DECOMPLETION OF CYCLOTOMIC PERFECTOID FIELDS IN POSITIVE CHARACTERISTIC

by

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Abstract. — Let E be a field of characteristic p. The group $\mathbf{Z}_p^\times$ acts on $E(\!(X)\!)$ by $a \cdot f(X) = f((1+X)^a - 1)$. This action extends to the X-adic completion $\widetilde{\mathbf{E}}$ of $\cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)$. We show how to recover $E(\!(X)\!)$ from the valued E-vector space $\widetilde{\mathbf{E}}$ endowed with its action of $\mathbf{Z}_p^\times$. To do this, we introduce the notion of super-Hölder vector in certain E-linear representations of $\mathbf{Z}_p$. This is a characteristic p analogue of the notion of locally analytic vector in p-adic Banach representations of p-adic Lie groups.

Contents

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Introduction

Let p be a prime number, and let E be a field of characteristic p. Let $\mathbf{E} = E(\!(X)\!)$, and let $\widetilde{\mathbf{E}}$ be the X-adic completion of $\cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)$. Note that if E is perfect, the field $\widetilde{\mathbf{E}}$ is perfectoid. The group $\mathbf{Z}_p^\times$ acts on $\mathbf{E}$ by $(a \cdot f)(X) = f((1+X)^a - 1)$. This action extends to $\cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)$ by $(a \cdot f)(X^{1/p^n}) = f((1+X^{1/p^n})^a - 1)$, and by continuity to $\widetilde{\mathbf{E}}$. The question that motivated this paper is the following.

Question. — *Can we recover $\cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)$ or even $E(\!(X)\!)$ from the data of the valued E-vector space $\widetilde{\mathbf{E}}$ endowed with the action of $\mathbf{Z}_p^\times$?*

In characteristic zero, it is possible to answer an analogous question by using Schneider and Teitelbaum's theory of locally analytic vectors in p-adic Banach representations of p-adic Lie groups. For characteristic p representations, there is no such theory. One of the main contributions of this article is to introduce a characteristic p analogue of locally analytic functions and vectors.

Let M be an E-vector space, endowed with a valuation val_M such that $\mathrm{val}_M(xm) = \mathrm{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M-adic topology. For example, we will consider $M = \mathbf{E}$ or $\widetilde{\mathbf{E}}$ with the X-adic valuation. We say that a function $f : \mathbf{Z}_p \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbf{R}$ such that $\mathrm{val}_M(f(x) - f(y)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $\mathrm{val}_p(x-y) \geqslant i$, for all $x, y \in \mathbf{Z}_p$ and $i \geqslant 0$. These super-Hölder functions are the characteristic p analogue of locally analytic functions $\mathbf{Z}_p \to \mathbf{Q}_p$. We prove an analogue of Mahler's theorem for continuous functions $f : \mathbf{Z}_p \to M$, and give a characterization of super-Hölder functions in terms of their Mahler expansions. This is a characteristic p analogue of a theorem of Amice.

Assume now that Γ is a group that is isomorphic to $\mathbf{Z}_p$ via a coordinate map c, and that M is endowed with an E-linear action of Γ by isometries. We say that $m \in M$ is a super-Hölder vector if the orbit map $z \mapsto c^{-1}(z) \cdot m$ is a super-Hölder function $\mathbf{Z}_p \to M$. This definition is a characteristic p analogue of the notion of locally analytic vector of a p-adic Banach representation of a p-adic Lie group. We let $M^{\Gamma\text{-sh},\lambda}$ denote the space of super-Hölder vectors for a given constant λ as in the definition above. We also let M^{sh} denote the set of super-Hölder vectors in M. Our main result is a complete answer to the question above. Consider $M = \widetilde{\mathbf{E}}$, endowed with the action of $\Gamma = 1 + p^k \mathbf{Z}_p$ for $k \geqslant 1$ (or $k \geqslant 2$ if $p = 2$).

Theorem. — *For all $n \geqslant 0$, we have $\widetilde{\mathbf{E}}^{(1+p^k\mathbf{Z}_p)\text{-sh},k-n} = E(\!(X^{1/p^n})\!)$.*

In particular, $\widetilde{\mathbf{E}}^{\mathrm{sh}} = \cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)$.

The main ingredients of the proof of this theorem are some simple computations in $E[\![X]\!]$, as well as Colmez' analogue of Tate traces for $\widetilde{\mathbf{E}}$.

We give several applications of our main result. First, we compute the perfectoid commutant of $\mathrm{Aut}(\mathbf{G}_{\mathrm{m}})$, namely the set of $u \in \widetilde{\mathbf{E}}^{\mathrm{val}_X>0}$ such that $u \circ \gamma_a = \gamma_a \circ u$ for all $a \in \mathbf{Z}_p^\times$, where $\gamma_a(X) = (1+X)^a - 1$. Using our main theorem, and a result of Lubin-Sarkis on the classical commutant of $\mathrm{Aut}(\mathbf{G}_{\mathrm{m}})$, we prove that such a u is of the form $\gamma_b(X^{p^n})$ for some $b \in \mathbf{Z}_p^\times$ and $n \in \mathbf{Z}$. Next we study (φ, Γ)-modules over $\mathbf{E}$. We prove that the action of Γ on a (φ,Γ)-module $\mathbf{D}$ is always super-Hölder, and deduce that $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} \mathbf{D})^{\mathrm{sh}} = (\cup_{n \geqslant 0} E(\!(X^{1/p^n})\!)) \otimes_{\mathbf{E}} \mathbf{D}$. This allows us to extend our computation of super-Hölder vectors to the finite extensions of $\mathbf{F}_p(\!(X)\!)$ provided by Fontaine and Wintenberger's theory of the field of norms. We finish this article with a computation that suggests that the theory of super-Hölder vectors could have some applications to the p-adic local Langlands correspondence.

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1. Super-Hölder functions and vectors

In this section, we define super-Hölder functions $\mathbf{Z}_p \to M$ and super-Hölder vectors in M when M is a representation of a group isomorphic to $\mathbf{Z}_p$. We prove an analogue of Mahler's theorem for continuous functions $\mathbf{Z}_p \to M$, and give a characterization of super-Hölder functions in terms of their Mahler expansions.

1.1. Super-Hölder functions. — We keep the notation of the introduction. Let M be an E-vector space, endowed with a valuation val_M such that $\mathrm{val}_M(xm) = \mathrm{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M-adic topology. For example, we will consider $M = E[\![X]\!]$ with the X-adic valuation.

Let $C^0(\mathbf{Z}_p, M)$ denote the space of continuous functions $f : \mathbf{Z}_p \to M$.

Definition 1.1. — We say that $f : \mathbf{Z}_p \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbf{R}$ such that $\mathrm{val}_M(f(x) - f(y)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $\mathrm{val}_p(x-y) \geqslant i$, for all $x, y \in \mathbf{Z}_p$ and $i \geqslant 0$.

We let $\mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$ denote the space of functions such that $\mathrm{val}_M(f(x)-f(y)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $\mathrm{val}_p(x-y) \geqslant i$, for all $x, y \in \mathbf{Z}_p$ and $i \geqslant 0$, and $\mathcal{H}^\lambda(\mathbf{Z}_p, M) = \cup_{\mu \in \mathbf{R}} \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$ and $\mathcal{H}(\mathbf{Z}_p, M) = \cup_{\lambda \in \mathbf{R}} \mathcal{H}^\lambda(\mathbf{Z}_p, M)$.

For example, if $M = E[\![X]\!]$ with $\mathrm{val}_M = \mathrm{val}_X$, then $[a \mapsto (1+X)^a] \in \mathcal{H}^{0,0}(\mathbf{Z}_p, M)$. Indeed, $(1+X)^a - (1+X)^{a+p^i b} = (1+X)^a(1-(1+X^{p^i})^b) \in X^{p^i} E[\![X]\!]$ if $i \geqslant 0$.

Remark 1.2. — The space $\mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$ is closed in $C^0(\mathbf{Z}_p, M)$.

Remark 1.3. — If $\alpha : \mathbf{Z}_p \to \mathbf{Z}_p$ is an isometry, then $f : \mathbf{Z}_p \to M$ belongs to $\mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$ if and only if $f \circ \alpha \in \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$

Proposition 1.4. — *Suppose that M is a ring, and that $\mathrm{val}_M(mm') \geqslant \mathrm{val}_M(m) + \mathrm{val}_M(m')$ for all $m, m' \in M$. If $c \in \mathbf{R}$, let $M_c = M^{\mathrm{val}_M \geqslant c}$.*

1. *If $f \in \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M_c)$ and $g \in \mathcal{H}^{\lambda,\nu}(\mathbf{Z}_p, M_d)$, and $\xi = \min(\mu + d, \nu + c)$, then $fg \in \mathcal{H}^{\lambda,\xi}(\mathbf{Z}_p, M_{c+d})$.*
2. *If $\lambda, \mu \in \mathbf{R}$, then $\mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M_0)$ is a subring of $C^0(\mathbf{Z}_p, M)$.*
3. *If $\lambda \in \mathbf{R}$, then $\mathcal{H}^\lambda(\mathbf{Z}_p, M)$ is a subring of $C^0(\mathbf{Z}_p, M)$*
4. *If $d \geqslant 1$, we see $\mathrm{GL}_d(M)$ as a subset of the valued E-vector space $\mathrm{M}_d(M)$. If $\lambda, \nu \in \mathbf{R}$ and $Q \in \mathcal{H}^\lambda(\mathbf{Z}_p, \mathrm{GL}_d(M))$ are such that $\mathrm{val}_M(\det Q(x)) \leqslant \nu$ for all $x \in \mathbf{Z}_p$, then $Q^{-1} \in \mathcal{H}^\lambda(\mathbf{Z}_p, \mathrm{GL}_d(M))$.*

Proof. — Items (2) and (3) follow from item (1), which we now prove. If $x, y \in \mathbf{Z}_p$, then

$$(fg)(x) - (fg)(y) = (f(x) - f(y))g(x) + (g(x) - g(y))f(y),$$

which implies the claim. We now prove (4). If $d = 1$, then

$$Q^{-1}(y) - Q^{-1}(x) = \frac{Q(x) - Q(y)}{Q(x)Q(y)},$$

which implies the claim. If $d \geqslant 1$, we can write $Q^{-1} = {}^t\mathrm{co}(Q) \cdot \det(Q)^{-1}$, and the claim results from (3), and (4) applied to $d = 1$. $\square$

Remark 1.5. — Take $u \in X + X^2 E[\![X]\!]$, and let $u^{\circ n}$ be u composed with itself n times. Sen's theorem ([**Sen69**], theorem 1) implies that $\mathrm{val}_X(u^{\circ p^k}(X) - X) \geqslant p^k$ if $k \geqslant 0$, so that $\mathrm{val}_X(u^{\circ x} - u^{\circ y}) \geqslant p^i$ if $\mathrm{val}_p(x-y) \geqslant i$. This implies that the map $\mathbf{Z}_{\geqslant 0} \to X + X^2 E[\![X]\!]$, given by $n \mapsto u^{\circ n}$, extends to a super-Hölder function on $\mathbf{Z}_p$.

1.2. Super-Hölder vectors. — We now assume that M is endowed with an E-linear action by isometries of a group Γ, where Γ is isomorphic to $\mathbf{Z}_p$, via a coordinate map c. If $m \in M$, let $\mathrm{orb}_m : \Gamma \to M$ denote the function defined by $\mathrm{orb}_m(a) = a \cdot m$, so that $\mathrm{orb}_m \circ c^{-1}$ is a function $\mathbf{Z}_p \to M$.

Definition 1.6. — Let $M^{\Gamma\text{-sh},\lambda,\mu}$ denote the set of $m \in M$ such that $\mathrm{orb}_m \circ c^{-1} \in \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$, and let $M^{\Gamma\text{-sh},\lambda}$ and $M^{\Gamma\text{-sh}}$ be the corresponding sub-E-vector spaces of M.

This definition should be seen as a characteristic p analogue of the locally analytic vectors of a Banach representation of a p-adic Lie group, as defined in §7 of [**ST03**]. The requirement that Γ acts by isometries is the analogue of the condition that the norm be invariant.

Remark 1.7. — We assume that Γ acts by isometries on M, but not that Γ acts continuously on M, namely that $\Gamma \times M \to M$ is continuous. However, let M^{cont} denote the set of $m \in M$ such that $\mathrm{orb}_m \circ c^{-1} : \mathbf{Z}_p \to M$ is continuous. It is easy to see that M^{cont} is a closed sub-E-vector space of M, and that $\Gamma \times M^{\mathrm{cont}} \to M^{\mathrm{cont}}$ is continuous (compare with §3 of [**Eme17**]). We then have $M^{\mathrm{sh}} \subset M^{\mathrm{cont}}$.

Lemma 1.8. — *We have $m \in M^{\Gamma\text{-sh},\lambda,\mu}$ if and only if $\mathrm{val}_M(g \cdot m - m) \geqslant p^\lambda \cdot p^i + \mu$ for all $g \in \Gamma$ such that $c(g) \in p^i\mathbf{Z}_p$.*

Proof. — Since Γ acts by isometries, we have $\mathrm{val}_M(hg \cdot m - h \cdot m) = \mathrm{val}_M(g \cdot m - m)$ for all $g, h \in \Gamma$. $\square$

Lemma 1.9. — *The space $M^{\Gamma\text{-sh},\lambda,\mu}$ is a closed sub-E-vector space of M.*

Lemma 1.10. — *If $k \geqslant 0$ and $\Gamma' = c^{-1}(p^k\mathbf{Z}_p)$, then $g \mapsto c(g)/p^k$ is a coordinate on Γ', and $M^{\Gamma\text{-sh},\lambda} = M^{\Gamma'\text{-sh},\lambda+k}$.*

Proof. — It is clear that $M^{\Gamma\text{-sh},\lambda} \subset M^{\Gamma'\text{-sh},\lambda+k}$. Conversely, let $C = \{1, \ldots, p^k - 1\}$. If $m \in M^{\Gamma'\text{-sh},\lambda+k,\mu}$, let $\nu = \min_{c(h) \in C} \mathrm{val}_M(h \cdot m - m)$. If $g \in \Gamma \setminus \Gamma'$, we can write $g = g_k h$ with $c(h) \in C$ and $g_k \in \Gamma'$. We then have $g \cdot m - m = (g_k \cdot h \cdot m - g_k \cdot m) + (g_k \cdot m - m)$ so that $\mathrm{val}_M(g \cdot m - m) \geqslant \min(\mu, \nu)$.

This implies that $m \in M^{\Gamma\text{-sh},\lambda,\mu'}$ with $\mu' = \min(\mu, \nu) - p^{k+\lambda}$. $\square$

In particular, the space $M^{\Gamma'\text{-sh}}$ does not depend on the choice of open subgroup $\Gamma' \subset \Gamma$, and we denote it by M^{sh}.

Proposition 1.11. — *Suppose that M is a ring, and that $g(mm') = g(m)g(m')$ and $\mathrm{val}_M(mm') \geqslant \mathrm{val}_M(m) + \mathrm{val}_M(m')$ for all $m, m' \in M$ and $g \in \Gamma$.*

1. *If $v \in \mathbf{R}$ and $m, m' \in M^{\Gamma\text{-sh},\lambda,\mu} \cap M^{\mathrm{val}_M \geqslant v}$, then $m \cdot m' \in M^{\Gamma\text{-sh},\lambda,\mu+v}$;*
2. *If $m \in M^{\Gamma\text{-sh},\lambda,\mu} \cap M^\times$, then $1/m \in M^{\Gamma\text{-sh},\lambda,\mu-2\,\mathrm{val}_M(m)}$.*

Proof. — Item (1) follows from prop 1.4 and lemma 1.8. Item (2) follows from

$$g\left(\frac{1}{m}\right)-\frac{1}{m}=\frac{m-g(m)}{g(m)m}.$$

□

Remark 1.12. — One can extend the definition of super-Hölder vectors to the setting of a p-adic Lie group G acting by isometries on a valued E-vector space M as follows (the details are in our paper *Super-Hölder vectors and the field of norms*). Let P be a nice enough open pro-p subgroup of G. We say that $m \in M$ is super-Hölder if and only if there exists $\lambda, \mu \in \mathbf{R}$ and $e > 0$ such that $\mathrm{val}_M(g \cdot m - m) \geqslant p^{\lambda+ei} + \mu$ whenever $g \in P^{p^i}$, for all $i \geqslant 0$. Juan Esteban Rodríguez Camargo pointed out to us that there is a similar purely metric characterization of locally analytic vectors for a p-adic Lie group acting on a Banach space.

1.3. Mahler's theorem. — In this section, we prove a characteristic p analogue of Mahler's theorem for continuous functions $\mathbf{Z}_p \to \mathbf{Q}_p$. We then give a characterization of super-Hölder functions in terms of their Mahler expansions. If $z \in \mathbf{Z}_p$ and $n \geqslant 0$, then $\binom{z}{n} \in \mathbf{Z}_p$ and we still denote by $\binom{z}{n}$ its image in $\mathbf{F}_p$.

Theorem 1.13. — *If $\{m_n\}_{n\geqslant 0}$ is a sequence of M such that $m_n \to 0$, the function $f : \mathbf{Z}_p \to M$ given by $f(z) = \sum_{n\geqslant 0} \binom{z}{n} m_n$ belongs to $C^0(\mathbf{Z}_p, M)$. We have $m_n = (-1)^n \sum_{i=0}^n (-1)^i \binom{n}{i} f(i)$ and $\inf_{z\in\mathbf{Z}_p} \mathrm{val}_M(f(z)) = \inf_{n\geqslant 0} \mathrm{val}_M(m_n)$.*

Conversely, if $f \in C^0(\mathbf{Z}_p, M)$, there exists a unique sequence $\{m_n(f)\}_{n\geqslant 0}$ such that $m_n(f) \to 0$ and such that $f(z) = \sum_{n\geqslant 0} \binom{z}{n} m_n(f)$.

Proof. — Our proof follows Bojanic's proof (cf [**Boj74**]) of Mahler's theorem. The first part of the theorem is easy: f is continuous since it is a uniform limit of continuous functions, and if $f(z) = \sum_{n\geqslant 0} \binom{z}{n} m_n$, then $\mathrm{val}_M(f(z)) \geqslant \inf_{n\geqslant 0} \mathrm{val}_M(m_n)$. The fact that $m_n = (-1)^n \sum_{i=0}^n (-1)^i \binom{n}{i} f(i)$ is a classical exercise, given that $f(k) = \sum_{j=0}^k \binom{k}{j} m_j$ for all $k \geqslant 0$, and it implies that $\mathrm{val}_M(m_n) \geqslant \inf_{z\in\mathbf{Z}_p} \mathrm{val}_M(f(z))$ for all n. In order to show the converse, it is enough to show that if f is continuous and $m_n(f) = (-1)^n \sum_{i=0}^n (-1)^i \binom{n}{i} f(i)$, then $m_n(f) \to 0$. Indeed, the functions f and $z \mapsto \sum_{n\geqslant 0} \binom{z}{n} m_n(f)$ are then two continuous functions on $\mathbf{Z}_p$ with the same values on $\mathbf{Z}_{\geqslant 0}$, so that they are equal.

We now show that $m_n(f) \to 0$. If $s \geqslant 0$, there exists t such that if $\mathrm{val}_p(x-y) \geqslant t$ then $\mathrm{val}_M(f(x) - f(y)) \geqslant s$, as f is uniformly continuous. Take $n \geqslant p^t$ and write $n = qp^t + r$

with $0 \leqslant r < p^t$ and $q \geqslant 1$. Writing $i = a + jp^t$, we get

$$m_n(f) = \sum_{a=0}^{p^t-1} \sum_{j=0}^{q} (-1)^{n+a+jp^t} \binom{n}{a+jp^t} f(a+jp^t).$$

As we are in characteristic p, Lucas' theorem implies that $\binom{n}{a+jp^t} = \binom{r}{a}\binom{q}{j}$, so that:

$$m_n(f) = \sum_{a=0}^{p^t-1} (-1)^{n+a} \binom{r}{a} \left(\sum_{j=0}^{q} (-1)^j \binom{q}{j} f(a+jp^t) \right).$$

As $\left(\sum_{j=0}^{q}(-1)^j\binom{q}{j}\right) \cdot f(a) = 0$, and $\mathrm{val}_M(f(a+jp^t) - f(a)) \geqslant s$ for all j, we get that $\mathrm{val}_M(m_n(f)) \geqslant s$ if $n \geqslant p^t$. □

We now give a characterization of super-Hölder functions in terms of their Mahler expansions.

Proposition 1.14. — *If $f \in C^0(\mathbf{Z}_p, M)$, then $f \in \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$ if and only if for all $i \geqslant 0$, we have $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $n \geqslant p^i$.*

Proof. — Take $f \in C^0(\mathbf{Z}_p, M)$ such that $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $n \geqslant p^i$. Recall that if $a \in \mathbf{Z}_p$ and $i \geqslant 1$, then for all $j < p^i$ we have $\binom{a}{j} = \binom{a+p^i}{j}$ in $\mathbf{F}_p$. If $z \in \mathbf{Z}_p$ and $i \geqslant 1$, then

$$f(z+p^i) - f(z) = \sum_{n \geqslant 0} m_n(f) \left(\binom{z+p^i}{n} - \binom{z}{n} \right) = \sum_{n \geqslant p^i} m_n(f) \left(\binom{z+p^i}{n} - \binom{z}{n} \right).$$

Since $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^i + \mu$ whenever $n \geqslant p^i$, the formula above implies that $\mathrm{val}_M(f(x+p^i) - f(x)) \geqslant p^\lambda \cdot p^i + \mu$. Iterating this, we get that $\mathrm{val}_M(f(x+kp^i) - f(x)) \geqslant p^\lambda \cdot p^i + \mu$ for all $k \in \mathbf{Z}_{\geqslant 0}$. By continuity, this implies that $\mathrm{val}_M(f(y) - f(x)) \geqslant p^\lambda \cdot p^i + \mu$ for all $x, y \in \mathbf{Z}_p$ such that $\mathrm{val}_p(y-x) \geqslant i$.

Assume now that $f \in \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M)$. We prove that for all $i \geqslant 0$ and $n \geqslant p^i$, we have $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^i + \mu$. Fix $i \geqslant 0$ and take $a \in \{0, \ldots, p^i - 1\}$. Define a function g on $\mathbf{Z}_p$ by $g(z) = f(a + p^i z) - f(a)$. By hypothesis, we have $\mathrm{val}_M(g(z)) \geqslant p^\lambda \cdot p^i + \mu$ for all z. This implies that $\mathrm{val}_M(m_n(g)) \geqslant p^\lambda \cdot p^i + \mu$ for all n. We now compute $m_n(g)$. We have

$$\begin{aligned} g(z) &= \sum_{n \geqslant 0} \left(\binom{a+p^i z}{n} - \binom{a}{n} \right) m_n(f) \\ &= \sum_{n \geqslant p^i} \left(\binom{a+p^i z}{n} - \binom{a}{n} \right) m_n(f) = \sum_{n \geqslant p^i} \binom{a+p^i z}{n} m_n(f), \end{aligned}$$

since $a \leqslant p^i - 1$. If we write $n = t + p^i\ell$, with $0 \leqslant t \leqslant p^i - 1$ and $\ell \geqslant 1$, then $\binom{a+p^iz}{n} = \binom{a}{t}\binom{z}{\ell}$. This implies that

$$g(z) = \sum_{t=0}^{p^i-1} \sum_{\ell \geqslant 1} \binom{a}{t}\binom{z}{\ell} m_{t+p^i\ell}(f),$$

which gives $m_\ell(g) = \sum_{t=0}^{p^i-1} \binom{a}{t} m_{t+p^i\ell}(f)$ for all $\ell \geqslant 1$. This now implies that

$$\mathrm{val}_M \left(\sum_{t=0}^{p^i-1} \binom{a}{t} m_{t+p^i\ell}(f) \right) \geqslant p^\lambda \cdot p^i + \mu$$

for all $\ell \geqslant 1$ and $a \in \{0, \dots, p^i - 1\}$. The matrix $\left(\binom{a}{t}\right)_{0 \leqslant a,t \leqslant p^i-1}$ is unipotent with integral coefficients. Hence for a given $\ell \geqslant 1$, the above inequality implies that $\mathrm{val}_M(m_{a+p^i\ell}(f)) \geqslant p^\lambda \cdot p^i + \mu$ for all $a \in \{0, \dots, p^i - 1\}$. Writing $n \geqslant p^i$ as $n = a + p^i\ell$, we get $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda \cdot p^i + \mu$ for all $n \geqslant p^i$. □

Remark 1.15. — Let $\mathcal{W}^{\lambda,\mu}(\mathbf{Z}_p, M)$ denote the set of $f \in C^0(\mathbf{Z}_p, M)$ such that $\mathrm{val}_M(m_n(f)) \geqslant p^\lambda n + \mu$ for all $n \geqslant 0$.

Prop 1.14 implies that $\mathcal{W}^{\lambda,\mu}(\mathbf{Z}_p, M) \subset \mathcal{H}^{\lambda,\mu}(\mathbf{Z}_p, M) \subset \mathcal{W}^{\lambda-1,\mu}(\mathbf{Z}_p, M)$.

Prop 1.14 and remark 1.15 strengthen the analogy between our definition of super-Hölder functions and the classical theory of locally analytic functions. Indeed, if $f : \mathbf{Z}_p \to \mathbf{Q}_p$ is a continuous function, and if $f(z) = \sum_{n \geqslant 0} \binom{z}{n} m_n(f)$ is its Mahler expansion, then by a result of Amice ([**Ami64**], see corollary I.4.8 of [**Col10**]), f is locally analytic if and only if there exists $\lambda, \mu \in \mathbf{R}$ such that $\mathrm{val}_p(m_n(f)) \geqslant p^\lambda \cdot n + \mu$ for all $n \geqslant 0$.

Remark 1.16. — Daniel Gulotta pointed out to us that Gulotta (in §3 of [**Gul19**]), as well as Johansson and Newton (in §3.2 [**JN19**]), had defined a generalization of locally analytic functions, for functions valued in certain general Tate $\mathbf{Z}_p$-algebra. When $p = 0$ in the algebra, their definition is equivalent to our definition of super-Hölder functions.

2. Decompletion of cyclotomic perfectoid fields

Let $\mathbf{E}^+ = E[\![X]\!]$. For $n \geqslant 0$, let $\mathbf{E}_n^+ = E[\![X^{1/p^n}]\!]$, so that $\mathbf{E}^+ = \mathbf{E}_0^+$. Let $\mathbf{E}_\infty^+ = \cup_{n \geqslant 0} \mathbf{E}_n^+$ and let $\widetilde{\mathbf{E}}^+$ be the X-adic completion of $\mathbf{E}_\infty^+$. We denote by $\mathbf{E}$, $\mathbf{E}_n$, $\mathbf{E}_\infty$, $\widetilde{\mathbf{E}}$ the fields $\mathbf{E}^+[1/X]$, $\mathbf{E}_n^+[1/X]$, $\mathbf{E}_\infty^+[1/X]$, $\widetilde{\mathbf{E}}^+[1/X]$ respectively. The ring $\widetilde{\mathbf{E}}^+$ is the ring of integers of the field $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[1/X]$. If E is perfect, then $\widetilde{\mathbf{E}}$ is perfectoid.

2.1. The action of $\mathbf{Z}_p^\times$. — The group $\mathbf{Z}_p^\times$ acts continuously by isometries on each $\mathbf{E}_n^+$ by the formula $a \cdot X^{1/p^n} = (1+X^{1/p^n})^a - 1$. This action is compatible when n varies, extends to the fields $\mathbf{E}_n$, and extends by continuity to $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{E}}$.

Remark 2.1. — If $E = \mathbf{F}_p$, then $\widetilde{\mathbf{E}}$ is the tilt of $\widehat{\mathbf{Q}_p(\mu_{p^\infty})}$ (see §3.3 for more details). The group $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ is isomorphic to $\mathbf{Z}_p^\times$ via the cyclotomic character χ_{cyc}, and acts on $\widetilde{\mathbf{E}}$ by $g(f) = \chi_{\mathrm{cyc}}(g) \cdot f$.

If $k \geqslant 1$ (or $k \geqslant 2$ if $p = 2$), let $\Gamma_k = 1 + p^k \mathbf{Z}_p$. The natural coordinate on Γ_k is given by $1 + p^k a \mapsto \log_p(1+p^k a)/p^k$. It differs from the coordinate $1 + p^k a \mapsto a$ (which is not a group homomorphism) by an isometry. By remark 1.3, the definition of $(\widetilde{\mathbf{E}}^+)^{\Gamma_k\text{-sh},\lambda,\mu}$ does not depend on the choice of one of those coordinates, and we use $1 + p^k a \mapsto a$.

Proposition 2.2. — *We have* $\mathbf{E}_n^+ = (\mathbf{E}_n^+)^{\Gamma_k\text{-sh},k-n,0}$.

Proof. — We have $(1+X^{1/p^n})^{1+p^{k+i}b} = (1+X^{1/p^n}) \cdot (1+X^{p^{k+i-n}})^b$, so that

$$\mathrm{val}_X((1+X^{1/p^n})^{1+p^{k+i}b} - (1+X^{1/p^n})) \geqslant p^{k-n} \cdot p^i.$$

This implies that $X^{1/p^n} \in (\mathbf{E}_n^+)^{\Gamma_k\text{-sh},k-n,0}$. The claim now follows from prop 1.11 and lemma 1.9. □

Taking $n = 0$ in prop 2.2, we find that $E[\![X]\!] = E[\![X]\!]^{\Gamma_k\text{-sh},k}$. Let $\mathbf{E} = \mathbf{E}^+[1/X]$.

Corollary 2.3. — *We have* $\mathbf{E} = \mathbf{E}^{\Gamma_k\text{-sh},k}$.

Proof. — This follows from prop 2.2 and prop 1.11. □

Proposition 2.4. — *If* $\varepsilon > 0$, *then* $E[\![X]\!]^{\Gamma_k\text{-sh},k+\varepsilon} \subset E[\![X^p]\!]$.

Proof. — Take $f(X) \in E[\![X]\!]$. There is a power series $h(Y,Z) \in E[\![Y,Z]\!]$ such that

$$f(Y+Z) = f(Y) + Z \cdot f'(Y) + Z^2 \cdot h(Y,Z).$$

If $m \geqslant 0$, this implies that

$$\begin{aligned} f((1+X)^{1+p^m} - 1) &= f(X + X^{p^m}(1+X)) \\ &= f(X) + X^{p^m}(1+X) \cdot f'(X) + \mathrm{O}(X^{2p^m}). \end{aligned}$$

If $f(X) \notin E[\![X^p]\!]$, then $f'(X) \neq 0$. Let $\mu = \mathrm{val}_X(f'(X))$. The above computations imply that $\mathrm{val}_X((1+p^{i+k}) \cdot f(X) - f(X)) = p^{i+k} + \mu$ for $i \gg 0$. This implies the claim. □

Corollary 2.5. — *We have* $(\mathbf{E}_\infty^+)^{\Gamma_k\text{-sh},k-n} = \mathbf{E}_n^+$.

Proof. — Take $f(X^{1/p^m}) \in (\mathbf{E}_\infty^+)^{\Gamma_k\text{-sh},k-n}$ where $f(X) \in E[\![X]\!]$. Since $\mathrm{val}_X(h^p) = p \cdot \mathrm{val}_X(h)$ for all $h \in \widetilde{\mathbf{E}}^+$, we have $f^{p^m}(X) \in (\mathbf{E}_\infty^+)^{\Gamma_k\text{-sh},k+m-n}$, where $f^{p^m}(X) \in E[\![X]\!]$ is $f^{p^m}(X) = f(X^{1/p^m})^{p^m}$. If $m \geqslant n+1$, then prop 2.4 implies that $f^{p^m}(X) \in E[\![X^p]\!]$, so that $f(X) = g(X^p)$, and $f(X^{1/p^m}) = g(X^{1/p^{m-1}})$. This implies the claim. □

2.2. Tate traces. — We recall some constructions of Colmez (see §8.2 of [**Col08**]). For $m \geqslant 0$ let $I_m = p^{-m}\mathbf{Z} \cap [0,1)$, and let $I = \cup_m I_m$. Note that if $i \in I_m$, then $(1+X)^i \in \mathbf{E}_m^+$.

Lemma 2.6. — *The elements $(1+X)^i$, $i \in I_m$, form a basis of $\mathbf{E}_m^+$ over $\mathbf{E}_0^+$.*

Proof. — See lemma 8.2 of [**Col08**]. Colmez works with $E = \mathbf{F}_p$, but the proofs are the same with arbitrary coefficients. □

Proposition 2.7. — *Any $f \in \widetilde{\mathbf{E}}^+$ can be written uniquely as $\sum_{i \in I}(1+X)^i a_i(f)$, with $a_i(f) \in \mathbf{E}_0^+$, and $a_i(f) \to 0$. Moreover, $\mathrm{val}_X(f) - 1 < \inf_{i \in I} \mathrm{val}_X(a_i(f)) \leqslant \mathrm{val}_X(f)$.*

Proof. — See props 4.10 and 8.3 of [**Col08**]. □

In particular, for all $i \in I$, the map $\widetilde{\mathbf{E}}^+ \to \mathbf{E}_0^+$, given by $f \mapsto a_i(f)$ is continuous.

Proposition 2.8. — *There exists a family $\{T_n\}_{n \geqslant 0}$ of continuous maps $T_n : \widetilde{\mathbf{E}}^+ \to \mathbf{E}_n^+$ satisfying the following properties:*

1. *The restriction of T_n to $\mathbf{E}_n^+$ is the identity map.*
2. *We have $T_n(f) \to f$ as $n \to +\infty$.*
3. *We have $\mathrm{val}_X(T_n(f)) \geqslant \mathrm{val}_X(f) - 1$ for all n.*
4. *Each T_n is $\mathbf{Z}_p^\times$-equivariant.*

Proof. — If $f = \sum_{i \in I}(1+X)^i a_i(f)$, let $T_n(f) = \sum_{i \in I_n}(1+X)^i a_i(f)$. With this definition, the first property is immediate. The second and third one follow from prop 2.7.

For the last one, observe that if $i \in I$ and $g \in \mathbf{Z}_p^\times$, then $g \cdot (1+X)^i = (1+X)^{ig}$ so $g \cdot (1+X)^i$ can be written uniquely as $(1+X)^{\sigma_g(i)} u_{i,g}(X)$ with $\sigma_g(i) \in I$ and $u_{i,g}(X) \in \mathbf{E}_0^+$. The map σ_g induces a bijection from I_m to itself for all m. Take $f \in \widetilde{\mathbf{E}}^+$, and write $f = \sum_{i \in I}(1+X)^i a_i(f)$. We have $g \cdot f = \sum_{i \in I}(1+X)^{\sigma_g(i)} u_{i,g}(X)(g \cdot a_i(f))$, so that $T_n(g \cdot f) = \sum_{i \in I_n}(1+X)^{\sigma_g(i)} u_{i,g}(X)(g \cdot a_i(f)) = g \cdot T_n(f)$. □

2.3. Decompletion of $\widetilde{\mathbf{E}}$. — We now prove that $\widetilde{\mathbf{E}}^{\mathrm{sh}} = \mathbf{E}_\infty$. More precisely, we have the following result.

Theorem 2.9. — *We have $\widetilde{\mathbf{E}}^{\Gamma_k\text{-sh},k-m} = \mathbf{E}_m$ for all $m \geqslant 0$, and $\widetilde{\mathbf{E}}^{\mathrm{sh}} = \mathbf{E}_\infty$.*

Proposition 2.10. — *If $f \in (\widetilde{\mathbf{E}}^+)^{\Gamma_k\text{-sh},\lambda,\mu}$, then $T_n(f) \in (\mathbf{E}_n^+)^{\Gamma_k\text{-sh},\lambda,\mu-1}$.*

Proof. — If $g \in \Gamma_k$, then $g(T_n(f)) - T_n(f) = T_n(g(f) - f)$ so that

$$\mathrm{val}_X(g(T_n(f)) - T_n(f)) = \mathrm{val}_X(T_n(g(f) - f)) \geqslant \mathrm{val}_X(g(f) - f) - 1$$

by prop 2.8. This implies the claim. □

Proof of theorem 2.9. — Take $f \in (\widetilde{\mathbf{E}}^+)^{\Gamma_{k\text{-sh},k-m}}$. By prop 2.10, we have $T_n(f) \in (\mathbf{E}_n^+)^{\Gamma_{k\text{-sh},k-m}}$ for all $n \geqslant 0$. By coro 2.5, $T_n(f) \in \mathbf{E}_m^+$ for all n. Since $T_n(f) \to f$ as $n \to +\infty$, we have $f \in \mathbf{E}_m^+$.

Hence $(\widetilde{\mathbf{E}}^+)^{\Gamma_{k\text{-sh},k-m}} = \mathbf{E}_m^+$, and this implies the theorem by prop 1.11. □

3. Applications

We now give several applications of the fact that $\widetilde{\mathbf{E}}^{\mathrm{sh}} = \mathbf{E}_\infty$.

3.1. The perfectoid commutant of $\mathrm{Aut}(\mathbf{G}_{\mathrm{m}})$**.** — In this section, we assume that $E = \mathbf{F}_p$. If $a \in \mathbf{Z}_p^\times$, let $\gamma_a(X) = (1+X)^a - 1 \in \mathbf{F}_p[\![X]\!]$. Note that if $f \in \widetilde{\mathbf{E}}$, then $a \cdot f = f \circ \gamma_a$. If $u \in \widetilde{\mathbf{E}}^+$ is such that $\mathrm{val}_X(u) > 0$, the series $\gamma_a \circ u$ converges in $\widetilde{\mathbf{E}}^+$. If $u = \gamma_b(X^{p^n})$ for some $b \in \mathbf{Z}_p^\times$ and $n \in \mathbf{Z}$, then $u \circ \gamma_a = \gamma_a \circ u$ for all $a \in \mathbf{Z}_p^\times$.

Theorem 3.1. — *If $u \in \widetilde{\mathbf{E}}^+$ is such that $\mathrm{val}_X(u) > 0$ and $u \circ \gamma_a = \gamma_a \circ u$ for all $a \in \mathbf{Z}_p^\times$, then there exists $b \in \mathbf{Z}_p^\times$ and $n \in \mathbf{Z}$ such that $u(X) = \gamma_b(X^{p^n})$.*

Recall that a power series $f(X) \in \mathbf{F}_p[\![X]\!]$ is separable if $f'(X) \neq 0$. If $f(X) \in X \cdot \mathbf{F}_p[\![X]\!]$, we say that f is invertible if $f'(0) \in \mathbf{F}_p^\times$, which is equivalent to f being invertible for composition (denoted by $\circ$). We say that $w(X) \in X \cdot \mathbf{F}_p[\![X]\!]$ is nontorsion if $w^{\circ n}(X) \neq X$ for all $n \geqslant 1$. The following is a reformulation of lemma 6.2 of [**Lub94**].

Lemma 3.2. — *Let $w(X) \in X + X^2 \cdot \mathbf{F}_p[\![X]\!]$ be an invertible nontorsion series, and let $f(X) \in X \cdot \mathbf{F}_p[\![X]\!]$ be a separable power series. If $w \circ f = f \circ w$, then f is invertible.*

Lemma 3.3. — *If $u \in \widetilde{\mathbf{E}}^+$ is such that $\mathrm{val}_X(u) > 0$ and $u \circ \gamma_a = \gamma_a \circ u$ for all $a \in \mathbf{Z}_p^\times$, then $u \in (\widetilde{\mathbf{E}}^+)^{\mathrm{sh}}$.*

Proof. — The group $\mathbf{Z}_p^\times$ acts on $\widetilde{\mathbf{E}}^+$ by $a \cdot u = u \circ \gamma_a$, so we need to check that the function $a \mapsto \gamma_a \circ u$ is super-Hölder. This is clear since $\gamma_a(u) = \sum_{n \geqslant 1} \binom{a}{n} u^n$ and $\mathrm{val}_X(u) > 0$. □

Proof of theorem 3.1. — Take $u \in \widetilde{\mathbf{E}}^+$ such that $\mathrm{val}_X(u) > 0$ and $u \circ \gamma_a = \gamma_a \circ u$ for all $a \in \mathbf{Z}_p^\times$. By lemma 3.3 and theorem 2.9, there exists $m \geqslant 0$ such that $u \in \mathbf{E}_m^+$. Hence there is an $n \in \mathbf{Z}$ such that $f(X) = u(X^{1/p^n})$ belongs to $X \cdot \mathbf{F}_p[\![X]\!]$ and is separable. Take $g \in 1 + p\mathbf{Z}_p$ such that g is nontorsion, and let $w(X) = \gamma_g(X)$ so that $u \circ w = w \circ u$. We also have $f \circ w = w \circ f$. By lemma 3.2, f is invertible. Since $f \circ \gamma_a = \gamma_a \circ f$ for all

$a \in \mathbf{Z}_p^\times$, theorem 6 of [**LS07**] implies that $f \in \mathrm{Aut}(\mathbf{G}_\mathrm{m})$. Hence there exists $b \in \mathbf{Z}_p^\times$ such that $f(X) = \gamma_b(X)$. This implies the theorem. □

3.2. Decompletion of (φ, Γ)-modules. — Let $\Gamma_k = 1 + p^k \mathbf{Z}_p$ with $k \geqslant 1$, as in §2.1. Let M be a finite-dimensional $\mathbf{E}$-vector space with a continuous semi-linear action of Γ_k.

Proposition 3.4. — *There is an $\mathbf{E}^+$-lattice in M that is stable under Γ_k.*

Proof. — Choose any lattice M_0^+ of M. The map $\pi : \Gamma_k \times M \to M$ is continuous, so there is an open subgroup H of Γ_k and an $n \geqslant 0$ such that $\pi^{-1}(M_0^+)$ contains $H \times X^n M_0^+$. In particular, $h(m) \in X^{-n} M_0^+$ for all $h \in H$ and $m \in M_0^+$. Since H is open in the compact group Γ_k, it is of finite index, and there exists $d \geqslant n$ such that $g(m) \subset X^{-d} M_0^+$ for all $g \in \Gamma_k$ and $m \in M_0^+$. The space $M^+ = \sum_{g \in \Gamma_k} g(M_0^+)$ is an $\mathbf{E}^+$-module such that $M_0^+ \subset M^+ \subset X^{-d} M_0^+$, so that M^+ is a lattice of M. It is clearly stable under Γ_k. □

Choosing such an $\mathbf{E}^+$-lattice in M defines a valuation val_M on M, such that Γ_k acts on M by isometries. We make such a choice, and we can therefore define M^{sh} and $M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ as in definition 1.6. We say that the action of Γ_k on M is super-Hölder if $M = M^{\mathrm{sh}}$.

Lemma 3.5. — *The space $M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ does not depend on the choice of Γ_k-stable lattice of M. If $\lambda \leqslant k$ then $M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ is sub-$\mathbf{E}$-vector space of M.*

Proof. — The first assertion results from the fact that if we choose two $\mathbf{E}^+$-lattices M_1^+ and M_2^+ in M, then there exists a constant C such that $|\mathrm{val}_1 - \mathrm{val}_2| \leqslant C$.

Next, recall that by coro 2.3, $\mathbf{E} = \mathbf{E}^{\Gamma_k\text{-}\mathrm{sh},k}$. If $m \in M^{\mathrm{sh},\lambda}$, $f \in \mathbf{E}$, and $g \in \Gamma_k$, then $g(fm) - fm = g(f)(g(m) - m) + (g(f) - f)m$, so that $fm \in M^{\mathrm{sh},\lambda}$ by lemma 1.8. □

Lemma 3.5 implies that M^{sh} is a sub-$\mathbf{E}$-vector space of M. We say that a basis of M is good if it generates a lattice that is stable under Γ_k.

Proposition 3.6. — *Take $\lambda \leqslant k$ and fix a good basis of M. We have $M = M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ if and only if the map $\Gamma_k \to \mathrm{M}_n(\mathbf{E}^+)$, given by $g \mapsto \mathrm{Mat}(g)$, is in $\mathcal{H}^\lambda(\Gamma_k, \mathrm{M}_n(\mathbf{E}^+))$.*

Proof. — We fix a good basis $(m_1, \dots, m_n)$ of M, and work with the corresponding valuation val_M on M. By lemma 3.5, we have $M = M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ if and only if $m_j \in M^{\Gamma_k\text{-}\mathrm{sh},\lambda}$ for all j. We have $g \cdot m_j = \sum_{i=1}^n \mathrm{Mat}(g)_{i,j} m_i$ by definition of $\mathrm{Mat}(g)$. Hence if $g, h \in \Gamma_k$, then $g \cdot m_j - h \cdot m_j = \sum_{i=1}^n (\mathrm{Mat}(g)_{i,j} - \mathrm{Mat}(h)_{i,j}) m_i$. This implies that if $\ell \geqslant 0$ and $\mu \in \mathbf{R}$, then $\mathrm{val}_M(g \cdot m_j - h \cdot m_j) \geqslant p^{\lambda+\ell} + \mu$ if and only if $\mathrm{val}_X(\mathrm{Mat}(g) - \mathrm{Mat}(h)) \geqslant p^{\lambda+\ell} + \mu$. This implies the claim. □

If M is a finite-dimensional $\mathbf{E}$-vector space with a semi-linear action of Γ_k, then $\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M$ is a finite-dimensional $\widetilde{\mathbf{E}}$-vector space with a semi-linear action of Γ_k. If M is super-Hölder, there exists $m_0 = m_0(M) \geqslant 0$ such that $M = M^{\Gamma_k\text{-sh},k-m_0}$

Proposition 3.7. — *If M is super-Hölder and $m \geqslant m_0(M)$, then $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\Gamma_k\text{-sh},k-m} = \mathbf{E}_m \otimes_{\mathbf{E}} M$.*

Proof. — By the same argument as in the proof of lemma 3.5, we see that for $m \geqslant m_0$, $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\Gamma_k\text{-sh},k-m}$ is a sub-$\mathbf{E}_m$-vector space of $\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M$. The space $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\Gamma_k\text{-sh},k-m}$ contains M, and therefore also $\mathbf{E}_m \otimes_{\mathbf{E}} M$. This proves one inclusion.

We now prove that $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\Gamma_k\text{-sh},k-m} \subset \mathbf{E}_m \otimes_{\mathbf{E}} M$. Fix a good basis $(m_1, \dots, m_n)$ of M, the corresponding valuation val_M on $\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M$, and $m \geqslant m_0$. Take $x = \sum_{i=1}^n x_i m_i \in \widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M$ and write $g(x) = \sum_{i=1}^n f_i(g) m_i$. We have $x \in (\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\Gamma_k\text{-sh},k-m}$ if and only if $f_i \in \mathcal{H}^{k-m}(\Gamma_k, \widetilde{\mathbf{E}})$ for all i. In addition, $g(x) = \sum_{i,j} g(x_i) \operatorname{Mat}(g)_{j,i} m_j$. Hence $f_j : g \mapsto \sum_{i=1}^n g(x_i) \operatorname{Mat}(g)_{j,i}$ belongs to $\mathcal{H}^{k-m}(\Gamma_k, \widetilde{\mathbf{E}})$ for all j. We have $g(x_\ell) = \sum_{j=1}^n f_j(g)(\operatorname{Mat}(g)^{-1})_{\ell,j}$. By props 3.6 and 1.4, $[g \mapsto g(x_\ell)] \in \mathcal{H}^{k-m}(\Gamma_k, \widetilde{\mathbf{E}})$ and therefore $x_\ell \in \widetilde{\mathbf{E}}^{\Gamma_k\text{-sh},k-m} = \mathbf{E}_m$ for all ℓ. □

Corollary 3.8. — *If M is super-Hölder, then $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} M)^{\mathrm{sh}} = \mathbf{E}_\infty \otimes_{\mathbf{E}} M$.*

The field $\mathbf{E} = E(\!(X)\!)$ is equipped with its action of $\mathbf{Z}_p^\times$ and with the E-linear Frobenius map φ given by $\varphi(f)(X) = f(X^p)$. Let $\Gamma = \Gamma_k$ with $k \geqslant 1$. A (φ, Γ)-module $\mathbf{D}$ over $\mathbf{E}$ is a finite-dimensional $\mathbf{E}$-vector space, endowed with commuting, semi-linear actions of φ and Γ, such that the action of Γ is continuous and such that $\operatorname{Mat}(\varphi)$ is invertible (in any basis of $\mathbf{D}$).

Proposition 3.9. — *If $\mathbf{D}$ is a (φ, Γ)-module over $\mathbf{E}$, then $\mathbf{D} = \mathbf{D}^{\Gamma_k\text{-sh},k}$.*

Lemma 3.10. — *If $\ell \geqslant 1$ and $\lambda, \mu \in \mathbf{R}$, then $\mathcal{H}^{\lambda,\mu}(\Gamma_\ell, \mathrm{M}_n(\mathbf{E}^+))$ is a ring, that is stable under φ.*

Proof. — The first claim follows from prop 1.4. The second one follows from the fact that if $M \in \mathrm{M}_n(\mathbf{E}^+)$, then $\mathrm{val}_X(\varphi(M)) \geqslant \mathrm{val}_X(M)$. □

Proof of of proposition 3.9. — Choose a good basis $(d_1, \dots, d_n)$ of $\mathbf{D}$. We can replace $(d_1, \dots, d_n)$ by $(X^s d_1, \dots, X^s d_n)$ for some $s \geqslant 0$, and assume that $P = \operatorname{Mat}(\varphi) \in \mathrm{M}_n(\mathbf{E}^+)$. Take $r \geqslant 1$ such that $X^r P^{-1} \in X\,\mathrm{M}_n(\mathbf{E}^+)$. Let G_g be the matrix of $g \in \Gamma$. By continuity of the map $\Gamma \to \mathrm{GL}_n(\mathbf{E}^+)$, $g \mapsto G_g$, there exists $\ell \geqslant k$ such that for all $g \in \Gamma_\ell$, we have $\mathrm{val}_X(G_g - \mathrm{Id}) \geqslant r$. Write $G_g = \mathrm{Id} + X^r H_g$ with $H_g \in \mathrm{M}_n(\mathbf{E}^+)$.

By definition of r, we have $X^r g(P)^{-1} \in X\,\mathrm{M}_n(\mathbf{E}^+)$, so that if $Q_g = X^{r(p-1)}g(P)^{-1}$, then $Q_g \in X\,\mathrm{M}_n(\mathbf{E}^+)$. The commutation relation between φ and Γ_ℓ gives $P\varphi(G_g) = G_g g(P)$ for all $g \in \Gamma_\ell$. Therefore, $P\varphi(\mathrm{Id} + X^r H_g) = (\mathrm{Id} + X^r H_g)g(P)$, so that

$$Pg(P)^{-1} - \mathrm{Id} = X^r(H_g - P\varphi(H_g)Q_g).$$

This implies that $Pg(P)^{-1} - \mathrm{Id} \in X^r\,\mathrm{M}_n(\mathbf{E}^+)$. Let

$$f(g) = H_g - P\varphi(H_g)Q_g = X^{-r}\left(Pg(P)^{-1} - \mathrm{Id}\right).$$

Recall that $Q_g, f(g) \in \mathrm{M}_n(\mathbf{E}^+)$ for all $g \in \Gamma_\ell$, and that (compare with (4) of prop 1.4)

$$Q_g = X^{r(p-1)}g(P)^{-1} = X^{r(p-1)}g({}^t\mathrm{co}(P))g(\det(P)^{-1})$$

and

$$f(g) = X^{-r}\left(Pg({}^t\mathrm{co}(P))g(\det(P)^{-1}) - \mathrm{Id}\right).$$

By props 1.11 and 2.2, and lemma 3.10, there exists $\mu \in \mathbf{R}$ such that $g \mapsto Q_g$ and $g \mapsto f(g)$ belong to $\mathcal{H}^{\ell,\mu}(\Gamma_\ell, \mathrm{M}_n(\mathbf{E}^+))$.

Let $f_0 = f$ and for $i \geqslant 1$, let $f_i : \Gamma_\ell \to \mathrm{M}_n(\mathbf{E}^+)$ be the function

$$g \mapsto P\varphi(P)\cdots\varphi^{i-1}(P)\cdot\varphi^i(f(g))\cdot\varphi^{i-1}(Q_g)\cdots\varphi(Q_g)Q_g.$$

Since $P \in \mathrm{M}_n(\mathbf{E}^+)$, lemma 3.10 implies that $f_i \in \mathcal{H}^{\ell,\mu}(\Gamma_\ell, \mathrm{M}_n(\mathbf{E}^+))$. In addition, $\mathrm{val}_X(Q_g) \geqslant 1$, so that $\mathrm{val}_X(\varphi^{i-1}(Q_g)\cdots\varphi(Q_g)Q_g) \geqslant (p^i-1)/(p-1)$. Hence $\sum_{i \geqslant 0} f_i$ converges in $\mathcal{H}^{\ell,\mu}(\Gamma_\ell, \mathrm{M}_n(\mathbf{E}^+))$, and we let $T(f)$ be its limit.

We have $T(f)(g) = H_g$. This implies that $g \mapsto H_g$ belongs to $\mathcal{H}^{\ell,\mu}(\Gamma_\ell, \mathrm{M}_n(\mathbf{E}^+))$, and hence so does $g \mapsto G_g = \mathrm{Id} + X^r H_g$.

We therefore have $\mathbf{D} = \mathbf{D}^{\Gamma_\ell\text{-sh},\ell}$, so that $\mathbf{D} = \mathbf{D}^{\Gamma_k\text{-sh},k}$ by lemma 1.10. □

Corollary 3.11. — *If $\mathbf{D}$ is a (φ,Γ)-module over $\mathbf{E}$, then $(\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} \mathbf{D})^{\Gamma_k\text{-sh},k-m} = \mathbf{E}_m \otimes_{\mathbf{E}} \mathbf{D}$ for $m \geqslant 0$.*

We now prove the following result, which generalizes prop 3.9. Note that the underlying constants are not as good as in the case of a (φ,Γ)-module.

Proposition 3.12. — *If M is a finite-dimensional $\mathbf{E}$-vector space with a continuous semi-linear action of Γ_k, then $M = M^{\mathrm{sh}}$.*

Proof. — Choose a good basis of M. Let $f(g)$ denote the matrix of $g \in \Gamma$ in this basis. If $\ell \geqslant 1$, there exists $k \geqslant \ell + 1$ such that $f(g) \in \mathrm{Id} + X^{p^\ell}\,\mathrm{M}_n(\mathbf{E}^+)$ for all $g \in 1 + p^k\mathbf{Z}_p$. Write $f(g) = \mathrm{Id} + X^{p^\ell}H$. The cocycle formula gives

$$f(g^p) = (\mathrm{Id} + X^{p^\ell}H)(\mathrm{Id} + g(X^{p^\ell}H))\cdots(\mathrm{Id} + g^{p-1}(X^{p^\ell}H)).$$

Prop 2.2, with $n=0$, implies that $g^m(X^{p^\ell}H) \equiv X^{p^\ell}H \bmod X^{p^k}$ for all $0 \leqslant m \leqslant p-1$. Hence $f(g^p) \equiv (\mathrm{Id} + X^{p^\ell}H)^p \bmod X^{p^k}$. This implies that $f(g^p) \equiv \mathrm{Id} + X^{p^{\ell+1}}H^p \bmod X^{p^k}$ so that $f(g^p) = \mathrm{Id} \bmod X^{p^{\ell+1}}$ since $k \geqslant \ell+1$.

Since $(1+p^k\mathbf{Z}_p)^p = 1+p^{k+1}\mathbf{Z}_p$, the above computation implies by induction on i that $f(1+p^{k+i}\mathbf{Z}_p) \subset \mathrm{Id} + X^{p^{\ell+i}}\mathrm{M}_n(\mathbf{E}^+)$ for all $i \geqslant 0$.

This implies that $M = M^{\Gamma_k\text{-sh},\ell,0}$ by lemma 1.8. □

Corollary 3.13. — *Let N be an $\mathbf{E}$-vector space, with a compatible valuation and a semi-linear action of Γ_k by isometries. Let N^{fin} denote the set of $x \in N$ that belong to a finite dimensional $\mathbf{E}$-vector space stable under Γ_k, in analogy with classical Sen theory.*

Prop 3.12 implies that $N^{\mathrm{fin}} \subset N^{\mathrm{sh}}$. In particular, if $N = \widetilde{\mathbf{E}}$, then $\widetilde{\mathbf{E}}^{\mathrm{fin}} = \widetilde{\mathbf{E}}^{\mathrm{sh}} = \mathbf{E}_\infty$.

3.3. The field of norms. — Let K be a finite extension of $\mathbf{Q}_p$. Let $K_n = K(\mu_{p^n})$ and let $K_\infty = \cup_{n\geqslant 0} K_n$. The field of norms of the extension $K(\mu_{p^\infty})/K$ is defined and studied in [**Win83**]. It is the set of sequences $\{x_n\}_{n\geqslant 0}$ where $x_n \in K_n$ and $\mathrm{N}_{K_{n+1}/K_n}(x_{n+1}) = x_n$ for all $n \geqslant 0$. This set has a natural structure of a field of characteristic p whose residue field is that of K_∞ (§2.1 of ibid), which we denote by $\mathbf{E}_K$. If $K = \mathbf{Q}_p$, then $\mathbf{E}_{\mathbf{Q}_p} = \mathbf{F}_p(\!(X)\!)$, where $X = \{x_n\}_{n\geqslant 0}$ with $x_n = 1-\zeta_{p^n}$ for $n \geqslant 1$. When K is a finite extension of $\mathbf{Q}_p$, $\mathbf{E}_K$ is a finite separable extension of $\mathbf{E}_{\mathbf{Q}_p}$ of degree $[K_\infty : (\mathbf{Q}_p)_\infty]$ (§3.1 of ibid).

Let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, so that Γ_K is isomorphic to an open subgroup of $\mathbf{Z}_p^\times$ via the cyclotomic character χ_{cyc}. The group Γ_K acts naturally on $\mathbf{E}_K$, and if $g \in \Gamma_K$, then $g(X) = (1+X)^{\chi_{\mathrm{cyc}}(g)} - 1$. Let $\varphi : \mathbf{E}_K \to \mathbf{E}_K$ denote the map $y \mapsto y^p$. Let $\widetilde{\mathbf{E}}_K$ denote the X-adic completion of $\cup_{n\geqslant 0}\varphi^{-n}(\mathbf{E}_K)$. In particular, $\widetilde{\mathbf{E}}_{\mathbf{Q}_p} = \widetilde{\mathbf{E}}$ in the notation of §2, and $\widetilde{\mathbf{E}}_K$ is the tilt of $\widehat{K}_\infty$ (§4.3 of ibid and §3 of [**Sch12**]).

Lemma 3.14. — *We have $\varphi^{-n}(\mathbf{E}_K) = \mathbf{E}_n \otimes_{\mathbf{E}} \mathbf{E}_K$ for all n, and $\widetilde{\mathbf{E}}_K = \widetilde{\mathbf{E}} \otimes_{\mathbf{E}} \mathbf{E}_K$.*

Proof. — The extensions $\mathbf{E}_n/\mathbf{E}$ and $\mathbf{E}_K/\mathbf{E}$ are linearly disjoint since the first is purely inseparable and the second is separable. By comparing degrees, we get the first claim. It implies that $\widetilde{\mathbf{E}} \otimes_{\mathbf{E}} \mathbf{E}_K \to \widetilde{\mathbf{E}}_K$ is surjective, and the second claim follows, since $[\widetilde{\mathbf{E}}_K : \widetilde{\mathbf{E}}] = [\mathbf{E}_K : \mathbf{E}] = [K_\infty : (\mathbf{Q}_p)_\infty]$. □

Corollary 3.15. — *We have $\widetilde{\mathbf{E}}_K^{\mathrm{sh}} = \cup_{n\geqslant 0}\varphi^{-n}(\mathbf{E}_K)$.*

Proof. — This follows from lemma 3.14 and coro 3.11, as $\mathbf{E}_K$ is a (φ, Γ_K)-module over $\mathbf{E}$, and $\cup_{n\geqslant 0}\varphi^{-n}(\mathbf{E}_K) = \mathbf{E}_\infty \otimes_{\mathbf{E}} \mathbf{E}_K$. □

Remark 3.16. — In characteristic zero, $\hat{K}_\infty$ is a p-adic Banach representation of Γ_K, and by theorem 3.2 of [**BC16**], K_∞ is the space $\hat{K}_\infty^{\mathrm{la}}$ of locally analytic vectors in $\hat{K}_\infty$.

3.4. The p-adic local Langlands correspondence. — We now prove a result that suggests that the theory of super-Hölder vectors could have some applications to the p-adic local Langlands correspondence. In order to avoid too many technicalities, we consider only the simplest example. Recall that if $f \in \mathbf{E}^+$, there exist $f_0, \ldots, f_{p-1} \in \mathbf{E}^+$ such that $f = \sum_{i=0}^{p-1} \varphi(f_i)(1+X)^i$. We define $\psi(f) = f_0$. The map $\psi : \mathbf{E}^+ \to \mathbf{E}^+$ has the following properties: $\psi(f\varphi(h)) = h\psi(f)$ if $f, h \in \mathbf{E}^+$ and $\psi \circ g = g \circ \psi$ if $g \in \mathbf{Z}_p^\times$.

Let $M = \varprojlim_\psi \mathbf{E}^+$ be the set of sequences $m = (m_0, m_1, \ldots)$ with $m_i \in \mathbf{E}^+$ and $\psi(m_{i+1}) = m_i$ for all $i \geqslant 0$. The space M is endowed with an action of $\mathbf{Z}_p^\times$ given by $(g \cdot m)_i = g \cdot m_i$ and the structure of an $\mathbf{E}^+$-module given by $(f(X)m)_i = \varphi^i(f(X))m_i$. Following Colmez, we could extend these structures to an action of the Borel subgroup $\mathrm{B}_2(\mathbf{Q}_p)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ on M, and this idea is an important step in the construction of the p-adic local Langlands correspondence. The representation M is then the dual of most of the restriction to $\mathrm{B}_2(\mathbf{Q}_p)$ of a parabolic induction. However, we don't use this here.

Let val_X be the X-adic valuation on M: $\mathrm{val}_X(m)$ is the max of the $n \geqslant 0$ such that $m \in X^n M$. The space M is separated and complete for the X-adic topology, although this is not the natural topology on M (the natural topology is induced by the product topology $\varprojlim_\psi \mathbf{E}^+ \subset \prod \mathbf{E}^+$. The action of $\mathbf{Z}_p^\times$ on M is not continuous for the X-adic topology: $M \neq M^{\mathrm{cont}}$ in the notation of remark 1.7).

We have an injection $i : \mathbf{E}^+ \to M$, given by $i(f) = (f, \varphi(f), \varphi^2(f), \ldots)$.

Proposition 3.17. — *We have $M^{\Gamma_k\text{-sh},k} = i(\mathbf{E}^+)$.*

Proof. — Recall that if $m \in M$ and $f(X) \in \mathbf{E}$, then $(f(X)m)_j = \varphi^j(f(X))m_j$ for all $j \geqslant 0$. We have $\mathrm{val}_X(\varphi^j(f(X))) = p^j \mathrm{val}_X(f(X))$. In particular, if $m \in M^{\Gamma_k\text{-sh},k}$, then $m_j \in (\mathbf{E}^+)^{\Gamma_k\text{-sh},k+j}$. The results of §2.1 imply that $m_j \in \varphi^j(\mathbf{E}^+)$. If $m_j = \varphi^j(f_j)$, the ψ-compatibility implies that $f_j = f_0$ for all $j \geqslant 0$. This implies the claim. □

A generalization of prop 3.17 to representations of $\mathrm{B}_2(\mathbf{Q}_p)$ obtained from (φ, Γ)-modules using Colmez' construction shows that using the theory of super-Hölder vectors, we can recover the (φ, Γ)-module giving rise to such a representation of $\mathrm{B}_2(\mathbf{Q}_p)$. One of the main results of [**BV14**] is that every infinite dimensional smooth irreducible E-linear representation of $\mathrm{B}_2(\mathbf{Q}_p)$ having a central character comes from a (φ, Γ)-module by Colmez' construction. Is it possible to reprove this result using super-Hölder vectors?

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