VANISHING OF $H^2(G_K, V)$

by

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A

Let V be a $\mathbf{Q}_p$-linear representation of G_K. In this appendix we prove the following theorem.

Theorem A.1. — *If V is semistable and all its Hodge-Tate weights are ≥ 2, then $H^2(G_K, V) = 0$.*

Let $\mathrm{D}(V)$ be Fontaine's (φ, Γ)-module attached to V [**Fon90**]. It comes with a Frobenius map φ and an action of Γ_K. Let $H_K = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$ and let $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{nr}})$. The injectivity of the restriction map $H^2(G_K, V) \to H^2(G_L, V)$ for L/K finite allows us to replace K by a finite extension, so that we can assume that $H_K I_K = G_K$ and that $\Gamma_K \simeq \mathbf{Z}_p$. Let γ be a topological generator of Γ_K. Recall (§I.5 of [**CC99**]) that we have a map $\psi : \mathrm{D}(V) \to \mathrm{D}(V)$.

Ideally, our proof of this theorem would go as follows. We use the Hochschild-Serre spectral sequence

$$H^i(G_K/I_K, H^j(I_K, V|_{I_K})) \Rightarrow H^{i+j}(G_K, V)$$

and, interpreting Galois cohomology in terms of (φ, Γ)-modules, we compute that $H^2(I_K, V|_{I_K}) = 0$ and $H^1(I_K, V|_{I_K}) = \hat{K}^{\mathrm{nr}} \otimes_K D_{\mathrm{dR}}(V)$. We conclude since, by Hilbert 90, $H^1(G_K/I_K, H^1(I_K, V|_{I_K})) = 0$. However, we do not, in general, have Hochschild-Serre spectral sequences for continuous cohomology. We mimic thus the above argument with direct computations on continuous cocycles (again using (φ, Γ)-modules). Laurent Berger is grateful to Kevin Buzzard for discussions related to the above spectral sequence.

Lemma A.2. — 1. *If V is a representation of G_K, then there is an exact sequence*

$$0 \to \mathrm{D}(V)^{\psi=1}/(\gamma - 1) \to H^1(G_K, V) \to (\mathrm{D}(V)/(\psi - 1))^{\Gamma_K} \to 0;$$

2. *We have* $H^2(G_K, V) = \mathrm{D}(V)/(\psi-1,\gamma-1)$.

Proof. — See I.5.5 and II.3.2 of [**CC99**]. □

Lemma A.3. — *We have* $\mathrm{D}(V|_{I_K})/(\psi-1)=0$

Proof. — Since $V|_{I_K}$ corresponds to the case when k is algebraically closed, see the proof of Lemma VI.7 of [**Ber01**]. □

Let γ_I denote a generator of $\Gamma_{\widehat{K}^{\mathrm{nr}}}$.

Lemma A.4. — *The natural map* $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1) \to (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}$ *is an isomorphism if* $V^{I_K}=0$.

Proof. — This map is part of the six term exact sequence that comes from the map γ_I-1 applied to $0\to \mathrm{D}(V|_{I_K})^{\psi=1}\to \mathrm{D}(V|_{I_K}) \xrightarrow{\psi-1} \mathrm{D}(V|_{I_K})\to 0$. Its kernel is included in $\mathrm{D}(V|_{I_K})^{\gamma_I=1}$ which is 0, since $V^{I_K}=0$ (note that the inclusion $(\widehat{K}^{\mathrm{nr}}\otimes V)^{G_K}\subseteq (\widehat{\mathcal{E}}^{\mathrm{nr}}\otimes V)^{G_K}=\mathrm{D}(V)^{G_K}$ is an isomorphism). □

Suppose that $x\in \mathrm{D}(V)/(\psi-1,\gamma-1)$. If $\tilde{x}\in\mathrm{D}(V)$ lifts x, then Lemma A.3 gives us an element $y\in\mathrm{D}(V|_{I_K})$ such that $(\psi-1)y=\tilde{x}$. Define a cocycle $\delta(x)\in Z^1(G_K/I_K, \mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1))$ by $\delta(x):\overline{g}\mapsto (g-1)(y)$ if $g\in G_K$ lifts $\overline{g}\in G_K/I_K$.

Proposition A.5. — *If* $V^{I_K}=0$, *then the map*

$$\delta : \mathrm{D}(V)/(\psi-1,\gamma-1)\to H^1(G_K/I_K, (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1})$$

is well-defined and injective.

Proof. — We first check that $\delta(x)(g)\in(\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}$. We have $(\psi-1)(g-1)(y)=(g-1)(x)$. If we write $g=ih\in I_K H_K$, then $(g-1)x=(ih-1)x=(i-1)x\in(\gamma_I-1)\mathrm{D}(V|_{I_K})$ since γ_I-1 divides the image of $i-1$ in $\mathbf{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. This implies that $\delta(x)(g)\in(\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}$.

We now check that $\delta(x)$ does not depend on the choices. If we choose another lift $g'\in G_K$ of $\overline{g}\in G_K/I_K$, then $g'=ig$ for some $i\in I_K$ and $(g'-1)y-(g-1)y=(i-1)gy\in(\gamma_I-1)\mathrm{D}(V|_{I_K})$ since γ_I-1 divides the image of $i-1$ in $\mathbf{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. If we choose another y' such that $(\psi-1)y'=\tilde{x}$, then $y-y'\in\mathrm{D}(V|_{I_K})^{\psi=1}$ so that δ and δ' are cohomologous. Finally, if $\tilde{x}'$ is another lift of x, then $\tilde{x}'-\tilde{x}=(\gamma-1)a+(\psi-1)b$ with $a,b\in\mathrm{D}(V)$. We can then take $y'=y+b+(\gamma_G-1)c$ where $(\psi-1)c=a$. We then have $(g-1)y'=(g-1)y+(g-1)b+(\gamma_G-1)(g-1)c$. Since $G_K=I_KH_K$, we can write $g=ih$ and $(g-1)b=(i-1)b$. Using $G_K=I_KH_K$ once again, we see that $I_K\to G_K/H_K$ is surjective, so that we can identify γ_I and γ_G. The resulting cocycle is then cohomologous to $\delta(x)$. This proves that δ is well-defined.

We now prove that δ is injective. If $\delta(x)=0$, then using Lemma A.4 there exists $z\in \mathrm{D}(V|_{I_K})^{\psi=1}$ such that $\delta(x)(\overline{g})$ is the image of $(g-1)(z)$ in $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1)$. This implies that $(g-1)(y-z)\in(\gamma_I-1)\mathrm{D}(V|_{I_K})^{\psi=1}$. Applying $\psi-1$ gives $(g-1)\tilde{x}=0$ so that $\tilde{x}\in \mathrm{D}(V)^{G_K}\subset V^{I_K}=0$. The map δ is therefore injective. □

Lemma A.6. — *If V is semistable and the weights of V are all ≥ 2, then* $\exp_V : \mathrm{D}_{\mathrm{dR}}(V|_{I_K})\to H^1(I_K,V)$ *is an isomorphism.*

Proof. — Apply Thm. 6.8 of [**Ber02**] to $V|_{I_K}$. □

Proof of Theorem A.1. — We can replace K by K_n for $n\gg 0$ and use the fact that if $H^2(G_{K_n},V)=0$, then $H^2(G_K,V)=0$ since the restriction map is injective. In particular, we can assume that $H_K I_K=G_K$ and that Γ_K is isomorphic to $\mathbf{Z}_p$. By item (2) of Lemma A.2, we have $H^2(G_K,V)=\mathrm{D}(V)/(\psi-1,\gamma-1)$, and so by Proposition A.5 above, it is enough to prove that

$$H^1(G_K/I_K,(\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1})=0.$$

Lemma A.4 tells us that $(\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}=\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1)$. Since $\mathrm{D}(V|_{I_K})/(\psi-1)=0$ by Lemma A.3, item (1) of Lemma A.2 tells us that $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma-1)=H^1(I_K,V)$.

The map $\exp_V:\mathrm{D}_{\mathrm{dR}}(V|_{I_K})\to H^1(I_K,V)$ is an isomorphism by Lemma A.6, and this isomorphism commutes with the action of G_K since it is a natural map. We therefore have $H^1(I_K,V)=\widehat{K}^{\mathrm{nr}}\otimes_K \mathrm{D}_{\mathrm{dR}}(V)$ as G_K-modules. It remains to observe that the cocycle $\delta(x)\in Z^1(G_K/I_K,\widehat{K}^{\mathrm{nr}}\otimes_K\mathrm{D}_{\mathrm{dR}}(V))$ is continuous and that $H^1(G_K/I_K,\widehat{K}^{\mathrm{nr}})=0$ by taking a lattice, reducing modulo a uniformizer of K, and applying Hilbert 90. □

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