Z}$, then the representation $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ does not arise from an overconvergent modular eigenform of finite slope. The theorem also provides examples of representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ whose restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$ is trianguline but which are not themselves trianguline even though $\mathbf{Q}_{p^2}$ is an unramified extension of $\mathbf{Q}_p$.

It is not hard to analytify our constructions and hence to get a two-dimensional representation over $\mathbf{Z}_p\{T\}$. An analogue of theorem A then gives a corresponding family of (φ,Γ)-modules over $\mathbf{Z}_p\{T\}$ and theorem B tells us about the trianguline locus for that family. Note that one can twist $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ by a character of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and this way one obtains a three-dimensional family of representations, sitting inside the usually five-dimensional (see §9 of [**FM95**]) universal deformation space of a given mod p representation. These families are in some sense orthogonal to the ones constructed in [**BLZ04**]. Can one combine them to get an explicit family over some four-dimensional space?

1. A family of dihedral representations

We start by constructing the representations $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ in a way which shows that they are actually defined over $\mathbf{Q}_p$. Let $\chi_2 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2}) \to \mathbf{Z}_{p^2}^\times$ be the character attached to the Lubin-Tate module over $\mathbf{Z}_{p^2}$ for the uniformizer p of $\mathbf{Q}_{p^2}$ and denote by $\mathbf{Q}_{p^2}^{\mathrm{LT}}$ the fixed field of $\ker(\chi_2)$. If $\sigma : \mathbf{Z}_{p^2} \to \mathbf{Z}_{p^2}$ denotes the absolute Frobenius, then the group $\mathrm{Gal}(\mathbf{Q}_{p^2}^{\mathrm{LT}}/\mathbf{Q}_p)$ is naturally isomorphic to $\mathbf{Z}_{p^2}^\times \rtimes \mathbf{Z}/2\mathbf{Z}$, where the map $\mathbf{Z}/2\mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}_{p^2}^\times)$ is given by $\varepsilon \mapsto \sigma^\varepsilon$.

Every element $x \in \mathbf{Z}_{p^2}^\times$ can be written in a unique way as $x = \omega(x)\langle x\rangle$ where $\omega(x)^{p^2-1} = 1$ and $\langle x\rangle \in 1 + p\mathbf{Z}_{p^2}$. If $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$, then we define $\omega_2(g) = \omega(\chi_2(g))$ and $\langle g\rangle_2 = \langle \chi_2(g)\rangle$. If $r \in \mathbf{Z}/(p^2-1)\mathbf{Z}$ and $s \in \mathbf{Z}_p$, then we have a character $\eta_{r,s} : \mathbf{Z}_{p^2}^\times \to \mathbf{Z}_{p^2}^\times$ given by $x \mapsto \omega(x)^r\langle x\rangle^s$ where

$$\begin{aligned}\langle x\rangle^s &= (1 + (\langle x\rangle - 1))^s \\ &= \sum_{k\geqslant 0}\binom{s}{k}(\langle x\rangle - 1)^k \in 1 + p\mathbf{Z}_{p^2}.\end{aligned}$$

If $d \in \mathbf{Z}_p$ is some element such that $\mathbf{Z}_{p^2} = \mathbf{Z}_p[\sqrt{d}]$, then we have a homomorphism $\mathbf{Z}_{p^2}^\times \rtimes \mathbf{Z}/2\mathbf{Z} \to \mathrm{GL}_2(\mathbf{Z}_p)$ given by

$$(x + y\sqrt{d}, 0) \mapsto \begin{pmatrix} x & dy \\ y & x\end{pmatrix} \quad\text{and}\quad (x + y\sqrt{d}, 1) \mapsto \begin{pmatrix} x & -dy \\ y & -x\end{pmatrix}.$$

By composing this map and $\eta_{r,s}$ we get a representation

$$\rho_{r,s} : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_2(\mathbf{Z}_p),$$

whose underlying $\mathbf{Z}_p$-module we denote by $T_r(s)$. We also let $V_r(s) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_r(s)$.

Lemma 1.1. — *If $s_1 = s_2 \bmod p^k$, then $T_r(s_1) = T_r(s_2) \bmod p^{k+1}$.*

Proof. — If $s_1 = s_2 \bmod p^k$, then $(1 + (\langle x\rangle - 1))^{s_1} = (1 + (\langle x\rangle - 1))^{s_2} \bmod p^{k+1}$ for all $x \in \mathbf{Z}_{p^2}^\times$ and therefore the same is true of ρ_{r,s_1} and ρ_{r,s_2}. □

Proposition 1.2. — *We have $\mathbf{Q}_{p^2} \otimes_{\mathbf{Q}_p} V_r(s) = \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$.*

Proof. — In a suitable basis of $\mathbf{Q}_{p^2} \otimes_{\mathbf{Q}_p} V_r(s)$, the restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$ of this representation is isomorphic to $\omega_2(\cdot)^r\langle\cdot\rangle_2^s \oplus \sigma(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$. If $s = 0$ and $r = 0 \bmod p+1$, then the proposition is clear; otherwise, the two characters $\omega_2(\cdot)^r\langle\cdot\rangle_2^s$ and $\sigma(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ are distinct so that $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2(\cdot)^r\langle\cdot\rangle_2^s)$ is irreducible by a suitable version of Mackey's criterion (or an explicit computation) and the proposition follows from Frobenius reciprocity. □

2. Crystalline periods for Lubin-Tate groups

The character χ_2 extends to a map $\chi_2 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathbf{Z}_{p^2}^\times$ which is no longer a character but satisfies the formula $\chi_2(gh) = g(\chi_2(h))\chi_2(g)$. Let $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{dR}}$ be the rings of periods constructed in [**Fon94a**] and let $t_2 \in \mathbf{B}_{\mathrm{cris}}^+$ be the element t_E constructed in §9.3 of [**Col02**] for $E = \mathbf{Q}_{p^2}$ and $\pi_E = p$.

Proposition 2.1. — *The element t_2 has the following properties :*

1. *if $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then $g(t_2) = \chi_2(g)t_2$;*
2. *$\varphi^2(t_2) = pt_2$;*
3. *$t_2 \in \mathrm{Fil}^1 \setminus \mathrm{Fil}^2 \mathbf{B}_{\mathrm{dR}}$ and $\varphi(t_2) \in \mathrm{Fil}^0 \setminus \mathrm{Fil}^1 \mathbf{B}_{\mathrm{dR}}$.*

Proof. — Properties (2) and (3) are proved in §2.4 of [**Col08a**]. As for property (1), we use the notations of §9 of [**Col02**]. The element t_E is defined as $L_E(\omega_E)$ where ω_E is constructed so that $g(\omega_E) = [\chi_2(g)](\omega_E)$ and L_E is the logarithm of the Lubin-Tate group, which implies (1). □

In particular, if $h \in \mathbf{Z}$, then the space $W_h = \mathbf{Q}_{p^2} \cdot t_2^h$ is a $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable subspace of $\mathbf{B}_{\mathrm{cris}}$ and hence a two-dimensional $\mathbf{Q}_p$-linear representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Lemma 2.2. — *We have $\mathbf{Q}_{p^2} \otimes_{\mathbf{Q}_p} W_h = \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^h)$.*

Proof. — The lemma is immediate if $h = 0$, so let us assume that $h \neq 0$. The restriction of W_h to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$ contains the characters χ_2^h and $\sigma(\chi_2)^h$ and since $\chi_2^h \neq \sigma(\chi_2)^h$, the induced representation $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^h)$ is irreducible and the lemma follows from Frobenius reciprocity. □

Note that in the definition $W_h = \mathbf{Q}_{p^2} \cdot t_2^h$, the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on $\mathbf{Q}_{p^2}$ is semilinear, while in lemma 2.2 above we extend scalars to get $\mathbf{Q}_{p^2} \otimes_{\mathbf{Q}_p} W_h$ but there the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on $\mathbf{Q}_{p^2}$ is linear. The following result is well-known, see for instance proposition 5.16 of [**Fon04**] and the remark which follows.

Proposition 2.3. — *If $h \in \mathbf{Z}$, then W_h is a crystalline representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and if $h \leqslant -1$, then $\mathrm{D}_{\mathrm{cris}}(W_h) = \mathbf{Q}_p \cdot e \oplus \mathbf{Q}_p \cdot f$ where*

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & 1 \\ p^{-h} & 0 \end{pmatrix} \quad \textit{and} \quad \mathrm{Fil}^i \mathrm{D}_{\mathrm{cris}}(W_h) = \begin{cases} \mathrm{D}_{\mathrm{cris}}(W_h) & \textit{if } i \leqslant 0, \\ \mathbf{Q}_p \cdot e & \textit{if } 1 \leqslant i \leqslant -h, \\ \{0\} & \textit{if } 1-h \leqslant i. \end{cases}$$

Proof. — The dual of $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^h)$ is naturally isomorphic to $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^{-h})$ and hence $W_h^* = W_{-h}$ so that $\mathrm{D}_{\mathrm{cris}}(W_h) = \mathrm{Hom}(W_{-h}, \mathbf{B}_{\mathrm{cris}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$. The "inclusion map" $e : W_{-h} \to \mathbf{B}_{\mathrm{cris}}$ is one such element and the map $f = p^h\varphi \circ e$, which is given by $f : \alpha \cdot t_2^{-h} \mapsto p^h\sigma(\alpha) \cdot \varphi(t_2^{-h})$ is another one which is linearly independent. The fact that

$\varphi^2(t_2) = pt_2$ gives us the matrix of φ in the basis e, f. The fact that $t_2, \varphi(t_2) \in \mathrm{Fil}^0 \mathbf{B}_{\mathrm{dR}}$ implies that $\mathrm{Fil}^0 \mathrm{D}_{\mathrm{cris}}(W_h) = \mathrm{D}_{\mathrm{cris}}(W_h)$, and the fact that $t_2 \in \mathrm{Fil}^1 \setminus \mathrm{Fil}^2 \mathbf{B}_{\mathrm{dR}}$ implies that $\mathrm{Fil}^{-h} \mathrm{D}_{\mathrm{cris}}(W_h) = \mathbf{Q}_p \cdot e$ and $\mathrm{Fil}^{1-h} \mathrm{D}_{\mathrm{cris}}(W_h) = \{0\}$. □

Note that there is a similar result if $h \geqslant 1$, but in that case the non-trivial line of the filtration is generated by f.

3. Interlude : some p-adic analysis

Before moving on to the construction of (φ, Γ)-modules, we prove a few simple results concerning functions belonging to the Robba ring $\mathcal{R}_E$ where E is a finite extension of $\mathbf{Q}_p$. Recall that if $r < 1$, then $\mathcal{R}_E^{\dagger,r}$ is defined to be the set of functions $f(X) = \sum_{k \in \mathbf{Z}} a_k X^k$ with $a_k \in E$ which converge on the annulus $r \leqslant |X|_p < 1$ and that $\mathcal{R}_E = \cup_{r<1} \mathcal{R}_E^{\dagger,r}$. This ring is endowed with a Frobenius φ given by $\varphi(f)(X) = f((1+X)^p - 1)$, and with an action of $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ given by $\gamma(f)(X) = f((1+X)^{\chi(\gamma)} - 1)$.

The subset $\mathcal{E}_E^\dagger$ of $\mathcal{R}_E$ consisting of those functions $f(X)$ for which the sequence $\{a_k(f)\}$ is bounded is a subfield of $\mathcal{R}_E$, and we write $\mathcal{O}_{\mathcal{E}_E}^\dagger$ for the subring of $\mathcal{E}_E^\dagger$ consisting of those functions for which $|a_k(f)|_p \leqslant 1$ for all k. If $E = \mathbf{Q}_p$ then we drop the subscript E from the notation. These rings were studied by Lazard in [**Laz62**] and more recently by Kedlaya in [**Ked05**].

Let $Q = \varphi(X)/X$ and for $n \geqslant 1$, let $Q_n = \varphi^{n-1}(Q)$ so that Q_n is the minimal polynomial of $\zeta_{p^n} - 1$ over $\mathbf{Q}_p$. If $r < 1$, we define $n(r)$ to be the smallest integer n such that $|\zeta_{p^n} - 1|_p \geqslant r$. We also let $t = \log(1+X)$ so that $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$, and recall that we have the Weierstrass product formula $t = X \cdot \prod_{n=1}^{\infty} Q_n/p$. Recall also that by an argument analogous to that of lemma I.3.2 of [**Ber04**], we have the following result.

Lemma 3.1. — *Every principal ideal of $\mathcal{R}_E^{\dagger,r}$ which is stable under Γ is generated by an element of the form $\prod_{n \geqslant n(r)} (Q_n/p)^{a_n}$ for some a_n in $\mathbf{Z}_{\geqslant 0}$.*

We now prove a few results which are used in the remainder of the paper.

Lemma 3.2. — *The map $f(X) \mapsto \varphi^2(f(X))/f(X)$ from $1 + X\mathbf{Z}_p[\![X]\!]$ to itself is bijective.*

Proof. — The map is injective because if $f(X) = 1 + a_k X^k + \mathcal{O}(X^{k+1})$ with $a_k \neq 0$, then $\varphi^2(f(X)) = 1 + p^{2k} a_k X^k + \mathcal{O}(X^{k+1})$ so that $\varphi^2(f(X)) = f(X)$ if and only if $f(X) = 1$. Let us now prove surjectivity. If $f(X) \in 1 + X\mathbf{Z}_p[\![X]\!]$, then $\varphi^n(f(X)) \to 1$ as $n \to \infty$ and the product $\prod_{n=0}^{\infty} \varphi^{2n}(f(X))$ converges to $g(X) \in 1 + X\mathbf{Z}_p[\![X]\!]$ such that $\varphi^2(g(X))/g(X) = f(X)$. □

Corollary 3.3. — *If $\gamma\in\Gamma$, then there exist two uniquely determined power series $f_\gamma(X)\in 1+X\mathbf{Z}_p[\![X]\!]$ and $g_\gamma(X)\in 1+pX\mathbf{Z}_p[\![X]\!]$ such that*

$$\frac{\varphi^2(f_\gamma(X))}{f_\gamma(X)}=\frac{\gamma(Q_2)}{Q_2}\quad\textit{and}\quad\frac{\varphi^2(g_\gamma(X))}{g_\gamma(X)}=\frac{\gamma(Q_2/Q^p)}{Q_2/Q^p}.$$

Proof. — We have $\gamma(Q_2)/Q_2\in 1+X\mathbf{Z}_p[\![X]\!]$ and $Q_2/Q^p\in 1+p\mathcal{O}_{\mathcal{E}}^\dagger$ so that

$$\frac{\gamma(Q_2/Q^p)}{Q_2/Q^p}\in 1+X\mathbf{Z}_p[\![X]\!]\cap 1+p\mathcal{O}_{\mathcal{E}}^\dagger=1+pX\mathbf{Z}_p[\![X]\!].$$

The corollary then follows from lemma 3.2, and the fact that $g_\gamma(X)\in 1+pX\mathbf{Z}_p[\![X]\!]$ follows from the explicit construction for the inverse of $\varphi^2(\cdot)/(\cdot)$. □

Note that since both $f_\gamma(X)$ and $g_\gamma(X)$ are uniquely determined, we have $f_{\gamma\eta}(X)=f_\gamma(X)\gamma(f_\eta(X))$ and $g_{\gamma\eta}(X)=g_\gamma(X)\gamma(g_\eta(X))$ if $\gamma,\eta\in\Gamma$.

Lemma 3.4. — *We have $(t^{-1}\mathcal{R}_E)^{\varphi^2=p^{-2}}=E\cdot t^{-1}$.*

Proof. — If $t^{-1}f(X)\in(t^{-1}\mathcal{R}_E)^{\varphi^2=p^{-2}}$ then $f(X)\in\mathcal{R}_E^{\varphi^2=1}=E$. □

Let $\partial:\mathcal{R}_E\to\mathcal{R}_E$ be the operator defined by $\partial f(X)=(1+X)df(X)/dX$. If $f\in\operatorname{Frac}\mathcal{R}_E^{\dagger,r}$ and if $n\geqslant n(r)$ and if f has at most a simple pole at $\zeta_{p^n}-1$, we define $\operatorname{res}_n(f)$ to be the value at $\zeta_{p^n}-1$ of $f\cdot Q_n/\partial Q_n$ so that $\operatorname{res}_n(f)$ is the residue of f at $\zeta_{p^n}-1$ multiplied by a suitable constant chosen so that $\operatorname{res}_n(\partial Q_n/Q_n)=1$.

Lemma 3.5. — *If $f\in\mathcal{R}_E^{\dagger,r}$ and $n\geqslant n(r)$, then $\operatorname{res}_n(\partial f/f)\in\mathbf{Z}$.*

Proof. — If $g(X)\in\mathcal{R}_E^{\dagger,r}$ is nonzero at $\zeta_{p^n}-1$ then $\partial g/g$ has no pole at $\zeta_{p^n}-1$ and hence $\operatorname{res}_n(\partial g/g)=0$. If $f(X)\in\mathcal{R}_E^{\dagger,r}$ then we can write $f(X)=Q_n(X)^a g(X)$ where $g(X)\in\mathcal{R}_E^{\dagger,r}$ is nonzero at $\zeta_{p^n}-1$ and a is the order of vanishing of $f(X)$ at $\zeta_{p^n}-1$ so that $\operatorname{res}_n(\partial f/f)=a\operatorname{res}_n(\partial Q_n/Q_n)+\operatorname{res}_n(\partial g/g)=a\in\mathbf{Z}$. □

4. A family of dihedral (φ,Γ)-modules

We now construct the (φ,Γ)-modules over $\mathcal{R}$ attached to the representations $V_r(s)$. Since both Q_2/Q^p and $g_\gamma(X)$ belong to $1+p\mathcal{O}_{\mathcal{E}}^\dagger$, we have $(Q_2/Q^p)^u\in 1+p\mathcal{O}_{\mathcal{E}}^\dagger$ and $g_\gamma(X)^u\in 1+p\mathcal{O}_{\mathcal{E}}^\dagger$ if $u\in\mathbf{Z}_p$.

Definition 4.1. — Given $j\in\mathbf{Z}$, we define a (φ,Γ)-module $\mathrm{D}_j^0(u)$ over $\mathcal{O}_{\mathcal{E}}^\dagger$ by $\mathrm{D}_j^0=\mathcal{O}_{\mathcal{E}}^\dagger\cdot e\oplus\mathcal{O}_{\mathcal{E}}^\dagger\cdot f$ where

$$\operatorname{Mat}(\varphi)=\begin{pmatrix}0&1\\ Q_2^j(Q_2/Q^p)^u&0\end{pmatrix}$$

and

$$\operatorname{Mat}(\gamma)=\begin{pmatrix}\chi(\gamma)^j\langle\chi(\gamma)\rangle^u\varphi(f_\gamma(X)^j g_\gamma(X)^u)&0\\ 0&\chi(\gamma)^j\langle\chi(\gamma)\rangle^u f_\gamma(X)^j g_\gamma(X)^u\end{pmatrix}.$$

We then extend scalars to $\mathcal{R}$ to get an étale (φ,Γ)-module $\mathrm{D}_j(u) = \mathcal{R} \otimes_{\mathcal{O}_{\mathcal{E}}^\dagger} \mathrm{D}_j^0(u)$.

Lemma 4.2. — *If $u_1 = u_2 \bmod p^k$, then $\mathrm{D}_j^0(u_1) = \mathrm{D}_j^0(u_2) \bmod p^{k+1}$.*

Proof. — This follows from the definition above and the fact that both Q_2/Q^p and $g_\gamma(X)$ belong to $1 + p\mathcal{O}_{\mathcal{E}}^\dagger$ so that if $u_1 = u_2 \bmod p^k$, then $(Q_2/Q^p)^{u_1} = (Q_2/Q^p)^{u_2} \bmod p^{k+1}$ and $g_\gamma(X)^{u_1} = g_\gamma(X)^{u_2} \bmod p^{k+1}$. □

Theorem 4.3. — *We have $\mathrm{D}_j(u) = \mathrm{D}(V_j(j+(p+1)u))$ if $u \in \mathbf{Z}_p$.*

Proof. — By lemmas 1.1 and 4.2, it is enough to check the isomorphism for u belonging to a p-adically dense subset of $\mathbf{Z}_p$, and we do so for those $u \in (p-1)\mathbf{Z}$ such that $h = j + (p+1)u \leqslant -1$, so that $V_j(j+(p+1)u) = V_h(h) = W_h$. Using the fact that $\varphi^2(X) = Q_2QX$, we find that in the basis $(e', f') = (\varphi(X)^{-u-j}e, X^{-u-j}f)$, we have

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & 1 \\ Q^{-h} & 0 \end{pmatrix},$$

and using the fact that $\gamma(X)/X = \chi(\gamma)(1+\mathrm{O}(X))$ and $\gamma(Q_n)/Q_n = 1 + \mathrm{O}(X)$, we find

$$\mathrm{Mat}(\gamma) = \begin{pmatrix} 1+\mathrm{O}(X) & 0 \\ 0 & 1+\mathrm{O}(X) \end{pmatrix}.$$

In particular, $\mathbf{Z}_p[\![X]\!] \cdot e' \oplus \mathbf{Z}_p[\![X]\!] \cdot f'$ is a Wach module as defined in [**Ber04**, §III.4]. By proposition III.4.2 and theorem III.4.4 of [**Ber04**], the (φ,Γ)-module $\mathrm{D}_j(u)$ attached to this Wach module corresponds to a crystalline representation V such that in the basis $(\overline{e}', \overline{f}')$ of $\mathrm{D}_{\mathrm{cris}}(V)$ we have

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & 1 \\ p^{-h} & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{Fil}^i \mathrm{D}_{\mathrm{cris}}(V) = \begin{cases} \mathrm{D}_{\mathrm{cris}}(V) & \text{if } i \leqslant 0, \\ \mathbf{Q}_p \cdot \overline{e}' & \text{if } 1 \leqslant i \leqslant -h, \\ \{0\} & \text{if } 1-h \leqslant i. \end{cases}$$

By proposition 2.3 and the fact that $V \mapsto \mathrm{D}_{\mathrm{cris}}(V)$ is fully faithful by theorem 5.3.5 of [**Fon94b**], we have $V = W_h$ and so $\mathrm{D}_j(u) = \mathrm{D}(V_j(j+(p+1)u))$. □

5. Determination of the trianguline points

Colmez has defined (see definitions 4.1 and 3.4 of [**Col08b**]) a p-adic representation to be trianguline if its attached (φ,Γ)-module over $\mathcal{R}$ is an iterated extension of (φ,Γ)-modules of rank 1, after possibly extending scalars. This important definition was made in the context of the p-adic local Langlands correspondence. For a survey about trianguline representations, see [**Ber11**].

In this last chapter, we determine which of the representations $V_r(s)$ are trianguline. The key point is the result below.

Proposition 5.1. — *If $\mathrm{D}_j(u)$ is trianguline, then $(p+1)u \in \mathbf{Z}$.*

Proof. — By definition 3.4 and proposition 3.1 of [**Col08b**], $\mathrm{D}_j(u)$ is trianguline if and only if there exist some finite extension $E/\mathbf{Q}_p$, a continuous character $\delta : \mathbf{Q}_p^\times \to E^\times$ and $\alpha, \beta \in \mathcal{R}_E$ such that (here e and f are those of definition 4.1)

$$\gamma(\alpha e + \beta f) = \delta(\gamma)(\alpha e + \beta f) \text{ if } \gamma \in \Gamma \quad \text{and} \quad \varphi(\alpha e + \beta f) = \delta(p)(\alpha e + \beta f).$$

Given the formulas of definition 4.1, the first condition implies that the ideals of $\mathcal{R}_E$ generated by α and β are each stable under the action of Γ, and the second condition implies that β satisfies the equation

$$\varphi^2(\beta) Q_2^j \left(\frac{Q_2}{Q^p}\right)^u = \delta(p)^2 \beta.$$

Let $\mu(X) \in \mathcal{R}_E$ be the power series $\mu(X) = \prod_{n=1}^\infty Q_{2n}/p$ so that we have

$$\varphi^2(\mu) Q_2 = p\mu \quad \text{and} \quad \varphi^2(\mu^{p+1} X^p) \frac{Q_2}{Q^p} = p^{p+1} \mu^{p+1} X^p.$$

If we apply the map $\partial(\cdot)/(\cdot)$ to the above three equations, bearing in mind that $\partial \circ \varphi^2 = p^2 \varphi^2 \circ \partial$, we get

$$\begin{aligned} (1 - p^2\varphi^2) \frac{\partial \beta}{\beta} &= j \frac{\partial Q_2}{Q_2} + u \frac{\partial (Q_2/Q^p)}{Q_2/Q^p}, \\ (1 - p^2\varphi^2) \frac{\partial \mu}{\mu} &= \frac{\partial Q_2}{Q_2}, \\ (1 - p^2\varphi^2) \frac{\partial (\mu^{p+1} X^p)}{\mu^{p+1} X^p} &= \frac{\partial (Q_2/Q^p)}{Q_2/Q^p}, \end{aligned}$$

so that

$$(1 - p^2\varphi^2) \left(\frac{\partial \beta}{\beta} - j \frac{\partial \mu}{\mu} - u \frac{\partial (\mu^{p+1} X^p)}{\mu^{p+1} X^p} \right) = 0.$$

Since the ideal of $\mathcal{R}_E$ generated by β is stable under the action of Γ, lemma 3.1 implies that the only possible zeroes of β are the $\zeta_{p^n} - 1$ and the same is true for μ, so that $\partial\beta/\beta$ and $\partial\mu/\mu$ and $\partial(\mu^{p+1}X^p)/\mu^{p+1}X^p$ all belong to $t^{-1}\mathcal{R}_E$ since logarithmic derivatives have only simple poles. The above equation and lemma 3.4 imply that there exists $c \in E$ such that

$$\frac{\partial \beta}{\beta} - j \frac{\partial \mu}{\mu} - u \frac{\partial (\mu^{p+1} X^p)}{\mu^{p+1} X^p} = \frac{c}{t}.$$

We now fix a radius r such that $\beta \in \mathcal{R}_E^{\dagger,r}$ and apply the residue maps res_n of §3 for $n \geqslant n(r)$. If n is odd, then $\mu(X)$ has no zero nor pole at $\zeta_{p^n} - 1$ and hence $\mathrm{res}_n(\partial\mu/\mu) = 0$, which implies that $\mathrm{res}_n(c/t) = \mathrm{res}_n(\partial\beta/\beta) \in \mathbf{Z}$ by lemma 3.5 and therefore $c \in \mathbf{Z}$ since $\mathrm{res}_n(1/t) = \mathrm{res}_n(\partial t/t) = 1$. If n is even, then by applying once more res_n to the above equation we get $\mathrm{res}_n(\partial\beta/\beta) - j - (p+1)u = c$ and since both c and $\mathrm{res}_n(\partial\beta/\beta)$ belong to $\mathbf{Z}$, we also have $(p+1)u \in \mathbf{Z}$. □

Theorem 5.2. — *The representation $V_r(s)$ is trianguline if and only if $s \in \mathbf{Z}$ and $r = s \bmod p+1$.*

Proof. — If $s \in \mathbf{Z}$ and $r - s = (p+1)k$, then

$$\mathbf{Q}_{p^2} \otimes_{\mathbf{Q}_p} V_r(s) = \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2^{r-s}\chi_2^s) = \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^s) \otimes \omega_1^k$$

where $\omega_1 = \omega_2^{p+1} = \omega(\chi(\cdot))$. By proposition 2.3, $\mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\chi_2^s)$ is crystalline and hence trianguline by theorem 0.8 of [**Col08b**] so that $V_r(s)$ is trianguline if $s \in \mathbf{Z}$ and $r = s \bmod p+1$.

Assume now that $V_r(s)$ is trianguline. By combining theorem 4.3 and proposition 5.1, we see that $s \in \mathbf{Z}$ so that we have $V_r(s) = \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}(\omega_2^{r-s}\chi_2^s)$ and $V_r(s)$ is potentially crystalline. By theorem 0.8 of [**Col08b**], a trianguline p-adic representation is potentially crystalline if and only if its restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\zeta_{p^n}))$ is crystalline for some $n \gg 0$. By restricting $V_r(s)$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$, we see that the fixed field of ω_2^{r-s} must lie in $\mathbf{Q}_{p^2}(\zeta_{p^n})$ for some $n \gg 0$ and hence that $r - s$ is divisible by $p+1$. □

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