Cours ENSL:
Big Data – Streaming, Sketching, Compression

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Introduction

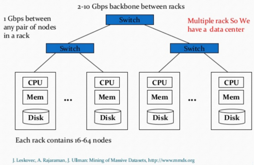
Positioning: Memory Aware Complexity of Algorithms

- w.r.t. traditional courses on algorithms
  - Exact algorithms for polynomial problems
  - Approximation algorithms for NP-Complete problems
  - Potentially exponential algorithms for difficult problems (going through an ILP for example)
- Here, we will consider extreme contexts
  - not enough space to transmit input data (sketching) or
  - not enough space to store the data stream (streaming)
  - not enough time to use an algorithm other than a linear complexity one
- Compared to the more "classical" context of algorithms:
  - we aim at solving simple problems and
  - we are looking for approximate solutions only because we have very strong time or space constraints.
- Disclaimer: it is not my research topic, but I like to look at the sketching/streaming papers and I am happy to teach it to you!

Application Context 1: Internet of Things (IoT)

- Connected objects, which take measurements
- The goal is to aggregate data.
- Processing can be done either locally, or on their way (fog computing), or in a data center (cloud computing).
- We must be very energy efficient
  - because objects are often embedded without power supply.
- Energy cost: Communication is the main source of energy consumption, followed by memory movements (from storage), followed by computations (which are inexpensive)
- A good solution is to do as many local computations as possible!
  - but it is known to be difficult (distributed algorithms)
  - especially when the complexity is not linear (e.g. think about quadratic complexity)
- Solution:
  - compress information locally (and on the fly)
  - only send the summaries; summaries must contain enough information!

Application Context 2: Datacenters



- Aggregate construction
- except the network (we can have several levels + infiniband), everything is "linear"
- the distance between certain nodes/data is very large but a strong proximity with certain data stored on disk
- with 1,000 nodes with 1TB of disk and a link at 400 MB/s, we have 1 PB and 400 GB/s (higher than with a HPC system)
- **provided the data is loaded locally !**
- for 25 TF/s ($10^3$25GFs seti@home) in total, ratio 60 (HPC system 40 000)
- in practice, dedicated to linear algorithms and very inefficient for other classes.
- In both contexts, there is a strong need to have data driven algorithms (where placement is imposed by data) whose complexity is linear

Outline of the lectures

- Keywords:
  - Compression, Hashing, Randomized Approximation Algorithms

1. Lecture 1: Two basic theoretical problems
   - Lecture 2: with known lower and upper + randomized and deterministic bounds
2. Lecture 3: Big Data example: Plagiarism detection
   - randomized algorithm + Locality Sensitive Hashing
3. Lecture 4: Randomized Linear Algebra
   - compression beyond Singular Value Decompositions for very large matrices

- Shared Problems
  - Not enough space to store input data
  - Not enough space/time to implement something else than low (linear) complexity algorithms
  - Need for very cheap (online) but dedicated compression algorithm

Sketching – Streaming

Sketching - Streaming – Context

- large volume of data generated in a distributed way
  - to be processed locally and compressed before transmission.
- Types of compression?
  - lossless compression
  - compression with losses
  - compression with losses, but tightly controlled loss for a specific function (sketching)
- + we are going to do online (on the fly) compression (streaming)

On-the-fly compression dedicated to a function f

- Let X be a stream of numbers (temperatures from a sensor)
- Easy problems?
  - examples: $\min, \max, \sum$, mean value median?
  - Constraint: compress data and linearize computations
- How?
  - The solution is often to switch to **randomized approximation algorithms**.

Compression associated to a specific function f

- More formally, given f and a stream X,
- we want to compress the data X but still be able to compute $\simeq f(X)$.
- Sketching: we are looking for C_f and g such that
    - the storage space $C_f(X)$ is small (compression)
    - from $f(X)$, we can recover $f(X)$, ie $g(C_f(X)) \simeq f(X)$
- Streaming: additional difficulty, the update is performed on the fly.
    - we cannot compute $C_f(\{X, y\})$ from $\{X, y\}$
    - because we cannot store $\{X, y\}$
    - so we need another function h such that . $h(C_f(X), \{y\}) = C_f(\{X, y\})$
- and one last difficulty:
- very often, it is impossible to do in deterministic and exact / deterministic and approximate
- but only with a randomized and approximation algorithm.
- How to write this ?
    - We are looking for an estimator Z such that for given α and ϵ
    - $Pr(|Z - f(X)| \geq \epsilon f(X)) \leq \alpha$. How to read this?
        - the probability of making a mistake by a ratio greater than ϵ (as small as you want)
        - is smaller than α (as small as you want)

Count the number of visits

Example: count the number of visits / packets

- Context
  - a sensor/router sees packets / visits passing through,....
  - you just want to maintain elementary statistics (number of visits, number of visits over the last 1 hour, standard deviations)
  - Here, we simply want to count the number of visits
- What storage is necessary if we have n visits? $\log n$ bits. Why ? Pigeonhole principle. If we have strictly less than $\log n$ bits, then we have two events (among the n) that will be coded in the same way.
- What happens if we only allow an approximate answer (say, to a factor of ρ <2)? you need at least $\log \log n$ bits. Why ? sketch of the proof: if we use $t < \log \log n$ bits, then we will be able to distinguish less than $\log n$ different groups and you can estimate how many groups are needed to count $\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, ..., 7\}$.
- We will look for a randomized and approximated solution
  - Let us set α and ϵ
  - we are looking for an algorithm that computes $\tilde{n}$, an approximation of n
  - that only uses $K \log \log n$ bits storage
  - and such that $Pr(|\tilde{n} - n| \geq \epsilon n) \leq \alpha$
  - K must be a constant...not necessarily a small constant for now!

Crash Course in probabilities

- Z random variable with positive values
- $E(Z)$ is the expectation of Z
- definitions and properties ?
  - $E(Z) = \int \lambda P(Z = \lambda) d\lambda$ or $E(Z) = \sum_j jP(Z = j)$
  - $E(Z) = \int P(Z \geq \lambda) d\lambda$ or $E(Z) = \sum_j P(Z \geq j)$
  - $E(aX + bY) = aE(X) + bE(Y)$
  - total probabilities (with conditioning) $E(Z) = \sum_j E(ZIY = j)P(Y = j)$
- To measure the distance from Z to $E(Z)$, we use the variance $V(Z)$
  - Definition?
  - $V(Z) = E((Z - E(Z))^2) = E(Z^2) - E(Z)^2$
  - Properties:
  - $V(aZ) = a^2 V(Z)$
  - In general, $V(X + Y) \neq V(X) + V(Y)$ (but it is true if X and Y are independent random variables)
- How to measure the difference between Z to $E(Z)$?
  1. Markov: $Pr(Z \geq \lambda) \leq E(Z)/\lambda$
  2. Chebyshev: $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq \frac{V(Z)}{\lambda^2 E(Z)^2}$
  3. Chernoff: If $Z_1, \ldots, Z_n$ are Independent Bernouilli rv with $p_i \in [0.1]$ and $Z = \sum Z_i$, then $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq 2\exp(\frac{-\lambda^2 E(Z)}{3})$.

Morris Algorithm: Counting the number of events

- Step 1: Find an estimator Z
  - Z must be small (of order of $\log \log n$)
  - we need to define an additional function g
  - such that $E(g(Z)) = n$
- Morris algorithm
  - $Z \to 0$
  - At each event, $Z \to Z + 1$ with probability $1/2^Z$
  - When queried, return $g(Z) = 2^Z - 1$
- What is the space complexity to implement Morris' algorithm?
- What is the time complexity in the worst case? What is the expected complexity of a step?
- Prove the correctness: $E(2^{Z_n} - 1) = n$ (note Z_n the random variable that denotes Z after n events) Hint: by induction, assuming that $E(2^{Z_n}) = n + 1$ and showing that $E(2^{Z_{n+1}}) = n + 2$
- How to find a probabilistic guarantee of the type $Pr(|f(Z_n) = \tilde{n} - n| \geq \epsilon n) \leq \alpha$? Hint Prove $E(2^{2Z_n}) = 3/2n^2 + 3/2n + 1$.
- Conclusion? Is this unexpected ?

From Morris to Morris+ and Morris+++

- 2nd step: How to get a useful bound?
- Objective: to reduce the variance (the expectation is already what we want). How to do it?
  - Classic idea: do the same experience many times and average them
- Morris algorithm +
  - Morris is used to compute independent $Z_n^{(1)}, Z_n^{(2)}, \ldots, Z_n^{(K)}$
  - On demand, compute $Y_n = \frac{\sum_{i=1}^{K}(2^{Z_n^{(i)}})}{K} - 1$.
- Questions:
  - Which space complexity to implement Morris+'s algorithm?
  - What time complexity?
  - Establish the correctness: $E(2^{Y_n} - 1) = n$
  - What is the new guarantee obtained with Chebyshev? How many counters should be maintained?
- How can we do even better?
  - Morris++ = Morris+(1/3) and median
  - proof with Chernoff: If $Z_1, \ldots, Z_n$ are Independent Bernouilli rv with $p_i \in [0.1]$ and $Z = \sum Z_i$, then $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq 2\exp(\frac{-\lambda^2 E(Z)}{3})$.

How to count the number of unique visitors

2nd example: how to count the number of unique visitors

Context

- It is assumed that visitors are identified by their address ($i_k \in [1, n]$)
- We observe a flow of m visits $i_1, \ldots, i_m$ with $i_k \in [1, n]$
- How many different visitors ?
- Deterministic and trivial algorithms:
  - if n is small, if n is big... and in front of what?
  - solution in n:n bit array
  - solution in $m \log n$: we keep the whole stream!
- We will see a bit later
  - that we cannot do better with exact and deterministic algorithms
  - that we cannot do better with approximated and deterministic algorithms
- How to do if you cannot store n bits
  - but only $O(\log^k n)$ for a certain k?
- we will see that it is again possible by using both randomization and approximation.
- and that no deterministic exact or deterministic approximation can do it with this space constraint.

Idealized algorithm (1) – Flajolet Martin

We will start with an idealized algorithm (which cannot be implemented in practice).

- Let us choose a random h function from $[1, n]$ to $[0, 1]$
- Why idealized?
  - Problem 1: to store such a random function, you must define the images for in each of the n points... at least $\Omega(n)$ bits
  - Problem 2: and in addition we would have to store real values!
  - We will come back to these two problems in a moment....
  - Let us assume for now that storing such a function costs $\Theta(1)$
- How do you keep track of the number of unique visitors?
- We will keep $Z \longrightarrow \min_{i \in \text{stream}} h(i)$. Intuition?
  - If you see the same visitor k times, it won't change Z
  - If we see t different visitors, then the values taken by h split $[0, 1]$ in $t + 1$ intervals...and all should have the same size in expectation... and this size is $\frac{1}{t+1}$ including the first !
- so you should return $\frac{1}{Z} - 1$!

Idealized algorithm (2) – Flajolet Martin

Proof of correctness

- Let's prove that $E(Z) = \frac{1}{t+1}$.
- $E(Z) = \int_0^{+\infty} P(Z \geq \lambda) d\lambda$.
  - Show that $E(Z) = \frac{1}{t+1}$
  - How to continue? by calculating the variance and applying Chebychev
  - Prove that $E(Z^2) = \frac{2}{(t+1)(t+2)}$
  - There is still one foolishness not to be said.... $E(1/Z) \neq 1/E(Z)$
  - Intuition: if we can control closely Z and $\frac{1}{t+1}$, $1/Z - 1$ will be close to t
- FM+
  - Let us maintain $q = \frac{1}{\epsilon^2 \eta}$ FM instances.
  - Z_i is the value produced by FM_i
  - What to return? $Y = \frac{1}{(\sum_1^q Z_i)/q} - 1$
  - $E(\frac{\sum_1^q Z_i}{q}) = \frac{1}{t+1}$
  - $V(\frac{\sum_1^q Z_i}{q}) = \frac{t}{q(t+1)^2(t+2)} < \frac{E(Z))^2}{q}$
  - Claim 1: $P(IY - \frac{1}{t+1} I \geq \frac{\epsilon}{t+1}) \leq \eta$
  - Claim 2: $P(I\frac{1}{Y} - 1 - tI \geq \Theta(\epsilon) t) \leq \eta$
- FM++
  - choose $\eta = \frac{1}{3}$adapt ϵ, instantiate K copies of Y $Y_1, \ldots, Y_K$
  - output median$\{\frac{1}{Y_i}\}$ Ok for $K = \lceil 36 \log(\frac{1}{\delta}) \rceil$

Toward a Non Idealized Version. A crucial tool: hashing functions

- We used the set of all possible functions (too large set, too large storage for one function)
- To make it practical, we will consider a large (not too large) family of functions $\mathcal{H}$ from $[1, p] \to [1, p]$
- How to define the quality of a family $\mathcal{H}$?
- Notion of k-wise independence
    - $\forall i_1, \ldots, i_k, \forall j_1, \ldots, j_k$, $i_k \neq i_l$, and if we pick a random h function in $\mathcal{H}$, then
    - $P(h(i_1) = j_1$ and $h(i_k) = j_k)$ $1/p^k$
    - a larger k provides a "better" family
- Examples:
    1. the set of all functions from $[1, p] \to [1, p]$ is Ok.
        - What k, what storage cost?
        - $f(1) \to p$ choices,..., $f(p) \to p$ choices
        - Problem: expensive, $p \log p$ bits are necessary for one function
    2. with the polynomials $\mathcal{H}^k_{\text{poly}}$ of degree $k-1$ in F_p
        - evaluation cost? for degree k, k mult & and adds
        - independence? how many polynomials such that $(h(i_1) = j_1$ and $h(i_k) = j_k$
        - exactly one, Lagrange polynomial: $P = \sum_{r=1}^{k} \frac{\prod_{l \neq r}(X - i_l)}{\prod_{l \neq r}(i_r - i_l)} \times j_r$
        - choice? picking a function at random in $\mathcal{H}^k_{\text{poly}} \to$ choose k coefficients.
        - and thus the family $\mathcal{H}^k_{\text{poly}}$ is $k-$independent

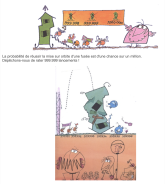
Why do we need randomization and approximation?

- Because a deterministic algorithm needs at least $\Omega(n)$ bits
- How to prove this? We assume $n = \Theta(m)$
    - Let us consider the state of the memory of the algorithm after seeing $i_1, \ldots, i_m$
    - We need to prove that there is enough information in what is stored
    - so as to differentiate 2^n distinct elements
    - Remark: you can add as many computations as you want !

    - Input X, let us denote by $C_f(X)$ the state on the memory
    - What can be computed using $C_f(X)$ (and only $C_f(X)$)?
    - we can compute $h(C_f(X))$ and $h(C_f(X), \{y\}) = C_f(X \bigcup \{y\})$
    - do it for all possible y values (visitors)...
    - If y was in the stream, then $h(C_f(X), \{y\}) = h(C_f(X))$ otherwise $h(C_f(X), \{y\}) = h(C_f(X) + 1!$
    - In $C_f(X)$, there is enough information to distinguish 2^n possible vectors (all visitors vectors)
    - and thus n bits are needed!

Why do we need randomization and approximation?

- Because a deterministic approximation algorithm (say 1.1-approx) needs at least $\Omega(n)$ bits
- Let us suppose that there exists a collection $\mathcal{C}$ of subsets of n such that
  - $|\mathcal{C}|$ is large ($\geq \exp(n/10^4)$)
  - $\forall S \in \mathcal{C}, |S| = n/100$ (sets are large)
  - $\forall S_1, S_2 \in \mathcal{C}^2, |S_1 \bigcap S_2| \leq n/2000$ (intersections are small)
- General idea
  - Let us assume that we have presented to the algorithm one of the sequences of $\mathcal{C}$
  - Then, we can find back which one!
  - just by trying exhaustively all $\#\mathcal{C}$ sequences with $C_f(X)$
  - Since we know how to differentiate exponentially many ($\exp(n/10^4)$) elements, we need $\Omega(n)$ bits



- We still need to prove that such a set $\mathcal{C}$ exists !
  - n visitors numbered from 1 to n split into $n/100$ packets of 100 visitors
  - In $S_i, \forall i$ we randomly choose one visitor per packet
  - we build $\exp(n/10^4)$ such sets S_i.
  - easy: What is their size? $n/100$
  - we need to check that $\forall i, j,\ i \neq j, |S_i \bigcap S_j| \leq n/2000$
  - How to do this ?it is enough to prove that the P(it works) is > 0
  - Why does it work ? $Y_{i,j}$ number of collisions between S_i and S_j
  - $E(Y_{i,j})$? $Pr(Y_{i,j} > n/2000)$? $Pr(\exists i, jt.q. Y_{i,j} > n/2000)$?

Non Idealized FM (1)

- Step1: find a $O(1)$-approximation $\tilde{t}$ of t in $O(\log n)$ bits, ie a constant C such that $\frac{t}{C} \leq \tilde{t} \leq Ct$ with constant probability (say $\frac{2}{3}$) **this is the subject of your homework !**

Non Idealized FM (2)

- Playing with constants, let us assume that Step1 provides a 32-approximation with probability $\frac{2}{3}$, then perform K experiments and take the median to have 32-approx with large probability
- To obtain a stronger approximation, we rely on the following technique
- let us chose g in a 2 wise family from $[n]$ to $[n]$.
  1. Imagine that we consider $\log n$ sets, with $\mathcal{S}_j$ contains the elements i of the stream s.t. $lsb(g(i)) = j$.
  2. we know $\tilde{t}$ (close to t), let us denote by Z the size of $\mathcal{S}_j$ when $2^{j+1} \simeq \tilde{t}\epsilon^2$
  3. and let consider $U = 2^{j+1}Z$ in this case
- $E(U) = 2^{j+1}E(Z) = t$, $V(U_i) = 2^{2j+2}Var(Z) \leq t2^{j+1}$
- so that (Chebychev) $P(IU - tI \geq \epsilon t) \leq \frac{t2^{j+1}}{\epsilon^2 t^2} = \frac{2^{j+1}}{\epsilon^2 \tilde{t}} \frac{\tilde{t}}{t} \leq C'$
- Then, we use several hashing functions and take the average value to obtain an error with arbitrarily small probability
- Not completely finished ! Is this algorithm implementable this time with small space ?
- No, because $\mathcal{S}_0$ is very large for instance ! But the maximum value we are expecting in "interesting" $\mathcal{S}_j$ is $\frac{t}{2^{j+1}} = \frac{\tilde{t}}{2^{j+1}} \frac{t}{\tilde{t}} \leq \frac{C}{\epsilon^2}$
- Thus, we can "only" remember the first $\frac{C}{\epsilon^2}$ is each set !
- Overall space complexity ???

Note on Non Idealized FM (3)

- Technique called Geometric sampling
- n elements in the stream, $k \leq n$ distinct elements (with respect to some property)
- Store $\log n$ sub-streams, where S_0 stores $1/2$ of the elements (distinct wrt the property), S_1 stores $1/4$ of the elements,... $S_{\log k}$ stores (close to) 1 element, $S_{\log n}$ a priori stores nothing if $k << n$
- Suppose that when there are l elements in one of the sets, we can find a good estimation of k where typically l is of order $\frac{1}{\epsilon^2}$
- Then, we bound all the sets to store less than $10l$ elements (they are useless after that)
- if we have a constant approximation of k (obtained elsewhere), then we know in which set we should look at.

Finding Similar Itemsets

General Idea

- 2 type of difficulties related to
  - the number of objects: N objects $\longrightarrow N^2$ comparisons
  - the objects themselves : large texts,...
- Applications
  - pages with a lot of text in common (mirror sites, approximate mirror)
  - plagiarism (today)
  - group news that deal with the same event
  - Amazon, Netflix: users with the same taste
  - dual: Amazon, Netflix: products with the same fans
- we will concentrate on texts, the first step only is application specific
  - Order of magnitude: 10^6 documents, size a few MB not huge (a few TB)
  - Distributed over a datacenter: large number of nodes $10^3 - 10^6$ nodes
- Two goals:
  - avoid moving data between the nodes (small and shared bandwidth)
  - avoid performing 10^{12} comparisons: both for time and data movements

Techniques

- 3 steps
  1. Shingling : conversion of a large text into a (large) set
  2. Min-hashing: assign to each text a (small) similarity-preserving signature
  3. Locality Sensitive Hashing: detect suspect pairs by collision detection
- randomized approximation algorithm $\longrightarrow$ errors: false positive and false negative
- what is crucial in our context ? Complexity: we want to deal with linear complexity algorithms only !
- Remark: we assume that the output is (at most) of linear size (otherwise, we have no chance !)

Shingling

- $k-$shingle
  - sequence of k successive characters in the text
  - $\{abcab\}$ et $k = 2$?
- Observation (admitted) close texts $\longrightarrow$ a lot of shingles in common
- A text: represented by the set of shingles it contains
- How many shingles ? with $k = 10$
- The data structure should enable to perform comparisons easily...
  - 30^{10} shingles $= 2^{49}$
  - size if stored as a vector of bits 70 TB
  - what happens in practice? Solution? shingles $\longrightarrow$ tokens
  - first use of hashing functions: adapt the size, control collisions

Min-hashing

- Each text is associated to a set of items
- We need to define a similarity between sets.
- Jaccard Similarity: $Sim(C_1, C_2) = \frac{C_1 \cap C_2}{C_1 \cup C_2}$
- $d(C_1, C_2) = 1 - Sim(C_1, C_2)$ is a distance (proof later)
  - one vector per document
  - one row per token token
- How to compute the similarity between two documents ?
- Problems:
  - we do not want to deal with N^2 pairs
  - we cannot centralize all N pairs at a single node

Minhashing

- Let us suppose that (C_1, C_2) are stored at the same place
- Goal: build a small similarity preserving signature for each document.
- General Idea: build a random game whose expected value (to win) is $Sim(C_1, C_2)$
- Do we really need to have (C_1, C_2) at the same place to play the game ?
- Do we really need permutations ?
- How many hash functions do we need in order to obtain a good precision ?

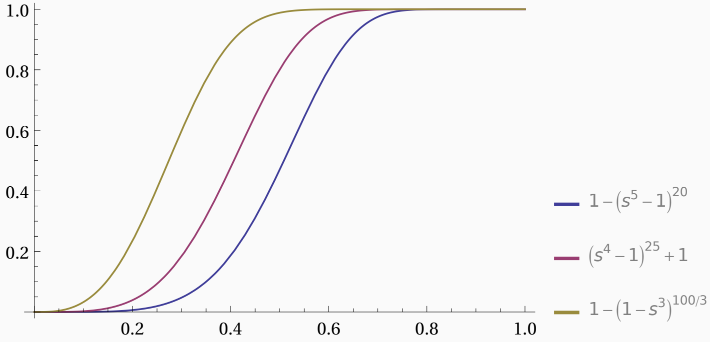
Locality Sensitive Hashing

- So far: we have a very compact summary of each document (250×4B integers$= 1$kB)
- Last step: given a suspicion threshold $s \in [0, 1]$, return all pairs (C_1, C_2) such that $Sim(C_1, C_2) > s$
- Without doing all comparisons!
- Order of magnitude:
  - 10^6 documents $\longrightarrow 1GO$, Ok en mémoire
  - 10^{12} comparisons with 10^{-6}s per comparison 12 days)-;
- Goal: go from quadratic to linear complexity
- using hash functions again and collision detection
- now, we want close vectors to collide, and distant vectors not to collide

locally sensitive hashing

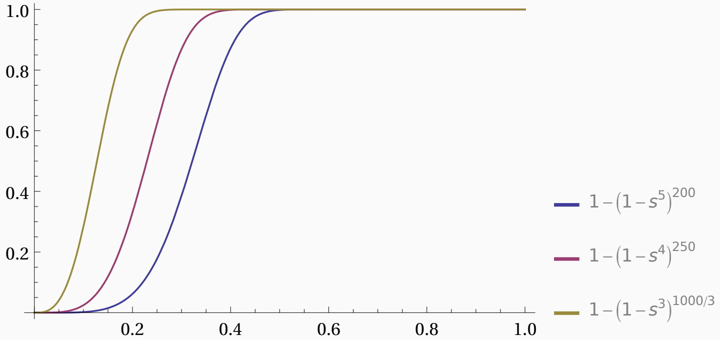
- split the summary (250 integers) into b blocks of size r ($rb = 250$)
- let h_k be the hash function associated to the k-th block
- collision: (almost) only if both vectors coincide on this block. Solution: use a large number of buckets (with respect to 10^6) $\longrightarrow 10^9$ is Ok in practice, very few false positive.
- a pair (C_1, C_2) is suspicious if $\exists k, h_k(B_k^1) = h_k(B_k^2)$ where B_k^i is the $k-$th block of C_i
- what happens ?
    - if r is too small ? too many false positive
    - if r is too big ? we will miss similar itemsets and get false negative
- Given r (and thus b) and $s = Sim(C_1, C_2)$, what is the probability that a collision occurs ?

with $N = 100$ and $r = 3, 4, 5$



false positive ? false negative ?

with $N = 1000$ and $r = 3, 4, 5$



Practical implementation ?

- distributed documents.
- keep everything local (until the computation of signatures)
- keep everything local (compute the hashing of each block) 1 document + 1 block $\longrightarrow$ 4B !
- gather all information related to one block number to the same node (4B + 1B for the index) $\longrightarrow$ 5MB
- detect all suspicious pairs (very few and send them to a specific node)...
- very few communications !

Locality Sensitive Hashing and Nearest Neighbors Search

($k-$) nearest neighbors

- Metric space with distance, set P of points
- preprocessing allowed on P
- Query: given a point, find its (k) closest neighbor(s)
  - example for spam classification: start with a huge annotated emails
  - one word = one item
  - return the k closest emails, majority vote to determine if it is spam or not
- Approach # 1: no preprocessing, just look through all possible items
  - space $O(dn)$
  - query $O(dn)$
- Approach # 2: if d=1
  - space $O(n)$ just keep the boundaries
  - query $O(\log n)$ just a basic binary search
- Approach # 3: if d=2
  - Voronoï diagrams: space $O(n)$ and computing cost $O(n \log n)$
  - query time easy (locate the cell)
  - As dimension increase, the description increases exponentially with d
- all exact (known) approaches in high dimension either have
  - exponential space $O(n^d)$
  - or exponential query time !
  - (same for $kd-$trees)
  - in very. large dimension, the naive algorithm is not that bad !

$c-$approximate nearest neighbors

- Given a set of P points, construct a data structure such that
    - on query q, we return p in P such that
    - $d(p, q) \leq c \min_{p' \in P} \ d(p', q)$
- (r_1, r_2)PLEB problem: point location in equal balls
    - given a set P of points and r_1, r_2
    - construct a data structure to answer as follows
    - If $\exists p \in P$ st $d(p, q) \leq r_1$, return YES and any $p' \in P$ s.t. $d(p', q) \leq r_2$
    - If there is no $p \in P$ st $d(p, q) \leq r_2$, return NO
    - elif don't care what algorithm returns

Locality Sensitive Hashing (Indyk, Motwani)

- Usually, we want hashing functions to "shuffle" items as much as possible
  - When writing $P(h(i_1) = j_1 \ \& \ h(i_2) = j_1) = 1/p^2$, we say that the distance between images should not depend on the distance between initial points
  -
- Here, we want to detect "collisions"
  - we want close points to have a high probability to collide
  - we want distant points to have a low probability to collide
  - just as in the context of plagiarism.
- General idea
  - hash items into many different buckets (with different functions)
  - declare that there is a collision if two items fall into of the buckets
- Formal definition $\mathcal{H}$ a family of hash function $U \longrightarrow S$
- (where U is the set of points, S the set of buckets)
- is said to be (r_1, r_2, p_1, p_2) locality sensitive if
  - If $d(p, q) \leq r_1$, then $P_{h \in \mathcal{H}}(h(p) == h(q)) \geq p_1$ and
  - If $d(p, q) \geq r_2$, then $P_{h \in \mathcal{H}}(h(p) == h(q)) \leq p_2$
- of course $r_1 < r_2$, $p_1 > p_2$

Example

- Let $H^d = \{0,1\}^d$ equipped with Hamming distance (number of different coordinates)
- Let $\mathcal{H} = \{h_i, \forall i$, where $h_i(b_1, \ldots, b_d) = b_i\}$
- $\mathcal{H}$ is $(r, cr, 1 - r/d, 1 - cr/d)$ locality sensitive
    - if p, q are at distance at most r, they have at least $(d - r)$ coordinates in common and thus a probability at least $1 - r/d$ to be hashed similarly,
    - if p, q are at distance at least cr, they have at most $(d - cr)$ coordinates in common and thus a probability at most $1 - cr/d$ to be hashed similarly.

Master Theorem of LSH

Theorem
Suppose $\exists(r_1, r_2, p_1, p_2)$*-LSH family, then there is an algorithm for* (r_1, r_2)*-PLEB with answer queries with constant probability (it might be wrong), and that uses space* $O(dn + n^{1+\rho})$ *and query time* $O(n^{\rho})$ *(evaluation of hash functions), where* $\rho = \frac{\log(1/p_1)}{\log(1/p_2)}$ *(complexity decreases when* ρ *decreases, ie when* $p_2 << p_1$*).*

Sketch of the proof — Algorithm

- let (k, l) be parameters (t.b.d. later), let G be a family of hash functions from U to S^k (new buckets), $g(p) = (h_{g_1}(p), \ldots, h_{g_k}(p))$ each h_{g_i} being randomly chosen in H.
- Preprocessing:
  - (1) choose $g_1, \ldots, g_l$ (other parameter) independently from G
  - (2) for each $p \in P$, store $g_1(p), \ldots, g_l(p)$
- On query
  - (1) search the points of P in $g_1(q), \ldots, g_l(q)$, but stop after the first $2l$ points in (the unlikely) case there are more than $2l$.
  - (2) If there is one point p such that $d(p, q) \leq r_2$ return it and return YES, otherwise return NO

Proof (continued)

- Intuition (1): if q and p are "close", then one the g_i will send them into the same k-bin.
- Intuition (2): it is unlikely that they are $2l$ distant (useless) points in the set (that would prevent to find the useful point)
- (2) There are at most $2l-1$ points st $d(p,q) > r_2$ and $\exists j,\ g_j(p) = g_j(q)$ with constant probability
  - Let $k = \log_{1/p_2} n$, what is the expected number of points st (2) holds ?
  - If $d(p,q) > r_2$ then for each h, the probability of collision is at most p_2
  - so the expected number of times p is written in any g_j is p_2^k, and the expected number of times it is written for a given g is lp_2^k
  - and the number of "bad points" written for g is therefore at most $nlp_2^k = l$
  - Markov says $P(X > \lambda) < E(X)/\lambda$ so $P(\text{more than } 2l \text{ bad points}) \leq 1/2$
- (1) If $p \in P$ with $d(p,q) \leq r_1$, then $\exists j,\ g_j(p) = g_j(q)$ with constant probability
  - If $d(p,q) \leq r_1$ then for each h, the prob of collision for h is at least p_1
  - the probability of collisions in one bucket is p_1^k
  - the probability of a collision in at least one of the l buckets is $1-(1-p_1^k)^l$ $= 1-(1-n^\rho)^l$
  - choice of l ? if we set $l = n^\rho$ then the probability is at least $1-1/e \simeq 0.63$.
- to increase the probability, use the classical and tricks
- Check space and time complexities

Conclusion on sketching/streaming/compression

- Goal: data flow X and a function to evaluate f
- streaming: maintain a summary $C_f(X)$ enough to compute $f(X) =\simeq g(C_f(X))$
  - Solution: Use approximation randomized algorithms
  - $\forall \epsilon, \delta,\ Pr(\text{relative error} \geq \epsilon) \leq \delta$
  - enough (and often necessary) to change space complexities (from $\log n \to \log\log n$, $n \to \log n$, from n^d to $n^\rho d$)
  - at the price of sometimes large constants
  - but constants are pessimistic
  - and very small ϵ and α are not always required (plagiarism)
- General Idea:
  - Do less communications (same a lot of energy, time)
  - But more local computations (cheap)
  - crucial for IoT and datacenters
- Method:
  - Find an estimator Z tel que $E(Z) =$ what we want to estimate
  - go for $+$ and $++$ versions to control the probability
  - hash functions are a very powerful and versatile tool:
    - to shuffle potentially correlated entries (Unique Visitors)
    - to adapt the size of sets (Plagiarism)
    - to create short summaries (Min-Hashing)
    - to detect close items (Locality Sensitive Hashing)