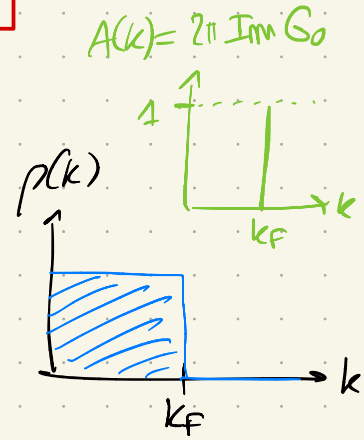
# Journal club: Luttinger's theorem

## I. Brief reminder: Fermi liquids




- Free fermions: $G_0(p, i\omega) = \dfrac{1}{i\omega - \epsilon_p + \mu}$

$$N_f = 2 \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G_0(p, i\omega) e^{i\omega \cdot 0^+}$$

$$= 2 \sum_p \Theta(-\epsilon_p + \mu) = \frac{2L^d}{(2\pi)^d} \times \mathrm{Vol}(FS) .$$

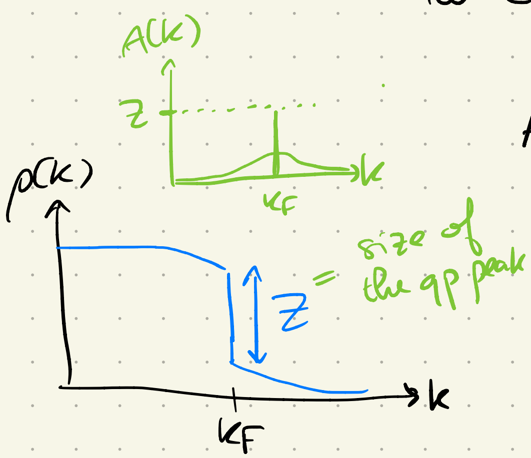
without interactions

Not a surprise (Pauli exclusion principle).

easy: 


- Interacting fermions: all the usual stuff with Dyson's equation, etc, and $G(p, i\omega) = \dfrac{1}{i\omega - \epsilon_p^0 + \mu - \Sigma(p, i\omega)}$

self-energy → Re: energy shift → Im: decay rate.




Assume there is still a FS, i.e.:

long-lived qps.

- $\exists\, E_F$ such that $\mathrm{Im}\,\Sigma(\vec{k}, E_F - \mu) = 0 \quad \underline{\underline{\forall \vec{k}}}$
- $\exists\, k_F$ (or more generally a surface) such that $G^{-1}(k_F, E_F - \mu) = 0$ and changes sign between inside/outside the surface.

In particular, $E_F = \varepsilon^0_{k_F} + \mathrm{Re}\,\Sigma(k_F, E_F - \mu)$ defines the surface.

Now, question: what is the size of this FS? I.e: is the $k_F$ with interactions the same as without interactions?

Answer (Luttinger's thm): it remains the same.

↳ Vol(FS without interactions) = Vol(FS with interactions).

↳ This looks quite boring.

↳ But actually very interesting sometimes (e.g. HFL).

## II. Main idea of the "historical" demonstration

$$G^{-1}(p, i\omega) = i\omega - \epsilon^0_p + \mu - \Sigma(p, i\omega)$$

$$i\frac{d}{d\omega}\ln G(p, i\omega) = G + iG\frac{d}{d\omega}\Sigma(p, i\omega)$$

$-iG\frac{d}{d\omega}(\text{this})$

**Quasiparticles' population:**

NB: by definition, $N_{qp} = N_f$.

$$N_{qp} = 2\sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G(p, i\omega) e^{i\omega \cdot 0^+}$$

$$= 2\sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega\cdot 0^+}\Big\{ \underbrace{i\frac{d}{d\omega}\ln G(p, i\omega)}_{\text{"easy"}^* \;=\; \Theta(-\epsilon^0_p + \mu - \mathrm{Re}\,\Sigma(p, \epsilon_F))} - \underbrace{iG\frac{d}{d\omega}\Sigma(p, i\omega)}_{\text{goal: show that } \int \text{this} = 0.}\Big\}$$

→ Luttinger's thm will be proven, provided that the second term cancels.
since we will have $N_{qp} = \frac{2L^d}{(2\pi)^d}\,\mathrm{Vol}(\text{FS with interactions})$.

**Key idea:** there exists a functional $\Phi_{LW}[G]$ ← Luttinger-Ward such that:

(i) $\Sigma(p, i\omega) = \dfrac{\delta \Phi_{LW}[G]}{\delta G(p, i\omega)}$.

and (ii) $\Phi[G(p, i\omega + i\Omega)] = \Phi[G(p, i\omega)]$ : "Ward condition"

Then, $$\int \frac{d\omega}{2\pi} G(p, i\omega)\frac{\partial}{\partial\omega}\frac{\delta\Phi_{LW}}{\delta G(p, i\omega)} \overset{\text{ibp}}{=} -\int\frac{d\omega}{2\pi}\underbrace{\frac{\partial G(p, i\omega)}{\partial\omega}\frac{\delta\Phi_{LW}}{\delta G(p, i\omega)}}_{=\frac{\partial}{\partial\Omega}\Phi_{LW}[G(p, i\Omega)]\big|_{\Omega=\omega}} = 0$$

**Questions that remain:**

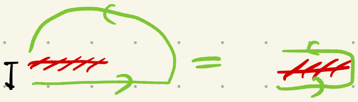
→ Does this $\Phi_{LW}$ indeed exist?

→ What does the "Ward condition" mean physically?

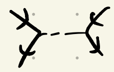
* Only depends upon the analytic properties of ln, actually.

$\xi\uparrow$  $= 0$ → hence $-2i\pi\Theta(-\xi)$

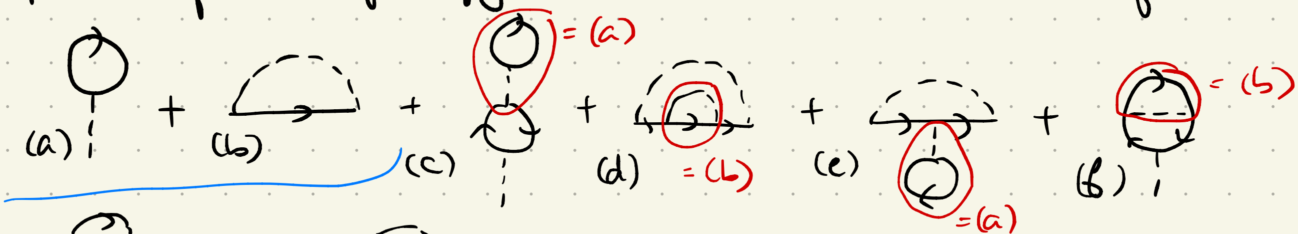
↑

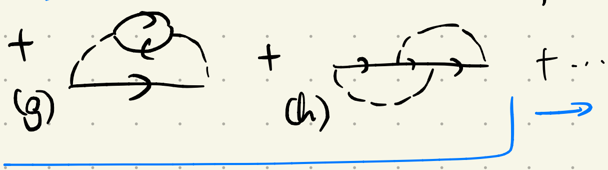
$\xi\downarrow$  $= -2i\pi$

# III – Construction of $\Phi_{LW}[G]$

**Recall**: expansion of the self-energy: consider  interactions for instance.

$\Sigma(p,i\omega) =$

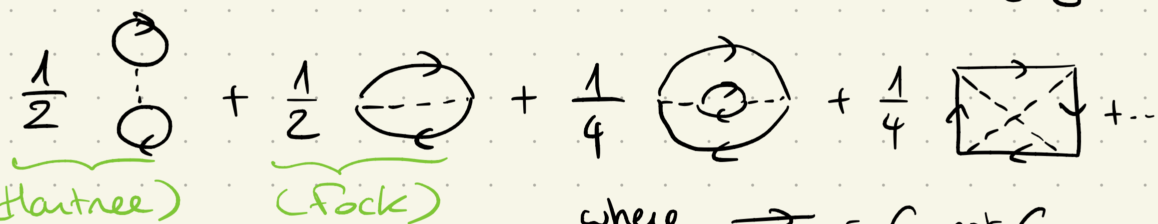



→ "skeleton diagrams"

$\Sigma(p,i\omega) = $ sum(skeleton diagrams where $\xrightarrow[G_0]{}$ is replaced by $\xrightarrow[G]{}$).

**Define**: $\Phi[G] =$



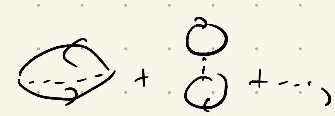

where $\rightarrow = G$, not $G_0$.

(NB there is a close relationship between $\Phi[G]$ and the energy functional of the electron liquid – see e.g. Giuliani, Vignale).

And one can check that, by construction, $\Sigma(p,i\omega) = \dfrac{\delta\Phi[G]}{\delta G(p,i\omega)}$.

As for the Ward condition:

Physically, actually quite obvious, $i\omega \to i\omega + i\Omega$ just shifts the origin of energies, so $\Phi[G]$, which is very much like the total energy of the liquid, is conserved up to a cst.

Mathematically, even more obvious: in $\Phi[G] =$ ,
all the fermion loops are closed, which means that one integrates over $\int_{-\infty}^{+\infty} d\omega$, so obviously $\Phi[G]$ is invariant under a shift of $\omega$.

# IV - About $U(1)$ symmetry and the Ward condition

→ Starting point: $L = \sum_p c^\dagger_{p\sigma}\left(\frac{\partial}{\partial\tau} + \varepsilon^0_p - \mu\right)c_{p\sigma}$ + Interactions.

One "implicit" assumption of the FL theory is the conservation of particle number by the interactions (cf adiabatic concept).

I.e. global $U(1)$ symmetry: $\begin{cases} c_{p\sigma} \mapsto c_{p\sigma}e^{i\theta} \\ c^\dagger_{p\sigma} \mapsto c^\dagger_{p\sigma}e^{-i\theta}\end{cases}$ leaves $L$ invariant (including the Interactions).

→ Consequence: time-dependent $U(1)$ sym $\begin{cases} c_{p\sigma}(\tau) \mapsto c_{p\sigma}(\tau)e^{i\Omega\tau} \\ c^\dagger_{p\sigma}(\tau) \mapsto c^\dagger_{p\sigma}(\tau)e^{-i\Omega\tau}\end{cases}$
leaves $L$ invariant up to a shift of $\mu$ "gauge transfo."

Thus $G(p,i\omega) = \int e^{i\omega\tau}\langle T c_p(\tau)c^\dagger_p(0)\rangle$

with the new $\int d\tau\, L$ and new measure (obv. invar.)

$$\mapsto \int e^{i\omega\tau}\langle T c_p(\tau)e^{i\Omega\tau}c^\dagger_p(0)\rangle = G(p, i\omega + i\Omega).$$

→ There is a set of correlation functions which are conserved by this transformation (basically all the "gauge-invar. quantities indep. of $\mu$")

The conservation laws are the Ward identities.

There is one such law for $\Phi[G]$.

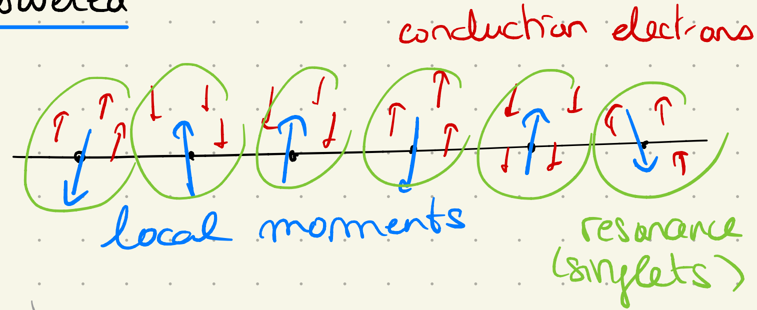
$\Longrightarrow$ $\Phi[G(p,i\omega)] = \Phi[G(p, i\omega + i\Omega)]$ is thus understood as a Ward identity for the global $U(1)$ symmetry of the FL Lagrangian.

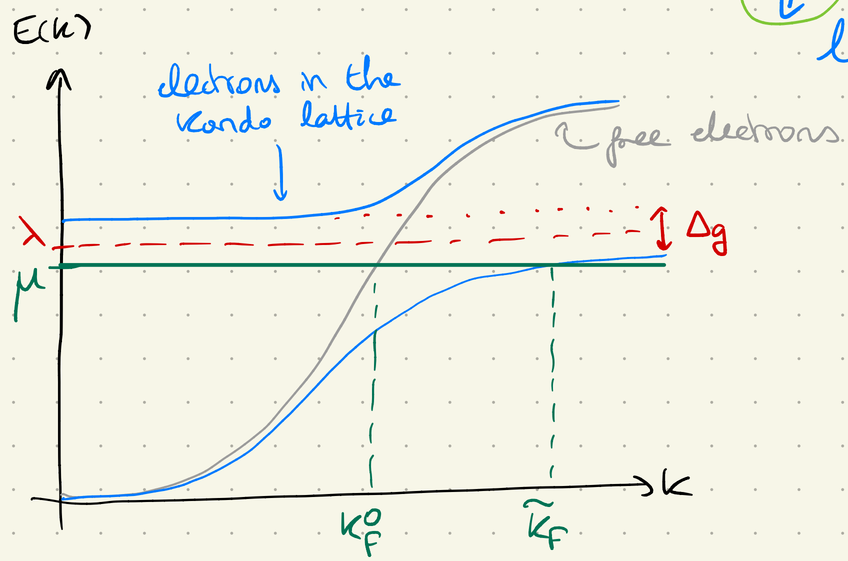
Luttinger's theorem

| Number conservation | $\longleftrightarrow$ | Volume conservation |
|---|---|---|
| ↑ | Ward identity | ↑ |
| Sym. of the Lagrangian | | Sym. of the correlations. |

# V - Questions that remained unanswered

Physics of the Kondo lattice:




$\lambda \sim$ chemical potential for the local moments

$\Delta g =$ gap opened by the resonance

$\mu =$ chemical potential of the conduction electrons.

- $k_F^0$ : "small" FS of the free electrons; $m_* \propto \left|\frac{d^2 E_F}{dk^2}\right|^{-1}$ light
- $\tilde{k}_F$ : "large" FS of the renormalized fermions. $m_*$ large : heavy fermions.

But Luttinger's thm tells us that the FS volume should be conserved!
→ is this a breakdown of Luttinger's thm?

The Kondo effect is known to be nonperturbative (e.g. $T_K = D e^{-\frac{1}{J\rho}}$)

Kondo temperature; conduction bandwidth; coupling; $e^-$ density

→ is Luttinger's thm true perturbatively but fails nonperturbatively?

→ or is there a nonperturbative proof of the theorem?

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Note about LSM: "equivalent" to Luttinger in 1D (see other paper by M.O.)

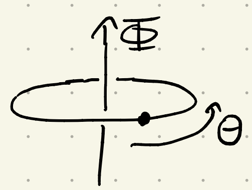
cf Jordan-Wigner and the XXZ chain.

Mapping to fermions (interacting as soon as $J_z \neq 0$) with filling

$\nu = m_z + \frac{1}{2}$ shows the equivalence.

(And LSM actually generalize this to $S \in \mathbb{N} + \frac{1}{2}$, with $\nu = m_z + S$).

# VI. A (very) brief reminder: Aharonov-Bohm effect

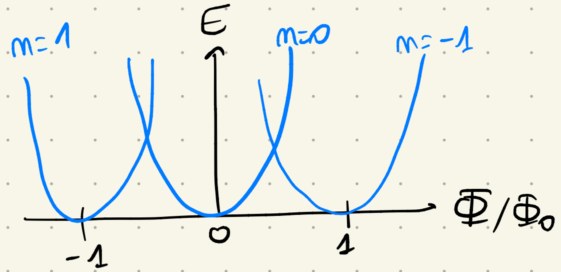



Particle constrained to move along a ring.

$$H = \frac{1}{2m}(p+eA)^2 = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial\theta} + \frac{e\Phi}{2\pi}\right)^2 .$$

$\left(\Phi_0 = \frac{h}{e}\right)$

Eigenfunctions: $\psi_m = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, $m \in \mathbb{Z}$ ; eigenvalues: $E_m = \frac{1}{2m}\left(m + \frac{\Phi}{\Phi_0}\right)^2$




Spectral flow: take $\Phi \to \Phi_0$.

This takes $H \mapsto H' = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial\theta} + e\frac{\Phi}{2\pi} + 1\right)$

and $\psi_m \mapsto \psi_m' = \psi_m e^{-i\int_0^\theta \vec{A}\cdot d\vec{r}} = e^{-i\frac{e}{\hbar}\frac{\Phi_0}{2\pi}\theta}\psi_m$

$= e^{-i\theta}\psi_m = \psi_{m-1}$

such that $H'\psi_m' = E_m \psi_m'$. *

# VII. The proof by Oshikawa

- Main idea: recycle an argument about the LSM theorem in dimension $D$.

Take a system of size $L_1 \times \cdots \times L_D$ with periodic b.c. ("$D$-dim tore").

Then insert a flux along axis $x_1$, in the gauge $A_1 = \Phi/L_1$, adiabatically from $\Phi = 0$ to $\Phi = \Phi_0$.

$H(0)|\psi_0\rangle = E_0|\psi_0\rangle$

This evolves $H(\Phi=0) \mapsto H(\Phi_0)$, and the g.s. $|\psi_0\rangle \mapsto |\psi_0'\rangle$

and the reminder hereabove tells us that $H(\Phi_0)|\psi_0'\rangle = E_0|\psi_0'\rangle$. *

Now imagine there is a unitary $\hat{U}$ such that $U H(\Phi_0) U^{-1} = H(0)$.

Then, $H(0)\hat{U}|\psi_0'\rangle = \hat{U} H(\Phi_0)|\psi_0'\rangle = E_0 \hat{U}|\psi_0'\rangle$.

↳ in VI, $U(\theta) = e^{i\theta}$

So we've shown that in the gauge choice where $H$ is fixed, insertion of a flux $\Phi_0$ sends $\boxed{|\psi_0\rangle \to U|\psi_0'\rangle}$.

⤳ in the case of VI, $U\psi_m' = \psi_m$ quite boring*

* Indeed, LT for (VI) states "$2\pi \times 1 \in 2\pi\mathbb{Z}$" not wow.
[1 particle on the ring]

- <u>How does one build $\hat{U}$?</u>

Action of the flux insertion upon particles: $\begin{cases} c_{r,\sigma} \mapsto e^{i2\pi x_1/L_1} c_{r,\sigma} \\ c^\dagger_{r,\sigma} \mapsto e^{-i2\pi x_1/L_1} c^\dagger_{r,\sigma} \end{cases}$.

$\Rightarrow$ Choose $\hat{U} = \exp\left(i\frac{2\pi}{L_1}\sum_{\vec{r},\sigma} x_1 m_{\vec{r},\sigma}\right)$.

One can check it takes $U c_{r\sigma} U^{-1} \mapsto c_{r,\sigma}$ and $U c^\dagger_{r,\sigma} U^{-1} \mapsto c^\dagger_{r,\sigma}$ independently.

As long as $H(0)$ is built out of $c^\dagger_s$ and $c_s$, it ensures $U H(\Phi_0) U^{-1} = H(0)$.

- <u>Momentum transformation</u>: one had $\underbrace{\hat{P}_x}_{\text{generator of transl.}; \ T_x = e^{iP_x}}|\Psi_0\rangle = P_x^0|\Psi_0\rangle$ $\quad (x \equiv x_1)$

(assumption of the FL: translation sym. is not broken).

Since $[H(\Phi), P_x] = 0 \ \forall \Phi$, $\hat{P}_x|\Psi_0'\rangle = P_x^0|\Psi_0'\rangle$ as well. $\rightsquigarrow$ ⚠ Very different* from VI: that's because $P_x^0 + 2\pi \equiv P_x^0$. *(at first sight)

But one can check $\hat{U}^{-1} e^{i\hat{P}_x}\hat{U} = e^{i\hat{P}_x}\exp\left(i\frac{2\pi}{L_1}\sum_{\vec{r},\sigma} m_{\vec{r},\sigma}\right)$

Thus $e^{iP_x} U|\Psi_0'\rangle = \hat{U} e^{i\hat{P}_x}|\Psi_0'\rangle\, e^{i\frac{2\pi}{L_1}\sum m_{r\sigma}} = \exp\left(iP_x^0 + i\frac{2\pi}{L_1}\sum_{r\sigma} m_{r\sigma}\right) U|\Psi_0'\rangle$

$\Rightarrow \hat{P}_x(U|\Psi_0'\rangle) = P_x^0 + 2\pi \underbrace{\nu}_{\text{particle number per unit cell}} L_2 L_3 \cdots L_D$

- <u>FL hypothesis</u>: there are $N_F^{(L)}$ quasiparticles occupying $V_F^{(L)} = \frac{(2\pi)^D N_F^{(L)}}{L_1 \cdots L_D}$

NB this is very different from the "historical proof": we do not use the adiabatic concept leading to $\rho_{qp} = \rho_e$, but instead we define a qp as the "unit of matter" occupying a unit volume of the FS.

Put more clearly: $\begin{cases} \text{Luttinger: we know } N_F^{(L)} = N_F^0\text{; now show } \frac{V_F^{(L)}}{N_F^{(L)}} = \frac{V_F^0}{N_F^0}. \\ \text{Oshikawa: we know } \frac{V_F^{(L)}}{N_F^{(L)}} = \frac{V_L^0}{N_F^0}\text{, now show } N_F^{(L)} = N_F^0. \end{cases}$

And minimal coupling tells us that $k_1 \to k_1 + eA_1 = k_1 + 2\pi/L_1$ $\rightarrow$ now this is similar to VI.

Thus $\Delta P_x = N_F^{(L)}\cdot\frac{2\pi}{L_1}$. And recall this is also $2\pi\nu L_2 \cdots L_D$.

- <u>Identification</u>: $\frac{2\pi}{L_1}(\overbrace{N_F^{(L)}}^{= N_F^0} - \nu L_1 \cdots L_D) = 2\pi \times \text{integer}$ (since momentum is defined mod $2\pi$.)

Provided the $L_i$ are incommensurate, $N_F^{(L)} - \nu L_1 \cdots L_D = L_1 L_2 \cdots L_D \times \text{integer}$

$$\Rightarrow \boxed{\frac{V_F^{(L)}}{(2\pi)^D} - \nu = m \in \mathbb{Z}}$$

Luttinger's thm

$m$ = number of filled bands.

## VIII. The Kondo lattice: Oshikawa's bold move

$= c^{+}_{j\alpha}\vec{\sigma}_{\alpha\beta}c_{j\beta}$

local qubit

Now take $H_{KL} = -\sum_{ij} t_{ij} c^{+}_{i\sigma} c_{j\sigma} + U \sum_j n_{j\uparrow} n_{j\downarrow} + \sum_j J^K_j \vec{s}_j \cdot \vec{S}_j$

and a fictitious, spin-dependent, flux insertion $c^{(+)}_{j\uparrow} \to c^{(+)}_{j\uparrow} \exp[(-)i 2\pi x_1/L_1]$.

And Oshikawa writes "we must now use $U_\uparrow = \exp\left(i \frac{2\pi}{L_1} \sum_r x_1 (n_{r,\uparrow} + S^z_r)\right)$".

(The result being $\frac{2V_F}{(2\pi)^D} - \nu = 2 N_s S$ consistent with the HFL theory).

But this means that implicitly, he consider that the flux insertion must also take $S^{\pm}_r \to S^{\pm}_r e^{\mp i 2\pi x_1/L_1}$; i.e. that somehow, the local qubits $\vec{S}_j$ are charged under the same $U(1)$ as the electrons. Why?

## IX. Why should spins become fermions?

How come that localized spins can "put more fermions into the FS"?

### (1) Back to Anderson's model.

Conduction electrons $c^{+}_{k\sigma}$ coupled to a localized orbital $d_\sigma$.

$$\hat{H}_{c\text{-}d} = \sum_{k\sigma}\left(\epsilon_k c^{+}_{k\sigma} c_{k\sigma} + (V_k d^{+}_\sigma c_{k\sigma} + h.c.)\right) + \sum_\sigma \epsilon_\sigma n_{d\sigma} + U n_{d\uparrow} n_{d\downarrow}$$

and recall that 2nd-order perturbation theory: $\hat{H}_{c\text{-}d} \mapsto \hat{H}_K$

where $\hat{H}_K = \sum_{k\sigma} \epsilon_{k\sigma} c^{+}_{k\sigma} c_{k\sigma} + 2 J_K \hat{\vec{S}}_d \cdot \hat{\vec{s}}_c(\vec{r}=\vec{0})$

$\vec{0}$ = position of $d$

$$\vec{S}_d = \sum_{\alpha\beta} d^{+}_\alpha \vec{\sigma}_{\alpha\beta} d_\beta$$

$\hat{S}^{+}_d \equiv \hat{d}^{+}, \quad \hat{S}^{-}_d \equiv \hat{d}$

→ The fact that $\hat{S}^{\pm}_d$ are charged like the conduction electrons

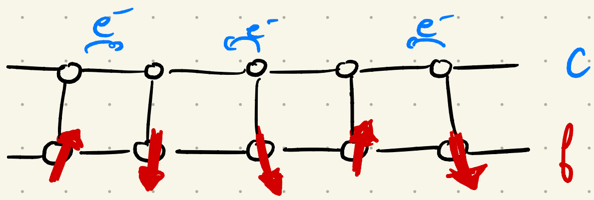
fishy is inherited from their being the spin of the d-electron.

## (2) Fractionalize the spin

There's something fishy in the previous argument: we are not supposed to know that our spins $\vec{S}_j$ are those of hidden d-electrons.
"Oshikawa's theorem" should hold for any qubit $\vec{S}_j$, even a nuclear spin!

Solution: assume that the qubit $\vec{S}_j$, regardless of its physical nature, is fractionalized into fermions, i.e. write $\vec{S}_j = f^\dagger_{j\alpha} \vec{\sigma}_{\alpha\beta} f_{j\beta}$ ← "fictitious" fermions
and the constraint $f^\dagger_{i\sigma} f_{i\sigma} = 1 \ \forall i \longrightarrow U(1)_{gauge}$.

Back to the Kondo lattice: $c^\dagger$ with density $p$, $f^\dagger$ with density $1$.




$$\boxed{\mathcal{H}_{KL} = \sum_p \epsilon_p c^\dagger_{p\sigma} c_{p\sigma} + J_k \sum_i c^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} c_{i\beta} \vec{S}_i}$$

→ Hybridization $c^\dagger_{i\alpha} c_{i\beta} f^\dagger_{i\beta} f_{i\alpha}$ with group $U(1) \times U(1)_{gauge}$.

→ Now our $f^\dagger$ fermions are either localized d-electrons, or some kind of localized fictitious fermions, and in both cases they are charged under a $U(1)_{gauge}$ ensuring there is one fermion per site.

→ Question: only the $c^\dagger$ fermions are mobile, and they have density $p$ → where does the large heavy FS come from?
↳ of density $1+p$.

# X – When does the FS actually become large?

$\mathcal{H}_{hyb} = (c^\dagger_{j\beta} f_{j\beta})(f^\dagger_{j\alpha} c_{j\alpha})$ with gauge group $U(1)_c \times U(1)_f$.

- First case: $V_{j\beta} = \langle c^\dagger_{j\beta} f_{j\beta} \rangle \neq 0$ spontaneous sym. breaking.

  $\to U(1)_c \times U(1)_f$ broken down to $U(1)$.

(Note $c^\dagger_{j\beta} f_{j\beta}$ corresponds to a singlet formation, $\downarrow\uparrow$.)

Both $c^\dagger$ and $f^\dagger$ are charged under the remaining $U(1)$.

The GS is a Slater determinant of $(c, f)$, with density $p+1$.

This is the Kondo physics i.e. what Oshikawa wanted to achieve.

- Second case: $V_{j\beta} = 0$ and $U(1)_c \times U(1)_f$ remains unbroken.

Then $\vec{S}_j$ is not charged like the conduction electrons, the GS is (Slater $(c)$)$\times$(Slater$(f)$) and the FS density is $p$.

It seems boring at first sight, maybe that's why Oshikawa did not mention this case.

NB actually there is some interesting physics there (FL* criticality, ...)

That's all folks!