SUPER-HÖLDER FUNCTIONS AND VECTORS

by

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1. Motivation

Let $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})$ be the cyclotomic extension of $\mathbf{Q}_p$. The Galois group $\Gamma = \mathrm{Gal}(K_\infty/\mathbf{Q}_p)$ is isomorphic to $\mathbf{Z}_p^\times$ via the cyclotomic character. The action of Γ on K_∞ extends to a continuous action of Γ on $\widehat{K}_\infty$. How can we recover K_∞ from the p-adic Banach representation $\widehat{K}_\infty$ of Γ? The space K_∞ is the space of smooth vectors $\widehat{K}_\infty^{\mathrm{sm}} = \{x \in \widehat{K}_\infty$ such that $\mathrm{Stab}(x)$ is open in $\Gamma\}$. The space K_∞ is also (see [**BC16**]) the space of locally analytic vectors $\widehat{K}_\infty^{\mathrm{la}} = \{x \in \widehat{K}_\infty$ such that the orbit map $\gamma \mapsto \gamma(x)$ is a locally analytic function on $\Gamma\}$.

Let $\mathbf{E} = \mathbf{F}_p(\!(X)\!)$ and $\mathbf{E}_n = \mathbf{F}_p(\!(X^{1/p^n})\!)$ for $n \geq 0$ and $\mathbf{E}_\infty = \cup_{n\geq 0}\mathbf{E}_n$ and let $\widetilde{\mathbf{E}}$ be the X-adic completion of $\mathbf{E}_\infty$. The group $\Gamma = \mathbf{Z}_p^\times$ acts on $\mathbf{E}$ by $a \cdot f(X) = f((1+X)^a - 1)$, and this action extends to $\widetilde{\mathbf{E}}$. The motivation for our work was the following analogue of the above question: how can we recover $\mathbf{E}_\infty$ from the valued $\mathbf{F}_p$-representation $\widetilde{\mathbf{E}}$ of Γ? One can prove that $\widetilde{\mathbf{E}}^{\mathrm{sm}} = \mathbf{F}_p$, so smooth vectors are not enough. In order to answer the question, we define super-Hölder functions, that seem to be a characteristic p analogue of locally analytic functions.

2. Super-Hölder functions

Let G be a uniform pro-p-group of rank d and let $G_i = G^{p^i}$ for $i \geq 0$ (for example, one could take $G = \mathbf{Z}_p^d$, so that $G_i = p^i\mathbf{Z}_p^d$). Let M be an $\mathbf{F}_p$-vector space, equipped with a valuation val_M for which it is separated and complete. We say that a function $f : G \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbf{R}$ and $e > 0$ such that $\mathrm{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$.

We let $\mathcal{H}_e^{\lambda,\mu}(G,M)$ denote the corresponding space of functions. For example, the map $\mathbf{Z}_p \to \mathbf{F}_p[\![X]\!]$ given by $a \mapsto (1+X)^a$ belongs to $\mathcal{H}_1^{0,0}(\mathbf{Z}_p, \mathbf{F}_p[\![X]\!])$.

These super-Hölder functions seem to be the analogue in characteristic p of locally analytic functions. As further evidence, take $G = \mathbf{Z}_p$ and let M be as above. If $\{m_n\}_{n\geq 0}$ is a sequence of M with $m_n \to 0$, the map $z \mapsto \sum_{n\geq 0} \binom{z}{n} m_n$ defines a continuous function $\mathbf{Z}_p \to M$. Conversely, every continuous function $\mathbf{Z}_p \to M$ can be written in this way in one and only one way. Such a function is then in $\mathcal{H}_e^{\lambda,\mu}(\mathbf{Z}_p, M)$ if and only if $\mathrm{val}_M(m_n) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $n \geq p^i$, for all $i \geq 0$. This criteria (see §1.3 of [**BR22**]) is the analogue of a criteria of Amice characterizing locally analytic functions in terms of their Mahler expansion.

3. Super-Hölder vectors

We now assume that M is endowed with an $\mathbf{F}_p$-linear action of G by isometries. We say that $m \in M$ is a super-Hölder vector if the orbit map $g \mapsto g(m)$ is a super-Hölder function $G \to M$. We denote by $M^{G\text{-}e\text{-sh},\lambda,\mu}$ the elements for which the orbit map is in $\mathcal{H}_e^{\lambda,\mu}(G,M)$. Let $M^{G\text{-}e\text{-sh},\lambda} = \cup_\mu M^{G\text{-}e\text{-sh},\lambda,\mu}$ and $M^{G\text{-}e\text{-sh}} = \cup_\lambda M^{G\text{-}e\text{-sh},\lambda}$. If H is an open uniform subgroup of G, note that $M^{G\text{-}e\text{-sh}} = M^{H\text{-}e\text{-sh}}$.

We can now answer the above question. Let $M = \widetilde{\mathbf{E}}$, with $\mathrm{val}_M = \mathrm{val}_X$, and let $G = 1 + p^k\mathbf{Z}_p$ with $k \geq 1$ (or $k \geq 2$ if $p = 2$). Theorem 2.9 of [**BR22**] now says that $\widetilde{\mathbf{E}}^{1+p^k\mathbf{Z}_p\text{-1-sh}} = \mathbf{E}_\infty$. More precisely, $\widetilde{\mathbf{E}}^{1+p^k\mathbf{Z}_p\text{-1-sh},k-n} = \mathbf{E}_n$ for $n \geq 0$. The proof of this result in [**BR22**] uses Colmez' analogue in $\widetilde{\mathbf{E}}$ of Tate's normalized trace maps. In [**BR23**], we prove a more general result that implies the above one: see §5 of this report.

4. (φ,Γ)-modules

Let $\Gamma = \mathbf{Z}_p^\times$. In this report, a (φ,Γ)-module is a finite dimensional $\mathbf{F}_p(\!(X)\!)$-vector space $\mathbf{D}$, endowed with a semilinear injective Frobenius map $\varphi : \mathbf{D} \to \mathbf{D}$ (acting by $f(X) \mapsto f(X^p)$ on $\mathbf{F}_p(\!(X)\!)$), and a compatible action of Γ. These objects correspond, via Fontaine's equivalence (see [**Fon90**]), to $\mathbf{F}_p$-linear representations of $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{alg}}/\mathbf{Q}_p)$. Such an object has a Γ-stable lattice, which allows us to define an X-adic valuation on $\mathbf{D}$. Proposition 3.9 of [**BR22**] says that $\mathbf{D} = \mathbf{D}^{1+p^k\mathbf{Z}_p\text{-1-sh},k}$.

Let ψ be the usual map on $\mathbf{D}$, defined by $\psi(y) = y_0$ if one writes $y \in \mathbf{D}$ as $y = \sum_{i=0}^{p-1}(1+X)^i\varphi(y_i)$ with $y_i \in \mathbf{D}$. Following Colmez (see [**Col10**]), let $\mathbf{D}^+$ be the set of $x \in \mathbf{D}$ such that $\{\varphi^i(x)\}_{i\geq 0}$ is bounded, and let $\mathbf{D}^\sharp$ be the largest sub $\mathbf{F}_p[\![X]\!]$-module of finite rank of $\mathbf{D}$ that is stable under ψ and on which ψ is surjective. For example, if $\mathbf{D} = \mathbf{F}_p(\!(X)\!)$, then $\mathbf{F}_p(\!(X)\!)^+ = \mathbf{F}_p[\![X]\!]$ and $\mathbf{F}_p(\!(X)\!)^\sharp = X^{-1} \cdot \mathbf{F}_p[\![X]\!]$. Let $M = \varprojlim_\psi \mathbf{D}^\sharp = \{(y_0, y_1, \ldots)$ where $y_i \in \mathbf{D}^\sharp$ and $\psi(y_{i+1}) = y_i$ for all $i \geq 0\}$.

The space M is an $\mathbf{F}_p[\![X]\!]$-module; we can define an X-adic valuation on it. The group Γ acts on M by isometries (note: the X-adic topology on M is not the natural topology of M, and the action of Γ on M is not continuous for the X-adic topology). There is a map $i : \mathbf{D}^+ \to M$ given by $y \mapsto (y, \varphi(y), \varphi^2(y), \ldots)$. We then have $M^{1+p^k\mathbf{Z}_p\text{-1-sh},k} = i(\mathbf{D}^+)$. When $\mathbf{D} = \mathbf{F}_p(\!(X)\!)$, this result is proved in §3.4 of [**BR22**]. The $\mathbf{D} \mapsto \varprojlim_\psi \mathbf{D}^\sharp$ construction is an important part of the construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, and the previous result shows that we can "invert" this construction using super-Hölder vectors.

5. The field of norms

We now explain how super-Hölder vectors allow us to recover the field of norms of certain extensions by decompleting their tilt. This material is in [**BR23**]. Let K be a finite extension of $\mathbf{Q}_p$, and let K_∞ be an almost totally ramified Galois extension of K, whose Galois group Γ is a p-adic Lie group of dimension ≥ 1. Such an extension is then deeply ramified (equivalently, $\widehat{K}_\infty$ is perfectoid) and also strictly arithmetically profinite (see [**Win83**]). One can then attach two objects to K_∞/K. The first object is the field $\widetilde{\mathbf{E}}_{K_\infty}$, the fraction field of $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$ (now called the tilt of $\widehat{K}_\infty$). This is a perfect valued field of characteristic p, on which Γ acts by isometries.

The second object is the field of norms. Let $\mathcal{E} = \{E/K$ such that E/K is finite and $E \subset K_\infty\}$. Let $X_K(K_\infty) = \varprojlim_{\mathrm{N}_{F/E}} E = \{(x_E)_{E \in \mathcal{E}}$ with $x_E \in E$ and such that $\mathrm{N}_{F/E}(x_F) = x_E$ whenever $E \subset F\}$. The set $X_K(K_\infty)$ can be given (see [**Win83**]) a natural structure of a valued field of characteristic p, on which Γ acts by isometries. It is then isomorphic to $k_{K_\infty}(\!(\pi)\!)$ where k_{K_∞} is the residue field of K_∞ and π is a norm compatible sequence of uniformizers. Furthermore (see ibid), there is a natural map $X_K(K_\infty) \to \widetilde{\mathbf{E}}_{K_\infty}$, and $\widetilde{\mathbf{E}}_{K_\infty}$ is the completion of the perfection $\cup_{n \geq 0} X_K(K_\infty)^{1/p^n}$ of $X_K(K_\infty)$.

Theorem A of [**BR23**] says that $\cup_{n \geq 0} X_K(K_\infty)^{1/p^n} = \widetilde{\mathbf{E}}_{K_\infty}^{\Gamma\text{-}d\text{-sh}}$. In the "cyclotomic" case, with $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})$, we have $d = 1$ and $X_K(K_\infty) = \mathbf{F}_p(\!(X)\!)$ and $\widetilde{\mathbf{E}}_{K_\infty} = \widetilde{\mathbf{E}}$ and the action of Γ on $\widetilde{\mathbf{E}}$ is the one coming from $a \cdot f(X) = f((1+X)^a - 1)$. Hence the result above implies the answer to the question formulated at the beginning.

6. Examples

Here are two examples of super-Hölder functions with interesting properties.

6.1. A locally analytic function that has a nonisolated zero is locally constant at this point. Here is a function $f : \mathbf{Z}_p \to \mathbf{F}_p[\![X]\!]$ that is super-Hölder and has a nonisolated zero but is nowhere locally constant.

Set $f(0) = 0$ and if $a \in \mathbf{Z}_p^\times$ and $i \geq 0$, let $f(p^i a) = ((1+X)^a - (1+X))^{p^i}$.

6.2. If $\alpha \in \mathbf{Z}_{\geq 1}$, then $\sum_{n \geq 0} X^{p^{n\alpha}+p^{-n}} \in \mathbf{F}_p[\![X]\!]$ is a super-Hölder vector for the action of $1+2p\mathbf{Z}_p$ on $\mathbf{F}_p[\![X]\!]$ with $e = \alpha/(1+\alpha)$, but not for $e > \alpha/(1+\alpha)$.

References

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