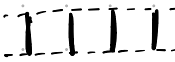
About the LSM theorem

A theorem about ground states of spin systems.

I. Context: what this theorem is about

Two basic questions about a ground state: is it gapped? is it unique?

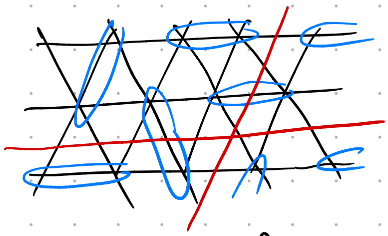
Let's see what happens with a few examples.

- Ising ferromagnet: gapped, twofold degenerate.
- Heisenberg FM: gapless, continuous $SO(3)$ degeneracy.
  (and the same with any continuous spontaneous sym-broken GS)
- The $S=\frac{1}{2}$ XY chain: we know it is mapped by Jordan-Wigner onto a theory of free fermions $\rightarrow$ unique, gapless GS.
- Actually the $S=\frac{1}{2}$ XXZ chain is also solvable (cf LSM paper), idem.
- The other extreme case: the Bethe chain: again, unique, gapless.
- The AKLT chain ($S=1$ ~Heisenberg): unique, gapped GS
  ↳ solvable "parent hamiltonian".
- Shastry-Sutherland (basically  ← AF rungs): gapped, unique.
- Majumdar-Ghosh (basically  ← AF): gapped, twofold degeneracy.

↳ (just like AKLT, parent hamiltonians and the GS is built from singlets).

- RVB state of the $S=\frac{1}{2}$ triangular lattice: gapped, topological (4-fold?) degeneracy.

Indeed:



you cannot rearrange the singlets locally so as to change the * number of ⬭ crossing either of the red lines

* parity of the

- Toric code: gapped, topological (4-fold) degeneracy.
- Kitaev's honeycomb model: gapless, unique. (then gapped phase ≡ toric code).
- Quantum spin ice: gapless, unique.

With all these examples, it appears that there is a unique, gapped GS only when the spin per unit cell is $S=1$ (or at least an integer).

This is consistent with Haldane's conjecture for spin chains: Heisenberg $\begin{cases} S \in \mathbb{N}+\frac{1}{2} : \text{gapless, unique} \\ S \in \mathbb{N} : \text{gapped, unique} \end{cases}$

(supported by RG arguments)

This is the idea of the LSM theorem.

II. Statement of the LSM theorem

1) Without an external field

The ground state of a system of spins with spin $S \notin \mathbb{N}$ per unit cell and S^z_{tot} conserved (i.e. $U(1)$ symmetry) cannot simultaneously (a) be unique and (b) have a finite gap to all excitations.

the technical reason for this will be given in the "proof" section; now, I don't really know of an intuitive justification.

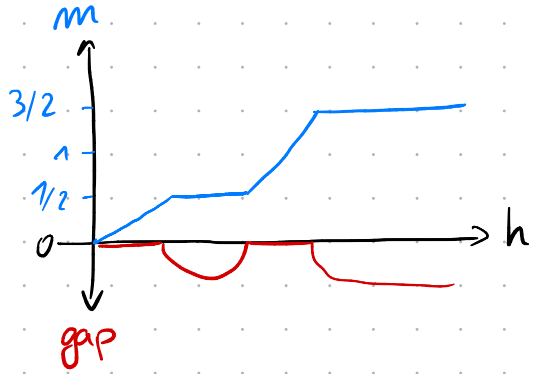
But considering this proof is identical to that of Luttinger's thm by Oshikawa, where the $U(1)$ symmetry is of course a key ingredient, we know this must be very important.
Maybe a better understanding when we discuss parton interpretations!

2) Generalization with a $\neq 0$ magnetic field

Replace "$S \notin \mathbb{N}$" by "$S-m \notin \mathbb{Z}$" with m the magnetization / unit cell.

Experimental consequence: magnetization plateaux.

(here $S = 3/2$ per unit cell)



Interpretation: having $\chi \neq 0$ means that there are excitations with arbitrarily low energy which can change the magnetization value.

So $\chi \neq 0 \Longrightarrow$ gapless.
$(\Leftarrow)$

Now let's see the proof of the thm (original version)

Oshikawa's version: well, see my note about Luttinger's thm :)

III. Proof of the theorem in $d=1$.

- **Idea**: the operation of $e^{i\frac{\theta}{2}\hat{S}^z}$ (i.e. rotation by θ around z) is, by assumption, a symmetry of the system.

So if we rotate all $\vec{S}_i$ by the same angle θ, we are still in the GS.
(we assume the GS is unique, then show it is gapless).

Now the idea is to build a "twisted" state, by rotating spins slowly, step by step; then check that this state has vanishingly small energy.

Define $\hat{U} = \exp\left(-\sum_{j=1}^{L} \frac{2\pi}{L} j \, \hat{S}_j^z\right)$; it has the effect defined above:

$$\uparrow\uparrow\uparrow\uparrow\cdots\uparrow\uparrow\uparrow\uparrow \xrightarrow{\hat{U}} \uparrow\nearrow\nearrow\nearrow\cdots\nwarrow\nwarrow\uparrow\uparrow$$

It builds a kink which is diluted over the whole system length.

- One can check that thanks to this "dilution", the energy cost vanishes.

In fact, one can show $\langle\psi| U^\dagger H U - H |\psi\rangle = O(1/L)$ for $|\psi\rangle = $ GS.

To do so, use Hadamard's lemma: $e^{A} H e^{-A} = H + [A,H] + \frac{1}{2!}[A,[A,H]] + \dots$

Since $|\psi\rangle$ is an eigenstate of H, the $[A,H]$ term does not contribute.

$$[A,H] = \left[\frac{2\pi}{L}\sum_j j S_j^z , H\right] \overset{U(1)\text{ sym}}{\propto} \frac{1}{L}\left[\sum_j j S_j^z , \sum_k S_k^+ S_{k+1}^- + hc\right]$$

$$= \frac{1}{L}\sum_k O(1)\, S_k^+ S_{k+1}^- + hc.$$

not rigorous: there could be nnn exchange as well.

(Indeed, S^+S^- may change $+j$ into $-j$ but simultaneously $-(j+1)$ into $+(j+1)$ so the global energy change is only $O(1)$.)

This argument uses the $U(1)$ sym and works as well for any local spin-spin interaction.

$$[A,[A,H]] \propto \frac{1}{L^2}\left[\sum_j j S_j^z , \sum_k S_k^+ S_{k+1}^- + hc\right] = \frac{1}{L^2}\sum_k O(1)\, S_k^+ S_{k+1}^- + hc$$

$$\Rightarrow \langle\psi| \hat{U}^\dagger H U - H |\psi\rangle = O(1/L^2)\, O(E_0).$$

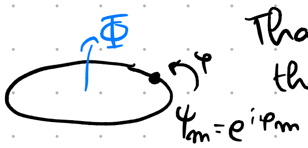
This is valid in any dimension; now, $E_0 = O(L^d)$, so that our "dilute kink" state has vanishing energy only in $d=1$.

We will see later how Oshikawa extends the argument to $d \geq 2$.

- Now we have to check that the state we have built is truly $\neq$ from the GS; we will check $U|\psi\rangle \perp |\psi\rangle$.

To do so, we will check that $U|\psi\rangle$ has a different eigenstate than $|\psi\rangle$ for some hermitian observable which commutes with H.

$\rightarrow$ choose $\hat{T}_x$ since the system has translational invariance along $\hat{x}$.



That's the same idea as, when proving Luttinger's thm, the topological pumping. $\hat{U} = e^{i\varphi}$: $\psi_m \mapsto \psi_{m+1}$ etc.

Here we have $\hat{U}^\dagger \hat{T}_x \hat{U} = e^{2i\pi\left(S_L^z - \frac{1}{L}\sum_j S_j^z\right)} \hat{T}_x \doteq e^{2i\pi(S-m)}$.

which does not yield the same eigenvalues as $\hat{T}_x$, provided $S-m \notin \mathbb{Z}$.

This concludes the proof of LSM's thm in $d=1$.

IV. A few remarks: how deep does it go?

1) Mapping spins $\leftrightarrow$ particle and $U(1)$ symmetry.

- Recall: Abrikosov fermions: two species $c_{i\uparrow}, c_{i\downarrow}$ such that

$$S_i^+ = c_{i\uparrow}^\dagger c_{i\downarrow}, \quad S_i^- = c_{i\downarrow}^\dagger c_{i\uparrow}, \quad 2S_i^z = c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow},$$

and a local constraint $c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} = 1 \ \forall i$

enforced by a $U(1)$ gauge theory.

- Here, we have a global symmetry, since S_{tot}^z is preserved. This means $\hat{H}$ contains only $S_i^+ S_j^-$ and S_i^z combinations, this means a global $U(1)$ symmetry $\begin{cases} c_{i\uparrow}^\dagger \to e^{i\Lambda} c_{i\uparrow}^\dagger \\ c_{i\downarrow}^\dagger \to e^{-i\Lambda} c_{i\downarrow}^\dagger \end{cases}$.

- Therefore (quite intuitive) we can absorb the gauge charge of, say, the $c_{i\downarrow}^\dagger$ fermions, so that only the $c_{i\uparrow}^\dagger$ fermions are charged under a $U(1)$ gauge field: basically "spins-up are charged" now.

- This explains how come the proof of LSM's thm is so similar to that of Luttinger's: it is really a thm about charged fermions!

2) Extension: commensurability.

Note that one can apply $\hat{U}$ twice, thus building a "2-kink" state. We thus build yet another "quasi-GS" provided that $\bar{S}m \notin \mathbb{Z}/q$.

→ We see clearly the link to Luttinger's thm, commensurability etc.

3) Extension to higher dimensions (L→∞ limit already taken)

→ More "handwaving" arguments without explicit energy evaluation.

Idea: our operator $\hat{U}$ is exactly the one which you would apply to cancel a flux insertion of 2π from the hamiltonian (but not the evolved states!).

i.e. $\begin{cases} |\Psi_0\rangle \\ H \end{cases} \xrightarrow[\text{insertion of } 2\pi]{\text{adiabatic flux}} \begin{cases} |\Psi_0'\rangle \\ H' \end{cases} \xrightarrow[\text{("gauge choice")}]{\text{apply } \hat{U}} \begin{cases} U|\Psi_0'\rangle \\ H \text{ again} \\ \overset{\shortparallel}{U^+H'U} \end{cases}$

- Argument 1: the flux insertion is adiabatic, so that if there is a gap, it sent $|\Psi_0\rangle$ (a GS of H) into $|\Psi_0'\rangle$ a GS of H'.

So, shifting gauge choice: $U|\Psi_0'\rangle$ must be a GS of $U^+H'U = H$.

- Rk: the LSM proof discussed the "energy" (H "eigenvalue") of $U|\Psi_0\rangle$.
↳ (it was not an eigenstate actually...)

Here we don't need that!

- Argument 2: $\hat{P}_x$ (or $\hat{T}_x$) commutes with the hamiltonian all along the flux insertion process, so $|\Psi_0\rangle$ and $|\Psi_0'\rangle$ have the same eigenvalue under $\hat{P}_x$ (or $\hat{T}_x$). So the argument we used with $U|\Psi_0\rangle$ in the LSM proof works identically with $U|\Psi_0'\rangle$ here.

→ With these "adiabatic arguments", the thm is "proved" in any dimension d! But the role of $U(1)$ sym is somewhat hidden.